# On differential geometry

Dr Yann Golanski[1] and Dr Chris Wood [2]

2003-2004

[1]yg2@york.ac.uk
[2]cmw4@york.ac.uk

# Contents

# List of Tables

# List of Figures

# Chapter 1

# Introduction

**What is Differential Geometry?** Differential geometry is the study of curves and surfaces as found by differential calculus. Gauss was the first to define the curvature of a surface at a point and to formulate said curvature in terms of partial differentials. Riemann later extended differential geometry to any type of space in any number of dimensions.

**Why is Differential Geometry Interesting?** Differential geometry deals with curves and surfaces. Historically, it was developed to answer Euler's fifth postulate – the parallel postulate. However, work by Gauss and Riemann soon revealed a much richer geometry non-euclidean in nature. The latter became a foundation of the theory of general relativity [2]. In turn this led to field theory. Table 1.1 describe equivalence relationships and invariance for different classes of objects.

| **Object** | **Equivalent relation** | **Invariants** |
| --- | --- | --- |
| Vector Space | Isomorphism | Dimension |
| Smooth Curves in Space | Congruence | Curvature and Torsion |
| Smooth Surfaces in Space | Congruence | First and Second Fundamental Forms |

Table 1.1: Invariance classification of some geometric objects.

**Reference material** The recommended textbook for this course is *J. McCleary*'s Geometry from a differential viewpoint [3]. Another book that students may find useful is *A. Pressley*'s Elementary differential geometry [4]. The University of York library has a whole section on differential geometry labeled **S 6.7**. It includes *M. doCarmo*'s Differential geometry of surfaces and curves [1] which is available in the library under S6.7CAR.

**Lectures plan** There will be two lectures given by Dr Golanski every week on Tuesdays and Thursdays and one examples class given by Dr Wood on Fridays. The breakdown is as follows:

1. **Section I**: *On the geometry of space curves*
   - **Lecture 1**: Smooth paths, regularity, arc length.
   - **Lecture 2**: Reparametrisation, curvature.
   - **Lecture 3**: Planar paths, signed curvature, the fundamental theorem of plane curves.
   - **Lecture 4**: Torsion, Frenet frame.
   - **Lecture 5**: Frenet formulas, congruence, fundamental theorem of space curves.
2. **Section II**: *On smooth surfaces*
   - **Lecture 6**: Charts and Atlases.
   - **Lecture 7**: Differentials of smooth maps, the chain rule, tangent planes.
   - **Lecture 8**: Regular value theorem.
   - **Lecture 9**: Smooth mappings of smooth surfaces, diffeomorphisms.
3. **Section III**: *On the geometry of surfaces*
   - **Lecture 10**: Riemannian metric, $(E, F, G)$.
   - **Lecture 11**: Arc lengths, angles, areas.
   - **Lecture 12**: Local isometries, the $(E, F, G)$-lemma.
   - **Lecture 13**: Shape operator, normal curvatures, self-adjointness and principal curvatures.

- **Lecture 14**: Euler's theorem, Gauss and mean curvature, elliptic, hyperbolic and parabolic points.
- **Lecture 15**: The second fundamental form, $(e,\ f,\ g)$.
- **Lecture 16**: Theorema Egregium, rigid motions and congruence, congruent surfaces have conjugate shape operators.
- **Lecture 17**: $(e,\ f,\ g)$-lemma, Gauss-Weingarten formulas.
- **Lecture 18**: Bonnet's theorem.

**Additional help** Students are encouraged to either email or see either Dr Golanski (yg2@york.ac.uk or in G/005) or Dr Wood (cmw4@york.ac.uk or G/128) if they have any queries about the course.

**Comments** The notes were typed in LaTeXby Dr Golanski from Dr Wood's course. All errors herein are his and not Dr Wood. Please report any such errors as soon as you find them.

# Chapter 2

# On the geometry of space curves

## 2.1 Smooth paths

**Definition 2.1.0.1.** A smooth path is a smooth map $p : I \to \mathbb{R}^3$, where $I = (a, b) \in \mathbb{R}$ allowing $a$ and $b$ to be $\infty$.

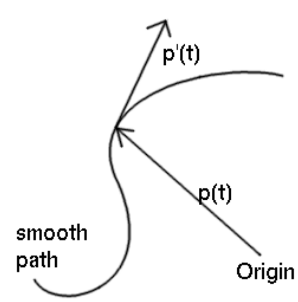



Figure 2.1: A smooth path and its velocity vector.

Since the path is smooth, it is infinitely differentiable. If

$$p(t) = \big(x(t), y(t), z(t)\big) \tag{2.1}$$

then

$$\frac{dp(t)}{dt} = \Big(\frac{dx(t)}{dt}, \frac{dy(t)}{dt}, \frac{dz(t)}{dt}\Big) \tag{2.2}$$

or

$$p'(t) = \big(x'(t), y'(t), z'(t)\big) \tag{2.3}$$

is the **tangent vector** also known as the **velocity vector**.

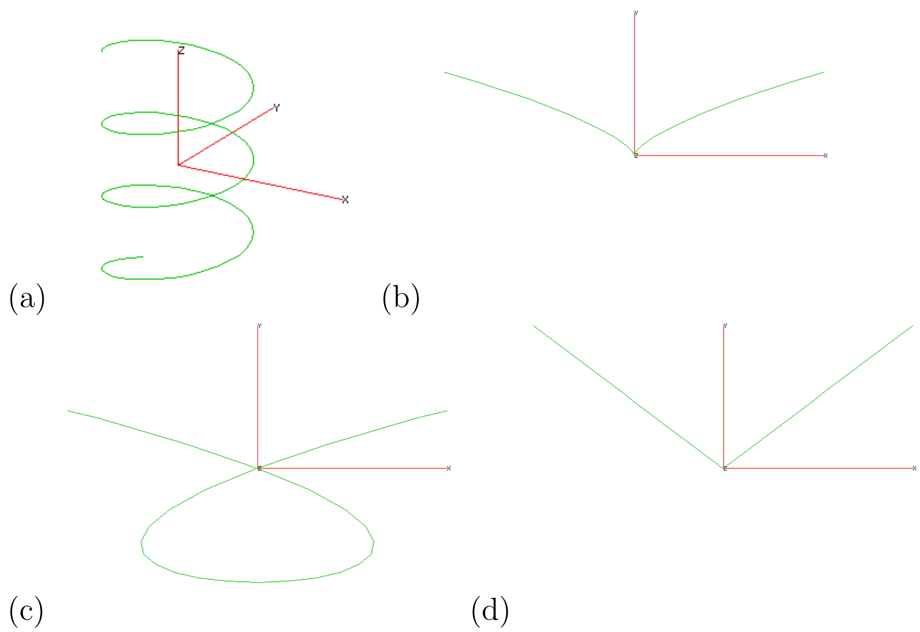



Figure 2.2: Examples: (a) helix, (b) $p(t) = (t^3, t^2, 0)$, (c) $p(t) = (t^3 - 4t, t^2 - 4, 0)$, (d) $p(t) = (t, |t|, 0)$

**Example 1.** *Let* $p(t) = (a\cos(t), a\sin(t), bt)$ *and* $a, b > 0$*. This path describe a right handed helix on the cylinder* $x^2 + y^2 = a^2$*.*

**Example 2.** *Let* $p(t) = (t^3, t^2, 0)$*. Note that* $p'(t) = (3t^2, 2t, 0)$ *so that* $p'(0) = (0, 0, 0)$*.*

**Example 3.** *Let* $p(t) = (t^3 - 4t, t^2 - 4, 0)$ *be a* ***crunodal*** *cubic. A crunode is a singular point at which a curve intersects itself such that there are two different tangents at the point. Here,* $p'(t) = (3t^2 - 4, 2t, 0) \neq 0$ *but* $p'(2) = p'(-2) = 0$*.*

**Example 4.** *Let* $p(t) = (t, |t|, 0)$ *which is not smooth at* $t = 0$*. Note that*

$$q(t) = \begin{cases} (e^{\frac{-1}{t^2}}, e^{\frac{-1}{t^2}}, 0) \ \textit{if}\ t \geq 0 \\ (-e^{\frac{-1}{t^2}}, e^{\frac{-1}{t^2}}, 0) \ \textit{if}\ t \leq 0 \end{cases}$$

*is a smooth path with the same image as $p(t)$ but $q^n(0) = 0$ for $n = 1, 2, 3, ...$*

**Definition 2.1.0.2.** A point $p(t_0)$ is regular if $p'(t_0) \neq 0$

**Definition 2.1.0.3.** A point $p(t_0)$ is singular or critical if $p'(t_0) = 0$

At regular points there's a well defined tangent line as seen in figure 2.1. A smooth path is **regular** if all its points are regular.

## 2.2 Arc Length

**Definition 2.2.0.4.** The arc length from $p(t_0)$ to $p(t)$ for $t_0 < t$ is defined as

$$s(t) = \int_{t_0}^{t} |p'(u)| du \tag{2.4}$$

Note that $s(t)$ is also meaningful when $t \leq t_0$ when of course $s(t) \leq 0$. In fact, $s(t)$ is a smooth function on $I$.

**Example 5.** *Let $p(t) = (a\cos t, a\sin t, bt)$ then $p'(t) = (-a\sin t, a\cos t, b)$. The arc length from the origin is given by $s(t) = \int_{t_0}^{t} \sqrt{a^2 + b^2} dt = \sqrt{a^2 + b^2} t$. In particular the arc length of one pitch is $s(2\pi) = 2\pi\sqrt{a^2 + b^2}$ and taking $b = 0$ gives the circumference of circle of radius $a$ is $2\pi a$.*

**Definition 2.2.0.5.** If $s(t) = t - t_0$ then $p(t)$ is called an arc length parametrisation

In this case,

$$t - t_0 = \int_{t_0}^{t} |p'(u)| du \Rightarrow 1 = |p'(u)| \tag{2.5}$$

Conversely if

$$|p'(u)| = 1 \Rightarrow \int_{t_0}^{t} |p'(u)| du = t - t_0 \tag{2.6}$$

Thus if $p(t)$ is an arc length parametrisation then $|p'(t)| = 1$. So we call such $p(t)$ a **unit speed path** and write it as $p(s)$ instead of $p(t)$.

If $p(t)$ is regular then $s'(t) = |p'(t)| \neq 0$, hence $s(t)$ has a smooth inverse. We can write $t = t(s)$. Then $p(t) = p(t(s)) = \tilde{p}(s)$ and we have reparametrised the path by the arc length.

Note that

$$\tilde{p}'(s) = \frac{dp}{dt}\frac{dt}{ds} = \frac{\frac{dp}{dt}}{\frac{ds}{dt}} = \frac{p'(t)}{s'(t)} = \frac{p'(t)}{|p'(t)|} \tag{2.7}$$

thus

$$|\tilde{p}'(s)| = 1 \tag{2.8}$$

Which is as it should from the above.

**Example 6.** *A helix is defined by* $p(t) = (a\cos(t), a\sin(t), bt)$ *and* $a, b > 0$ *and thus* $s(t) = ct$ *where* $c = \sqrt{a^2 + b^2}$ *since we are measuring the arc length from the origin. Hence* $t(s) = \frac{s}{c}$ *and an arc length reparametrisation is* $\tilde{p}(s) = \big(a\cos(s/c), a\sin(s/c), b(s/c)\big)$*.*

**Definition 2.2.0.6.** If $q(u) = p\big(t(u)\big)$ for some smooth invertible function $t(u)$ say that $q(u)$ is a reparametrisation of $p(t)$.

Let $\tilde{p}(t) = p(-t)$ then $\tilde{p} : (-b, -a) \to \mathbb{R}^3$ is called the **opposite** (or **reverse**) path.

**Definition 2.2.0.7.** If $t'(u) > 0$ then $p(t)$ and $q(u)$ have the same orientation.

**Definition 2.2.0.8.** If $t'(u) < 0$ then $p(t)$ and $q(u)$ have the opposite orientation.

**Example 7.** *Suppose* $p(s)$ *is unit speed and* $q(u) = p\big(s(u)\big)$ *is a unit speed reparametrisation. Then* $q'(u) = p'\big(s(u)\big)s'(u)$ *so* $s'(u) = \pm 1$*. Thus* $s(u) = \pm u + c, c \in \mathbb{R}$*.*

*Therefore if* $p(t)$ *is regular with unit speed reparametrisation* $\tilde{p}(s)$ *then any other unit speed reparametrisation looks like:*

$$\begin{cases} \tilde{p}(s + c), \textit{same orientation} \\ \tilde{p}(-s + c), \textit{opposite orientation} \end{cases} \quad \textit{for some } c \in \mathbb{R}$$

## 2.3 Curvature

Let $p(s)$ be a unit speed path.

**Definition 2.3.0.9.** $T(s) = p'(s)$ is the unit tangent vector

**Definition 2.3.0.10.** $K(s) = T'(s) = p''(s)$ is the curvature vector.

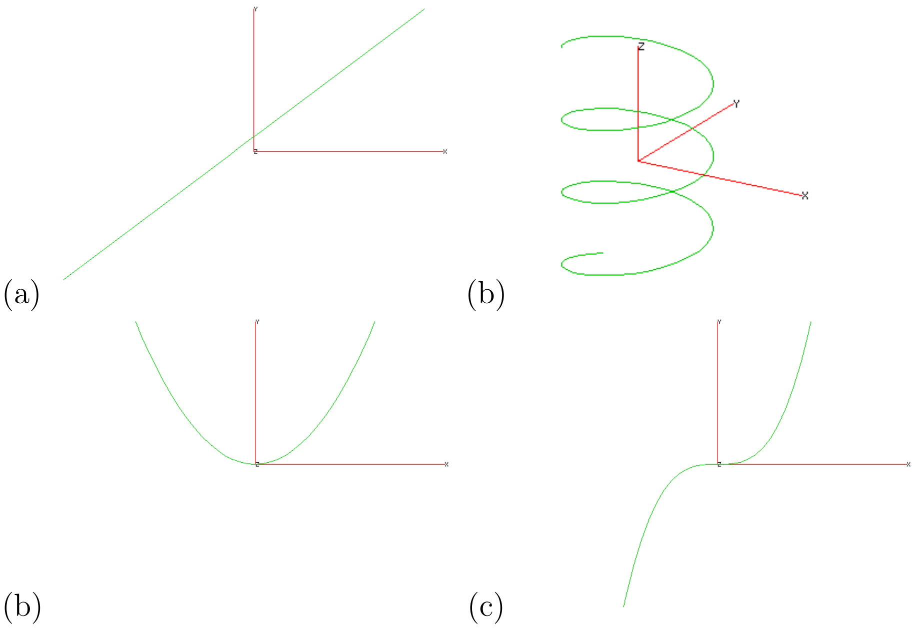



Figure 2.3: Examples: (a) straight line, (b) helix, (c) parabola, (d) Cubic

Note that $K \cdot T = 0$, that $\kappa(s) = |K(s)| \geq 0$ and that the curvature $\kappa(s)$ measures rate of change of direction.

**Example 8 (Straight lines).** *Let $p(s) = (s, as + b, 0)$ where $a$ and $b$ are constant vectors then $p'(s) = (1, a, 0)$ and $p''(s) = (0, 0, 0)$ so $\kappa(s) = 0$. Conversely, if the curvature is zero, then $p(s)$ describes a straight line.*

**Example 9 (Helices).** *Let*

$$\begin{aligned} p(s) &= \Big(a\cos\big(\frac{s}{c}\big), a\sin\big(\frac{s}{c}\big), b\frac{s}{c}\Big) \quad \textit{where} \quad c = \sqrt{a^2+b^2} \\ T(s) &= \Big(-\frac{a}{c}\sin\big(\frac{s}{c}\big), \frac{a}{c}\cos\big(\frac{s}{c}\big), \frac{b}{c}\Big) \\ K(s) &= \Big(-\frac{a}{c^2}\cos(\frac{s}{c}), -\frac{a}{c^2}\sin(\frac{s}{c}), 0\Big) \\ \kappa(s) &= \frac{a}{c^2} = \frac{a}{a^2+b^2} = const \end{aligned}$$

*In particular if $b = 0$ (circle of radius $a > 0$) then $\kappa(s) = 1/a$.*

**Definition 2.3.0.11.** If $\kappa(s) \neq 0$ then $\frac{1}{\kappa(s)}$ is the radius of curvature at $p(s_0)$.

Let $p(t)$ be any regular path.

**Definition 2.3.0.12.** $\kappa(t)$ (respectively $K(t)$) is the curvature (respectively curvature vector) of any unit speed reparametrisation at $s = s(t)$.

Note that by example 7, those are well defined.

