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Collocation Methods for Volterra Integral and Related Functional Equations

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15

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Preface

The principal aims of this monograph are (i) to serve as an introduction and a guide to the basic principles and the analysis of collocation methods for a broad range of functional equations, including initial-value problems for ordinary and delay differential equations, and Volterra integral and integro-differential equations; (ii) to describe the current 'state of the art' of the field; (iii) to make the reader aware of the many (often very challenging) problems that remain open and which represent a rich source for future research; and (iv) to show, by means of the annotated list of references and the Notes at the end of each chapter, that Volterra equations are not simply an 'isolated' small class of functional equations but that they play an (increasingly) important – and often unexpected! – role in time-dependent PDEs, boundary integral equations, and in many other areas of analysis and applications.

The book can be divided in a natural way into four parts:

- In Part I we focus on collocation methods, mostly in piecewise polynomial spaces, for first-kind and second-kind Volterra integral equations (VIEs, *Chapter 2*), and Volterra integro-differential equations (*Chapter 3*) possessing *smooth solutions*: here, the regularity of the solution on the interval of integration essentially coincides with that of the given data. This situation is similar to the one encountered in initial-value problems for ordinary differential equations. Hence, *Chapter 1* serves as an introduction to collocation methods applied to initial-value problems for ODEs: this will allow us to acquire an appreciation of the richness of these methods and their analysis for more general functional equations encountered in subsequent chapters of this book.
- Part II deals with Volterra integral and integro-differential equations containing *delay arguments*. For non-vanishing delays (*Chapter 4*), smooth data will in general no longer lead to solutions with comparable regularity on the entire

interval of integration, and hence optimal orders of convergence in collocation approximations comparable to those seen in the previous chapters can only be attained by a careful choice of the underlying meshes. For equations with (vanishing) proportional delays (*Chapter 5*) the situation is completely different. Here, the solution inherits the regularity of the given data, but on uniform meshes the analysis of the attainable order of superconvergence is much more complex, due to the 'overlap' between the collocation points and their images under the given delay function. This is not yet completely understood, and a number of problems remain open.

- In Part III we study collocation methods for Volterra integral equations (*Chapter 6*) and integro-differential equations (*Chapter 7*) with *weakly singular kernels*. The presence of these kernel singularities gives rise to a singular behaviour (different in nature from the non-smooth behaviour encountered in Chapter 4) of the solutions at the initial point of the interval of integration, and at the primary discontinuity points if there is a non-vanishing delay: typically, the first- or second-order derivatives of the solutions, or (in the case of first-kind Volterra integral equations) the solution itself, are unbounded at these points. Thus, a decrease in the order of convergence can only be avoided either by introducing suitably graded meshes, or by switching to appropriate non-polynomial spline spaces, reflecting the nature of this singular behaviour. This insight is then combined with results gained in Chapter 4 when turning, at the end of Chapters 6 and 7, to collocation methods for Volterra equations possessing weakly singular kernels *and* delay arguments.
- In Part IV (*Chapters 8 and 9*) we shall have reached the current 'frontier' in the analysis of collocation methods when considering their use for solving integral-algebraic equations (IAEs, which may be viewed as differential-algebraic equations (DAEs) with memory terms, or as 'abstract' DAEs in an infinite-dimensional setting) and singularly perturbed Volterra integral and integro-differential equations. It is known from the numerical analysis of DAEs that the 'direct' application of collocation (even for index-1 problems) will in general not yield the 'expected' convergence (and stability) behaviour since very often the given problem is not 'numerically well formulated'. But while this is now well understood for DAEs, we have a far way to go when analysing collocation methods for suitably reformulated IAEs. Thus, much of Chapter 8 consists of a look into the future. Chapter 9 adds some additional dimensions to this outlook: it points to a number of – to me – promising and important directions of research that may contain the keys to obtaining deeper insight into a number of the open problems we met in previous chapters.

It will become apparent that the number of unanswered questions and open problems becomes larger as we move through the chapters. For example, the analysis of asymptotic stability of collocation solutions for most classes of Volterra integral and functional differential equations is still in its infancy (I believe that relatively little essential progress has been made since Pieter van der Houwen and I wrote down a similar observation in the preface of our 1986 book), and this lack of progress and new results is reflected in the fact that the present monograph deals with this topic only peripherally. It has also become clear from recent advances in the analysis of the asymptotic properties of numerical solutions to ordinary differential equations (Hairer and Wanner (1996)), dynamical systems (Stuart and Humphries (1996)), and delay differential equations (Bellen and Zennaro (2003)), that the study of the analogous properties of collocation methods for more general functional differential and integral equations will eventually have to be treated in a separate monograph.

Most chapters begin with a section reviewing the relevant elementary theory of the class of equations to be discretised by collocation. It goes without saying that a thorough understanding of the theoretical aspects of a given functional equation is imperative since a successful analysis of its discretisation will often be inspired, and thus helped along, by insight into the essential features in the analysis of the given equation and the corresponding discrete analogue derived by collocation.

At the end of each chapter the reader will find exercises and extensive notes. The *Exercises* range from 'hands-on' problems (intended to illustrate and complement the theory of the respective chapter) to research topics of various degree of difficulty, and these will often include important unsolved problems. The purpose of the *Notes* is twofold: they contain remarks complementing the contents of the given chapter (giving, e.g., the sources of original results), and they point out papers on related topics not treated in the book.

The list of *References* tries to be representative, without being exhaustive, of the developments in the research on collocation methods over the last 80 years or so. Moreover, it includes many papers on the analysis and application of collocation methods to types of functional equations not treated in this book. The intent of these references is to guide the reader to work that describes results and mathematical techniques whose analogues and application are, in my view, of potential interest for Volterra integral and related functional differential equations, and they may thus yield the motivation for future research work. In order to make this extensive bibliography more useful and give it a certain guiding role, many of its items have been annotated, so as to enhance the Notes given at the end of each chapter: the brief comments are either cross-references to related work, give an idea of the main content of a paper, or point to books and survey articles containing large bibliographies complementing the one given in this monograph.

As mentioned above, the bibliography lists also many papers and books dealing with topics where exciting work is currently being carried but which, due to limitations of space (and lack of expertise on my part) are not included in this book. Among these topics are *spectral and pseudo-spectral methods* (which appear to be very promising for Volterra equations but whose theory remains to be developed); *sequential (collocation based) regularisation methods* for first-kind VIEs; the numerical treatment of *Volterra equations occurring in control theory*; and *a posteriori error estimation* and the design of *adaptive collocation methods* (especially for problems with non-smooth solutions). I hope that these additional references, while not directly relevant to the text of the monograph, and the accompanying notes will encourage the reader to have a closer look at these important topics.

This monograph is intended for researchers in numerical and applied analysis, for 'users' of collocation methods in the physical sciences and in engineering, and as an introduction to collocation methods for senior undergraduate and graduate students.

Since the exercise section of each chapter contains a rich list of *open problems*, the book may also serve as a source of topics for M.Sc. and Ph.D. theses.

Prerequisites: Senior-level courses in linear algebra, the theory of ordinary differential equations, and numerical analysis (especially numerical quadrature and the numerical solution of ODEs). A knowledge of elementary functional analysis will prove helpful in Chapter 8.

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1

The collocation method for ODEs: an introduction

A collocation solution u_h to a functional equation (for example an ordinary differential equation or a Volterra integral equation) on an interval I is an element from some finite-dimensional function space (the collocation space) which satisfies the equation on an appropriate finite subset of points in I (the set of collocation points) whose cardinality essentially matches the dimension of the collocation space. If initial (or boundary) conditions are present then u_h will usually be required to fulfil these conditions, too.

The use of polynomial or piecewise polynomial collocation spaces for the approximate solution of boundary-value problems has its origin in the 1930s. For initial-value problems in ordinary differential equations such collocation methods were first studied systematically in the late 1960s: it was then shown that collocation in continuous piecewise polynomial spaces leads to an important class of implicit (high-order) Runge–Kutta methods.

1.1 Piecewise polynomial collocation for ODEs

1.1.1 Collocation-based implicit Runge–Kutta methods

Consider the initial-value problem

$$y'(t) = f(t, y(t)), \quad t \in I := [0, T], \quad y(0) = y_0, \tag{1.1.1}$$

and assume that the (Lipschitz-) continuous function $f : I \times \Omega \subset \mathbb{R} \to \mathbb{R}$ is such that (1.1.1) possesses a unique solution $y \in C^1(I)$ for all $y_0 \in \Omega$. Let

$$I_h := \{t_n : 0 = t_0 < t_1 < \ldots < t_N = T\}$$

be a given (not necessarily uniform) mesh on I, and set $\sigma_n := (t_n, t_{n+1}]$, $\bar{\sigma}_n := [t_n, t_{n+1}]$, with $h_n := t_{n+1} - t_n$ $(n = 0, 1, \ldots, N-1)$. The quantity

$h := \max\{h_n : 0 \le n \le N-1\}$ will be called the *diameter* of the mesh I_h; in the context of time-stepping we will also refer to h as the *stepsize*. Note that we have, in rigorous notation,

$$t_n = t_n^{(N)}, \quad \sigma_n := \sigma_n^{(N)}, \quad h_n = h_n^{(N)} \ (n = 0, 1, \ldots, N-1), \quad \text{and} \quad h = h^{(N)}.$$

However, we will usually suppress this dependence on N, the number of subintervals corresponding to a given mesh I_h, except occasionally in the convergence analyses where $N \to \infty$, $h = h^{(N)} \to 0$, so that $Nh^{(N)}$ remains uniformly bounded.

The solution y of the initial-value problem (1.1.1) will be approximated by an element u_h of the piecewise polynomial space

$$S_m^{(0)}(I_h) := \{v \in C(I) : v|_{\bar{\sigma}_n} \in \pi_m \ (0 \le n \le N-1)\}, \tag{1.1.2}$$

where π_m denotes the space of all (real) polynomials of degree not exceeding m. It is readily verified that $S_m^{(0)}(I_h)$ is a linear space whose dimension is

$$\dim S_m^{(0)}(I_h) = Nm + 1$$

(a description of more general piecewise polynomial spaces will be given in Section 2.2.1). This approximation u_h will be found by *collocation*; that is, by requiring that u_h satisfy the given differential equation on a given suitable finite subset X_h of I, and coincide with the exact solution y at the initial point $t = 0$. It is clear that the cardinality of X_h, the *set of collocation points*, will have to be equal to Nm, and the obvious choice of X_h is to place m distinct collocation points in each of the N subintervals $\bar{\sigma}_n$. To be more precise, let X_h be given by

$$X_h := \{t = t_n + c_i h_n : 0 \le c_1 < \ldots < c_m \le 1 \ (0 \le n \le N-1)\}. \tag{1.1.3}$$

For a given mesh I_h, the *collocation parameters* $\{c_i\}$ completely determine X_h. Its cardinality is

$$|X_h| = \begin{cases} Nm & \text{if } 0 < c_1 < \ldots < c_m \le 1 \ (\text{or } 0 \le c_1 < \ldots < c_m < 1), \\ N(m-1)+1 & \text{if } 0 = c_1 < c_2 < \ldots < c_m = 1 \ (m \ge 2). \end{cases}$$

The collocation solution $u_h \in S_m^{(0)}(I_h)$ for (1.1.1) is defined by the *collocation equation*

$$u_h'(t) = f(t, u_h(t)), \quad t \in X_h, \quad u_h(0) = y(0) = y_0. \tag{1.1.4}$$

If u_h corresponds to a set of collocation points with $c_1 = 0$ and $c_m = 1$ $(m \ge 2)$, it lies (if it exists on I) in the smoother space $S_m^{(0)}(I_h) \cap C^1(I) =: S_m^{(1)}(I_h)$ of dimension $N(m-1)+2$ whenever the given function f in (1.1.1) is continuous. This follows readily by considering the collocation equation (1.1.4) at

$t = t_{n-1} + c_m h_{n-1} =: t_n^-$ and at $t = t_n + c_1 h_n =: t_n^+$: taking the difference and using the continuity of f leads to

$$u_h'(t_n^+) - u_h'(t_n^-) = 0, \quad n = 1, \ldots, N-1,$$

and this is equivalent to u_h' being continuous at $t = t_n$.

In order to obtain more insight into this piecewise polynomial collocation method, and to exhibit its recursive nature, we now derive the computational form of (1.1.4). This will reveal that the collocation equation (1.1.4) represents the stage equations of an *m-stage continuous implicit Runge–Kutta method* for the initial-value problem (1.1.1) (compare also the original papers by Guillou and Soulé (1969), Wright (1970), or the book by Hairer, Nørsett and Wanner (1993).

Here, and in subsequent chapters of the book, it will be convenient (and natural) to work with the local Lagrange basis representations of u_h. Since $u_h'|_{\sigma_n} \in \pi_{m-1}$, we have

$$u_h'(t_n + vh_n) = \sum_{j=1}^{m} L_j(v) Y_{n,j}, \quad v \in (0, 1], \quad Y_{n,j} := u_h'(t_n + c_j h_n), \quad (1.1.5)$$

where the polynomials

$$L_j(v) := \prod_{k \neq j}^{m} \frac{v - c_k}{c_j - c_k} \qquad (j = 1, \ldots, m),$$

denote the Lagrange fundamental polynomials with respect to the (distinct) collocation parameters $\{c_i\}$. Setting $y_n := u_h(t_n)$ and

$$\beta_j(v) := \int_0^v L_j(s)ds \quad (j = 1, \ldots, m),$$

we obtain from (1.1.5) the local representation of $u_h \in S_m^{(0)}(I_h)$ on $\bar{\sigma}_n$, namely

$$u_h(t_n + vh_n) = y_n + h_n \sum_{j=1}^{m} \beta_j(v) Y_{n,j}, \quad v \in [0, 1]. \qquad (1.1.6)$$

The unknown (derivative) approximations $Y_{n,i}$ $(i = 1, \ldots, m)$ in (1.1.6) are defined by the solution of a system of (generally nonlinear) algebraic equations obtained by setting $t = t_{n,i} := t_n + c_i h_n$ in the collocation equation (1.1.4) and employing the local representations (1.1.5) and (1.1.6). This system is

$$Y_{n,i} = f\left(t_{n,i}, y_n + h_n \sum_{j=1}^{m} a_{i,j} Y_{n,j}\right), \quad (i = 1, \ldots, m), \qquad (1.1.7)$$

where we have defined $a_{i,j} := \beta_j(c_i)$.

We see that the equations (1.1.6) and (1.1.7) define, as asserted above, a *continuous implicit Runge–Kutta (CIRK) method* for the initial-value problem (1.1.1): its m stage values $Y_{n,i}$ are given by the solution of the nonlinear algebraic systems (1.1.7), and (1.1.6) defines the approximation u_h for each $t \in \bar{\sigma}_n$ $(n = 0, 1, \dots, N-1)$. This local representation may be viewed as the natural interpolant in π_m on $\bar{\sigma}_n$ for the data $\{(t_n, y_n),\ (t_{n,i}, Y_{n,i})\ (i = 1, \dots, m)\}$.

It thus follows that such a continuous implicit RK method contains an embedded 'classical' (discrete) m-stage implicit Runge–Kutta method for (1.1.1): it corresponds to (1.1.6) with $v = 1$,

$$y_{n+1} := u_h(t_n + h_n) = y_n + h_n \sum_{j=1}^{m} b_j Y_{n,j} \quad (n = 0, 1, \dots, N-1), \tag{1.1.8}$$

with $b_j := \beta_j(1)$, and the stage equations (1.1.7).

If $m \geq 2$ and if the collocation parameters $\{c_i\}$ are such that

$$0 = c_1 < c_2 < \dots < c_m = 1,$$

then $t_{n,1} = t_n$ implies $Y_{n,1} = f(t_n, y_n)$, and the CIRK method (1.1.6), (1.1.7) reduces to

$$u_h(t_n + vh_n) = y_n + h_n \beta_1(v) f(t_n, y_n) + h_n \sum_{j=2}^{m} \beta_j(v) Y_{n,j}, \quad v \in [0, 1], \tag{1.1.9}$$

and

$$Y_{n,i} = f\left(t_{n,i}, y_n + h_n a_{i,1} f(t_n, y_n) + h_n \sum_{j=2}^{m} a_{i,j} Y_{n,j}\right) \quad (i = 2, \dots, m). \tag{1.1.10}$$

Moreover, since $c_m = 1$, we obtain

$$Y_{n,m} = f\left(t_{n+1}, y_n + h_n b_1 f(t_n, y_n) + h_n \sum_{j=2}^{m} b_j Y_{n,j}\right).$$

Example 1.1.1 $u_h \in S_1^{(0)}(I_h)$ $(m = 1)$, with $c_1 =: \theta \in [0, 1]$:
Since $L_1(v) \equiv 1$ and $\beta_1(v) = v$ (hence $a_{1,1} = \theta$ and $b_1 = 1$), (1.1.6) reduces to

$$u_h(t_n + vh_n) = y_n + h_n v Y_{n,1}, \quad v \in [0, 1],$$

with $Y_{n,1}$ defined by the solution of

$$Y_{n,1} = f(t_n + \theta h_n, y_n + h_n \theta Y_{n,1}).$$

These equations may be combined into a single one (by setting $v = 1$ in the expression for $u_h(t_n + vh_n)$ and solving for $Y_{n,1}$); the resulting method is the *continuous θ-method* for (1.1.1),

$$u_h(t_n + vh_n) = (1 - v)y_n + vy_{n+1}, \quad v \in [0, 1].$$

where

$$y_{n+1} = y_n + h_n f(t_n + \theta h_n, (1 - \theta)y_n + \theta y_{n+1})$$

implicitly defines y_{n+1}.

This family of continuous one-stage Runge–Kutta methods contains the *continuous implicit Euler method* ($\theta = 1$) and the *continuous implicit midpoint method* ($\theta = 1/2$). For $\theta = 0$ we obtain the *continuous explicit Euler method*. Due to its importance in the time-stepping of (spatially) semidiscretised parabolic PDEs (or PVIDEs) we state the continuous implicit midpoint method for the *linear* ODE

$$y'(t) = a(t)y(t) + g(t), \quad t \in I,$$

with a and g in $C(I)$. Setting $\theta = 1/2$ we obtain

$$y_{n+1} = y_n + \frac{h_n}{2} a(t_n + h_n/2)[y_n + y_{n+1}] + g(t_n + h_n/2)(n = 0, 1, \ldots, N-1),$$

or, using the notation $t_{n+1/2} := t_n + h_n/2$,

$$\left(1 - \frac{h_n}{2} a(t_{n+1/2})\right) y_{n+1} = \left(1 + \frac{h_n}{2} a(t_{n+1/2})\right) y_n + h_n g(t_{n+1/2}). \quad (1.1.11)$$

Observe the difference between (1.1.11) and the *continuous trapezoidal method*: the latter corresponds to collocation in the space $S_2^{(0)}(I_h)$, with $c_1 = 0,\ c_2 = 1$ being the Lobatto points; it is described in Example 1.1.2 below ($m = 2$).

Example 1.1.2 $u_h \in S_2^{(0)}(I_h)$ ($m = 2$), with $0 \leq c_1 < c_2 \leq 1$:
It follows from $L_1(v) = (c_2 - v)/(c_2 - c_1),\ L_2(v) = (v - c_1)/(c_2 - c_1)$ that

$$\beta_1(v) = \frac{v(2c_2 - v)}{2(c_2 - c_1)}, \qquad \beta_2(v) = \frac{v(v - 2c_1)}{2(c_2 - c_1)}.$$

Hence, $b_1 = \beta_1(1) = (2c_2 - 1)/(2(c_2 - c_1)),\ b_2 = \beta_2(1) = (1 - 2c_1)/(2(c_2 - c_1))$. The resulting continuous two-stage Runge–Kutta method thus reads:

$$u_h(t_n + vh_n) = y_n + h_n\{\beta_1(v)Y_{n,1} + \beta_2(v)Y_{n,2}\}, \quad v \in [0, 1],$$

where

$$Y_{n,i} = f(t_{n,i}, y_n + h_n\{a_{i,1}Y_{n,1} + a_{i,2}Y_{n,2}\}) \quad (i = 1, 2).$$

We present three important special cases:

- *Gauss points* $c_1 = (3-\sqrt{3})/6,\ c_2 = (3+\sqrt{3})/6$:
  We obtain

$$\beta_1(v) = v(1+\sqrt{3}(1-v))/2, \qquad \beta_2(v) = v(1-\sqrt{3}(1-v))/2,$$

  and

$$A := \left[\, a_{i,j} \,\right] = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} - \frac{\sqrt{3}}{6} \\ \frac{1}{4} + \frac{\sqrt{3}}{6} & \frac{1}{4} \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

  The discrete version of this *two-stage implicit Runge–Kutta–Gauss method* (of order 4; cf. Section 1.1.3, Corollary 1.1.6) was introduced by Hammer and Hollingsworth (1955) and generalised by Kuntzmann in 1961 (see Ceschino and Kuntzmann (1963) for details).
- *Radau II points* $c_1 = 1/3,\ c_2 = 1$:
  Here, we have

$$\beta_1(v) = 3v(2-v)/4, \qquad \beta_2(v) = 3v(v-2/3)/4,$$

  and

$$A = \begin{bmatrix} \frac{5}{12} & -\frac{1}{12} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix}, \quad b = \begin{bmatrix} \frac{3}{4} \\ \frac{1}{4} \end{bmatrix}.$$

  This represents the continuous *two-stage Radau IIA method.*
- *Lobatto points* $c_1 = 0,\ c_2 = 1$ ($\Longrightarrow\ u_h \in S_2^{(1)}(I_h)$):
  The continuous weights are now

$$\beta_1(v) = v(2-v)/2, \qquad \beta_2(v) = v^2/2,$$

  and hence

$$A = \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad b = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

  This yields the *continuous trapezoidal method*: it can be written in the form

$$u_h(t_n + vh_n) = y_n + \frac{h_n}{2}\left(v(2-v)Y_{n,1} + v^2 Y_{n,2}\right), \quad v \in [0,1],$$

  with

$$Y_{n,1} = f(t_n, y_n), \quad Y_{n,2} = f(t_{n+1}, y_n + (h_n/2)\{Y_{n,1} + Y_{n,2}\}).$$

  (See also Hammer and Hollingsworth (1955).)

For the linear ODE $y'(t) = a(t)y(t) + g(t)$ the stage equation assumes the form

$$\left(1 - \frac{h_n a(t_{n+1})}{2}\right) Y_{n,2} = \left(1+\frac{h_n a(t_n)}{2}\right) a(t_{n+1})y_n + \frac{h_n a(t_{n+1})}{2} g(t_n) + g)t_{n+1}).$$

Remark Other examples of (discrete) RK methods based on collocation, including methods corresponding to the Radau I points ($c_1 = 0,\ c_2 = 2/3$ when $m = 2$), may be found for example in the books by Butcher (1987, 2003), Lambert (1991), and Hairer and Wanner (1996).

There is an alternative way to formulate the above continuous implicit Runge–Kutta method (1.1.6),(1.1.7). Setting

$$U_{n,i} := y_n + h_n \sum_{j=1}^{m} a_{i,j} Y_{n,j} \quad (i = 1, \ldots, m),$$

we obtain the *symmetric* formulation

$$u_h(t_n + v h_n) = y_n + h_n \sum_{j=1}^{m} \beta_j(v) f(t_{n,j}, U_{n,j}), \quad v \in [0, 1], \tag{1.1.12}$$

with

$$U_{n,i} = y_n + h_n \sum_{j=1}^{m} a_{i,j} f(t_{n,j}, U_{n,j}) \quad (i = 1, \ldots, m). \tag{1.1.13}$$

Here, the unknown stage values $U_{n,i}$ represent aproximations to the solution y at the collocation points $t_{n,i}$ $(i = 1, \ldots, m)$. For $v = 1$, (1.1.12) yields the symmetric analogue of (1.1.8),

$$y_{n+1} = y_n + h_n \sum_{j=1}^{m} b_j f(t_{n,j}, U_{n,j}); \tag{1.1.14}$$

if $c_m = 1$ we have $y_{n+1} = U_{n,m}$.

For later reference, and to introduce notation needed later, we also write down the above CIRK method (1.1.6), (1.1.7) for the *linear* initial-value problem

$$y'(t) = a(t)y(t), \quad t \in I, \quad y(0) = y_0,$$

where $a \in C(I)$. Setting $A := (a_{i,j}) \in L(\mathbb{R}^m)$, $\beta(v) := (\beta_1(v), \ldots, \beta_m(v))^T \in \mathbb{R}^m$, and $\mathbf{Y}_n := (Y_{n,1}, \ldots, Y_{n,m})^T \in \mathbb{R}^m$, the CIRK method can be written in the form

$$u_h(t_n + v h_n) = y_n + h_n \beta^T(v) \mathbf{Y}_n, \quad v \in [0, 1], \tag{1.1.15}$$

with $\mathbf{Y}_n$ given by the solution of the linear algebraic system

$$[\mathcal{I}_m - h_n A_n]\mathbf{Y}_n = \operatorname{diag}(a(t_{n,i}))\mathbf{e} \cdot y_n \quad (n = 0, 1, \ldots, N-1). \tag{1.1.16}$$

Here, $\mathcal{I}_m$ denotes the identity in $L(\mathbb{R}^m)$, $A_n := \text{diag}(a(t_{n,i}))A$, and $\mathbf{e} := (1, \ldots, 1)^T \in \mathbb{R}^m$.

The derivation of the analogue of (1.1.15),(1.1.16) corresponding to the symmetric formulation (1.1.12),(1.1.13) of the CIRK method is left as an exercise (Exercise 1.10.1).

The classical conditions for the existence and uniqueness of a solution $y \in C^1(I)$ to the initial-value problem (1.1.1) (see, e.g. Hairer, Nørsett and Wanner (1993, Sections I.7–I.9) assure the existence and uniqueness of the collocation solution $u_h \in S_m^{(0)}(I_h)$ to (1.1.1) or its linear counterpart for all $h := \max_{(n)} h_n$ in some interval $(0, \bar{h})$, provided that f_y is bounded (or a and g lie in $C(I)$ when the ODE is $y' = a(t)y + g(t)$). In the latter case, the existence of such an $\bar{h}$ follows from the Neumann Lemma which states that $(\mathcal{I}_m - h_n A_n)^{-1}$ is uniformly bounded for all sufficiently small $h_n > 0$, so that $h_n||A_n|| < 1$ for some (operator) matrix norm. We shall give the precise formulation of this result in in Chapter 3 (Theorem 3.2.1) for VIDEs which contains the version for ODEs as a special case.

It is clear that not every implicit Runge–Kutta method can be obtained by collocation as described above (see, for example, Nørsett (1980), Hairer, Nørsett and Wanner (1993)): a necessary condition is clearly that the parameters c_i are distinct. The framework of *perturbed collocation* (Nørsett (1980), Nørsett and Wanner (1981); see also Section 1.2 below) encompasses all implicit) Runge–Kutta methods. There is also an elegant connection between continuous Runge–Kutta methods and *discontinuous collocation methods* (Hairer, Lubich and Wanner (2002, pp. 31–34)). The following result (which can be found in Hairer, Nørsett and Wanner (1993, p. 212)) characterises those implicit Runge–Kutta methods that are collocation-based.

Theorem 1.1.1 *The m-stage implicit Runge–Kutta method defined by (1.1.7) and (1.1.8), with distinct parameters c_i and order at least m, can be obtained by collocation in $S_m^{(0)}(I_h)$, as described above, if and only if the relations*

$$\sum_{j=1}^{m} a_{i,j} c_j^{\nu-1} = \frac{c_i^\nu}{\nu}, \quad \nu = 1, \ldots, m \quad (i = 1, \ldots, m),$$

hold.

The **proof** of this result is left as an exercise. Recall that a (discrete) Runge–Kutta method for (1.1.1) is said to be of *order p* if

$$|y(t_1) - y_1| \le Ch^p$$

for all sufficiently smooth $f = f(t, y)$ in (1.1.1). The next section will reveal that the collocation solution $u_h \in S_m^{(0)}(I_h)$ to (1.1.1) is of global order $p \geq m$ on I.

1.1.2 Convergence and global order on I

Suppose that the collocation equation (1.1.4) defines a unique collocation solution $u_h \in S_m^{(0)}(I_h)$ for all sufficiently small mesh diameters $h \in (0, \bar{h})$. What are the optimal values of p_ν and p_ν^* ($\nu = 0, 1$) in the (global and local) error estimates

$$||y^{(\nu)} - u_h^{(\nu)}||_\infty := \sup_{t\in I} |y^{(\nu)}(t) - u_h^{(\nu)}(t)| \leq C_\nu h^{p_\nu} \tag{1.1.17}$$

and

$$||y^{(\nu)} - u_h^{(\nu)}||_{h,\infty} := \max_{t\in I_h\setminus\{0\}} |y^{(\nu)}(t) - u_h^{(\nu)}(t)| \leq C_\nu h^{p_\nu^*}, \tag{1.1.18}$$

respectively? These values depend of course on the regularity of the solution y of the initial-value problem (1.1.1). For arbitrarily regular y we will refer to the largest attainable p_ν ($\nu = 0, 1$) as the *(optimal) orders of global (super-) convergence* (on the interval I) of u_h and u_h', respectively, and the corresponding p_ν^* will be called the *(optimal) orders of local superconvergence* (at the mesh points $I_h \setminus \{0\}$) of u_h and u_h', provided $p_\nu^* > p_\nu$.

In order to introduce the essential ideas underlying the answer to the above question regarding the optimal orders, we first present the result on global convergence for the *linear* initial-value problem

$$y'(t) = a(t)y(t) + g(t), \quad t \in I, \quad y(0) = y_0. \tag{1.1.19}$$

Theorem 1.1.2 *Assume that*

(a) the given functions in (1.1.19) satisfy $a, g \in C^m(I)$;
(b) the collocation solution $u_h \in S_m^{(0)}(I_h)$ for the initial-value problem (1.1.19) corresponding to the collocation points X_h is defined by (1.1.15), (1.1.16);
(c) $\bar{h} > 0$ is such that, for any $h \in (0, \bar{h})$, each of the linear systems (1.1.16) has a unique solution.

Then the estimates

$$||y - u_h||_\infty := \max_{t\in I} |y(t) - u_h(t)| \leq C_0||y^{(m+1)}||_\infty h^m \tag{1.1.20}$$

and

$$||y' - u_h'||_\infty := \sup_{t\in I} |y'(t) - u_h'(t)| \leq C_1||y^{(m+1)}||_\infty h^m, \tag{1.1.21}$$

hold for $h \in (0, \bar{h})$ and any X_h with $0 \leq c_1 < \ldots < c_m \leq 1$. The constants C_ν depend on the collocation parameters $\{c_i\}$ but are independent of h, and the exponent m of h cannot in general be replaced by $m+1$.

Proof Assumption (a) implies that $y \in C^{m+1}(I)$ and hence $y' \in C^m(I)$. Thus we have, using Peano's Theorem (Corollary 1.8.2 with $d = m$) for y' on $\bar{\sigma}_n$,

$$y'(t_n + vh_n) = \sum_{j=1}^{m} L_j(v)Z_{n,j} + h_n^m R_{m+1,n}^{(1)}(v), \quad v \in [0,1], \tag{1.1.22}$$

with $Z_{n,j} := y'(t_{n,j})$. The Peano remainder term and Peano kernel are given by

$$R_{m+1,n}^{(1)}(v) := \int_0^1 K_m(v,z) y^{(m+1)}(t_n + zh_n)dz, \tag{1.1.23}$$

and

$$K_m(v,z) := \frac{1}{(m-1)!}\left\{(v-z)_+^{m-1} - \sum_{k=1}^{m} L_k(v)(c_k - z)_+^{m-1}\right\}, \quad v \in [0,1].$$

Integration of (1.1.22) leads to

$$y(t_n + vh_n) = y(t_n) + h_n \sum_{j=1}^{m} \beta_j(v)Z_{n,j} + h_n^{m+1} R_{m+1,n}(v), \quad v \in [0,1], \tag{1.1.24}$$

where

$$R_{m+1,n}(v) := \int_0^v R_{m+1,n}^{(1)}(s)ds$$

(see also Exercise 1.10.3).

Recalling the local representation (1.1.6) of the collocation solution u_h on $\bar{\sigma}_n$, and setting $\mathcal{E}_{n,j} := Z_{n,j} - Y_{n,j}$, the collocation error $e_h := y - u_h$ on $\bar{\sigma}_n$ may be written as

$$e_h(t_n + vh_n) = e_h(t_n) + h_n \sum_{j=1}^{m} \beta_j(v)\mathcal{E}_{n,j} + h_n^{m+1} R_{m+1,n}(v), \quad v \in [0,1], \tag{1.1.25}$$

while

$$e_h'(t_n + vh_n) = \sum_{j=1}^{m} L_j(v)\mathcal{E}_{n,j} + h_n^m R_{m+1,n}^{(1)}(v), \quad v \in (0,1], \tag{1.1.26}$$

with $e_h'(t_{n,i}) = \mathcal{E}_{n,i} + h_n^m R^{(1)}_{m+1,n}(c_i)$. Since e_h is continuous in I, and hence at the mesh points, we also have the relation

$$e_h(t_n) = e_h(t_{n-1} + h_{n-1}) = e_h(t_{n-1}) + h_{n-1} \sum_{j=1}^{m} b_j \mathcal{E}_{n-1,j} + h_{n-1}^{m+1} R_{m+1,n-1}(1)$$

$(n = 1, \ldots, N-1)$, with $b_j := \beta_j(1)$. The fact that $e_h(0) = 0$ yields

$$e_h(t_n) = \sum_{\ell=0}^{n-1} h_\ell \sum_{j=1}^{m} b_j \mathcal{E}_{\ell,j} + \sum_{\ell=0}^{n-1} h_\ell^{m+1} R_{m+1,\ell}(1) \quad (n = 1, \ldots, N-1). \tag{1.1.27}$$

We are now ready to establish the estimates in Theorem 1.1.2: since the collocation error satisfies

$$e_h'(t_{n,i}) = a(t_{n,i}) e_h(t_{n,i}), \quad i = 1, \ldots, m \ \ (0 \le n \le N-1), \tag{1.1.28}$$

with $e_h(t_n) = e_h(t_{n-1} + h_{n-1})$, it follows from (1.1.25) and (1.1.26) that

$$\mathcal{E}_{n,i} = a(t_{n,i}) \left(e_h(t_n) + h_n \sum_{j=1}^{m} \beta_j(c_i) \mathcal{E}_{n,j} + h_n^{m+1} R_{m+1,n}(c_i) \right) - h_n^m R^{(1)}_{m+1,n}(c_i)$$

$(i = 1, \ldots, m)$. Recalling that $\beta_j(c_i) = a_{i,j}$ and employing (1.2.24), this becomes

$$\mathcal{E}_{n,i} = a(t_{n,i}) \left(\sum_{\ell=0}^{n-1} h_\ell \sum_{j=1}^{m} b_j \mathcal{E}_{\ell,j} + h_n \sum_{j=1}^{m} a_{i,j} \mathcal{E}_{n,j} \right) + \rho_{n,i} \quad (i = 1, \ldots, m), \tag{1.1.29}$$

where the remainder terms $\rho_{n,i}$ are defined by

$$\rho_{n,i} := a(t_{n,i}) \left\{ \sum_{\ell=0}^{n-1} h_\ell^{m+1} R_{m+1,\ell}(1) + h_n^{m+1} R_{m+1,n}(c_i) \right\} - h_n^m R^{(1)}_{m+1,n}(c_i). \tag{1.1.30}$$

Set $\mathbf{b} := (b_1, \ldots, b_m)^T$ and define $\boldsymbol{\rho}_n := (\rho_{n,1}, \ldots, \rho_{n,m})^T$. It then follows from the above equation (1.2.29) that $\boldsymbol{\mathcal{E}}_n := (\mathcal{E}_{n,1}, \ldots, \mathcal{E}_{n,m})^T$ is the solution of the linear algebraic system

$$[\mathcal{I}_m - h_n A_n] \mathcal{E}_n = \operatorname{diag}(a(t_{n,i}))\mathbf{e} \sum_{\ell=0}^{n-1} h_\ell b^T \boldsymbol{\mathcal{E}}_\ell + \boldsymbol{\rho}_n, \tag{1.1.31}$$

where, as in (1.1.16), we have set $A_n := \operatorname{diag}(a(t_{n,i}))A$. This system has the same structure (due to the choice of the local representation for y' and y) as the linear system (1.1.16) defining $\mathbf{Y}_n$ in the representation (1.1.15), except that now the role of y_n is assumed by $e_h(t_n)$ (which can be expressed in the recursive form (1.1.27)). The matrices on the left-hand side of (1.1.31) coincide with those in (1.1.16); hence, all have bounded inverses whenever $h = \max_{(n)} h_n \in (0, \bar{h})$, for some $\bar{h} > 0$. That is, there exists a constant $D_0 < \infty$ so that the uniform bound

$$||(\mathcal{I}_m - h_n A_n)^{-1}||_1 \leq D_0 \quad (n = 0, 1, \ldots, N-1)$$

holds. Here, for $B \in L(\mathbb{R}^m)$, $||B||_1$ denotes the matrix (operator) norm induced by the ℓ^1-norm in $\mathbb{R}^m$. If we define

$$A_0 := ||a||_\infty, \quad M_{m+1} := ||y^{(m+1)}||_\infty, \quad k_m := \max_{v \in [0,1]} \int_0^1 |K_m(v, z)| dz,$$

then, by (1.1.30),

$$||\boldsymbol{\rho}_n||_1 \leq A_0[k_m M_{m+1} m \sum_{\ell=0}^{n-1} h_\ell^{m+1} + h_n^{m+1} k_m M_{m+1}] + h_n^m k_m M_{m+1} \leq \rho M_{m+1} h^m,$$

with obvious meaning of ρ. Using the above estimates in equation (1.1.31) (solved for $\boldsymbol{\mathcal{E}}_n$) and defining $\bar{b} := \max_{(j)} |b_j|$, we readily see that

$$||\boldsymbol{\mathcal{E}}_n||_1 \leq D_0 \left(A_0 m \bar{b} \sum_{\ell=0}^{n-1} h_\ell ||\boldsymbol{\mathcal{E}}_\ell||_1 + \rho M_{m+1} h^m \right),$$

which we write as

$$||\boldsymbol{\mathcal{E}}_n||_1 \leq \gamma_0 \sum_{\ell=0}^{n-1} h_\ell ||\boldsymbol{\mathcal{E}}_\ell||_1 + \gamma_1 M_{m+1} h^m \quad (n = 0, 1, \ldots, N-1), \qquad (1.1.32)$$

where the meaning of the positive constants γ_0 and γ_1 is again clear.

The inequality (1.1.32) is a *generalised discrete Gronwall inequality* (see Corollary 2.1.18)); its solution is bounded by

$$\begin{aligned} ||\boldsymbol{\mathcal{E}}_n||_1 &\leq \gamma_1 M_{m+1} h^m \exp\left(\gamma_0 \sum_{\ell=0}^{n-1} h_\ell \right) \\ &\leq \gamma_1 M_{m+1} h^m \exp(\gamma_0 T) \quad (n = 0, 1, \ldots, N-1). \end{aligned}$$

In other words, there exists a constant $B < \infty$ so that, uniformly for $h \in (0, \bar{h})$,

$$||\boldsymbol{\mathcal{E}}_n||_1 \leq B M_{m+1} h^m \quad \text{for } n = 0, 1, \ldots, N-1.$$

Recall now the local representations (1.1.25) and (1.1.26) for e_h' and e_h: for $n = 0, 1, \ldots, N-1$ and $v \in [0, 1]$, they yield the estimates

$$\begin{aligned} |e_h'(t_n + vh_n)| &\leq \Lambda_m ||\boldsymbol{\mathcal{E}}_n||_1 + h^m M_{m+1} k_m \leq \Lambda_m B M_{m+1} h^m + M_{m+1} k_m h^m \\ &=: C_1 M_{m+1} h^m, \end{aligned}$$

and

$$\begin{aligned}|e_h(t_n + vh_n)| &\leq |e_h(t_n)| + h\bar{\beta}||\mathcal{E}_n||_1 + h^{m+1}M_{m+1}k_m \\ &\leq \bar{b}\sum_{\ell=0}^{n-1} h_\ell||\mathcal{E}_\ell||_1 + h^m M_{m+1}k_m T + h\bar{\beta}||\mathcal{E}_n||_1 + h^m M_{m+1}k_m T \\ &\leq (\bar{b}BT + k_m T + \bar{\beta}Bh + k_m h)M_{m+1}h^m \\ &=: C_0 M_{m+1}h^m,\end{aligned}$$

where $\Lambda_m := \max_{(j)} ||L_j||_\infty$ and $\bar{\beta} := \max_{(j)} ||\beta_j||_\infty$. This establishes the desired estimates of Theorem 1.1.2. We note that Guillou and Soulé (1969) derived these estimates for the *first subinterval* $\bar{\sigma}_0$.

We have presented the proof of the global convergence estimates in Theorem 1.1.2 in some detail because, as we shall soon see, analogous global collocation error estimates for various types of Volterra integral and integro-differential equations can be established along very similar lines. In other words, the key to the proof of such results consists in a *suitable local representation (on σ_n) of the solution y* of the given integral or integro-differential equation *which reflects* (i) the *regularity of y*, and (ii) the *choice of the (local) basis* employed in the *representation of the piecewise polynomial collocation solution u_h*. Since the latter is most conveniently chosen to be the local Lagrange basis, the Peano Kernel Theorem is clearly the appropriate tool for the local representation of y (or y'), especially if the exact solution does not have full regularity.

Remark The above proof reveals that we could have stated Theorem 1.1.2 under weaker regularity conditions on y: if assumption (a) is replaced by $a, g \in C^d(I)$, with $1 \leq d < m$ (implying $y \in C^{d+1}(I)$), then its proof can be trivially modified to show that now $u_h \in S_m^{(0)}(I_h)$ satisfies only

$$||y^{(\nu)} - u_h^{(\nu)}||_\infty \leq C_\nu ||y^{(d+1)}||_\infty h^d \quad (\nu = 0, 1). \tag{1.1.33}$$

Compare also Theorem 3.2.4 which contains the above result as a special case.

For certain choices of the collocation parameters $\{c_i\}$ we obtain *global superconvergence* on I; that is, the estimate (1.1.20) holds with m replaced by $m + 1$, as is made precise in the following theorem.

Theorem 1.1.3 *Assume that the assumptions (b), (c) of Theorem 1.1.2 hold and let (a) be replaced by $a, g \in C^d(I)$, with $d \geq m + 1$. If the m collocation parameters $\{c_i\}$ are subject to the orthogonality condition*

$$J_0 := \int_0^1 \prod_{i=1}^m (s - c_i)ds = 0, \tag{1.1.34}$$

then the corresponding collocation solution $u_h \in S_m^{(0)}(I_h)$ *satisfies, for* $h \in (0, \bar{h})$,

$$||y - u_h||_\infty \leq Ch^{m+1}, \tag{1.1.35}$$

with C *depending on the collocation parameters and on* $||y^{(m+2)}||_\infty$ *but not on* h*. The exponent* $m+1$ *cannot, in general, be replaced by* $m+2$*. For the derivative* u_h' *we attain only* $||y' - u_h'||_\infty = \mathcal{O}(h^m)$.

We remind the reader that the orthogonality condition (1.1.34) implies that the interpolatory m-point quadrature formula over [0, 1] whose abscissas are the collocation parameters c_i possesses the *higher degree of precision* of (at least) m, while for arbitrary $\{c_i\}$ the degree of precision is only $m-1$ (see, for example, Davis and Rabinowitz (1984), Atkinson (1989), or Plato (2002)). This orthogonality condition is often written in the form

$$J_0 = \int_0^1 M_m(s)ds = 0,$$

where (see also Lemma 1.1.12)

$$M_m(s) := \frac{1}{m!}\prod_{i=1}^{m}(s - c_i), \quad s \in [0, 1],$$

denotes the so-called *collocation polynomial* associated with the collocation parameters $\{c_i\}$.

Proof Let

$$\delta_h(t) := -u_h'(t) + f(t, u_h(t)), \quad t \in I, \tag{1.1.36}$$

denote the *defect* (or: *residual*) associated with the collocation solution $u_h \in S_m^{(0)}(I_h)$ to the initial-value problem (1.1.1). By definition of the collocation solution the defect δ_h vanishes on the set X_h:

$$\delta_h(t) = 0 \quad \text{for all} \;\; t \in X_h.$$

Moreover, the uniform convergence of u_h and u_h' established in Theorem 1.1.2 implies the uniform boundedness (as $h \to 0$) of δ_h on I, as well as that of its derivatives of order not exceeding d (compare also Exercise 1.10.4).

Consider now the linear ODE (1.1.19): it follows from (1.1.36) that the collocation error $e_h = y - u_h$ satisfies the equation

$$\delta_h(t) = e_h'(t) - a(t)e_h(t), \quad t \in I.$$

Hence, using the estimates in Theorem 1.1.2 and the notation in its proof we readily derive the estimate

$$||\delta_h||_\infty \leq C_1||y^{(m+1)}||_\infty h^m + a_0 C_0||y^{(m+1)}||_\infty h^m \leq D_1 M_{m+1} h^m, \quad (1.1.37)$$

and this holds for any choice of the $\{c_i\}$. On the other hand, the collocation error e_h solves the initial-value problem

$$e_h'(t) = a(t)e_h(t) + \delta_h(t), \quad t \in I, \quad e_h(0) = 0,$$

whose solution is given by

$$e_h(t) = r(t, 0)e_h(0) + \int_0^t r(t, s)\delta_h(s)ds = \int_0^t r(t, s)\delta_h(s)ds, \quad t \in I. \quad (1.1.38)$$

The function $r = r(t, s)$ denotes the 'resolvent' (or: resolvent kernel) of the ODE (1.1.19)):

$$r(t, s) := \exp\left(\int_s^t a(v)dv\right), \quad \text{with } r \in C^{m+1}(D),$$

where $D := \{(t, s) : \ 0 \leq s \leq t \leq T\}$. For $t = t_n + vh_n \in \bar{\sigma}_n$ the integral term on the right-hand ide of (1.1.38) may be written as

$$\begin{aligned} \int_0^t r(t, s)\delta_h(s)ds &= \sum_{\ell=0}^{n-1} h_\ell \int_0^1 r(t, t_\ell + sh_\ell)\delta_h(t_\ell + sh_\ell)ds \\ &\quad + h_n \int_0^v r(t, t_n + sh_n)\delta_h(t_n + sh_n)ds \\ &=: \sum_{\ell=0}^{n-1} h_\ell \int_0^1 \phi_n(t_\ell + sh_\ell)ds + h_n \int_0^v \phi_n(t_n + sh_n)ds. \end{aligned}$$

Suppose now that each of the integrals over $[0, 1]$ is approximated by the interpolatory m-point quadrature formula with abscissas $\{c_i\}$,

$$\int_0^1 \phi_n(t_\ell + sh_\ell)ds = \sum_{j=1}^{m} b_j \phi_n(t_\ell + c_j h_\ell) + E_n^{(\ell)}(v), \quad v \in [0, 1] \quad (\ell < n). \quad (1.1.39)$$

Here, terms $E_n^{(\ell)}(v)$ denote the quadrature errors induced by these quadrature approximations. By assumption (1.1.34) each of these quadrature formulas has degree of precision m, and thus the Peano Theorem for quadrature (Corollary 1.8.4, with $d = m+1,\ p = m$) implies that the quadrature errors can be bounded by

$$|E_n^{(\ell)}(v)| \leq Q_\ell h_\ell^{m+1}, \quad v \in [0,1] \quad (\ell < n),$$

because the defect δ_h is in C^{m+1} on each subinterval σ_n and has a bounded derivative $\delta_h^{(m+1)}$ on $\bar{\sigma}_n$ (see Exercise 1.10.4). This follows from (1.1.36), with $f(t,y) = a(t)y + g(t)$, and the assumed regularity of a and g (which is inherited by $r(t,s)$). Due to the special choice of the quadrature abscissas, we have $\phi_n(t_\ell + c_j h_\ell) = 0$, because $\delta_h(t) = 0$ whenever $t \in X_h$. Hence, the equation (1.1.38) reduces to

$$e_h(t_n + vh_n) = \sum_{\ell=0}^{n-1} h_\ell E_n^{(\ell)}(v) + h_n \int_0^v r(t_n + vh_n, t_n + sh_n)\delta_h(t_n + sh_n)ds, \tag{1.1.40}$$

$v \in [0,1],\ 0 \leq n \leq N-1$. This leads to the estimate

$$|e_h(t_n + vh_n)| \leq \sum_{\ell=0}^{n-1} h_\ell Q_\ell h_\ell^{m+1} + h_n r_0 ||\delta_h||_\infty, \tag{1.1.41}$$

and so, by (1.1.37) and with $r_0 := \max_{t \in I} \int_0^t |r(t,s)|ds$, to

$$|e_h(t_n + vh_n)| \leq h^{m+1} Q \sum_{\ell=0}^{N-1} h_\ell + h r_0 D_1 M_{m+1} h^{m+1},$$
$$v \in [0,1] \ (0 \leq n \leq N-1).$$

The constant $Q := \max\{Q_\ell : 0 \leq \ell < n \leq N-1\}$ depends on $||y^{(m+2)}||_\infty$. Since this is true uniformly in v and n, the assertion of Theorem 1.1.3 that $||e_h||_\infty \leq Ch^{m+1}$ follows.

Remark In the above proof (cf. (1.1.38)) the representation of the collocation error in terms of the resolvent r of the (homogeneous) ODE and the subsequent quadrature argument already give an indication that a much higher order of convergence may be attained at the *mesh points* $t = t_n$ (*local superconvergence* on I_h). Details will be given in the next section, and it will be shown in Sections 2.2.5 and Section 3.2.4 that the principle underlying the analysis of the attainable order of global and local superconvergence extends to Volterra integral and integro-differential equations, as well as to Volterra functional equations with non-vanishing delays (Chapter 4).

1.1.3 Local superconvergence results on I_h

We observed in the proof of Theorem 1.1.3 on global superconvergence of the collocation solution u_h that there is a close link between the attainable (optimal) order on I and the degree of precision of the m-point interpolatory quadrature formula whose abscissas are the collocation parameters $\{c_i\}$. The reason (cf. (1.1.41) and (1.1.38)) that the order of *global* superconvergence cannot exceed $p = m + 1$ is given by the fact that on $I \setminus X_h$ the defect δ_h is in general only $\mathcal{O}(h^m)$. If, however, we restrict e_h to the points of the mesh I_h then, by (1.1.40) with $v = 0$, the inequality (1.1.41) reduces to

$$|e_h(t_n)| \leq \sum_{\ell=0}^{n-1} h_\ell Q_\ell h_\ell^{m+1}, \quad 1 \leq n \leq N, \tag{1.1.42}$$

where $Q_\ell := \max\{|E_n^{(\ell)}(v)| : v \in [0, 1]\}$ $(\ell < n)$. Since the exponent in h_ℓ^{m+1} reflects the degree of precision of the quadrature formulas governed by (1.1.34), we are able to replace these terms by $h_\ell^{m+\kappa}$ with $0 \leq \kappa \leq m$, provided that the collocation parameters $\{c_i\}$ satisfy the more general orthogonality condition

$$J_\nu := \int_0^1 s^\nu \prod_{i=1}^{m} (s - c_i) ds = 0, \quad \nu = 0, \ldots, \kappa - 1, \tag{1.1.43}$$

with $J_\kappa \neq 0$, and the solution y has the appropriate regularity. This condition – which says that the collocation polynomial $M_m(s)$ is orthogonal with respect to the polynomial space $\pi_{\kappa-1}$ – implies that the m-point interpolatory quadrature formula with the m distinct abscissas $\{c_i\}$ has degree of precision $m + \kappa$ (see, e.g. Davis and Rabinowitz (1984)). In other words, the quadrature argument that formed the basis of the the proof of Theorem 1.1.3 now shows that 1.1.42 can be replaced by

$$|e_h(t_n)| \leq \sum_{\ell=0}^{n-1} h_\ell Q_\ell h_\ell^{m+\kappa} \leq h^{m+\kappa} QT \quad (h_\ell \leq h \in (0, \bar{h});\ \kappa \leq m), \tag{1.1.44}$$

uniformly for $v \in [0, 1]$ and $1 \leq n \leq N$. Thus we have

Theorem 1.1.4 *Assume:*

(a) *The solution of the initial-value problem (1.1.1) lies in $C^{m+\kappa}(I)$, for some κ with $1 \leq \kappa \leq m$ and value as specified in (b) below.*
(b) *The m distinct collocation parameters $\{c_i\}$ are chosen so that the general orthogonality condition (1.1.43) holds, with $J_\kappa \neq 0$.*

Then, for all meshes I_h with $h \in (0, \bar{h})$, the collocation solution $u_h \in S_m^{(0)}(I_h)$ corresponding to the collocation points X_h based on these $\{c_i\}$ satisfies

$$\max\{|y(t) - u_h(t)| : t \in I_h\} \leq C_0 h^{m+\kappa}, \tag{1.1.45}$$

where C_0 depends on the collocation parameters and on $||y^{(m+\kappa+1)}||_\infty$ but not on h.
Moreover, if $c_m = 1$, then

$$\max\{|y'(t) - u_h'(t)| : t \in I_h \backslash \{0\}\} = \mathcal{O}(h^{m+\kappa}),$$

too. For $c_m < 1$ we only have $e_h'(t_n) = \mathcal{O}(h^m)$ $(n = 1, \ldots, N)$.

Proof For linear IVPs, $f(t, y) = a(t)y + g(t)$, with $a, g \in C^{m+\kappa}(I)$, the proof is obvious from the remarks preceding Theorem 1.1.3. Its extension to *nonlinear* initial-value problems (1.1.1) will be studied in Section 1.1.4.

Corollary 1.1.5 *For $\kappa = m$ the (unique) set $\{c_i\}$ of collocation parameters satifying the orthogonality conditions (1.1.43) is given by the Gauss (–Legendre) points, i.e. the zeros of the (shifted) Legendre polynomial $P_m(2s - 1)$, and for these points we have*

$$\max\{|y(t) - u_h(t)| : t \in I_h\} \leq C h^{2m},$$

while $\max\{|y'(t) - u_h'(t)| : t \in I_h \backslash \{0\} = \mathcal{O}(h^m)$ *only.*

Remark It was shown by Kuntzmann in 1961 (see Kuntzmann and Ceschino (1963)) and by Butcher (1964) that 'classical' (discrete) m-stage implicit Runge–Kutta–Gauss methods have order of convergence $p = 2m$ (see also Hammer and Hollingsworth (1955) for the case $m = 2$). The above result for the corresponding *continuous* m-stage Runge–Kutta–Gauss methods was established by Guillou and Soulé (1969) and by Wright (1970); see also the 1979 paper by Nørsett and Wanner, and the book by Hairer, Nørsett and Wanner (1993).

In applications one is often interested in obtaining collocation solutions that approximate the solution y *and* its derivative y' on the mesh I_h with the same (high) order. As we have shown above, this will not be true for collocation at the Gauss points (for which $c_m < 1$). This can be seen from the differentiated form of (1.1.38) at $t = t_n$,

$$e_h'(t_n) = r(t_n, t_n)\delta_h(t_n) + \int_0^{t_n} \frac{\partial r(t_n, s)}{\partial t} \delta_h(s) ds$$

with $r(t, t) = 1$: while the quadrature argument employed to establish (1.1.44) can be aplied to the integral term, (1.1.37) shows that $\delta_h(t_n) = \mathcal{O}(h^m)$ only unless t_n $(1 \le n \le N)$ is a collocation point. In the linear case (1.1.19) it follows from

$$e_h'(t) = a(t)e_h(t) + \delta_h(t), \quad t \in X_h,$$

that the order of $e_h'(t)$ matches the one of $e_h(t)$ at $t = t_n$ if and only if $\delta_h(t_n) = 0$; that is, when $c_m = 1$. (An analogous argument shows that this is also true for nonlinear problems; see Section 1.1.4.) Thus, $\kappa \le m - 1$. This observation yields the following two corollaries on 'balanced' optimal local superconvergence.

Corollary 1.1.6 *Let $\kappa = m - 1$ and assume that the collocation parameters $\{c_i\}$ are the Radau II points, that is, the zeros of $P_m(2s - 1) - P_{m-1}(2s - 1)$. Then the collocation solution $u_h \in S_m^{(0)}(I_h)$ has the property that*

$$\max_{t \in I_h \setminus \{0\}} |e_h^{(\nu)}(t)| \le C_\nu h^{2m-1} \quad (\nu = 0, 1), \tag{1.1.46}$$

for all meshes I_h with $h \in (0, \bar{h})$.

If we consider smooth collocation solutions $u_h \in S_m^{(1)}(I_h)$ $(m \ge 2)$, corresponding to collocation parameters with $c_1 = 0$ and $c_m = 1$ (compare the remark preceding Theorem 1.1.1), then the optimal local order cannot exceed $2(m - 1)$:

Corollary 1.1.7 *Let the $\{c_i\}$ be the Lobatto points ($\kappa = m - 2$, with $m \ge 2$), given by the zeros of $s(s - 1)P'_{m-1}(2s - 1)$. Then the collocation error e_h corresponding to the collocation solution $u_h \in S_m^{(1)}(I_h)$ satisfies*

$$\max_{t \in I_h \setminus \{0\}} |e_h^{(\nu)}(t)| \le C_\nu h^{2(m-1)} \quad (\nu = 0, 1) \tag{1.1.47}$$

for all $h \in (0, \bar{h})$.

The following section will reveal that all these superconvergence results remain true for nonlinear initial-value problems.

1.1.4 Nonlinear initial-value problems

If the function $f = f(t, y)$ describing the initial-value problem $y'(t) = f(t, y(t))$ is such that the solution y exists uniquely on I and is in $C^{m+1}(I)$, then the global convergence result of Theorem 1.1.2 remains valid for such nonlinear equations: the role of $a(t_{n,i})$ in the error equation (1.1.28) is now assumed by $f_y(t_{n,i}, \cdot)$, where the second argument comes from the application of the mean-value theorem (i.e. the linear version of Taylor's Theorem). The details of the proof are left as an exercise.

In order to extend the superconvergence results of Theorems 1.1.3 and 1.1.4 to *nonlinear* initial-value problems (1.1.1) we may either employ a linearisation argument in the equation for the collocation error,

$$e_h'(t) = f(t, y(t)) - \{f(t, u_h(t)) - \delta_h(t)\}, \quad t \in I, \tag{1.1.48}$$

where $u_h(t) = y(t) - e_h(t)$, and then use a 'perturbed' counterpart of the resolven representation of (1.1.38); or we may resort to the *nonlinear variation-of-constants formula* of Gröbner and Alekseev (see, e.g. Hairer, Nørsett and Wanner (1993, pp. 96–97)). We will choose the first approach and then comment briefly on the second one.

Assuming that $f_{yy}(t, y)$ is bounded for $(t, y) \in I \times \Omega$, we may write

$$f(t, y(t)) - f(t, y(t) - e_h(t)) = f_y(t, y(t))e_h(t) - (1/2)f_{yy}(t, w(t))e_h^2(t),$$

where, by Taylor's Theorem, $w(t) := y(t) - \theta e_h(t),\ \theta \in (0, 1)$. Thus, the error equation (1.1.48) assumes the form

$$e_h'(t) = a_1(t)e_h(t) + a_2(t)e_h^2(t) + \delta_h(t), \quad t \in I, \quad e_h(0) = 0, \tag{1.1.49}$$

where $a_1(t) := f_y(t, y(t))$ and $a_2(t) := -(1/2)f_{yy}(t, w(t))$. Setting $r_1(t, s) := \exp(\int_s^t a_1(v)ds)$, the solution of this perturbed linear initial-value problem is given by

$$e_h(t) = \int_0^t r_1(t, s)\big(\delta_h(s) + a_2(s)e_h^2(s)\big)ds, \quad t \in I, \tag{1.1.50}$$

in analogy to (1.1.38). Hence, recalling the quadrature argument of the proof of Theorem 1.1.3 and the global error estimate of Theorem 1.1.2 we obtain, with $A_2 := ||a_2||_\infty$,

$$\begin{aligned} |e_h(t_n)| &\le \sum_{\ell=0}^{n-1} h_\ell Q_\ell h_\ell^{m+\kappa} + A_2||e_h^2||_\infty \\ &\le QTh^{m+\kappa} + A_2(C_0h^m)^2 = \mathcal{O}(h^{m+\kappa}), \end{aligned}$$

since $\kappa \le m$. This completes the proof.

As we mentioned above, another – more elegant – way of extending the convergence estimates (1.1.45) to nonlinear problems is based on a nonlinear version of (1.1.38). This is the nonlinear variation-of-constants formula for (1.1.48) due to Alekseev (1961) and Gröbner (1960) (see, in addition to the reference mentioned above, the 1973 paper by Wanner and Reitberger, also for historical references, and Nørsett and Wanner (1981)).

Theorem 1.1.8 *Let $y = y(t)$ be the solution of the initial-value problem*

$$y' = f(t, y), \quad t \in I, \quad y(0) = y_0, \tag{1.1.51}$$

and let $w = w(t)$ *be an approximate solution to* y *with the same initial value, that is,*

$$w' = f(t, w) - g(t, w), \quad t \in I, \quad w(0) = y_0, \tag{1.1.52}$$

for some g. *If* f_y *exists and is continuous on* $I \times \mathbb{R}$, *and if* $g = g(t, w)$ *is (piecewise) continuous, then*

$$w(t) = y(t) + \int_0^t \Phi(t, s, w(s))g(s, w(s))ds, \quad t \in I. \tag{1.1.53}$$

Here, $\Phi(t, s, w(s)) := (\partial/\partial w)y(t, s, w(s))$ *denotes the partial derivative of the solution* y *passing through* $(s, w(s))$ *with respect to the initial values* $w(s)$.

A nice **proof** of this result can be found in Hairer, Nørsett and Wanner (1993, pp. 96–97). The application of this result is now obvious: the role of w in (1.1.53) is assumed by the collocation solution u_h, and the initial-value problem (1.1.52) is given by

$$u_h'(t) = f(t, u_h(t)) - \delta_h(t), \quad t \in I, \quad u_h(0) = y_0$$

(recall also (1.1.48)), where the defect $\delta_h(t)$ depends by definition on u_h. The quadrature argument introduced in the proofs of Theorems 1.1.3 and 1.1.4 can now be used in (1.1.53) in exactly the same way, supported by our knowledge of the regularity of the integrand.

1.1.5 Collocation for 'integrated' ODEs

When establishing existence and uniqueness results for an initial-value problem of the form

$$y'(t) = f(t, y(t)), \quad t \in I := [0, T], \quad y(0) = y_0, \tag{1.1.54}$$

one resorts to its *integrated form*,

$$y(t) = y_0 + \int_0^t f(s, y(s))ds, \quad t \in I, \tag{1.1.55}$$

and then applies Picard iteration. Suppose now that we use the Volterra integral equation (1.1.55) as the basis for obtaining collocation approximations to the solution y of (1.1.54). Denote by

$$S_{m-1}^{(-1)}(I_h) := \{v : \ v|_{\sigma_n} \in \pi_{m-1} \ (0 \le n \le N-1)\}$$

the space of piecewise polynomials of degree $m - 1 \ge 0$ which may be discontinuous at the interior points $t_1, \ldots, t_{N-1}$ of the mesh I_h (see also Section 2.2.1

for additional details of these collocation spaces). Since the dimension of this linear space is

$$\dim S_{m-1}^{(-1)}(I_h) = Nm = \dim S_m^{(0)}(I_h) - 1,$$

we may employ the same set X_h of collocation points given by (1.1.3), as there is now no prescribed initial condition to be satisfied.

The collocation solution $v_h \in S_{m-1}^{(-1)}(I_h)$ for (1.1.55) is given locally by

$$v_h(t_n + vh_n) = \sum_{j=1}^{m} L_j(v)V_{n,j}, \quad v \in (0, 1], \quad \text{with} \;\; V_{n,j} := v_h(t_n + c_j h_n),$$

and is defined by the collocation equation

$$v_h(t) = y_0 + \int_0^t f(s, v_h(s))ds, \quad t \in X_h. \tag{1.1.56}$$

Setting

$$F_n := \int_0^{t_n} f(s, v_h(s))ds = \sum_{\ell=0}^{n-1} h_\ell \int_0^1 f(t_\ell + sh_\ell, v_h(t_\ell + sh_\ell))ds, \tag{1.1.57}$$

and $t = t_{n,i} := t_n + c_i h_n$, (1.1.56) may be written in the form

$$\begin{aligned} V_{n,i} &= y_0 + F_n + h_n \int_0^{c_i} f(t_n + sh_n, v_h(t_n + sh_n))ds \\ &= y_0 + F_n + h_n \int_0^{c_i} f(t_n + sh_n, \sum_{j=1}^{m} L_j(s)V_{n,j})ds \quad (i = 1, \dots, m). \end{aligned} \tag{1.1.58}$$

We now introduce the *iterated collocation solution* v_h^{it} corresponding to the collocation solution v_h for (1.1.55): it is defined by

$$v_h^{it}(t) := y_0 + \int_0^t f(s, v_h(s))ds, \quad t \in I.$$

For $t \in \bar{\sigma}_n$ it can be written as

$$\begin{aligned} v_h^{it}(t_n + vh_n) &= y_0 + \int_0^{t_n + vh_n} f(s, v_h(s))ds \\ &= y_0 + F_n + h_n \int_0^{v} f(t_n + sh_n, \sum_{j=1}^{m} L_j(s)V_{n,j})ds, \end{aligned} \tag{1.1.59}$$

where $v \in [0, 1]$ and

$$y_0 + F_n = v_h^{it}(t_n).$$

Note that $v_h^{it} \in C(I)$, in contrast to v_h itself.

In general, the integrals occurring in the above collocation equations (1.1.58) and (1.1.59) cannot be found analytically and thus will have to be approximated by suitable quadrature formulas. Suppose that these quadrature formulas are *interpolatory m-point quadrature rules* whose abscissas coincide with, or are based on, the collocation parameters $\{c_i\}$. Hence,

$$\int_0^1 f(t_\ell + sh_\ell, v_h(t_\ell + sh_\ell))ds \doteq \sum_{j=1}^{m} b_j f(t_\ell + c_j h_\ell, v_h(t_\ell + c_j h_\ell)) \quad (\ell < n),$$

and

$$\int_0^{c_i} f(t_n + sh_n, v_h(t_n + sh_n))ds \doteq \sum_{j=1}^{m} a_{i,j} f(t_n + c_j h_n, v_h(t_n + c_j h_n)),$$

where $a_{i,j} := \beta_j(c_i)$ and $b_j := \beta_j(1)$ (cf. (1.1.7)). Due to the presence in general of quadrature errors the so-discretised collocation equation generates a 'perturbed' collocation solution in the same space, $\hat{v}_h \in S_{m-1}^{(-1)}(I_h)$, and corresponding iterated collocation solution $\hat{v}_h^{it}$: they are given respectively by

$$\hat{v}_h(t_n + vh_n) = \sum_{j=1}^{m} L_j(v)\hat{V}_{n,j}, \quad v \in (0, 1], \tag{1.1.60}$$

with $\hat{V}_{n,j} := \hat{v}_h(t_{n,j})$ defined by the solution of the algebraic system

$$\hat{V}_{n,i} = y_0 + \hat{F}_n + h_n \sum_{j=1}^{m} a_{i,j} f(t_{n,j}, \hat{V}_{n,j}) \quad (i = 1, \dots, m), \tag{1.1.61}$$

where

$$\hat{F}_n := \sum_{\ell=0}^{n-1} h_\ell \sum_{j=1}^{m} b_j f(t_{\ell,j}, \hat{V}_{\ell,j}),$$

and by

$$\hat{v}_h^{it}(t_n + vh_n) := y_0 + \hat{F}_n + h_n \sum_{j=1}^{m} \beta_j(v) f(t_{n,j}, \hat{V}_{n,j}), \quad v \in [0, 1]. \tag{1.1.62}$$

Setting $\hat{W}_{n,i} := f(t_{n,i}, \hat{V}_{n,i})$ we may write (1.1.62) as

$$\hat{v}_h^{it}(t_n + vh_n) = \hat{v}_h^{it}(t_n) + h_n \sum_{j=1}^{m} \beta_j(v)\hat{W}_{n,j}, \quad v \in [0, 1], \tag{1.1.63}$$

with

$$\hat{W}_{n,i} = f(t_{n,i}, \hat{v}_h^{it}(t_n) + h_n \sum_{j=1}^{m} a_{i.j}\hat{W}_{n,j}) \quad (i = 1, \dots, m). \tag{1.1.64}$$

Comparing the last two equations with (1.1.6) and (1.1.7), the analogous ones for u_h, a simple induction argument shows that $Y_{n,i} = \hat{W}_{n,i}$ for all i and n, and hence

$$u_h(t) = \hat{v}_h^{it}(t) \quad \text{for all} \quad t \in I.$$

There is a way of avoiding the use of quadrature approximations when employing collocation for the integrated IVP (1.1.55). Suppose the integral equation (1.1.55) is written in *implicitly linear* form: defining $z(t) := f(t, y(t))$, it becomes

$$y(t) = y_0 + \int_0^t z(s)ds, \quad t \in I, \tag{1.1.65}$$

where $z(t)$ is the solution of the implicitly linear Volterra integral equation

$$z(t) = f(t, y_0 + \int_0^t z(s)ds), \quad t \in I \tag{1.1.66}$$

(which is a simple example of a *Volterra–Hammerstein equation*; see Section 2.3). We now approximate the solution z of this nonlinear integral equation (1.1.66) by the collocation solution $z_h \in S_{m-1}^{(-1)}(I_h)$, using the same collocation points X_h as before. With the local representation

$$z_h(t_n + vh_n) = \sum_{j=1}^{m} L_j(v)Z_{n,j}, \quad v \in (0, 1],$$

the corresponding collocation equation becomes

$$Z_{n,i} := z_h(t_{n,i}) = f(t_{n,i}, y_0 + \int_0^{t_{n,i}} z_h(s)ds) \quad (i = 1, \ldots, m),$$

or

$$Z_{n,i} = f(t_{n,i}, y_0 + \Phi_n + h_n \sum_{j=1}^{m} a_{i,j} Z_{n,j}), \quad a_{i,j} := \beta_j(c_i), \tag{1.1.67}$$

with

$$\Phi_n := \int_0^{t_n} z_h(s)ds.$$

Once z_h is known we obtain the approximation y_h to the solution y of (1.1.55) by setting

$$\begin{aligned} y_h(t_n + vh_n) &:= y_0 + \int_0^{t_n + vh_n} z_h(s)ds \\ &= y_0 + \Phi_n + h_n \sum_{j=1}^{m} \beta_j(v)Z_{n,j}, \quad v \in [0, 1]. \end{aligned} \tag{1.1.68}$$

Note that

$$y_h(t_n) = y_0 + \int_0^{t_n} z_h(s)ds = y_0 + \Phi_n.$$

How are these approximations, v_h, v_h^{it}, $\hat{v}_h$, $\hat{v}_h^{it}$, and y_h, related to $u_h \in S_m^{(0)}(I_h)$, the 'direct' collocation approximation to the solution y of (1.1.54)? It is clear from the above analysis that, in general, $v_h \neq u_h$ and, especially, $v_h^{it} \neq u_h$ ('wrong' function space!). But, as the comparison of (1.1.63), (1.1.64) with (1.1.66), (1.1.65) and (1.1.6), (1.1.7) readily reveals, the following is true.

Theorem 1.1.9 *Let $u_h \in S_m^{(0)}(I_h)$ denote the 'direct' collocation solution to the initial-value problem 1.1.54), and let y_h and $\hat{v}_h^{it}$ be the implicitly linear collocation aproximations to (1.1.55) defined by (1.1.66) and (1.1.68), respectively. Then, for all sufficiently small $h > 0$,*

$$u_h(t) = y_h(t) = \hat{v}_h^{it}(t), \quad t \in I.$$

If in (1.1.54) we have $f(t, y) = ay$ for some constant $a \neq 0$, then the interpolatory quadrature formulas used in the discretisation of (1.1.58) and (1.1.59) are *exact*. This leads to the following

Corollary 1.1.10 *Under the assumptions of Theorem 1.1.9 we have, for (1.1.54) with $f(t, y) = ay$,*

$$u_h(t) = v_h^{it}(t), \quad t \in I.$$

Remarks

1. Local superconvergence results for collocation-based implicit Runge–Kutta methods applied to the integrated form of the given initial-value problem were first derived by Axelsson (1969) for the Radau and Lobatto points.

   The reader is also referred to the results in Theorems 5.3.5 and 5.3.6 (for $q = 1$) which reveal more explicitly, and in a more general setting, the connection between $u_h(t)$, $v_h(t)$ and $v_h^{it}(t)$ at $t = h$.
2. The nonlinear Volterra integral operator of (1.1.55) is a special case of a *Volterra–Hammerstein* integral operator. Its general form is

$$(\mathcal{H}y)(t) := \int_0^t K(t, s)G(s, y(s))ds, \quad t \in I,$$

   where G is a (usually smooth) function from $I \times \Omega \subset \mathbb{R} \to \mathbb{R}$. We shall study collocation methods for this important class of nonlinear second-kind Volterra integral equations in Section 2.3.3 (for bounded kernels $K(t, s)$), Section 4.3.4 (VH equations with non-vanishing delays), and Section 6.2.9 (VH equations with weakly singular kernels).

1.1.6 Padé approximations to $\exp(z)$

There is a close connection between the attainable order of local superconvergence (and the asymptotic stability) of the collocation solution $u_h \in S_m^{(0)}(I_h)$ on I_h and certain *Padé approximants* to the exponential function $f(z) = \exp(z)$ – which, for $z := ah$, is the solution at $t = h$ of the initial-value problem

$$y'(t) = ay(t), \quad y(0) = 1.$$

Since this connection will also play a role in Chapter 5 we briefly describe its main points. Additional details may be found in, e.g. Iserles and Nørsett (1991) and Hairer and Wanner (1996).

Definition Let $f = f(z)$ be a complex function that is analytic at $z = 0$ and denote, for given non-negative integers k, ℓ, by $\pi_{k/\ell}$ the set of all rational functions of the form P/Q where P and Q (with $Q(0) = 1$) are polynomials of degree not exceeding k and ℓ, respectively. A function $R_{k/\ell} \in \pi_{k/\ell}$ is called a $[k/\ell]$-*Padé approximant* to f if

$$R_{k/\ell}(z) - f(z) = \mathcal{O}(z^{p^*+1}) \quad \text{near} \;\; z = 0,$$

with

$$p^* := \max\{\rho : \;\; R \in \pi_{k/\ell} \quad \text{so that} \quad R(z) - f(z) = \mathcal{O}(z^{\rho+1})\}.$$

It can be shown that $p^* \geq k + \ell$ implies that the Padé approximant $R_{k/\ell}$ is unique. This is in particular the case for $f(z) = \exp(z)$: here, $p^* = k + \ell$. In the following we will use the notation $R_{k/\ell}$ to denote rational functions in $\pi_{k/\ell}$ which are $[k/\ell]$-Padé approximants while $R_{k,\ell}$ will be a generic element of $\pi_{k/\ell}$.

The following lemma describes the general form of Padé approximants to $\exp(z)$.

Lemma 1.1.11 *Let k and ℓ be given non-negative integers. Then the $[k,\ell]$-Padé approximant to $f(z) = \exp(z)$ is given by*

$$R_{k/\ell}(z) = P_{k,\ell}(z)/Q_{k,\ell}(z), \tag{1.1.69}$$

with

$$P_{k,\ell}(z) := \sum_{j=0}^{k} \frac{k!(\ell + k - j)!}{(k - j)!(k + \ell)!} \frac{z^j}{j!},$$

and

$$Q_{k,\ell}(z) := \sum_{j=0}^{\ell} \frac{\ell!(\ell + k - j)!}{(\ell - j)!(k + \ell)!} \frac{(-z)^j}{j!}.$$

Suppose now that we solve the linear ODE

$$y'(t) = ay(t), \quad t \in I, \quad y(0) = y_0 = 1, \tag{1.1.70}$$

by collocation in $S_m^{(0)}(I_h)$, with uniform mesh I_h and collocation points X_h described by $\{c_i : 0 < c_1 < \ldots < c_m \leq 1\}$. It follows from the collocation equation

$$u_h'(t) = au_h(t), \quad t \in X_h, \quad u_h(0) = 1, \tag{1.1.71}$$

and its computational form for the subinterval $[0, t_1 = h]$,

$$U_{0,i} = ay_0 + ah \sum_{j=1}^{m} a_{i,j} U_{0,j} \quad (i = 1, \ldots, m) \tag{1.1.72}$$

(cf. (1.1.16)), that the value of

$$u_h(t_0 + vh) = u_h(vh) = 1 + h \sum_{j=1}^{m} L_j(v) U_{0,j}, \quad v \in [0, 1],$$

at $t = t_1 = h$ can be expressed in the form

$$u_h(h) = p_m(z)/q_m(z) =: R_{m,m}(z), \quad z := ah,$$

where the right-hand side is a rational function whose numerator p_m and denominator q_m are polynomials of degree not exceeding m. This rational function is obviously an approximation (or, more precisely, an interpolant) to the exact solution $y(h) = \exp(z)$ of (1.1.70) at $t = h$. For special choices of the collocation parameters $\{c_i\}$ the rational approximant is a *Padé approximant* to $\exp(z)$. We mention the two most important cases:

1. If the collocation parameters $\{c_i\}$ are the *Gauss points* (corresponding to the zeros of the shifted Legendre polynomial $P_m(2s - 1)$), then the resulting rational approximation $R_{m,m}(z)$ is the the Padé approximant is $R_{m/m}(z)$ for $\exp(z)$.
2. For the *Radau II* points (given by the zeros of $P_m(2s - 1) - P_{m-1}(2s - 1)$) we obtain the Padé approximant $R_{m,m}(z) = R_{(m-1)/m}(z)$.

We summarise these facts in the next lemma; we shall return to its result (and its proof) in Section 5.2.3.

Lemma 1.1.12 *Let $M(s) = M_m(s) := (1/m!) \prod_{i=1}^{m} (s - c_i)$ denote the collocation polynomial associated with the collocation parameters $\{c_i\}$. Then the value of the collocation solution $u_h \in S_m^{(0)}(I_h)$ to (1.1.70) at $t = h$ is given by* *the rational function*

$$R_{m,m}(z) := \frac{P_{m,m}(z)}{Q_{m,m}(z)} = \frac{\sum_{j=0}^{m} M^{(m-j)}(1)z^j}{\sum_{j=0}^{m} M^{(m-j)}(0)z^j}, \tag{1.1.73}$$

with $z := ah$. *If the* $\{c_i\}$ *are the* m *Gauss points, then* $R_{m,m}(z)$ *is the* $[m/m]$-*Padé approximant* $R_{m/m}(z)$ *to* $y(h) = \exp(z)$*, as described in Lemma 1.1.11: it is given by*

$$P_{m,m}(z) = \sum_{j=0}^{m} \frac{m!(2m-j)!z^j}{(m-j)!(2m)!j!}, \tag{1.1.74}$$

$$Q_{m,m}(z) = \sum_{j=0}^{m} \frac{m!(2m-j)!(-z)^j}{(m-j)!(2m)!j!}, \tag{1.1.75}$$

and hence

$$y(h) - u_h(h) = \mathcal{O}(h^{2m+1}).$$

If the $\{c_i\}$ *are the Radau II points, then*

$$R_{m-1/m}(z) = P_{m-1,m}(z)/Q_{m-1,m}(z),$$

with the polynomials $P_{m-1,m}(z)$ *and* $Q_{m-1,m}(z)$ *obtained from Lemma 1.1.11 by replacing* k *by* $m-1$ *and* ℓ *by* m*. We now have*

$$y(h) - u_h(h) = \mathcal{O}(h^{2m}).$$

Analogous results hold for the *Radau I points* (zeros of $P_m(2s-1) + P_{m-1}(2s-1)$, yielding $R_{m/(m-1)}(z)$) and the *Lobatto points* (zeros of $s(s-1)P'_{m-1}(2s-1)$; leading to $R_{(m-2)/m}(z)$). Details and proofs of these classical results may be found in in the books by Lambert (1991), Iserles and Nørsett (1991, pp. 48–51), Strehmel and Weiner (1992, pp. 75–76), Hairer and Wanner (1996), or in the papers by Guillou and Soulé (1969), Axelsson (1969), and Wright (1970). The comprehensive theory of so-called C-polynomials underlying the above result is is due to Nørsett (1975); see also the generalisation in Iserles (1981).

We shall see in Chapter 5 (Theorem 5.2.7 and Theorem 5.2.8) that the result of Lemma 1.1.12 will no longer be valid if the ODE $y'(t) = ay(t)$ is replaced by a (seemingly closely related) delay differential equation with proportional (vanishing) delay,

$$y'(t) = by(qt), \quad 0 < q < 1, \quad t \geq 0.$$

1.2 Perturbed collocation methods

We have seen at the beginning of Section 1.1 that the (continuous) implicit m-stage Runge–Kutta methods generated by collocation for (1.1.1) in $S_m^{(0)}(I_h)$ form a proper subset of all implicit m-stage RK Methods. Since this insight leads to a very elegant analysis of the optimal superconvergence properties of methods from this subset, there arises the question of whether a similar approach is possible for other implicit RK methods. Nørsett (1980) and Nørsett and Wanner (1981) introduced such a framework in the form of *perturbed collocation methods* which we will briefly describe in his section.

Definition

(a) For given (real) polynomials

$$N_j(t) := \frac{1}{j!}\sum_{i=0}^{m}(p_{i,j} - \delta_{i,j})t^i \quad (j = 1, \ldots, m), \tag{1.2.1}$$

the operator $P_{n,h} : \pi_m \to \pi_m$ defined by

$$(P_{n,h}z)(t) := z(t) + \sum_{j=1}^{m} N_j((t - t_n)/h_n)z^{(j)}(t_n)h_n^j, \quad t \in \bar{\sigma}_n \ (0 \le n \le N - 1), \tag{1.2.2}$$

is called a *perturbation operator* with respect to the mesh $I_h = \{t_n : 0 = t_0 < t_1 < \ldots < t_N = T\}$.

(b) Let $\{c_i\}$ be a given set of m distinct points in [0, 1] and let $P_{n,h}$ be the perturbation operator introduced in (1.2.2). The *perturbed collocation method* corresponding to $P_{n,h}$ consists in finding $u_h \in S_m^{(0)}(I_h)$ so that

$$\begin{aligned} u_h(t_n) &= y_n, \\ u_h'(t_n + c_ih_n) &= f(t_n + c_ih_n, (P_{n,h}u_h)(t_n + c_ih_n) \quad (i = 1, \ldots, m), \\ y_{n+1} &:= u_h(t_n + h_n) \quad (n = 0, 1, \ldots, N - 1). \end{aligned} \tag{1.2.3}$$

Remark The choice $N_j(t) \equiv 0$ $(j = 1, \ldots, m)$ obviously reduces (1.2.3) to the 'classical' collocation method (1.1.7),(1.1.8), since $P_{n,h}$ is now the identity operator.

We observe also that if to each polynomial $N_j(t)$ we add an arbitrary constant multiple of the collocation polynomial $M_m(s) := (1/m!)\prod_{k=1}^{m}(s - c_k)$ (corresponding to $t = t_n + sh_n \in \bar{\sigma}_n$), the method (1.2.3) remains unchanged because $M_m(s)$ vanishes for each $s = c_i$.

Theorem 1.2.1 *The perturbed collocation method (1.2.3) is equivalent to an implicit RK method of the form*

$$Y_{n,i} = f(t_n + c_i h_n, y_n + h_n \sum_{j=1}^{m} a_{i,j} Y_{n,j}) \quad (i = 1, \dots, m), \tag{1.2.4}$$

$$y_{n+1} = y_n + h_n \sum_{i=1}^{m} b_i Y_{n,i} \quad (n = 0, 1, \dots, N-1),$$

where the matrix $A := (a_{i,j}) \in L(\mathbb{R}^m)$ and the vector $\mathbf{b}^T := (b_1, \dots, b_m)$ are now given by

$$A = \hat{V}_m P_m J_m V_m^{-1} \tag{1.2.5}$$

and

$$\mathbf{b}^T = (1, 1, \dots, 1) J_m V_m^{-1}. \tag{1.2.6}$$

Here,

$$V_m := \begin{pmatrix} 1 & c_1 & \dots & c_1^{m-1} \\ 1 & c_2 & \dots & c_2^{m-1} \\ \vdots & & & \vdots \\ 1 & c_m & \dots & c_m^{m-1} \end{pmatrix},$$

$\hat{V}_m$ is the rectangular matrix formed by augmenting V_m by a new last column $(c_1^m, \dots, c_m^m)^T$, and

$$P_m := \begin{pmatrix} 1 & p_{0,1} & \dots & p_{0,m} \\ 0 & p_{1,1} & \dots & p_{1,m} \\ \vdots & \vdots & & \vdots \\ 0 & p_{m,1} & \dots & p_{m,m} \end{pmatrix}, \qquad J_m := \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 1/2 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & 1/m \end{pmatrix}.$$

The entries $p_{i,j}$ in P_m are the coefficients occurring in (1.2.1).

The **proof** of this result is straightforward. It, and the one for Theorem 1.1.2, can also be found in Nørsett and Wanner (1981).

We have the following converse of Theorem 1.2.1, in which a RK method will be called *interpolatory* if (1.2.6) holds.

Theorem 1.2.2 *Consider any m-stage interpolatory RK method with distinct parameters $\{c_i\}$. Then this method is equivalent to a perturbed collocation method (1.2.3).*

We conclude this brief description of perturbed collocation methods with a result on the attainable order (Nørsett and Wanner (1981)).

Theorem 1.2.3 *Assume that for given integers l and $\kappa \leq m$ we have:*

(a) $N_j(t) \equiv 0$ *for* $j = 1, \ldots, l-1$, *and* N_j *has exact degree* j *for* $j = l, \ldots, m$;

(b) $\int_0^1 s^\nu N_j(s) ds = 0$ *for* $j = l, \ldots, m$; $\nu = 0, \ldots, m + \kappa - j - 1$;

(c) $\int_0^1 s^\nu M_m(s) ds = 0$ *for* $\nu = 0, \ldots, \kappa - 1$;

(d) $2l \geq m + \kappa$.

Then the perturbed collocation method has order $p^* \geq m + \kappa$ on the mesh I_h.

The **proof** can again be based on the nonlinear variation-of-constants formula of Alekseev and Gröbner (cf. Theorem 1.1.8): now, the defect is given by

$$\delta_h(t) := -u_h'(t) + f(t, (P_{n,h}u_h)(t)), \quad t \in \bar{\sigma}_n.$$

However, the argument (degree of (piecewise) regularity of δ_h, etc.) is rather more complex than in the classical case. The reader is referred to Nørsett and Wanner (1981) for details; see also Exercise 1.10.7 for the case of a linear ODE.

1.3 Collocation in smoother piecewise polynomial spaces

1.3.1 Divergence of collocation solutions

What can be said about the aproximation properties of collocation solutions u_h that lie in smooth collocation spaces $S_\mu^{(d)}(I_h)$ with $\mu \geq 2$ and $1 \leq d < \mu$? It was shown by Loscalzo and Talbot (1967) (compare also Loscalzo (1968, 1969), Hung (1970), and Schoenberg's 1974 survey paper) that collocation in the 'classical' spline space $S_4^{(3)}(I_h)$ (which corresponds, in the notation of Section 2.1, to $S_{m+d}^{(d)}(I_h)$ with $m = 1$, $d = 3$) at the collocation points $t_n + c_1 h_n$ based on the single collocation parameter $c_1 = 1$ is *divergent*. On the other hand, Callender (1971) proved that collocation in $S_\mu^{(1)}(I_h$ ($\mu \geq 2$) leads to convergent collocation solutions when the $\{c_i\}$ are equidistant: $c_i = i/(\mu - 1)$ ($i = 1, \ldots, \mu - 1$).

Piecewise polynomial collocation methods where some (or all) of the collocation parameters coalesce were briefly considered by Guillou and Soulé (1969, pp. 24–26). Important related work was carried out by Kastlunger and Wanner (1972) on implicit Turán–Runge–Kutta methods; see also Chapter II.13

of Hairer, Nørsett and Wanner (1993). A comprehensive divergence and convergence analysis was provided by Mülthei in the late 1970s and the early 1980s (see especially Mülthei (1979, 1980a)).

Here, we summarise his results on the *divergence* of piecewise polynomial collocation solutions u_h, including Hermite-type methods where some of the collocation points have multiplicity greater than one. See also Nørsett (1984) for a good overview and numerous examples.

Let $u_h \in S_\mu^{(d)}(I_h)$ where the mesh I_h is supposed to be uniform. The dimension of this linear space is $N(\mu - d) + (d + 1)$ (see Section 2.2.1). Set $q := \mu - d$ (this integer is sometimes called the *defect* of the piecewise polynomial spline space $S_\mu^{(d)}(I_h)$), and let $c_1, \dots, c_r$, with $0 < c_1 < \dots < c_r = 1$, be given collocation parameters with multiplicities $\delta_i \geq 1$, where

$$\sum_{i=1}^{r} \delta_i = q = \mu - d.$$

Instead of using local representations of u_h on $\bar{\sigma}_n$ based on Hermite canonical polynomials (see, e.g. Hairer, Nørsett and Wanner (1993, pp. 274–276)), it will be more convenient for our purpose to write

$$u_h(t_n + v h_n) = \sum_{l=0}^{d} \frac{y_h^{(l)} h_n^l}{l!} v^l + \sum_{j=d+1}^{\mu} \alpha_{n,j} v^j, \quad v \in [0, 1],$$

with $y_n^{(l)} := u_h^{(l)}(t_n)$. The collocation equation for u_h at $t_{n,i} := t_n + c_i h_n \in \sigma_n$ is

$$u_h^{(\nu)}(t_{n,i}) = \Phi^{(\nu-1)}(t_{n,i}, u_h(t_{n,i})), \quad \nu = 1, \dots, \delta_i; \quad i = 1, \dots, r,$$

where

$$\Phi^{(k)}(t, y) := \Phi_t^{(k-1)}(t, y) + \Phi_y^{(k-1)}(t, y) f(t, y), \quad k \geq 1; \ \Phi^{(0)}(t, y) := f(t, y).$$

In the methods of Loscalzo and Talbot (1967) we have $d = \mu - 1$ and $c_1 = 1$; hence $q = 1, \ \delta_1 = 1$ and $r = 1$. The generalisation encompasses two possibilities:

(I) $q = \mu - d > 1$, but $c_1 = 1$ with $\delta_1 = q$: this corresponds to *Hermite collocation* at $t = t_{n+1}$ $(n = 0, \dots, N - 1)$.
(II) $q = \mu - d > 1$, with $1 < r \leq q, \ \delta_i \geq 1$. If $r = \mu - d$ then all the parameters c_i have multiplicity one.

We consider first the case (I) where $c_1 = \dots = c_r = 1$, generalising the original approach by Loscalzo and Talbot. In the first paper of Mülthei (1979) the following general result was proved.

Theorem 1.3.1 *Let $u_h \in S_\mu^{(d)}(I_h)$ be the collocation solution to (1.1.1) corresponding to (Hermite) collocation at the collocation points*

$t = t_{n+1} (n = 0, \dots, N-1)$, *each having multiplicity* $q := \mu - d$. *The* u_h *is divergent, as* $h \to 0$ $(Nh = T)$ *whenever*

$$\mu \geq 2(q+1), \quad \textit{or, equivalently, if} \quad d \geq q+2.$$

For collocation in the classical spline space $S_4^{(3)}(I_h)$ (corresponding to $\mu = 4,\ d = 3$ and hence to $q = 1$) collocation at $t = t_{n+1}$ ($c_1 = 1$) leads to a divergent collocation solution u_h. This is the result due to Loscalzo and Talbot (1967). More generally, we have:

Corollary 1.3.2 *If* $u_h \in S_\mu^{(\mu-1)}(I_h)$ $(q = 1)$ *and* $c_1 = 1$, *then* u_h *diverges, as* $h \to 0$, *whenever* $\mu \geq 4$.

If there are interior collocation points present, the divergence/convergence of the collocation solution may or may not depend on the location of these points. The following general divergence result was proved in Mülthei (1980b); it uses the above assumptions and notation.

Theorem 1.3.3 *(i) Assume that* $d \geq q + 1 + \delta_{r,1}$, *where* $\delta_{i,j}$ *denotes the Kronecker symbol. Then the collocation solution* $u_h \in S_\mu^{(d)}(I_h)$ *is divergent, regardless of the location of the (interior) collocation parameters.*

(ii) If we have $c_i = i/r$ $(i = 1, \dots, r-1)$ *and* $\delta_{r-i} \leq \delta_i$ $(i = 1, \dots, \lfloor (r-1)/2 \rfloor)$, *then* u_h *is divergent whenever* $d \geq \delta_r + 2$.

The last theorem (Mülthei (1980b)) shows that for non-equally spaced collocation parameters $\{c_i\}$ the convergence/divergence of the collocation solution will in general depend on their location in (0, 1).

Theorem 1.3.4 *Assume that the degree of regularity* d *in* $S_\mu^{(d)}(I_h)$ *satisfies* $d = \delta_r + 1$ (≥ 2), *with* $r > 1$. *Then the collocation solution* u_h *is divergent if*

$$\prod_{i=1}^{r-1} \left(\frac{1 - c_i}{c_i} \right)^{\delta_i} > 1.$$

Example 1.3.1

(i) $r = 1,\ q = 1,\ c_1 = 1$: Method of Loscalzo and Talbot (1967) (see also Loscalzo (1968, 1969)).

(ii) $r = 1,\ q > 1$ $(d < \mu - 1)$, $c_1 = 1$: Method of Mülthei, analysed in his first three papers of 1980.

(iii) $d = 1,\ c_i = i/r$ $(i = 1, \dots, r)$, $\delta_i = 1$ for all i: the convergence of collocation solutions in $S_\mu^{(1)}(I_h)$ with (simple) equidistant collocation parameters was studied by Callender (1971).

Example 1.3.2 $u_h \in S_{2m}^{(m)}(I_h)$:
For $c_1 = 1,\ \delta_1 = q = m$ $(r = 1)$, Mülthei (1980a, II) showed that u_h is convergent. However, Hermite collocation in the smoother space $S_{2m}^{(m+1)}(I_h$ $(q =$

$m - 1$) leads to divergent approximations, since $d = m + 1 = q + 2$ (Theorem 1.3.1).

Example 1.3.3 $u_h \in S_4^{(2)}(I_h)$:
Here we have $\mu = 4$, $d = q = 2$, and the dimension of this linear space is $2N + 3$. If the collocation parameters are chosen so that $0 < c_1 < c_2 = 1$, then the collocation solution u_h is divergent whenever

$$\frac{1 - c_1}{c_1} > 1,$$

that is, when $c_1 < 1/2$.

Example 1.3.4 $u_h \in S_5^{(2)}(I_h)$:
This space corresponds to $\mu = 5$, $d = 2$, $q = 3$, and its dimension is $3N + 3$. For

$$0 < c_1 < c_2 < c_3 = 1,$$

we have $r = 3$ and $d = 2 = \delta_r + 1$. The collocation solution is divergent whenever

$$\frac{(1 - c_1)(1 - c_2)}{c_1 c_2} > 1.$$

The last two examples reveal that collocation in $S_\mu^{(2)}(I_h)$ ($\mu \geq 4$), with the $\mu - 2$ parameters $\{c_i\}$ given by the *Radau II points*, leads to divergence.

We summarise this general divergence result in the following corollary.

Corollary 1.3.5 *Let* $u_h \in S_\mu^{(2)}(I_h)$ ($\mu \geq 4$) *be the collocation solution corresponding to the* $\mu - 2$ *Radau II points* $\{c_i\}$ *in* $(0, 1]$. *Then* u_h *is divergent.*

Proof The Radau II points are the zeros of $P_{\mu-2}(2s - 1) - P_{\mu-3}(2s - 1)$. The corresponding points $c_i^I := 1 - c_{\mu-1-i}$ $(i = 1, \ldots, \mu - 2)$ are the Radau I points (given by the zeros of $P_{\mu-2}(2s - 1) + P_{\mu-3}(2s - 1)$). Thus, we may write

$$\prod_{i=1}^{r-1} \frac{1 - c_i}{c_i} = \prod_{i=1}^{r-1} \frac{c_{i+1}^I}{c_i} > 1.$$

The assertion follows since the $\{c_i\}$ interlace with the $\{c_i^I\}$:

$$0 = c_1^I < c_1 < c_2^I < \cdots < c_{\mu-2}^I < c_{\mu-2} = 1,$$

and by Theorem 1.3.4 (with $\delta_i = 1$ and $r = q = \mu - 2$).

1.4 Higher-order ODEs

Let $k \geq 2$ be a given integer and consider the initial-value problem

$$y^{(k)}(t) = f(t, y(t), y'(t), \ldots, y^{(k-1)}(t)), \quad t \in I := [0, T], \qquad (1.4.1)$$
$$y^{(\nu)}(0) = y_0^{(\nu)} \quad (\nu = 0, 1, \ldots, k - 1).$$

The comments in Section 1.1.1 motivating the use of the 'natural' collocation space $S_m^{(0)}(I_h)$ when $k = 1$ imply that we will now seek the collocation solution for (1.4.1) in the smooth piecewise polynomial space

$$S_{m+d}^{(d)}(I_h) := \{v \in C^d(I) : \ v|_{\bar{\sigma}_n} \in \pi_{m+d} \ (0 \le n \le N-1)\}$$

with $d = k - 1 \ge 1$ (see also Section 2.2.1). The dimension of this linear vector space is

$$\dim S_{m+d}^{(d)}(I_h) = Nm + d + 1 = Nm + k.$$

Let X_h, the set of collocation points in I defined in (1.1.3). The collocation solution u_h in this space for (1.4.1) is thus defined by

$$\begin{aligned} u_h^{(k)}(t) &= f(t, u_h(t), u_h'(t), \dots, u_h^{(k-1)}(t)), \quad t \in X_h, \\ u_h^{(\nu)}(0) &= y_0^{(\nu)} \quad (\nu = 0, 1, \dots, k-1). \end{aligned} \tag{1.4.2}$$

Setting $y_n^{(\nu)} := u_h^{(\nu)}(t_n)$ $(y_n := y_n^{(0)})$, $Y_{n,j} := u_h^{(k)}(t_{n,j})$ and

$$u_h^{(k)}(t_n + vh_n) = \sum_{j=1}^{m} L_j(v) Y_{n,j}, \quad v \in (0, 1],$$

the local Lagrange representation of $u_h^{(\nu)}$ $(\nu = k-1, \dots, 0)$ on $\bar{\sigma}_n$ is given by

$$u_h^{(\nu)}(t_n + vh_n) = \sum_{\ell=0}^{k-\nu-1} \frac{y_n^{(\nu+\ell)}}{\ell!}(h_n v)^\ell + h_n^{k-\nu} \sum_{j=1}^{m} \beta_{\nu,j}(v) Y_{n,j}, \quad v \in [0, 1], \tag{1.4.3}$$

where we have defined

$$\beta_{\nu,j}(v) := \int_0^v \frac{(v-s)^{k-\nu-1}}{(k-\nu-1)!} L_j(s) ds. \tag{1.4.4}$$

For $\nu = 0$, (1.4.3) yields

$$u_h(t_n + vh_n) = \sum_{\ell=0}^{k-1} \frac{y_n^{(\ell)}}{\ell!}(h_n v)^\ell + h_n^k \sum_{j=1}^{m} \beta_{0,j}(v) Y_{n,j}, \quad v \in [0, 1]. \tag{1.4.5}$$

This allows us to write down the computational form of the collocation equation (1.4.2) corresponding to the m collocation points $t = t_{n,i} \in \bar{\sigma}_n$. However, instead of doing this in complete generality we illustrate this for the important case $k = 2$.

Illustration *Continuous m-stage Runge–Kutta–Nyström method ($k = 2$):* Consider (1.4.1) with $k = 2$,

$$y''(t) = f(t, y(t), y'(t)), \quad t \in I.$$

It follows from the collocation equation on $\bar{\sigma}_n$,

$$Y_{n,i} = f(t_{n,i}, u_h(t_{n,i}), u'_h(t_{n,i})), \quad i = 1, \ldots, m, \tag{1.4.6}$$

that the components of the vector $Y_n := (\, Y_{n,1}, \ldots, Y_{n,m}\,)^T$ are given by the solution of the nonlinear algebraic system

$$Y_{n,i} = f\left(t_{n,i}, y_n + h_n v y_n^{(1)} + h_n^2 \sum_{j=1}^{m} \beta_{0,j}(c_i) Y_{n,j}, y_n^{(1)} + h_n \sum_{j=1}^{m} \beta_{1,j}(c_i) Y_{n,j}\right) \tag{1.4.7}$$

($i = 1, \ldots, m$). Once the solution $\mathbf{Y}_n := (\, Y_{n,1}, \ldots, Y_{n,m}\,)^T$ has been computed the values of u_h and u'_h on $\bar{\sigma}_n$ are determined by

$$u_h(t_n + v h_n) = y_n + h_n v y_n^{(1)} + h_n^2 \sum_{j=1}^{m} \beta_{0,j}(v) Y_{n,j}, \quad v \in [0, 1], \tag{1.4.8}$$

and

$$u'_h(t_n + v h_n) = y_n^{(1)} + h_n \sum_{j=1}^{m} \beta_{1,j}(v) Y_{n,j}, \quad v \in [0, 1], \tag{1.4.9}$$

with

$$\beta_{1,j}(v) := \int_0^v L_j(s) ds \quad \text{and} \quad \beta_{0,j}(v) := \int_0^v (v - s) L_j(s) ds.$$

We now state the global and local convergence theorems for the collocation solution $u_h \in S_{m+d}^{(d)}(I_h)$ ($d = k - 1$) for the linear version of (1.4.1),

$$y^{(k)}(t) = \sum_{\nu=0}^{k-1} a_\nu(t) y^{(\nu)}(t) + g(t), \quad t \in I. \tag{1.4.10}$$

As in the case $k = 1$ these results remain valid for the nonlinear problem (1.4.1), provided it has a sufficiently regular solution on I (see also Chapter 3 and the remark following Theorem 1.4.3). The first theorem describes the attainable order of *global convergence* for arbitrarily chosen collocation points.