**Example 10 (Straight lines).** $\kappa(t) = 0$

**Example 11 (Helix).** $\kappa(t) = \frac{a}{a^2+b^2}$

**Example 12 (Parabola).** *Let* $p(t) = (t, \frac{1}{2}t^2, 0)$ *then measure the arc length from the origin* $s(t) = \int_0^t \sqrt{1+u^2}du = ... = \frac{1}{2}\Big(\sinh^{-1}(t) + t\sqrt{1+t^2}\Big)$ *then get* $t(s)$*. This is very impractical and tedious.*

**Example 13 (Cubic).** *Let* $p(t) = (t, \frac{1}{3}t^3, 0)$ *then measure the arc length from the origin as above to give* $s(t) = \int_0^t \sqrt{1+u^4}du$ *which is an elliptic integral.*

However, because $\kappa(t)$ and $K(t)$ involve only derivatives of $p\big(t(s)\big)$ the chain rule and the inverse function theorem can be used to show

**Theorem 2.3.0.1.** *The curvature* $\kappa$ *can be written as:*

$$\kappa(t) = \frac{|p'(t)|^2 p''(t) - \big(p'(t)\cdot p''(t)\big)p'(t)}{|p'(t)|^4} = \frac{|p'(t)\times p''(t)|}{|p'(t)|^3} \tag{2.9}$$

*Proof.* See exercise 5 on sheet I. □

Note that for any $a, b \in \mathbb{R}^3$ the following holds true $|a\times b|^2 = |a|^2|b|^2\sin^2\Theta = |a|^2|b|^2 - (a\cdot b)^2$. It is now a lot easier to deal with parabolic and cubic paths as in the last two examples.

**Example 14 (Once more a parabola).** *Let* $p(t) = (t, \frac{1}{2}t^2, 0)$ *therefore*

$$\begin{aligned} p'(t) &= (1, t, 0) \\ p''(t) &= (0, 1, 0) \\ |p'(t)|^2 &= 1 + t^2 \\ |p''(t)|^2 &= 1 \\ p'(t)\cdot p''(t) &= t \end{aligned}$$

*and hence*

$$\kappa(t) = \frac{1}{(1+t^2)^{3/2}}$$

*Note that $\kappa(t)$ is maximum when $t - 0$.*

**Definition 2.3.0.13.** A point where $\kappa'(t) = 0$ is called a **vertex** (plural vertices).

**Example 15 (Once more a cubic).** *Let $p(t) = (t, \frac{1}{3}t^3, 0)$ therefore*

$$\begin{aligned} p'(t) &= (1, t^2, 0) \\ p''(t) &= (0, 2t, 0) \\ |p'(t)|^2 &= 1 + t^4 \\ |p''(t)|^2 &= 4t^2 \\ p'(t) \cdot p''(t) &= 2t^3 \end{aligned}$$

*and hence*

$$\kappa(t) = \frac{2|t|}{(1+t^4)^{3/2}}$$

**Definition 2.3.0.14.** A point where $\kappa(t) = 0$ is called an **inflection**.

## 2.4 Planar curves

### 2.4.1 Definition

From now on, assume that $p(s) = \big(x(s), y(s)\big)$. Let $N(s)$ be the unit vector such that $\big(T(s), N(s)\big)$ is a positively oriented orthonormal basis of $\mathbb{R}^2$. Note that $T(s) = \big(\cos\psi(s), \sin\psi(s)\big)$ and $N(s) = \big(-\sin\psi(s), cos\psi(s)\big)$ as seen in figure 2.4. Since $T \cdot T' = 0$ we have $T'(s) = \kappa(s)N(s)$ for some $\kappa(s) \in \mathbb{R}$.

**Definition 2.4.1.1.** The smooth function $\kappa : s \mapsto \kappa(s)$ is called the signed curvature (or directed curvature).

Note that since $|\kappa| = |T'|$, $|\kappa|$ is the curvature as defined previously. The convention is that $\kappa$ always denotes the signed curvature. From now on, assume that the curvature is always directed unless specified otherwise.

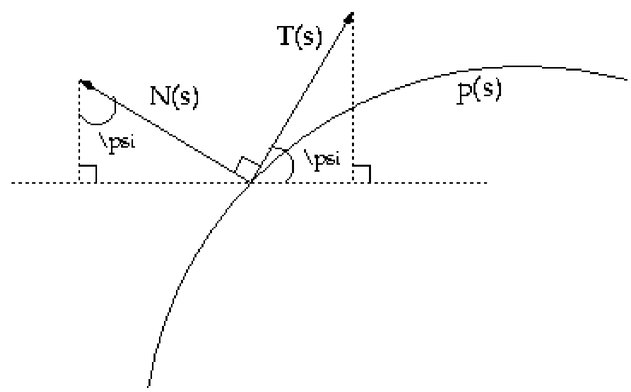



Figure 2.4: A planar curve and its normal and tangent vectors.

### 2.4.2 Features

#### Direction

Since $T' = (-\sin(\psi)\psi', \cos(\psi)\psi')$ we see that $\kappa = \psi'$ hence if $\kappa(s) > 0$ if the path curves **towards** $N(s)$ and $\kappa(s) < 0$ if the path curves **away from** $N(s)$.

#### Orientation

If the orientation of the path is reversed, expect $\kappa$ to change sign. Indeed if $\tilde{p}(s) = p(-s)$ then $\tilde{T}(s) = -T(-s)$ and $\tilde{N}(s) = -N(-s)$. Hence $\tilde{\kappa}(s) = \tilde{T}'(s) \cdot \tilde{N}(s) = -T'(s) \cdot N(-s) = -\kappa(-s)$

#### "Magic formula"

Since $T = (x', y')$, $T' = (x'', y'')$ and $N = (-y', x')$ so $\kappa = T' \cdot N = x'y'' - x''y'$. Note that this is only correct for **unit speed paths**. Now let $p(t)$ be any regular planar path.

**Definition 2.4.2.1.** $\kappa(t)$ is the signed curvature of any unit speed reparametrisation with the **same orientation** at $t = t(s)$

The Magic formula is:

$$\kappa = \frac{x'y'' - x''y'}{|p'|^3} \tag{2.10}$$

The proof can be found on page 68 of [3].

**Example 16.** *Cubic:* $p(t) = (t, \frac{1}{3}t^3)$, $p'(t) = (1, t^2)$ *and* $p''(t) = (0, 2t)$ *hence* $\kappa = \frac{2t}{(1+t^4)^{\frac{3}{2}}}$.

**Example 17 (Semi-cubic parabola).** *Let* $p(t) = (\frac{1}{3}t^3, \frac{1}{2}t^2)$, $p'(t) = (t^2, t)$ *and* $p''(t) = (2t, 1)$ *hence*

$$\kappa(t) = \frac{-t^2}{|t|^3(1+t^2)^{\frac{3}{2}}} = \frac{-1}{|t|(1+t^2)^{\frac{3}{2}}} < 0$$

**Example 18 (An extra example).** $p(t) = (\frac{1}{4}t^4, \frac{1}{2}t^2 + \frac{1}{3}t^3)$, $p'(t) = (t^3, t + t^2)$ *and* $p''(t) = (3t^2, 1 + 2t)$ *hence*

$$\kappa = \frac{-t^3}{|t|^3} \frac{2+t}{\left((1+t)^2 + t^4\right)^{\frac{3}{2}}} = \begin{cases} -\frac{2+t}{\left((1+t)^2+t^4\right)^{\frac{3}{2}}} & , \textit{if } t < 0 \\ +\frac{2+t}{\left((1+t)^2+t^4\right)^{\frac{3}{2}}} & , \textit{if } t > 0 \end{cases}$$

*Note the following two points:*

- *Close to the origin,* $\lim_{(t \to 0, t<0)} \kappa = 2$ *and* $\lim_{(t \to 0, t>0)} \kappa = -2$.
- *There is an inflection at* $t = -2$, *with* $\kappa(t) > 0$ *for* $t < -2$.

Suppose $\kappa(s)$ is given. Since $\kappa = \psi'$ it follows that $\psi = \int \kappa(s)ds$. Changing the constant of integration produces a **rotation** of the curve. Since $T(s) = p'(s)$, it follows that $p(s) = \left(\int \cos\psi(s)ds, \int \sin\psi(s)ds\right)$. Changing the constant of integration produces a **translation** of the curve.

**Definition 2.4.2.2.** A rotation of the plane, followed by a translation, is called a **proper rigid motion**.

**Theorem 2.4.2.1 (Fundamental theorem of plane curves).** *If* $\kappa(s)$ *is any smooth function, there exists a smooth planar path* $p(s)$*, parametrized by arc length, whose signed curvature is* $\kappa(s)$*. Any two such paths* ***differ by a proper rigid motion****.*

*Proof.* This theorem is already proved by the paragraph introducing the definition of a proper rigid motion. □

**Example 19.** *Find a planar path with* $\kappa(s) = \frac{1}{1+s^2}$.

*We have* $\psi(s) = \tan^{-1}(s)$ *by setting the constant of integration to zero. Thus*

$$T(s) = (\cos\psi, \sin\psi) = \left(\frac{1}{\sqrt{1+s^2}}, \frac{s}{\sqrt{1+s^2}}\right)$$

*So*

$$x(s) = \int \frac{ds}{\sqrt{1+s^2}} = \int du \text{ if } u = \sinh^{-1}(s)$$

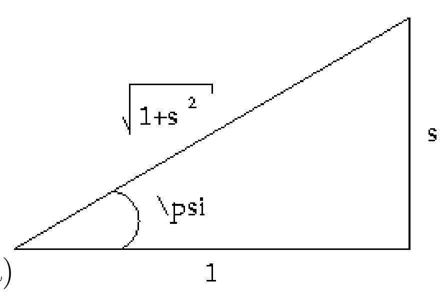
*such that* $s = \sinh u$ *and* $ds = \cosh u du$. *Thus*

$$y(s) = \int \frac{sds}{\sqrt{1+s^2}} = \sqrt{1+s^2}$$

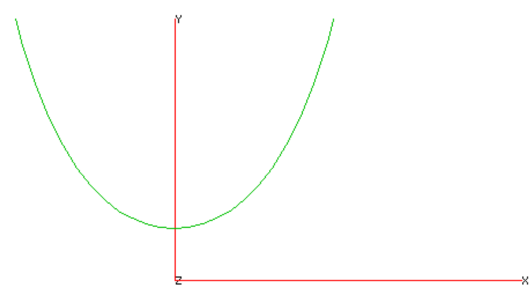
*by setting all the constant of integration to zero once more. To identify the curve, set* $t = sinh^{-1}(s)$ *then*

$$\begin{aligned} p(s) &= \left(\sinh^{-1} s, \sqrt{1+s^2}\right) \\ &= \left(t, \sqrt{1+\sinh^{-1}(s)^2 t}\right) \\ &= \left(t, \cosh(t)\right) \end{aligned}$$

*which is a* ***catenary***.




(a)




(b)

Figure 2.5: (a) Angles on a triangle and (b) a catenary.

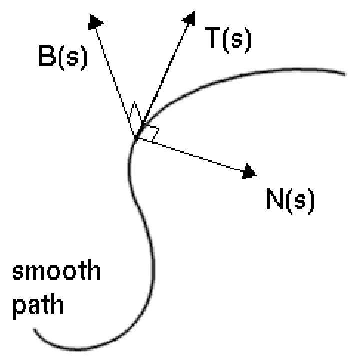



Figure 2.6: The binormal vector.

## 2.5 Torsion: back into space

### 2.5.1 Torsion

The example of a helix and a circle show that curvature does **not** determine the shape of a space curve. Therefore there is a need for another invariant measure. Suppose that $\forall s \in \mathbb{R}$, $p(s)$ is a unit speed path with $\kappa(s) \neq 0$ (IE $p''(s) \neq 0$) then we can normalise the curvature vector and define the following.

**Definition 2.5.1.1.** The **principle normal vector** $N(s)$ is defined as

$$N(s) = \frac{p''(s)}{|p''(s)|} = \frac{T'(s)}{\kappa(s)} \tag{2.11}$$

Thus

$$T'(s) = \kappa(s)N(s) \tag{2.12}$$

**Definition 2.5.1.2.** The **osculating plane** is the plane spanned by $\Big(T(s), N(s)\Big)$.

Let $B(s)$ be the unique unit vector such that $\Big(T(s), N(s), B(s)\Big)$ is a right handed orthonormal basis – $B(s) = T(s) \times N(s)$.

**Definition 2.5.1.3.** $\Big(T(s), N(s), B(s)\Big)$ is called the **Frenet frame**

Just as $|T'(s)| = \kappa(s)$ measures the rate at which $p$ pulls away from its tangent line, so $|B'(s)|$ measures the rate at which $p$ pulls away from its osculating plane. Since $\forall s \in \mathbb{R}\ |B(s)| = 1$, $B'(s) \perp B(s)$ IE $B'(s)$ lies in the osculating plane. Also $B'(s) = T'(s) \times N(s) + T(s) \times N'(s) = T(s) \times N'(s)$ so $B'(s) \perp T(s)$. Hence $B'(s) = \tau(s)N(s)$ for some function $\tau : I \mapsto \mathbb{R}$

**Definition 2.5.1.4.** The smooth function $\tau(s) = B'(s) \cdot N(s)$ is called the **torsion** of $p(s)$.

Note that $|\tau(s)| = |B'(s)|$.
Consider the following, if

$$\tilde{p}(s) = p(-s) \tag{2.13}$$

then

$$\tilde{T}(s) = \tilde{p}'(s) = -p'(-s) = -T(s) \tag{2.14}$$

$$\tilde{N}(s) = \frac{\tilde{T}'(s)}{\tilde{\kappa}(s)} = \frac{T'(-s)}{\kappa(s)} = N(-s) \tag{2.15}$$

$$\tilde{B}(s) = \tilde{T}(s) \times \tilde{B}(s) = -B(-s) \tag{2.16}$$

and

$$\tilde{B}'(s) = B'(-s) \tag{2.17}$$

Hence

$$\tilde{\tau}(s) = \tilde{B}'(s) \cdot \tilde{N}(s) = B'(-s) \cdot N(-s) = \tau(-s) \tag{2.18}$$

Informally, torsion is invariant under change of orientation.

Suppose that $\tau(s) = 0$, $\forall s \in \mathbb{R}$ then $B'(s) = 0$, $\forall s \in \mathbb{R}$, so $B(s)$ is a constant, say $B$. Thus $\big(p(s) \cdot B\big)' = p'(s) \cdot B = T(s) \cdot B = 0$. Hence $p(s)$ lies in a plane normal to $B$ which by definition is the osculating plane.

Conversely, suppose $p(s)$ lies in a plane: $p(s) \cdot b = \mathit{const}$ for some constant unit vector $b$. Then $0 = p'(s) \cdot b = T(s) \cdot b$, $0 = T'(s) \cdot b = \kappa(s)N(s) \cdot b$. So $B(s) = \pm b$ hence $B'(s) = 0$ and $\tau = 0$.

**Example 20 (Helix again).**

$$p(s) = \Big(a\cos(\frac{s}{c}), a\sin(\frac{s}{c}), \frac{bs}{c}\Big),$$

$$T(s) = \frac{1}{c}\Big(-a\sin(\frac{s}{c}), a\cos(\frac{s}{c}), b\Big)$$

*and*

$$N(s) = \frac{T'(s)}{\kappa(s)} = \Big( -\cos(\frac{s}{c}), -\sin(\frac{s}{c}), 0 \Big)$$

*Note that* $\kappa(s) = \frac{a}{c^2}$ *never vanishes unless* $a = 0$.

$$B(s) = T(s) \times N(s) = \begin{bmatrix} -a\sin(\frac{s}{c}) & -\cos(\frac{s}{c}) & \overrightarrow{i} \\ a\cos(\frac{s}{c}) & -\sin(\frac{s}{c}) & \overrightarrow{j} \\ b & 0 & \overrightarrow{k} \end{bmatrix} = \Big( \frac{b}{c}\sin(\frac{s}{c}), -\frac{b}{c}\cos(\frac{s}{c}), a \Big)$$

*Therefore*

$$B'(s) = \frac{1}{c^2}\Big( b\cos(\frac{s}{c}), b\sin(\frac{s}{c}), 0 \Big)$$

*And finally*

$$\tau(s) = B'(s) \cdot N(s) = -\frac{b}{c^2}$$

*Observe the following three points:*

1. $(T, N, B)$ *is orthonormal,*
2. $\tau(s) = 0$ *if and only if* $b = 0$ *in which case the helix is a circle*

### 2.5.2 Formula for torsion

Recall that $B = T \times N$ so $B' = T' \times N + T \times N' = T \times N'$. Therefore $\tau = B' \cdot N = [T, N', N] = -[T, N, N']$, the triple scalar product. The latter is defined as $[a, b, c] = (a \times b) \cdot c = a \cdot (b \times c)$.

Now

$$T = p' \tag{2.19}$$

$$N = \frac{T'}{\kappa} = \frac{p''}{\kappa} \tag{2.20}$$

and

$$N' = \frac{\kappa p''' - p''\kappa'}{\kappa^2} \tag{2.21}$$

So a new formula for the torsion follows

**Definition 2.5.2.1.** The torsion can be written as

$$\tau = \frac{-1}{\kappa^2}[p', p'', p'''] \tag{2.22}$$

Note that because of the minus sign, some people prefer to define the torsion as $\tau = -B' \cdot N$.