Theorem 1.4.1 *Assume that the given functions a_ν ($0 \le \nu \le k - 1$) and g in the linear ODE (1.4.10) are m times continuously differentiable on I. Then for all sufficiently small $h > 0$ and any $\{c_i\}$ we have the estimates*

$$||y^{(\nu)} - u_h^{(\nu)}||_\infty \le C_\nu h^m \quad (\nu = 0, 1, \ldots, k - 1)$$

and

$$\sup_{t \in I} |y^{(k)}(t) - u_h^{(k)}(t)| \le C_k h^m.$$

For certain special sets $\{c_i\}$ we obtain *global superconvergence* on I, as described in the following theorem.

Theorem 1.4.2 *Assume that the given functions a_ν $(0 \le \nu \le k-1)$ and g in (1.4.10) are in $C^d(I)$ with $d \ge m+1$, and let the $\{c_i\}$ be chosen such that the orthogonality condition*

$$J_0 := \int_0^1 \prod_{i=1}^{m} (s - c_i) ds = 0$$

holds. Then the collocation solution $u_h \in S_{m+d}^{(d)}(I_h)$ $(d = k-1)$ satisfies, for all sufficiently small $h > 0$,

$$||y^{(\nu)} - u_h^{(\nu)}||_\infty \le C_\nu h^{m+1} \quad (\nu = 0, 1, \ldots, k-1).$$

While the collocation solution u_h and its derivatives $u_h', \ldots, u_h^{(k-1)}$ are globally superconvergent on I, with order $p^* = m+1$, we only have $\mathcal{O}(h^m)$-convergence for $u_h^{(k)}$ on I.

This result suggests (recalling the proof for the case $k = 1$) that *local superconvergence*, of order up to $2m$, at the *mesh points* is also possible.

Theorem 1.4.3 *Let $a_\nu \in C^{m+\kappa}(I)$ $(0 \le \nu \le k-1)$, $g \in C^{m+\kappa}(I)$, for some κ with $1 \le \kappa \le m$, and assume that the $\{c_i\}$ satisfy*

$$J_\ell := \int_0^1 s^\ell \prod_{i=1}^{m} (s - c_i) ds = 0, \quad \ell = 0, 1, \ldots, \kappa - 1,$$

with $J_\kappa \ne 0$. Then for all sufficiently small mesh diameters $h > 0$ the collocation solution $u_h \in S_{m+k-1}^{(k-1)}(I_h)$ and its derivatives $u_h^{(\nu)}$ $(\nu = 1, \ldots, k-1)$ are superconvergent on the mesh I_h:

$$\max_{t \in I_h} |y^{(\nu)}(t) - u_h^{(\nu)}(t)| \le C_\nu h^{m+\kappa} \quad (\nu = 0, 1, \ldots, k-1).$$

In particular, $\kappa = m$ (implying that the $\{c_i\}$ are the m Gauss points in $(0, 1)$) leads to

$$\max_{t \in I_h} |y^{(\nu)}(t) - u_h^{(\nu)}(t)| \le C_\nu h^{2m} \quad (\nu = 0, 1, \ldots, k-1).$$

If $\kappa = m-1$ and $c_m = 1$ (corresponding to the Radau II points in $(0, 1]$), then local superconvergence of order $2m-1$ holds also for $u_h^{(k)}$ at the points $I_h \setminus \{0\}$: we now have

$$\max_{t \in I_h \setminus \{0\}} |y^{(\nu)}(t) - u_h^{\nu}(t)| \le C_\nu h^{2m-1} \quad (\nu = 0, 1, \ldots, k-1, k).$$

We will see in Chapter 3 that these results can be viewed as corollaries to analogous statements for *Volterra integro-differential equations* of order $k \ge 2$,

$$y^{(k)}(t) = f(t, y(t), y'(t), \ldots, y^{(k-1)}(t)) + (\mathcal{V}^{(k} y)(t),$$

where

$$(\mathcal{V}^{(k)}y)(t) := \int_0^t K(t, s, y(s), y'(s), \ldots, y^{(k)}(t)).$$

The details will be presented in Section 3.2.5 (Theorems 3.2.12 and 3.2.13).

1.5 Multistep collocation

We have seen in Section 1.2 that if $u_h \in S_m^{(0)}(I_h)$ is obtained by collocation at the Gauss points then it is locally superconvergent (on I_h) of order $p^* = 2m$. Since the numerical implementation of the collocation method will become rather expensive for large m and, especially, for systems of ODEs resulting from the semidiscretisation in space of (parabolic) PDEs, there arises the question of 'cheaper' collocation methods of comparable order. The *multistep collocation methods* (introduced by Lie (1990) and Lie and Nørsett (1989) in the late 1980s; see also Hairer and Wanner (1996, pp. 270–278)) – which contain as special cases the one-leg methods of Dahlquist (1983) and the BDF methods – represent a possible alternative. These methods form themselves a particular class of so-called *general linear methods* introduced by Butcher (see, e.g. Butcher (1987, Chapter 4) or Hairer and Wanner (1996, pp. 290–295)).

A μ-step collocation method is based on piecewise polynomials $u_h \in S_{m+\mu-1}^{(0)}(I_h)$ $(\mu \geq 2)$, and u_h is defined by the *μ-step collocation equations*

$$u_h(t_\ell) = y_\ell \quad (\ell = n - \mu + 1, \ldots, n), \tag{1.5.1}$$

$$u_h'(t_n + c_i h) = f(t_n + c_i h, u_h(t_n + c_i h)) \quad (i = 1, \ldots, m), \tag{1.5.2}$$

where we have assumed for simplicity that the underlying mesh I_h is uniform. On the interval $[t_{n-\mu+1}, t_{n+1}]$ the collocation solution is described by

$$u_h(t_n + vh) = \sum_{k=1}^{\mu} \phi_k(v) y_{n-\mu-k} + h \sum_{j=1}^{m} \psi_j(v) Y_{n,j} \tag{1.5.3}$$

with $Y_{n,j} := u_h'(t_n + c_j h)$. The functions ϕ_k and ψ_j are the canonical Hermite polynomials (observe that the above problem may be viewed as an *incomplete Hermite interpolation problem* for u_h and u_h') defined by

$$\left\{ \begin{array}{l} \phi_k(\ell) := \delta_{\ell,k} \\ \phi_k'(c_i) := 0 \end{array} \right\} \quad \text{and} \quad \left\{ \begin{array}{l} \psi_j(\ell) := 0 \\ \psi_j(c_i) := \delta_{i,j} \end{array} \right\}.$$

It is a consequence of (incomplete) Hermite (–Birkhoff) interpolation theory (see, e.g. Lorentz et al. (1983)) that, in contrast to one-step collocation, the multistep collocation solution need not exist. An analysis of this problem of existence and uniqueness is given in Lie and Nørsett (1989) and in Hairer and Wanner (1996). However, Lie and Nørsett have shown that there exist μ-step collocation methods whose (optimal) order of local superconvergence is given by $p^* = 2m + \mu - 1 > 2m$. They correspond to sets $\{c_i\}$ which are the abscissas of μ-step Gauss quadrature formulas (see, e.g. Krylov (1962)). The global convergence of μ-step collocation methods is due to Lie (1990). Examples of such μ-step collocation methods, especially for $\mu = 2$, are presented in Lie and Nørsett (1989, pp. 77–78). Here, we mention without proof the following result which represents the μ-step analogue of the local superconvergence result (1.1.45) in Theorem 1.1.4:

Theorem 1.5.1 *For given collocation parameters $\{c_i : 0 \le c_1 < \ldots < c_m \le 1\}$ let $M_m(s) := (1/m!)\prod_{i=1}^{m}(s - c_i)$. Assume that $\mu \ge 2$ and define the determinants $D_\nu^{(\mu)}$ by*

$$D_\nu^{(\mu)} := \begin{bmatrix} \int_{-1}^{0} s^\nu M_m(s)ds & \cdots & \int_{-1}^{0} s^{\nu+\mu-1} M_m(s)ds \\ \vdots & \ddots & \vdots \\ \int_{-(\mu-1)}^{0} s^\nu M_m(s)ds & \cdots & \int_{-(\mu-1)}^{0} s^{\nu+\mu-1} M_m(s)ds \\ \int_{0}^{1} s^\nu M_m(s)ds & \cdots & \int_{0}^{1} s^{\nu+\mu-1} M_m(s)ds \end{bmatrix}.$$

Then the μ-step collocation solution defined by (1.5.1)–(1.5.3), if it exists, has local order $p^* = m + \mu - 1 + \kappa$ $(\kappa \le m)$ on I_h if, and only if, the $\{c_i\}$ are such that

$$D_\nu^{(\mu)} = 0 \quad \textit{for} \quad \nu = 0, 1, \ldots, \kappa - 1.$$

Note that for $\mu = 1$ the above theorem reduces to the first part of Theorem 1.1.4.

As Lie and Nørsett (1989) have shown, this result can be derived either by a suitable adaptation of the Alekseev–Gröbner (nonlinear) variation-of-constants formula, or by an algebraic approach based on the interpolation conditions underlying the method. The latter leads to the following alternative characterisation of the order of local superconvergence.

Theorem 1.5.2 *The μ-step collocation method based on the collocation parameters $\{c_i\}$ possesses the order $p^* = m + \mu - 1 + \kappa$ on I_h if*

$$\frac{d}{ds}\left(\rho_\mu(s)p(s)\right)\Big|_{s=c_i} = 0 \quad (i = 1, \ldots, m) \quad \textit{for all} \quad p \in \pi_{\kappa-1}.$$

Here,

$$\rho_\mu(s) := \prod_{i=-1}^{\mu-1} (s+i).$$

1.6 The discontinuous Galerkin method for ODEs

It was shown by Lesaint and Raviart (1974) that there is a close connection between (collocation based) implicit Runge–Kutta methods and the *discontinuous Galerkin (dG) method* for (1.1.1). In order to describe the dG method we introduce the following notation. For a given mesh I_h let

$$[\phi]_n := \phi(t_n^+) - \phi(t_n^-)$$

denote the jump of the function ϕ at the (interior) mesh point $t = t_n$, and set

$$V(I_h) := \{\phi \in L^2(I) :\ \phi|_{\sigma_n} \text{ is continuous and bounded}\}.$$

The *weak form* of the (scalar) ODE (1.1.1) is then given by: find $y \in C^1(I)$ so that, for each $\phi \in V(I_h)$,

$$\sum_{n=0}^{N-1} \int_{\sigma_n} [y'(t) - f(t, y(t))]\phi(t)dt + \sum_{n=1}^{N-1} [y]_n \phi(t_n^+) + y(t_0^+)\phi(t_0^+) = y_0\phi(t_0^+). \tag{1.6.1}$$

(An analogous definition holds for *systems* of the form (1.1.1.): if $y \in \mathbb{R}^d$ then the above products are replaced by the corresponding inner products in $\mathbb{R}^d$.) Equation (1.6.1) forms the basis for the dG method: given the finite-dimensional subspace $V_h(I_h) = S_m^{(-1)}(I_h)$ of $V(I_h)$ we wish to find $u_h \in V_h(I_h)$ so that, for all $\phi \in V_h(I_h)$,

$$\sum_{n=0}^{N-1} \int_{\sigma_n} [u_h'(t) - f(t, u_h(t))]\phi(t)dt + \sum_{n=1}^{N-1} [u_h]_n \phi(t_n^+) + u_h(t_0^+)\phi(t_0^+) = y_0\phi(t_0^+). \tag{1.6.2}$$

The approximation u_h defined in this way is called the discontinuous Galerkin solution to the initial-value problem (1.1.1) in the space $S_m^{(-1)}(I_h)$. Its existence and uniqueness can be established in complete analogy to that for the collocation solution (see, e.g. Johnson (1988) or Schötzau and Schwab (2000) for a general analysis); this will also become clear from the subsequent discussion.

Although (1.6.2) appears to be a 'global' equation on I, we will now show that it represents in fact a *time-stepping method* similar to the collocation method. This computational form of (1.6.2) says: find a polynomial

$u|_{\sigma_n} =: u_{n,h} \in \pi_m(\sigma_n)$ so that

$$\int_{\sigma_n} [u'_{n,h}(t) - f(t, u_{n,h}(t))]\phi(t)dt + u_{n,h}(t_n^+)\phi(t_n^+) = u_{n,h}(t_n^-)\phi(t_n^-) \qquad (1.6.3)$$

holds for all $\phi \in \pi_m(\sigma_n)$ and $n = 0, 1, \dots, N-1$. Recall that $u_{0,h}(t_0^-) = y_0$.

Suppose now that the integrals in (1.6.3) are approximated by interpolatory $(m+1)$-point quadrature formulas with abscissas $t_{n,j} := t_n + c_j h_n$ $(0 =: c_0 < c_1 < \dots < c_m \le 1)$ and weights w_j $(j = 0, 1, \dots, m)$. We denote the resulting *discretised dG solution* in $S_m^{(-1)}(I_h)$ by $\hat{u}_h$ and write $\hat{u}_{n,h}$ for its restriction to the subinterval σ_n. The fully discretised version of (6.1.3) is then given by

$$\begin{aligned} h_n \sum_{j=0}^{m} w_j [\hat{u}'_{n,h}(t_{n,j}) - f(t_{n,j}, \hat{u}_{n,h}(t_{n,j}))]\phi(t_{n,j}) + \hat{u}_{n,h}(t_n^+)\phi(t_n^+) \\ - \hat{u}_{n,h}(t_n^-)\phi(t_n^-) = 0, \qquad (1.6.4) \end{aligned}$$

for all $\phi \in \pi_m(\sigma_n)$. For ease of notation we will omit the subscript n in $\hat{u}_{n,h}$. Let

$$\hat{y}_n := \hat{u}_h(t_n^-), \quad \hat{U}_{n,0} := \hat{u}_h(t_n^+), \quad \hat{U}_{n,j} := \hat{u}_h(t_{n,j}) \; (j = 1, \dots, m),$$

and let $L_j(v)$ be the jth Lagrange canonical polynomial (of degree $m-1$) corresponding to the points $\{c_i : i = 1, \dots, m\}$. Moreover, denote by $\{\phi_j : j = 0, 1, \dots, m\}$ a (canonical) basis for $\pi_m(\sigma_n)$ so that

$$\phi_i(t_n + c_j h_n) = \delta_{i,j} \quad (i, j = 0, 1, \dots, m).$$

Since the restriction of $\hat{u}'_h$ to σ_n is a polynomial of degree $m-1$ we may write

$$\hat{u}'_h(t_n + vh_n) = \sum_{j=1}^{m} L_j(v)\hat{u}'_h(t_{n,j}), \quad v \in (0, 1].$$

We also have the identity

$$\hat{u}_h(t_n + vh_n) = \hat{u}_h(t_n^+) + h_n \int_0^v \hat{u}'_h(t_n + sh_n)ds, \quad v \in (0, 1]. \qquad (1.6.5)$$

For $\phi = \phi_0$, (1.6.4) reduces to

$$h_n w_0[\hat{u}'_h(t_{n,0}) - f(t_{n,0}, \hat{u}_h(t_{n,0})] + \hat{u}_h(t_n^+) - \hat{u}_h(t_n^-) = 0,$$

with non-vanishing quadrature weight w_0, and this furnishes

$$\hat{U}_{n,0} = \hat{y}_n + h_n w_0 \left(f(t_{n,0}, \hat{U}_{n,0}) - \sum_{j=1}^{m} L_j(c_0)\hat{u}'_h(t_{n,j}) \right). \qquad (1.6.6)$$

For $\phi = \phi_i$ $(i = 1, \dots, m)$, with $\phi_i(t_{n,j}) = \delta_{i,j}$, we obtain from (1.6.4) the equations

$$w_i[\hat{u}'_h(t_{n,i}) - f(t_{n,i}, \hat{u}_h(t_{n,i}))] = 0,$$

where $w_i \neq 0$ $(i = 1, \ldots, m)$. This result can be used in (1.6.6) to produce

$$\hat{U}_{n,0} = \hat{y}_n + h_n w_0 f(t_{n,0}, \hat{U}_{n,0}) - h_n \sum_{j=1}^{m} w_0 L_j(c_0) f(t_{n,j}, \hat{U}_{n,j}), \tag{1.6.7}$$

where we have defined $\hat{U}_{n,j} := \hat{u}_h(t_{n,j})$. Since the identity (1.6.5) allows us to write

$$\hat{U}_{n,i} = \hat{u}_h(t_n^+) + h_n \sum_{j=1}^{m} \beta_j(c_i) f(t_{n,j}, \hat{U}_{n,j}), \tag{1.6.8}$$

with $\beta_j(v)$ as in Section 1.1 (cf. (1.1.6)) and $\beta_j(c_i) =: a_{i,j}$, it follows from (1.6.7) that

$$\hat{U}_{n,i} = \hat{y}_n + h_n w_0 f(t_{n,0}, \hat{u}_h(t_n^+)) + h_n \sum_{j=1}^{m} [a_{i,j} - w_0 L_j(c_0)] f(t_{n,j}, \hat{U}_{n,j}) \tag{1.6.9}$$

$(i = 1, \ldots, m)$. The equations (1.6.7) and (1.6.9) form a system of $m + 1$ nonlinear algebraic equations for $\hat{\mathbf{U}}_n := (\hat{U}_{n,0}, \hat{U}_{n,1}, \ldots, \hat{U}_{n,m})^T \in \mathbb{R}^{m+1}$, with $\hat{U}_{n,0} := \hat{u}_h(t_n^+)$: they closely resemble the ones corresponding to collocation at the points $\{t_{n,0}, t_{n,1}, \ldots, t_{n,m}\}$.

We now show that these equations may indeed be interpreted as the stage equations of an implicit $(m + 1)$-stage Runge–Kutta method. Let $b_j := \beta_j(1)$ $(j = 1, \ldots, m)$, and observe that

$$b_j = \int_0^1 L_j(s)ds = \sum_{k=0}^{m} w_k L_j(c_k) = w_0 L_j(c_0) + w_j,$$

because our interpolatory $(m + 1)$-point quadrature formula is exact for polynomials of degree not exceeding m. This leads to the relationship

$$b_j - w_0 L_j(c_0) = w_j,$$

and hence, by (1.6.5), to the time-stepping equation

$$\hat{y}_{n+1} := \hat{u}_h(t_{n+1}^-) = \hat{y}_n + h_n \sum_{j=0}^{m} w_j f(t_{n,j}, \hat{U}_{n,j}). \tag{1.6.10}$$

We have thus shown that the *discretised dG method* in $S_m^{(-1)}(I_h)$ described by (1.6.10), (1.6.7), (1.6.9) represents an *implicit Runge–Kutta method* with $m + 1$ *stages* for the initial-value problem (1.1.1).

1.7 Spectral and pseudo-spectral methods

Spectral methods (which have their origins in the numerical solution of boundary-value problems; see, e.g. Mercier (1989), Funaro (1992), Fornberg (1996), Boyd (2000), and Trefethen (2000)) are based on finite expansions, in terms of orthogonal functions, approximating the unknown solution. These approximating series employ Fourier expansions or expansions involving certain orthogonal polynomials. If algebraic or orthogonal polynomials are used, and if the unknown coefficients are determined by *collocation* (at feasible points in I) then the method is called a *pseudo-spectral method*. One of their most prominent features is the exponential convergence of the resulting approximations.

Numerical evidence shows (Kauthen (1998), personal communication) that (pseudo-)spectral methods represent a class of numerical methods for Volterra integral and integro-differential equations that are potentially superior to the piecewise collocation methods described in the following chapters. This is intuitively not surprising, given their success in the numerical treatment of (ordinary and partial) differential equations. However, their analysis remains to be carried out.

1.8 The Peano theorems for interpolation and quadrature

In this section we briefly review two important special cases of the celebrated *Peano Kernel Theorem* due to Giuseppe Peano (1913). The books by Stroud (1974), Davis (1975), and Powell (1981) offer good introductions to this important tool in analysis.

We start with Peano's Theorem on the representation of the *interpolation error*.

Theorem 1.8.1 *Assume:*

(a) *For given abscissas* $a \le \xi_1 < \ldots < \xi_m \le b$, *let*

$$e_m(f;t) := f(t) - \sum_{j=1}^{m} L_j(t) f(\xi_j), \quad t \in [a,b]$$

denote the error between f *and the Lagrange interpolation polynomial of degree* $m-1$ *with respect to the given points* $\{\xi_j\}$;

(b) $f \in C^d[a,b]$ *with* $1 \le d \le m$.

Then $e_m(f;t)$ *possesses the integral representation*

$$e_m(f;t) = \int_a^b K_d(t,s) f^{(d)}(s)\,ds, \quad t \in [a,b], \tag{1.8.1}$$

where the Peano kernel K_d *is given by*

$$K_d(t,s) := \frac{1}{(d-1)!}\left\{(t-s)_+^{d-1} - \sum_{k=1}^{m} L_k(t)(\xi_k - s)_+^{d-1}\right\} .$$

Here, $(t-s)_+^p := 0$ *for* $t < s$ *and* $(t-s)_+^p := (t-s)^p$ *for* $t \geq s$.

Proofs of this important result (as well as of Theorem 1.8.3) may be found for example in Stroud (1974) or Powell (1981); consult also the Notes in Section 1.11 for remarks on more general (abstract) versions of Peano's Theorem.

If the Peano kernel $K_d(t,\cdot)$ has *constant sign* in $[a,b]$, we may use the Mean-Value Theorem for integrals to write the above error representation (1.8.1) as

$$e_m(f;t) = f^{(d)}(\xi)\int_a^b K_d(t,s)ds, \quad t \in [a,b],$$

for some $\xi \in [a,b]$, and this permits the derivation of error bounds of the type

$$|e_m(f;t)| \leq C_d||f^{(d)}||_\infty; \quad 1 \leq d \leq m.$$

In the context of estimating global errors in piecewise polynomial collocation methods the role of f will be taken either by y or by y', and we have $[a,b] = [t_n, t_{n+1}]$, $\xi_j = t_n + c_j h_n$ $(0 \leq c_1 < \ldots < c_m \leq 1)$. In view of these applications we state

Corollary 1.8.2 *Under the assumptions of Theorem 1.8.1 and with* $[a,b] = [t_n, t_{n+1}]$, $t = t_n + vh_n$ $(v \in [0,1], h_n := t_{n+1} - t_n)$, $\xi_j = t_n + c_j h_n$ $(i = 1, \ldots, m)$ *the interpolation error*

$$e_m(f;t) := f(t_n + vh_n) - \sum_{j=1}^{m} L_j(v)f(t_n + c_j h_n), \quad v \in [0,1], \tag{1.8.2}$$

can be expressed in the form

$$e_m(f;t_n + vh_n) = h_n^d \int_0^1 K_d(v,z)f^{(d)}(t_n + zh_n)dz, \quad v \in [0,1], \tag{1.8.3}$$

where

$$K_d(v,z) := \frac{1}{(d-1)!}\left\{(v-z)_+^{d-1} - \sum_{k=1}^{m} L_k(v)(c_k - z)_+^{d-1}\right\} .$$

An analogous result exists for the representation of the error in (weighted) *quadrature formulas* of the form

$$Q_m(f) := \sum_{j=1}^{m} w_j f(\xi_j), \quad a \leq \xi_1 < \ldots < \xi_m \leq b, \tag{1.8.4}$$

approximating the integral $Q(f) := \int_a^b w(t)f(t)dt$, where the weight function w is assumed to satisfy $w \in L^1[a, b]$.

Theorem 1.8.3 *Assume:*

(a) The quadrature formula $Q_m(f)$ defined in (1.8.4) has degree of precision $p \geq 1$.
(b) $E_m(f) := Q(f) - Q_m(f)$, with $Q_m(f)$.
(c) $f \in C^d[a, b]$, with $1 \leq d \leq p + 1$.

Then there exists a function $K_d = K_d(s)$ (the Peano kernel of $Q_m(f)$) so that

$$E_m(f) = \int_a^b K_d(s) f^{(d)}(s) ds. \tag{1.8.5}$$

Moreover,

$$K_d(s) = \frac{1}{(d-1)!} \left\{ \int_a^b w(z)(z - s)_+^{d-1} dz - \sum_{k=1}^{m} w_k(\xi_k - s)_+^{d-1} \right\}.$$

The constant

$$e_d := \int_a^b |K_d(s)| ds$$

is often called the *Peano error constant*. As in the case of interpolation, the above result is the basis for classical bounds for the quadrature errors if the Peano kernel $K_d(s)$ does not change its sign on (a, b). A good discussion of when this occurs may be found in Stroud (1974, pp. 168–182).

In view of later applications we also state the counterpart of Corollary 1.8.2, namely

Corollary 1.8.4 *Let $[a, b] = [t_n, t_{n+1}]$ $x_j = t_n + c_j h_n$ $(0 \leq c_1 < \ldots < c_m \leq 1,\ h_n = t_{n+1} - t_n)$. Then, under the assumptions of Theorem 1.8.3, the quadrature error*

$$E_m(f) := \int_0^1 w(t_n + sh_n) f(t_n + sh_n) ds - \sum_{j=1}^{m} w_j f(t_n + c_j h_n)$$

can be expressed in integral form, namely

$$E_m(f) = h_n^d \int_0^1 K_d(s) f^{(d)}(t_n + sh_n) ds. \tag{1.8.6}$$

Here,

$$K_d(s) = \frac{1}{(d-1)!} \left\{ \int_0^1 w(t_n + sh_n)(z - s)_+^{d-1} dz - \sum_{k=1}^{m} w_k(c_k - s)_+^{d-1} \right\}.$$

1.9 Preview: Collocation for Volterra equations

Formally, the collocation approach described for initial-value problems in ODEs is readily extended to integral equations or integro-differential equations of Volterra type,

$$y(t) = g(t) + (\mathcal{V}_\alpha y)(t), \quad t \in I := [t_0, T], \tag{1.9.1}$$

or

$$y'(t) = f(t, y(t)) + (\mathcal{V}_\alpha y)(t), \quad t \in I, \tag{1.9.2}$$

where $\mathcal{V}_\alpha$ denotes a Volterra integral operator given by

$$(\mathcal{V}_\alpha y)(t) := \int_{t_0}^{t} (t-s)^{-\alpha} k(t, s, y(s))ds \quad (0 \le \alpha < 1),$$

and to delay problems, for example to

$$y'(t) = f(t, y(t), y(\theta(t))) + (\mathcal{W}_{\theta,\alpha} y)(t), \quad t \in I. \tag{1.9.3}$$

Here, the delay $\tau(t)$ in the lag function $\theta(t) = t - \tau(t)$ may be non-vanishing, $\tau(t) \ge \tau_0 > 0$ for $t \in I$, or vanishing, when $\theta(t) = qt = t - (1-q)t$ $(0 < q < 1)$ with $t_0 = 0$. The delay integral operator $\mathcal{W}_{\theta,\alpha}$ has the form

$$(\mathcal{W}_{\theta,\alpha} y)(t) := \int_{\theta(t)}^{t} (t-s)^{-\alpha} k_2(t, s, y(s), y'(s))ds.$$

In the case of (1.9.1) we will study, in Chapter 2 ($\alpha = 0$) and Chapter 6 ($\alpha \in (0, 1]$)), the convergence properties of collocation solutions in $S_{m-1}^{(-1)}(I_h)$ and corresponding iterated collocation solutions,

$$u_h^{it}(t) := g(t) + (\mathcal{V}_\alpha y)(t), \quad t \in I,$$

thus generalising the basic results of Section 1.1.5. Solutions of classical and functional Volterra integro-differential equations (1.9.2) will be approximated in the now familiar collocation space $S_m^{(0)}(I_h)$. However, numerous new problems and questions will be encountered:

- The presence of the 'memory terms' corresponding to the Volterra integral operators will in general necessitate a second discretisation step, consisting of suitable quadrature processesses for the integral terms in the 'exact' collocation equations. How does this affect the (order of) convergence of the resulting 'discretised' collocation solution?
- Weakly singular kernels lead, for smooth data, to solutions y which have very low regularity at $t = t_0$ (unbounded y' for (1.9.1); unbounded y'' for (1.9.2)). Thus, the use of uniform meshes yields collocation solutions with low order

of convergence, regardless of the choice of m or the collocation parameters $\{c_i\}$. Are there ways to restore the optimal orders we had for smooth solutions, by a suitable choice of the mesh (reflecting the non-smooth behaviour of y), or by working in a different (non-polynomial) collocation space?

- The presence of non-vanishing delays θ in a given Volterra equation necessitates a judicious choice of the mesh if u_h is to exhibit the optimal order of convergence, since – for reasons different from those above – the exact solution y will have low regularity at certain ‘primary discontinuity points’ induced by θ.
- Vanishing delays like $\theta(t) = qt$ $(0 < q < 1)$ on $I = [0, T]$ increase the complexity of the analysis (especially of local superconvergence properties of u_h very significantly, due to (initial) ‘overlap’ of the collocation points $t_n + c_i h_n$ and the points $q(t_n + c_i h_n)$ $(i = 1, \ldots, m)$. We will see that the classical local superconvergence results (e.g. ‘$\mathcal{O}(h^{2m})$ for the Gauss points’) are no longer true: the reason underlying this fact is that the solutions to such delay problems can no longer be represented by a variation-of-constants formula.
- First-kind Volterra integral equations are known to be (mildly) ill-posed and hence, again not surprisingly, collocation solutions in piecewise polynomial spaces will no longer be convergent for arbitrary $\{c_i\}$. This fact will have important implications in the convergence analysis for ‘mixed’ systems of Volterra equations (now usually referred to as *integral-algebraic equations*; cf. Gear (1990)), consisting of second-kind Volterra integral equations or Volterra integro-differential equations, and one or more Volterra integral equations of the first kind (Chapter 8). Here, we may paraphrase the title of Petzold’s 1982 paper, by saying that ‘IAEs are not VIEs’.

1.10 Exercises

Exercise 1.10.1 Derive the symmetric form (1.1.12), (1.1.13) of the CIRK method corresponding to $u_h \in S_m^{(0)}(I_h)$ for $y'(t) = a(t)y(t)$, with $a \in C(I)$. State it also in the case where $c_1 = 0,\ c_m = 1$ $(m \geq 2)$.

Exercise 1.10.2 Prove Theorem 1.1.1.

Exercise 1.10.3 Derive the exact form of the integrated Peano remainder term $R_{m+1,n}(v)$ in (1.1.24).

Exercise 1.10.4 Show that, under the regularity assumptions of Theorem 1.1.4, the defect δ_h has derivatives $\delta^{(\nu)}$ $(\nu = 1, \ldots, m + \kappa)$ that are smooth in σ_n and *uniformly bounded* on $\bar{\sigma}_n$ $(n = 0, 1, \ldots, N - 1)$.

Exercise 1.10.5 Prove the nonlinear variation-of-constants formula of Gröbner and Alekseev.

Exercise 1.10.6 Give the details of the proof for the nonlinear counterpart of Theorem 1.1.2.

Exercise 1.10.7 Prove Theorem 1.2.3 for the linear initial-value problem

$$y'(t) = ay(t), \quad t \in I, \quad y(0) = y_0.$$

Exercise 1.10.8 Prove that the collocation solution $u_h \in S_4^{(3)}(I_h)$, with uniform I_h, for $y'(t) = ay(t)$ is divergent.

Exercise 1.10.9 Use Hermite canonical polynomials to derive the collocation equations for $u_h \in S_\mu(^d)(I_h)$ approxiating the solution of (1.1.1). The resulting implicit method is an example of an r-stage, d-derivative method (see Hairer, Nørsett and Wanner **?**, pp. 274–276). In particular, find the method corresponding to $\mu = 4, \ d = 2$.

Exercise 1.10.10 Give the proof of Theorems 1.4.2 and 1.4.3 when $k = 2$.

Exercise 1.10.11 Assume that the initial-value problem 1.4.1 is solved by collocation in $S_{m+d}^{(d)}(I_h)$ $(d = k - 1)$, and suppose that solution of the equivalent initial-value problem for the system consisting of k first-order ODEs is approximated by collocation in $S_m^{(0)}(I_h)$, using the same set X_h of collocation points. Discuss the connection between the approaches (order of superconvergence, etc.).

Exercise 1.10.12 Show that the two-step collocation method has order $p^* = m + 1 + \mu$ if, and only if,

$$\int_0^2 M_m s^\ell ds = \gamma \int_0^1 M_m(s) s^\nu ds, \quad \nu = 1, \ldots, \mu.$$

Here,

$$\gamma := \int_0^2 M_m(s) ds \ / \int_0^1 M_m(s) ds.$$

(Compare Nørsett (1980, pp. 128–129).)

Exercise 1.10.13 Derive the continuous two-stage Runge–Kutta–Nyström method corresponding (i) to the Gauss points; (ii) the Radau II points.

Exercise 1.10.14 Derive the discontinuous Galerkin method of Section 1.6 for $m = 1$ and $m = 2$, using the Gauss points and the Radau II points.

1.11 Notes

The principal aim of the notes given at the end of each chapter is to complement the annotated bibliography. While providing additional information on results cited in that chapter (including their history), they will focus mostly on references dealing with topics not discussed in the book and should be read 'hand-in-hand' with the bibliography.

1.1: Piecewise polynomial collocation for ODEs
Collocation methods were introduced for *boundary-value problems* in linear partial differential equations by Kantorovich in 1934. The analysis of their convergence properties has its origin in the work by Karpilovskaya of 1953 and 1963, and – especially – by Vainikko from 1965 onwards, while the 'modern' analysis of collocation methods in spaces of piecewise polynomials began with the papers by Russell and Shampine (1972) and by de Boor and Swartz (1973). An excellent historical survey, accompanied by a chronological list of references, was given by Matthäus (1980). The book by Ascher, Mattheij and Russell (1995, pp. 213–226) contains a good introduction to collocation methods for two-point BVPs; see also the papers by Ascher and Weiss (1983, 1984) and by Auzinger, Koch and Weinmüller (2002).

The systematic study of collocation methods for *initial-value problems* in ODEs, Volterra integral and integro-differential equations, and other types of functional differential equations has its origin, respectively, in the late 1960s, the early 1970s, and the early 1980s. For initial-value problems in ordinary (first-order) differential equations collocation by continuous piecewise polynomials leads to a subclass of (continuous) implicit Runge–Kutta methods for which results on optimal superconvergence and asymptotic stability properties were well understood by the early 1980s (in fact, related superconvergence results date back to work by Kuntzmann (1961), Butcher (1964), and – especially – by Guillou and Soulé (1969) and Wright (1970)). The subsequent extension of the analysis of collocation methods to more general functional differential and integral equations (Tavernini (1971, 1978), Bellen (1984), Buhmann and Iserles (1991, 1992, 1993), Iserles (1994c, 1997b), Brunner (1997a)) quickly revealed that many of the results obtained for ODEs do either not carry over to many of these equations, or their proofs have presented (and are still presenting) formidable challenges to numerical analysts (see Chapter 5).

The connection between collocation in $S_m^{(0)}(I_h)$ and certain classes of (high-order) Runge–Kutta methods was first observed by Guillou and Soulé (1969) and by Wright (1970). Compare also Nørsett and Wanner (1979) and the relevant sections in, e.g. the monographs by Butcher (1987), Hairer, Nørsett and

Wanner (1993), Hairer and Wanner (1996); and see Lambert (1991), Iserles and Nørsett (1991), and Iserles (1996).

Superconvergence results were also given by Axelsson (1969) (for the Radau and Lobatto points) and Wouk (1976). Wanner (1976) uses the collocation framework to obtain an elegant derivation of A-stability properties. Collocation for ODEs with periodic solutions was studied by L.-Q. Zhang (1991, 1992) (see also the references in these papers); see also Engelborghs, Luzyanina, in 't Hout and Roose (2000) and Engelborghs and Doedel (2002) on collocation for delay DEs with periodic solutions.

Various properties of collocation matrices are described in Wright (1984), Gerard and Wright (1984), Ahmed and Wright (1985); see also Russell and Sun (1997) and D. Sloan (2003) (properties of pseudo-spectral differentiation matrices).

Continuous (and natural extensions of) Runge–Kutta methods are analysed in Zennaro (1985, 1986, 1988); see also Chapter 5 in Bellen and Zennaro (2003). Collocation methods in more general settings (DEs on Lie groups) are discussed in Zanna (1999) and in the illuminating survey by Iserles *et al.* (2000); compare also the monograph by Hairer, Lubich and Wanner (2002). The 1996 monograph by Stuart and Humphries is the authoritative source for their analysis in the framework of dynamical systems; see also Stuart and Peplow (1991) on the dynamics of the θ-method.

The solvability of the system of nonlinear algebraic equations (1.1.7) or (1.1.10) arising in implicit Runge–Kutta methods, and how the order is affected by the iterative process, has been studied by many researchers: see, for example, Liu and Kraaijevanger (1988) (and references), Spijker (1994), Jackson, Kvaernø and Nørsett (1996), and Hairer and Wanner (1996), pp. 215–224.

A very comprehensive survey of superconvergence results (from ODEs and PDES to IEs and IDEs) is given by Křížek and Neittaanmäki (1998); see also their earlier paper of (1987) for an impression of the rapid development of this area of numerical analysis.

1.2: Perturbed collocation methods

The question on how to 'embed' general Runge-Kutta methods in a collocation framework led Nørsett (1980) and Nørsett and Wanner (1981) to introduce perturbed collocation methods. Compare also Nørsett (1984).

1.3: Collocation in smoother piecewise polynomial spaces

Because of the considerable interest in cubic (and more general natural) spline functions in the early 1960s it is not surprising that collocation spaces with high regularity were used in most of the early papers on piecewise polynomial collocation for ODEs, namely in Loscalzo and Talbot (1967), Loscalzo (1968,

1969) (see also Schoenberg's review paper of 1974) and in the 1970s in, e.g. Hung (1970), Callender (1971), Micula (1972), and in the book by Micula (1978, pp. 184–200). However, collocation spaces of high regularity tend to lead to divergent approximations: this was first observed by Loscalzo and Talbot (1967), Loscalzo (1968), and Hung (1970). The comprehensive analysis of the convergence/divergence properties of polynomial spline collocation solutions for ODEs is due to Mülthei who, in a series of papers between 1979 and 1982 provided a complete divergence/convergence theory which also encompassed collocation with multiple c_i. Compare also the related paper by Werner and Hilgers (1986) on nonlinear spline collocation. Related earlier results were given by A. Pahnutov (1975) and by B.I. Kvasov (1973); compare also *MR 80i:65085ab*.

1.4: Higher-order ODEs
Spline collocation methods for initial-value problems in kth-order ODEs were first studied by Micula (1974) for $k = 2$; see also Micula (1978) and its references. The convergence and stability properties are analysed in detail (for $k = 2$) in Kramarz (1978, 1980), van der Houwen, Sommeijer and Cong (1991), Coleman (1992), Coleman and Duxbury (2000), and Paternoster (2000). See also Aguilar and Brunner (1988) and Brunner (1988a, 1988b) for local superconvergence results in kth-order VIDEs, from which analogous results for kth-order ODEs follow.

1.5: Multistep collocation
The papers by Lie (1990) and Lie and Nørsett (1989) contain a complete analysis of multistep collocation methods, and in particular of their superconvergence properties; see also Nguyen Cong and Mitsui (1996). The monograph by Hairer and Wanner (1996, pp. 270–278) should be consulted for a concise review of convergence results for multistep Runge–Kutta and collocation methods. In his 1983 paper Dahlquist comments on the connection between one-leg and (multistep) collocation methods.

1.6: The discontinuous Galerkin method for ODEs
The origins of dG methods for ODEs can be traced back to the early 1970s: see Hulme (1972a, 1972b), Lesaint and Raviart (1974) (connection with implicit Runge–Kutta methods), and Nørsett (1974). Results dealing with superconvergence properties are given in the fundamental papers by Delfour, Hager and Trochu (1981) and Delfour and Dubeau (1986); see also Bensebah and Dubeau (1997). The theory of a priori and, especially, a posteriori error estimates and corresponding adaptive mesh selection is analysed in depth in Johnson (1988), Estep (1995), in the surveys by Eriksson, Estep, Hansbo and Johnson (1995a, 1995b), and in the book by the same authors (1996). Compare also Thomée

(1997, Ch. 12) for a good introduction to dG methods, in the context of time-stepping for parabolic PDEs.

Schötzau and Schwab (2000, 2001) present the definitive analysis of *hp*-versions of dG methods; see also the survey paper by Cockburn, Karniadakis and Shu (2000) and its extensive list of references (including the history of the subject).

1.7: Spectral and pseudospectral methods
The literature on spectral and pseudospectral methods has grown rapidly in the last ten years. The reader will find good introductions in, e.g. the books by Mercier (1989), Funaro (1992), Fornberg (1996), Boyd (2000), and Trefethen (2000); see also the surveys by Tadmore (1987) and the recent paper by D. Sloan (2003) on properties of matrices ocurring in spectral differentiation.

A closely related class of methods are the sinc methods introduced by Stenger; see his 1993 monograph and his survey papers of 1995 and 2000.

1.8: The Peano theorems for interpolation and quadrature
Giuseppe Peano's paper appeared in 1913. Good introductions to (and examples for) Peano's remainder theory can be found in Stroud (1974), Davis (1975), Powell (1981), and Davis and Rabinowitz (1984, pp. 285–295). Its use in the analysis of linear multistep methods for ODEs is discussed in Hairer, Nørsett and Wanner (1993, pp. 375–377).

A more general setting for Peano's remainder theory is given in Sard (1963). De Marchi and Vianello (1996, 1997) discuss abstract versions of the Peano theorems for vector-valued functions and normed spaces. Fractional versions of the Peano theorems are due to Diethelm (1997a, 1997b, 1999).

2

Volterra integral equations with smooth kernels

Piecewise polynomial collocation methods for Volterra integral equations of the first and second kind introduce a number of aspects not present when solving ODEs. The first is that the collocation solution for a second-kind VIE does no longer exhibit $\mathcal{O}(h^{2m})$ superconvergence at the mesh points if collocation is at the Gauss points: this optimal order is recovered only in the iterated collocation solution. For VIEs of the first kind, the collocation solution is convergent only under certain 'stability constraints' on the collocation parameters, and local superconvergence at the mesh points cannot occur. Secondly, the collocation equations are in general not yet in a form amenable to numerical computation, due to the presence of the memory term given by the Volterra integral operator. Thus, another discretisation step, based on appropriate quadrature approximations, is necessary to obtain the fully discretised collocation scheme.

In order to make the book largely self-contained we will begin this and each of the subsequent chapters with a brief introduction to those aspects of the theory of Volterra integral and more general Volterra functional equations that will play a role in the analysis of the corresponding collocation solutions.

2.1 Review of basic Volterra theory (I)

2.1.1 Linear VIEs of the second kind

Let $\mathcal{V}: \ C(I) \to C(I)$ denote the linear Volterra integral operator defined by

$$(\mathcal{V}\phi)(t) := \int_0^t K(t,s)\phi(s)ds, \quad t \in I := [0,T] \ (T < \infty), \tag{2.1.1}$$

where the kernel $K = K(t, s)$ is continuous on $D := \{(t, s) : \ 0 \le s \le t \le T\}$. The integral equation

$$y(t) = g(t) + (\mathcal{V}y)(t), \quad t \in I, \tag{2.1.2}$$

is a (linear) *Volterra integral equation (VIE) of the second kind* for the unknown function $y = y(t)$; $g = g(t)$ is a given continuous function on I. It will be assumed throughout the book that the given functions, and hence the solution, are real-valued.

The classical theory of linear VIEs is due to Vito Volterra (1896a): the starting point in his Nota I was the problem of 'inverting the integral'

$$(\mathcal{V}y)(t) = g(t), \quad t \in I, \quad g(0) = 0, \tag{2.1.3}$$

in $C(I)$. Using the terminology suggested by Lalesco (1908, p. 126), the problem consists in solving a *Volterra integral equation of the first kind*. Volterra showed that under certain conditions on its kernel K (see Section 2.1.4) the first-kind VIE (2.1.3) is equivalent to a second-kind equation to which *Picard iteration* (introduced in Picard (1890)) can be applied. This iteration process leads, via the Neumann series associated with the kernel K in (2.1.1), to the *resolvent kernel* and hence to the 'resolvent representation' of the solution y.

To be more precise, let $y_0(t) := g(t)$ and define the infinite sequence $\{y_n(t)\}$ associated with (2.1.2) by

$$y_n(t) := g(t) + (\mathcal{V}y_{n-1})(t), \quad t \in I, \quad n \ge 1. \tag{2.1.4}$$

A straightforward induction argument shows that the iterates $y_n(t)$ can be expressed in terms of the *iterated kernels* $K_n = K_n(t, s)$ $(n \ge 1)$, namely,

$$y_n(t) = g(t) + \int_0^t \left(\sum_{\nu=1}^{n} K_\nu(t, s) \right) g(s)ds, \quad n \ge 1, \tag{2.1.5}$$

where $K_1(t, s) := K(t, s)$ and

$$K_n(t, s) := \int_s^t K_1(t, v)K_{n-1}(v, s)dv \quad (n \ge 2). \tag{2.1.6}$$

The iterated kernels also satisfy a relationship more general than (2.1.6), as the following result (first established in Volterra (1896a) (Nota I, p. 316) shows.

Lemma 2.1.1 *Let $K \in C(D)$. Then for any integer r with $1 \le r < n$ $(n \ge 2)$,*

$$K_n(t, s) = \int_s^t K_r(t, v)K_{n-r}(v, s)dv, \quad (t, s) \in D. \tag{2.1.7}$$

Proof The above assertion is obviously true for $r = 1$, since $K_1 = K$. Thus, assuming it holds for n, a simple induction argument establishes the result (2.1.7) for $n + 1$. The details are left as an exercise (Exercise 2.5.1).

Remark If we associate with a given iterated kernel K_n the Volterra operator $\mathcal{V}_n : C(I) \to C(I)$ defined by

$$(\mathcal{V}_n\phi)(t) := \int_0^t K_n(t, s)\phi(s)ds, \quad n \geq 1,$$

then the result of Lemma 2.1.1 may be stated in a more general way, by saying that the Volterra integral operators $\mathcal{V}_n$ *commute*:

$$\mathcal{V}_r \circ \mathcal{V}_{n-r} = \mathcal{V}_{n-r} \circ \mathcal{V}_r \quad 1 \leq r < n \ (n \geq 2).$$

Consider now (2.1.6): for $K \in C(D)$, with $\bar{K} := \max\{|K(t, s)| : (t, s) \in D\}$, an induction argument readily yields the uniform bounds

$$|K_n(t, s)| \leq \bar{K}^n \frac{(t - s)^{n-1}}{(n - 1)!} \leq \bar{K}^n \frac{T^{n-1}}{(n - 1)!}, \quad (t, s) \in D \quad (n \geq 1).$$

Thus it follows that the *Neumann series* generated by the given kernel K and by (2.1.6),

$$\sum_{n=1}^{\infty} K_n(t, s) = \lim_{\nu \to \infty} \sum_{n=1}^{\nu} K_n(t, s) =: R(t, s), \quad (t, s) \in D, \tag{2.1.8}$$

converges absolutely and uniformly in D. Hence its limit $R(t, s)$, the so-called *resolvent kernel* associated with the given kernel $K(t, s)$, is continuous on D. This uniform convergence also implies that $R(t, s)$ satisfies

$$R(t, s) = K(t, s) + \sum_{n=2}^{\infty} K_n(t, s) = K(t, s) + \sum_{n=2}^{\infty} \int_s^t K(t, v)K_{n-1}(v, s)dv,$$

which we can write, by (2.1.6) and (2.1.8), as

$$R(t, s) = K(t, s) + \int_s^t K(t, v)R(v, s)dv, \quad (t, s) \in D. \tag{2.1.9}$$

An equivalent equation may be obtained by recalling the result of Lemma 2.1.1 ($r = n - 1$): we readily find

$$R(t, s) = K(t, s) + \int_s^t R(t, v)K(v, s)dv, \quad (t, s) \in D. \tag{2.1.10}$$

In the modern theory of linear Volterra integral equations the resolvent kernel is usually introduced by means of the above *resolvent equations*. We summarise this in the following

Definition 2.1.1 *Let $K \in C(D)$. Then the (unique) resolvent kernel $R = R(t,s)$ corresponding to the given kernel K in the linear Volterra integral equation (2.1.2) is (formally) defined by either of the resolvent equations (2.1.9) and (2.1.10).*

Remark Good introductions to the resolvent theory (including the uniqueness of the solution of the resolvent equations in $C(D)$ and more general spaces like $L^p(D)$) may be found in Miller (1971a, Chapter IV) and Corduneanu (1991, Section 4.2). The most general treatments are those in Gripenberg, Londen and Staffans (1990) (see in particular Sections 2.3–2.5 and Chapter 6) and in Prüss (1993) where the abstract resolvent theory is presented in Chapters I.1, I.2, and III.10. See also the papers by Grimmer (1982), Grimmer and Pritchard (1983), and Grimmer and Prüss (1985).

The existence and uniqueness of solutions to the linear Volterra integral equations (2.1.2) is established in Theorem 2.1.2. This result is due to Volterra and can be found in his Nota I of 1896.

Theorem 2.1.2 *Let $K \in C(D)$, and let R denote the resolvent kernel associated with K. Then for any $g \in C(I)$ the second-kind Volterra integral equation (2.1.2) has a unique solution $y \in C(I)$, and this solution is given by*

$$y(t) = g(t) + \int_0^t R(t,s)g(s)ds, \quad t \in I. \tag{2.1.11}$$

Proof Replace t in the VIE (2.1.2) by v, then multiply the equation by $R(t,v)$ and integrate with respect to v over the interval $[0,t]$. Using Dirichlet's formula and the resolvent equation (2.1.10) we obtain

$$\begin{aligned}\int_0^t R(t,v)y(v)dv &= \int_0^t R(t,v)g(v)dv + \int_0^t R(t,v)\left(\int_0^v K(v,s)y(s)ds\right)dv \\ &= \int_0^t R(t,s)g(s)ds + \int_0^t \left(\int_s^t R(t,v)K(v,s)dv\right)y(s)ds \\ &= \int_0^t R(t,s)g(s)ds + \int_0^t (R(t,s) - K(t,s))\,y(s)ds,\end{aligned}$$

implying that

$$(\mathcal{V}y)(t) = \int_0^t K(t,s)y(s)ds = \int_0^t R(t,s)g(s)ds, \quad t \in I.$$

The resolvent representation (2.1.11) now follows by substituting the above relation in (2.1.2). Thus, (2.1.11) defines a solution $y \in C(I)$ for (2.1.2). In order to show that, under the assumptions of Theorem 2.1.2, this solution is unique, assume that $z \in C(I)$ is also a solution. Since

$$z(v) = g(v) + (\mathcal{V}z)(v), \quad v \in I,$$

multiplication of both sides by $R(t, v)$ and integration with respect to v over $[0, t]$ leads to

$$\begin{aligned}\int_0^t R(t,v)z(v)dv &= \int_0^t R(t,v)g(v)dv + \int_0^t \left(\int_s^t R(t,v)K(v,s)dv\right) z(s)ds \\ &= \int_0^t R(t,v)g(v)dv + \int_0^t (R(t,s) - K(t,s))\, z(s)ds.\end{aligned}$$

Here, we have again employed Dirichlet's formula and the second resolvent equation (2.1.10). By (2.1.11) the above equation thus reduces to

$$0 = [y(t) - g(t)] - \int_0^t K(t,s)z(s)ds = [y(t) - g(t)] - [z(t) - g(t)] = 0,$$
$$t \in I.$$

The uniqueness of the solution y given by (2.1.11) can also be established directly via the integral form of Gronwall's Lemma (see Section 2.1.8). If y and z are two (continuous) solutions of (2.1.2) it follows that

$$y(t) - z(t) = (\mathcal{V}(y - z))(t), \quad t \in I.$$

Hence, assuming again that $|K(t, s)| \le \bar{K}$ in D,

$$|y(t) - z(t)| \le \bar{K} \int_0^t |y(s) - z(s)|ds, \quad t \in I.$$

By Lemma 2.1.14 this yields

$$|y(t) - z(t)| \le 0 \cdot \exp(\bar{K}t) = 0 \quad \text{for all} \quad t \in I,$$

thus verifying that $y(t) = z(t)$ for all $t \in I$.

Remark In contrast to the linear *Fredholm integral equation* of the second kind,

$$y(t) = g(t) + \lambda(\mathcal{F}y)(t), \quad t \in I := [0, T],$$

with

$$(\mathcal{F}y)(t) := \int_0^T K(t,s)y(s)ds$$

and $g \in C(I)$, $K \in C(I \times I)$, the Volterra integral equation

$$y(t) = g(t) + \lambda(\mathcal{V}y)(t), \quad t \in I,$$

possesses a unique solution $y \in C(I)$ for *any* (real or complex) parameter λ. This follows directly from the above analysis of the Neumann series (2.1.8). Alternatively, if we define

$$||\mathcal{V}|| := \sup_{\phi \neq 0} \frac{||\mathcal{V}\phi||_\infty}{||\phi||_\infty} = \max_{t \in I} \int_0^t |K(t,s)|ds \leq \bar{K}T,$$

and recall that $||\mathcal{V}\phi||_\infty \leq ||\mathcal{V}|| \cdot ||\phi||_\infty$, we find

$$||\mathcal{V}_n|| \leq \frac{\bar{K}^n T^n}{n!} \quad (n \geq 1);$$

hence, the inverse of the linear operator $\mathcal{I} - \lambda\mathcal{V}: \ C(I) \to C(I)$ exists as a bounded linear operator for any kernel $K \in C(D)$ and any complex number λ. In other words, the *spectrum* of $\mathcal{V}$, $\sigma(\mathcal{V})$ (that is, the set of values λ for which the operator $\mathcal{I} - \lambda\mathcal{V}$ is not invertible in $C(I)$) is *empty*. This is in general not true for $\mathcal{I} - \lambda\mathcal{F}$ (see, e.g. Fredholm (1903), Pogorzelski (1966), Cochran (1972), Gohberg and Goldberg (1980), or Kress (1999)).

Thus, the VIE $y = g + \lambda\mathcal{V}y$, possesses a unique solution $y \in C(I)$ for any $g \in C(I)$ and any $\lambda \in \mathbb{R}$ (or in $\mathbf{C}$).

The following regularity result is an immediate consequence of the definition of the iterated kernels of K and the resolvent kernel R, since $K \in C^m(D)$ implies $K_n \in C^m(D)$ for all $n \geq 2$ and hence, by the uniform convergence of the Neumann series, $R \in C^m(D)$.

Theorem 2.1.3 *Assume that $K \in C^m(D)$. Then its resolvent R has the same degree of regularity, namely $R \in C^m(D)$. Thus, for any $g \in C^m(I)$ the solution of the Volterra integral equation (2.1.2) satisfies $y \in C^m(I)$.*

It is often advantageous to represent the solution of the linear VIE (2.1.2) in a form resembling the familiar variation-of-constant formula for a linear first-order ODE. To derive this alternative representation, we first observe that the special Volterra integral equation

$$y(t) = g(t) + \int_0^t a(s)y(s)ds, \quad t \in I, \tag{2.1.12}$$

with $g \in C^1(I)$ and $a \in C(I)$, is equivalent to the initial-value problem

$$y'(t) = a(t)y(t) + g'(t), \quad t \in I, \quad y(0) = g(0), \tag{2.1.13}$$

whose solution is given by

$$y(t) = \exp\left(\int_0^t a(v)dv\right) g(0) + \int_0^t \exp\left(\int_s^t a(v)dv\right) g'(s)ds.$$

If we define

$$U(t,s) := \exp\left(\int_s^t a(v)dv\right), \quad (t,s) \in D,$$

we obtain

$$y(t) = U(t,0)y(0) + \int_0^t U(t,s)g'(s)ds, \quad t \in I, \tag{2.1.14}$$

the well-known representation of the solution of the initial-value problem (2.1.13). On the other hand, we have seen that the resolvent kernel R associated with the kernel $K(t,s) := a(s)$ in (2.1.12) satisfies the resolvent equation (2.1.9),

$$R(t,s) = a(s) + \int_s^t a(v)R(v,s)dv, \quad (t,s) \in D, \tag{2.1.15}$$

and hence,

$$\frac{\partial R(t,s)}{\partial t} = a(t)R(t,s), \quad \text{with } R(s,s) = a(s), \quad s \in I. \tag{2.1.16}$$

This initial-value problem possesses the (unique) solution

$$R(t,s) = a(s)\exp\left(\int_s^t a(v)dv\right), \quad (t,s) \in D.$$

In other words, we have shown that for the *special* Volterra integral equation (2.1.12),

$$\frac{\partial U(t,s)}{\partial s} = -R(t,s), \quad (t,s) \in D. \tag{2.1.17}$$

We will now prove that the variation-of-constants formula (2.1.14) can be extended to the general linear Volterra integral equation (2.1.2). It can be found in Bownds and Cushing (1973); compare also Brunner and van der Houwen (1986, pp. 13–14).

Theorem 2.1.4 *Assume that $g \in C^1(I)$ and $K \in C(D)$. Then the (unique) solution $y \in C(I)$ of the Volterra equation (2.1.2) may be expressed in the form*

$$y(t) = U(t,0)g(0) + \int_0^t U(t,s)g'(s)ds, \quad t \in I, \tag{2.1.18}$$

where $U(t,s)$ is the (unique) continuous solution of

$$U(t,s) = 1 + \int_s^t K(t,v)U(v,s)dv, \quad (t,s) \in D. \tag{2.1.19}$$

Moreover, $U(t,s)$ is related to the resolvent kernel $R(t,s)$ of the given kernel $K(t,s)$ by

$$-\frac{\partial U(t,s)}{\partial s} = R(t,s), \quad (t,s) \in D. \tag{2.1.20}$$

Proof Consider first the right-hand side of (2.1.18): since we have $U(t,t) = 1$ on I, integration by parts yields, for $t \in I$,

$$U(t,0)g(0) + \{g(t) - U(t,0)g(0) - \int_0^t \frac{\partial U(t,s)}{\partial s} g(s)ds\}$$
$$= g(t) - \int_0^t \frac{\partial U(t,s)}{\partial s} g(s)ds.$$

Thus, (2.1.18) can be written as

$$y(t) = g(t) - \int_0^t \frac{\partial U(t,s)}{\partial s} g(s)ds, \quad t \in I. \tag{2.1.21}$$

Since y is the unique solution of (2.1.2), comparison of (2.1.11) and (2.1.20) shows that the statement (2.1.20) in Theorem 2.1.4 is true. This can also be seen by observing that the right-hand side of (2.1.19) is continuously differentiable with respect to s:

$$\frac{\partial U(t,s)}{\partial s} = -K(t,s)U(s,s) + \int_s^t K(t,v)\frac{\partial U(v,s)}{\partial s} dv$$
$$= -K(t,t) + \int_s^t K(t,v)\frac{\partial U(v,s)}{\partial s} dv, \quad (t,s) \in D.$$

Multiplying the above equation by (-1) reveals that the resulting equation has the form of the resolvent equation (2.1.9): since that equation is uniquely solvable on D, we must have $(\partial/\partial s)U(t,s) = -R(t,s)$, $(t,s) \in D$.

For certain classes of linear Volterra integral equations it is not necessary to resort to Volterra's classical approach to establish the existence and uniqueness of continuous solutions. Here, we briefly mention Volterra equations corresponding to Volterra operators $\mathcal{V}$ with *finite rank* and described by *degenerate* (or: *finitely decomposable*) *kernels* of the form

$$K(t,s) = \sum_{i=1}^{r} A_i(t)B_i(s), \quad \text{with} \quad A_i,\ B_i \in C(I).$$

Setting

$$z_i(t) := \int_0^t B_i(s)y(s)ds \quad (i = 1, \ldots, r),$$

the integral equation can then be written as an equivalent initial-value problem for a system of linear ordinary differential equations for

$\mathbf{z}(t) := (z_1(t), \ldots, z_r(t))^T$, namely,

$$z_i'(t) = B_i(t)\left(g(t) + \sum_{j=1}^{r} A_j(t)z_j(t)\right) \quad (i = 1, \ldots, r), \quad t \in I,$$

with initial condition $\mathbf{z}(0) = 0$. Since the functions describing this system are continuous in I, there exists a unique solution $\mathbf{z} \in C^1(I)$ satisfying the given initial condition. It then follows from (2.1.2) with the above degenerate kernel that the original integral equation possesses a unique solution $y \in C(I)$ which is given by

$$y(t) = g(t) + \sum_{i=1}^{r} A_i(t)z_i(t) =: g(t) + (\mathbf{A}(t))^T \mathbf{z}(t), \quad t \in I.$$

2.1.2 Linear convolution equations

The resolvent kernel corresponding to the linear Volterra integral equations (2.1.2) with *convolution kernel* $K(t, s) = k(t - s)$,

$$y(t) = g(t) + \int_0^t k(t - s)y(s)ds, \quad t \in I := [0, T], \tag{2.1.22}$$

inherits the convolution structure of $k(t - s)$: we have $R(t, s) =: \rho(t - s)$. This is readily seen from the Picard iteration process applied to (2.1.22): according to (2.1.4) and (2.1.6) the iterated kernels corresponding to $k(t - s)$ are given by

$$k_n(t - s) = \int_0^{t-s} k(t - s - v)k_{n-1}(v)dv \quad (n \geq 2), \quad k_1(t - s) := k(t - s),$$

leading to the (absolutely and uniformly convergent) Neumann series

$$\rho(t - s) := \sum_{n=1}^{\infty} k_n(t - s), \quad 0 \leq t - s \leq T.$$

It also follows that the resolvent equations (2.1.9) and (2.1.10) assume the forms

$$\rho(z) = k(z) + \int_0^z k(z - v)\rho(v)dv, \quad z \in I, \tag{2.1.23}$$

and

$$\rho(z) = k(z) + \int_0^z \rho(z - v)k(v)dv, \quad z \in I, \tag{2.1.24}$$

respectively, with $z := t - s$, and Theorem 2.1.2 for (2.1.22) may be restated as

Theorem 2.1.5 *Let $k \in C(I)$. The for any $g \in C(I)$ the convolution integral equation (2.1.22) possesses a unique solution $y \in C(I)$ which is given by*

$$y(t) = g(t) + \int_0^t \rho(t-s)g(s)ds, \quad t \in I. \tag{2.1.25}$$

Here, the resolvent kernel ρ is defined by the resolvent equation (2.1.23) or (2.1.24).

The following result is often useful in applications (we shall return to a slightly more general version in Section 6.1.2) because it establishes a connection between the solution of the general convolution equation (2.1.22) and one with a particular (constant) forcing function g (Bellman and Cooke (1963)).

Theorem 2.1.6 *Consider the linear convolution equations*

$$y(t) = g(t) + \int_0^t k(t-s)y(s)ds, \quad t \in I, \tag{2.1.26}$$

and

$$w(t) = 1 + \int_0^t k(t-s)w(s)ds, \quad t \in I. \tag{2.1.27}$$

Assume that $g \in C^1(I)$, and $k \in C(I)$. Then the (unique) solutions $y \in C(I)$ and $w \in C(I)$ of (2.1.26) and (2.1.27) are related by

$$y(t) = g(0)w(t) + \int_0^t w(t-s)g'(s)ds, \quad t \in I. \tag{2.1.28}$$

If the solution w is in $C^1(I)$ then we also have

$$y(t) = w(0)g(t) + \int_0^t w'(t-s)g(s)ds, \quad t \in I.$$

Proof Exercise 2.5.5.

An alternative approach to solving linear Volterra integral equations possessing convolution kernels is by means of *Laplace transform techniques*. We shall not pursue this here; the interested reader is referred to the classical monograph by Doetsch (1974); see also Guy and Salès (1991, pp. 23–32) for a detailed description and numerous examples.

2.1.3 Systems of linear VIEs

Systems (of usually very large dimension) arise naturally in the spatial semidiscretisation of partial Volterra integral equations. A typical example is given by the equation

$$-\nabla^2 u(t,x) = f(t,x) - \int_0^t k(t-s)\nabla^2 u(s,x)ds, \quad t \in I := [0,T], \quad x \in \Omega,$$
$$u(t,x) = 0, \quad x \in \partial\Omega \quad (t \in I),$$

which occurs as a mathematical model in linear quasi-static visco-elasticity problems (see, e.g. Shaw, Warby and Whiteman (1997) and Shaw and Whiteman (2001), and their lists of references). Here, $\Omega \subset \mathbb{R}^d$ is open and bounded, with (piecewise) smooth boundary $\partial\Omega$. Spatial approximation of the differential operator based on finite element (or, for simple geometries, finite difference) techniques leads to a system of VIEs of the form

$$\mathbf{y}(t) = \mathbf{g}(t) + \int_0^t \mathbf{K}(t,s)\mathbf{y}(s)ds, \quad t \in I, \tag{2.1.29}$$

where $\mathbf{y}(t) := (\, y_1(t), \ldots, y_M(t)\,)^T \in \mathbb{R}^M$, $\mathbf{g}(t) := (\, g_1(t), \ldots, g_M(t)\,)^T \in \mathbb{R}^M$ and

$$\mathbf{K}(t,s) := \begin{bmatrix} K_{i,j}(t,s) \\ (i,j = 1,\ldots,M) \end{bmatrix} \in L(\mathbb{R}^M)$$

with $\mathbf{g}$ and $\mathbf{K}$ continuous on I and D, respectively. The theory on the existence and uniqueness of a continuous solutions $\mathbf{y}$ follows in a straightforward way from the theory developed in Section 2.1.1 (and was already established by Volterra (1896c)). In particular, the resolvent kernel $\mathbf{R} = \mathbf{R}(t,s) \in L(\mathbb{R}^m)$ satisfies the resolvent equations

$$\mathbf{R}(t,s) = \mathbf{K}(t,s) + \int_s^t \mathbf{K}(t,v)\mathbf{R}(v,s)ds, \quad (t,s) \in D,$$

and

$$\mathbf{R}(t,s) = \mathbf{K}(t,s) + \int_s^t \mathbf{R}(t,v)\mathbf{K}(v,s)ds, \quad (t,s) \in D.$$

(cf. (2.1.9), (2.1.10)).

The following theorems on the existence and uniqueness, and the regularity of solution are thus readily proved: the iterated kernel matrices $\mathbf{K}_n(t,s)$ are defined similar to (2.1.6), and the uniform upper bounds for their norms $||\mathbf{K}_n(t,s)||_\infty$ $(n \geq 1)$ on D are obtained in complete analogy to the scalar case, leading again to absolute and uniform convergence of the Neumann series (cf. (2.1.8)),

$$\mathbf{R}(t,s) := \sum_{n=1}^{\infty} \mathbf{K}_n(t,s), \quad (t,s) \in D,$$

and hence to the fact that $\mathbf{R}$ inherits the regularity of $\mathbf{K}$.

Theorem 2.1.7 *Assume that* $\mathbf{K} \in C(D)$*, and let* $\mathbf{R}$ *denote its resolvent kernel. Then for every* $\mathbf{g} \in C(I)$ *the system of second-kind Volterra integral equations (2.1.29) possesses a unique solution* $\mathbf{y} \in C(I)$*, and this solution is given by*

$$\mathbf{y}(t) = \mathbf{g}(t) + \int_0^t \mathbf{R}(t, s)\mathbf{g}(s)ds, \quad t \in I. \tag{2.1.30}$$

If $\mathbf{g} \in C^m(I)$ *and* $\mathbf{K} \in C^m(D)$ $(m \geq 1)$*, then the solution* $\mathbf{y}$ *of the linear system (2.1.29) lies in* $C^m(I)$*.*

We shall return to these results in Chapter 3 (Section 3.1.2) when discussing neutral Volterra integro-differential equations of the form

$$y^{(k)}(t) = \sum_{\nu=0}^{k-1} a_\nu(t)y^{(\nu)}(t) + g(t) + \int_0^t \sum_{\nu=0}^{k} K_\nu(t, s)y^{(\nu)}(s)ds \quad (k \geq 2).$$

It will be seen that such an equation can be rewritten as a system of $k + 1$ second-kind VIEs with a (sparse) kernel matrix $\mathbf{K} \in L(\mathbb{R}^{k+1})$. This observation, and the result of Theorem 2.1.7, will then play a key role in the derivation of optimal superconvergence estimates for collocation solutions $u_h \in S^{(d)}_{m+d}(I_h)$ $(d = k - 1)$ for the above VIDE.

2.1.4 Linear VIEs of the first kind

The general theory of integral equations with variable upper limit of integration was estabilshed by Volterra (1896a, Nota I). As we mentioned in Section 2.1.1 he studied the solvability of the first-kind integral equation

$$(\mathcal{V}y)(t) := \int_0^t K(t, s)y(s)ds = g(t), \quad t \in I := [0, T], \quad \text{with} \quad g(0) = 0, \tag{2.1.31}$$

under appropriate assumptions on g and the kernel K. Here is Volterra's classical result.

Theorem 2.1.8 *Assume that* K *satisfies* $K \in C(D)$, $\partial K/\partial t \in C(D)$*, and* $|K(t, t)| \geq k_0 > 0$ *for* $t \in I$*. Then for any* $g \in C^1(I)$ *with* $g(0) = 0$ *the integral equation (2.1.31) has a unique solution* $y \in C(I)$*.*

Proof Clearly, the condition that $g(0) = 0$ is necessary for y to be continuous at $t = 0$. The assumptions for K and g permit the differentiation of both sides of (2.1.31), yielding

$$K(t, t)y(t) + \int_0^t \frac{\partial K(t, s)}{\partial t} y(s)ds = g'(t), \quad t \in I.$$

Since $K(t,t)$ does not vanish in I, (2.1.31) is equivalent to the linear second-kind Volterra integral equation

$$y(t) = g_1(t) + \int_0^t K_1(t,s)y(s)ds, \quad t \in I, \tag{2.1.32}$$

where the functions $g_1 \in C(I)$ and $K_1 \in C(D)$ describing this equation are defined by

$$g_1(t) := g'(t)/K(t,t) \quad \text{and} \quad K_1(t,s) := -(\partial K(t,s)/\partial t)/K(t,t).$$

The proof is now completed by appealing to Theorem 2.1.2.

Remark In Section 6.1.6 (Theorem 6.1.16 with $\alpha = 0$) we will see an extension of this classical result to nonlinear first-kind VIEs of *Hammerstein type*,

$$\int_0^t K(t,s)G(s,y(s))ds = g(t), \quad t \in I.$$

Under appropriate regularity assumptions this problem is equivalent to an *implicit* VIE of the second-kind,

$$G(t,y(t)) = g_1(t) - \int_0^t K_1(t,s)G(s,y(s))ds,$$

where g_1 and K_1 have the same meaning as in the proof of Theorem 2.1.8.

Example 2.1.1 The assumption that $K(t,t)$ be non-zero for $t \in I$ is of course not necessary for (2.1.31) to possess a unique continuous solution in I. As an example, consider the kernel

$$K(t,s) = \frac{(t-s)^{r-1}}{(r-1)!}, \quad r \geq 1 \ (r \in \mathbb{N}),$$

which vanishes identically along the line $t = s$. However, equation (2.1.31) with this kernel and with

$$g \in C^r(I), \quad g^{(j)}(0) = 0 \ (j = 0, \ldots, r-1),$$

possesses the unique continuous solution $y(t) = g^{(r)}(t)$, $t \in I$. This can be readily verified either by direct substitution or by differentiating both sides of the integral equation r times.

Example 2.1.2 Isolated zeros of $K(t,s)$ in I can lead to non-uniqueness, as the following example shows. Let

$$K(t,s) = 2t - 3s, \quad g(t) = t^2 \quad (t \in I = [0,T]).$$

Here, $K(0,0) = 0$ and $K(t,t) < 0$ whenever $t > 0$. It is easily verified that $y(t) = 1 + \gamma t$ is a (real) continuous solution for any $\gamma \in \mathbb{R}$.

A detailed discussion of the connection between first-kind VIEs with kernels having $K(0,0)=0$ (and $K(t,t)\neq 0$ when $t>0$) and differential equations of Fuchsian type may also be found in the 1927 survey paper by Davis. The first study of such VIEs is due to Volterra (1896b).

The equivalence between (2.1.31) and (2.1.32) allows us also to derive the following regularity result.

Theorem 2.1.9 *Let $m \geq 0$ and assume that*

(a) $g \in C^{m+1}(I)$, with $g(0)=0$, and
(b) $K \in C^{m+1}(D)$, with $|K(t,t)| \geq k_0 > 0$ for all $t \in I$.

Then the unique solution of the first-kind Volterra integral equation (2.1.31) lies in the space $C^m(I)$.

Remarks

1. First-kind Volterra integral equations with convolution kernel,

$$\int_0^t k(t-s)y(s)ds = g(t), \quad t \in I = [0,T],$$

with $k \in C^1(I)$ and $k(0) \neq 0$ fall of course within the framework of Theorems 2.1.8 and 2.1.9. Like their second-kind conterparts in Section 2.1.2 they can also be solved by Laplace transform techniques. The books by Krasnov, Kissélev and Makarenko (1977) and Guy and Salès (1991) contain details and numerous examples.
2. The book by Srivastava and Buschman (1977) deals with linear first-kind Volterra integral equations possessing special kernels of convolution type; it contains an extensive list of equations whose solution can be found explicitly.

Can the solution of a linear first-kind VIE be represented in terms of a 'resolvent kernel', in analogy to linear VIEs of the second kind? As we shall see in more detail in Section 6.1.5, Niels Henrik Abel showed in his papers of 1823 and 1826 that the solution of the first-kind integral equation with *weakly singular kernel*,

$$\int_0^t (t-s)^{-\alpha} y(s)ds = g(t), \quad t \in I := [0,T] \quad (0<\alpha<1), \tag{2.1.33}$$

where $g \in C^1(I)$, can be written as

$$y(t) = \frac{d}{dt}\left(\int_0^t R(t-s;\alpha)g(s)ds\right), \quad t \in (0,T]. \tag{2.1.34}$$

Here, the resolvent kernel $R(\cdot\,;\alpha)$ has the form

$$R(t-s;\alpha) := \gamma_\alpha (t-s)^{\alpha-1}, \quad 0 \le s < t \le T,$$

with $\gamma_\alpha := \sin(\alpha\pi)/\pi (= 1/[\Gamma(\alpha)\Gamma(1-\alpha)])$. This is reminiscent of the result in Corollary 6.1.4.

Does an analogous resolvent representation for the solution of the first-kind VIE (2.1.31) exists if the kernel K is *smooth* on D? This problem was studied in detail by Gripenberg 1980 (see also the monograph by Gripenberg, Londen and Staffans (1990, pp. 156–167)). He showed that in this case no such function in the classical sense exists: in order for an analogue of (2.1.34) to be true, the resolvent kernel R has to be a *measure*. However, as can be seen in the monograph just mentioned, the general resolvent theory for linear first-kind VIEs is far from being completely understood.

We conclude this section by briefly looking at the *ill-posed* nature of first-kind Volterra integral equations. A detailed treatment, including feasible numerical (e.g. collocation based) methods for such problems, is beyond the scope of this book. The interested reader may wish to consult the excellent survey paper by Lamm (2000) and, in particular, the sections and references to sequential (i.e. 'Volterra type') regularisation methods studied by her and her collaborators. Compare also Ring and Prix (2000) and Ring (2001) for a complementary analysis.

As we have seen earlier, the solution of the (linear) first-kind VIE (2.1.31) does not depend continuously on the given data K and g. If $g \in C^1(I)$, with $g(0) = 0$, and $K \in C^1(D)$, with $|K(t,t)| \ge k_0 > 0$ on I, then the equation can be transformed into an equivalent second-kind equation with solution $y \in C(I)$. However, for a small change from $g(t)$ to $g_\varepsilon(t) := g(t) + \varepsilon g_1(t)$ ($\varepsilon \ne 0$), with $g_1 \in C^1(I),\ g_1(0) \ne 0$, this is no longer possible, and the solution y_ε is no longer continuous on I. The following definition describes a measure of the *degree of ill-posedness* (see Lamm (2000)).

Definition 2.1.2 The Volterra integral operator $\mathcal{V}$ given by (2.1.31) is said to be ν-*smoothing* if there exists an integer $\nu \ge 1$ for which the kernel K of $\mathcal{V}$ satisfies

(a) $$\left.\frac{\partial^j K(t,s)}{\partial t^j}\right|_{s=t} = 0, \quad t \in I, \quad j = 0, 1, \ldots, \nu - 2;$$

(b) $$\left.\frac{\partial^{\nu-1} K(t,s)}{\partial t^{\nu-1}}\right|_{s=t} = k_\nu \ne 0, \quad t \in I;$$

(c) $$\frac{\partial^\nu K}{\partial t^\nu} \in C(D).$$

If $\left.\dfrac{\partial^j K(t,s)}{\partial t^j}\right|_{s=t} = 0$ for $t \in I$ and *all* $j \in \mathbb{N}_0$, then $\mathcal{V}$ is called an *infinitely smoothing* Volterra operator.

The Volterra equation $\mathcal{V}y = g$ is a ν-*smoothing problem* if $\mathcal{V}$ is a ν-smoothing operator and $g \in C^{\nu}(I)$.

Example 2.1.3 If $K(t,s) \equiv 1$ on D, then

$$(\mathcal{V}y)(t) = \int_0^t y(s)ds, \quad t \in I,$$

is a *one-smoothing* operator. The corresponding first-kind VIE describes the process of *differentiating* the function g: $y(t) = g'(t),\ t \in I$.

Example 2.1.4 Let $r \in \mathbb{N}$. Then the Volterra operator given by

$$(\mathcal{V}y)(t) : \int_0^t \frac{(t-s)^{r-1}}{(r-1)!} y(s)ds, \quad t \in I,$$

is r-*smoothing*. We have seen in Example 2.1.1 that the solution of $\mathcal{V}y = g$ is $y(t) = g^{(r)}(t),\ t \in I$, provided we have $g^{(j)}(0) = 0,\ j = 0, \ldots, r-1$.

Remark The degree of ill-posedness – which increases as ν increases – can also be characterised by the *singular values* $\{\sigma_j\}$ of $\mathcal{V}$ (see, e.g. Lamm (2000, p. 75) for references). These singular values behave like $\mathcal{O}(1/j^{\nu})$ as $j \to \infty$.

Example 2.1.5 The *inverse heat conduction problem* (often referred to as the *sideways heat equation*; cf. Eldén (1983) or the references in Lamm (2000, pp. 76–77)) can be formulated as a first-kind VIE whose kernel is

$$K(t,s) = k(t-s) = \frac{1}{2\sqrt{\pi}} t^{-3/2} \exp(-1/(4t)).$$

It is easily seen that the corresponding $\mathcal{V}$ has $\nu = \infty$: it is an *infinitely smoothing* Volterra operator. This is of course consistent with the fact that solving the sideways heat equation is a severely ill-posed problem.

2.1.5 Nonlinear VIEs

Before dealing with general nonlinear VIEs we briefly consider the nonlinear VIE with *degenerate kernel*,

$$y(t) = g(t) + \int_0^t \sum_{i=1}^{r} A_i(t) b_i(s, y(s))ds, \quad t \in I := [0, T]. \qquad (2.1.35)$$

If the functions defining the kernel are continuous then we may formally transform (2.1.35) into a system of nonlinear ordinary differential equations for

$$z_i(t) := \int_0^t b_i(s, y(s))ds \quad (i = 1, \ldots, r).$$

This system of ODEs has the form

$$z_i'(t) = b_i(s, g(s) + \sum_{i=1}^{r} A_i(t)z_i(t)) \quad t \in I \quad (i = 1, \ldots, r),$$

with initial conditions $z_i(0) = 0$. We may thus appeal to the theory of nonlinear ODEs to establish the existence and uniqueness of a solution $\mathbf{z} \in C^1(I)$, with $\mathbf{z} := (\, z_1, \ldots, z_r \,)^T$. The unique solution $y \in C(I)$ of (2.1.35) is then given by

$$y(t) = g(t) + \int_0^t \sum_{i=1}^{r} A_i(t)z_i(s)ds, \quad t \in I.$$

Consider now the general nonlinear Volterra integral equation

$$y(t) = g(t) + \int_0^t k(t, s, y(s))ds =: g(t) + (\mathcal{V}y)(t), \quad t \in I. \tag{2.1.36}$$

For this equation Picard iteration assumes the form

$$y_{n+1}(t) := g(t) + (\mathcal{V}y_n)(t), \quad t \in I \quad (n \geq 0), \tag{2.1.37}$$

with $y_0(t) := g(t)$. The following (local) existence theorem (see, e.g. Miller (1971a)) generalises the classical result for

$$y(t) = y_0 + \int_0^t f(s, y(s))ds, \quad t \in I,$$

which is the integrated form of the initial-value problem $y'(t) = f(t, y(t)),\ y(0) = y_0$ and whose kernel k does not depend on t. To state this result we will adopt the notation

$$D := \{(t, s) :\ 0 \leq s \leq t \leq T\},$$
$$\Omega_B := \{(t, s, y) :\ (t, s) \in D,\ y \in \mathbb{R} \text{ and } |y - g(t)| \leq B\},$$

and we set $M_B := \max\{|k(t, s, y)| :\ (t, s, y) \in \Omega_B\}$.

Theorem 2.1.10 *Assume:*

(a) $g \in C(I)$;
(b) $k \in C(\Omega_B)$;
(c) *K satisfies the Lipschitz condition*

$$|k(t, s, y) - k(t, s, z)| \leq L_B|y - z| \quad \textit{for all } (t, s, y),\ (t, s, z) \in \Omega_B.$$

Then:

(i) The Picard iterates $y_n(t)$ *exist for all* $n \geq 1$*. They are continuous on the interval* $I_0 := [0, \delta_0]$*, where*

$$\delta_0 := \min\{T, B/M_B\},$$

and they converge uniformly on I_0 *to a solution* $y \in C(I_0)$ *of the nonlinear Volterra integral equation (2.1.36).*

(ii) This solution y *is the unique continuous solution on* I_0*.*

Proof

• *Uniqueness:* Suppose that (2.1.36) possesses two continuous solutions y_1 and y_2 on the interval I_0. Hence, by (c),

$$\begin{aligned} |y_1(t) - y_2(t)| &\leq \int_0^t |k(t, s, y_1(s)) - k(t, s, y_2(s)|ds \\ &\leq L_B \cdot \int_0^t |y_1(s) - y_2(s)|ds, \quad t \in I_0. \end{aligned}$$

It follows from the continuity of $|y_1 - y_2|$ and from Lemma 2.1.14 (Section 2.1.8) that

$$|y_1(t) - y_2(t)| \leq 0 \cdot \exp(L_B t) = 0, \quad t \in I_0.$$

Hence, $||y_1 - y_2||_{0,\infty} := \max_{t \in I_0} |y_1(t) - y_2(t)| = 0$, implying that the two solutions are identical on I_0.

• *Existence:* We begin by showing that the Picard iterates defined in (2.1.37) satisfy

$$|y_n(t) - g(t)| \leq M_B t \leq B \quad \text{for all} \quad t \in I_0.$$

Since this assertion is certainly true for $n = 0$, assume it holds for n. This implies that $(t, s, y_n(s)) \in \Omega_B$ when $t \in I_0$. Hence, $k(t, s, y_n(s))$ is well defined and we have

$$|k(t, s, y_n(s))| \leq M_B \quad \text{for} \quad (t, s) \in D.$$

This yields

$$|y_{n+1}(t) - g(t)| \leq \int_0^t |k(t, s, y_n(s))|ds \leq M_B t \leq B, \quad t \in I_0.$$

Thus, $y_{n+1}(t)$ is defined on I_0, and it follows from the continuity of g and k on I and Ω_B that $y_{n+1} \in C(I_0)$.

We now prove that the sequence $\{y_n(t)\}$ defined by Picard iteration (2.1.37) is a Cauchy sequence on I_0. To this end, let $z_n(t) := y_{n+1}(t) - y_n(t)$. It is readily verified that

$$|z_n(t)| \le \frac{M_B L_B^n t^{n+1}}{(n+1)!}, \quad t \in I_0 \quad (n \ge 0).$$

Therefore,

$$y_{n+m}(t) - y_n(t) = \sum_{j=0}^{m-1} [y_{n+j+1}(t) - y_{n+j}(t)]$$

implies that, for all $t \in I_0$,

$$|y_{n+m}(t) - y_n(t)| \le \sum_{j=0}^{m-1} |z_{n+j}(t)| \le M_B \sum_{j=0}^{m-1} \frac{L_B^{n+j} t^{n+j+1}}{(n+j+1)!} = M_B \sum_{j=n+1}^{m+n} \frac{L_B^{j-1} t^j}{j!}.$$

Thus, $\lim_{n\to\infty} y_n(t) =: y(t)$ uniformly on I_0, with limit $y \in C(I_0)$. Using the Lipschitz condition for $k(t, s, y)$ with respect to y (assumption (c)), we obtain

$$\left| \int_0^t [k(t, s, y_n(s)) - k(t, s, y(s))] ds \le L_B \int_0^t |y_n(s) - y(s)| ds \right| \longrightarrow 0, \; t \in I_0,$$

as $n \to \infty$. This allows us to carry out the final step in the existence proof, namely to show that y solves the nonlinear integral equation (2.1.36) in I_0:

$$\begin{aligned} y(t) &= \lim_{n\to\infty} y_n(t) = g(t) + \lim_{n\to\infty} \int_0^t k(t, s, y_{n-1}(s)) ds \\ &= g(t) + \int_0^t k(t, s, \lim_{n\to\infty} y_{n-1}(s)) ds = g(t) + (\mathcal{V}y)(t), \quad t \in I_0. \end{aligned}$$

The proof is now complete.

Does the solution exist (and remain continuous) beyond $t = \delta_0$? Setting $z(t) := y(t + \delta_0)$ the given VIE (2.1.36) can be written in 'shifted' form, namely,

$$z(t) = g_0(t) + \int_0^t k_0(t, s, z(s)) ds, \quad t \ge 0,$$

where

$$g_0(t) = g(t + \delta_0) + \int_0^{\delta_0} k(t + \delta_0, s, y(s)) ds$$

and

$$k_0(t, s, z) := k(t + \delta_0, s + \delta_0, z), \quad t + \delta_0 \le T.$$

Since g_0 and k_0 satisfy the hypotheses of Theorem 2.1.10, we may deduce the existence of a (unique) continuous solution z on some interval $[0, \delta_1]$, with $\delta_1 > 0$, and this implies that the solution y of the orginal VIE (2.1.36) has been continued continuously to $[\delta_0, \delta_1]$. How far this process can be continued obviously depends on the constants B and M_B. A more detailed analysis of the question regarding the continuation of solutions to nonlinear VIEs can be found in, e.g. the books by Miller (1971a, pp. 30–33) or Burton (1983, pp. 66–89).

We conclude this discussion with an example showing that a solution cannot always be continued to an arbitrary interval.

Example 2.1.6 *VIEs with blow-up solutions*

It is well known that solutions of the initial-value problem

$$y'(t) = \lambda y(t) + \epsilon(y(t))^p, \quad t \geq 0 \quad (\lambda \leq 0, \ \epsilon > 0, \ p > 1)$$
$$y(0) = y_0 > 0,$$

may not exist for all $t > 0$; in other words, the solution cannot always be continued to any finite interval. This initial-value problem is of course equivalent to the nonlinear VIE

$$y(t) = y_0 + \int_0^t \{\lambda y(s) + \epsilon(y(s))^p\} ds, \quad t \geq 0, \tag{2.1.38}$$

which may be viewed as a particular case of the 'semilinear' VIE (2.1.40) considered below. The following result shows that the solution may *blow up* in *finite time*: there exists $T_b < \infty$ so that $\lim_{t \to T_b^-} y(t) = +\infty$. Since the VIE (2.1.38) corresponds to a *Bernoulli differential equation*, the proof of Theorem 2.1.11 is elementary.

Theorem 2.1.11 *Assume that $\lambda < 0$, $\epsilon > 0$, $p > 1$ and $y_0 > 0$. Then the (unique) continuous solution of (2.1.38) is (formally) described by*

$$y(t) = \left(\frac{1}{y_0^{1-p} \exp(-\lambda(p-1)t) - (\epsilon/\lambda)(1 - \exp(-\lambda(p-1)t)} \right)^{1/(p-1)}. \tag{2.1.39}$$

For given λ and p it blows up in finite time T_b if, and only if, the initial value y_0 is such that

$$y_0 > (-\lambda/\epsilon)^{1/(p-1)}.$$

The blow-up time is then

$$T_b = \frac{1}{\lambda(p-1)} \ln \left(1 + \frac{\lambda}{\epsilon y_0^{p-1}} \right).$$

Remark Blow-up solutions for the more general VIE

$$y(t) = y_0 + \int_0^t k(t-s))G(y(s))ds,$$

with L^1 kernel k and nonlinearity G satisfying, respectively,

$$k(t) \geq bt \ (b > 0) \quad \text{and} \quad \liminf_{t\to\infty} \frac{G(z)}{z^p} > 0$$

are discussed in Miller (1971a, pp. 46–51). Related blow-up results can be found in, e.g. Okrasiński (1991), Mydlarczyk (1994, 1996, 1999), Bushell and Okrasiński (1996), Olmstead (2000); see also the survey paper by Roberts (1998) and its list of references.

In Chapter 6 (Exercise 6.6.24) we shall encounter an analogous result for a more general nonlinear VIE with weakly singular kernel that arises in, e.g. combustion theory.

As we have seen above, in applications nonlinear VIEs often occur in 'perturbed' (or: 'semi-linear') form,

$$\begin{aligned} y(t) &= g(t) + \int_0^t K(t,s)\{y(s) + H(s, y(s))\}ds, \\ &=: g(t) + (\mathcal{V}y)(t) + (\mathcal{H}y)(t), \quad t \in I, \end{aligned} \tag{2.1.40}$$

where $\mathcal{V}$ is our usual linear Volterra integral operator (2.1.1) corresponding to the kernel K and where $\mathcal{H}$ denotes the *Volterra–Hammerstein* integral operator defined by

$$(\mathcal{H}y)(t) := \int_0^t K(t,s)H(s, y(s))ds;$$

here, $K \in C(D)$ and H is a ('small') smooth function. If (2.1.40) possesses a unique solution $y \in C(I)$, this equation may be rewritten in the form of a nonlinear variation-of-constant representation in which the integral term corresponding to $\mathcal{H}$ is viewed as a perturbation of the linear VIE described by $\mathcal{V}$. This is made precise in the following theorem (Grossman and Miller (1970, 1973)).

Theorem 2.1.12 *Suppose that the nonlinear integral equation (2.1.40) has a unique solution $y \in C(I)$, and let $H : I \times \mathbb{R} \to \mathbb{R}$ be (Lipschitz) continuous. Then the solution of this equation may be written as*

$$y(t) = y_\ell(t) + \int_0^t R(t,s)H(s, y(s))ds, \quad t \in I. \tag{2.1.41}$$

Here, y_ℓ denotes the (unique) solution of the linear part of (2.1.40) and is given by

$$y_\ell(t) = g(t) + \int_0^t R(t,s)g(s)ds, \quad t \in I,$$

with $R = R(t, s)$ denoting the resolvent kernel corresponding to the given kernel $K = K(t, s)$.

Proof Setting

$$Q(t) := (\mathcal{H}y)(t) = \int_0^t K(t, s)H(s, y(s))ds,$$

and applying Theorem 2.1.2 to the 'linear' integral equation

$$y(t) = g(t) + Q(t) + \int_0^t K(t, s)y(s)ds, \quad t \in I,$$

we obtain

$$\begin{aligned}
y(t) &= g(t) + Q(t) + \int_0^t R(t, s)\{g(s) + Q(s)\}ds \\
&= g(t) + \int_0^t R(t, s)g(s)ds + [Q(t) + \int_0^t R(t, s)Q(s)ds] \\
&= y_\ell(t) + \int_0^t K(t, s)H(s, y(s))ds \\
&\quad + \int_0^t \left(\int_v^t R(t, s)K(s, v)ds\right) H(v, y(v))dv \\
&= y_\ell(t) + \int_0^t R(t, s)H(s, y(s))ds.
\end{aligned}$$

Here, we have made use of the resolvent equation (2.1.10) (read from right to left) to replace the integral involving the product $R(t, s)K(s, v)$ by the difference of R and K.

The nonlinear second-kind Volterra integral equation (2.1.40) is a particular case of a more general Volterra–Hammerstein integral equation often encountered in applications (see, e.g. Brunner (1991) and its references).

Definition 2.1.3 The nonlinear Volterra integral operator $\mathcal{H}$ given by

$$(\mathcal{H}y)(t) := \int_0^t K(t, s)G(s, y(s))ds, \quad t \in I. \tag{2.1.42}$$

is called a *Volterra–Hammerstein operator*. Here, $G : I \times \mathbb{R} \to \mathbb{R}$ is smooth, while the kernel function $K = K(t, s)$ may be continuous (bounded) or weakly singular (the latter case will be considered in Section 6.1.4). The corresponding second-kind Volterra integral equation,

$$y(t) = g(t) + (\mathcal{H}y)(t), \quad t \in I, \tag{2.1.43}$$

is a *Volterra–Hammerstein* integral equation of the second kind.

Note that in the above equation (2.1.40) the nonlinearity G is given by $G(s, y) = y + H(s, y)$.

Remark In his 1930 paper A. Hammerstein analysed the solvability of the nonlinear *Fredholm-type* integral equation

$$y(t) = g(t) + \int_0^T K(t, s)G(s, y(s))ds, \quad t \in I.$$

Therefore such equations now carry the name of Hammerstein. Hammerstein's analysis was continued by Niemytzki (1934); see also the books by Tricomi (1957), Krasnosel'skii and Zabreiko (1984), and Corduneanu (1991, pp. 86–88) for the theory of Hammerstein equations of Fredholm or Volterra type.

In Section 2.3.3 we shall see that it will frequently be advantageous to rewrite a nonlinear second-kind Volterra integral equation of Hammerstein form (2.1.43) as follows. Define the *Niemytzki operator* (or: substitution operator) $\mathcal{N}$ by

$$(\mathcal{N}\phi)(t) := G(t, \phi(t)), \quad t \in I,$$

and set $z(t) := (\mathcal{N}y)(t)$. The original Volterra–Hammerstein equation (2.1.43) then becomes

$$y(t) = g(t) + (\mathcal{V}\mathcal{N}y)(t), \quad t \in I, \tag{2.1.44}$$

and can thus be written as an *implicitly linear* integral equation for z,

$$z(t) = G(t, g(t) + (\mathcal{V}z)(t)), \quad t \in I, \tag{2.1.45}$$

where $\mathcal{V}$ denotes the *linear* Volterra integral operator with kernel $K(t, s)$,

$$(\mathcal{V}z)(t) := \int_0^t K(t, s)z(s)ds, \quad t \in I.$$

If (2.1.45) has a unique solution $z \in C(I)$ then the (unique) solution $y \in C(I)$ of the original Volterra–Hammerstein equation is obtained by

$$y(t) = g(t) + (\mathcal{V}z)(t), \quad t \in I. \tag{2.1.46}$$

It is shown in Krasnosel'skii and Zabreiko (1984, p. 143) that, under suitable assumptions on the (smooth) nonlinearity G, there is a one-to-one correspondence between (continuous) solutions of (2.1.43) and (2.1.45); hence, if (2.1.43) possesses a unique solution $y \in C(I)$ then (2.1.45) has a unique solution $z \in C(I)$. Compare also Corduneanu (1991, p. 153). General results on the existence of solutions to Hammerstein integral equations can also be found in Dolph and Minty (1964) and in Brezis and Browder (1975).

2.1.6 Volterra–Fredholm integral equations

One of the most prominent sources of Volterra–Fredholm integral equations of the second kind is mathematical population dynamics (see, e.g. Thieme (1977, 1979), Diekmann (1978)). Typically, such a VFIE has the form

$$u(t,x) = g(t,x) + (\mathcal{T}u)(t,x), \quad t \in I := [0,T], \quad x \in \Omega, \tag{2.1.47}$$

with Volterra–Fredholm integral operator $\mathcal{T} : C(I \times \Omega) \to C(I \times \Omega)$ defined by

$$(\mathcal{T}u)(t,x) := \int_0^t \int_\Omega K(t,s,x,\xi)G(u(s,\xi))d\xi ds. \tag{2.1.48}$$

Here, Ω denotes a (closed) bounded region in $\mathbb{R}^d$ $(d = 1, 2, 3)$ with (piecewise) smooth boundary $\partial\Omega$. In applications one often has

$$K(t,s,x,\xi) = k(t-s)H(x,\xi),$$

where k represents a (positive) memory kernel.

In this section we briefly show, by means of (2.1.47) corresponding to the *linear* Volterra–Fredholm operator (2.1.48) with $G(u) = u$, that the Volterra part of the integral operator 'dominates' the FIE, in the sense that the Neumann series generated by the Picard iteration process converges absolutely and uniformly on $I \times \Omega$ (in other words, the spectrum of $\mathcal{T}$ consists only of $\{0\}$). The following result, as well as detailed proofs, can be found in Kauthen (1989a, 1989b); see also Pachpatte (1986), and Brunner (1990) for the nonlinear case.

Theorem 2.1.13 *Assume:*

(a) $g \in C(I \times \Omega)$*;*
(b) $K \in C(D \times \Omega^2)$*, where* $D := \{(t,s) : 0 \le s \le t \le T\}$ *and* $\Omega^2 := \Omega \times \Omega$.

Then the linear VFIE

$$u(t,x) = g(t,x) + \int_0^t \int_\Omega K(t,s,x,\xi)u(s,\xi)d\xi ds, \quad (t,x) \in I \times \Omega, \tag{2.1.49}$$

possesses a unique solution $u \in C(I \times \Omega)$*. This solution is given by*

$$u(t,x) = g(t,x) + \int_0^t \int_\Omega R(t,s,x,\xi)g(s,\xi)d\xi ds, \quad (t,x) \in I\Omega. \tag{2.1.50}$$

The resolvent kernel $R \in C(D \times \Omega^2)$ associated with the kernel $K(t, s, x, \xi)$ is the limit of the Neumann series for K and solves the resolvent equations

$$R(t, s, x, \xi) = K(t, s, x, \xi) + \int_0^t \int_\Omega K(t, v, x, z) R(v, s, z, \xi) dz\, dv$$
$$= K(t, s, x, \xi) + \int_0^t \int_\Omega R(t, v, x, z) K(v, s, z, \xi) dz\, dv$$

on $D \times \Omega^2$.

Proof The proof is a straightforward adaptation of the arguments introduced at the beginning of Section 1.1.1. We define the sequence $\{u_n(t, x)\}$ $(n \geq 1)$ by the Picard iteration process applied to (2.1.49) with $u_0(t, x) := g(t, x)$, and then show that the resulting iterated kernels $K_n(t, s, x, \xi)$ can be (uniformly) bounded by

$$|K_n(t, s, x, \xi)| \leq K_0 \frac{(K_0 T |\Omega|)^{n-1}}{(n-1)!} \quad (n \geq 1)$$

on $D \times \Omega^2$. Here, we have set $K_0 := \max\{|K(t, s, x, \xi)| : (t, s) \in D, (x, \xi) \in \Omega^2\}$; $|\Omega|$ denotes the 'volume' of the spatial domain Ω. (Recall that this is in sharp contrast to Picard iteration for *Fredholm* integral equations where the majorant series for the Neumann series is a geometric series.) We leave the details of the proof as a simple exercise.

Analogous existence and uniqueness results for the nonlinear VFIE (2.1.47) can be found in, e.g. Diekmann (1978), Thieme (1979); see also Zhao (2003).

The reformulation of the initial-boundary-value problem for the linear heat equation in a two-dimensional spatial domain Ω with boundary $\partial\Omega$ by single-layer techniques leads to a *Volterra–Fredholm integral equation of the first kind*; its generic form is

$$\int_0^t \int_0^1 K(t - s, \mathbf{x}(\theta) - \mathbf{x}(\phi)) u(s, \phi) d\phi\, ds = g_\Gamma(t, \theta), \quad (t, \theta) \in I \times \mathbb{R}. \tag{2.1.51}$$

Here, $\mathbf{x}(\theta)$ is a smooth 1-periodic parametric representation of the boundary curve $\Gamma := \partial\Omega$, and g_Γ represents the function describing the given boundary condition on $I \times \partial\Omega$. Since the kernel in (2.1.51) possesses a weak singularity at $t = s$ a more detailed discussion of this first-kind VFIE belongs more appropriately in Chapter 6.

Details can be found, e.g. in the papers by Iso and Onishi (1991) and Hamina and Saranen (1994), and in the doctoral dissertation by Hämäläinen (1998). The book by Atkinson (1997a) contains a good introduction to the basic theory and the numerical treatment of boundary integral equations.

2.1.7 Volterra integral equations in $\mathbb{R}^2$

Consider first the second-kind VIE

$$u(x, y) = g(x, y) + \int_0^x \int_0^y K(x, \xi, y, \eta)u(\xi, \eta)d\eta \, d\xi,$$
$$(x, y) \in \Omega := [0, X] \times [0, Y], \tag{2.1.52}$$

with $g \in C(\Omega)$ and $K \in C(D_2)$, where we have set

$$D_2 := \{(x, \xi, y, \eta) : \; 0 \le \xi \le x \le X, \; 0 \le \eta \le y \le Y\}.$$

The existence and uniqueness of continuous solutions was discussed by Volterra (1986c), Lalesco (1912), Volterra (1913); see also Kowalewski (1930, pp. 83–90). Picard iteration is again the principal tool in the proof of the following result, the analogue of Theorem 2.1.2.