**Example 21.** *Check that for* $p(s) = \left(a\cos(\frac{s}{c}), a\sin(\frac{s}{c}), \frac{bs}{c}\right)$ *the torsion is* $\frac{-1}{\kappa^2}[p', p'', p'''] = \frac{-b}{c^2}$.

If $p(t)$ is not unit speed then

$$\tau = \frac{-1}{\kappa^2}\frac{[p', p'', p''']}{|p'|^6} = \frac{-[p', p'', p''']}{|p' \times p''|^2} \tag{2.23}$$

## 2.6 Frenet formulas

### 2.6.1 Definition

By definition

$$T' = \kappa N \tag{2.24}$$
$$B' = \tau N \tag{2.25}$$

What happens to $N'$? Since $|N| = 1$ we have $N' \perp N$, so $N' = aT + bB$ for $(a, b) \in \mathbb{R}$. Now

$$a = N' \cdot T = -N \cdot T' = -\kappa \tag{2.26}$$
$$b = N' \cdot B = -N \cdot B' = -\tau \tag{2.27}$$

**Definition 2.6.1.1.** The **Frenet formulas** are defined

$$T' = \kappa N \tag{2.28}$$
$$N' = -\kappa T - \tau B \tag{2.29}$$
$$B' = \tau N \tag{2.30}$$

Sometimes it is convenient to write these in matrix form:

$$(T'N'B') = (TNB)\begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} = (TNB)M \tag{2.31}$$

Note that for any $w \in \mathbb{R}^3$ we have $(w\cdot T' \; w\cdot N' \; w\cdot B') = (w\cdot T \; w\cdot N \; w\cdot B)M$. Note that $M$ is **skew symetric** which means that $M^T = -M$.

### 2.6.2 Congruence of curves

**Definition 2.6.2.1.** The transformation $R : \mathbb{R}^3 \to \mathbb{R}^3$ is a **rigid motion** if $R(w) = L(w) + C$, $\forall w \in \mathbb{R}^3$. Where $L$ is an orthogonal linear transformation of $\mathbb{R}^3$ (IE an orthogonal $3 \times 3$ matrix) and $C$ is a constant vector.

Note that if $L$ is orthogonal then $Lv \cdot Lw = v \cdot w$, $\forall v, w \in \mathbb{R}^3$.
Suppose that $p(s)$ is unit speed and define $\tilde{p}(s) = R\big(p(s)\big)$. Note that

$$\tilde{p}'(s) = \frac{d}{ds}\tilde{p}(s) = \frac{d}{ds}\Big[L\big(p(s)\big) + C\Big] = L\frac{dp}{ds} = L\big(p'(s)\big) \tag{2.32}$$

Thus $|\tilde{p}'(s)| = |L\big(p'(s)\big)| = |p'(s)| = 1$ since $L$ is orthogonal. Hence $\tilde{p}(s)$ is also a unit speed path.

**Definition 2.6.2.2.** Say two unit speed paths $p(s)$ and $q(s)$ are **congruent** if $q(s) = R\big(p(s)\big)$ for some rigid motion $R$.

This is an **equivalence relation**. Define the Frenet frames $(T, N, B)$ for $p(s)$ and $(\tilde{T}, \tilde{N}, \tilde{B})$ for $\tilde{p}(s)$. By equation 2.32:

$$\tilde{T}' = L(T') \Leftrightarrow \tilde{\kappa} = |\tilde{T}'| = |T'| = \kappa \tag{2.33}$$

and

$$\tilde{N} = \frac{\tilde{T}'}{\kappa} = \frac{L(T')}{\kappa} = L\Big(\frac{T'}{\kappa}\Big) = L(N) \tag{2.34}$$

Now, $(LT, LN, LB)$ is orthogonal because $L$ is orthogonal. It is right handed if $\det(L) = 1$. It is left handed if $\det(L) = -1$. Hence

$$\tilde{B} = \begin{cases} LB & \text{if } \det L = 1 \\ -LB & \text{if } \det L = -1 \end{cases} \tag{2.35}$$

and

$$\tilde{\tau} = \tilde{B}' \cdot \tilde{N} = \begin{cases} L\big(B'\big) \cdot L\big(N\big) = B' \cdot N = \tau & \text{if } \det L = 1 \\ -L\big(B'\big) \cdot L\big(N\big) = -B' \cdot N = -\tau & \text{if } \det L = -1 \end{cases} \tag{2.36}$$

**Definition 2.6.2.3.** If $\det L = 1$, $R$ is a **proper** rigid motion and $\tilde{p} = R \circ p$ is **properly congruent** to $p$

**Definition 2.6.2.4.** If $\det L = -1$, $R$ is a **improper** rigid motion and $\tilde{p} = R \circ p$ is **improperly congruent** to $p$

### 2.6.3 Serret-Frenet theorem

**Theorem 2.6.3.1 (Serret-Frenet).** *Suppose $p, \tilde{p} : I \to \mathbb{R}^3$ are unit speed paths with same curvature ($\tilde{\kappa}(s) = \kappa(s) \neq 0$) and torsion ($\tilde{\tau}(s) = \tau(s)$) for all $s$. Then $p, \tilde{p}$ are* ***properly congruent****. If they have opposite torsions ($\tilde{\tau}(s) = -\tau(s)$) then they are* ***improperly congruent****.*

*Proof.* For each $s \in I$, we can define a rigid motion $R_s$ as follows:

$$L_s \ : \quad \begin{aligned} T(s) &\mapsto \tilde{T}(s) \\ N(s) &\mapsto \tilde{N}(s) \\ B(s) &\mapsto \tilde{B}(s) \end{aligned}$$
$$C_s = \tilde{p}(s) - L_s\big(p(s)\big)$$

Then $R_s\big(p(s)\big) = \tilde{p}(s)$. We want to show that $R$ is constant – that it is independent of $s$. Note first that if suffice to show that $L_s$ is constant. For if $L_s = L$, $\forall s \in \mathbb{R}$ then

$$\begin{aligned} C_s = \tilde{p}(s) - L\big(p(s)\big) \Rightarrow \frac{dC_s}{ds} &= \tilde{p}'(s) - L\big(p'(s)\big) \\ \frac{dC_s}{ds} &= \tilde{T}(s) - L\big(T(s)\big) \\ \frac{dC_s}{ds} &= 0 \end{aligned} \tag{2.37}$$

Now, for all $w \in \mathbb{R}^3$ we have:

$$\begin{aligned} L_s(w) &= L_s\big((w\cdot T)T + (w\cdot N)N + (w\cdot B)B\big) \\ &= (w\cdot T)\tilde{T} + (w\cdot N)\tilde{N} + (w\cdot B)\tilde{B} \\ &= (\tilde{T}\ \tilde{N}\ \tilde{B}\ ) \begin{pmatrix} w\cdot T \\ w\cdot N \\ w\cdot B \end{pmatrix} \end{aligned} \tag{2.38}$$

So

$$\begin{aligned} \frac{d}{ds}L_s(w) &= (\tilde{T}'\ \tilde{N}'\ \tilde{B}'\ ) \begin{pmatrix} w\cdot T \\ w\cdot N \\ w\cdot B \end{pmatrix} + (\tilde{T}\ \tilde{N}\ \tilde{B}\ ) \begin{pmatrix} w\cdot T' \\ w\cdot N' \\ w\cdot B' \end{pmatrix} \\ &= (\tilde{T}\ \tilde{N}\ \tilde{B}\ )(\tilde{M} + M^t) \begin{pmatrix} w\cdot T \\ w\cdot N \\ w\cdot B \end{pmatrix} \\ &= 0 \end{aligned} \tag{2.39}$$

where $M$ is defined as the skew-symmetric matrix

$$M = \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \tag{2.40}$$

and hence $\tilde{M} = M$. □

Given a smooth $\kappa, \tau \in I \mapsto \mathbb{R}^3$ with $\kappa > 0$, it can be shown that a smooth unit speed path $p : I \mapsto \mathbb{R}^3$ **exists** whose curvature is $\kappa$ and torsion is $\tau$. This follows from the existence theorem for ordinary differential equations – see handout.

This conclude the study of the geometry of space curves.

# Chapter 3

# On smooth surfaces

We **do** need to define a smooth surface $S \subset \mathbb{R}^3$. Any such definition should include

1. Graphs of functions of two variables.

2. Parametric surfaces – see Vector Calculus II.

The best known surface: $S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ is **not** of these types. We will give a definition due to J. Milnor (1960).

## 3.1 Fundamental concepts

If $a \in \mathbb{R}^n$, $r > 0$, then $B_r(a) = \{x \in \mathbb{R}^n : |x - a| < r\}$ is the open ball of radius $r$ and centre $a$. Of course, $|x - a| = \sqrt{(x_1 - a_1)^2 + ... + (x_n - a_n)^2}$.

**Definition 3.1.0.1.** The subset $U \subset \mathbb{R}^n$ is open if for each $a \in U$ there exist $r > 0$ such that $B_r(a) \subset U$.

**Definition 3.1.0.2.** An open set containing $a$ is called a **neighbourhood** of $a$. See figure 3.1

**Definition 3.1.0.3.** A function $f : U \to \mathbb{R}^n$ is **smooth at** $a$ if partial derivatives of all orders exist in a neighbourhood of $a$ and are continuous at $a$.

**Definition 3.1.0.4.** Say that $f$ is smooth on $U$ if $f$ is smooth at all $a \in U$.

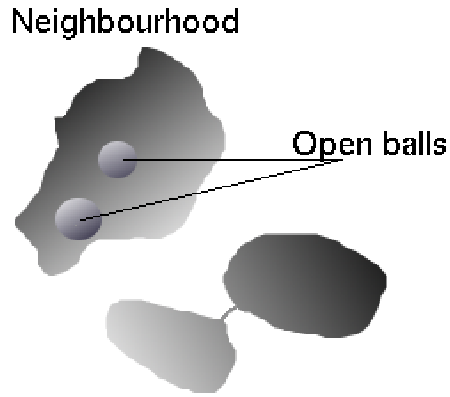



Figure 3.1: A neighbourhood and a none open subset.

The above definitions obviously require $U$ to be open. However, we want to extend the definition to functions $f : D \to \mathbb{R}^m$ where $D \subset \mathbb{R}^n$ is not necessary open.

**Example 22.** *Consider the following:*

- $D = [a, b] \subset \mathbb{R}$, *avoiding the use of one-side limits at end points.*
- $D = \{(x, y) : x^2 + y^2 \leq 1\}$.
- $D$ *is a curve in space.*
- *Ultimately,* $D$ *is a surface in space.*

**Definition 3.1.0.5.** Suppose $f : D \to \mathbb{R}^m$ with $D \subset \mathbb{R}^n$ and $a \in D$. If $U$ is a neighbourhood of $a$ and $\tilde{f} : U \to \mathbb{R}^m$ such that

$$\tilde{f} \mid U \cap D = f \mid U \cup D \tag{3.1}$$

then $\tilde{f}$ is called a **local extension** of $f$ at $a$.

**Example 23.** $f : [0, 1] \to \mathbb{R}$*;* $x \mapsto x^2$ *has two local extensions at* $x = 0$ *and* $x = 1$.

**Example 24.** $g : [0, 1] \to \mathbb{R}$*;* $x \mapsto \sqrt{x}$ *has a local extension at* $x = 1$ *but what about* $x = 0$*?*

**Definition 3.1.0.6.** $f : D \to \mathbb{R}^m$ is **smooth** at $a \in D$ if there exists a local extension of $f$ at $a$ which is smooth at $a$.

**Example 25.** *In examples 23 and 24, $f$ is smooth at $0, 1$ and $g$ is smooth at $1$ but not at $0$.*

**Definition 3.1.0.7.** Say $f$ smooth on $D$ if $f$ is smooth $\forall a \in D$.

To appreciate the next step towards a definition of a surface, consider first the following analogy from vector space theory. $V$ and $V'$ are **isomorphic** (the same as far as linear algebra is concerned) if there exist smooth linear maps $f : V \to V'$ and $g : V' \to V$ such that:

$$g\big(f(v)\big) = v \;\forall v \in V \tag{3.2}$$

$$f\big(g(v')\big) = v' \;\forall v' \in V' \tag{3.3}$$

Or

$$V \underset{g}{\overset{f}{\rightleftarrows}} V' \tag{3.4}$$

if $f$ and $g$ are linear functions. Now, if $D \subset \mathbb{R}^n$ and $D' \subset \mathbb{R}^m$ with

$$D \underset{g}{\overset{f}{\rightleftarrows}} D' \tag{3.5}$$

then $D$, $D'$ are said to be **diffeomorphic** and $f$, $g$ are said to be **diffeomorphisms** . For example, the open disk and the open hemisphere are diffeomorphic. The essential feature of a surface is its "two-dimensionalness" and the "standard models" with this property are open subsets of $\mathbb{R}^2$ hence

**Definition 3.1.0.8.** A **(smooth) surface patch** is a subset $D \subset \mathbb{R}^3$ which is diffeomorphic to an open subset $U \subset \mathbb{R}^2$.

$$\mathbb{R}^3 \supset D \underset{p}{\overset{\phi}{\rightleftarrows}} U \in \mathbb{R}^2 \tag{3.6}$$

**Definition 3.1.0.9.** Suppose that $S \subset \mathbb{R}^3$. A surface patch of the form: $D = S \cap V$ where $V \in \mathbb{R}^3$ open, is called a **chart** in S. $\phi$ is called a **chart map** and $p$ is called a **coordinate map** (or **local parametrisation**).

**Definition 3.1.0.10.** If each point of $S$ lie in a chart, $S$ is called a **smooth surface**.

**Definition 3.1.0.11.** A collection of charts whose union is $S$ is called an **atlas** for $S$.

Note that with this definition a surface patch is a surface, just take $V = \mathbb{R}^3$ for example.

**Example 26 (Graph).** *Let $f : U \to \mathbb{R}$, where $U \subset \mathbb{R}^2$ is open. Define $S_f = \{(u, v, f(u, v)) : (u, v) \in U\}$. Claim that $S_f$ is a surface patch because:*

- *Define $p(u, v) = \big(u, v, f(u, v)\big)$ thus $p : U \to S_f$ is smooth because $f$ is smooth. This is the coordinate map.*
- *Define $\phi$ such that $\phi(x, y, z) = (x, y)$ thus $\phi : S_f \to U$ is smooth (because it extends in an obvious way to $\mathbb{R}^3$) with $\phi = p^{-1}$.*

**Example 27 ($S^2$ aka Sphere).** *By definition,*

$$S^2 = \Big\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\Big\}$$

*Take $D = \{(x, y, z) \in S^2 : z > 0\}$. Then $D = S_f$ where $f(u, v) = \sqrt{1 - (u^2 + v^2)}$ so $D$ is a surface patch. Furthermore $D = S^2 \cap \{(x, y, z) : z > 0\}$ so $D$ is a chart of $S$. Here, $V$ is the open upper half space.*

*We can find other charts of this type and hence make an atlas for $S^2$. Thus $S^2$ is a smooth surface.*

**Example 28 ($S^2$ again).** *Every point on a sphere has a "geographic coordinate". $p(u, v) = \big(\cos(u)\cos(v), \sin(u)\cos(v), \sin(v)\big)$ then $p : U \to D = S^2 - C$ is smooth and a bijection. $C$ is the "Greenwich meridian" and $U = (0, 2\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2})$.*

*Define: $\phi : D \to U$; $(x, y, z) \mapsto \big(\theta(x, y), \arcsin(z)\big)$ so $\phi = p^{-1}$. Where $\theta(x, y) = \arctan(\frac{y}{x})$. $\phi$ is smooth because it extends to $\{(x, y, z) : |z| < 1\} \setminus \{(x, 0, y) : x \geq 0\} = V$.*

*Therefore $D$ is a surface patch. Furthermore, $D = S^2 \cap V$, so $D$ is a chart. Combine with one other similar chart to obtain an atlas.*

**Example 29 ($S^2$ yet again).** *Stereographic coordinates – See example class and Complex Calculus class.*

## 3.2 Some advanced calculus (revision)

### 3.2.1 Jacobian matrix

Suppose that $U \subset \mathbb{R}^n$ open and $f : U \mapsto \mathbb{R}^m$ is smooth at $a \in U$.

**Definition 3.2.1.1.** Define $J_f(a)$ to be the $m \times n$ **Jacobian** matrix whose $ij^{th}$ entry is

$$D_j f_i(a) = \frac{\partial f_i}{\partial x_j}(a) \tag{3.7}$$

**Definition 3.2.1.2.** Denote the corresponding linear map $\mathbb{R}^n \to \mathbb{R}^m$ by $DF(a)$ or $df(a)$ the **derivative** or **differential** of $f$ at $a$.