Theorem 2.1.14 *Assume that $K \in C(D_2)$. Then for any $g \in C(\Omega)$ the integral equation (2.1.52) possesses a unqique solution $u \in C(\Omega)$. This solution has the representation*

$$u(x, y) = g(x, y) + \int_0^x \int_0^y R(x, \xi, y, \eta)g(\xi, \eta)d\eta \, d\xi, \quad (x, y) \in \Omega.$$

Here, $R = R(x, \xi, y, \eta)$ denotes the resolvent kernel corresponding to the given kernel K; it inherits the regularity of K.

In Volterra (1896c), the sequel to his four *Note* of 1896, Volterra studied the two-dimensional first-kind integral equation

$$\int_0^x \int_0^y K(x, \xi, y, \eta)u(\xi, \eta)d\eta \, d\xi = g(x, y), \quad (x, y) \in \Omega, \tag{2.1.53}$$

where the given functions g and K are assumed to possess continuous (partial) derivatives on their respective domains Ω and D_2, with $g(0, y) = 0$ $(y \in [0, Y])$, $g(x, 0) = 0$ $(x \in [0, X])$, and $|K(x, x, y, y)| \ge k_0 > 0$ for $x \in [0, X]$, $y \in [0, Y]$. He showed that under these hypotheses the above 'multiple integral' can be 'inverted' and the (unique) solution $u \in C(\Omega)$ is given by the solution of a second-kind equation of a form more general than (2.1.52),

$$u(x, y) = g_1(x, y) + \int_0^x H_{1,0}(x, \xi, y)u(\xi, y)d\xi + \int_0^y H_{0,1}(x, y, \eta)u(x, \eta)d\eta$$
$$+ \int_0^x \int_0^y H_{1,1}(x, \xi, y, \eta)u(\xi, \eta)d\eta \, d\xi, \quad (x, y) \in \Omega, \tag{2.1.54}$$

with obvious meaning of the kernels $H_{1,0}$, $H_{0,1}$ and $H_{1,1}$. This equation is obtained by differentiating (2.1.53) with respect to x, followed by differentiation with respect to y.

Second-kind VIEs of the form (2.1.53) also arise in the analysis of a special second-order hyperbolic initial-value problem, known as the *Goursat problem*:

$$\frac{\partial^2 u(x, y)}{\partial x \partial y} = a(x, y)u(x, y) + \phi(x, y), \quad (x, y) \in \Omega,$$

with $u(0, y) = \alpha(y)$, $u(x, 0) = \beta(x)$ on $[0, Y]$ and $[0, X]$, respectively. Details can be found in, e.g. Goursat (1942), Moore (1961), and Dzyadyk (1995); see also McKee, Tang and Diogo (2000) and the Notes at the end of this chapter for additional references.

2.1.8 Comparison theorems

We begin with a slight generalisation of the classical result by Gronwall (1919). Its proof can be found for example in Quarteroni and Valli (1997, pp. 13–14).

Lemma 2.1.15 *Let* $I := [0, T]$ *and assume that* $z, g \in C(I)$, $k \in C(I)$, *with* $k(t) \geq 0$. *If* z *satisfies the inequality*

$$z(t) \leq g(t) + \int_0^t k(s)z(s)ds, \quad t \in I, \tag{2.1.55}$$

then

$$z(t) \leq g(t) + \int_0^t k(s)g(s) \cdot \exp\left(\int_s^t k(v)dv\right) ds \quad \text{for all } t \in I. \tag{2.1.56}$$

If g *is non-decreasing on* I *the above inequality reduces to*

$$z(t) \leq g(t) \cdot \exp\left(\int_0^t k(s)ds\right) \quad \text{for all } t \in I. \tag{2.1.57}$$

Remark Gronwall's original result is obtained by setting $k(s) = k_0 > 0$ and $g(t) = at$ with $a \geq 0$. We note also that the continuous function $k = k(s)$ can be replaced by an *unbounded*, but integrable, function, for example by $k(t - s) = (t - s)^{-\alpha}$ with $0 < \alpha < 1$. We shall return to this generalisation in Section 6.1.8 (Theorem 6.1.17).

We now turn to some representative comparison theorems for solutions of Volterra inequalities. Good treatments of such results are given in the papers by Beesack (1969, 1985a) and in Miller (1971a).

Theorem 2.1.16 *Assume that $g \in C(I)$ and $K \in C(D)$, with $g(t) \geq 0$ and $K(t,s) \geq 0$ on I and D, respectively. Let $R = R(t,s)$ be the resolvent kernel corresponding to $K = K(t,s)$. If $z \in C(I)$ satisfies the Volterra inequality*

$$z(t) \leq g(t) + \int_0^t K(t,s)z(s)ds, \quad t \in I,$$

then

$$z(t) \leq g(t) + \int_0^t R(t,s)g(s)ds, \quad t \in I,$$

and $R(t,s) \geq K(t,s) \geq 0$ for all $(t,s) \in D$.

This result is readily proved, by observing that the non-negativity of K is inherited by its iterated kernels K_n (cf. (2.1.7)) and hence, by the uniform convergence of the Neumann series, by the resolvent kernel R. This also implies that

$$R(t,s) = \sum_{n=1}^{\infty} K_n(t,s) \geq K_1(t,s) = K(t,s) \geq 0, \quad (t,s) \in D.$$

A more general comparison result is presented in Theorem 2.1.17. Its proof, as well as variants of this result (including extension to VIEs in the L^2 setting), can be found in Beesack (1969, 1975); see also Pachpatte (1998). The books by Miller (1971a) and Cochran (1972) contain nonlinear analogues of Theorem 2.1.16.

Theorem 2.1.17 *Assume:*

(a) The functions $g_i \in C(I)$ $(i = 1, 2)$ satisfy $|g_1(t)| \leq g_2(t)$ on I.
(b) An analogous inequality holds for the kernels $K_i \in C(D)$:

$$|K_1(t,s)| \leq K_2(t,s), \quad (t,s) \in D.$$

Then the (unique) solutions of the two integral equations

$$y_i(t) = g_i(t) + \int_0^t K_i(t,s)y_i(s)ds, \quad t \in I \quad (i = 1, 2),$$

are related by

$$|y_1(t)| \leq y_2(t) + |g_1(t)| - g_2(t), \quad t \in I.$$

If g_1 and K_1 are non-negative on I and D, respectively, the absolute value signs in the above inequality can be dropped.

Additional, and more general, comparison theorems can also be found in Section 9.8 of the monograph by Gripenberg, Londen and Staffans (1990).

2.1.9 Discrete Volterra equations and and discrete Gronwall inequalities

Theorem 2.1.18 *Assume that* $\{k_j\}$ $(j \geq 0)$ *is a given non-negative sequence and the sequence* $\{\varepsilon_n\}$ *satisfies* $\varepsilon_0 \leq \rho_0$ *and*

$$\varepsilon_n \leq \rho_0 + \sum_{j=0}^{n-1} q_j + \sum_{j=0}^{n-1} k_j \varepsilon_j, \quad n \geq 1, \tag{2.1.58}$$

with $\rho_0 \geq 0$, $q_j \geq 0$ $(j \geq 0)$. *Then*

$$\varepsilon_n \leq \left(\rho_0 + \sum_{j=0}^{n-1} q_j\right) \exp\left(\sum_{j=0}^{n-1} k_j\right), \quad n \geq 1. \tag{2.1.59}$$

Proofs of this result, as well as numerous variants, can be found in, e.g. Schmidt (1976), McKee (1982a), Beesack (1985), Brunner and van der Houwen (1986, Ch. 1), Quarteroni and Valli (1997, pp. 14–15).

In most applications (arising in the discretisation of second-kind VIEs or VIDEs by *one-step methods*) we have $q_n = 0$ $(n \geq 0)$.

Corollary 2.1.19 *Let* $\{\varepsilon_j\}$ *and* $\{k_j\}$ *satisfy the assumptions stated in Theorem 2.1.18. If* $q_n = 0$ *for all* $n \geq 0$, *then (2.1.57) implies*

$$\varepsilon_n \leq \exp\left(\sum_{j=0}^{n-1} k_j\right) \rho_0, \quad n \geq 1. \tag{2.1.60}$$

A more general version of the above result (see also Dixon and McKee (1986)) was given by Norbury and Stuart in the first of their two 1987 papers. It is deals with the inequality

$$\varepsilon_n \leq h \sum_{j=0}^{n} k_{n,j} \varepsilon_j + \gamma \quad (n \geq 0), \tag{2.1.61}$$

with $k_{n,j} \geq 0$ and $\gamma > 0$. We define the 'discrete iterated kernels' associated with the $k_{n,j}$ by

$$k_{n,j}^{(1)} := \frac{|k_{n,j}|}{|1 - hk_{n,n}|} \quad (0 \leq j \leq n),$$

$$k_{n,j}^{(\mu)} := h \sum_{\ell=j+1}^{n-1} k_{n,\ell}^{(1)} k_{\ell,j}^{(\mu-1)} \quad (\mu \geq 2;\ 0 \leq j \leq n).$$

Let $\Phi^{(\mu)} = \Phi^{(\mu)}(\cdot, \cdot)$ be a function satisfying

$$k_{n,j}^{(\mu)} \leq \Phi^{(\mu)}(nh, (j+1)h), \quad 0 \leq j \leq n-1.$$

Theorem 2.1.20 *Assume:*

(a) $hk_{n,n} \neq 0$ *for all* $n \geq 0$.

(b) $q_n := \sum_{j=0}^{n-1} k_{n,j}^{(1)}$ *is bounded independent of* h.

(c) There exists a positive integer μ_0 *not depending on* h*, so that* $\Phi^{(\mu_0)}(t,t)$ *exists, with*

$$\frac{\partial \Phi^{(\mu_0)}(t,s)}{\partial t} \geq 0 \quad and \quad \frac{\partial \Phi^{(\mu_0)}(t,s}{\partial s} \leq 0.$$

(d) $\int_0^t \Phi^{(\mu)}(t,s)ds$ *exists, regardless of the value of* h.

Then the inequality (2.1.60) implies that

$$|\varepsilon_n| \leq C\gamma \quad (n \geq 0),$$

with some constant C *not depending on* h *or* n.

Remark A comprehensive treatment of (systems of) discrete Volterra equations of the form

$$\mathbf{x}_n = \mathbf{g}_j + \sum_{j=0}^{n-1} k(n,j,\mathbf{x}_j), \quad n = 1, 2, \ldots,$$

can be found in the monograph by Elaydi (1999); see also Song and Baker (2003).

2.2 Collocation for linear second-kind VIEs

2.2.1 Meshes and piecewise polynomial spaces

Let

$$I_h := \{t_n = t_n^{(N)} : \ 0 = t_0^{(N)} < t_1^{(N)} < \ldots < t_N^{(N)} = T\}$$

denote a mesh (or: grid) on the given interval $I := [0,T]$ and set, as in Section 1.1.1,

$$\sigma_n^{(N)} := (t_n^{(N)}, t_{n+1}^{(N)}, \quad h_n^{(N)} := t_{n+1}^{(N)} - t_n^{(N)}, \quad h^{(N)} := \max_{(n)} h_n^{(N)},$$
$$h_{min}^{(N)} := \min_{(n)} h_n^{(N)}.$$

Four types of meshes (or, more precisely, mesh sequences) will be used in this and the following chapters:

- *Uniform mesh I_h:*

$$h_n^{(N)} = h_{min}^{(N)} = h^{(N)} = T/N \quad (n = 0, 1, \ldots, N).$$

- *Quasi-uniform mesh I_h:*

$$h_n^{(N)}/h_{min}^{(N)} \leq h^{(N)}/h_{min}^{(N)} \leq \gamma \quad \text{for all} \quad N \in \mathbb{N}.$$

  This implies that

$$Nh_n^{(N)} \leq Nh^{(N)} \leq \gamma Nh_{min}^{(N)} \leq \gamma T \quad \text{for all} \quad N \in \mathbb{N}. \tag{2.2.1}$$

  We shall encounter quasi-uniform meshes in Section 2.4.2 when analysing the convergence of collocation solutions for first-kind VIEs.
- *Graded mesh I_h:*

$$t_n^{(N)} := (n/N)^r T \quad (n = 0, 1, \ldots, N), \quad r > 1. \tag{2.2.2}$$

  The real number r is called the *grading exponent* (or: scaling parameter). For $r = 1$ such a mesh reduces of course to a uniform one.
  Observe that for $r > 1$ a graded mesh is *not quasi-uniform*, because we have

$$\frac{h^{(N)}}{h_{min}^{(N)}} = \frac{h_{N-1}^{(N)}}{h_0^{(N)}} = \frac{rN^{-1}(1-\theta/N)^{r-1}}{N^{-r}} = rN^{r-1}(1-\theta/N)^{r-1},$$

  with $0 < \theta < 1$. Hence, $h^{(N)}/h_{min}^{(N)} \to \infty$ as $N \to \infty$, whenever $r > 1$, but $\lim_{N\to\infty} h^{(N)} = 0$.
  Graded meshes will play an important role in Chapters 6 and 7, in the analysis of the attainable order of collocation solutions for Volterra equations with weakly singular kernels.
- *Geometric mesh I_h:*

$$t_n^{(N)} := \gamma^{N-n} T \quad (n = 0, 1, \ldots, N), \quad \text{with} \quad 0 < \gamma < 1.$$

  The mesh parameter γ will depend on N, m, such that $h^{(N)} = h_{N-1}^{(N)} = (1-\gamma)T \to 0$, as $N \to \infty$.
  These meshes will be used in Chapter 5 when analysing optimal local superconvergence of collocation solutions for functional equations with (vanishing) proportional delays.

For ease of notation we will, as in Chapter 1, usually suppress the superscript N in $t_n^{(N)}$, $h_n^{(N)}$ (etc.), except possibly in convergence analyses where we shall be dealing with sequences of meshes corresponding to $N \to \infty$ and $h^{(N)} \to 0$.

Definition 2.2.1 For a given mesh I_h the piecewise polynomial space $S_{\mu}^{(d)}(I_h)$, with $\mu \geq 0, \ -1 \leq d < \mu$, is given by

$$S_{\mu}^{(d)}(I_h) := \{v \in C^d(I): \ v|_{\sigma_n} \in \pi_{\mu} \ (0 \leq n \leq N-1)\}.$$

Here, π_{μ} denotes the space of (real) polynomials of degree not exceeding μ. It is readily verified that $S_{\mu}^{(d)}(I_h)$ is a (real) linear vector space whose dimension is given by

$$\dim S_{\mu}^{(d)}(I_h) = N(\mu - d) + d + 1.$$

Remark The particular piecewise polynomial space $S_{m+d}^{(d)}(I_h)$ corresponding to $\mu = m + d$ with $m \geq 1$ and $d \geq -1$ will play a central role in this book. Since its dimension is

$$\dim S_{m+d}^{(d)}(I_h) = Nm + (d+1), \qquad (2.2.3)$$

it may be viewed as the 'natural' collocation space for the approximation of solutions to initial-value problems for ODEs or Volterra equations: as we already indicated in Chapter 1, the choice of the degree of regularity d will be governed by the number of prescribed initial conditions, while the term Nm suggests that m (distinct) collocation points are to be placed in each of the N subintervals σ_n. Thus, the 'natural' choice of d in (2.2.3) is as follows:

- For *Volterra integral equations* (no initial condition) we choose $d = -1$; hence, the natural collocation space will be $S_{m-1}^{(-1)}(I_h)$. Its dimension is Nm.
- For *first-order ODEs* or *Volterra integro-differential equations* (one initial condition) we use $d = 0$, and the preferred collocation space is $S_m^{(0)}(I_h)$, with dimension equal to $Nm + 1$.
- For *ODEs* or *VIDEs of order k* with $k \geq 2$ (k initial conditions) the natural collocation space is $S_{m+k-1}^{(k-1)}(I_h)$, corresponding to the choice $d = k - 1$. The dimension of this space is $Nm + k$.

Remark In the computational use of piecewise collocation methods in $S_{m+d}^{(d)}(I_h)$, the value of m will usually not exceed $m = 4$. Hence, the obvious candidate for the local representation of the collocation solution on σ_n will be the *local Lagrange basis* corresponding to the m (distinct) collocation parameters $\{c_i\}$.

2.2.2 Piecewise polynomial collocation methods in $S_{m-1}^{(-1)}(I_h)$

As in Section 2.1.1 let the linear Volterra integral operator $\mathcal{V}: C(I) \to C(I)$ be given by

$$(\mathcal{V}\phi)(t) := \int_0^t K(t,s)\phi(s)ds, \quad t \in I := [0,T], \tag{2.2.4}$$

where $K \in C(D)$ $(D := \{(t,s): 0 \le s \le t \le T\})$, and let $g \in C(I)$ be a given function. The solution of the Volterra integral equation

$$y(t) = g(t) + (\mathcal{V}y)(t), \quad t \in I, \tag{2.2.5}$$

will be approximated by collocation in the piecewise polynomial space

$$S_{m-1}^{(-1)}(I_h) := \{v : v|_{\sigma_n} \in \pi_{m-1} \ (0 \le n \le N-1)\},$$

corresponding to the choice $\mu = m-1,\ d = -1$ in Definition 2.2.1 and possessing the dimension $\dim S_{m-1}^{(-1)}(I_h) = Nm$. This collocation solution u_h is defined by the *collocation equation* for (2.2.5),

$$u_h(t) = g(t) + (\mathcal{V}u_h)(t), \quad t \in X_h, \tag{2.2.6}$$

where (see (1.1.3)) the set of collocation points,

$$X_h := \{t_n + c_i h_n : 0 \le c_1 \le \ldots \le c_m \le 1 \ (n = 0, 1, \ldots, N-1)\}, \tag{2.2.7}$$

is determined by the given mesh I_h and the given collocation parameters $\{c_i\} \subset [0,1]$. Note that for $m \ge 2$ the choice $c_1 = 0$ and $c_m = 1$ implies

$$u_h \in S_{m-1}^{(-1)}(I_h) \cap C(I) = S_{m-1}^{(0)}(I_h),$$

with $\dim S_{m-1}^{(0)}(I_h) = N(m-1)+1$. This means that u_h assumes the initial value $u_h(t_{0,1}) = u_h(0) = g(0)$.

The *iterated collocation solution* u_h^{it} corresponding to the collocation solution u_h is defined by

$$u_h^{it}(t) := g(t) + (\mathcal{V}u_h)(t), \quad t \in I. \tag{2.2.8}$$

It trivially satisfies

$$u_h^{it}(t) = u_h(t) \quad \text{for all} \quad t \in X_h.$$

It will be seen below (Theorem 2.2.5 and, especially, Theorem 2.2.6) that u_h^{it} will often exhibit a higher order of convergence ('superconvergence') than u_h itself, for example if the collocation parameters $\{c_i\}$ are given by the Gauss points: globally (on I) a gain of one order is obtained, while locally (at the mesh points) the order is twice the global order of u_h. We observe also that for continuous data we have $u_h^{it} \in C(I)$, in contrast to u_h.

As we mentioned at the end of Section 2.1.1, a convenient computational form of the collocation equation (2.2.6) is obtained when employing local Lagrange basis functions: setting

$$L_j(v) := \prod_{k\neq j}^{m} \frac{v - c_k}{c_j - c_k} \quad (v \in [0, 1]) \text{ and}$$
$$U_{n,j} := u_h(t_n + c_j h_n) \quad (j = 1, \dots, m),$$

the restriction of the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ to the subinterval $\sigma_n := (t_n, t_{n+1}]$ can be written as

$$u_h(t) = u_h(t_n + vh_n) = \sum_{j=1}^{m} L_j(v)U_{n,j}, \quad v \in (0, 1]. \tag{2.2.9}$$

Thus, for $t = t_{n,i} := t_n + c_i h_n$ the collocation equation (2.2.6) assumes the form

$$u_h(t) = g(t) + \int_0^{t_n} K(t, s)u_h(s)ds + h_n \int_0^{c_i} K(t, t_n + sh_n)u_h(t_n + sh_n)ds.$$

Expressed in terms of the 'stage values' $\{U_{n,j}\}$ it is

$$U_{n,i} = g(t_{n,i}) + F_n(t_{n,i}) + h_n \sum_{j=1}^{m} \left(\int_0^{c_i} K(t_{n,i}, t_n + sh_n)L_j(s)ds \right) U_{n,j} \tag{2.2.10}$$

$(i = 1, \dots, m)$, where

$$F_n(t) := \int_0^{t_n} K(t, s)u_h(s)ds = \sum_{\ell=0}^{n-1} h_\ell \int_0^1 K(t, t_\ell + sh_\ell)u_h(t_\ell + sh_\ell)ds \tag{2.2.11}$$

denotes the *lag term* (or: history term) corresponding to the collocation solution on $[0, t_n]$. If we set $t = t_{n,i}$ in (2.2.11) and employ the local representation (2.2.9) we may write

$$\begin{aligned} F_n(t_{n,i}) &= \sum_{\ell=0}^{n-1} h_\ell \int_0^1 K(t_{n,i}, t_\ell + sh_\ell)u_h(t_\ell + sh_\ell)ds \\ &= \sum_{\ell=0}^{n-1} h_\ell \sum_{j=1}^{m} \left(\int_0^1 K(t_{n,i}, t_\ell + sh_\ell)L_j(s)ds \right) U_{\ell,j}. \end{aligned}$$

Let $\mathbf{U}_n := (U_{n,1}, \dots, U_{n,m})^T, \quad \mathbf{g}_n := (g(t_{n,1}), \dots, g(t_{n,m})^T, t_{n,m})^T$, and define the matrices in $L(\mathbb{R}^m)$,

$$B_n^{(\ell)} := \begin{pmatrix} \int_0^1 K(t_{n,i}, t_\ell + sh_\ell)L_j(s)ds \\ (i, j = 1, \dots, m) \end{pmatrix} \quad (0 \le \ell < n \le N - 1), \tag{2.2.12}$$

and

$$B_n := \begin{pmatrix} \int_0^{c_i} K(t_{n,i}, t_n + sh_n) L_j(s) ds \\ (i, j = 1, \ldots, m) \end{pmatrix}. \tag{2.2.13}$$

The collocation equation (2.2.5) then assumes the form

$$[\mathcal{I}_m - h_n B_n]\mathbf{U}_n = \mathbf{g}_n + \mathbf{G}_n \quad (n = 0, 1, \ldots, N-1), \tag{2.2.14}$$

with

$$\mathbf{G}_n := (F_n(t_{n,1}), \ldots, F_n(t_{n,m}))^T = \sum_{\ell=0}^{n-1} h_\ell B_n^{(\ell)} \mathbf{U}_\ell.$$

Here, $\mathcal{I}_m$ denotes again the identity matrix in $L(\mathbb{R}^m)$.

Theorem 2.2.1 *Assume that g and K in the Volterra integral equation (2.2.5) are continuous on their respective domains I and D. Then there exists an $\bar{h} > 0$ so that for any mesh I_h with mesh diameter $h \in (0, \bar{h})$ each of the linear algebraic systems (2.2.14) has a unique solution $\mathbf{U}_n$ $(n = 0, 1, \ldots, N-1)$. Hence the collocation equation (2.2.6) defines a unique collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ for (2.2.5), with local representation on σ_n given by (2.2.9).*

Proof Since the kernel K of the Volterra operator $\mathcal{V}$ is continuous on D, the elements of the matrices B_n $(n = 0, 1, \ldots, N-1)$ are all bounded. The Neumann Lemma (cf. Ortega (1972, p. 26) or Atkinson (1989, p. 492)) then shows that the inverse of the matrix $\mathcal{B}_n := \mathcal{I}_m - h_n B_n$ exists whenever $h_n||B_n|| < 1$ for some matrix norm. This clearly holds whenever h_n is sufficiently small. In other words, there is an $\bar{h} > 0$ so that for any mesh I_h with $h := \max\{h_n : 0 \le n \le N-1\} < \bar{h}$, each matrix $\mathcal{B}_n$ has a uniformly bounded inverse. The assertion of Theorem 2.2.1 now follows.

When the collocation solution on the subinterval σ_n has been computed, the *iterated collocation solution* for $t = t_n + vh_n \in \bar{\sigma}_n := [t_n, t_{n+1}]$ is given by

$$u_h^{it}(t) = g(t) + F_n(t) + h_n \sum_{j=1}^{m} \left(\int_0^v K(t, t_n + sh_n) L_j(s) ds \right) U_{n,j}, \tag{2.2.15}$$

with lag term $F_n(t)$ as in (2.2.11).

Example 2.2.1 $u_h \in S_0^{(-1)}(I_h)$ $(m = 1)$, $\quad 0 < c_1 =: \theta \le 1$:
Here, $u_h(t_n + vh_n) = U_{n,1}$ for all $v \in (0, 1]$. Setting $y_{n+1} := U_{n,1}$ the collocation solution is then determined by the equation

$$\left(1 - h_n \int_0^\theta K(t_{n,1}, t_n + sh_n) ds\right) y_{n+1} = g(t_{n,1}) + F_n(t_{n,1}) \tag{2.2.16}$$

$(n = 0, 1, \ldots, N-1)$, with $t_{n,1} = t_n + \theta h_n$ and with lag term given by

$$F_n(t_{n,1}) = \sum_{\ell=0}^{n-1} h_\ell \left(\int_0^1 K(t_{n,1}, t_\ell + s h_\ell) ds \right) y_{\ell+1}.$$

For $t = t_n + v h_n$ $(v \in [0, 1])$ the corresponding *iterated collocation solution* is then

$$u_h^{it}(t) = g(t) + F_n(t) + v h_n \left(\int_0^1 K(t, t_n + s v h_n) ds \right) y_{n+1}. \qquad (2.2.17)$$

Example 2.2.2 $u_h \in S_1^{(-1)}(I_h)$ $(m = 2)$, $0 < c_1 < c_2 \leq 1$:
Since the Lagrange fundamental polynomials corresponding to the two collocation parameters are

$$L_1(s) = (c_2 - s)/(c_2 - c_1) \quad \text{and} \quad L_2(s) = (s - c_1)/(c_2 - c_1),$$

the matrix $B_n \in L(\mathbb{R}^2)$ in (2.2.13) has the elements

$$(B_n)_{i,1} = \frac{1}{c_2 - c_1} \int_0^{c_i} K(t_{n,i}, t_n + s h_n)(c_2 - s) ds \quad (i = 1, 2)$$

and

$$(B_n)_{i,2} = \frac{1}{c_2 - c_1} \int_0^{c_i} K(t_{n,i}, t_n + s h_n)(s - c_1) ds \quad (i = 1, 2).$$

Moreover,

$$(B_n^{(\ell)})_{i,1} = \frac{1}{c_2 - c_1} \int_0^1 K(t_{n,i}, t_\ell + s h_\ell)(c_2 - s) ds \quad (i = 1, 2),$$

and

$$(B_n^{(\ell)})_{i,2} = \frac{1}{c_2 - c_1} \int_0^1 K(t_{n,i}, t_\ell + s h_\ell)(s - c_1) ds \quad (i = 1, 2).$$

The collocation solution is now determined by the resulting system (2.2.14) and the local Lagrange representation (2.2.9) with $m = 2$, and (2.2.15) then yields the iterated collocation solution on $\bar{\sigma}_n$.

2.2.3 The fully discretised collocation equation

We have seen in Section 1.1.1 that when an initial-value problem for an ODE is solved by collocation in a piecewise polynomial space, for example in $S_m^{(0)}(I_h)$, then the resulting collocation equation is completely discretised and thus in a form feasible for numerical computation. When applying the collocation method to Volterra integral (or integro-differential) equations, this is in general not true: in the collocation equation (2.2.10) and the lag term (2.2.11) corresponding to the VIE (2.2.5) the integrals cannot, in general, be found analytically but have to be approximated by suitable *numerical quadrature formulas*. Thus, the *fully discretised version* of (2.2.9) will have the form

$$\hat{u}_h(t) = g(t) + (\hat{\mathcal{V}}_h \hat{u}_h)(t), \quad t \in X_h, \tag{2.2.18}$$

where $\hat{\mathcal{V}}_h$ denotes some discretisation of the original Volterra integral operator $\mathcal{V}$ in (2.2.6). While in principle these integrals can be approximated to any desired accuracy, the nature of the Volterra integral operator (memory term!) makes such an approach prohibitively expensive, especially in long-time integration problems and when solving systems of VIEs. On the other hand we have to make sure that the quadrature formulas are *order preserving*; that is, they are such that the order of the resulting quadrature errors will (at least) match the order of convergence of the *exact* collocation solution defined by (2.2.6), either globally (on I), or at the mesh points I_h. We shall see below, when carrying out the detailed error and convergence analyses, that this can be achieved if we choose *interpolatory* m-point quadrature formulas whose abscissas are given by, or based on, the m collocation parameters $\{c_i\}$. To be more precise, we shall in the following employ the quadrature approximations

$$(\hat{Q}_n^{(\ell)} u_h)(t) := \sum_{j=1}^{m} b_j K(t, t_\ell + c_j h_\ell) U_{\ell,j} \quad (\ell < n) \tag{2.2.19}$$

and

$$(\hat{Q}_n u_h)(t) := v \sum_{j=1}^{m} b_j K(t, t_n + v c_j h_n) u_h(t_n + v c_j h_n) \tag{2.2.20}$$

for the integrals

$$(Q_n^{(\ell)} u_h)(t) := \int_0^1 K(t, t_\ell + s h_\ell) u_h(t_\ell + s h_\ell) ds \quad (\ell < n),$$

and

$$\begin{aligned}(Q_n u_h)(t) &:= \int_0^v K(t, t_n + s h_n) u_h(t_n + s h_n) ds \\ &= v \int_0^1 K(t, t_n + s v h_n) u_h(t_n + s v h_n) ds,\end{aligned}$$

respectively, where $t = t_n + vh_n \in \sigma_n$ and $b_j := \int_0^1 L_j(s)ds$. Note that, for any such t, (2.2.20) can also be written as

$$(\hat{Q}_n u_h)(t) = v \sum_{j=1}^{m} \left(\sum_{k=1}^{m} b_k K(t, t_n + c_k v h_n) L_j(c_k v) \right) U_{n,j}. \tag{2.2.21}$$

The *fully discretised collocation equation* is obtained from the *exact collocation equation* (2.2.10) by replacing the integrals by the above quadrature approximations, disregarding the quadrature errors induced by this secondary discretisation process. We shall denote the resulting *discretised collocation solution* by $\hat{u}_h$: it is, of course, still an element of our space $S_{m-1}^{(-1)}(I_h)$, but in general we now have $\hat{u}_h \neq u_h$. In analogy to (2.2.9) the local representation of $\hat{u}_h$ on σ_n is

$$\hat{u}_h(t_n + vh_n) = \sum_{j=1}^{m} L_j(v)\hat{U}_{n,j} \quad v \in (0, 1], \quad \text{with} \quad \hat{U}_{n,j} := \hat{u}_h(t_n + c_j h_n). \tag{2.2.22}$$

Thus, the fully discretised collocation equation is

$$\hat{U}_{n,i} = g(t_{n,i}) + \hat{F}_n(t_{n,i}) + h_n(\hat{Q}_n \hat{u}_h)(t_{n,i}) \quad (i = 1, \dots, m), \tag{2.2.23}$$

where $(\hat{Q}_n \hat{u}_h)(t_{n,i})$ is defined in (2.2.21) and the fully discretised lag term $\hat{F}_n$ has the form

$$\hat{F}_n(t) := \sum_{\ell=0}^{n-1} h_\ell (\hat{Q}_n^{(\ell)} \hat{u}_h)(t) = \sum_{\ell=0}^{n-1} h_\ell \left(\sum_{j=1}^{m} b_j K(t, t_\ell + c_j h_\ell) \hat{U}_{\ell,j} \right). \tag{2.2.24}$$

In analogy to (2.2.14) we may write the discretised collocation equation (2.2.23) in the more concise form

$$[\mathcal{I}_m - h_n \hat{B}_n]\hat{\mathbf{U}}_n = \mathbf{g}_n + \hat{\mathbf{G}}_n \quad (n = 0, 1, \dots, N-1), \tag{2.2.25}$$

with

$$\hat{\mathbf{G}}_n := (\ \hat{F}_n(t_{n,1}), \dots, \hat{F}_n(t_{n,m})\ )^T = \sum_{\ell=0}^{n-1} h_\ell \hat{B}_n^{(\ell)} \hat{\mathbf{U}}_\ell.$$

Here, $\hat{\mathbf{U}}_n := (\ \hat{U}_{n,1}, \dots, \hat{U}_{n,m}\ )^T \in \mathbb{R}^m$; the matrices $\hat{B}_n$ and $\hat{B}_n^{(\ell)}$ in $L(\mathbb{R}^m)$, defined respectively by

$$\hat{B}_n := \begin{pmatrix} c_i \sum_{k=1}^{m} b_k K(t_{n,i}, t_n + c_i c_k h_n) L_j(c_i c_k) \\ (i, j = 1, \dots, m) \end{pmatrix}$$

and

$$\hat{B}_n^{(\ell)} := \begin{pmatrix} b_j K(t_{n,i}, t_\ell + c_j h_\ell) \\ (i, j = 1, \dots, m) \end{pmatrix} \quad (\ell < n),$$

represent the discretised versions of B_n and $B_n^{(\ell)}$ in (2.2.13), (2.2.12).

Theorem 2.2.2 *Let the assumptions of Theorem 2.2.1 hold. Then there exists an $\hat{h} > 0$ so that for any mesh I_h with mesh diameter h satisfying $h \in (0, \hat{h})$ there exists a unique discretised collocation approximation $\hat{u}_h \in S_{m-1}^{(-1)}(I_h)$ defined by the unique solutions $\hat{\mathbf{U}}_n$ of the linear algebraic systems (2.2.25) $(n = 0, 1, \dots, N-1)$ and the local representations (2.2.22).*

The **proof** of Theorem 2.2.2 closely resembles the one for Theorem 2.2.1: since, for any fixed m, the weights $\{b_j\}$ of the interpolatory m-point quadrature formulas underlying the discretised collocation equations (2.2.23) are bounded, it follows from the Neumann Lemma that each matrix $\hat{\mathcal{B}}_n := \mathcal{I}_m - h_n \hat{B}_n$ $(n = 0, 1, \dots, N-1)$ in (2.2.25) possesses a uniformly bounded inverse whenever $h_n < \hat{h}$, for some suitable $\hat{h} > 0$, where in general $\hat{h} \neq \bar{h}$.

For $t = t_n + v h_n \in \sigma_n$ the corresponding *discretised iterated collocation solution* $\hat{u}_h^{it}$ corresponding to $\hat{u}_h \in S_{m-1}^{(-1)}(I_h)$ in (2.2.18) is defined by

$$\hat{u}_h^{it}(t_n + vh_n) := g(t_n + vh_n) + \hat{F}_n(t_n + vh_n) + h_n(\hat{Q}_n \hat{u}_h)(t_n + vh_n),$$
$$v \in [0, 1]. \tag{2.2.26}$$

The following two illustrations are the discrete counterparts of the exact collocation methods described in Examples 2.2.1 and 2.2.2.

Example 2.2.3 $\hat{u}_h \in S_0^{(-1)}(I_h)$, $0 < c_1 =: \theta \leq 1$:
Setting $\hat{y}_{n+1} := \hat{u}_h(t_n + vh_n) = \hat{U}_{n,1}$, equation (2.2.25) yields

$$(1 - \theta h_n K(t_{n,1}, t_n + \theta^2 h_n))\hat{y}_{n+1} = g(t_{n,1}) + \hat{F}_n(t_{n,1}) \tag{2.2.27}$$

$(n = 0, 1, \dots, N-1)$, with $t_{n,1} = t_n + \theta h_n$, and

$$\hat{F}_n(t_{n,1}) = \sum_{\ell=0}^{n-1} h_\ell K(t_{n,1}, t_\ell + \theta h_\ell)\hat{y}_{\ell+1}.$$

The corresponding discretised iterated collocation solution at $t = t_n + vh_n$ $(v \in [0, 1])$ is then given by

$$\hat{u}_h^{it}(t) = g(t) + \sum_{\ell=0}^{n-1} h_\ell K(t, t_\ell + \theta h_\ell)\hat{y}_{\ell+1} + v h_n K(t, t_n + v\theta h_n)\hat{y}_{n+1}. \tag{2.2.28}$$

Example 2.2.4 $\hat{u}_h \in S_1^{(-1)}(I_h)$ $(m = 2)$, $0 < c_1 < c_2 \leq 1$:
We see from Example 2.2.2 that the discretised matrices $\hat{B}_n \in L(\mathbb{R}^2)$ in (2.2.26)

possess the elements

$$(\hat{B}_n)_{i,1} = \frac{c_i}{c_2 - c_1}[b_1 K(t_{n,i}, t_n + c_i c_1 h_n)(c_2 - c_i c_1) + b_2 K(t_{n,i}, t_n + c_i c_2 h_n) c_2 (1 - c_i)]$$

and

$$(\hat{B}_n)_{i,2} = \frac{c_i}{c_2 - c_1}[b_1 K(t_{n,i}, t_n + c_i c_1 h_n) c_1 (c_i - 1) + b_2 K(t_{n,i}, t_n + c_i c_2 h_n)(c_i c_2 - c_1)]$$

($i = 1, 2$), with quadrature weights

$$b_1 = \frac{2c_2 - 1}{2(c_2 - c_1)}, \quad b_2 = \frac{1 - 2c_1}{2(c_2 - c_1)}.$$

The stage values $\hat{U}_{n,1}$ and $\hat{U}_{n,2}$ in the local representation of $\hat{u}_h$ on σ_n (cf. (2.2.22) with $m = 2$) are given by the solution of the linear algebraic system (2.2.25).

2.2.4 Global convergence results

Theorem 2.2.3 *Assume:*

(a) The given functions describing the Volterra integral equation (2.2.5) satisfy $K \in C^m(D)$ and $g \in C^m(I)$.
(b) $u_h \in S_m^{(-1)}(I_h)$ is the collocation solution to (2.2.5) defined by (2.2.6) with $h \in (0, \bar{h})$.

Then

$$||y - u_h||_\infty := \sup_{t \in I} |y(t) - u_h(t)| \leq C||y^{(m)}||_\infty h^m \tag{2.2.29}$$

holds for any set X_h of collocation points with $0 \leq c_1 < \ldots < c_m \leq 1$. The constant C depends on the $\{c_i\}$ but not on h.

Since the dependence of the error bounds on certain derivatives of the exact solution y will become apparent in the course of the proof, we will usually no longer state this dependence explicitly in subsequent convergence theorems.

Proof The proof of course follows closely the one for Theorem 1.1.2, except that now there is no continuity constraint at the the mesh points $t_1, \ldots, t_{N-1}$. Since assumption (a) implies $y \in C^m(I)$ we may resort to Peano's Theorem (Corollary 1.8.2 with $d = m$) to write

$$y(t_n + vh_n) = \sum_{j=1}^{m} L_j(v) Y_{n,j} + h_n^m R_{m,n}(v), \quad v \in [0, 1], \quad \text{with } Y_{n,j} := y(t_{n,j}). \tag{2.2.30}$$

Here, we have

$$R_{m,n}(v) := \int_0^1 K_m(v,z)y^{(m)}(t_n + zh_n)dz,$$

and

$$K_m(v,z) = \frac{1}{(m-1)!}\left\{(v-z)_+^{m-1} - \sum_{k=1}^m L_k(v)(c_k - z)_+^{m-1}\right\}, \quad z \in [0,1].$$

Thus, it follows from (2.2.9) that the collocation error $e_h := y - u_h$ possesses the local representation

$$e_h(t_n + vh_n) = \sum_{j=1}^m L_j(v)\mathcal{E}_{n,j} + h_n^m R_{m,n}(v), \quad v \in (0,1], \tag{2.2.31}$$

with $\mathcal{E}_{n,j} := Y_{n,j} - U_{n,j}$, and it satisfies the equation

$$e_h(t_{n,i}) = (\mathcal{V}e_h)(t_{n,i}), \quad i = 1,\dots,m \quad (0 \le n \le N-1). \tag{2.2.32}$$

Its right-hand side is

$$\begin{aligned}(\mathcal{V}e_h)(t_{n,i}) &= \int_0^{t_n} K(t_{n,i},s)e_h(s)ds + h_n\int_0^{c_i} K(t_{n,i}, t_n + sh_n)e_h(t_n + sh_n)ds \\ &= \sum_{\ell=0}^{n-1} h_\ell \int_0^1 K(t_{n,i}, t_\ell + sh_\ell)\left(\sum_{j=1}^m L_j(s)\mathcal{E}_{\ell,j} + h_\ell^m R_{m,\ell}(s)\right)ds \\ &\quad + h_n\int_0^{c_i} K(t_{n,i}, t_n + sh_n)\left(\sum_{j=1}^m L_j(s)\mathcal{E}_{n,j} + h_n^m R_{m,n}(s)\right)ds.\end{aligned}$$

Hence, we obtain a system of linear equations for $\boldsymbol{\mathcal{E}}_n := (\,\mathcal{E}_{n,1},\dots,\mathcal{E}_{n,m}\,)^T \in \mathbb{R}^m$, namely

$$\begin{aligned}&\mathcal{E}_{n,i} - h_n\sum_{j=1}^m\left(\int_0^{c_i} K(t_{n,i}, t_n + sh_n)L_j(s)ds\right)\mathcal{E}_{n,j} \\ &= \sum_{\ell=0}^{n-1} h_\ell \sum_{j=1}^m\left(\int_0^1 K(t_{n,i}, t_\ell + sh_\ell)L_j ds\right)\mathcal{E}_{\ell,j} \\ &\quad + \sum_{\ell=0}^{n-1} h_\ell^{m+1}\int_0^1 K(t_{n,i}, t_\ell + sh_\ell)R_{m,\ell}(s)ds \\ &\quad + h_n^{m+1}\int_0^{c_i} K(t_{n,i}, t_n + sh_n)R_{m,n}(s)ds \quad (i = 1,\dots,m).\end{aligned}$$

Introducing the vectors $\boldsymbol{\rho}_n^{(\ell)}$ and $\boldsymbol{\rho}_n$ in $\mathbb{R}^m$ by

$$\boldsymbol{\rho}_n^{(\ell)} := \left(\int_0^1 K(t_{n,i}, t_\ell + sh_\ell) R_{m,\ell}(s) ds \quad (i = 1, \ldots, m) \right)^T \quad (\ell < n)$$

and

$$\boldsymbol{\rho}_n := \left(\int_0^{c_i} K(t_{n,i}, t_n + sh_n) R_{m,n}(s) ds \quad (i = 1, \ldots, m) \right)^T,$$

and recalling the definition of the matrices $B_n^{(\ell)}$ and B_n in Section 2.2.2 (cf. (2.2.12), (2.2.13)) this linear algebraic system may be written more concisely as

$$[\mathcal{I}_m - h_n B_n]\boldsymbol{\mathcal{E}}_n = \sum_{\ell=0}^{n-1} h_\ell B_n^{(\ell)} \boldsymbol{\mathcal{E}}_\ell + \sum_{\ell=0}^{n-1} h_\ell^{m+1} \boldsymbol{\rho}_n^{(\ell)} + h_n^{m+1} \boldsymbol{\rho}_n \quad (0 \le n \le N-1). \tag{2.2.33}$$

We observe that it closely resembles (2.2.14): both systems are described by the same matrix $\mathcal{B}_n := \mathcal{I}_m - h_n B_n$, while the role of $\mathbf{g}_n$ is now assumed by the sum of the remainder term vectors. Moreover, a glimpse at (1.1.31), the ODE analogue of (2.2.33), reveals that the terms in (1.1.31) emanating from the continuity requirements at the mesh points are here replaced by the terms reflecting the memory term $\mathcal{V}e_h$.

It thus follows from the proof of Theorem 2.2.1 that we have again the uniform bound

$$||(\mathcal{I}_m - h_n B_n)^{-1}||_1 \le D_0 \quad (n = 0, 1, \ldots, N-1),$$

for all mesh diameters h with $h \in (0, \bar{h})$. Assume that $||B_n^{(\ell)}||_1 \le D_1$ for $0 \le \ell < n \le N-1$, and set

$$||\boldsymbol{\rho}_n^{(\ell)}||_1 \le m K_0 k_m M_m \quad (\ell < n), \quad ||\boldsymbol{\rho}_n||_1 \le m K_0 k_m M_m.$$

In analogy to the notation employed in the proof of Theorem 1.1.2, we define

$$M_m := ||y^{(m)}||_\infty, \quad k_m := \max_{v \in [0,1]} \int_0^1 |K_m(v, z)| dz,$$

and

$$\bar{K} := \max_{t \in I} \int_0^t |K(t, s)| ds = ||\mathcal{V}||_\infty$$

(the *(operator) norm* of the Volterra integral operator $\mathcal{V}$). Then, from (2.2.33),

$$||\boldsymbol{\mathcal{E}}_n||_1 \le D_0 D_1 \sum_{\ell=0}^{n-1} h_\ell ||\boldsymbol{\mathcal{E}}_\ell||_1 + D_0[m\bar{K} k_m M_m \sum_{\ell=0}^{n-1} h_\ell^{m+1} + h_n^{m+1} m\bar{K} k_m M_m],$$

and hence

$$||\mathcal{E}_n||_1 \leq \gamma_0 \sum_{\ell=0}^{n-1} h_\ell ||\mathcal{E}_\ell||_1 + \gamma_1 M_m h^m, \quad n = 0, 1, \ldots, N-1, \qquad (2.2.34)$$

where $\gamma_0 := D_0 D_1$, $\gamma_1 := m D_0 \bar{K} k_m (T + h)$. The above generalised discrete Gronwall inequality has the same form as the one encountered in the proof of Theorem 1.1.2, and so we obtain the estimate

$$||\mathcal{E}_n||_1 \leq B M_m h^m, \quad n = 0, 1, \ldots, N-1.$$

Using the local error representation (2.2.31) this yields, setting $\Lambda_m := \max_{(j)} ||L_j||_\infty$,

$$|e_h(t_n + v h_n)| \leq \Lambda_m ||\mathcal{E}_n||_1 + h^m k_m M_m \leq (\Lambda_m B + k_m) M_m h^m,$$

uniformly for $v \in [0, 1]$ and $0 \leq n \leq N-1$. The is equivalent to the estimate $||e_h||_\infty \leq C||y^{(m)}||_\infty h^m$, as asserted in (2.2.30).

Remarks

1. An important problem that, to my knowldege, remains open concerns the determination of an 'optimal', computable value of the error constant C in Theorem 2.2.3, especially in long-time integration. (See, however, the implementation of the collocation method on adaptive meshes discussed in Blom and Brunner (1987, 1991).)
2. The convergence analysis of *general one-step methods* for second-kind Volterra integral equation is given in Hairer, Lubich and Nørsett (1983).

The above proof shows that, as for ODEs, *lower regularity in the solution* leads to a corresponding lower order of global convergence. We summarise this result in Theorem 2.2.4 whose proof resorts again to Peano's Theorem where now m is replaced $d < m$.

Theorem 2.2.4 *Suppose that (a) in Theorem 2.2.3 is replaced by the weaker assumption $g \in C^d(I),\ K \in C^d(D)$), with $1 \leq d < m$, implying that $y \in C^d(I)$. Then*

$$||y - u_h||_\infty \leq C||y^{(d)}||_\infty h^d. \qquad (2.2.35)$$

The analysis in Section 1.1.3 has given a first indication that global (and local) superconvergence results for ODEs may not necessarily carry over to second-kind Volterra integral equations. On the other hand, they might hold for the *iterated collocation solution*. This is made precise in the following theorem, the counterpart of Theorem 1.2.3 (Brunner and Yan (1996)), for global superconvergence.

Theorem 2.2.5 *Assume:*

(a) $g \in C^{m+1}(I)$ *and* $K \in C^{m+1}(D)$*;*
(b) $u_h \in S_{m-1}^{(-1)}(I_h)$ $(h \in (0, \bar{h}))$ *is the collocation solution for (2.2.5), with collocation parameters* $\{c_i\}$ *satisfying the orthogonality condition*

$$J_0 := \int_0^1 \prod_{i=1}^{m} (s - c_i) ds = 0.$$

Then the iterated collocation solution,

$$u_h^{it}(t) := g(t) + (\mathcal{V} u_h)(t), \quad t \in I,$$

is globally superconvergent on I*, with*

$$||y - u_h^{it}||_\infty \leq C h^{m+1}, \tag{2.2.36}$$

where C *is depends on the* $\{c_i\}$ *and on* $||y^{(m+1)}||_\infty$ *but not on* h*.*

Proof In analogy to (1.1.36) we define the *defect* (or residual) associated with the collocation solution u_h to the VIE (2.2.5) by

$$\delta_h(t) := -u_h(t) + g(t) + (\mathcal{V} u_h)(t), \quad t \in I,$$

with $\delta_h(t) = 0$ whenever $t \in X_h$. Under the regularity assumptions for g and K it is piecewise C^{m+1}, and Theorem 2.2.3 implies that it has uniformly bounded derivatives on each subinterval σ_n (Exercise 2.5.14). Since $e_h = y - u_h$, with y denoting the solution of (2.2.5), we also have

$$\delta_h(t) = e_h(t) - (\mathcal{V} e_h)(t), \quad t \in I. \tag{2.2.37}$$

Thus, it follows from (2.2.29) and (2.2.37) that

$$||\delta_h||_\infty \leq ||e_h||_\infty + \bar{K} ||e_h||_\infty \leq C(1 + \bar{K}) h^m := D_1 h^m,$$

with

$$\bar{K} := ||\mathcal{V}||_\infty = \max_{t \in I} \int_0^t |K(t, s)| ds,$$

The defect and the collocation error are related by equation (2.2.37) which we write as

$$e_h(t) = \delta_h(t) + (\mathcal{V} e_h)(t), \quad t \in I, \tag{2.2.38}$$

and the iterated collocation error $e_h^{it} := y - u_h^{it}$ has the property that

$$e_h^{it}(t) = e_h(t) - \delta_h(t), \quad t \in I.$$

Thus, denoting by $R = R(t, s)$ the resolvent kernel of $K(t, s)$ we know from Section 2.1.1 (Theorem 2.1.1) that the solution of the Volterra equation (2.2.38) is given by

$$e_h(t) = \delta_h(t) + \int_0^t R(t, s)\delta_h(s)ds, \quad t \in I. \tag{2.2.39}$$

(We note that since δ_h is only *piecewise continuous* (but bounded) on I, (2.2.38) and the representation of e_h given by (2.2.39) are to be interpreted in the corresponding way, that is, for each subinterval σ_n.) The above thus implies

$$e_h^{it}(t) = \int_0^t R(t, s)\delta_h(s)ds, \quad t \in I. \tag{2.2.40}$$

Formally this reminds us immediately of equation (1.1.38) which we encountered in the proof of the global superconvergence result for ODEs. Therefore, the arguments based on replacing the various integrals over subintervals making up a given interval $[0, t_n + vh_n]$ ($v \in [0, 1]$) carry over to the present situation, where the role of $r(t, s)$ is assumed by $R(t, s)$. More precisely, it is easy to see that instead of (1.1.41) we now obtain

$$|e_h^{it}(t_n + vh_n)| \leq \sum_{\ell=0}^{n-1} h_\ell Q_\ell h_\ell^{m+1} + h_n \bar{R}||\delta_h||_\infty \leq h^{m+1}(QT + \bar{R}D_1), \tag{2.2.41}$$

uniformly for $v \in [0, 1]$ and $0 \leq n \leq N - 1$, with

$$\bar{R} := \max_{t \in I} \int_0^t |R(t, s)|ds.$$

This readily leads to the completion of the proof of Theorem 2.2.5.

2.2.5 Local superconvergence results

Will collocation using the collocation points X_h corresponding to the m Gauss points $\{c_i\}$ yield a collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ to the Volterra integral equation (2.2.5) for which

$$\max_{t \in I_h \setminus \{0\}} |y(t) - u_h(t)| = \mathcal{O}(h^{2m})$$

holds, in analogy to the result of Corollary 1.1.5 for ODEs? A first indication that this will *not* be true can already be found in Section 1.1.5 (Corollary 1.1.10): we showed that collocation in $S_{m-1}^{(-1)}(I_h)$ for the integrated form of a linear ODE will not coincide with the 'direct' collocation solution in $S_m^{(0)}(I_h)$ for the given ODE when the collocation parameters are the Gauss points. These observations suggest the conjecture that the above local superconvergence result will be true if u_h is replaced by the *iterated* collocation solution u_h^{it}. The following theorem shows that this is indeed so.

Theorem 2.2.6 *Assume that the given functions in (2.2.5) satisfy* $g \in C^{m+\kappa}(I)$ *and* $K \in C^{m+\kappa}(D)$ *for some integer* κ *with* $1 \le \kappa \le m$. *If the collocation parameters* $\{c_i\}$ *are chosen so that the orthogonality conditions*

$$J_\nu := \int_0^1 s^\nu \prod_{i=1}^m (s - c_i)ds = 0, \quad \nu = 0, \ldots, \kappa - 1, \tag{2.2.42}$$

hold, with $J_\kappa \neq 0$, *then the (optimal) order estimate*

$$\max_{t \in I_h \setminus \{0\}} |y(t) - u_h^{it}(t)| \le Ch^{m+\kappa} \tag{2.2.43}$$

is true whenever $h \in (0, \bar{h})$.

If, in addition to (2.2.42), we have $c_m = 1$, *then local superconvergence is obtained for the collocation solution* u_h *itself:*

$$\max_{t \in I_h \setminus \{0\}} |y(t) - u_h(t)| \le Ch^{m+\kappa}. \tag{2.2.44}$$

Here, κ *cannot exceed* $m - 1$.

Proof Recall (2.2.40): for $t = t_n$ the expression for the iterated collocation error can be written as

$$e_h^{it}(t_n) = \sum_{\ell=0}^{n-1} h_\ell \int_0^1 R(t_n, t_\ell + sh_\ell)\delta_h(t_\ell + sh_\ell)ds \quad (n = 1, \ldots, N).$$

Hence, a glimpse at the corresponding expression (1.1.38) in Section 1.1.2 reveals that the arguments used in the proof of Theorem 1.1.4 carry over to the present situation, leading straightforwardly to the assertions of Theorem 2.2.6.

Corollary 2.2.7 *Let* $\kappa = m$. *Collocation at the corresponding Gauss points yields*

$$\max_{t \in I_h \setminus \{0\}} |y(t) - u_h^{it}(t)| \le Ch^{2m},$$

but only

$$\max_{t \in I_h \setminus \{0\}} |y(t) - u_h(t)| = \mathcal{O}(h^m).$$

Note the second estimate is a consequence of the estimate $|\delta_h(t_n)| = \mathcal{O}(h^m)$ whenever $c_m < 1$.

Corollary 2.2.8 *If* $\kappa = m - 1$ *and* $c_m = 1$, *the* $\{c_i\}$ *are the Radau II points, leading to*

$$\max_{t \in I_h \setminus \{0\}} |y(t) - u_h(t)| \le Ch^{2m-1}.$$

Corollary 2.2.9 *Let* $m \geq 3$ *and assume that* $0 = c_1 < c_2 < \ldots < c_m = 1$ *are the Lobatto points. Then the optimal order of local superconvergence for the corresponding continuous collocation solution* $u_h \in S_{m-1}^{(0)}(I_h)$ *is* $p^* = 2(m-1)$. *For* $m = 2$ *the local order on* I_h *coincides with the global order on* I, *namely* $p^* = p = m = 2$.

The above analysis gives rise to the following question regarding *repeated iterated collocation*: if we define $u_{1,h}^{it} := u_h^{it}$ and

$$u_{\mu+1,h}^{it} := g(t) + (\mathcal{V}u_{\mu,h}^{it})(t), \quad t \in I \quad (\mu \geq 1),$$

what can be said about the resulting order and, more importantly, the behaviour of the error constants? It is clear that the order of (local) superconvergence cannot be increased. To understand this, observe first that it follows from

$$u_{2,h}^{it}(t) := g(t) + (\mathcal{V}u_{1,h}^{it})(t), \quad t \in I,$$

that

$$e_{2,h}^{it}(t) = \int_0^t [R(t,s) - K(t,s)]\delta_h(s)ds = e_{1,h}^{it}(t) - (\mathcal{V}\delta_h)(t), \quad t \in I,$$

with $e_{1,h}^{it}h := e_h^{it}$, and this yields

$$e_{\mu+1,h}^{it}(t) = e_{\mu,h}^{it}(t) - \int_0^t K_\mu(t,s)\delta_h(s)ds, \quad t \in I. \tag{2.2.45}$$

Here, K_μ denotes the μth iterated kernel of the given kernel K (cf. (2.1.6)). Hence, an induction argument readily leads to

Theorem 2.2.10 *For* $\mu \geq 1$ *the* μth *iterate* $u_{\mu,h}^{it}$ *of the collocation solution* $u_h \in S_{m-1}^{(-1)}(I_h)$ *induces an error* $e_{\mu,h}^{it} := y - u_{\mu,h}^{it}$ *which has the representation*

$$e_{\mu,h}^{it}(t) = \int_0^t R_\mu(t,s)\delta_h(s)ds, \quad t \in I.$$

Here we have set

$$R_\mu(t,s) := \sum_{n=\mu}^{\infty} K_n(t,s).$$

In Section 2.1.1 we derived uniform bounds for the iterated kernels $K_n(t,s)$. These bounds – which involve the factor $1/n!$ – will form the basis for obtaining more concrete answers to the above question. We leave this as an exercise (Exercise 2.5.20).

A result related to that in Theorem 2.2.9 can be found in Brunner, Lin and Yan (1996): it deals with *iterative correction* techniques (see also Q. Lin (1979),

Q. Lin, Sloan and Xie (1990), Q. Lin and Shi (1993), Q. Lin, Zhang and Yan (1997), and Q. Lin and Zhang (1997) for related correction techniques). We mention also that Porter and Stirling (1993) studied the convergence properties of repeated iterated *Galerkin solutions* to second-kind Fredholm integral equations.

2.2.6 Optimal orders for the discretised collocation solutions

Do the discretised collocation solution $\hat{u}_h$ and the corresponding iterate $\hat{u}_h^{it}$ possess the same order as the exact collocation approximations u_h and u_h^{it}; in particular, do the local superconvergence results of Theorem 2.2.6 remain valid for $\hat{u}_h$ and/or $\hat{u}_h^{it}$? The answer is in the affirmative if, as in Section 2.2.3, the quadrature processes employ interpolatory m-point quadrature formulas whose abscissas are based on the collocation parameters $\{c_i\}$.

In order to understand this, observe first that we have

$$|y(t) - \hat{u}_h(t)| \leq |y(t) - u_h(t)| + |u_h(t) - \hat{u}_h(t)| =: e_h(t) + z_h(t), \quad t \in I. \tag{2.2.46}$$

Global and local estimates for the exact collocation error e_h were established in Sections 2.2.4 and 2.2.5. In order to analyse the perturbation z_h induced by the quadrature processes, recall that u_h and $\hat{u}_h \in S_{m-1}^{(-1)}(I_h)$ are, respectively, the exact and discretised collocation solutions to (2.2.5) defined by

$$u_h(t) = g(t) + (\mathcal{V}u_h)(t), \quad t \in X_h,$$

and

$$\hat{u}_h(t) = g(t) + (\mathcal{V}_h\hat{u}_h)(t), \quad t \in X_h,$$

corresponding to

$$(\mathcal{V}u_h)(t_{n,i}) = F_n(t_{n,i}) + h_n(Q_n u_h)(t_{n,i})$$

and

$$(\mathcal{V}_h\hat{u}_h)(t_{n,i}) = \hat{F}_n(t_{n,i}) + h_n(\hat{Q}_n\hat{u}_h)(t_{n,i}).$$

Here, the lag terms are given by

$$F_n(t_{n,i}) = \sum_{\ell=0}^{n-1} h_\ell(Q_n^{(\ell)}u_h)(t_{n,i}),$$

and

$$\hat{F}_n(t_{n,i}) = \sum_{\ell=0}^{n-1} h_\ell(\hat{Q}_n\hat{u}_h)(t_{n,i})$$

(cf. (2.2.19) and (2.2.20)). We will denote by $E_n^{(\ell)}(t_{n,i})$ $(\ell < n)$ and $E_n(t_{n,i})$ the quadrature errors associated with these quadrature approximations $(\hat{Q}_n^{(\ell)}\hat{u}_h)(t_{n,i})$ and $(\hat{Q}_n\hat{u}_h)(t_{n,i})$. This allows us to write

$$(\hat{Q}_n^{(\ell)}\hat{u}_h)(t_{n,i}) = (Q_n^{(\ell)}\hat{u}_h)(t_{n,i}) - E_n^{(\ell)}(t_{n,i}) \quad (\ell < n),$$

and

$$(\hat{Q}_n\hat{u}_h)(t_{n,i}) = (Q_n\hat{u}_h)(t_{n,i}) - E_n(t_{n,i}).$$

Thus, setting

$$z_h(t_n + vh_n) := u_h(t_n + vh_n) - \hat{u}_h(t_n + vh_n) = \sum_{j=1}^{m} L_j(v)Z_{n,j}, \quad v \in (0, 1], \tag{2.2.47}$$

with $Z_{n,j} := U_{n,j} - \hat{U}_{n,j}$, it follows from the above collocation equations, upon replacing the quadrature approximations by the difference between the exact integrals and the quadrature errors, that the vector $\mathbf{Z}_n := (Z_{n,1}, \ldots, Z_{n,m})^T$ solves the linear algebraic system

$$Z_{n,i} = \sum_{\ell=0}^{n-1} h_\ell (Q_n^{(\ell)} z_h)(t_{n,i}) + h_n (Q_n z_h)(t_{n,i}) + \epsilon_n(t_{n,i}) \quad (i = 1, \ldots, m), \tag{2.2.48}$$

where

$$\epsilon_n(t_{n,i}) := \sum_{\ell=0}^{n-1} h_\ell E_n^{(\ell)}(t_{n,i}) + h_n E_n(t_{n,i}).$$

This algebraic system can be written as

$$[\mathcal{I}_m - h_n B_n]\mathbf{Z}_n = \sum_{\ell=0}^{n-1} h_\ell B_n^{(\ell)} \mathbf{Z}_\ell + \epsilon_n, \tag{2.2.49}$$

with $\epsilon_n := (\epsilon_n(t_{n,1}), \ldots, \epsilon_n(t_{n,m}))^T$. Its structure is similar to that of (2.2.33), and hence it leads again to a discrete Gronwall inequality analogous to (2.2.34), except that now the non-homogeneous term is governed by an upper bound for $||\epsilon_n||_1$. If the quadrature formulas have degree of precision p (where $p \geq m - 1$), then

$$||\epsilon_n||_1 \leq \sum_{j=1}^{m} \left(\sum_{\ell=0}^{n-1} h_\ell \cdot Q_{n,\ell} h_\ell^p + Q_{n,n} h_n^p \right) \leq Q h^p$$

$(n = 0, 1, \ldots, N - 1)$, provided the kernel K is sufficiently regular. The error constants $Q_{n,\ell}$ follow from the Peano Kernel Theorem, and we have used the fact that $\sum_{\ell=0}^{n-1} h_\ell \leq T$ $(n \leq N)$.

The analysis is now readily completed: since $||\mathbf{Z}_n||$ satisfies the discrete Gronwall inequality

$$||\mathbf{Z}_n||_1 \leq \gamma_0 h \sum_{\ell=0}^{n-1} ||\mathbf{Z}_\ell||_1 + \gamma_1 h^p \quad (n = 0, 1, \ldots, N-1),$$

it follows that $||\mathbf{Z}_n||_1 \leq \gamma h^p$ and so, by (2.2.47),

$$|z_h(t_n + vh_n)| \leq \sum_{j=1}^{m} |L_j(v)| \cdot |Z_{n,j}| \leq \Lambda_m ||\mathbf{Z}_n||_1 \leq \Lambda_m \gamma h^p,$$

where $\Lambda_m := \max_{(j)} ||L_j||_\infty$. This estimate is valid uniformly for $v \in [0, 1]$ and $n = 0, 1, \ldots, N-1$, for all meshes I_h with $h \in (0, \bar{h})$.

It is obvious that this perturbation analysis can also be used to deal with the effect of the full discretisation on the *iterated collocation error*: if we define $z_h^{it}(t) := u_h^{it}(t) - \hat{u}_h^{it}(t)$ then it follows from the definition of the exact iterated collocation solution and its discretised counterpart (recall (2.2.17) and (2.2.26)) that the above equation (2.2.48) is to be replaced by

$$z_h^{it}(t_n + vh_n) = \sum_{\ell=0}^{n-1} h_\ell (Q_n^{(\ell)} z_h)(t_n + vh_n) + h_n (Q_n z_h)(t_n + vh_n) + \epsilon_n(t_n + vh_n),$$

with $v \in [0, 1]$, and this leads to the expected order results, both globally on I and locally on the mesh points I_h, thus extending Theorems 2.2.5 and 2.2.6 to $||y - \hat{u}_h^{it}||_\infty$ and $\max_{(I_h)} |y(t) - \hat{u}_h^{it}(t)|$, respectively. Moreover, the same ideas can be used to deal with the fully discretised collocation methods for other types of Volterra integral and (functional) integro-differential equations. In the following chapters we will usually not explicitly state the convergence results corresponding to fully discretised collocation solutions since they can easily be obtained by adapting the above analysis. We will, however, summarise the insight obtained presently in Theorem 2.2.11: it represents the discrete analogue of Theorems 2.2.5 and 2.2.6.

Theorem 2.2.11 *Assume that the collocation equation defining the exact collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ and the corresponding iterated collocation solution u_h^{it} for (2.2.5) are discretised by using interpolatory m-point quadrature formulas whose absissas are based on the collocation parameters $\{c_i\}$. Then the resulting discretised collocation approximations $\hat{u}_h$ and $\hat{u}_h^{it}$ have the same (optimal) global and local optimal convergence properties as u_h and u_h^{it}.*

Remark Discretised collocation methods, in the past often called *block methods*, were analysed by Weiss (1972a) and by de Hoog and Weiss (1975). The connection between these methods and (exact) collocation methods was studied in Brunner (1977).

2.2.7 Divergence of collocation solutions in smoother collocation spaces

We have observed in Section 1.1.5 (Theorem 1.1.9 and Corollary 1.1.10) that if $u_h \in S_m^{(0)}(I_h)$ is the collocation solution for the initial-value problem $y'(t) = ay(t),\ t \in I,\ y(0) = y_0$, and if v_h^{it} denotes the iterated collocation solution corresponding to $v_h \in S_{m-1}^{(-1)}(I_h)$ for the integrated form of the initial-value problem,

$$y(t) = y_0 + \int_0^t ay(s)ds, \quad t \in I,$$

then $u_h(t) = v_h^{it}$ for all $t \in I$. We will now show that this result remains true for more general, *smooth* piecewise polynomial collocation spaces $S_\mu^{(d)}(I_h\ (d < \mu)$. An important consequence of this result will be that it allows us to extend Mülthei's divergence theory for collocation solutions for ODEs (described in Section 1.3.1) to second-kind VIEs.

Theorem 2.2.12 *Let μ and d be any positive integers satisfying $d < \mu$, and let*

$$X_h^{(d)} := \{t_n + c_i h_n :\ 0 < c_1 < \ldots < c_{\mu-d} \le 1\ (n = 0, 1, \ldots, N-1)\}$$

denote the set of collocation points associated with the mesh I_h and the given $\mu - d$ (distinct) collocation parameters $\{c_i\}$. If $u_h \in S_\mu^{(d)}(I_h)$ is the collocation solution defined by

$$u_h'(t) = au_h(t), \quad t \in X_h^{(d)}, \quad u_h(0) = y_0 \quad (a \neq 0), \tag{2.2.50}$$

and

$$v_h^{it}(t) := y_0 + \int_0^t av_h(s)ds, \quad t \in I, \tag{2.2.51}$$

is the iterate of the collocation solution $v_h \in S_{\mu-1}^{(d-1)}(I_h)$ for

$$y(t) = y_0 + \int_0^t ay(s)ds, \quad t \in I, \tag{2.2.52}$$

then

$$v_h^{it}(t) = u_h(t) \quad \textit{for all} \quad t \in I.$$

Proof Let the local representation of the collocation solution $u_h \in S_\mu^{(d)}(I_h)$ for the initial-value problem (2.2.50) be

$$u_h(t_n + vh_n) = \sum_{\nu=0}^{d} \frac{h_n^\nu y_n^{(\nu)}}{\nu!} v^\nu + \sum_{j=d+1}^{\mu} \alpha_{n,j} v^j, \quad v \in [0, 1].$$

We have $u_h^{(\nu)}(t_n) = y_n^{(\nu)}$ ($\nu = 0, 1, \ldots, d$), with $y_n^{(\nu)} := u_h^{(\nu)}(t_{n-1} + h_{n-1})$ and given initial values $u_h^{(\nu)}(0) := y^{(\nu)}(0)$. Since

$$u_h'(t_n + vh_n) = h_n^{-1} \left(\sum_{\nu=1}^{d} \frac{h_n^\nu y_n^{(\nu)}}{(\nu-1)!} v^{\nu-1} + \sum_{j=d+1}^{\mu} j\alpha_{n,j} v^{j-1} \right), \quad v \in [0, 1],$$

the collocation equation (2.2.50) for u_h at $t = t_{n,i}$ can be written in the form

$$\sum_{j=d+1}^{\mu} \left(jc_i^{j-1} - h_n a c_i^j \right) \alpha_{n,j} = -\sum_{\nu=1}^{d} \frac{h_n^\nu y_n^{(\nu)}}{(\nu-1)!} c_i^{\nu-1} + h_n a \sum_{\nu=0}^{d} \frac{h_n^\nu y_n^{(\nu)}}{\nu!} c_i^\nu$$

($i = 1, \ldots, \mu - d$). Somewhat more concisely, it reads

$$V_n^{(d)} \alpha_n = h_n a y_n \mathbf{e} - \left[\begin{array}{c} \sum_{\nu=1}^{d} \left(\frac{c_i^{\nu-1}}{(\nu-1)!} - h_n a \frac{c_i^\nu}{\nu!} \right) h_n^\nu y_n^{(\nu)} \\ (i = 1, \ldots, \mu - d) \end{array} \right], \tag{2.2.53}$$

where

$$V_n^{(d)} := \left(\begin{array}{c} jc_i^{j-1} - h_n a c_i^j \\ (i = 1, \ldots, \mu - d;\, j = d+1, \ldots, \mu) \end{array} \right) \in L(\mathbb{R}^{\mu-d})$$

and $\boldsymbol{\alpha}_n := (\alpha_{n,d+1}, \ldots, \alpha_{n,\mu})^T \in \mathbb{R}^{\mu-d}$.

Consider now the collocation solution $v_h \in S_{\mu-1}^{(d-1)}(I_h)$ for the VIE (2.2.52) (the integrated form of the initial-value problem for the ODE). We choose as its local representation on σ_n,

$$v_h(t_n + vh_n) = \sum_{\nu=0}^{d-1} \frac{h_n^\nu v_n^{(\nu)}}{\nu!} v^\nu + \sum_{j=d}^{\mu-1} (j+1)\gamma_{n,j} v^j, \quad v \in (0, 1]$$

which implies that $v_h^{(\nu)}(t_n) = v_n^{(\nu)}$, $\nu = 0, 1, \ldots, d-1$. It is then easily seen that the collocation equation at $t = t_{n,i}$,

$$v_h(t_{n,i}) = F_n + h_n a \int_0^{c_i} v_h(t_n + sh_n) ds \quad (i = 1, \ldots, \mu - d),$$

with lag term

$$F_n := y_0 + \int_0^{t_n} a v_h(s) ds,$$

after some algebraic manipulations assumes the form

$$\sum_{j=d}^{\mu-1}\left((j+1)c_i^j - h_n a c_i^{j+1}\right)\gamma_{n,j} = F_n - \sum_{\nu=0}^{d-1}\frac{h_n^\nu v_n^{(\nu)}}{\nu!}c_i^\nu + h_n a\sum_{\nu=0}^{d-1}\frac{h_n^\nu v_n^{(\nu)}}{(\nu+1)!}c_i^{\nu+1} \tag{2.2.54}$$

$(i = 1, \dots, \mu - d)$. Thus, the counterpart of (2.2.53) is given by

$$W_n^{(d)}\gamma_n = F_n\mathbf{e} - \begin{bmatrix} \sum_{\nu=0}^{d-1}\left(\frac{c_i^\nu}{\nu!} - h_n a\frac{c_i^{\nu+1}}{(\nu+1)!}\right)h_n^\nu v_n^{(\nu)} \\ (i = 1, \dots, \mu - d) \end{bmatrix}, \tag{2.2.55}$$

with

$$W_n^{(d)} := \begin{bmatrix} (j+1)c_i^j - h_n a c_i^{j+1} \\ (i = 1, \dots, \mu - d;\, j = d, \dots, \mu - 1) \end{bmatrix} \in L(\mathbb{R}^{\mu-d})$$

and $\gamma_n := (\,\gamma_{n,d}, \dots, \gamma_{n,\mu-1}\,)^T \in \mathbb{R}^{\mu-d}$.

The definition of the *iterated* collocation solution at $t = t_n + vh_n$,

$$v_h^{it}(t_n + vh_n) := F_n + h_n a\int_0^v v_h(t_n + sh_n)ds, \quad v \in [0, 1],$$

reveals that the value of the lag term F_n is of course (as in Section 1.1.5)

$$F_n = v_h^{it}(t_n) \quad (n = 1, \dots, N),$$

since the kernel of the VIE does not depend on t.

In order to bring the proof to its conclusion we first observe that:

(i) The matrices $V_n^{(d)}$ and $W_n^{(d)}$ on the left-hand sides of (2.2.53) and (2.2.54) are identical: $V_n^{(d)} = W_n^{(d)}$ for $n = 0, 1, \dots, N - 1$.
(ii) $u_h^{(\nu)}(0) = y_0^{(\nu)} = (v_h^{it})^{(\nu)}(0)$ $(\nu = 0, 1, \dots, d)$.
(iii) Since $y'(t) = ay(t)$, we have $y^{(\nu)}(0) = ay^{(\nu-1)}(0)$ $(\nu = 1, \dots, d)$.
(iv) The following relationship between u_h and v_h^{it} is obvious:

Lemma 2.2.13 *For* $v \in [0, 1]$ *and* $n = 0, 1, \dots, N - 1$,

$$u_h(t_n + vh_n) - v_h^{it}(t_n + vh_n) = y_n - v_h^{it}(t_n)\sum_{\nu=1}^{d}\left(y_n^{(\nu)} - h_n a v_n^{(\nu-1)}\right) + \frac{h_n^\nu v^\nu}{\nu!} + \sum_{j=d+1}^{\mu}\left(\alpha_{n,j} - h_n a\gamma_{n,j-1}\right)v^j.$$

If we now compare the solutions α_n and γ_n of the linear algebraic systems (2.2.53) and (2.2.55) and use the result of the above lemma, together with a straightforward induction argument (and an obvious change of the variables of summation j in (2.4.54) and ν in the first term on its right-hand side), we find that for all n,

$$u_h^{(\nu)}(t_n) = y_n^{(\nu)} = (v_h^{it})^{(\nu)}(t_n) \quad (n = 1, \ldots, N-1),$$

and hence, $\boldsymbol{\alpha}_n = \boldsymbol{\gamma}_n$, that is,

$$\alpha_{n,j} = h_n a \gamma_{n,j-1} \quad (j = d+1, \ldots, \mu;\ n = 0, 1, \ldots, N-1).$$

By combining all these observations we can now easily convince ourselves that the assertion of Theorem 2.2.11 is indeed true.

Remark As we indicated in Section 1.3.1, an obvious alternative to the above local representations of u_h and v_h are the ones based on the *Hermite* canonical polynomials with respect to the $\{c_i\}$ (see Exercise 2.5.26).

Corollary 2.2.14 *Assume that the collocation solution $u_h \in S_\mu^{(d)}(I_h)$ for the initial-value problem for $y'(t) = ay(t)$ is divergent. Then both the iterated collocation solution v_h^{it} and the underlying collocation solution $v_h \in S_{\mu-1}^{(d-1)}(I_h)$, based on the same (distinct) collocation points $X_h^{(d)}$, for the second-kind VIE (2.2.5) are also divergent.*

Proof We simply note that

$$|e_h^{it}(t)| = |\mathcal{V}e_h)(t)| \leq K_0 ||e_h||_\infty, \quad t \in I,$$

where

$$K_0 := \max_{t \in I} \int_0^t |K(t,s)| ds = |a| T.$$

We described divergence results for collocation solutions to ODEs (due mainly to Loscalzo and Talbot (1967) and Mülthei (1979, 1980a)) in Section 1.3.1. Hence, Theorem 2.2.11 and Corollary 2.2.13 provide the basis for the analogous *divergence theory for second-kind VIEs*. The first convergence and divergence results for second-kind VIEs are due to Hung (1970) and to El Tom (1971, 1974, 1976). A more general (stability) analysis can be found in the recent papers by Oja (2001a, 2001b) and Oja and Saveljeva (2002).

The equivalence result of Theorem 2.2.11 and Corollary 2.2.13 allows us to establish the complete *divergence theory* of piecewise polynomial collocation methods for second-kind VIEs. We state a number of representative results, starting with the case of *simple* parameters $\{c_i\}$. Similar results on collocation using parameters with higher multiplicity will be given in Section 2.2.8. Reverting to our standard notation, using again u_h, instead of v_h, to denote a collocation solution to the VIE (2.2.5), we conclude this section by stating the analogue of Theorem 1.3.4. The results for Hermite-type collocation (corresponding to Theorem 1.3.1 and Corollary 1.3.2) will be given in Section 2.2.8.

The analogue of Corollary 1.3.2 for second-kind VIEs is due to Hung (1970).

Theorem 2.2.15 *The collocation solution $u_h \in S_3^{(2)}(I_h)$ for (2.2.5), with $c_1 = 1$, is divergent.*

A more general result is given in Theorem 2.2.16: it corresponds to Theorem 1.3.4 for ODEs.

Theorem 2.2.16 *Let $u_h \in S_{\mu-1}^{(d-1)}(I_h)$ $(d < \mu)$ be the collocation solution for the second-kind VIE (2.2.5), with collocation parameters $0 < c_1 < \ldots < c_r = 1$ possessing multiplicities $\delta_i = 1$ (hence $r = \mu - d$). If $d \geq 2$ then u_h is divergent whenever*

$$\prod_{i=1}^{r-1} \frac{1 - c_i}{c_i} > 1.$$

Proof The above divergence result is implied by Theorem 1.3.4 ($\delta_i = 1$ for all i, and $d = 2 = \delta_r + 1$) and by Corollary 2.2.14.

Corollary 2.2.17 *If $u_h \in S_{\mu-1}^{(1)}(I_h)$ is the collocation solution corresponding to the $\mu - 2$ Radau II points, then u_h is divergent.*

Proof This divergence result follows from Corollary 1.3.5 and Corollary 2.2.14.

2.2.8 Hermite-type collocation methods

Assume now that some of the collocation parameters c_i have multiplicities $\delta_i > 1$. In analogy to Section 1.3, the collocation solution $u_h \in S_{\mu-1}^{(d-1)}(I_h)$ is now determined by the collocation equation

$$u_h^{(\nu)}(t_{n,i}) = g^{(\nu)}(t_{n,i}) + \frac{d}{dt}(\mathcal{V}u_h)(t_{n,i}), \quad \nu = 0, \ldots, \delta_i - 1; \quad i = 1, \ldots, r, \tag{2.2.56}$$

with initial values $u_h^{(\nu)}(0) = y^{(\nu)}(0)$ $(\nu = 0, \ldots, d-1)$. Setting

$$u_h(t_n + vh_n) = \sum_{l=0}^{d-1} \frac{y_n^{(l)} h_n^l}{l!} v^l + \sum_{j=d}^{\mu-1} (j+1)\gamma_{n,j} v^j, \quad v \in [0, 1],$$

with $y_n^{(l)} := u_h^{(l)}(t_n)$ (cf. Section 1.3.1), the proof of Theorem 2.2.11 is readily adapted to encompass this more general situation; hence the equivalence result remains valid. This allows us to deduce the following divergence statements.

Theorem 2.2.18 *Let $u_h \in S_{\mu-1}^{(d-1)}(I_h)$ $(1 \leq d < \mu)$ be the collocation solution for the VIE (2.2.5), corresponding to the collocation parameters satisfying $0 < c_1 < \ldots < c_{r-1} < c_r = 1$ and possessing multiplicities $\delta_1, \ldots, \delta_r$, so that $\delta_i > 1$ for at least one i.*

If d is such that $d = \delta_r + 1$ (≥ 2) then u_h is divergent whenever

$$\prod_{i=1}^{r-1} \left(\frac{1 - c_i}{c_i} \right)^{\delta_i} > 1.$$

Theorem 2.2.19 *The Hermite collocation solution $u_h \in S_{2m-1}^{(m)}(I_h)$, with $q = m - 1$ and $c_1 = 1$, is divergent for second-kind VIEs.*

The **proof** is a consequence of the second part in Example 1.3.2.

Remarks

1. Esser (1978) showed that Hermite collocation for second-kind VIEs in $S_{2m-1}^{(m-1)}(I_h)$ (with $c_1 = 1$ having multiplicity $\delta_1 = m$) is convergent.
2. Mülthei's *convergence results* for ODEs (Mülthei (1980a, 1980b, 1980c, 1982a)) do not necessarily imply analogous convergence results for VIEs, since a linear second-kind VIE is in general not reducible to a first-order ODE. Except for some partial results by Oja et al. mentioned above, the convergence analysis of piecewise polynomial collocation solutions to VIEs is still incomplete.

2.2.9 Multidimensional VIEs of the second kind

We have seen in Section 2.1.7 that the second-kind VIE,

$$u(x, y) = g(x, y) + (\mathcal{V}u)(x, y), \quad (x, y) \in \Omega := [0, X] \times [0, Y], \quad (2.2.57)$$

with Volterra integral operator $\mathcal{V} : C(\Omega) \to C(\Omega)$ given by

$$(\mathcal{V}\phi)(x, y) := \int_0^x \int_0^y K(x, \xi, y, \eta)\phi(\xi, \eta) d\eta \, d\xi,$$

possesses a unique solution $u \in C(\Omega)$ whenever g and K are continuous on Ω and $D_2 := \{(x, \xi, y, \eta) : \; 0 \leq \xi \leq x \leq X, \; 0 \leq \eta \leq y \leq Y\}$, respectively. Moreover, as in the one-dimensional case it can be represented by a variation-of-constants formula (resolvent representation), as shown in Theorem 2.1.14. This suggests that the convergence results derived in Section 2.2 will also carry over to collocation solutions and their iterates for the above VIE.

We will give a brief description of the collocation method and a summary of the corresponding (super-) convergence results; details and proofs are along familiar lines and can also be found in Brunner and Kauthen (1989).

Let

$$\begin{aligned}
I_h &:= \{x_j : 0 = x_0 < x_1 < \ldots < x_M = X\}, \\
J_k &:= \{y_l : 0 = y_0 < y_1 < \ldots < y_N = Y\}, \\
h_j &:= x_{j+1} - x_j, \quad k_l := y_{l+1} - y_l; \quad h := \max_{(j)} h_j, \quad k := \max_{(l)} k_l,
\end{aligned}$$

and set $\Omega_{h,k} := I_h \times J_k$. The collocation space will be

$$S^{(-1)}_{m-1,\mu-1}(\Omega_{h,k}) := \{v : v|_{\sigma_{j,l}} \in \pi_{m-1,\mu-1}\ (0 \le j \le M-1,\ 0 \le l \le N-1)\},$$

where $\sigma_{j,l} := (x_j, x_{j+1}] \times (y_l, y_{l+1}]$ and $\pi_{m-1,\mu-1} := \pi_{m-1} \otimes \pi_{\mu-1}$. The dimension of this linear space is $MNm\mu$, and hence the set of collocation points, $Z_{h,k} := X_h \times Y_k$, will be based on the sets

$$\begin{aligned}
X_h &:= \{x_j + c_i h_j : 0 \le c_1 < \ldots < c_m \le 1\ (0 \le j \le M-1)\}, \\
Y_k &:= \{y_l + d_i k_l : 0 \le d_1 < \ldots < d_\mu \le 1\ (0 \le l \le N-1)\}.
\end{aligned}$$

The collocation solution $u_{h,k}$ to (2.2.57) is the element of $S^{(-1)}_{m-1,\mu-1}(\Omega_{h,k})$ that satisfies the collocation equation

$$u_{h,k}(x,y) = g(x,y) + (\mathcal{V}u_{h,k})(x,y), \quad (x,y) \in Z_{h,k}, \tag{2.2.58}$$

and the corresponding *iterated collocation solution* is determined by

$$u^{it}_{h,k}(x,y) := g(x,y) + (\mathcal{V}u_{h,k})(x,y), \quad (x,y) \in \Omega. \tag{2.2.59}$$

Note that $u^{it}_{h,k} \in C(\Omega)$; this is true for $u_{h,k}$ only if $m \ge 2$ and $c_1 = d_1 = 0,\ c_m = d_\mu = 1$.

We leave the derivation of the computational forms of the above collocation equations to the reader: since the collocation space is a tensor-product space, the local representation of $u_{h,k}$ on a subregion $\sigma_{j,l}$ is the tensor product of the two familiar one-dimensional local Lagrange representations.

Theorem 2.2.20 *Assume that $g \in C^d(\Omega)$ and $K \in C^d(D_2)$, for some $d \ge 0$.*

(i) There exists $\bar{H} > 0$ so that (2.2.58) defines a unique collocation solution $u_{h,k} \in S^{(-1)}_{m-1,\mu-1}(\Omega_{h,k})$ for all meshes $\Omega_{h,k}$ with $h, k \in (0, \bar{H})$.

(ii) If $d \ge m$, then

$$\sup\{|u(x,y) - u_{h,k}(x,y)| : (x,y) \in \Omega\} \le CH^\rho, \quad \rho := \min\{m, \mu\}.$$

Here, $H := \max\{h,k\}$, and the estimate is true for arbitrary collocation parameters $\{c_i\}$ and $\{d_i\}$.

(iii) Suppose that the collocation parameters are chosen so that the orthogonality conditions

$$J_\ell^1 := \int_0^1 s^\ell \prod_{i=1}^{m}(s-c_i)ds = 0 \quad (\ell = 0, \ldots, \kappa_1 - 1, \quad \textit{with} \quad \kappa_1 \le m),$$

$$J_\nu^2 := \int_0^1 s^\nu \prod_{i=1}^{\mu}(s-d_i)ds = 0 \quad (\nu = 0, \ldots, \kappa_2 - 1, \quad \textit{with} \quad \kappa_2 \le \mu)$$

hold.

If $d \ge \max\{m+\kappa_1, \mu+\kappa_2\}$, *then*

$$\max\{|u(x_j, y_l) - u_{h,k}^{it}(x_j, y_l)| : \ 1 \le j, l \le M-1, N-1\} \le CH^{\rho^*},$$

with $\rho^* := \min\{m+\kappa_1, \mu+\kappa_2\}$. *In particular, if* $m = \mu$ *and the two sets* $\{c_i\}$ *and* $\{d_i\}$ *are the Gauss points, then we have* $\rho^* = 2m$.

Analogous results are true for the discretised collocation solutions $\hat{u}_{h,k}$ and $\hat{u}_{h,k}^{it}$, provided the (tensor product) quadrature formulas are of interpolatory type and based on the collocation parameters $\{c_i\}$ and $\{d_i\}$, respectively. See also Stroud (1971) for details and expressions for the quadrature errors (and the underlying two-dimensional version of the Peano Kernel Theorem).

2.2.10 Comparison with collocation for Fredholm integral equations

In order to gain a better perspective on the superconvergence results for Volterra integral equations, we briefly look at collocation methods for linear Fredholm integral equations of the second kind,

$$y(t) = g(t) + \lambda(\mathcal{F}y)(t), \quad t \in I := [0, T], \tag{2.2.60}$$

where λ denotes a (real or complex) parameter and where the Fredholm integral operator $\mathcal{F}: \ C(I) \to C(I)$ is defined by

$$(\mathcal{F}\phi)(t) := \int_0^T K(t,s)\phi(s)ds, \quad K \in C(I \times I).$$

We will assume that λ^{-1} is not in the spectrum $\sigma(\mathcal{F})$ of the Fredholm integral operator $\mathcal{F}$; that is, $\mathcal{I} - \lambda\mathcal{F}$ is invertible in $C(I)$.

The collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ for (2.2.60) is determined by

$$u_h(t) = g(t) + \lambda(\mathcal{F}u_h)(t), \quad t \in X_h, \tag{2.2.61}$$

and the corresponding *iterated collocation solution* is found by

$$u_h^{it}(t) := g(t) + \lambda(\mathcal{F}u_h)(t), \quad t \in I. \tag{2.2.62}$$

Using again the local representation

$$u_h(t_n + vh_n) = \sum_{j=1}^{m} L_j(v)U_{n,j}, \quad v \in (0, 1], \quad \text{with} \quad U_{n,j} := u_h(t_{n,j}),$$

and setting

$$B_n^{(\ell)} := \begin{pmatrix} \int_0^1 K(t_{n,i}, t_\ell + sh_\ell)L_j(s)ds \\ (i, j = 1, \ldots, m) \end{pmatrix} \quad (\ell,\ n = 0, 1, \ldots, N-1),$$

$\mathbf{g}_n := (g(t_{n,1}), \ldots, g(t_{n,m}))^T$, the linear algebraic system for the vectors $\mathbf{U}_n \in \mathbb{R}^m$ $(n = 0, 1, \ldots, N-1)$ can be written as

$$\begin{bmatrix} \mathcal{I}_m - \lambda h_0 B_0^{(0)} & -\lambda h_1 B_0^{(1)} & \ldots & -\lambda h_{N-1} B_0^{(N-1)} \\ \vdots & \vdots & & \vdots \\ -\lambda h_0 B_{N-1}^{(0)} & -\lambda h_1 B_{N-1}^{(1)} & \ldots & \mathcal{I}_m - \lambda h_{N-1} B_{N-1}^{(N-1)} \end{bmatrix} \begin{bmatrix} \mathbf{U}_0 \\ \vdots \\ \mathbf{U}_{N-1} \end{bmatrix} = \begin{bmatrix} \mathbf{g}_0 \\ \vdots \\ \mathbf{g}_{N-1} \end{bmatrix}.$$

The invertibility of the $Nm \times Nm$ block matrix now depends, in contrast to Volterra integral equations, not only on h but also on λ: it is guaranteed if $|\lambda| \cdot ||\mathcal{F}|| < 1$, where

$$||\mathcal{F}|| := \max_{t \in I} \int_0^T |K(t, s)|ds \leq \bar{K}T,$$

assuming that $|K(t, s| \leq \bar{K}$ on $I \times I$.

The convergence analysis of collocation methods for Fredholm integral equations of the second kind dates back to the work of Kadner (1960, 1967); of the many papers dealing with superconvergence properties of piecewise collocation solutions (also in Galerkin methods) we cite those by Sloan (1976, 1984, 1988a, 1990), Chandler (1979), Chatelin and Lebbar (1981), Brunner (1984a), Joe (1985a, 1985b); see also the monographs by Chatelin (1983), Golberg and Chen (1997), and Atkinson (1997a). As the following theorem shows, the fundamental difference between superconvergence results for Volterra and for Fredholm integral equations is that in the latter case, high-order (e.g. $\mathcal{O}(h^{2m})$-) convergence for u_h^{it} holds *globally* on I.

Theorem 2.2.21 *Assume:*

(a) $g \in C^{m+\kappa}(I),\ K \in C^{m+\kappa}(I \times I)$;
(b) $\lambda^{-1} \notin \sigma(\mathcal{F})$;

(c) $u_h \in S_m^{(-1)}(I_h)$ *is the collocation solution defined by (2.2.61), with associated iterated collocation solution* u_h^{it} *given by (2.2.62);*

(d) the collocation parameters $\{c_i\}$ *are subject to the orthogonality conditions*

$$J_\nu := \int_0^1 s^\nu \prod_{i=1}^m (s - c_i) ds = 0, \quad \nu = 0, \dots, \kappa - 1 \quad (\kappa \le m),$$

with $J_\kappa \neq 0$.

Then for all sufficiently small $h > 0$*, the collocation solution* u_h *is superconvergent on* X_h*:*

$$\max_{t \in X_h} |y(t) - u_h(t)| \le C_\kappa h^{m+\kappa},$$

and the iterated collocation solution exhibits the same order of superconvergence on the entire interval I*:*

$$||y - u_h^{it}||_\infty \le C_\kappa h^{m+\kappa}.$$

A proof of these results, along the lines of the one for VIEs (Section 2.2.5) may be found in Brunner (1984a); see also the survey paper by Brunner (1987). The analysis of the attainable order when the solution of (2.2.57) does not have full regularity was analysed in detail by Joe (1985a, 1985b).

Remark The above analysis and the results derived in Section 2.2.9 readily suggest that the global superconvergence result of Theorem 2.2.19 for u_h^{it} will remain valid for the analogous iterated collocation solution to

$$u(x, y) = g(x, y) + (\mathcal{F}u)(x, y), \quad (x, y) \in \Omega := [0, X] \times [0, Y],$$

with Fredholm integral operator $\mathcal{F} : C(\Omega) \to C(\Omega)$ given by

$$(\mathcal{F}\phi)(x, y) := \int_\Omega K(x, \xi, y, \eta)\phi(\xi, \eta) d\eta \, d\xi$$

and $K \in C(\Omega)$, provided $\lambda = 1$ is not in the spectrum of $\mathcal{F}$. Compare also Graham (1980) and Atkinson (1997a).

2.2.11 Collocation for Volterra–Fredholm integral equations

We have seen in Section 2.1.6 that in the analysis of 'mixed' integral equations of the form

$$u(t, x) = g(t, x) + (\mathcal{T}u)(t, x), \quad (t, x) \in I \times J := [0, T] \times [0, X], \quad (2.2.63)$$

with

$$(\mathcal{T}\phi)(t, x) := \int_0^t \int_J K(t, s, x, \xi)\phi(s, \xi) d\xi \, ds,$$

it is the Volterra part of the integral operator $\mathcal{T}$ that dominates the existence of solutions (Theorem 2.1.13). Hence, we intuitively expect that the (super-) convergence properties of (iterated) collocation solutions to (2.2.63) will also be governed to some extent by the Volterra integral operator making up $\mathcal{T}$. In order to state the precise convergence results, we adopt a notation similar to the one in Section 2.2.9: we set $\Omega_{\tau,h} := I_\tau \times J_h$, with

$$I_\tau := \{t_n : 0 = t_0 < t_1 < \ldots < t_n = T\}, \quad \tau_n := t_{n+1} - t_n,$$
$$J_h := \{x_j : 0 = x_0 < x_1 < \ldots < x_M = X\}, \quad h_j := x_{j+1} - x_j,$$

and $\tau := \max_{(n)} \tau_n$, $h := \max_{(j)} h_j$. The collocation space based on the mesh $\Omega_{\tau,h}$ will be

$$S^{(-1)}_{m-1,\mu-1}(\Omega_{\tau,h}) := \{v : v \in \pi_{m-1,\mu-1} \text{ on each subregion } (t_n, t_{n+1}] \times (x_j, x_{j+1}]\}.$$

Accordingly, we will work with the collocation points $X_\tau \times Y_h$, defined by

$$X_\tau := \{t_n + c_i \tau_n : 0 \le c_1 < \ldots < c_m \le 1 \ (n = 0, 1, \ldots, N-1)\},$$

and

$$Y_h := \{x_j + d_i h_j : 0 \le d_1 < \ldots < d_\mu \le 1 \ (j = 0, 1, \ldots, M-1)\}.$$

The collocation solution $u_{\tau,h} \in S^{(-1)}_{m-1,\mu-1}(\Omega_{\tau,h})$ to (2.2.63) and its iterate $u^{it}_{\tau,h}$ are then respectively defined by

$$u_{\tau,h}(t,x) = g(t,x) + (\mathcal{T} u_{\tau,h})(t,x), \quad (t,x) \in X_\tau \times Y_h, \tag{2.2.64}$$

and

$$u^{it}_{\tau,h}(t,x) := g(t,x) + (\mathcal{T} u_{\tau,h})(t,x), \quad (t,x) \in \Omega. \tag{2.2.65}$$

However, since in physical or biological problems leading to the VFIE (2.2.63) the independent variable t represents time, we will also consider the *continuous-time* collocation equation: for $t \in I$ it defines a collocation solution $V_h(t) \in S^{(-1)}_{\mu-1}(J_h)$ by means of

$$V_h(t,x) = g(t,x) + (\mathcal{T} V_h)(t,x), \quad x \in Y_h \quad (t \in I);$$

its iterate is

$$V^{it}_h(t,x) := g(t,x) + (\mathcal{T} V_h)(t,x), \quad x \in J \quad (t \in I).$$

The time-stepping scheme is then described by the collocation approximation $U_{\tau,h}$ to $V_h(t,x)$ with respect to t, using the space $S^{(-1)}_{m-1}(I_\tau)$ as the collocation space. We obviously have $U_{\tau,h} = u_{\tau,h}$.

The convergence properties of $V_h(t,x)$, $u_h(t,x)$ and their iterates $V_h^{it}(t,x)$, $u_h^{it}(t,x)$ were analysed by Kauthen (1989a, 1989b) (for (2.2.63)), and by Brunner (1990, 1991) (nonlinear VFIEs).

Theorem 2.2.22 *Let the functions describing the VIFE (2.2.63) be sufficiently regular on their respective domains, and assume that the collocation parameters* $\{c_i\}$ *and* $\{d_i\}$ *defining the sets* X_τ *and* Y_h *satisfy the orthogonality conditions stated in Theorem 2.2.14. Then:*

(i) $\sup\{|u(t,x) - V_h(t,x)| : \underline{x \in Y_h}, \ t \in I\} \le Ch^{\mu+\kappa_2}$, *with* $\kappa_2 \le \mu$.
(ii) $\max\{|u(t,x) - V_h^{it}(t,x)| : \underline{x \in J}, \ t \in I\} \le Ch^{\mu+\kappa_2}$.
(iii) $\max\{|u(t,x) - u_{\tau,h}(t,x)| : x \in J, \ \underline{t \in I_\tau \setminus \{0\}}\} \le CH^{\rho^*}$, *with* $\rho^* := \min\{m+\kappa_1, \mu+\kappa_2\}$ *and* $H := \max\{\tau, h\}$.

Remark The above results are also true for VFIEs of *Hammerstein* type, e.g. for (2.2.63) with integral operator $\mathcal{T}$ given by

$$(\mathcal{T}\phi)(t,x) := \int_0^t \int_J k(t-s)K(x,\xi)G(\phi(s,\xi))d\xi\, ds.$$

In such cases it may again be advantageous to use *implicitly linear collocation* (see Section 2.3.3 below) to approximate the solution. Compare also Brunner (1991) for details and additional comments.

2.3 Collocation for nonlinear second-kind VIEs

2.3.1 Global error analysis

The collocation error $e_h = y - u_h$ for the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ to the general nonlinear VIE

$$y(t) = g(t) + \int_0^t k(t,s,y(s))ds, \quad t \in I,$$

satisfies, at $t = t_n + c_i h_n \in X_h$,

$$\begin{aligned} e_h(t_{n,i}) &= (\mathcal{V}y)(t_{n,i}) - (\mathcal{V}u_h)(t_{n,i}) \\ &= \sum_{\ell=0}^{n-1} h_\ell \int_0^1 \big(k(t_{n,i}, t_\ell + sh_\ell, y(t_\ell + sh_\ell)) \\ &\quad - k(t_{n,i}, t_\ell + sh_\ell, u_h(t_\ell + sh_\ell))\big)ds \\ &\quad + h_n \int_0^{c_i} \big(k(t_{n,i}, t_n + sh_n, y(t_n + sh_n)) \\ &\quad - k(t_{n,i}, t_n + sh_n, u_h(t_n + sh_n))\big)ds. \end{aligned}$$

Since $u_h = y - e_h$, we may write this in the form

$$e_h(t_{n,i}) = \sum_{\ell=0}^{n-1} h_\ell \int_0^1 k_y(t_{n,i}, t_\ell + sh_\ell, z_\ell(s)) e_h(t_\ell + sh_\ell) ds$$
$$+ h_n \int_0^{c_i} k_y(t_{n,i}, t_n + sh_n, z_n(s)) e_h(t_n + sh_n) ds,$$

assuming that $k_y(t, s, \cdot)$ is continuous and bounded. The functions z_ℓ ($\ell \leq n$) are the arguments arising in the Taylor remainder terms. Hence, using the local representation of the collocation error e_h, the proof of Theorem 2.2.3 is readily extended to encompass nonlinear VIEs, and in particular the Hammerstein version (2.1.43) often found in applications,

$$y(t) = g(t) + \int_0^t K(t, s) G(s, y(s)) ds, t \in I$$

(cf. Section 2.3.3).

While leaving the details to the reader we just state that, under appropriate existence and regularity assumptions, the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ induces an error whose order is described by

$$||y - u_h||_\infty \leq Ch^m,$$

and this holds for any $\{c_i\}$ with $0 \leq c_1 < \ldots < c_m \leq 1$.

2.3.2 Local superconvergence results for nonlinear V2s

For nonlinear VIEs (2.1.36) with sufficiently regular solutions the global and local superconvergence results of Theorems 2.2.5 and 2.2.6 remain valid. This is not surprising because if we employ Taylor's theorem with quadratic remainder term, we may write the equation for the collocation error,

$$e_h(t) = \delta_h(t) + \int_0^t [k(t, s, y(s)) - k(t, s, u_h(s))] ds, \quad t \in I,$$

where $u_h = y - e_h$, in the form

$$e_h(t) = \delta_h(t) + \int_0^t H_0(t, s) e_h(s) ds + T_2(t), \quad t \in I. \qquad (2.3.1)$$

with $H_0(t, s) := k_y(t, s, y(s))$ and

$$T_2(t) := -\frac{1}{2} \int_0^t k_{yy}(t, s, \eta(s)) e_h^2(s) ds.$$

Here, $\eta(s) := y(t) - \theta e_h(s)$ for some θ with $0 < \theta < 1$. We deduce from the order of global convergence of u_h that

$$|T_2(t)| = \mathcal{O}(||e_h||_\infty^2) = \mathcal{O}(h^{2m}), \quad t \in I,$$

provided $f_{yy}(t, s, \cdot)$ is bounded on $D \times \Omega$ for some $\Omega \subset \mathbb{R}$. Denoting the resolvent kernel associated with the kernel $H_0(t, s)$ by $R_0(t, s)$, the solution of the linearised error equation can be written as

$$e_h(t) = \delta_h(t) + T_2(t) + \int_0^t R_0(t, s)\,(\delta_h(s) + T_2(s))\,ds, \quad t \in I.$$

Hence, the statements on the attainable order of $e_h(t)$ at $t = t_n$ $(n = 1, \dots, N)$ follow from the familiar quadrature arguments based on the degree of precision of the interpolatory quadrature formulas with the collocation points as abscissas. In other words, the superconvergence results of Theorems 2.2.5 and 2.2.6 remain valid for nonlinear VIEs.