Note that $df(a) : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, whose value at $h \in \mathbb{R}^n$ is denoted by $df(a)(h)$ or $df(a)[h]$.

**Example 30.**

$$f : U \to \mathbb{R}^3,\ U = B(0) \subset \mathbb{R}^2,$$

$$f(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$$

$$\begin{aligned} \frac{df}{du} &= \left(1, 0, \frac{-u}{\sqrt{1 - u^2 - v^2}}\right) \\ \frac{df}{dv} &= \left(0, 1, \frac{-v}{\sqrt{1 - u^2 - v^2}}\right) \end{aligned}$$

$$J_f(u, v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -u(1 - u^2 - v^2)^{-1/2} & -v(1 - u^2 - v^2)^{-1/2} \end{pmatrix}$$

$$df(u, v)(s, t) = \left(s, t, \frac{-us - vt}{\sqrt{1 - u^2 - v^2}}\right) \tag{3.8}$$

### 3.2.2 Chain rule

$$\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \xrightarrow{g} \mathbb{R}^p \tag{3.9}$$

$$\mathbb{R}^n \xrightarrow{g \circ f} \mathbb{R}^p \tag{3.10}$$

Assume that

1. $f$ is smooth at $a \in \mathbb{R}^n$, thus $df(a) : \mathbb{R}^n \to \mathbb{R}^m$ exists.
2. $g$ is smooth at $b = f(a) \in \mathbb{R}^m$, thus $dg(b) : \mathbb{R}^m \to \mathbb{R}^p$ exists.

then $gof$ is smooth at $a$ and

$$d(g \circ f)(a) = dg(b) \circ df(a) \tag{3.11}$$

### 3.2.3 Geometric consequence

**Theorem 3.2.3.1.** *Suppose that $\gamma(t)$ is a smooth path in $\mathbb{R}^n$ with $\gamma(0) = a$. Thus $f \circ \gamma(t)$ is a smooth path $\mathbb{R}^m$ with $f \circ \gamma(0) = b$. The tangent vectors $\gamma'(0)$ and $(f \circ \gamma)'(0)$ are related by*

$$(f \circ \gamma)'(0) = df(a)[\gamma'(0)] \tag{3.12}$$

*Proof.* See exercise II, question 4. □

## 3.3 Tangent space

**Preamble** For any point $p_0 \in S$ define $T_{p_0}S = \{c'(o) : c(t) \text{ smooth path in } S,\ c(0) = p_0\}$. Of course, $c : I \to \mathbb{R}^3$ is smooth with $c(t) \in S$, $\forall t \in I$.

Choose chart $D$ containing $p_0$ and set $a_0 = \phi(p_0) \in U$. If $\gamma(t)$ is a smooth path in $U$ with $\gamma(0) = a_0$ then $(p \circ \gamma)(t)$ is a smooth path in $S$ with $(p \circ \gamma)(0) = p_0$. Hence $(p \circ \gamma)'(0) \in T_{p_0}S$. But $(p \circ \gamma)'(0) = dp(a_0)[\gamma'(0)]$ so

$$Im\left(dp(a_0)\right) \in T_{p_0}S \tag{3.13}$$

We need to show that $T_{p_0}S$ is the image of $dp(a_0)$.

**Consequence** $T_{p_0}S$ is a vector space.

Now, since $\phi$ is smooth, there exists a local extension $\tilde{\phi}$ of $\phi$ at $o_0$ with $\tilde{\phi}$ smooth at $p_0$. We have $\tilde{\phi} \circ p = \phi \circ p = id_u$ so by the chain rule

$$d\tilde{\phi}(p_0) \circ dp(a_0) = d(id_u)(a_0) = id_{\mathbb{R}^2} \tag{3.14}$$

Note that for $id_u(u, v) = (u, v)$, so the Jacobian matrix is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Hence $d(id_u)(u, v)(s, t) = (s, t)$.

So, $dp(a_0) : \mathbb{R}^2 \to \mathbb{R}^3$ has a left inverse and hence is a injection.

**Conclusion** $T_{p_0}S$ is a two dimensional vector space.

**Example 31 ($S = S^2$).** *Let $p \in S^2$ be in the north hemisphere, then*

$$p = p(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$$

*for some $u^2 + v^2 < 1$. We already computed from equation 3.8.*

$$w = dp(u, v)(s, t) = \left(s, t, \frac{-us - vt}{\sqrt{1 - u^2 - v^2}}\right)$$

*Note that*

$$w \cdot p = us + vt - us - vt = 0$$

*so by dimensionality*

$$T_pS^2 = \left\{ w \in \mathbb{R}^3 : w \cdot p = 0 \right\} = p^\perp$$

*By applying the same argument in other charts, we note that*

$$T_pS^2 = p^\perp, \ \forall p \in S^2$$

## 3.4 More advanced calculus

### 3.4.1 Inverse function theorem

Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ is smooth at $a \in \mathbb{R}^n$ and $df(a) : \mathbb{R}^n \to \mathbb{R}^m$ is a linear **isomorphism** ($n = m$) thus for all $k \in \mathbb{R}^{n(=m)}$ the linear equation $df(a)(h) = k$ has a unique solution. What about the non-linear equation $f(x) = y$?

Note that the following result says that a unique solution exists and varies smoothly with $y$ provided that $y$ is "sufficiently close" to $b$.

**Theorem 3.4.1.1 (Inverse function theorem).** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be smooth at $a \in \mathbb{R}^n$ and $df(a) : \mathbb{R}^n \to \mathbb{R}^m$ an isomorphism – that is $n = m$. Then there exists neighborhoods $U$ of $a$ and $V$ of $b = f(a)$ such that $f : U \to V$ is a diffeomorphism.*

*Proof.* See Analysis I by S. Lang – need fix point theorem. □

### 3.4.2 Application: regular value theorem

There is an easy way to show that certain subset $S \in \mathbb{R}^3$ are smooth surfaces. These $S$ are of the form $S = \{(x, y, z) \in \mathbb{R}^3; f(x, y, z) = const\}$, where $f : \mathbb{R}^3 \to \mathbb{R}$, such as the unit sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1\}$

**Definition 3.4.2.1.** Let $f : \mathbb{R}^3 \to \mathbb{R}$ be smooth at $p_0 \in \mathbb{R}^3$. Say

1. $p_0$ is a **regular point** of $f$ if $df(p_0) \neq 0$.
2. $p_0$ is a **critical point** of $f$ if $df(p_0) = 0$.

The following results says that it is possible to smoothly change coordinates in a neighbourhood of a regular point so that $f$ becomes a linear function of new coordinates.

**Lemma 3.4.2.1 (Local linearisation lemma).** *If $p_0 \in \mathbb{R}^3$ is a regular point of $f : \mathbb{R}^n \to \mathbb{R}$ then there exists*

1. *a neighbourhood $V$ of $p_0$,*
2. *an open subset $W \subset \mathbb{R}^3$ and*
3. *diffeomorphism $\psi : W \to V$*

*such that one of the following holds for all $(x, y, z) \in W$:*

$$f\Big(\psi(x, y, z)\Big) = x \tag{3.15}$$

$$f\Big(\psi(x, y, z)\Big) = y \tag{3.16}$$

$$f\Big(\psi(x, y, z)\Big) = z \tag{3.17}$$

*Proof.* For convenience defined: $\pi_1(x,y,z) = x$, $\pi_2(x,y,z) = y$ and $\pi_3(x,y,z) = z$ and maps $F_1, F_2, F_3 : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$F_1(x,y,z) = \Big(f(x,y,z), y, z\Big) \tag{3.18}$$

$$F_2(x,y,z) = \Big(x, f(x,y,z), z\Big) \tag{3.19}$$

$$F_3(x,y,z) = \Big(x, y, f(x,y,z)\Big) \tag{3.20}$$

Thus $\pi_i \circ F_i = f$. Therefore simply need to show at least one $F_i$ is invertible on a neighbourhood of $P_0$. First note that $J_f(p_0) = \left(\frac{\partial f}{\partial x}\Big|_{p_0} \quad \frac{\partial f}{\partial y}\Big|_{p_0} \quad \frac{\partial f}{\partial z}\Big|_{p_0}\right)$. So, since $p_0$ is a regular point, at least one of these partial derivatives is not zero; say $\frac{\partial f}{\partial x}\Big|_{p_0}$.

Now,

$$J_{F_1}(p_0) = \begin{pmatrix} \frac{\partial f}{\partial x}\Big|_{p_0} & \frac{\partial f}{\partial y}\Big|_{p_0} & \frac{\partial f}{\partial z}\Big|_{p_0} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.21}$$

so $\det J_{F_1}(p_0) = \frac{\partial f}{\partial x}\Big|_{p_0} \neq 0$. Hence $dF_1(p_0)$ is an isomorphism and the result follows from the inverse function theorem – see theorem 3.4.1.1. □

Now, suppose $p_0$ is a regular point of $f : \mathbb{R}^3 \to \mathbb{R}$ where $f(p_0) = a \in \mathbb{R}$ and define

$$S = \Big\{p \in \mathbb{R}^3; f(p) = a\Big\} = f^{-1}(a) \tag{3.22}$$

which is the **level set**. Let $\psi : W \to V$ be as in lemma 3.4.2.1 with $f \circ \psi = \pi_3$, say. Thus $f = \pi_3 \circ \psi^{-1}$ on V. Define $D = S \cap V$, then for all $p \in D$:

$$a = f(p) = \pi_3 \circ \psi^{-1}(p) \tag{3.23}$$

Thus $\psi^{-1}(D) = W \cap P$ where $P$ is the plane $z = a$. Therefore $D$ is a chart in $S$. An atlas can be obtained for $S$ if **every** $p \in S$ is a regular point of $f$ in which case $S$ is called **a regular level set**. Alternatively, it is said that $a$ is a **regular value**.

**Theorem 3.4.2.2 (Regular value theorem).** *If $S$ is a regular level set of $f : \mathbb{R}^3 \to \mathbb{R}$ then $S$ is a smooth surface. Furthermore $T_pS = \ker df(p)$, $\forall p \in S$.*

*Proof.* We just need to show that $T_pS = ker df(p)$ and since the rest is proven by the local linearisation lemma 3.4.2.1. If $X \in T_pS$ then $X = c'(0)$ for some smooth path $c(t)$ in $S$ with $c(0) = p$. Now, $df(p)(x) = (f \circ c)'(0) = 0$ since $(f \circ c)'(t) = a$, Thus $T_pS \subset \ker df(p)$. Since $\ker df(p)$ is two dimensional, $T_pS = \ker df(p)$. □

Note that since

$$J(p_0) = \begin{pmatrix} \frac{\partial f}{\partial x}\Big|_{p_0} & \frac{\partial f}{\partial y}\Big|_{p_0} & \frac{\partial f}{\partial z}\Big|_{p_0} \end{pmatrix} \tag{3.24}$$

then

$$df(p_0)(u, v, w) = u\frac{\partial f}{\partial x}\Big|_{p_0} + v\frac{\partial f}{\partial y}\Big|_{p_0} + w\frac{\partial f}{\partial z}\Big|_{p_0} = (u, v, w) \cdot \nabla f(p_0) \tag{3.25}$$

So $df(p_0) = 0$ if and only if $\nabla f(p_0) = 0$ and $\ker df(p_0) = \nabla f(p_0)^{\perp}$.

**Example 32.** *Define $f : \mathbb{R}^3 \to \mathbb{R}$, $(x, y, z) \mapsto x^2 + y^2 + z^2$ then $S^2 = f^{-1}(1)$. Now*

$$\nabla f(x, y, z) = (2x, 2y, 2z)$$

*which vanishes if and only if $(x, y, z) = (0, 0, 0)$ or $f(x, y, z) = 0$. Therefore 1 is a regular value, so $S^2$ is a smooth surface. Moreover, if $p \in S^2$ then $T_pS^2 = \nabla f(p)^{\perp} = p^{\perp}$.*

## 3.5 Calculus on surfaces

### 3.5.1 Preamble

Let $S_1, S_2 \subset \mathbb{R}^3$ be smooth surfaces.

**Example 33.**

$$S_1 = S^2 = \left\{(x, y, z) : x^2 + y^2 + z^2 = 1\right\}$$

$$S_2 = Q^2 = Q^2(a, b, c) = \left\{(x, y, z) : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\right\}$$

*As an exercise to the reader: prove that $S_2$ is a smooth surface.*

**Definition 3.5.1.1.** Let $f : S^2 \to Q^2$ then $f$ is smooth if a smooth local extension exists at every point.

**Example 34.** *Take*

$$f : S^2 \to Q^2;\ (x, y, z) \to (ax, by, cz)$$

*is smooth since it extends to a smooth map* $\mathbb{R}^3 \to \mathbb{R}^3$*.*

**Definition 3.5.1.2.** If $f$ is a diffeomorphism, $S_1$ and $S_2$ are said to be **diffeomorphic**.

**Example 35.** $f : S^2 \to Q^2$ *has smooth inverse* $f^{-1} : Q^2 \to S^2;\ (x, y, z) \mapsto (\frac{x}{a}, \frac{x}{b}, \frac{x}{c})$*. Hence* $f$ *is a diffeomorphism and* $S^2$*,* $Q^2$ *are diffeomorphic.*

Let $\tilde{f}$ be a smooth local extension of $f$ at $p_0 \in S_1$. Then $d\tilde{f}(p_0) : \mathbb{R}^3 \to \mathbb{R}^3$ can be formed. If $X \in T_{p_0}S_1$ then a smooth path in $S_1$ can be written as $X = c'(0)$ with $c(0) = p_0$.

Hence

$$d\tilde{f}(p_0)(X) = (f \circ c)'(0) \in T_{p_0}S_2 \tag{3.26}$$

since $f \circ c(t)$ is a smooth path in $S_2$ with $f \circ c(0) = f(p)$.

### 3.5.2 Definitions

**Definition 3.5.2.1.** Define $df(p_0) : T_{p_0}S_1 \to T_{f(p_0)}S_2$ by

$$df(p_0)(X) = (f \circ c)'(0) \tag{3.27}$$

where $c(t)$ is any smooth path in $S$ with $c(0) = p_0$ and $c'(0) = X$. It follows from equation 3.26 that $df(p_0)$ is **linear**. Also, the chain rule for surfaces is immediate.

Suppose $D \subset S_1$ is a chart around $p_0 \in U$ where $U \subset \mathbb{R}^2$.

$$D \underset{p}{\overset{\phi}{\rightleftarrows}} U \tag{3.28}$$

with $\phi(p_0) = a$.

**Definition 3.5.2.2.** The coordinate lines through $p_0$ are defined as

$$p \circ \gamma(t) \tag{3.29}$$

$$p \circ \delta(t) \tag{3.30}$$

where

$$\gamma(t) = a + te_1 \text{ where } e_1 = (1, 0) \tag{3.31}$$

$$\delta(t) = a + te_2 \text{ where } e_2 = (0, 1) \tag{3.32}$$

The corresponding tangent vectors are given by

$$(p \circ \gamma)'(0) = \frac{d}{dt}\Big|_{t=0} p(a + te_1) = \frac{\partial p}{\partial u}\Big|_a \tag{3.33}$$

$$(p \circ \delta)'(0) = \frac{d}{dt}\Big|_{t=0} p(a + te_2) = \frac{\partial p}{\partial v}\Big|_a \tag{3.34}$$

so the partial derivatives $\frac{\partial p}{\partial u}|_a$ and $\frac{\partial p}{\partial v}|_a$ are members of $T_{p_0}S_1$.