Remark As in the case of nonlinear ODEs (Section 1.1.4) an alternative proof can be based on the nonlinear 'variation-of-constants formula' for the VIE (2.1.36) (see Beesack (1987)). Note, however, that the formula given originally in Bernfeld and Lord (1978) is not correct.

2.3.3 Hammerstein-type VIEs: implicitly linear collocation

We have seen at the end of Section 2.1.5 that the Volterra–Hammerstein integral equation

$$y(t) = g(t) + (\mathcal{H}y)(t), \quad t \in I := [0, T], \tag{2.3.2}$$

with

$$(\mathcal{H}y)(t) := \int_0^t K(t, s)G(s, y(s))ds,$$

can be rewritten in a form that leads to a computationally more attractive version of the collocation method. This form is based on the *Niemytzki operator* (or: substitution operator) $\mathcal{N}$,

$$z(t) := (\mathcal{N}y)(t) := G(t, y(t)), \quad t \in I, \tag{2.3.3}$$

which permits the recasting of (2.3.2) as an 'implicitly linear' Volterra integral equation,

$$z(t) = (\mathcal{N}(g + \mathcal{V}z)(t) = G(t, g(t) + (\mathcal{V}z)(t)), \quad t \in I, \tag{2.3.4}$$

with *linear* Volterra operator

$$(\mathcal{V}z)(t) := \int_0^t K(t,s)z(s)ds.$$

The solution to the original VIE (2.3.2) is then obtained by the *iteration*

$$y(t) = g(t) + (\mathcal{V}z)(t), \quad t \in I. \tag{2.3.5}$$

We now approximate z by the collocation solution $z_h \in S_{m-1}^{(-1)}(I_h)$,

$$z_h(t) = G(t, g(t) + (\mathcal{V}z_h)(t)), \quad t \in X_h, \tag{2.3.6}$$

and define the approximation y_h to the solution y of the original VIE by

$$y_h(t) := g(t) + (\mathcal{V}z_h)(t), \quad t \in I. \tag{2.3.7}$$

The computational form of the collocation equation (2.3.6) on σ_n uses the local representation

$$z_h(t_n + vh_n) = \sum_{j=1}^{m} L_j(v)Z_{n,j}, \quad v \in (0,1], \quad \text{with} \quad Z_{n,j} := z_h(t_{n,j}), \tag{2.3.8}$$

and is thus given by

$$Z_{n,i} = G\left(t_{n,i}, g(t_{n,i}) + F_n(t_{n,i}) + h_n \sum_{j=1}^{m} [\int_0^{c_i} K(t_{n,i}, t_n + sh_n)L_j(s)ds]Z_{n,j}\right) \tag{2.3.9}$$

($i = 1, \ldots, m$), with lag term

$$F_n(t) := \int_0^{t_n} K(t,s)z_h(s)ds, \quad t \in \sigma_n.$$

After z_h has been found we obtain the approximation y_h to the solution y of the given VIE (2.3.2) on $\bar{\sigma}_n$ by

$$\begin{aligned} y_h(t_n + vh_n) &= g(t_n + vh_n) + F_n(t_n + vh_n) \\ &\quad + h_n \sum_{j=1}^{m} \left(\int_0^{v} K(t_n + vh_n, t_n + sh_n)L_j(s)ds\right) Z_{n,j}, \end{aligned} \tag{2.3.10}$$

$v \in [0, 1]$.

In applications the kernel $K(t,s)$ is usually of convolution type, $K(t,s) = k(t-s)$, where the memory function k is often such that the integrals arising from

$$(\mathcal{V}z_h)(t) = \int_0^t k(t-s)z_h(s)ds$$

can be found analytically, thus avoiding the need for quadrature approximations. A more important advantage of implicitly linear collocation lies in the fact that, in contrast to direct collocation for (2.3.2), the integrals need not be re-computed for every iteration step when solving the nonlinear algebraic system (2.3.9), or when computing the approximation $y_h(t)$ for different values of $t \in I$ by (2.3.10).

Remark Implicitly linear collocation methods (called 'new collocation-type methods') were first introduced for Fredholm integral equations in Kumar and Sloan (1987) and Kumar (1987); see also Brunner (1991, 1992b), Frankel (1995) and Kaneko, Noren and Padilla (1997).

It turns out that the approximation y_h obtained by implicitly linear collocation and the iterated collocation solution u_h^{it} generated by 'direct' collocation are closely related and, for certain kernels, essentially identical. Thus, for judiciously chosen collocation parameters both approaches yield superconvergent approximations of the same global and local orders. This is made precise in the following theorem.

Theorem 2.3.1 *Assume that:*

(a) The given functions g, K and G in (2.3.2) are continuous on their domains, and G is such that the VHIE possesses a unique solution $y \in C(I)$.
(b) y_h is the approximation to the solution y of the Volterra–Hammerstein equation (2.3.2) obtained by implicitly linear collocation and defined by (2.3.6), (2.3.7).
(c) $\hat{u}_h^{it}$ denotes the discretised iterated collocation solution corresponding to the 'direct' collocation solution $\hat{u}_h \in S_{m-1}^{(-1)}(I_h)$ defined by the fully discretised collocation equation (2.3.11) below, using the same collocation points X_h as for the computation of $z_h \in S_{m-1}^{(-1)}(I_h)$ in (2.3.6).

Then

$$\hat{u}_h^{it}(t) = y_h(t) \quad \textit{for all} \quad t \in I.$$

Proof If we solve the given Volterra–Hammerstein equation (2.3.2) by 'direct' collocation, then the exact collocation equation is

$$u_h(t) = g(t) + (\mathcal{H}u_h)(t), \quad t \in I,$$

and the exact iterated collocation solution is found from

$$u_h^{it}(t) := g(t) + (\mathcal{H}u_h)(t), \quad t \in I.$$

For $t = t_{n,i}$ $(i = 1, \ldots, m)$ and $t = t_n + vh_n$ $(v \in [0, 1])$ these equations become, respectively,

$$U_{n,i} = g(t_{n,i}) + \Phi_n(t_{n,i}) + h_n \int_0^{c_i} K(t_{n,i}, t_n + sh_n) G(t_n + sh_n, \sum_{j=1}^{m} L_j(s) U_{n,j}) ds,$$

and

$$u_h^{it}(t_n + vh_n) = g(t_n + vh_n) + \Phi_n(t_n + vh_n) + h_n \int_0^{v} K(t_n + vh_n, t_n + sh_n) G(t_n + vh_n, \sum_{j=1}^{m} L_j(s) U_{n,j}) ds.$$

Here, we have set

$$\Phi_n(t_n + vh_n) := \int_0^{t_n} K(t_n + vh_n, s) G(s, u_h(s)) ds, \quad v \in [0, 1].$$

Consider now their fully discretised versions based on interpolatory m-point product quadrature formulas with weight function $K(\cdot, s)$ and abscissas given by the collocation points: according to Section 2.2.3 they are given by

$$\hat{U}_{n,i} = g(t_{n,i}) + \hat{\Phi}_n(t_{n,i}) + h_n \sum_{\nu=1}^{m} \left(\int_0^{c_i} K(t_{n,i}, t_n + sh_n) L_\nu(s) ds \right) G(t_{n,\nu}, \hat{U}_{n,\nu}) \quad (2.3.11)$$

and

$$\hat{u}_h^{it}(t_n + vh_n) = g(t_n + vh_n) + \hat{\Phi}_n(t_n + vh_n) + h_n \sum_{\nu=1}^{m} \left(\int_0^{v} K(t_n + vh_n, t_n + sh_n) L_\nu(s) ds \right) G(t_{n,\nu}, \hat{U}_{n,\nu}). \quad (2.3.12)$$

Let $\hat{V}_{n,i} := G(t_{n,i}, \hat{U}_{n,i})$. From (2.3.11) and (2.3.12) we thus obtain the equations

$$\hat{V}_{n,i} = G(t_{n,i}, g(t_{n,i}) + \hat{\Phi}_n(t_{n,i}) + h_n \sum_{\nu=1}^{m} \left(\int_0^{c_i} K(t_{n,i}, t_n + sh_n) L_\nu(s) ds \right) \hat{V}_{n,\nu}) \quad (2.3.13)$$

and

$$\hat{u}_h^{it}(t_n + vh_n) = g(t_n + vh_n) + \hat{\Phi}_n(t_n + vh_n) + h_n \sum_{\nu=1}^{m} \left(\int_0^{v} K(t_n + vh_n, t_n + sh_n) L_\nu(s) ds \right) \hat{V}_{n,\nu}. \quad (2.3.14)$$

We wish to show that $\hat{V}_{n,i} = Z_{n,i}$ $(i = 1, \ldots, m;\ n = 0, 1, \ldots, N-1)$, where the $Z_{n,i}$ are defined by the solution of the nonlinear algebraic system (2.3.9). This is readily verified by induction, using the obvious fact that the assertion is true for $n = 0$. We leave the remaining details of this simple proof to the reader.

In view of applications the kernel will usually be of *convolution type*, $K(t, s) = k(t - s)$, and hence the quadrature weights in (2.3.11)–(2.3.14) have an analogous structure, e.g.

$$w_{n,\nu}(v) := \int_0^v k((v - s)h_n)L_\nu(s)ds,\ v \in (0, 1].$$

For $K(t, s) \equiv 1$ we obtain the result of Theorem 1.1.9 for the special Volterra–Hammerstein integral equation arising in the integrated form of the initial-value problem $y'(t) = f(t, y(t))$, $y(0) = y_0$.

2.4 Collocation for first-kind VIEs

2.4.1 The exact collocation equations for $S_{m-1}^{(-1)}(I_h)$

Assume that the kernel K of the Volterra integral operator $\mathcal{V}:\ C(I) \to C(I)$ defined by

$$(\mathcal{V}\phi)(t) := \int_0^t K(t, s)\phi(s)ds, \quad t \in I := [0, T], \tag{2.4.1}$$

is in $C^1(D)$ and is strictly non-zero along the line $t = s$ of I; that is, $|K(t, t)| \geq k_0 > 0$ for all $t \in I$. According to Theorem 2.1.8 the first-kind Volterra integral equation

$$(\mathcal{V}y)(t) = g(t), \quad t \in I, \tag{2.4.2}$$

then possesses a unique solution $y \in C(I)$ for any $g \in C^1(I)$ with $g(0) = 0$.

Since $(\mathcal{V}y)(0) = 0$ the parameters $\{c_i\}$ underlying the set X_h of collocation points will have to satisfy

$$0 < c_1 < \ldots < c_m \leq 1,$$

both for the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ and the one in the continuous space $S_m^{(0)}(I_h)$. We will see, however, that these collocation solutions will converge uniformly on I to the solution y of (2.4.1), as $h \to 0$, only under rather stringent conditions on the $\{c_i\}$. In particular, while collocation at the Gauss points will yield a convergent collocation solution in $S_{m-1}^{(-1)}(I_h)$ (with possible order reduction), the collocation solution in $S_m^{(0)}(I_h)$ for the same collocation points turns out to be divergent.

The collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ to (2.4.1) is defined by the collocation equation

$$(\mathcal{V}u_h)(t) = g(t), \quad t \in X_h. \tag{2.4.3}$$

Its local representation is again

$$u_h(t_n + sh_n) = \sum_{j=1}^{m} L_j(v)U_{n,j}, \quad v \in (0, 1], \quad \text{with } U_{n,j} := u_h(t_{n,j}), \tag{2.4.4}$$

and hence $\mathbf{U}_n := (U_{n,1}, \ldots, U_{n,m})^T \in \mathbb{R}^m$ is defined by the solution of the linear algebraic system

$$B_n\mathbf{U}_n = h_n^{-1}[\mathbf{g}_n - \mathbf{G}_n] \quad (n = 0, 1, \ldots, N-1), \tag{2.4.5}$$

where

$$\mathbf{G}_n := (F_n(t_{n,1}), \ldots, F_n(t_{n,m}))^T = \sum_{\ell=0}^{n-1} h_\ell B_n^{(\ell)}\mathbf{U}_\ell,$$

in complete analogy to (2.2.14). The matrix $B_n \in L(\mathbb{R}^m)$ was introduced in (2.2.13) and has the form

$$B_n := \begin{pmatrix} \int_0^{c_i} K(t_{n,i}, t_n + sh_n)L_j(s)ds \\ (i, j = 1, \ldots, m) \end{pmatrix}. \tag{2.4.6}$$

Is this matrix non-singular, at least for sufficiently small values of h_n? We observe that its elements sample the kernel $K(t, s)$ only 'near' the boundary $t = s$ of D; hence, since $K \in C^1(D)$ we may employ Taylor's Theorem to write

$$\begin{aligned} K(t_{n,i}, t_n + sh_n) = K(t_n, t_n) + h_n[&c_i K_t(t_n + \theta_1 c_i h_n, t_n + \theta_2 sh_n) \\ &+ sK_s(t_n + \theta_1 c_i h_n, t_n + \theta_2 sh_n)], \end{aligned}$$

where $\theta_k \in (0, 1)$ $(k = 1, 2)$. Hence, the elements of the matrix B_n can be therefore be expressed in the form

$$\int_0^{c_i} K(t_{n,i}, t_n + sh_n)L_j(s)ds = K(t_n, t_n)a_{i,j} + \mathcal{O}(h_n), \quad i, j = 1, \ldots, m;$$

as in Section 1.1, we have set $a_{i,j} := \int_0^{c_i} L_j(s)ds$. This shows that for sufficiently small $h_n > 0$, and under the assumption that $|K(t, t)| \geq k_0 > 0, \ t \in I$, we have

$$B_n = \begin{pmatrix} \int_0^{c_i} K(t_n, t_n)L_j(s)ds \\ (i, j = 1, \ldots, m) \end{pmatrix} + \mathcal{O}(h_n),$$

where the first matrix on the right-hand side is non-singular for all $n = 0, 1, \dots, N-1$, due to the linear independence of the (local) basis functions $\{L_j\}$.

Theorem 2.4.1 *Assume that g and K in the first-kind Volterra integral equation (2.4.2) satisfy*

$$g \in C^1(I),\ g(0) = 0; \quad K \in C^1(D),\ |K(t,t)| \geq k_0 > 0,\ t \in I.$$

Then there exists an $\bar{h} > 0$ so that for all meshes I_h with diameter $h \in (0, \bar{h})$ each of the linear algebraic systems (2.4.5) possesses a unique solution $\mathbf{U}_n \in \mathbb{R}^m$. Hence, the collocation equation (2.4.3) defines a unique collocation solution $u_h \in S_{m-1}^{(0)}(I_h)$ which on σ_n is given by (2.4.4).

Remark A more general analysis of the existence and uniqueness of collocation solutions to first-kind Volterra integral equations in more general linear spaces can be found in Brunner (1976).

Example 2.4.1 $u_h \in S_0^{(-1)}(I_h)$ $(m = 1),\ 0 < c_1 \leq 1$:
The collocation equation follows immediately from Example 2.2.1 and now reads (setting again $\theta := c_1$)

$$\left(\int_0^\theta K(t_{n,1}, t_n + sh_n)ds\right) y_{n+1}$$
$$= h_n^{-1}[g(t_{n,1}) - F_n(t_{n,1})](n = 0, 1, \dots, N-1),$$

with $y_{n+1} := U_{n,1} = u_n(t_n + vh_n),\ v \in (0, 1]$, and

$$F_n(t) := \sum_{\ell=0}^{n-1} h_\ell \left(\int_0^1 K(t_{n,1}, t_\ell + sh_\ell)ds\right) y_{\ell+1} \quad (t \in \sigma_n).$$

We shall see in Theorem 2.4.2 below that this method yields a convergent collocation solution only if $\theta \in [1/2, 1]$; its (global) order is then $p = m = 1$. For $\theta = 1$ we obtain the *continuous midpoint method* (see also Example 2.4.4 for its fully discrete version): the name has its origin in the fact that, as we will see in Theorem 2.4.6, it exhibits *local superconvergence* of order $p^* = m + 1 = 2$ at the midpoints $t_{n+1/2} := t_n + h_n/2$ $(n = 0, 1, \dots, N-1)$ of the subintervals σ_n.

Example 2.4.2 $u_h \in S_1^{(-1)}(I_h)$ $(m = 2),\ 0 < c_1 < c_2 \leq 1$:
Here, the local representation of the collocation solution is

$$u_h(t_n + vh_n) = \frac{1}{c_2 - c_1}[(c_2 - v)U_{n,1} + (v - c_1)U_{n,2}], \quad v \in (0, 1].$$

The vector $\mathbf{U}_n := (U_{n,1}, U_{n,2})^T \in \mathbb{R}^2$ is the solution of the linear system

$$B_n U_n = h_n^{-1}[\mathbf{g}_n - \mathbf{G}_n]$$

(recall (2.4.5)) with the elements of the matrix $B_n \in L(\mathbb{R}^2)$ as in Example 2.2.2.

This collocation solution is convergent only if the collocation parameters are chosen so that $(1-c_1)(1-c_2) \leq c_1 c_2$ (see Theorem 2.4.2 below). The global order of convergence is $p = m = 2$ for, e.g. the points $c_1 = 1/3,\ c_2 = 1$ (the Radau II points) and $c_1 = 1/2,\ c_2 = 1$; in both cases we have $\rho_m = 0$ in (2.4.7). If c_1, c_2 are the Gauss points (for which we have $\rho_2 = +1$ in (2.4.7)) then we obtain only $\mathcal{O}(h)$-convergence.

2.4.2 Global convergence in $S_{m-1}^{(-1)}(I_h)$

We have seen in Section 2.2.4 that for *second-kind* Volterra integral equations the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ converges to the exact solution for any choice of the collocation parameters $\{c_i\}$ with $0 \leq c_1 < \ldots < c_m \leq 1$. This is no longer true for *first-kind* VIEs, as the following theorem shows.

Theorem 2.4.2 *Let $d \geq m$ and assume that*

(a) $g \in C^{d+1}(I)$, with $g(0) = 0$;
(b) $K \in C^{d+1}(D)$, with $|K(t,t)| \geq k_0 > 0$ for $t \in I$;
(c) $u_h \in S_{m-1}^{(-1)}(I_h)$ is the collocation solution defined by the collocation equation (2.4.3).

Then for all uniform meshes I_h with $h \in (0, \bar{h})$ the collocation solution u_h converges uniformly on I to the solution y of (2.4.2) if, and only if, the collocation parameters satisfy the condition

$$-1 \leq \rho_m := (-1)^m \prod_{i=1}^{m} \frac{1-c_i}{c_i} \leq 1. \tag{2.4.7}$$

The attainable global order of convergence is then described by

$$||y - u_h||_\infty = \begin{cases} \mathcal{O}(h^m) & \text{if } \rho_m \in [-1, 1), \\ \mathcal{O}(h^{m-1}) & \text{if } \rho_m = 1. \end{cases} \tag{2.4.8}$$

Remark Recall that in Section 1.1.2 (following Theorem 1.1.3) we introduced the *collocation polynomial* $M_m(s) := (1/m!) \prod_{i=1}^m (s - c_i)$ associated with the given collocation parameters $\{c_i\}$. Thus, the above theorem can be rephrased, to say that the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ is convergent of (global) order $p = m$ if, and only if,

$$-1 \leq \rho_m = \frac{M_m(1)}{M_m(0)} < 1. \tag{2.4.9}$$

Proof The collocation error $e_h := y - u_h$ is governed by the equations

$$(\mathcal{V} e_h)(t_{n,i}) = 0, \quad i = 1, \ldots, m \quad (0 \leq n \leq N-1).$$

Assume first that $n = 0$: using the local representation (2.2.31),

$$e_h(t_n + vh_n) = \sum_{j=1}^{m} L_j(v)\mathcal{E}_{n,j} + h_n^m R_{m,n}(v), \quad v \in (0, 1],$$

we obtain the linear algebraic system

$$\sum_{j=1}^{m} \left(\int_0^{c_i} K(t_{0,i}, t_0 + sh_0)L_j(s)ds \right) \mathcal{E}_{0,j} = \rho_{0,i} h_0^m \quad (i = 1, \dots, m),$$

where

$$\rho_{0,i} := -\int_0^{c_i} K(t_{0,i}, t_0 + sh_0)R_{m,0}(s)ds.$$

According to Theorem 2.4.1 this system possesses a unique solution since the left-hand side matrix B_0 is invertible whenever $h_0 \in (0, \bar{h})$. Hence we obtain the initial estimate

$$||\boldsymbol{\mathcal{E}}_0||_1 \le ||B_0^{-1}||_1 \cdot ||\boldsymbol{\rho}_0||_1 h_0^m,$$

implying that

$$|e_h(t_0 + vh_0)| \le \Lambda_m ||\boldsymbol{\mathcal{E}}_0||_1 + h_0^m m K_0 k_m M_m =: C_0 h_0^m, \quad v \in [0, 1]$$

(recall the proof of Theorem 2.2.3 and the notation employed there).

Assume now that $1 \le n \le N - 1$. In order to analyse the convergence of u_h we resort to the discrete analogue of differentiating the (continuous) error equation, namely,

$$\frac{1}{h_n}[\mathcal{V}e_h)(t_{n,i}) - (\mathcal{V}e_h)(t_{n-1,m})] = 0, \quad i = 1, \dots, m.$$

This can be written in the more explicit form

$$\begin{aligned}
&h_n \int_0^{c_i} K(t_{n,i}, t_n + sh_n)e_h(t_n + sh_n)ds \\
&= h_{n-1} \int_0^{c_m} K(t_{n-1,m}, t_{n-1} + sh_{n-1})e_h(t_{n-1} + sh_{n-1})ds \\
&\quad - h_{n-1} \int_0^{1} K(t_{n,i}, t_{n-1} + sh_{n-1})e_h(t_{n-1} + sh_{n-1})ds \\
&\quad - \sum_{\ell=0}^{n-2} h_\ell \int_0^{1} [K(t_{n,i}, t_\ell + sh_\ell) - K(t_{n-1,m}, t_\ell + sh_\ell)]e_h(t_\ell + sh_\ell)ds
\end{aligned}$$

$(i = 1, \dots, m)$. Due the assumed regularity of the kernel $K(t, s)$ we have

$$\begin{aligned}
&K(t_n + c_i h_n, t_\ell + sh_\ell) - K(t_{n-1} + c_m h_{n-1}, t_\ell + sh_\ell) \\
&\quad = c_i h_n K_t(\cdot, t_\ell + sh_\ell) + (1 - c_m)h_{n-1} K_t(\cdot, t_\ell + sh_\ell) + \mathcal{O}(h),
\end{aligned}$$

where the unspecified first arguments in the partial derivatives of K are those arising in the Taylor's remainder terms. Thus, we can write

$$\int_0^{c_i} K(t_{n,i}, t_n + sh_n)e_h(t_n + sh_n)ds$$
$$= -\frac{h_{n-1}}{h_n}\left(\int_0^1 K(t_{n,i}, t_{n-1} + sh_{n-1})e_h(t_{n-1} + sh_{n-1})ds\right.$$
$$\left. - \int_0^{c_m} K(t_{n-1,m}, t_{n-1} + sh_{n-1})e_h(t_{n-1} + sh_{n-1})ds\right)$$
$$- \sum_{\ell=0}^{n-2} \frac{h_\ell}{h_n} \int_0^1 [c_i h_n K_t(\cdot, t_\ell + sh_\ell) + (1 - c_m)h_{n-1}K_t(\cdot, t_\ell + sh_\ell)]e_h$$
$$(t_\ell + sh_\ell)ds. \tag{2.4.10}$$

For $K(t, s) \equiv 1$ on D (which we will assume later, without loss of generality, when discussing the case $c_m < 1$) this becomes

$$\int_0^{c_i} e_h(t_n + sh_n)ds = -\frac{h_{n-1}}{h_n}\int_{c_m}^1 e_h(t_{n-1} + sh_{n-1})ds \quad (i = 1, \ldots, m). \tag{2.4.11}$$

<u>Case I:</u> $c_m = 1$.
The error equation (2.4.10) reduces to

$$\int_0^{c_i} K(t_{n,i}, t_n + sh_n)e_h(t_n + sh_n)ds$$
$$= - c_i \sum_{\ell=1}^{n-1} h_\ell \int_0^1 K_t(\cdot, t_\ell + sh_\ell)e_h(t_\ell + sh_\ell)ds$$
$$- c_i h_0 \int_0^1 K_t(\cdot, t_0 + sh_0)e_h(t_0 + sh_0)ds$$

$(i = 1, \ldots, m)$. We have shown at the beginning of the proof that $e_h(t_0 + vh_0) = \mathcal{O}(h^m)$ for $v \in [0, 1]$. Thus, substitution of the local representations of e_h leads to the discrete Gronwall inequality

$$||\mathcal{E}_n||_1 \leq \gamma_0 h \sum_{\ell=1}^{n-1} ||\mathcal{E}_\ell||_1 + \gamma_1 h^m \quad (n = 1, \ldots, N - 1),$$

and hence to $||e_h||_\infty = \mathcal{O}(h^m)$, for any choice of the first $m - 1$ collocation parameters, $0 < c_1 < \ldots < c_{m-1} < 1$.

<u>Case II:</u> $c_m < 1$.
For ease of notation we will assume that $K(t, s) \equiv 1$ on D. The error equation

(2.4.11) then becomes, employing again the local representations of e_h,

$$\int_0^{c_i}\left(\sum_{j=1}^{m} L_j(s)\mathcal{E}_{n,j} + h_n^m R_{m,n}(s)\right) ds$$
$$= -\frac{h_{n-1}}{h_n}\int_{c_m}^{1}\left(\sum_{j=1}^{m} L_j(s)\mathcal{E}_{n-1,j} + h_{n-1}^m R_{m,n-1}(s)\right) ds.$$

If we introduce the matrices P and Q in $L(\mathbb{R}^m)$ by setting

$$P := \begin{pmatrix} \int_0^{c_i} L_j(s)ds \\ (i, j = 1, \ldots, m) \end{pmatrix} \quad \text{and} \quad Q := -\begin{pmatrix} \int_{c_m}^{1} L_j(s)ds \\ (i, j = 1, \ldots, m) \end{pmatrix},$$

where P is non-singular and Q has rank one, and define vectors $\mathbf{r}_n$ and $\boldsymbol{\rho}_{n-1}$ in $\mathbb{R}^m$ with components given by

$$r_{n,i} := -\int_0^{c_i} R_{m,n}(s)ds \quad \text{and} \quad \rho_{n-1,i} := -\int_{c_m}^{1} R_{m,n-1}(s)ds,$$

respectively, we arrive at the system of difference equations

$$P\boldsymbol{\mathcal{E}}_n = -\frac{h_{n-1}}{h_n}Q\boldsymbol{\mathcal{E}}_{n-1} + h_n^m\mathbf{r}_n + \frac{h_{n-1}}{h_n}h_{n-1}^m\boldsymbol{\rho}_{n-1} \quad (n = 1, \ldots, N-1). \tag{2.4.12}$$

Recall from Section 2.2.1 that for a sequence of quasi-uniform meshes I_h the ratio of any two stepsizes h_ℓ and h_n is bounded by some constant $\gamma < \infty$, uniformly in N.

Lemma 2.4.3 *Assume that* $0 < c_1 < \ldots < c_m < 1$. *Then the non-trivial eigenvalue* λ_1 *of the rank one matrix* $P^{-1}Q$ *is*

$$\lambda_1 = (-1)^m \prod_{i=1}^{m} \frac{1-c_i}{c_i}. \tag{2.4.13}$$

Proof Since the local bases $\{L_j(v) : j = 1, \ldots, m\}$ and $\{v^{j-1} : j = 1, \ldots, m\}$ of π_{m-1} are related via a linear transformation (the paper by de Boor (2001) illuminates many aspects related to such basis transformations), the matrix $P^{-1}Q$ is similar to

$$\tilde{P}^{-1}\tilde{Q} := -\begin{pmatrix} \int_0^{c_i} s^{j-1}ds \\ (i, j = 1, \ldots, m) \end{pmatrix}^{-1} \begin{pmatrix} \int_{c_m}^{1} s^{j-1}ds \\ (i, j = 1, \ldots, m) \end{pmatrix}.$$

The non-trivial eigenvalue of $\tilde{P}^{-1}\tilde{Q}$ is easily computed; it is given by (2.4.13). Compare also Brunner (1978) for details.

If the meshes I_h are *uniform* ($\gamma = 1$), it follows from the elementary theory of difference equations (see, e.g. Elaydi (1999)) that the solutions of the system of first-order difference equations (2.4.12) remain uniformly bounded (as $N \to \infty$) if, and only if, $|\lambda_1| = \rho_m \leq 1$. This completes the first part of the proof of Theorem 2.4.2.

Suppose now that $c_m < 1$. If $\rho_m \in [-1, 1)$ then the proof that this yields again $\mathcal{O}(h^m)$-convergence is almost identical to the one for $c_m = 1$, as (2.4.12) and the corresponding discrete Gronwall estimate show. If $\rho_m = 1$ then the reason for the resulting *order reduction* is found in the following lemma (see, e.g. Henrici (1962, p. 18)).

Lemma 2.4.4 *Assume that* $\{z_n\}$ $(0 \leq n \leq N)$ *is a sequence of non-negative numbers satisfying the inequality*

$$z_{n+1} \leq A z_n + B, \quad A,\ B \geq 0.$$

Then

$$z_n \leq A^n z_0 + \begin{cases} \dfrac{A^n - 1}{A - 1} & \text{if} \quad A \neq 1, \\ nB & \text{if} \quad A = 1. \end{cases}$$

Our situation corresponds to the case $A = 1$, where A assumes the role of ρ_m and where we use the fact that the spectral radius (which for the rank-one matrix $P^{-1}Q$ now equals one) is a lower bound for any matrix norm induced by a vector norm. Hence, the term nB corresponds to const $\cdot\, nh^m$ and is bounded by const $\cdot\, Nh \cdot h^{m-1} = CT \cdot h^{m-1}$.

Remark If the mesh sequence $\{I_h\}$ is *quasi-uniform*, then we have $h_\ell / h_n \leq \gamma$ for $0 \leq \ell < n \leq N - 1$. Hence, the solutions of the system of difference equations (2.4.12) (whose coefficients are now variable) remain uniformly bounded if

$$\gamma |\lambda_1| = \gamma \prod_{i=1}^{m} \frac{1 - c_i}{c_i} \leq 1.$$

Illustration *Numerical differentiation by collocation*

If the kernel of $\mathcal{V}$ in (2.4.3) is given by $K(t, s) \equiv 1$ on D, then the solution of the integral equation (4.4.2) is $y(t) = g'(t),\ t \in I$, for any $g \in C^1(I)$ with $g(0) = 0$. Hence, Examples 2.4.1–2.4.3 yield simple (global) numerical differentiation formulas of orders one and two. Since the kernel $K(t, s)$ has now the value one on D the integrals occurring in the collocation equations can be evaluated analytically, leading to the desired differentiation formulas on, e.g. the subinterval $\sigma_0 := (0, h]$ with $h = 1$. We leave their explicit derivation as an exercise (see also Brunner and van der Houwen (1986, Section 5.5.2)).

2.4.3 Collocation and global convergence in $S_m^{(0)}(I_h)$

Recall from Section 2.2.1 that dim $S_m^{(0)}(I_h) = Nm + 1$. Thus, in order to compute the collocation solution u_h in this *continuous* collocation space from the collocation equation (2.4.3) we need to prescribe an initial value; it can be obtained from the differentiated form of (2.4.2) and is given by

$$u_h(0) = y(0) = \frac{g'(0)}{K(0,0)} =: y_0. \tag{2.4.14}$$

The analogue of the collocation equation (2.4.3) is: find $u_h \in S_m^{(0)}(I_h)$ so that

$$(\mathcal{V}u_h)(t) = g(t) \quad \text{for all} \quad t \in X_h, \quad \text{with} \quad u_h(0) = y_0. \tag{2.4.15}$$

Let the local representation of u_h on σ_n be

$$u_h(t_n + vh_n) = \sum_{j=0}^{m} L_j(v)U_{n,j}, \quad v \in [0,1], \quad \text{with} \ \ U_{n,j} := u_h(t_n + c_j h_n), \tag{2.4.16}$$

where we have set $c_0 := 0$ and

$$L_0(v) := (-1)^m \prod_{k=1}^{m} \frac{v - c_k}{c_k}, \quad L_j(v) := \frac{v}{c_j} \prod_{k=0,k\neq j}^{m} \frac{v - c_k}{c_j - c_k} \ (j = 1, \ldots, m).$$

Hence, (2.4.16) can be written in the form

$$u_h(t_n + vh_n) = L_0(v)y_n + \sum_{j=1}^{m} L_j(v)U_{n,j}, \quad v \in [0,1]. \tag{2.4.17}$$

Here, $y_n := u_h(t_n) = u_h(t_{n-1} + h_{n-1})$ $(n = 1, \ldots, N-1)$, since the collocation solution u_h is continuous at the mesh points. Note also that $c_m = 1$ implies $U_{n,m} = u_h(t_n + h_n) = y_{n+1}$.

The collocation equation on σ_n now becomes

$$\int_0^{t_n} K(t_{n,i}, s)u_h(s)ds + h_n \int_0^{c_i} K(t_{n,i}, t_n + sh_n)u_h(t_n + sh_n)ds$$
$$= g(t_{n,i}) \quad (i = 1, \ldots, m),$$

or, by employing (2.4.17),

$$\sum_{j=1}^{m} \left(\int_0^{c_i} K(t_{n,i}, t_n + sh_n)L_j(s)ds \right) U_{n,j}$$
$$= h_n^{-1}[g(t_{n,i}) - F_n(t_{n,i})] - \left(\int_0^{c_i} K(t_{n,i}, t_n + sh_n)L_0(s)ds \right) y_n. \tag{2.4.18}$$

Setting

$$\rho_n := -\left(\int_0^{c_i} K(t_{n,i}, t_n + sh_n)L_0(s)ds \quad (i = 1, \dots, m)\right)^T$$

we are led to a linear algebraic system for $\mathbf{U}_n := (U_{n,1}, \dots, U_{n,m})^T$, namely

$$B_n\mathbf{U}_n = h_n^{-1}[\mathbf{g}_n - \mathbf{G}_n] + \rho_n y_n \quad (n = 0, 1, \dots, N-1). \tag{2.4.19}$$

The matrix B_n and the vectors $\mathbf{g}_n$ and $\mathbf{G}_n$ are as in (2.4.5).

Observe that the existence of a unique collocation solution $u_h \in S_m^{(0)}(I_h)$ is assured by Theorem 2.4.1 because in the systems of linear algebraic equations (2.4.19) we have the same coefficient matrices B_n as in (2.4.5).

Example 2.4.3 $u_h \in S_1^{(0)}(I_h),\ 0 < c_1 =: \theta \le 1$:
Here, we have

$$L_0(v) = (\theta - v)/\theta, \quad L_1(v) = v/\theta,$$

and

$$B_n = \left(\frac{1}{\theta}\int_0^{\theta} K(t_{n,1}, t_n + sh_n)s\,ds\right).$$

The resulting collocation method is thus described by

$$u_h(t_n + vh_n) = L_0(v)y_n + L_1(v)U_{n,1}, \quad v \in [0, 1],$$

with $U_{n,1}$ determined by the solution of

$$\begin{aligned} B_nU_{n,1} &= h_n^{-1}[g_{n,1} - F_n(t_{n,1})] \\ &\quad -\frac{1}{\theta}\left(\int_0^{\theta} K(t_{n,1}, t_n + sh_n)(\theta - s)ds\right) y_n \quad (n = 0, 1, \dots, N-1) \end{aligned}$$

where y_0 is given by (2.4.14).

For $\theta = 1$ we obtain the *exact continuous trapezoidal method:* here, $L_0(v) = 1 - v$ and $L_1(v) = v$, and hence

$$u_h(t_n + vh_n) = (1 - v)y_n + vy_{n+1}, \quad v \in [0, 1],$$

with

$$B_n = \int_0^1 K(t_{n+1}, t_n + sh_n)s\,ds.$$

It follows from Theorem 2.4.5 below that this method is convergent and its global order of convergence is $p = m + 1 = 2$ (since $c_1 = 1$).

The fully discretised (continuous) counterpart of this method will be derived in Example 2.4.5.

We have seen in Section 2.4.1 that collocation in the discontinuous collocation space $S_{m-1}^{(-1)}(I_h)$ does not yield a uniformly convergent collocation solution for any choice of the $\{c_i\}$. Thus, it is intuitively clear that for $u_h \in S_m^{(0)}(I_h)$ to be uniformly convergent on I the collocation parameters will have to obey an even more restrictive condition than (2.4.7), due to the continuity constraints now imposed on u_h at the interior mesh points. In fact, as Theorem 2.4.6 below shows, collocation at the *Gauss points* leads to a *divergent* u_h.

We first consider the case where $c_m = 1$. We know from Theorem 2.4.2 that if the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ is based on collocation parameters satisfying this condition, it will be uniformly convergent on I for any distinct $c_1, \ldots, c_{m-1}$ in $(0, 1)$. This is no longer true in the collocation space $S_m^{(0)}(I_h)$, as the following theorem makes clear (Brunner and van der Houwen (1986), Kauthen and Brunner (1997)).

Theorem 2.4.5 *Assume:*

(a) $g \in C^{m+3}(I)$, *with* $g(0) = 0$*;*
(b) $K \in C^{m+3}(D)$, *and* $|K(t,t)| \geq k_0 > 0$ *on* I*;*
(c) $u_h \in S_m^{(0)}(I_h)$ *is the collocation solution defined by (2.4.15) and (2.4.17), with uniform mesh* I_h.

If the collocation parameters $\{c_i\}$ *are so that* $0 < c_1 < \ldots < c_m = 1$, *then* u_h *converges uniformly (on* I*) to the solution* y *of (2.4.2) if, and only if,*

$$-1 \leq \rho_{m-1} := (-1)^m \prod_{i=1}^{m-1} \frac{1-c_i}{c_i} \leq 1. \tag{2.4.20}$$

For such collocation parameters the global order of convergence, as $h \to 0$ $(Nh = T)$, *is given by*

$$||y - u_h||_\infty = \begin{cases} \mathcal{O}(h^{m+1}) & \text{if } -1 \leq \rho_{m-1} < 1, \\ \mathcal{O}(h^m) & \text{if } \rho_{m-1} = 1. \end{cases} \tag{2.4.21}$$

In the case where $c_m < 1$ (for example if the $\{c_i\}$ are the *Gauss points*) the convergence analysis is much more complex. The following theorem gives a first indication of this fact.

Theorem 2.4.6 *Let* g *and* K *be subject to the assumptions in Theorem 2.4.5. If* $c_m < 1$ *and if the collocation parameters are symmetrical,*

$$c_i = c_{m+1-i}, \quad i = 1, \ldots, m,$$

then the collocation solution $u_h \in S_m^{(0)}(I_h)$ *to (2.4.2) does not converge uniformly to* y. *In particular, the collocation solution* $u_h \in S_1^{(0)}(I_h)$ *corresponding to* $c_1 = 1/2$ *and uniform* I_h *is divergent as* $h \to 0$.

Proof See Kauthen and Brunner (1997). The techniques employed in the proof are based on a connection between the collocation solution defined by equation (2.4.3) and the value of the stability function at infinity associated with (continuous) Runge–Kutta methods for (stiff) ODEs.

We note that there exist *non-symmetric* sets $\{c_i\}$ for which the collocation solution $u_h \in S_m^{(0)}(I_h)$ to (2.4.2) does converge uniformly on I. A detailed discussion of the construction and the numerical performance of such methods can be found in Kauthen and Brunner (1997, pp. 1449–1452).

2.4.4 Is local superconvergence on I_h possible?

Do there exist sets of collocation parameters $\{c_i\}$ in (0, 1] for which the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ exhibits *local superconvergence* of order $p^* > m$ at the mesh points? The next theorem shows that the answer is negative. However, local superconvergence of (optimal) order $p^* = m + 1$ can occur at certain non-mesh points.

Theorem 2.4.7 *Assume:*

(a) $g \in C^d(I)$, *with* $d \geq m + 2$ *and non-trivial* g *satisfying* $g(0) = 0$*;*
(b) $K \in C^d(D)$, *with* $d \geq m + 2$ *and* $|K(t, t)| \geq k_0 > 0,\ t \in I$*;*
(c) $u_h \in S_{m-1}^{(-1)}(I_h)$ *is the collocation solution defined by (2.4.5), with distinct collocation parameters* $c_i \in (0, 1]$.

Then:

(i) Local superconvergence of order $p^* > m$ *on* $I_h \setminus \{0\}$ *is not possible for* u_h.
(ii) If X_h *is based on collocation parameters* $\{c_i\}$ *given by the zeros of* $(s - 1)P_m'(2s - 1)$ *(the* $m + 1$ *Lobatto points minus the point* 0*), and if the set*

$$Y_h := \{t_n + d_i h_n :\ 0 < d_1 < \ldots < d_m < 1\ (0 \leq n \leq N - 1)\}$$

corresponds to the Gauss points $\{d_i\}$ *(the zeros of* $P_m(2s - 1)$*), then*

$$\max_{t \in Y_h} |y(t) - u_h(t)| \leq C h^{m+1} :$$

local superconvergence of order $p^* = m + 1$ *occurs at the Gauss (–Legendre) points in each subinterval* σ_n*. In particular, for odd values of* m *we have*

$$\max_{1 \leq n < N} |y(t_{n+\frac{1}{2}}) - u_h(t_{n+\frac{1}{2}})| \leq C h^{m+1}.$$

Remark Note that the set Y_h of points at which local superconvergence can occur is not unique (Exercise 2.5.19). Compare also Brunner (1978, 1979a, 1979b) and Eggermont (1982, 1983, 1986) for additional details.

Proof For $\{d_i\}$ with $0 < d_1 < \ldots < d_m \le 1$ we define

$$\bar{L}_j(v) := \prod_{k \neq j} \frac{v - d_k}{d_j - d_k} \quad (j = 1, \ldots, m),$$

and we set

$$u_h(t_n + vh_n) = \sum_{j=1}^{m} \bar{L}_j(v)\bar{U}_{n,j}, \quad v \in (0, 1], \quad \text{with} \quad \bar{U}_{n,j} := u_h(t_n + d_j h_n). \tag{2.4.22}$$

Using assumptions (a) and (b) we write the collocation error (remainder term) in the form

$$e_h(t_n + vh_n) = \sum_{j=1}^{m} \bar{L}_j(v)\bar{\mathcal{E}}_{n,j} + h_n^m \bar{M}_m(v)[y^{(m)}(t_n) + h_n\theta_n y^{(m+1)}(t_n + \eta_n v h_n)], \tag{2.4.23}$$

$v \in (0, 1]$, where $\bar{\mathcal{E}}_{n,j} := e_h(t_n + d_j h_n)$, $\theta_n, \eta_n \in (0, 1)$, and

$$\bar{M}_m(v) := \frac{1}{m!} \prod_{i=1}^{m} (v - d_i).$$

We wish to show that for certain choices of the collocation parameters $\{c_i\}$ there exist sets $\{d_i\}$ (with a prominent example specified in Theorem 2.4.7) so that $||\bar{\mathcal{E}}_n||_1 = \mathcal{O}(h^{m+1})$, and that under the constraint $d_m = 1$ we only obtain $||\bar{\mathcal{E}}_n||_1 = \mathcal{O}(h^m)$.

We know that the collocation error satisfies

$$(\mathcal{V}e_h)(t_{n,i}) = 0, \quad i = 1, \ldots, m \quad (0 \le n \le N - 1).$$

For $n = 0$ these equations reduce to

$$\int_0^{c_i} K(t_{0,i}, t_0 + sh_0)e_h(t_0 + sh_0)ds = 0 \quad (i = 1, \ldots, m),$$

and by (2.4.23) we obtain

$$\sum_{j=1}^{m} \left(\int_0^{c_i} K(t_{0,i}, t_0 + sh_0)\bar{L}_j(s)ds \right) \bar{\mathcal{E}}_{0,j} = h_0^m r_{0,1} + \mathcal{O}(h_0^{m+1}) \quad (i = 1, \ldots, m). \tag{2.4.24}$$

Here,

$$r_{0,i} := -\int_0^{c_i} K(t_{0,i}, t_0 + sh_0)\bar{M}_m(s)ds \cdot y^{(m)}(t_n).$$

Since K is continuously differentiable on D we may write

$$K(t_{n,i}, t_n + sh_n) = K(t_n, t_n) + c_i h_n K_t(\cdot, t_n + sh_n) = K(t_n, t_n) + \mathcal{O}(h_n)$$

(recall also Section 2.4.2), where by assumption, $K(t,t) \neq 0$ for $t \in I$. Hence, the linear algebraic system (2.4.24) possesses a unqiue solution $\bar{\mathcal{E}}_0$ whenever $h_0 > 0$ is sufficiently small (cf. (2.4.6) and Theorem 2.4.1). Moreover, since

$$r_{0,i} = -K(t_0,t_0)y^{(m)}(t_0) \cdot \int_0^{c_i} \bar{M}_m(s)ds + \mathcal{O}(h_0),$$

this solution satifies $||\bar{\mathcal{E}}_n||_1 = \mathcal{O}(h_0^{m+1})$ if, and only if,

$$\int_0^{c_i} \bar{M}_m(s)ds = 0, \quad i = 1, \ldots, m, \tag{2.4.25}$$

is true. It is easily seen that (2.4.25) can only hold if the sets $\{c_i\}$ (defining the collocation points) and $\{d_i\}$ (describing the 'evaluation points') *interlace*, that is, if

$$0 < d_1 < c_1 < d_2 < \ldots < d_m < c_m \leq 1.$$

Direct computation, using an elementary property of the Legendre polynomials, shows that

$$Q_{m+1}(v) := \int_0^v P_m(2s-1)ds = \text{const} \cdot [P_{m+1}(2v-1) - P_{m-1}(2v-1)]$$

(see also Ghizzetti and Ossicini (1970, pp. 62–63)). The zeros of $Q_{m+1}(v)$ are the $m+1$ *Lobatto points* in [0, 1] (including 0 and 1). If we denote these points by

$$0 =: c_0 < c_1 < \ldots < c_{m-1} < c_m = 1,$$

(2.4.25) is satisfied if we choose

$$\bar{M}_m(v) = a_m P_m(2v-1) = \frac{1}{m!}\prod_{i=1}^{m}(v-d_i),$$

with $a_m := 1/(1 \cdot 3 \cdots (2m-1)2^m)$. This also shows, by the above interlacing property, that $\mathcal{O}(h^{m+1})$-convergence at $t = t_1$ (corresponding to $d_m = 1$) is not possible.

We leave the extension of the above analysis to the subintervals σ_n with $1 \leq n \leq N-1$ to the reader. It uses as its starting point the equation (2.4.10), and (for $c_m = 1$) it leads to

$$\sum_{j=1}^{m}\left(\int_0^{c_i} K(t_{n,i}, t_n + sh_n)\bar{L}_j(s)ds\right)\bar{\mathcal{E}}_{n,j} = h_n^m r_{n,i} + \mathcal{O}(h_n^{m+1}), \quad i = 1, \ldots, m).$$

An analogous (negative) result holds for the collocation solution $u_h \in S_m^{(0)}(I_h)$. We we will not state it here but refer the interested reader to the 1997 paper by Kauthen and Brunner.

2.4.5 Fully discretised collocation for first-kind VIEs

Since we have discussed the fully discretised collocation method for second-kind VIEs in some detail in Section 2.2.3, we can be brief in dealing with the analogous discretisations for VIEs of the first kind. As before, the integrals in the collocation equations (2.4.5) and (2.4.19) are approximated by the interpolatory m-point quadrature formulas corresponding to the local abscissas $t_n + c_i h_n$. The resulting approximations then coincide with those obtained, in the early 1970s (Weiss (1972a), de Hoog and Weiss (1973a, 1973b), by 'block-by-block' methods (see also Brunner (1977, 1978) for the connection with discretised collocation methods). It is intuitively clear from the perturbation analysis of Section 2.2.3 that the discretised collocation solution $\hat{u}_h$ will also converge, provided the collocation parameters satisfy the stability condition (2.4.7) when $\hat{u}_h \in S_{m-1}^{(-1)}(I_h)$, and (2.4.20) for $\hat{u}_h \in S_m^{(0)}(I_h)$ with $0 < c_1 < \ldots < c_m = 1$.

Theorem 2.4.8 *Assume that g and K satisfy the assumptions stated in Theorem 2.4.4, and let $\hat{u}_h^{(-1)}(I_h)$ be the discretised collocation solution for (2.4.3) that is based on the interpolatory m-point quadrature formulas introduced in Section 2.2.3. Then the order results of Theorem 2.4.4 remain valid for $\hat{u}_h$.*

An analogous theorem – the discrete counterpart of Theorem 2.4.6 – can be stated for $\hat{u}_h \in S_m^{(0)}(I_h)$.

We conclude this discussion with three examples which play a role in many applications (and which, in Linz's work of the late 1960s, introduced – and are still introducing – many numerical analysts and users of computational mathematics to the numerical solution of first-kind VIEs). Their proper place within the general framework of (discretised) collocation, as described above, does, however, shed more light on their convergence analysis and their numerical implementation.

Example 2.4.4 $\hat{u}_h \in S_0^{(-1)}(I_h)$ $(m = 1)$, $0 < c_1 =: \theta \leq 1$:
Consider the fully discretised version of Example 2.4.1 corresponding to $\theta = 1$: using again interpolatory one-point collocation based on the abscissas $t_n + \theta h_n$, the discretised collocation solution in this space is determined by $\hat{y}_{n+1} := u_h(t_n + v h_n)$ $(v \in (0, 1])$ and

$$K(t_{n+1}, t_{n+1})\hat{y}_{n+1} = h_n^{-1}[g(t_{n+1}) - \hat{F}_n(t_{n+1})],$$

where

$$\hat{F}_n(t_{n+1}) = \sum_{\ell=0}^{n-1} h_\ell K(t_{n+1}, t_{\ell+1})\hat{y}_{\ell+1}.$$

Its global order of convergence is $p = m = 1$; on the set

$$Y_h := \{t_{n+1/2} := t_n + h_n/2 : \ n = 0, 1, \ldots, N-1\}$$

we have local superconvergence of order $p^* = m + 1 = 2$,

$$\max_{t \in Y_h} |y(t) - u_h(t)| \le C h^2$$

(see Theorem 2.4.6), a result well known in the classical literature on the numerical analysis of first-kind VIEs (see, e.g. Linz (1969b) for a discussion of the (discrete) midpoint method for first-kind VIEs).