Furthermore

$$(p \circ \gamma)'(0) = dp(a)[\gamma'(0)] = dp(a)(e_1) \tag{3.35}$$

$$(p \circ \delta)'(0) = dp(a)[\delta'(0)] = dp(a)(e_2) \tag{3.36}$$

Since $dp(a)$ is one to one, it follows that $\left(\frac{\partial p}{\partial u}\big|_a, \frac{\partial p}{\partial v}\big|_a\right)$ is a **basis** of $T_{p_0}S_1$. Now,

$$df(p_0)\left(\frac{\partial p}{\partial u}\Big|_a\right) = df(p_0)[(p \circ \gamma)'(0)] \tag{3.37}$$

$$= (f \circ \gamma)'(0) \tag{3.38}$$

$$= \frac{d}{dt}\Big|_{t=0} f \circ p(a + te_1) \tag{3.39}$$

$$= \frac{\partial (f \circ p)}{\partial u}\Big|_a \tag{3.40}$$

Similarly

$$df(p_0)\left(\frac{\partial p}{\partial v}\Big|_a\right) = \frac{\partial (f \circ p)}{\partial v}\Big|_a \tag{3.41}$$

**Example 36 (Plane and Cylinder).** *Define:*

$$S_1 = \Big\{(x, y, z) : y = 0\Big\}$$

$$S_2 = \Big\{(x, y, z) : x^2 + y^2 = 1\Big\}$$

*Both surfaces are smooth by the regular value theorem – see theorem 3.4.2.2. Define*

$$f : S_1 \to S_2;\ (x, 0, z) \mapsto (\cos x, \sin x, z)$$

*which is smooth. Let $S_1 = p(\mathbb{R}^2)$ where $p(u, v) = (u, 0, v)$ then*

$$\frac{\partial p}{\partial u} = (1, 0, 0)$$
$$\frac{\partial p}{\partial v} = (0, 0, 1)$$

*and*

$$df\Big(\frac{\partial p}{\partial u}\Big) = \frac{\partial (f \circ p)}{\partial u} = \frac{\partial}{\partial u}(\cos u, \sin u, v) = (-\sin u, cos u, 0)$$
$$df\Big(\frac{\partial p}{\partial v}\Big) = \frac{\partial (f \circ p)}{\partial v} = \frac{\partial}{\partial v}(\cos u, \sin u, v) = (0, 0, 1)$$

*Notice that these vectors are independent so*

$$df(p_0) = T_{p_0}S_1 \to T_{f(p0)}S_2$$

*$f$ is not a diffeomorphism because it is not one to one.*

*If $D \subset S_1$ is any vertical strip of width less than $2\pi$ and $D_2 = f(D_1) \subset S_2$ then $f : D_1 \to D_2$ is a diffeomorphism.*

**Definition 3.5.2.3.** A smooth map $f : S_1 \to S_2$ is a **local diffeomorphism** if for each $p_0 \subset S_1$ there exists charts $D_1 \in S_1$ containing $p_0$ and $D_2 \subset S_2$ containing $F(p_0)$ such that $f : D_1 \to D_2$ is a diffeomorphism.

This concludes the study of the differential topology of smooth surfaces.

# Chapter 4

# On the geometry of surfaces

## 4.1 Riemannian metric

Assume the following:

- $S \subset \mathbb{R}^3$ is a smooth surface.
- For each $p_0 \in S$, there exists a two dimensional vector space $T_{p_0}S$.
- The elements of the vector space $T_{p_0}S$ are usually denoted by $X, Y$.
- On each $T_{p_0}S$ there is an **inner product** $<,>_{p_0}$ defined as follows:

$$< X, Y >_{p_0} = X \cdot Y \tag{4.1}$$

**Definition 4.1.0.4.** The family of inner products is called the **Riemannian metric** (or **first fundamental form** - Gauss) of $S$ and is written as:

$$< X, X >_{p_0} = |X|^2 \tag{4.2}$$

$$< X, Y >_{p_0} = < X, Y > \tag{4.3}$$

Suppose $D$ is a chart about $p_0$ with $\phi(p_0) = a$. Then $\left(\frac{dp}{du}\Big|_a = p_u(a), \frac{dp}{dv}\Big|_a = p_v(a)\right)$ is a basis of $T_{p_0}S$ so we can write:

$$X = X^u p_u + X^v p_v \tag{4.4}$$

$$Y = Y^u p_u + Y^v p_v \tag{4.5}$$

with $X^u, X^v, Y^u, Y^v \in \mathbb{R}^6$. Thus

$$\begin{aligned} < X, Y > &=< X^u p_u + X^v p_v, Y^u p_u + Y^v p_v > && (4.6)\\ &= X^u Y^u |p_u|^2 + (X^u Y^v + X^v Y^u) < p_u, p_v > + X^v Y^v |p_v|^2 && (4.7)\\ &= (X^u X^v) \begin{pmatrix} |p_u|^2 & < p_u, p_v > \\ < p_u, p_v > & |p_v|^2 \end{pmatrix} \begin{pmatrix} Y^u \\ Y^v \end{pmatrix} && (4.8) \end{aligned}$$

Define: $E = |p_u|^2$, $F =< p_u, p_v >$ and $G = |p_v|^2$ then

$$< X, Y >= \begin{pmatrix} X^u X^v \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} Y^u \\ Y^v \end{pmatrix} \quad (4.9)$$

If we choose another chart about $p_0$ then $E, F$ and $G$ would be different but so too would $X^u, X^v, Y^u, Y^v$ – Exercises III, question 5.

**Example 37 (Plane).** *Suppose $p_0 \in \mathbb{R}^3$ and $X, Y \in \mathbb{R}^3$ are orthonormal. Define*

$$\begin{aligned} &S = \Big\{ p_0 + uX + vY : u, v \in \mathbb{R} \Big\},\\ &p : \mathbb{R}^2 \to S; (u, v) \mapsto p_o + uX + vY \ \textit{and}\\ &\phi : S \to \mathbb{R}^2; p \mapsto \big( (p - p_0) \cdot X, (p - p_0) \cdot Y \big). \end{aligned}$$

*Both $p$ and $\phi$ are clearly smooth. Thus, $S$ has a global chart. Furthermore:*

$$\frac{\partial p}{\partial u} = X$$
$$\frac{\partial p}{\partial v} = Y$$

*hence*

$$E = |X|^2 = 1$$
$$G = |Y|^2 = 1$$
$$F = X \cdot Y = 0$$

**Example 38 (Cylinder).** *Given*

$$S = \Big\{ (x, y, z) : x^2 + y^2 = 1 \Big\}$$

*define*

$$p(u, v) = (\cos u, \sin u, v)$$

*which is clearly smooth and one to one on the open set* $U = \{(u, v) : 0 < u < 2\pi\}$. *The image of* $p$ *is*

$$D = \{(x, y, z) \in S : x \neq 1\}$$

*and the inverse map is*

$$\phi(x, y, z) = (\theta(x, y), z);\ D \to U$$

$\phi$ *extends smoothly to* $\mathbb{R}^3 - P$ *where* $P$ *is the half plane* $y = 0, x > 0$. *Now,*

$$\frac{\partial p}{\partial u} = (-\sin u, \cos u, 0)$$
$$\frac{\partial p}{\partial v} = (0, 0, 1)$$

*So that*

$$E = |X|^2 = 1$$
$$G = |Y|^2 = 1$$
$$F = X \cdot Y = 0$$

*which are the same values as for a plane – see example 37.*

**Example 39 (Helicoid).**

$$S = \{(u \cos v, u \sin v, av) : u, v\ in \mathbb{R}\}$$

*Define:*

$$p(u, v) = (u \cos v, u \sin v, av); \mathbb{R}^2 \to \mathbb{R}^3$$
$$\phi(x, y, z) = \left(x \cos \frac{z}{a} + y \sin \frac{z}{a}, \frac{z}{a}\right)$$

*which are both smooth functions. So* $S$ *is a smooth surface with a global chart. Note that the coordinate lines* $u = const$ *are helices and* $v = const$ *are horizontal straight lines. Now,*

$$\frac{\partial p}{\partial u} = (\cos v, \sin v, 0)$$
$$\frac{\partial p}{\partial v} = (-u \sin v, u \cos v, a)$$

*So that*

$$E = \cos^2 v + \sin^2 v = 1$$
$$G = 0$$
$$F = u^2(\sin^2 v + \cos^2 v) + a^2 = u^2 + a^2$$

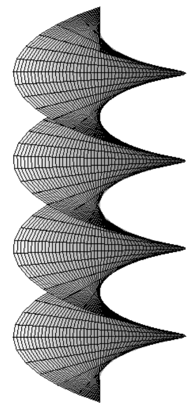


Figure 4.1: A helicoid.

## 4.2 Geometry using the Riemannian metric

### 4.2.1 Arc lengths

For reference see section 2.2. Suppose $c(t)$ is a smooth path in $S \in \mathbb{R}^3$. Then $c'(t) \in T_{c(t)}S$ and

$$s(t) = \int_{t_0}^{t} |c'(t)| dt \tag{4.10}$$

$$= \int_{t_0}^{t} \sqrt{< c'(t), c'(t) >} dt \tag{4.11}$$

Suppose that $c(t)$ lies in chart $D$. Define $\phi\big(c(t)\big) = \big(u(t), v(t)\big)$ a smooth path in $U$. Then

$$c(t) = p\big(u(t), v(t)\big) \tag{4.12}$$

and by the chain rule

$$c'(t) = \frac{\partial p}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial p}{\partial v}\frac{\partial v}{\partial t} \tag{4.13}$$

$$\tag{4.14}$$

hence

$$|c'(t)|^2 = E \cdot (u')^2 + 2Fu'v' + G \cdot (v')^2 \tag{4.15}$$

Note that this explains the classical notation for the Riemannian metric given by

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2 \tag{4.16}$$

### 4.2.2 Angles

Let $c(t)$, $\tilde{c}(t)$ be smooth paths in $S$ with $c(t_0) = \tilde{c}(t_0)$. The angle $\Phi$ of intersection is given by the usual formula:

$$\cos\Phi = \frac{c'(t_0)\cdot\tilde{c}'(t_0)}{\big|c'(t_0)\big|\big|\tilde{c}'(t_0)\big|} \tag{4.17}$$

$$= \frac{< c'(t_0), \tilde{c}'(t_0) >}{\big|c'(t_0)\big|\big|\tilde{c}'(t_0)\big|} \tag{4.18}$$

In terms of $E, F$ and $G$:

$$\cos\Phi = \frac{Eu'\tilde{u}' + F(u'\tilde{v}' + v'\tilde{u}') + Gv'\tilde{v}'}{\sqrt{E(u')^2 + 2Fu'v' + G(v')^2}\sqrt{E(\tilde{u}')^2 + 2F\tilde{u}'\tilde{v}' + G(\tilde{v}')^2}} \tag{4.19}$$

In particular, if $c(t)$ and $\tilde{c}(t)$ are **coordinate lines** then $c' = p_u$ and $\tilde{c}' = p_v$ so

$$\cos\Phi = \frac{F}{\sqrt{EG}} \tag{4.20}$$

Thus coordinate lines are orthogonal if and only if $F = 0, \forall(u, v)$. Such coordinates lines are said to be **orthogonal**. If, in addition $E = G$, coordinates are said to be **isothermal** or **conformal**. See exercises III, question 2 for angle preserving.

### 4.2.3 Areas

Suppose a chart $D \subset S$, $U \subset \mathbb{R}^2$ and $D \xrightarrow{\phi} U \xrightarrow{p} D$. Let $Q \subset U$ be the **closure** of a bounded open subset of $U$.

**Definition 4.2.3.1.** If $A \subset \mathbb{R}^n$ then $A \cup \partial A$ is called the **closure** of $A$ – it is the smallest closed set containing $A$.

Define $R = p(Q) \subset D \subset S$. Recall from Vector Calculus II

$$\text{area}(R) = \int\int_Q \Big|\frac{\partial p}{\partial u} \times \frac{\partial p}{\partial v}\Big| dudv \tag{4.21}$$

Since

$$|p_u \times p_v|^2 = |p_u|^2|p_v|^2 - (p_u \cdot p_v)^2 = EG - F^2 \tag{4.22}$$

we have

$$\text{area}(R) = \int\int_Q \sqrt{EG - F^2}\,dudv \tag{4.23}$$

**Remark** : One needs to check that if $\bar{p}$ is another chart for $D$ with $R = \bar{p}(\bar{Q})$, then

$$\int\int_{\bar{Q}} \sqrt{\bar{E}\bar{G} - \bar{F}^2}\,d\bar{u}d\bar{v} = \int\int_Q \sqrt{EG - F^2}\,dudv \tag{4.24}$$

**Example 40 (Area of the torus).** *Find the area of the torus $T^2(a, b)$, where $0 < b < a$. Recall*

$$T^2 = \Big\{\big(\cos u(a + b\cos v), \sin u(a + b\cos v), b\sin v\big); u, v \in \mathbb{R}\Big\}$$

*Hence a local coordinate map:*

$$p(u, v) = \big(\cos u(a + b\cos v), \sin u(a + b\cos v), b\sin v\big) \text{ for } 0 < u, v < 2\pi$$

*Thus*

$$U = (0, 2\pi) \times (0, 2\pi)$$
$$D = T^2 \setminus (C_1 \cup C_2)$$

*where $C_{1,2}$ are circles. Put*

$$Q_\epsilon = [\epsilon, 2\pi - \epsilon] \times [\epsilon, 2\pi - \epsilon]$$
$$R_\epsilon = p(Q_\epsilon)$$

*where $0 < \epsilon < \pi$.*

$$p_u = \big(-\sin u(a + b\cos v), \cos u(a + b\cos v), 0\big)$$
$$p_v = \big(-b\cos u\sin v, -b\sin u\sin v, b\cos v\big)$$

*hence*

$$E = (a + b\cos v)^2$$
$$F = 0$$
$$G = b^2$$
$$\sqrt{EG - F^2} = b(a + b\cos v)$$

*Therefore*

$$\begin{aligned} area(R_\epsilon) &= \int\int_{Q_\epsilon} b(a + b\cos v)dudv \\ &= b\int_\epsilon^{2\pi-\epsilon}\Big(\int_\epsilon^{2\pi-\epsilon}(a + b\cos v)dv\Big)du \\ &= 2b(\pi-\epsilon)\big[2a(\pi-\epsilon) + b\sin(2\pi-\epsilon) - b\sin\epsilon\big] \end{aligned}$$

*Finally*

$$\lim_{\epsilon\to 0} area(R_\epsilon) = 4\pi^2 ab = area(T^2)$$

### 4.2.4 Intrinsic geometric properties

**Definition 4.2.4.1.** Any "geometric property" of $S$ which depends only on the Riemannian metric ($E, F$ and $G$) is said to be **intrinsic**.

## 4.3 Local isometries

Having defined the Riemannian metric, we now seek to identify those smooth mappings of surfaces which preserve this additional structure.

### 4.3.1 Yet more calculus on surfaces

**Theorem 4.3.1.1 (chain rule).** *If $f : S_1 \to S_2$ is smooth at $p_0 \in S_1$ and $g : S_2 \to S_3$ is smooth at $q_0 = f(p_0) \in S_2$ then $g \circ f : S_1 \to S_3$ is smooth at $p_0$ and*

$$d(g\circ f)(p_0) = dg(q_0)\circ df(p_0) \tag{4.25}$$

*Proof.* Let $X \in T_{p_0}S_1$ and $X = c'(0)$ where $c(t)$ is a smooth path in $S_1$ with $c(0) = p_0$. Then

$$d(g\circ f)(p_0)[X] = \frac{d}{dt}(g\circ f\circ c)(0) \tag{4.26}$$

$$= dg(f(p_0))[df(c)] \tag{4.27}$$

$$= dg(q_0)\circ df(p_0)[X] \tag{4.28}$$

□

**Definition 4.3.1.1.** Smooth $f : S \to \bar{S}$ is a **local diffeomorphism** at $p_0$ if there exist charts $D$ about $p_0$ and $\bar{D}$ about $q_0 = f(p_0)$ such that $f : D \to \bar{D}$ is a diffeomorphism.

**Theorem 4.3.1.2 (Inverse function theorem for surfaces).** *$f$ is a local diffeomorphism at $p_0$ if and only if $df(p_0) : T_{p_0}S \to T_{q_0}\bar{S}$ is a linear isomorphism of vector spaces.*

*Proof.* See question 18 on Exercise III. □

Note that by this theorem, to show $f$ is a local diffeomorphism it suffices to examine its derivatives.

Say that $f$ is a local diffeomorphism if $f$ is a local diffeomorphism at each point.

**Definition 4.3.1.2.** A smooth $f : S \to \bar{S}$ is a **local isometry at** $p_0$ if $df(p_0) : T_{p_0}S \to T_{q_0}\bar{S}$ is a linear isometry of vector spaces.

This means

$$< df(p_0)[X], df(p_0)[Y] >=< X, Y > \ \forall X, Y \in T_{p_0}S \tag{4.29}$$

**Note:** It suffices to check equation 4.29 on all pairs of vector of a basis of $T_{p_0}S$. In particular, if $df(p_0)$ maps an orthonormal basis of $T_{p_0}S$ to an orthonormal basis of $T_{p_0}\bar{S}$ then $df(p_0)$ is a linear isometry and conversely. It follows that a linear isometry is a linear isomorphism. Hence if $f$ is a local isometry at $p_0$ then $f$ is a local diffeomorphism at $p_0$ by the inverse function theorem 4.3.1.2.

**Definition 4.3.1.3.** $f$ is a **local isometry** if $f$ is a local isometry at each point. If in addition $f$ is a diffeomorphism then $f$ is called an **isometry** and $S_1$, $S_2$ are said to be **isometric**.

**Example 41 ($S$ plane and $\bar{S}$ cylinder).** *Let $X, Y$ be orthonormal:*

$$S = \left\{p_o + uX + vY : u, v \in \mathbb{R}\right\}$$
$$\bar{S} = \left\{(\cos u, \sin u, v) : u, v \in \mathbb{R}\right\}$$

*Define $f : S \to \bar{S}$; $p_o + uX + vY \to (\cos u, \sin u, v)$. We have the global coordinates for $S$ given by*

$$p(u, v) = p_o + uX + vY$$

*with $p_u = X$ and $p_v = Y$ which is a orthonormal basis of $T_pS$. Recall*

$$df[p_u] = \frac{\partial(f \circ p)}{\partial u} = (-\sin u, \cos u, 0)$$
$$df[p_v] = \frac{\partial(f \circ p)}{\partial v} = (0, 0, 1)$$

*which is an orthonormal basis of $T_p\bar{S}$. Therefore $f$ is a local isometry. However, $f$ is not an isometry because it is not one to one.*

**Definition 4.3.1.4.** Suppose $f : S \to \bar{S}$ smooth. Charts $(\phi, p)$ and $(\bar{\phi}, \bar{p})$ are said to be $f$**-adapted** if $U = \bar{U}$ and $\bar{\phi} \circ f = \phi \Leftrightarrow f \circ p = \bar{p}$.

Examples 37 and 38 showed that the plane and the cylinder have charts with same Riemannian metric which is explained by

$$\begin{array}{ccc} D & \xrightarrow[f]{} & \bar{D} \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\bar{\phi}} \\ U & = & \bar{U} \end{array} \tag{4.30}$$

Note that if $\phi$ and $\bar{\phi}$ are $f$-adapted then $f : D \to \bar{D}$ is a diffeomorphism.

**Example 42 ($S$ plane and $\bar{S}$ cylinder again).** *Let*

$$f : S \to \bar{S};\ p_o + uX + vY \mapsto (\cos u, \sin u, v)$$

*Chart in $\bar{S}$:*

$$\bar{D} = \{(x, y, z) \in \bar{S} : x \neq 1\}$$
$$\bar{U} = (0, 2\pi) \times \mathbb{R}$$
$$\bar{p} : \bar{U} \to \bar{D}; (u, v) \mapsto (\cos u, \sin u, v)$$
$$\bar{\phi} : \bar{D} \to \bar{U}; (x, y, z) \mapsto (\Theta(x, y), z)$$

*Chart in S:*

$$\begin{aligned} D &= \{p_o + uX + vY : 0 < u < 2\pi, v \in \mathbb{R}\} \\ U &= (0, 2\pi) \times \mathbb{R} \\ p :&U \to D; (u, v) \mapsto p_o + uX + vY \\ \phi :&D \to U; (x, y, z) \mapsto \big((w - p_0) \cdot X, (w - p_0) \cdot Y\big) \end{aligned}$$

*Clearly $U = \bar{U}$ and by definition $f\big(p(u, v)\big) = \bar{p}(u, v)$ so these charts are $f$-adapted. Note that the chart in $S$ is a restriction of the standard global chart.*

### 4.3.2 $(E, F, G)$ lemma

**Lemma 4.3.2.1 ($(E, F, G)$-lemma).** *Suppose $f : S \to \bar{S}$ is smooth then $f$ is a local isometry at $p_0 \in S$ if and only if there exist $f$-adapted charts $D$ about $p_0$ and $\bar{D}$ about $f(p_0)$ such that*

$$(\bar{E}, \bar{F}, \bar{G}) = (E, F, G) \tag{4.31}$$

*at $\phi(p_0)$.*

*Proof.* For $f$-adapted charts we have:

$$\frac{\partial \bar{p}}{\partial u} = \frac{\partial (f \circ p)}{\partial u} = df[p_u] \tag{4.32}$$

$$\frac{\partial \bar{p}}{\partial v} = \frac{\partial (f \circ p)}{\partial v} = df[p_v] \tag{4.33}$$

Hence

$$\bar{E} = |df(p_u)|^2 \tag{4.34}$$

$$\bar{F} = < df(p_u), df(p_v) > \tag{4.35}$$

$$\bar{G} = |df(p_v)|^2 \tag{4.36}$$

Write $X_1 = p_u$ and $X_2 = p_v$ then

$$(\bar{E}, \bar{F}, \bar{G}) = (E, F, G) \text{ at } \phi(p_0) \tag{4.37}$$

$$\Leftrightarrow\ < df(p_0)[X_i], df(p_0)[X_j] > = < X_i, X_j > \ \forall i, j = 1, 2 \tag{4.38}$$

$$\Leftrightarrow df(p_0) \text{ a linear isometry} \tag{4.39}$$

$$\Leftrightarrow f \text{ a local isometry at } p_0 \tag{4.40}$$

It remains to show that if $f$ is a local isometry at $p_0$ then $f$-adapted charts exist. Since $f$ is a local diffeomorphism we can find charts $D$ about $p_0$ and $\bar{D}$ about $f(p_0)$ such that $f : D \to \bar{D}$ is a diffeomorphism.

If $\phi : D \to U$, redefine $\bar{\phi} : \bar{D} \to U$ by $\bar{\phi} = \phi \circ f^{-1}$. Then $\bar{\phi}$ is smooth and invertible with inverse $\bar{p} = f \circ p$ also smooth. □

## 4.4 Curvature

### 4.4.1 Orientable surfaces

Let $S \subset \mathbb{R}^3$ a smooth surface.

**Definition 4.4.1.1.** Say $S$ is **orientable** if there exists a "smooth unit normal" IE a smooth function $\xi : S \to \mathbb{R}^3$ satisfying $\forall p_0 \in S$:

$$\xi(p_0) \perp Tp_0S \tag{4.41}$$
$$|\xi(p_0)| = 1 \tag{4.42}$$

Note that not all surfaces are orientable, for example the Möbius band is none-orientable.

### 4.4.2 Shape operator

The derivative $d\xi(p_0) : T_{p_0}S \to \mathbb{R}^3$ measures the rate at which nearby tangent planes pull away from $T_{p_0}S$ – in the same way as the curvature for curves.

Since $|\xi| = 1$, we may regard $\xi$ as a map into $S^2$. Therefore

$$d\xi(p_0) : T_{p_0}S \to T_{\xi(p_0)}S^2 \tag{4.43}$$

Now,

$$T_{\xi(p_0)}S^2 = \xi(p_0)^{\perp} \tag{4.44}$$

and by definition

$$\xi(p_0)^{\perp} = T_{p_0}S \tag{4.45}$$

Therefore

$$d\xi(p_0) : T_{p_0}S \to T_{p_0}S \tag{4.46}$$

**Definition 4.4.2.1 (Shape operator).** The **shape operator** of $S$ at $p_0$ is the linear map/operator

$$A_{p_0} = T_{p_0}S \to T_{p_0}S \tag{4.47}$$

defined as

$$A_{p_0} = -d\xi(p_0) \tag{4.48}$$

## 4.5 Geometry of curves in $S$

Let $c(s)$ be a unit speed path in $S$. Recall that $c''(s)$ is the **curvature vector** and that $c''(s) \perp c'(s)$. Now $c'(s) = T(s) \subset T_{c(s)}S$. Define

$$V(s) = \xi\big(c(s)\big) \tag{4.49}$$

$$U(s) = V(s) \times T(s) \tag{4.50}$$

Thus $U(s) \subset T_{c(s)}S$.

**Definition 4.5.0.2.** $\big(T(s), U(s), V(s)\big)$ is a positively oriented orthonormal basis called the **Darboux frame**.

Write:

$$\underset{\text{curvature vector}}{\underset{\downarrow}{c''}} = \underset{\text{geodesic curvature vector}}{\underset{\downarrow}{\underbrace{(c''\cdot U)U}}} + \underset{\text{normal curvature vector}}{\underset{\downarrow}{\underbrace{(c''\cdot V)V}}} \tag{4.51}$$

**Definition 4.5.0.3.** Let $\kappa_g = c'' \cdot U$ be the **geodesic curvature** of $c$.

**Definition 4.5.0.4.** Let $\kappa_n = c'' \cdot V$ be the **normal curvature** of $c$.

Note that

$$\kappa_n = c'' \cdot V \tag{4.52}$$

$$= (c' \cdot V)' - c' \cdot V' \tag{4.53}$$

$$= (T \cdot V)' - c' \cdot V' \tag{4.54}$$

$$= 0 - c' \cdot (\xi \circ c)' \tag{4.55}$$

from the definition of $d\xi[c']$ hence

$$= -c' \cdot d\xi[c'] \tag{4.56}$$

$$= -T \cdot d\xi[T] \tag{4.57}$$

$$= < T, AT > \tag{4.58}$$

**Conclusion** If $X \in T_{p_0}S$ is a unit vector, then $< X, A_{p_0}X >$ is the normal curvature of any curve in $S$ through $p_0$ with direction $X$.

**Consequence** $\kappa_n$ depends only on the direction of the curve. Write $\kappa_n(X)$ for the normal curvature of any curve with direction $X$. This gives a nice geometric interpretation of $\kappa_n$. For the most obvious curve through $p_0$ in direction $X$ is the **normal section** obtained by cutting $S$ with the plane $P$ through $p_0$ spanned by $X$ and $\xi(p_0)$. Thus, $P$ is the osculating plane and so $\xi(p_0)$ is $\pm$ the principal normal. Hence $\kappa_n = \pm\kappa$ the **ordinary curvature**.

**Lemma 4.5.0.1.** *The shape operator is* ***self-adjoint*** *(or* ***symmetric****)*

$$< A_{p_0}X, Y >=< X, A_{p_0}Y > ; \forall X < Y \in T_{p_0}S \tag{4.59}$$

*Proof.* Choose a chart about $p_0$ and take $X_1 = \frac{\partial p}{\partial u}$ and $X_2 = \frac{\partial p}{\partial v}$ which is a basis of $T_{p_0}S$. It suffice to check equation 4.59 on a basis of $T_{p_0}S$, IE when $X = X_1$ and $Y = X_2$.

We have

$$A_{p_0}X_1 = -d\xi(p_0)\left[\frac{\partial p}{\partial u}\right] = -\frac{\partial(\xi \circ p)}{\partial u} \tag{4.60}$$

$$A_{p_0}X_2 = -d\xi(p_0)\left[\frac{\partial p}{\partial v}\right] = -\frac{\partial(\xi \circ p)}{\partial v} \tag{4.61}$$

hence

$$< A_{p_0}X_1, X_2 > = -(\xi \circ p)_u \cdot p_v \tag{4.62}$$

$$= (\xi \circ p) \cdot p_{uv} \tag{4.63}$$

$$= (\xi \circ p) \cdot p_{vu} \tag{4.64}$$

$$= -(\xi \circ p)_v \cdot p_u \tag{4.65}$$

$$=< A_{p_0}X_2, X_1 > \tag{4.66}$$

□

**Corollary 4.5.0.2.** *The shape operator is diagonalisable. IE there exists an orthonormal basis $(Z_1, Z_2)$ of $T_{p_0}S$ with $Z_1$ and $Z_2$ eigenvectors of $A_{p_0}$.*

$$A_{p_0}Z_1 = \kappa_1 Z_1 \tag{4.67}$$

$$A_{p_0}Z_2 = \kappa_2 Z_2 \tag{4.68}$$

*for some $\kappa_1, \kappa_2 \in \mathbb{R}$, eigenvalues of $A_{p_0}$.*

*Proof.* The proof is a standard one of linear algebra. □

Note that $\kappa_i =< A_{p_0} Z_i, Z_i >= \kappa_n(Z_i)$.

**Definition 4.5.0.5.** The unit eigenvectors $Z_i$ are called **principal directions**.

**Definition 4.5.0.6.** The unit eigenvalues $\kappa_i$ are called **principal curvatures**.

If the orientation of $S$ is reversed (IE choose $-\xi$ instead of $\xi$) then the shape operator changes sign and hence so do $\kappa_1$ and $\kappa_2$.

**Theorem 4.5.0.3 (Euler's theorem).**

$\kappa_1$ and $\kappa_2$ are the maximum and minimum normal curvatures.

*Proof.* Can write any unit vector $X \in T_{p_0}S$ as

$$X = \cos\theta Z_1 + \sin\theta Z_2 \tag{4.69}$$

then

$$A_{p_0} X = \cos\theta \kappa_1 Z_1 + \sin\theta \kappa_2 Z_2 \tag{4.70}$$

So

$$\kappa_n(X) =< A_{p_0} X, X >= \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta \tag{4.71}$$

thus, $\kappa_n$ always lies between $\kappa_1$ and $\kappa_2$ IE $\kappa_1$ and $\kappa_2$ are the maximum and minimum normal curvatures of $S$ at $p_0$. □

**Definition 4.5.0.7.** Say that $p_0$ is an **elliptic point** if $\kappa_1$ and $\kappa_2$ have same sign.

**Definition 4.5.0.8.** Say that $p_0$ is an **hyperbolic point** if $\kappa_1$ and $\kappa_2$ have opposite sign.

**Definition 4.5.0.9.** Say that $p_0$ is an **parabolic point** if precisely one of $\kappa_1$ and $\kappa_2$ vanishes

**Definition 4.5.0.10.** Say that $p_0$ is an **planar point** if $\kappa_1 = \kappa_2 = 0$

**Definition 4.5.0.11.** The **Gauss curvature of $S$ at $p_0$ is defined as**

$$K(p_0) = detA_{p_0} = \kappa_1\kappa_2 \tag{4.72}$$

**Definition 4.5.0.12.** The **mean curvature of $S$ at $p_0$ is defined as**

$$H(p_0) = \frac{1}{2}trA_{p_0} = \frac{1}{2}(\kappa_1 + \kappa_2) \tag{4.73}$$

Note that if the orientation of $S$ is reversed, then $H$ changes sign but $K$ does not. Thus

$$\begin{array}{lll} p_0 \text{ elliptic} & \Leftrightarrow & K(p_0) > 0 \\ p_0 \text{ hyperbolic} & \Leftrightarrow & K(p_0) < 0 \\ p_0 \text{ parabolic} & \Leftrightarrow & K(p_0) = 0\ H(p_0) \neq 0 \\ p_0 \text{ planar} & \Leftrightarrow & K(p_0) = H(p_0) = 0 \end{array} \tag{4.74}$$

To recover $\kappa_1$ and $\kappa_2$ from $K$ and $H$ note that the characteristic polynomial of $A_{p_0}$ is

$$x^2 - 2H(p_0)x + K(p_0) \tag{4.75}$$

since $kappa_1$ and $\kappa_2$ are the roots we have

$$\kappa_1, \kappa_2 = \frac{2h \pm \sqrt{4H^2 - 2K}}{2} \tag{4.76}$$

$$= H \pm \sqrt{H^2 - K} \tag{4.77}$$

**Definition 4.5.0.13.** $p_0$ is an **umbilic point** if $\kappa_1 = \kappa_2 \Leftrightarrow K = H^2$.

Note that, by Euler's theorem 4.5.0.3, all normal curvatures at any umbilic point $p_0$ are the same. It follows that an umbilic point is either elliptic or planar.

**Definition 4.5.0.14.** $p_0$ is a **minimal point** if $\kappa_1 = -\kappa_2$ ($\Leftrightarrow H = 0$). If all points of S are minimal, S is called a **minimal surface**.

Note that minimal points are either hyperbolic or planar.

**Definition 4.5.0.15.** $p_0$ is a **flat point** if $K = 0$, IE either $\kappa_1 = 0$, $\kappa_2 = 0$ or $\kappa_1 = \kappa_2 = 0$ .

Note that all flat points are either parabolic or planar.

## 4.6 Second fundamental form

Define $\alpha(X,Y) =< AX, Y >=< X, AY >$. Since $A$ is self adjoint, $\alpha$ is a **symmetric bilinear form**. Choose a chart,

$$\mathbb{R}^3 \supset D \underset{p}{\overset{\phi}{\rightleftarrows}} U \subset \mathbb{R}^2 \tag{4.78}$$

such that:

$$\xi\big((p(u,v)\big) = \frac{p_u \times p_v}{|p_u \times p_v|} \tag{4.79}$$

Note that if this is not the case, just swap $u$ and $v$ in $U$.

**Definition 4.6.0.16.** Let

$$e = \alpha(p_u, p_u) \tag{4.80}$$
$$f = \alpha(p_u, p_v) = \alpha(p_v, p_u) \tag{4.81}$$
$$g = \alpha(p_v, p_v) \tag{4.82}$$

in a similar way to the Riemannian metric.

Compute $e, f$ and $g$ as follows:

$$e =< Ap_u, p_u > \tag{4.83}$$
$$= - < \frac{\partial(\xi \circ p)}{\partial u}, \frac{\partial p}{\partial u} > \tag{4.84}$$
$$= (\xi \circ p) \cdot \frac{\partial^2 p}{\partial u^2} \text{ (dot product)} \tag{4.85}$$
$$= \frac{(p_u \times p_v) \cdot p_{uu}}{|p_u \times p_v|} \tag{4.86}$$
$$= \frac{[p_u, p_v, p_{uu}]}{\sqrt{EG - F^2}} \tag{4.87}$$

and

$$f =< Ap_u, p_v >= \dots \tag{4.88}$$
$$= \frac{[p_u, p_v, p_{uv}]}{\sqrt{EG - F^2}} \tag{4.89}$$

finally

$$g =< Ap_v, p_v >= \dots \tag{4.90}$$
$$= \frac{[p_u, p_v, p_{vv}]}{\sqrt{EG - F^2}} \tag{4.91}$$

where $[p_i, p_j, p_{ij}]$ is the triple scalar product. Note that $|p_u \times p_v|^2 = |p_u|^2|p_v|^2 - (p_u \cdot p_v)^2 = EG - F^2$. To compute $K$ and $H$, we need the $detA$ and $trA$.

Suppose the matrix of $A$ with respect to the basis $(p_u, p_v)$ is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where

$$Ap_u = ap_u + cp_v \tag{4.92}$$
$$Ap_v = bp_u + dp_v \tag{4.93}$$

Now

$$\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} < Ap_u, p_u > & < Ap_u, p_v > \\ < Ap_v, p_u > & < Ap_v, p_v > \end{pmatrix} \tag{4.94}$$
$$= \begin{pmatrix} < Ap_u, p_u > & < p_u, Ap_v > \\ < p_v, Ap_u > & < Ap_v, p_v > \end{pmatrix} \tag{4.95}$$
$$= \begin{pmatrix} a|p_u|^2 + c < p_u, p_v > & b|p_u|^2 + d < p_u, p_v > \\ a < p_u, p_v > +c|p_v|^2 & b < p_u, p_v > +d|p_v|^2 \end{pmatrix} \tag{4.96}$$
$$= \begin{pmatrix} aE + cF & bE + dF \\ aF + cG & bF + dG \end{pmatrix} \tag{4.97}$$
$$= \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{4.98}$$

therefore

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} e & f \\ f & g \end{pmatrix} \tag{4.99}$$

Recall that

$$\det \begin{pmatrix} E & F \\ F & G \end{pmatrix} = EG - F^2 = |p_u \times p_v|^2 \neq 0 \tag{4.100}$$

so the inverse matrix exists. Hence

$$
\begin{aligned}
K &= \det A && (4.101)\\
&= \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} && (4.102)\\
&= \frac{\det \begin{pmatrix} e & f \\ f & g \end{pmatrix}}{\det \begin{pmatrix} E & F \\ F & G \end{pmatrix}} && (4.103)\\
K &= \frac{eg - f^2}{EG - F^2} && (4.104)
\end{aligned}
$$

and

$$
\begin{aligned}
2H &= trA && (4.105)\\
&= tr \begin{pmatrix} a & b \\ c & d \end{pmatrix} && (4.106)\\
&= a + d && (4.107)
\end{aligned}
$$

Now,

$$
\begin{aligned}
\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \frac{1}{EG - F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \begin{pmatrix} e & f \\ f & g \end{pmatrix} && (4.108)\\
&= \frac{1}{EG - F^2} \begin{pmatrix} eG - fF & \dots \\ \dots & gE - fF \end{pmatrix} && (4.109)
\end{aligned}
$$

so

$$
H = \frac{eG - 2fF + gE}{2(EG - F^2)} \qquad (4.110)
$$

**Example 43 (Torus).** *Find $K$ and $H$ for the torus given the chart map:*

$$\begin{aligned}
p(u,v) &= \Big(\cos u(a+b\cos v), \sin u(a+b\cos v), b\sin v\Big),\ 0<b<a \\
p_u &= \Big(-\sin u(a+b\cos v), \cos u(a+b\cos v), 0\Big) \\
p_v &= \Big(-b\cos u\sin v, -b\sin u\sin v, b\cos v\Big) \\
p_{uu} &= \Big(-\cos u(a+b\cos v), -\sin u(a+b\cos v), 0\Big) \\
p_{uv} &= \Big(b\sin u\sin v, -b\cos u\sin v, 0\Big) = p_{vu} \\
p_{vv} &= \Big(-b\cos u\cos v, -b\sin u\cos v, -b\sin v\Big)
\end{aligned}$$

*therefore*

$$\begin{aligned}
E &= |p_u|^2 = (a+b\cos v)^2 \\
F &= p_u\cdot p_v = 0 \\
G &= |p_v|^2 = b^2 \ \textit{and} \\
\sqrt{EG-F^2} &= b(a+b\cos v)
\end{aligned}$$

*Hence*

$$\begin{aligned}
e &= \frac{[p_u,p_v,p_{uu}]}{\sqrt{EG-F^2}} \\
&= \frac{b(a+b\cos v)^2}{b(a+b\cos v)}\det\begin{pmatrix}-\sin u & \dots & -\cos u \\ \cos u & \dots & -\sin u \\ 0 & \cos v & 0\end{pmatrix} \\
&= -\cos v(a+b\cos v)
\end{aligned}$$

*Similarly*

$$\begin{aligned}
f &= 0 \\
g &= -b
\end{aligned}$$

*So*

$$eg-f^2 = b\cos v(a+b\cos v)$$

*hence*

$$K = \frac{eg-f^2}{EG-F^2} = \frac{\cos v}{b(a+b\cos v)} \tag{4.111}$$

*furthermore*

$$H = \frac{eG - 2fF + gE}{2(EG - F^2)} = \frac{-(a + 2b\cos v)}{a + b\cos v}$$

*and*

$$\begin{aligned}\kappa_1, \kappa_2 &= H \pm \sqrt{H^2 - K} \\ &= -\frac{\cos v}{a + b\cos v}, -\frac{1}{b}\end{aligned}$$

Note that this method always works. However, sometimes more direct methods are available by computing the shape operator. Some examples of those follow.

**Example 44 (Plane).**

$$S = \Big\{p_0 + uX + vY : u, v \in \mathbb{R}\Big\}$$

*with* $X, Y \in \mathbb{R}^3$ *orthonormal. Choose* $\xi(p) = X \times Y = const$*, hence* $d\xi(p) = 0$ *the zero map* $T_pS \to \mathbb{R}^3$, $\forall p \in S$. *Thus* $A_p = 0$ *so* $\kappa_1, \kappa_2 = 0$ *and* $K = H = 0$.

**Example 45 (Sphere).**

$$s = \Big\{(x, y, z) : x^2 + y^2 + z^2 = R^2\Big\}$$

*Choose* $\xi(p) = p/R$, $\forall p \in S$ *hence* $d\xi(p)[X] = X/R$, $\forall X \in T_pS$ *since* $\xi : \mathbb{R}^3 \to \mathbb{R}^3$ *is linear. Thus every tangent vector is an eigenvector of* $A_p$ *with eigenvalue* $-1/R$ *therefore all points of* $S$ *are umbilical points. Finally,* $\kappa_1 = \kappa_2 = -1/R$ *and* $H = -1/R$ *and* $K = 1/R^2 = H^2$.

**Example 46 (Cylinder).**

$$S = \Big\{(x, y, z) : x^2 + y^2 = R^2\Big\} = \Big\{(R\cos u, R\sin u, v) : u, v \in \mathbb{R}\Big\}$$

*Take*

$$\xi(x, y, z) = \frac{1}{R}(x, y, 0)$$

*hence given* $X = (X_1, X_2, X_3)$*:*

$$d\xi(p)[X] = \frac{1}{R}(X_1, X_2, 0)$$

*by linearity. One tangent vector is* $Z_1 = (0, 0, 1)$ *and*

$$AZ_1 = 0 \Leftrightarrow \kappa_1 = 0$$

*By Euler's theorem,* $Z_2 \perp Z_1$ *so* $Z_2 = (X_1, X_2, 0)$ *(where* $\xi \cdot Z_2 = xX_1 + yX_2 = 0$*) and*

$$AZ_2 = -\frac{1}{R} Z_2$$

*so*

$$\kappa_2 = -\frac{1}{R}$$

*so*

$$K = 0 \text{ and } H = -\frac{1}{2R}$$

**Example 47 (Cylinder again).**

$$S = \left\{ (x, y, z) : x^2 + y^2 = R^2 \right\} = \left\{ (R \cos u, R \sin u, v) : u, v \in \mathbb{R} \right\}$$

*Choose* $\xi(p) = (\cos u, \sin u, 0)$ *which is outwards pointing. Define* $p(u, v) = (R \sin u, R \cos u, 0)$*. One guesses that the Principal directions are*

$$\frac{1}{R} \frac{\partial p}{\partial u} = (-\sin u, \cos u, 0)$$
$$\frac{\partial p}{\partial v} = (0, 0, 1)$$

*and verify*

$$A(p_u) = -\frac{\partial (\xi \circ p)}{\partial u} = (\sin u, -\cos u, 0) = -\frac{p_u}{R}$$
$$A(p_v) = -\frac{\partial (\xi \circ p)}{\partial v} = (0, 0, 0) = 0 p_v$$

*Hence*

$$\kappa_1 = -\frac{1}{R} \text{ and } \kappa_2 = 0$$
$$K = 0 \text{ and } H = -\frac{1}{2R}$$

**Example 48 (Cylinder for the last time).** *Using* $P(u, v)$ *of the two example above, we can compute* $e, f, g$ *and hence* $K$ *and* $H$ *using the "standard method". The solution is left as an exercise for the reader.*

## 4.7 Gauss curvature

Recall

$$K = \frac{eg - f^2}{EG - F^2} \tag{4.112}$$

Using the formula for $e, f$ and $g$:

$$(EG - F^2)^2 K = [p_u, p_v, p_{uu}][p_u, p_v, p_{vv}] - [p_u, p_v, p_{uv}]^2 \tag{4.113}$$

Let $\begin{pmatrix} a & b & c \end{pmatrix}$ be a $3 \times 3$ matrix with vectors as columns and $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be a $3 \times 3$ matrix with vectors as rows. We can write:

$$(EG - F^2)^2 K = \det \begin{pmatrix} p_u & p_v & p_{uu} \end{pmatrix} \det \begin{pmatrix} p_u & p_v & p_{vv} \end{pmatrix} - \Big( \det \begin{pmatrix} p_u & p_v & p_{uv} \end{pmatrix} \Big)^2 \tag{4.114}$$

$$= \det \begin{pmatrix} p_u \\ p_v \\ p_{uu} \end{pmatrix} \det \begin{pmatrix} p_u & p_v & p_{vv} \end{pmatrix} - \det \begin{pmatrix} p_u \\ p_v \\ p_{uv} \end{pmatrix} \det \begin{pmatrix} p_u & p_v & p_{vu} \end{pmatrix} \tag{4.115}$$

$$= \det \left( \begin{pmatrix} p_u \\ p_v \\ p_{uu} \end{pmatrix} \begin{pmatrix} p_u & p_v & p_{vv} \end{pmatrix} \right) - \det \left( \begin{pmatrix} p_u \\ p_v \\ p_{uv} \end{pmatrix} \begin{pmatrix} p_u & p_v & p_{vv} \end{pmatrix} \right) \tag{4.116}$$

$$= \begin{vmatrix} E & F & p_u \cdot p_{vv} \\ F & G & p_v \cdot p_{vv} \\ p_u \cdot p_{uu} & p_v \cdot p_{uu} & p_{uu} \cdot p_{vv} \end{vmatrix} - \begin{vmatrix} E & F & p_u \cdot p_{uv} \\ F & G & p_v \cdot p_{uv} \\ p_u \cdot p_{uv} & p_v \cdot p_{uv} & |p_{uv}|^2 \end{vmatrix} \tag{4.117}$$

Notice that:

$$p_u \cdot p_{vv} = (p_u \cdot p_v)_v - p_{uv} \cdot p_v = F_v - \frac{1}{2} G_u \tag{4.118}$$

$$p_u \cdot p_{uv} = \frac{1}{2} E_v \tag{4.119}$$

$$\tag{4.120}$$

Hence

$$\begin{vmatrix} E & F & F_v - \frac{1}{2}G_u \\ F & G & \frac{1}{2}G_v \\ \frac{1}{2}E_u & F_u - \frac{1}{2}E_v & p_{uu}\cdot p_{vv} - |p_{uv}|^2 \end{vmatrix} - \begin{vmatrix} E & F & \frac{1}{2}E_v \\ F & G & \frac{1}{2}G_u \\ \frac{1}{2}E_v & \frac{1}{2}G_u & 0 \end{vmatrix} \tag{4.121}$$

Notice that:

$$p_{uu}\cdot p_{vv} = (p_u \cdot p_{vv})_u - p_u \cdot p_{vvu} = (F_v - \frac{1}{2}G_u)_u - p_u \cdot p_{vvu} \tag{4.122}$$

$$|p_{uv}|^2 = (p_u \cdot p_{uv})_v - p_u \cdot p_{uvv} = \frac{1}{2}E_{vv} - p_u \cdot p_{uvv} \tag{4.123}$$

so

$$p_{uu}\cdot p_{vv} - |p_{uv}|^2 = F_{vu} - \frac{1}{2}G_{uu} - \frac{1}{2}E_{vv} \tag{4.124}$$

**Conclusion** $K$ can be expressed solely in terms of $E, F$ and $G$. Thus $K$ is an intrinsic geometric quantity!

**Theorem 4.7.0.4 (Theorema egregium).** *If $f : S \to \bar{S}$ is a local isometry then $\bar{K}\big(f(p_0)\big) = K(p_0)$, $\forall p_0 \in S$.*

*Proof.* By the $(E, F, G)$ lemma, there exist charts about $p_0$ and $f(p_0)$ such that $(E, F, G) = (\bar{E}, \bar{F}, \bar{G})$. □

**Note 1** It is not true that $\bar{H}\big(f(p)\big) = H(p)$. For example, if $S$ is a plane and $\bar{S}$ is a cylinder we have $H = 0$ and $\bar{H} = \pm\frac{1}{2R}$ but there is a local isometry $f : S \to \bar{S}$.

**Note 2** Can have $f : S \to \bar{S}$ with $\bar{K}\big(f(p)\big) = K$, $\forall p \in S$ but $f$ is not a local isometry – see question 34, Exercise III. Thus the converse to the theorema egregium is false.

**Application** It's impossible to accurately map any part of the Earth's surface no matter how small it is. For a sphere has non-zero Gauss curvature whereas a plane has zero Gauss curvature.

## 4.8 Congruent surface and Bonnet's theorem

**Question** When do two surfaces have the "same shape"?

**Answers** Diffeomorphism is not the solution since a sphere and an ellipsoid are diffeomorphic. Neither is isometry since a plane and a parabolic cylinder are isometric. So, what is?

**Example 49 (Parabolic cylinder and plane).** *Let*

$$S = \{(x, y, \frac{1}{2}y^2) : x, y \in \mathbb{R}\} \text{ be the parabolic cylinder and}$$
$$\bar{S} = \{(x, y, 0) : x, y \in \mathbb{R}\} \text{ be the } xy\text{-plane.}$$

*Note that just "lifting" the points from the plane towards the parabolic cylinder will not preserve distances. We need to work with arc lengths. Let the mapping $f : S \to \bar{S}$ be*

$$f : (x, y, 0) \mapsto (x, \frac{1}{2}(\sinh^{-1} y + y\sqrt{(1 + y^2)}, 0)$$

*Recall that $\frac{1}{2}(\sinh^{-1} y + y\sqrt{(1 + y^2)})$ is the arc length of a parabola as seen in example 12. We can check that this is an isometry*

$$\begin{aligned}
p(u, v) &= (u, v, \frac{1}{2}v^2) \\
p_u &= (1, 0, 0) \\
p_v &= (0, 1, v) \\
df[p_u] &= \frac{\partial(f \circ p)}{\partial u} = (1, 0, 0) \\
df[p_v] &= \frac{\partial(f \circ p)}{\partial v} \\
&= \frac{\partial}{\partial v}(u, \frac{1}{2}(\sinh^{-1} v + v\sqrt{(1 + v^2)}, 0) \\
&= (0, \frac{1}{2}(\frac{1}{\sqrt{1 + v^2}} + \sqrt{1 + v^2} + \frac{v^2}{\sqrt{1 + v^2}}), 0) \\
&= (0, \frac{2 + 2v^2}{2\sqrt{1 + v^2}}, 0) \\
&= (0, \sqrt{1 + v^2}, 0)
\end{aligned}$$

*and finally*

$$\begin{aligned} |df[p_u]|^2 &= 1 = |p_u|^2 \\ |df[p_v]|^2 &= 1 + v^2 = |p_v|^2 \\ < df[p_u], df[p_v] > &= 0 =< p_u, p_v > \end{aligned}$$

*Hence the parabolic cylinder and the plane are isometric.*

**Recall** A **rigid motion** of $\mathbb{R}^3$ is a map $R : \mathbb{R}^3 \to \mathbb{R}^3$; $R(w) = L(w) + C$ where $L$ is an orthogonal transformation (IE $3 \times 3$ orthogonal matrix) and $C$ is a constant vector.

Note that an orthogonal transformation is a linear isometry of $\mathbb{R}^3$:

$$Lv \cdot Lw = v \cdot w, \ \forall v, w \in \mathbb{R}^3 \tag{4.125}$$

- If $S \subset \mathbb{R}^3$ is a smooth surface, then so is $\bar{S} = R(S)$. For if $\phi : D \to U$ is a chart in $S$ then $\bar{\phi} = \phi \circ R^{-1} : \bar{D} = R(D) \to U$ is a chart in $\bar{S}$.
- The restriction $f : R|S : S \to \bar{S}$ is an isometry:

$$\begin{aligned} < df[X], df[Y] > &= dR[X] \cdot dR[Y] && (4.126) \\ &= dL[X] \cdot dL[Y] && (4.127) \\ &= LX \cdot LY \ \text{ by linearity} && (4.128) \\ &= X \cdot Y \text{ by orthogonality} && (4.129) \\ &=< X, Y > && (4.130) \end{aligned}$$

  and $f$ is invertible with inverse $f^{-1} = R^{-1}|\bar{S}$. Such an isometry is called a **congruence** and $S, \bar{S}$ are said to be **congruent**.

- Charts $\phi : D \to U$ and $\bar{\phi} : \bar{D} \to U$ (as above) are $f$-adapted hence $(\bar{E}, \bar{F}, \bar{G}) = (E, F, G)$ on $U$ by the $(E, F, G)$-lemma.
- If $S$ is orientable with unit normal $\xi$ then $\bar{S}$ has unit normal:

$$\bar{\xi}\big(f(p)\big) = dR(p)[\xi(p)] = L\big(\xi(p)\big) \tag{4.131}$$

  hence $\bar{S}$ is also orientable. Note that the above equation contains

  1. $|\bar{\xi}| = 1$ and

2. $\bar{\xi}$ is normal to $\bar{S}$.

**Proposition 4.8.0.5.** *The shape operators of congruent surfaces are conjugate.*

$$\bar{A} \circ df = df \circ A \tag{4.132}$$

*Proof.* If $X \in T_pS$ then

$$\begin{aligned}
\bar{A}\big(df(X)\big) &= -d\bar{\xi}\big(df(X)\big) \quad \text{by definition of } \bar{A} && (4.133)\\
&= -d(\bar{\xi} \circ f)(X) \quad \text{by chain rule} && (4.134)\\
&= -d(L \circ \xi)(X) \quad \text{by equation 4.131} && (4.135)\\
&= -dL\big(d\xi(X)\big) \quad \text{by the chain rule again} && (4.136)\\
&= dL(AX) \quad \text{by definition of A} && (4.137)\\
&= dR(AX) \quad \text{because } R = L + C && (4.138)\\
&= df(AX) \quad \text{by definition of } f && (4.139)
\end{aligned}$$

□

**Corollary 4.8.0.6.** *Congruent surfaces have the same principal curvatures (IE $\bar{\kappa}_i\big(f(p)\big) = \kappa_i(p)$), hence the same Gauss and mean curvatures.*

*Proof.* Conjugate linear operators have the same eigenvalues. □

The converse of the above corollary is called **Bonnet's theorem**.

**Definition 4.8.0.17.** A smooth surface $S$ is said to be **path-connected** or more briefly **connected** if for every pair of points $p, q \in S$ there exists a smooth path $c(t)$ in $S$ with $c(0) = p$ and $c(1) = q$.

**Theorem 4.8.0.7 (Bonnet's theorem).** *If $f : S \to \bar{S}$ is a local isometry of oriented surfaces with $S$ connected and $df \circ A = \bar{A} \circ df$ then $f$ is a congruence.*

*Proof.* See section 4.10. □

**Lemma 4.8.0.8 ($(e, f, g)$-lemma).** *Suppose $f : S \to \bar{S}$ is a smooth map of oriented surfaces and a local isometry at $p_0 \in S$. Then $\bar{A} \circ df = df \circ A$ on $T_{p_0}S$ if and only if there exist $f$-adaptable charts $\phi : D \to U$ and $\bar{\phi} : \bar{D} \to U$ with $p_0 \in D$ such that $(\bar{e}, \bar{f}, \bar{g}) = (e, f, g)$ at $p_0$.*

*Proof.* Please differentiate between $f : S \to \bar{S}$ the mapping and $f = \alpha(p_u, p_v)$ from the second fundamental form.

For $f$-adapted charts:

$$\frac{\partial \bar{p}}{\partial u} = \frac{\partial (f \circ p)}{\partial u} = df\Big(\frac{\partial p}{\partial u}\Big) \tag{4.140}$$
$$\frac{\partial \bar{p}}{\partial v} = \frac{\partial (f \circ p)}{\partial v} = df\Big(\frac{\partial p}{\partial v}\Big) \tag{4.141}$$

At $\phi(p_0)$:

$$\bar{e} = \bar{\alpha}(\bar{p}_u, \bar{p}_u) \tag{4.142}$$
$$=< \bar{A}\bar{p}_u, \bar{p}_u > \text{ by definition of } \bar{\alpha} \tag{4.143}$$
$$=< \bar{A} \circ df[p_u], \bar{p}_u > \text{ by equation 4.140} \tag{4.144}$$

and

$$e = \alpha(p_u, p_u) \tag{4.145}$$
$$=< Ap_u, p_u > \tag{4.146}$$
$$=< df \circ A[p_u], \bar{p}_u > \text{ because } f \text{ is a local isometry} \tag{4.147}$$

Thus

$$\bar{e} - e =< (\bar{A} \circ df - df \circ A)[p_u], \bar{p}_u > \tag{4.148}$$
$$=< Bp_u, \bar{p}_u > \text{ say} \tag{4.149}$$

where

$$B : T_{p_0}S \to T_{f(p_0)}\bar{S},\ B = \bar{A} \circ df - df \circ A \tag{4.150}$$

Similarly for $f$ and $g$:

$$\bar{f} - f =< Bp_u, \bar{p}_v >=< Bp_v, \bar{p}_u > \text{ because } A \text{ and } \bar{A} \text{ are self adjoint} \tag{4.151}$$
$$\bar{g} - g =< Bp_v.\bar{p}_v > \tag{4.152}$$

The check that $< df[X], BY >=< BX, df[Y] >$ is left for the reader.

So $\bar{e} = e$ and $\bar{f} = f$ if and only if $Bp_u = 0$. Similarly $\bar{g} = g$ and $\bar{f} = f$ if and only if $Bp_v = 0$. Hence

$$(\bar{e}, \bar{f}, \bar{g}) = (e, f, g) \Leftrightarrow Bp_u = 0 = Bp_v \Leftrightarrow B = 0 \tag{4.153}$$

□

**Consequence** $(E, F, G)$-lemma, $(e, f, g)$-lemma and Bonnet's theorem imply that $E$, $F$, $G$, $e$, $f$ and $g$ determine $S$ locally up to congruence. Compare with the Serret-Frenet theorem for space curves – 2.6.3.1.

## 4.9 Gauss-Weingarten Formulas

Recall the Serret-Frenet formulas 2.6.1 for a unit speed curve:

$$\frac{d}{ds}\begin{pmatrix} T & N & B \end{pmatrix} = \begin{pmatrix} T' & N' & B' \end{pmatrix} = \begin{pmatrix} T & N & B \end{pmatrix}\begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \tag{4.154}$$

For a smooth surface, use chart to define a basis of $\mathbb{R}^3$ :

$$p_u = \frac{\partial p}{\partial u} \tag{4.155}$$

$$p_v = \frac{\partial p}{\partial v} \tag{4.156}$$

$$\xi \circ p = \frac{p_u \times p_v}{|p_u \times p_v|} \tag{4.157}$$

Thus $\frac{\partial p_u}{\partial u}$, $\frac{\partial p_v}{\partial v}$ and $\frac{\partial(\xi \circ p)}{\partial u}$ are linear combinations of $p_u$, $p_v$ and $\xi \circ p$ so we can write:

$$\frac{\partial}{\partial u}\begin{pmatrix} p_u & p_v & \xi \circ p \end{pmatrix} = \begin{pmatrix} p_u & p_v & \xi \circ p \end{pmatrix} P_1 \tag{4.158}$$

$$\frac{\partial}{\partial v}\begin{pmatrix} p_u & p_v & \xi \circ p \end{pmatrix} = \begin{pmatrix} p_u & p_v & \xi \circ p \end{pmatrix} P_2 \tag{4.159}$$

where $P_1$ and $P_2$ are $3 \times 3$ matrices. After some calculations $P_1 = Q^{-1}M$ and $P_2 = Q^{-1}N$ where

$$Q = \begin{pmatrix} E & F & 0 \\ F & G & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.160}$$

and

$$M = \begin{pmatrix} \frac{1}{2}E_u & \frac{1}{2}E_v & -e \\ F_u - \frac{1}{2}E_v & \frac{1}{2}G_u & -f \\ e & f & 0 \end{pmatrix} \tag{4.161}$$

$$N = \begin{pmatrix} \frac{1}{2}E_v & F_v - \frac{1}{2}G_u & -f \\ \frac{1}{2}G_u & \frac{1}{2}G_v & -g \\ f & g & 0 \end{pmatrix} \tag{4.162}$$

Note that

$$M + M^T = \begin{pmatrix} E_u & F_u & 0 \\ F_u & G_u & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{\partial Q}{\partial u} \tag{4.163}$$

and

$$N + N^T = \frac{\partial Q}{\partial v} \tag{4.164}$$

## 4.10 Proof of Bonnet's theorem

**Theorem 4.10.0.9 (Bonnet's theorem).** *If $f : S \to \bar{S}$ is a local isometry of oriented surfaces with $S$ connected and $df \circ A = \bar{A} \circ df$ then $f$ is a congruence.*

*Proof.* Compare this proof to the proof of the Serret-Frenet theorem 2.6.3.1.

For any $p \in S$, define a rigid motion $R_p = L_p + C_p$ as follows

$$L_p(X) = df(p)(X),\ \forall X \in T_pS \tag{4.165}$$

$$L_p(\xi) = \bar{\xi}\big(f(p)\big) \tag{4.166}$$

Thus for all $w \in \mathbb{R}^3$ we have

$$L_p(w) = df(p)[w - (w \cdot \xi)\xi] + (w \cdot \xi)\bar{\xi} \tag{4.167}$$

Note that $L_p : \mathbb{R}^3 \to \mathbb{R}^3$ is orthogonal as $df(p)$ is a linear isometry. Then define

$$C_p = f(p) - L_p(p) \tag{4.168}$$

to ensure that $R_p(p) = f(p)$. Call $R_p$ the **rigid approximation** to $f$ at $p$. Obtain the smooth maps:

$$L : S \to \big\{3 \times 3 \text{ matrices}\big\} \cong \mathbb{R}^9,\ p \mapsto L_p \tag{4.169}$$

$$C : S \to \mathbb{R}^3;\ p \mapsto C_p \tag{4.170}$$

The aim is to show that $dL$ and $dC$ are null so that $L$ and $C$ are constant. Note that we need to only show that $L$ is content. For if $L_p = \Lambda$, $\forall p \in S$ then

$$C(p) = C_p = f(p) - \Lambda(p) \tag{4.171}$$

hence

$$dC(p) = df(p) - d\Lambda(p) = df(p) - \Lambda = 0 \tag{4.172}$$

by equation 4.165.

Fix $p_0 \in S$ from now onwards. Since $f$ is a local isometry, we can find $f$-adapted charts $D$ (about $p_0$) and $\bar{D}$ (about $f(p_0)$) such that $(\bar{E}, \bar{F}, \bar{G}) = (E, F, G)$ by the $(E, F, G)$-lemma.

Recall that $f$-adapted means that $f \circ p = \bar{p}$. Thus $\bar{p}_u = df(p_u)$ and $\bar{p}_v = df(p_v)$. In general, write

$$X = X^u p_u + X^v p_v \tag{4.173}$$

where

$$\left.\begin{matrix} < X, p_u >= X^u E + X^v F \\ < X, p_v >= X^u F + X^v G \end{matrix}\right\} \Rightarrow \begin{pmatrix} < X, p_u > \\ < X, p_v > \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} X^u \\ X^v \end{pmatrix} \tag{4.174}$$

so

$$df(X) = X^u \bar{p}_u + X^v \bar{p}_v \tag{4.175}$$

$$= \begin{pmatrix} \bar{p}_u & \bar{p}_v \end{pmatrix} \begin{pmatrix} X^u \\ X^v \end{pmatrix} \tag{4.176}$$

$$= \begin{pmatrix} \bar{p}_u & \bar{p}_v \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} < X, p_u > \\ < X, p_v > \end{pmatrix} \tag{4.177}$$

Take $X = w - (w \cdot \xi)\xi$ then equation 4.167 reads:

$$L(p)[w] = \begin{pmatrix} \bar{p}_u & \bar{p}_v \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} w \cdot p_u \\ w \cdot p_v \end{pmatrix} + (w \cdot \xi)\bar{\xi} \tag{4.178}$$

$$= \begin{pmatrix} \bar{p}_u & \bar{p}_v & \bar{\xi} \end{pmatrix} Q^{-1} \begin{pmatrix} w \cdot p_u \\ w \cdot p_v \\ w \cdot \xi \end{pmatrix} \tag{4.179}$$

where $p = p(u,v)$. Then by Gauss-Weingarten:

$$\frac{\partial}{\partial u}L(p)[w] = \begin{pmatrix}\bar{p}_u & \bar{p}_v & \bar{\xi}\end{pmatrix}\bar{Q}^{-1}\bar{M}Q^{-1}\begin{pmatrix}w\cdot p_u\\ w\cdot p_v\\ w\cdot\xi\end{pmatrix} + \begin{pmatrix}\bar{p}_u & \bar{p}_v & \bar{\xi}\end{pmatrix}\frac{\partial Q^{-1}}{\partial u}\begin{pmatrix}w\cdot p_u\\ w\cdot p_v\\ w\cdot\xi\end{pmatrix} + \begin{pmatrix}\bar{p}_u & \bar{p}_v & \bar{\xi}\end{pmatrix}Q^{-1}M^TQ^{-1}\begin{pmatrix}w\cdot p_u\\ w\cdot p_v\\ w\cdot\xi\end{pmatrix} \tag{4.180}$$

Note that

$$\frac{\partial}{\partial u}\begin{pmatrix}w\cdot p_u & w\cdot p_v & w\cdot\xi\end{pmatrix} = \begin{pmatrix}w\cdot p_u & w\cdot p_v & w\cdot\xi\end{pmatrix}Q^{-1}M \tag{4.181}$$

So

$$\frac{\partial}{\partial u}\begin{pmatrix}w\cdot p_u\\ w\cdot p_v\\ w\cdot\xi\end{pmatrix} = M^TQ^{-1}\begin{pmatrix}w\cdot p_u\\ w\cdot p_v\\ w\cdot\xi\end{pmatrix} \tag{4.182}$$

Hence noting that $Q = Q^T$, $\bar{Q} = Q$ and $\bar{M} = M$:

$$\frac{\partial}{\partial u}L(p)[w] = \begin{pmatrix}\bar{p}_u & \bar{p}_v & \bar{\xi}\end{pmatrix}\Big(Q^{-1}(M+M^t)Q^{-1} + \frac{\partial Q^{-1}}{\partial u}\Big)\begin{pmatrix}w\cdot p_u\\ w\cdot p_v\\ w\cdot\xi\end{pmatrix} \tag{4.183}$$

Note that $M + M^T = \frac{\partial Q}{\partial u}$ and

$$\frac{\partial Q^{-1}}{\partial u} = \frac{\partial}{\partial u}\Big(Q^{-1}QQ^{-1}\Big) \tag{4.184}$$

$$= \frac{\partial Q^{-1}}{\partial u} + Q^{-1}\frac{\partial Q}{\partial u}Q^{-1} + \frac{\partial Q^{-1}}{\partial u} \tag{4.185}$$

$$\Rightarrow 0 = Q^{-1}\frac{\partial Q}{\partial u}Q^{-1} + \frac{\partial Q^{-1}}{\partial u} \tag{4.186}$$

Hence

$$\frac{\partial(L\circ p)}{\partial u} = dL\Big(\frac{\partial p}{\partial u}\Big) = 0 \tag{4.187}$$

Similarly

$$dL\left(\frac{\partial p}{\partial v}\right) \tag{4.188}$$

hence

$$dL = 0 \tag{4.189}$$

□

This concludes the study of the geometry of surfaces and this course.

# Bibliography


[1] M. P. Do Carmo. *Differential geometry of curves and surfaces.* Prentice-Hall, inc, Englewood Cliffs, New Jersey, USA, first edition, 1976.

[2] R. L. Faber. *Differential geometry and relativity theory: an introduction.* Marcel Dekker, INC, New Yoprk and Basel, first edition, 1983.

[3] J. McCleary. *Geometry from a differential viewpoint.* Cambridge Press, Cambridge, UK, first edition, 1994.

[4] A. Pressley. *Elementary differential geometry.* Springer, King's colledge, The Strand, London WC2R2LS, UK, first edition, 2001.

# Index