Example 2.4.5 $\hat{u}_h \in S_1^{(0)}(I_h)$ $(m = 1)$, $0 < c_1 =: \theta \le 1$:
It follows from Example 2.4.3 that the discretised collocation solution in the space of continuous piecewise linear polynomials is, on $\bar{\sigma}_n$,

$$\hat{u}_h(t_n + v h_n) = \frac{1}{\theta}[(\theta - v)\hat{y}_n + v U_{n,1}], \quad v \in [0, 1],$$

with $\hat{U}_{n,1}$ being the solution of

$$\frac{1}{2} K(t_{n,1}, t_{n,1}) \hat{U}_{n,1} = h_n^{-1}[g(t_{n,1}) - \hat{F}_n(t_{n,1})] - \frac{2\theta - 1}{2} K(t_{n,1}, t_n)\hat{y}_n.$$

The corresponding discretised lag term is given by

$$\hat{F}_n(t_{n,1}) = \frac{1}{2} \sum_{\ell=0}^{n-1} h_\ell [K(t_{n,1}, t_\ell)\hat{y}_\ell + K(t_{n,1}, t_{\ell+1})\hat{y}_{\ell+1}].$$

For $\theta = 1$ this fully discretised method becomes the *discretised* (continuous) *trapezoidal method*, described by

$$\hat{u}_h(t_n + v h_n) = (1 - v)\hat{y}_n + v\hat{y}_{n+1}, \quad v \in [0, 1],$$

and

$$\frac{1}{2} K(t_{n+1}, t_{n+1})\hat{y}_{n+1} = h_n^{-1}[g(t_{n+1}) - \hat{F}_n(t_{n+1})] - \frac{1}{2} K(t_{n+1}, t_n)\hat{y}_n.$$

Its order (globally, on I, and locally, on X_h) is $p = m + 1 = 2$.

2.4.6 Direct versus indirect collocation

We have seen in the previous sections that the collocation solution u_h in $S_{m+d}^{(d)}(I_h)$ $(d \in \{-1, 0\})$ is in general not locally superconvergent at the points of the mesh I_h. Thus, if a given first-kind Volterra integral equation can be converted into an equation of the second kind, it will be advantageous to use the latter as the basis for obtaining high-order collocation solutions to (2.4.2) since the superconvergence results of Section 2.2.3 and 2.2.4 will now apply.

It follows under the assumptions of Theorem 2.1.8 that, upon differentiation with respect to t, the first-kind VIE (2.4.2) can be rewritten as

$$y(t) = g_1(t) + \int_0^t K_1(t,s)y(s)ds, \quad t \in I, \tag{2.4.26}$$

with

$$g_1(t) := g'(t)/K(t,t), \quad K_1(t,s) := -(\partial K(t,s)/\partial t)/K(t,t)$$

(cf. (2.1.32)), with $|K(t,t)| \geq k_0 > 0$ for $t \in I$. If g_1 and K_1 are sufficiently regular, then collocation in $S_{m-1}^{(-1)}(I_h)$ yields local superconvergence of order $p^* = 2m - 1$ at the mesh points t_n $(n = 1, \ldots, N)$ when the Radau II points are chosen as the collocation parameters (Corollary 2.2.8). Note that the amount of linear algebra required for solving the linear algebraic systems (2.4.5) and (2.2.14) is the same, and the matrices $\mathcal{I}_m - h_n B_n$ in (2.2.14) are in general better conditioned than B_n in (2.4.5).

If collocation for (2.4.26) is at the Gauss points, then the *iterated collocation solution* for u_h yields local superconvergence of order $p^* = 2m$ at these mesh points with little additional computational cost. We observe again that the amount of linear algebra remains the same in both approaches.

The indirect collocation approach will be particularly advantageous when solving *nonlinear* first-kind VIEs, as we shall see in Section 2.4.8.

2.4.7 Adjoint first-kind Volterra integral equations

We will refer to the VIE

$$(\mathcal{V}^* y)(t) := \int_t^T K(s,t)y(s)ds = g(t), \quad t \in I := [0,T], \tag{2.4.27}$$

as the *adjoint* equation of (2.4.2).

Let $0 \leq c_1 < c_2 < \ldots < c_m < 1$ be the collocation parameters underlying the collocation points X_h and the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ for (2.4.27). The collocation equation now reads

$$(\mathcal{V}^* u_h)(t_{n,i}) = g(t_{n,i}) \quad i = 1, \ldots, m \quad (n = N-1, N-2, \ldots, 0).$$

Theorem 2.4.9 *Assume:*

(a) $g \in C^{m+1}(I)$, *with* $g(T) = 0$;
(b) $K \in C^{m+1}(D)$, *with* $|K(t,t)| \geq k_0 > 0$ *for* $t \in I$;
(c) $u_h \in S_{m-1}^{(-1)}(I_h)$ *is the collocation solution to the adjoint VIE (2.4.27) with respect to a uniform mesh* I_h.

Then u_h converges uniformly to the exact solution y on I,

$$\lim_{N \to \infty} ||y - u_h||_\infty = 0,$$

if, and only if, the collocation parameters satisfy the condition

$$-1 \leq \rho_m^* := (-1)^m \prod_{i=1}^{m} \frac{c_i}{1 - c_i} \leq 1. \tag{2.4.28}$$

We leave the proof (and the statement, analogous to (2.4.8) in Theorem 2.4.2, on the attainable order of convergence) as an exercise.

Remarks

1. If the collocation parameters are such that $c_1 = 0$, then we have $\rho_m^* = 0$, and the collocation solution is convergent for any choice of the remaining c_i in $(0, 1)$.
2. The stability condition (2.4.28) can also be written as

$$-1 \leq \frac{M_m(0)}{M_m(1)} \leq 1$$

(compare (2.4.7) and (2.4.9)), and we have $\rho_m^* = 1/\rho_m$. It is reminiscent of a similar condition for *adjoint collocation methods* in ODEs whose collocation parameters $\{c_i^*\}$ are given by

$$c_i^* = 1 - c_{m+1-i}, \quad i = 1, \ldots, m.$$

For details, see Section V.2.1 in Hairer, Lubich and Wanner (2002).

2.4.8 Nonlinear first-kind VIEs

Although nonlinear first-kind VIEs,

$$\int_0^t k(t, s, y(s))ds = g(t), \quad t \in I,$$

can in principle be solved by 'direct' collocation in $S_{m-1}^{(-1)}(I_h)$, it is often advantageous – as we have already briefly indicated in Section 2.4.6 – to use a somewhat different, 'indirect' approach. We will illustrate by considering nonlinear equations of *Hammerstein* type, namely,

$$(\mathcal{H}y)(t) := \int_0^t K(t, s)G(s, y(s))ds = g(t), \quad t \in I, \tag{2.4.29}$$

with $g(0) = 0$ and appropriately differentiable K and G.

The basis of this approach is the *differentiated form* of (2.4.29): it is an *implicit* VIE of the second kind, again of Hammerstein type,

$$G(t, y(t)) = g_1(t) + \int_0^t K_1(t, s)G(s, y(s))ds, \quad t \in I, \tag{2.4.30}$$

with g_1 and K_1 as in (2.4.26). We assume again that $|K(t, t)| \geq k_0 > 0$ on I. In analogy to Section 2.3.3 we rewrite this equation, by setting $z(t) := (\mathcal{N} y)(t) = G(t, y(t))$, as

$$z(t) = g_1(t) + \int_0^t K_1(t, s)z(s)ds, \quad t \in I. \tag{2.4.31}$$

If its solution z is known, the solution y of the original VIE can be found by solving the *nonlinear operator equation*

$$(\mathcal{N} y)(t) = G(t, y(t)) = z(t), \quad t \in I \tag{2.4.32}$$

for y in $C(I)$. In the collocation framework the problem becomes: find the collocation solution $z_h \in S_{m-1}^{(-1)}(I_h)$ satisfying the collocation equation corresponding to (2.4.31),

$$z_h(t) = g_1(t) + \int_0^t K_1(t, s)z_h(s)ds, \quad t \in X_h, \tag{2.4.33}$$

and then define the iterated collocation solution z_h^{it} by

$$z_h^{it}(t) := g_1(t) + \int_0^t K_1(t, s)z_h(s)ds, \quad t \in I. \tag{2.4.34}$$

For any given $t \in I$ the approximation $y_h(t)$ to the exact solution $y(t)$ is obtained by solving the nonlinear equation

$$(\mathcal{N} y_h)(t) = G(t, y_h(t)) = z_h^{it}(t). \tag{2.4.35}$$

Under suitable conditions guaranteeing the invertibility of the Niemytzki operator on $C(I)$ we obtain a unique approximation $y_h \in C(I)$ to the solution of the first-kind Volterra-Hammerstein equation (2.4.29).

2.4.9 Collocation in smoother piecewise polynomial spaces

Hung (1970) showed that the collocation solution $u_h \in S_2^{(1)}(I_h)$ to the first-kind VIE (2.4.1) is divergent when $c_1 = 1$. From what we have seen in Section 1.3.1 (Theorem 1.3.1) and Section 2.2.15 this result is perhaps not entirely surprising. However, the general 'Mülthei theory' on the divergence and convergence of smooth piecewise polynomial collocation solutions for Volterra integral equations of the first kind has not yet been established.

2.4.10 Multidimensional first-kind VIEs

The collocation approach described in Section 2.2.9 can be adapted to discretise the two-dimensional first-kind VIE first analysed by Volterra (1896c),

$$\int_0^x \int_0^y K(x,\xi,y,\eta)u(\xi,\eta)d\eta\, d\xi = g(x,y), \quad (x,y) \in \Omega := [0,X] \times [0,Y] \tag{2.4.36}$$

(recall Section 2.1.7). Suppose, as in Section 2.2.9, that its solution is approximated by the collocation solution $u_{h,k} \in S^{(-1)}_{m-1,\mu-1}(\Omega_{h,k})$, defined by the collocation equation

$$\int_0^x \int_0^y K(x,\xi,y,\eta)u_{h,k}(\xi,\eta)d\eta\, d\xi = g(x,y), \quad (x,y) \in X_{h,k},$$

where $X_{h,k} := X_h \times Y_k$ denotes the set of collocation points corresponding to

$$X_h := \{x_j + c_i\tau_j :\ 0 < c_1 < \ldots < c_m \le 1\ (0 \le j \le M-1)\},$$
$$Y_k := \{y_l + d_i h_l :\ 0 < d_1 < \ldots < d_\mu \le 1\ (0 \le j \le N-1)\}.$$

The convergence analysis for $m = \mu = 1$ and $c_1 = d_1$, with correspondingly discretised integrals occurring in the collocation equation (resulting in the Euler method), can be found in McKee, Tang and Diogo (2000). Ries (1988) in her diploma thesis generalised the first part of Theorem 2.4.2 to the two-dimensional first-kind VIE (2.4.36). We state her result but leave the proof as an exercise.

Theorem 2.4.10 *Suppose that g and K in (2.4.36) are such that the integral equation has a unique solution $u \in C^m(\Omega)$. If u is approximated by the collocation solution $u_h \in S^{(-1)}_{m-1,\mu-1}(\Omega_{h,k})$, with collocation points given by $X_{h,k}$, then $u_{h,k}$ converges uniformly to u on Ω if, and only if, the collocation parameters satisfy the conditions*

$$\prod_{i=1}^{m} \frac{1-c_i}{c_i} \le 1 \quad \text{and} \quad \prod_{i=1}^{\mu} \frac{1-d_i}{d_i} \le 1.$$

2.5 Exercises and research problems

Exercise 2.5.1 Prove Lemma 2.1.1.

Exercise 2.5.2 Let $\mathcal{V}$ be a linear (finite-rank) Volterra integral operator with kernel

$$K(t,s) = \sum_{i=1}^{r} A_i(t)B_i(s), \quad \text{with} \quad A_i,\ B_i \in C(I)\ (i = 1,\ldots,r).$$

What can be said about the resolvent kernel corresponding to K?

Exercise 2.5.3 Determine the resolvent kernel of the VIE

$$u(t,s) = g(t,s) + \int_0^t \int_0^s A(\tau)B(\sigma)u(\tau,\sigma)d\sigma\, d\tau,$$

where g, A, B are continuous functions.

Exercise 2.5.4 Let $K \in C(D)$, and assume that ε is a (small) non-zero constant. Consider the perturbed VIE

$$z(t) = g(t) + \int_0^t K_\varepsilon(t,s)z(s)ds, \quad t \in I,$$

where $K_\varepsilon(t,s) := K(t,s) + \varepsilon K_0(t,s)$, with $K_0 \in C(D)$. How are the resolvent kernels $R(t,s)$ and $R_\varepsilon(t,s)$ related? Are we justified to write

$$R_\varepsilon(t,s) = R(t,s) + \varepsilon R_0(t,s), \quad (t,s) \in D?$$

If so, what can be said about $R_0(t,s)$?

Exercise 2.5.5 Prove Theorem 2.1.6.

Exercise 2.5.6 Determine the solution of

$$y(t) = g(t) + \lambda \int_0^t \frac{(t-s)^r}{r!} y(s)ds$$

for $g \in C(I)$ and $r \in \mathbb{N}$.

Exercise 2.5.7 Consider the first-kind VIEs

$$\int_0^t (t-s)y(s)ds = t, \quad t \in I := [0,T],$$

$$\int_0^t \sin(t-s)y(s)ds = t+1, \quad t \in I,$$

$$\int_0^t (t-s)^2 y(s)ds = t^2(1+t), \quad t \in I.$$

Show that these equations do not have ‘classical’ (i.e. continuous) solutions on I. Solutions exist in the setting of (special) distributions: determine these solutions, and discuss their uniqueness. (Compare also Krasnov et al. (1977) for additional examples.)

Exercise 2.5.8 Consider the linear first-kind VIE $(\mathcal{V}y)(t) = g(t),\ t \in I := [0,T]$, with $g \in C^1(I),\ g(0) = 0$, and

$$K(t,s) = \sum_{i=1}^{r} A_i(t)B_i(s), \quad A_i,\ B_i \in C^1(I).$$

Assume that the A_i are linearly independent on I. When does this integral equation have a unique solution $y \in C(I)$?

Exercise 2.5.9 Do Exercise 2.5.8 for the nonlinear VIE

$$\int_0^t \sum_{i=1}^r A_i(t)b_i(s, y(s))ds = g(t), \quad t \in I.$$

Exercise 2.5.10

$$y(t) = \exp(-t/2) - (1/2)\int_0^t (t-s)^2 \exp(s-t)\{y(s) + [1+y(s)]^{-2}\}ds:$$

Show that a solution exists in $[0, \infty)$. Is the solution bounded? [Hint: Theorem 2.1.10 on 'nonlinear perturbations' of linear V2s.]

Exercise 2.5.11

(a) Formulate Theorem 2.1.9 for Volterra–Hammerstein integral equations.
(b) Do the same for equation (2.1.34).

Exercise 2.5.12 Provide the details in the proof of Theorem 2.1.13. In particular, show the uniqueness of the solution $u \in C(I \times \Omega)$.

Exercise 2.5.13 Extend the Gronwall-type result of Lemma 2.1.14 to the inequality

$$z(t) \le g(t) + \int_0^t A(t)B(s)z(s)ds, \quad t \in I,$$

where a and B are continous, non-negative functions on I. (See also Beesack (1975).)

Exercise 2.5.14 Show that, under the regularity assumptions of Theorem 2.2.5, the defect δ_h defined at the beginning of the proof of Theorem 2.2.5 has derivatives $\delta^{(\nu)}$ ($\nu = 1, \dots, m+1$) that are smooth in σ_n and *uniformly bounded* on $\bar{\sigma}_n$ ($n = 0, 1, \dots, N-1$).

Exercise 2.5.15 Recall the superconvergence result of Theorem 2.2.6 (and Corollaries 2.2.5 and 2.2.8). Is local superconvergence on X_h possible? (Compare also Theorem 2.2.19 for Fredholm integral equations.)

Exercise 2.5.16 (Research problem)
Establish global and local superconvergence results for the two-dimensional ('mixed') VIE (2.1.53).

Exercise 2.5.17 Discuss the analogue of two-step collocation (Section 1.5) to Volterra integral equations of the second kind. Is it true that if the collocation parameters $\{c_i\}$ are such that the two-step collocation solution in $S^{(0)}_{m+\mu-1}(I_h)$ for the ODE has optimal (local) order $p^* = 2m + \mu - 1$, then the same is true for the *iterated* two-step collocation solution associated with the collocation solution in $S^{(-1)}_{m+\mu-2}(I_h)$ for the VIE? In other words, does the analogue of Theorem 1.5.1 hold?

Exercise 2.5.18 Derive the numerical differentiation formulas corresponding to Examples 2.4.1–2.4.5. Find those of *optimal order* (cf. Theorem 2.4.6).

Exercise 2.5.19 Show that the set of parameters $\{c_i\}$ that lead to local superconvergence for first-kind VIEs at certain points $t_n + d_i h_n$ is not unique. Give such a set different from the one in Theorem 2.4.7.

Exercise 2.5.20 Use Theorem 2.2.10 to derive concrete error bounds for $e^{it}_{\mu,h}(t)$. Compute $||e^{it}_{\mu,h}||_\infty$ $(\mu = 1, 2, 3)$ for some test VIEs with known solutions.

Exercise 2.5.21 Describe collocation and iterated collocation for *systems* of second-kind VIEs (cf. Section 2.1.3).

Exercise 2.5.22 Consider the 'non-standard' VIE

$$y(t) = g(t) + \int_0^t k(t-s)G(y(t), y(s))ds, \quad t \in I.$$

Establish global and local superconvergence results for u^{it}_h corresponding to the collocation solution $u_h \in S^{(-1)}_{m-1}(I_h)$.

Exercise 2.5.23 (see Sloss and Blyth (1994))
Assume that the nonlinear VIE

$$y(t) = \sum_{l=1}^{r} b_l \left(g_l(t) + \int_0^t K_l(t,s)y(s)ds \right), \quad t \in I \quad (r \in \mathbb{N},\ r \geq 2),$$

is solved by collocation in $S^{(-1)}_{m-1}(I_h)$. Discuss the existence and uniqueness of the collocation solution, and analyse its global and local (super-) convergence properties.

Exercise 2.5.24 Prove Theorem 2.2.12 by using local representations based on the *Hermite canonical polynomials* with respect to the $\{c_i\}$ (with given multiplicities).

Exercise 2.5.25 Do the statements of Theorem 2.4.5 on the attainable order of local superconvergence in collocation solutions $u_h \in S^{(-1)}_{m-1}(I_h)$ for *first-kind* VIEs remain valid *discretised* collocations $\hat{u}_h$ in this space (cf. Section 2.4.5)?

Exercise 2.5.26 Prove the analogue of Theorem 2.4.6 (local superconvergence for VIE1) for the discretised collocation solution. In particular: $m = 1, t = t_{n+\frac{1}{2}}$.

Exercise 2.5.27 Prove Theorem 2.4.8 (necessary and sufficient condition for uniform convergence of collocation solution $u_h \in S^{(-1)}_{m-1}(I_h)$ for the *adjoint* first-kind VIE (2.4.27)). Establish also the analogue of (2.4.8) on the attainable global order.

Exercise 2.5.28 (Research problem)
Consider the general first-kind VIE

$$(\mathcal{V}_1 y)(t) + (\mathcal{V}_2^* y)(t) = g(t), \quad t \in I,$$

where

$$(\mathcal{V}_1 \phi)(t) := \int_0^t K_1(t,s)\phi(s)ds, \quad (\mathcal{V}_2^* \phi)(t) := \int_t^T K_2(s,t)\phi(s)ds.$$

Analyse the existence, uniqueness, and regularity of its solution, and derive convergence results for the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$.

Exercise 2.5.29 (Research problem)
Analyse the application of collocation in $S_{m-1}^{(-1)}(I_h)$ and corresponding iterated collocation (especially for $m = 1$ and $m = 2$) to nonlinear second-kind VIEs with *blow-up solutions* (see the Remark following Theorem 2.1.11). In particular, extend the approach based on the θ-method in Stuart and Floater (1990).

Exercise 2.5.30 Discuss the existence and uniqueness of solutions $y \in C(I)$ of the non-standard second-kind VIE,

$$y(t) = g(t) + \lambda y(t) \int_0^t K(t,s)y(s)ds, \quad t \in I := [0,T],$$

where $g \in C(I)$ and $K \in C(D)$, λ is a parameter (see also Nestell and Ghandehari (2000) in Corduneanu and Sandberg (2000, pp. 357–365).
Is global and local superconvergence for the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ and its iterate u_h^{it} possible?

Exercise 2.5.31 (Research problem)
Discuss the solvability of the system of nonlinear algebraic equations arising in the implicitly linear collocation equation (2.3.9), and analyse the effect of the stopping error in iterative methods, e.g. in Newton's method, on the attainable order of the method. This will generalise analogous investigations for ODEs, as given for example in Liu and Kraaijevanger (1988), Spijker (1994), Jackson, Kvaernø and Nørsett (1996); see also Hairer and Wanner (1996, pp. 215–224).

2.6 Notes

2.1: Basic Volterra theory (I)
The most comprehensive and advanced analyses of Volterra integral and integro-differential equations are contained in the monographs by Miller (1971a), Gripenberg, Londen and Staffans (1990), Corduneanu (1991), and Prüss (1993).

The name ‘integral equation’ appears to be due to Du Bois-Reymond (1888): in his paper on elliptic partial differential equations he says that ‘Ich schrieb diese Gleichungen nicht hin, als ob sie etwa das Problem lösten oder doch der Lösung näher führten, sie sollen nur ein Beispiel unter zahllosen sein, dafür, dass man bei Randwertproblemen der linearen partiellen Differentialgleichungen beständig vor dieselbe Gattung von Aufgaben gestellt wird, welche jedoch, wie es scheint, für die heutige Analysis im Allgemeinen unüberwindliche Schwierigkeiten darbieten. Ich meine die zweckmässig *Integralgleichungen* zu nennenden Aufgaben, welche darin bestehen, dass die zu bestimmende Function, ausser ihrem sonstigen Vorkommen, in ihnen unter bestimmten Integralen enthalten ist,... ’ (‘I write down these equations not as if they solve the problem or even carry it nearer to a solution; they serve only as examples of the fact that in the boundary value problem of linear partial differential equations one is continually faced by this type of problem which still, for the analysis of today, presents in general insurmountable difficulties. I propose to give to these very useful problems the name *integral equations* ... ’.) An English translation of most of this can be found in Davis (1926, p. 10). These ‘insurmountable difficulties’ were of course dealt with by Ivar Fredholm some twelve years later (see his main paper of 1903).

Picard introduced the iteration technique that now bears his name in his *mémoire* of 1890. Some of the results by Le Roux (1895) predate the ones by Volterra (1896a); however, his convergence analysis for the Neumann series was based on a geometric series and hence valid only under the condition that $\bar{K}T < 1$.

The paper by Lauricella (1908) gives a survey of the early developments of the theory of integral equations with variable upper limits of integration. While such integral equations were already considered by Liouville in the late 1830s (see, e.g. Dieudonné (1981)), the name ‘Volterra integral equation’ appears to have been coined by Lalesco (1908, p. 126), following a suggestion by his teacher, E. Picard.

Bôcher (1909, 2nd edn: 1913) was the first monograph dedicated to the theory of integral equations. Its publication was followed by the report by Bateman (1910) on the state of the art in their theory. (It is interesting to compare this account with the report by Walther and Dejon (1960), published 50 years later.)

The following books contain large sections on Volterra integral equations: Lalesco (1912) (pp. 5–18: Volterra theory, including V1s in $\mathbb{R}^2$; systems of V2s; nonlinear V2s (pp. 127–130); $(V1)_\alpha$ (pp. 103–111); *chronological bibliography* ($\leq$ 1911), Volterra (1913) (based on lectures given in Rome during 1909–1910: Volterra theory (pp. 34–101); ‘finite to infinite’ (corresponding to discretised collocation in $S_0^{(-1)}(I_h)$ with $c_1 = 0$ for V2), Vivanti (1929) (includes a comprehensive list of references, including Ph.D. theses), and

Kowalewski (1930). See also Schmeidler (1950), Pogorzelski (1966), Cochran (1972), and Zabreyko et al. (1975). Mingarelli (1983) analyses VIEs of Stieltjes type.

The papers by Klebanov and Sleeman (1996) and Väth (1998a, 1998b), and the book by Väth (1999) beautifully complement the above expositions of Volterra theory: they respectively introduce an axiomatic theory of VIEs and study abstract VIEs by means of topological and algebraic methods (Väth).

Because of practical implications, we also mention a number of contributions to the problem of deriving optimal estimates for the norm of the nth power of a linear Volterra operator (usually with $K(t, s) \equiv 1$); they are Halmos (1982, Chapter 20), Lao and Whitley (1997), Thorpe (1998), Little and Read (1998), and Kershaw (1999).

The detailed study of nonlinear Volterra integral equations of the second kind has its origin in the early 1950s, in the papers by Sato (1951, 1953), Mann and Wolf (1951), and Roberts and Mann (1951); see also Padmavally (1958). A detailed survey (including a comprehensive list of references) of the early developments of this theory can be found in Nohel (1964), Wouk (1964), and Nohel (1976). See also v. Wolfersdorf (2000). Miller (2000) describes the important role the group at the University of Wisconsin at Madison played in this.

The classical book on nonlinear VIEs is the one by Miller (1971a); see also Gripenberg, Londen and Staffans (1990) and Corduneanu (1991) (this book starts with an excellent overview of the many contributions to the subject). The reader may wish to look at the paper by Diekmann and Gils (1981) for a variation-of-constants formula for nonlinear VIEs with convolution kernels; see also Brauer (1972). We also mention Diekmann and van Gils (1984) for an illuminating discussion of invariant manifolds.

The existence and uniqueness of solutions to nonlinear VFIEs is studied in, e.g. Pachpatte (1986) and Zaghrout (1993).

Theoretical aspects of the Goursat problem are discussed in Dzyadyk (1995) (Section 5.3.1); see also Kowalewski (1930), Goursat (1942), and Törnig (1959), Moore (1961), Filippi and Stimberg (1968), and Dobner (1987) for additional references and numerical approaches to the problem.

The books by Bellman and Cooke (1963), Lakshmikantham and Leela (1969), Miller (1971a), Cochran (1972), Mitrinović, Pečarić and Fink (1991), and Bainov and Simeonov (1992) contain much material on Gronwall type inequalities and comparison theorems for Volterra equations; see also the papers by Beesack (1969, 1985b), Agarwal and Thandapani (1981), and Dixon and McKee (1984).

The first result on a discrete Gronwall inequality appears to occur in Mikeladze (1935, p. 259); more recent results can be found in, e.g. Jones (1964), Beesack (1975, 1985b), Schmidt (1976), McKee (1982a), Dixon and McKee (1986), and Brunner and van der Houwen (1986), Ch. 1.

Due to limitation of space we have not mentioned the *qualitative theory* of VIEs (and VIDEs), which may be said to have its origin in the celebrated results by Paley and Wiener (1934) (see also Exercise 3.5.3). The monograph by Gripenberg, Londen and Staffans (1990) should be consulted for a thorough exposition of this theorem (see pp. 45–63 and pp. 83–89). However, see also Nohel (1964, 1976), Tsalyuk (1969), Shea and Wainger (1975), and the survey paper by Tsalyuk (1979).

Linear and nonlinear VIEs with periodic solutions are studied in Friedman (1965); see also Miller (1971a) and Gripenberg, Londen and Staffans (1990).

Applications of VIEs:
The following books and survey papers contain sections dealing with various applications of Volterra integral equations in the physical and biological sciences: Schmeidler (1950), Bellman and Cooke (1963), Anselone (1964), Miller (1971a), Brunner (1982a), Burton (1983), Webb (1985), Okrasiński (1989), Corduneanu (1991), Guy and Salès (1991), Prüss (1993), Agarwal and O'Regan (2000), and Corduneanu and Sandberg (2000), Zhao (2003). Most of these also include extensive lists of references.

The following is a *selection* of papers dealing with specific applications of VIEs of the *second kind:*

- *Population dynamics, spread of epidemics:* Brauer (1975, 1976a), Diekmann (1978, 1979), Thieme (1977, 1979), Gripenberg (1981) ('non-standard' VIE), Brauer and Castillo-Chávez (2001) (see also for additional references).
- *Renewal equation:* Feller (1941), Karlin (1955), Bellman and Cooke (1963) (Chapters 7 and 8), Brauer (1976b).
- *Wave problems:* Levinson (1960) (superfluidity), Gilding (1993) (travelling wave analysis in nonlinear reaction-convection-diffusion problems), monograph by Kabanikhin and Lorenzi (1999) (identification problems for wave phenomena), Franco (1999) (nonlinear waves).
- *Water percolation:* Okrasiński (1978).
- *Semi-conductor devices:* Miller and Unterreiter (1992), Schmeiseer, Unterreiter and Weiss (1993), Unterreiter (1996) (models for switching behaviour of PN-diodes).
- *Inverse problems related to wave propagation:* A detailed discussion of Volterra (operator) integral equations arising in such problems and their regularisation is given in the book by Kabanikhin and Lorenzi. (See also the Notes to Chapters 3 and 6 on additional, related work by Lorenzi and his co-workers.)

- *Identification of memory kernels in viscoelasticity and heat conduction:* This problem was studied extensively by v. Wolfersdorf (1994), Unger and v. Wolfersdorf (1995), Janno and v. Wolfersdorf (1997a,b), Kiss (1999 / doctoral thesis). See also Berrone (1995) on the modelling of materials that may undergo a change of phase.
- *Viscoelasticity (partial VIEs):* Shaw, Warby and Whiteman (1994, 1996, 1997), Shaw and Whiteman (1997).

Applications of *first-kind VIEs* with bounded kernels:
Volterra's *Nota I* of 1896 may have been motivated by a first-kind integral equation he encountered in a problem in electrostatics (Volterra (1884)). A selection of more recent sources of applications of such functional equations is given below.

- The books by Sneddon (1972), Asanov (1998) and Bukhgeim (1999) contain numerous sources of applications of first-kind VIEs.
- Inverse problems in heat conduction: Beck, Backwell and St. Clair (1985), Eldén (1976), Lamm (2000 / survey paper with comprehensive list of references), Lamm and Scofield (2000).
- Peirce and Siebrits (1996): elastodynamic models / boundary integral equations.
- Davies and Duncan (2002, 2003): retarded potential equations.

2.2: Collocation for second-kind VIEs
In his book of 1913 (pp. 40–46) Volterra used a discretised version of the linear second-kind VIE to establish the existence of a unqiue continuous solution for the latter. The underlying quadrature formula is based on the left rectangular rule, and the resulting discrete version version of the VIE may be viewed as discretised collocation in $S_0^{(-1)}(I_h)$, with $c_1 = 1$. The idea of employing collocation-type approximations in $S_1^{(0)}(I_h)$ for the numerical solution of VIEs is due to Huber (1939); his aproach was extended by Wagner (1954). See also the paper by Kaspšickaja (1969) where special polynomial collocation spaces are used.

The books by Baker (1977), Linz (1985), and Brunner and van der Houwen (1986) contain a wealth of information (and extensive bibliographies) on the numerical treatment of Volterra integral equations. See also the survey papers by Bernier (1945), Noble (1964, 1977), Baker (1982, 1997, 2000), Brunner (1982a, 1987, 1999b).

The papers by Brunner (1984c, 1986b) review the historical development of numerical methods – particularly collocation methods – for second-kind VIEs. In Section 2.2.6 we analysed the error $u_h - \hat{u}_h$ that results when the exact collocation equation is replaced by its fully discretised counterpart. If the kernel of the Volterra integral operator $\mathcal{V}$ is *highly oscillatory*, $K(t,s) = \exp(i\omega(t-s))H(t,s)$, with $\omega \gg 1$, this analysis will be misleading when $h > 0$ is fixed: interpolatory quadrature based on the collocation points will in general introduce large errors depending on ω. A powerful alternative to such classical quadrature formulas is described in Iserles (2004); it employs an elegant variant of Filon-type quadrature. The application to the derivation of feasible fully discretised collocation equations remains to be investigated.

As we already indicated in the Preface, an attractive alternative to piecewise polynomial collocation methods is given by *pseudo-spectral methods*. Of the papers dealing with these methods we mention the ones by Elnagar and Kazemi-Dekhordi (1996) and Elnagar and Razzaghi (1996) on Volterra–Hammerstein equations; see also their references.

An early contribution to the numerical analysis of (nonlinear) two-dimensional VIEs of the second kind is by Bel'tyukov and Kuznechikhina (1976): they design and analyse a class of Runge–Kutta methods (extending Bel'tyukov's method of 1965; see Brunner and van der Houwen (1986, Chapter 4)). See also the papers by Singh (1976) and Mureşan (1984). The paper by Schaback (1974) contains may pertinent remarks on multi-dimensional spline collocation.

More recent papers on such VIEs are by, e.g. Brunner and Kauthen (1989), G. Han and Zhang (1994a), Luo and Hu (1995), G. Han et al. (2000), and G. Han and Wang (2001).

Second-kind integral equations of ('mixed') *Volterra–Fredholm type* arise for example in the modelling of the spread of epidemics (cf. Thieme (1977, 1979), Diekmann (1978), Pachpatte (1986), and Brunner (1990) for additional references). The numerical treatment of such functional equations has been studied by many authors; see, e.g. Haçia (1979, 1996, 1999), Kauthen (1989a, 1989b), Brunner (1990, 1991), Han and Zhang (1994b), Han (1995), and Hadizadeh (2003).

Divergence of classical (full-continuity) cubic spline collocation solution: convergent variants given by Hung (1970), Netravali (1973) and by Oja and Saveljeva (2001); in latter paper: one of the initial conditions is replaced by a right-hand boundary condition. The analysis for the collocation spaces $S_{m+d}^{(d)}(I_h)$ given by Danciu (1995) is, as pointed out in *MR 99g:65125a* and in Oja (2001a, 2002b), unfortunately flawed.

Fredholm integral equations of the second kind: Nyström's paper on the numerical solution of Fredholm integral equations dates from 1928. Kadner (1960, 1967) initiated the interest in collocation methods for second-kind FIEs (see also Dejon (1962) on the choice of the collocation points). One of the subsequent key papers is the one by Sloan (1976) in which the notion of the iterated collocation (and Galerkin) solution is introduced; compare also Sloan

(1984) (variants of the Galerkin method) and Schock (1985) (analysis of possible rates of convergence). Surveys of superconvergence results for FIEs can be found in Chatelin and Lebbar (1981), the monograph by Chatelin (1983), in Brunner (1987), and in Sloan (1988a, 1990). The reader should also consult the papers by Joe (1985a, 1985b) on exact and discretised collocation methods, and their optimal convergence estimates. Similar analyses, especially for nonlinear FIEs, can be found in Atkinson and Bogomolny (1987) (Galerkin methods) and Atkinson and Flores (1993) (collocation methods). See also the survey paper of Atkinson (1992).

The paper by Graham, Joe and Sloan (1985) gives an illuminating comparison (with respect to regularity requirements and attainable orders of convergence) of iterated Galerkin methods with iterated collocation methods for second-kind Fredholm integral equations. See also Sloan (1990, pp. 63–64) for a concise survey of these results. It would be of considerable interest to carry out a similar study for Volterra integral equations.

The authoritative book on the numerical analysis of FIEs is Atkinson (1997a); it contains extensive sections on piecewise polynomial collocation solutions.

2.3: Collocation for nonlinear second-kind VIEs
A general analysis of fully discretised collocation methods can be found in Brunner (1992a). Implicitly linear collocation methods are analysed in Brunner (1992b); see also Brunner (1991).

2.4: Collocation for first-kind VIEs
Many of the common numerical methods described in, e.g. Linz (1969b), Weiss (1972a), de Hoog and Weiss (1973b, 1973c) and McAlevey (1987) can be interpreted as fully discretised collocation methods in $S_{m-1}^{(-1)}(I_h)$ or $S_m^{(0)}(I_h)$; see Brunner (1977) and Brunner and van der Houwen (1986, Chapter 5). The diploma theses of Rothe (1982) and Ries (1988) analyse various aspects of piecewise poynomial collocation methods for VIEs of the first kind with smooth kernels.

The numerical solution of two-dimensional VIEs of the first kind by block methods was studied by Ten Men Yan (1979): it corresponds to discretised collocation in $S_{0,0}^{(-1)}(\Omega_{h,k})$ with $c_1 = d_1 = 1$ and leads (as in the one-dimensional case) to local superconvergence of order $\mathcal{O}(h^2)$ at the midpoints points $x_{j+1/2}$, $y_{l+1/2}$ of a uniform mesh.

We have already mentioned (end of Section 2.1.4) the sequential regularisation approaches by Lamm et al. (see e.g. the surveys by Lamm (2000, 2003); also Ring (2001)): some of these methods are based on collocation techniques. Plato and Vainikko (1990) study the regularisation of general projection methods. A different approach to regularisation for first-kind VIEs is discussed in Brunner and Sizikov (1998).

Galerkin methods and adaptivity

Galerkin-type methods for second-kind VIEs have been analysed by, e.g. Lin, Thomée and Wahlbin (1991), Bedivan and Fix (1997, 1998), and Brunner, Lin and Zhang (1998) (see also for additonal references). The reader should also consult the important paper by Graham, Joe and Sloan (1985) in which the relative merits of Galerkin and collocation methods for second-kind Fredholm integral equations are studied. An analogous comparison of these methods for second-kind VIEs would be valuable, too.

An important topic for future research relates to a posteriori error estimates for (iterated) collocation solutions and adaptive mesh selection. There are now a number of papers in which this problem has been studied for (continuous and discontinuous) Galerkin methods: we cite especially Shaw and Whiteman (1996a, 1997, 2000a, 2001b, 2001).

Postprocessing methods

There is an extensive literature on methods aimed at improving the accuracy/order of a computed numerical (quadrature, collocation, or Galerkin) solution to a VIE (or a Fredholm integral equation). Hock (1979, 1980, 1981) studied extrapolation techniques based on simple quadrature methods (which can be interpreted as discretised collocation methods). The research work of Lin Qun (Chinese Academy of Sciences) from the late 1970s onwards was the starting point for many more recent papers. A detailed treatment of the mathematical framework and their computational applications of such postprocessing methods would easily fill a separate monograph. (The monograph by Marchuk and Shaidurov (1983) deals with extrapolation methods for VIEs of the first and second kind.)

Extrapolation techniques applied to simple quadrature methods for second-kind VIEs were suggested by Noble (1964) and, especially, by Hock (1979, 1980, 1981). Of the numerous papers on iterative correction and multilevel correction techniques we mention the ones by Lin and Lü (1984), Lin, Sloan and Xie (1990), Xiang (1991), Lin and Shi (1993), Han (1993, 1994a, 1994b, 1994c), Han and Zhang (1994a, 1995), Han (1995) (for VFIEs), Luo and Hu (1995), Brunner, Lin and Yan (1996), Zhou (1997), Lin and Zhou (1997a, 1997b), Lin, Zhang and Yan (1998a,b), Hu (1998a), Brunner, Y. Lin and Zhang (1998), Zhang, Y. Lin and Rao (2000), Luo (2000), Han et al. (2000), Han and Wang (2001).

Extrapolation methods for first-kind VIEs were studied by Linz (1969b), Hung (1970) (for collocation in $S_1^{(0)})(I_h)$), Eggermont (1985, 1986) (for collocation solutions), and McAlevey (1987). Zhou (1991) presents a detailed analysis of extrapolation methods for collocation solutions (see also Zhou (1997) for multiparameter error resolution techniques).

3

Volterra integro-differential equations with smooth kernels

In 1909 Volterra wrote (following the study of the modelling of hysteresis problems) that one is led '... ad equazioni che hanno tipo misto, cioé in parte quello delle equazioni differenziali a derivate parziale ed in parte quello delle equazioni integrali. Mi permetto perciò di chiamarle *equazioni integro-differenziali*.' He then used such 'equations of mixed type', namely linear integro-differential equations involving Volterra integral operators, as models describing heredity effects (see Volterra (1913, pp. 138–162)). Related, but more general (non-linear) versions became famous in Volterra's work, starting around 1926, on the growth of single-species or interacting populations. At the end of his 1909 paper (p. 174) he added, however, a cautionary note when he observed that '... *il problema della risoluzione delle equazioni integro-differenziali costituisce in generale un problema essenzialmente distinto dai problemi delle equazioni differenziali e da quelli ordinarii delle equazioni integrali'* [his italics].

Although such functional equations may be viewed formally as ODEs perturbed by a 'memory' term given by a Volterra integral operator, the analysis of collocation methods will be more complex (perhaps not 'essentially distinct' – except when it comes to the analysis of qualitative properties) than simply a synthesis of the techniques employed in Chapters 1 and 2. The convergence results we establish in this chapter will of course yield those of Chapter 1 as special cases.

3.1 Review of basic Volterra theory (II)

3.1.1 Linear VIDEs

Consider the initial-value problem for a linear first-order Volterra integro-differential equation (VIDE),

$$y'(t) = a(t)y(t) + g(t) + (\mathcal{V}y)(t), \quad t \in I, \quad y(0) = y_0, \tag{3.1.1}$$

where $\mathcal{V} : C(I) \to C(I)$ denotes the linear Volterra integral operator introduced in (2.1.1),

$$(\mathcal{V}\phi)(t) := \int_0^t K(t,s)\phi(s)ds, \quad t \in I,$$

and where $a, g \in C(I), \;\; K \in C(D)$ are given (real-valued) functions. It is clear from the analysis presented in Section 2.1.1 that the result on the existence and uniqueness of a solution to (3.1.1) can readily be obtained by rewriting the above initial-value problem (which generalizes the initial-value problem for a linear ODE) as a second-kind Volterra integral equation,

$$y(t) = g_0(t) + \int_0^t H(t,s)y(s)ds, \quad t \in I, \tag{3.1.2}$$

where

$$g_0(t) := y_0 + \int_0^t g(s)ds, \quad H(t,s) := a(s) + \int_s^t K(v,s)dv, \tag{3.1.3}$$

to which Theorem 2.1.1 can be applied. This will also allow us to introduce the notion of the (differential) resolvent kernel associated with the given functions a and K (which describe the homogeneous part of the VIDE in (3.1.1)) and to derive the corresponding resolvent equations. We first state

Theorem 3.1.1 *Assume that $a, g \in C(I)$ and $K \in C(D)$. Then for any initial value $y_0 \in \mathbb{R}$ the VIDE (3.1.1) possesses a unique solution $y \in C^1(I)$ satisfying $y(0) = y_0$. Moreover, there exists a unique function $r = r(t,s)$, the (differential) resolvent kernel, with $r \in C^1(D)$, so that this solution can be written as*

$$y(t) = r(t,0)y_0 + \int_0^t r(t,s)g(s)ds, \quad t \in I. \tag{3.1.4}$$

Proof Let $Q(t,s)$ denote the resolvent kernel of the kernel $H(t,s)$ in the integral equation (3.1.2). According to Section 2.1.1, Q solves the resolvent equation (2.1.10),

$$Q(t,s) = H(t,s) + \int_s^t Q(t,v)H(v,s)dv, \quad (t,s) \in D, \tag{3.1.5}$$

and the (unique) solution of (3.1.2) is given by

$$y(t) = g_0(t) + \int_0^t Q(t,s)g_0(s)ds, \quad t \in I. \tag{3.1.6}$$

Using the above definitions of g_0 and H we obtain

$$y(t) = \left(1 + \int_0^t Q(t,s)ds\right) y_0 + \int_0^t \left(1 + \int_s^t Q(t,v)dv\right) g(s)ds.$$

This shows that the desired function r in (3.1.4) is given by

$$r(t,s) := 1 + \int_s^t Q(t,v)dv, \quad (t,s) \in D; \tag{3.1.7}$$

its uniqueness, and hence that of y, follow from the uniqueness of the resolvent kernel Q and from Theorem 2.1.2. Note that $r \in C(D)$, with $\partial r(t,s)/\partial s = -Q(t,s) \in C(D)$, and we have $r(t,t) = 1$ for all $t \in I$. This completes the proof.

These observations reveal that the resolvent $r(t,s)$ associated with the linear VIDE (3.1.1) satisfies

$$\begin{aligned}\frac{\partial r(t,s)}{\partial s} &= -Q(t,s) = -H(t,s) - \int_s^t Q(t,v)H(v,s)dv \\ &= -a(s) - \int_s^t K(v,s)dv - \int_s^t Q(t,v)\left(a(s) + \int_s^v K(z,s)dz\right)dv \\ &= -\left(1 + \int_s^t Q(t,v)dv\right)a(s) - \int_s^t \left(1 + \int_v^t Q(t,z)dz\right)K(v,s)dv,\end{aligned}$$

and hence, by (3.1.7),

$$\frac{\partial r(t,s)}{\partial s} = -r(t,s)a(s) - \int_s^t r(t,v)K(v,s)dv, \quad (t,s) \in D. \tag{3.1.8}$$

The resolvent kernel $r(t,s)$ can also be defined by the (unique) solution of an *adjoint resolvent equation*, in complete analogy to the situation for second-kind VIEs. We summarise this result in the following theorem.

Theorem 3.1.2 *Assume that $a \in C(I)$ and $K \in C(D)$. Then the resolvent kernel $r = r(t,s)$ of the linear VIDE (3.1.1) is the (unique) solution of the resolvent equation (3.1.8), corresponding to $r(t,t) = 1$ for $t \in I$. It also solves the adjoint resolvent equation,*

$$\frac{\partial r(t,s)}{\partial t} = r(t,s)a(t) + \int_s^t K(t,v)r(v,s)dv, \quad (t,s) \in D, \tag{3.1.9}$$

with $r(s,s) = 1$ for $s \in I$.

We leave it to the reader to prove the second part of Theorem 3.1.2 (see Exercise 3.5.1).

Corollary 3.1.3 *The resolvent equations associated with the special VIDE*

$$y'(t) = g(t) + (\mathcal{V}y)(t), \quad t \in I,$$

are

$$\frac{\partial r(t,s)}{\partial s} = -\int_s^t r(t,v)K(v,s)dv, \quad (t,s) \in D,$$

and

$$\frac{\partial r(t,s)}{\partial t} = \int_s^t K(t,v)r(v,s)dv, \quad (t,s) \in D,$$

with $r(t,t) = 1$ $(t \in I)$ *and* $r(s,s) = 1$ $(s \in I)$*, respectively. If* $K(t,s) = k(t-s)$ *then the resolvent* r *inherits the convolution structure of the kernel.*

The next theorem, the counterpart of Theorem 2.1.3, deals with the *regularity* of the solution to the linear VIDE (3.1.1).

Theorem 3.1.4 *Assume that* $a,\ g \in C^m(I)$ *and* $K \in C^m(D)$*. Then for any* y_0 *the solution* y *of the linear VIDE (3.1.1) lies in the space* $C^{m+1}(I)$*.*

Proof This regularity result can be proved either by applying Theorem 2.1.3 to the second-kind VIE (3.1.2) (with $m+1$ replacing m, due to the additional regularity in g_0 and H), or by showing that the resolvent r lies in $C^{m+1}(D)$ and using this fact in the representation (3.1.4). Details are left to the reader.

In Section 3.1.3 we shall describe analogous results for various nonlinear counterparts of the VIDE (3.1.1), including the generic nonlinear VIDE

$$y'(t) = f(t, y(t)) + \int_0^t k(t,s,y(s))ds. \tag{3.1.10}$$

Similar to Section 2.1, the *semilinear* VIDE

$$y'(t) = a(t)y(t) + g(t) + \int_0^t K(t,s)\,(y(s) + H(s,y(s)))ds \tag{3.1.11}$$

represents a first step towards more general nonlinear problems: here, the linear Volterra integral operator $\mathcal{V}$ has been perturbed by the *Hammerstein* term

$$(\mathcal{H}y)(t) := \int_0^t K(t,s)H(s,y(s))ds$$

(recall (2.1.40); see also the 1970 paper by Grossman and Miller).

Theorem 3.1.5 *Assume that the initial-value problem for the semilinear VIDE (3.1.11) possesses a unique solution* $y \in C^1(I)$*, and let*

$$y_\ell(t) := r(t,0)y_0 + \int_0^t r(t,s)g(s)ds, \quad t \in I,$$

denote the solution of the linear VIDE

$$y'(t) = a(t)y(t) + g(t) + (\mathcal{V}y)(t), \quad y(0) = y_0.$$

Then y and y_ℓ are related by

$$y(t) = y_\ell(t) - \int_0^t \left(r(t,s)a(s) + \frac{\partial r(t,s)}{\partial s} \right) G(s, y(s))ds, \quad t \in I. \quad (3.1.12)$$

Here, $r(t,s)$ denotes the resolvent kernel associated with a and K describing the linear part of (3.1.11).

Proof Setting $Q(t) := g(t) + (\mathcal{H}y)(t)$, the semilinear VIDE (3.1.11) becomes

$$y'(t) = a(t)y(t) + Q(t) + (\mathcal{V}y)(t), \quad t \in I.$$

According to Theorem 3.1.1 the solution of this 'linear' VIDE is formally given by

$$y(t) = r(t,0)y_0 + \int_0^t r(t,s)Q(s)ds, \quad t \in I.$$

The representation of y in Theorem 3.1.5 now follows readily by resorting to the resolvent equation (3.1.8) and writing

$$\int_s^t r(t,v)K(v,s)dv = -r(t,s)a(s) - \frac{\partial r(t,s)}{\partial s}.$$

3.1.2 Neutral and higher-order VIDEs

If the kernel $k = k(t,s,y)$ in the VIDE (3.1.10) also depends on y', that is, if the VIDE has the form

$$y'(t) = f(t, y(t)) + \int_0^t k(t, s, y(s), y'(s))ds, \quad t \in I, \quad (3.1.13)$$

then such a functional equation is often (but not quite properly) referred to as a *neutral* (first-order) VIDE. It is a particular case of the kth-order VIDE (3.1.14) we shall study below. In Chapters 4 and 8 we shall meet another class of neutral (delay) VIDEs for which

$$\frac{d}{dt}\left(y(t) - \int_0^{\theta(t)} k(t, s, y(s))ds \right) = f(t, y(t), y'(t)),$$

with $\theta(t) \le t$, is a typical example.

Let now $k \geq 2$ be a given integer and consider the initial-value problem for the general (nonlinear) neutral VIDE,

$$y^{(k)}(t) = f(t, y(t), y'(t), \ldots, y^{(k-1)}(t)) + (\mathcal{V}y)(t), \quad t \in I, \quad (3.1.14)$$
$$y^{(\nu)}(0) = y_0^{(\nu)} \quad (\nu = 0, 1, \ldots, k-1),$$

where we now define

$$(\mathcal{V}y)(t) := \int_0^t k(t, s, y(s), y'(s), \ldots, y^{(k)}(s))ds.$$

We will often use the linear counterpart of this VIDE, described by

$$f(t, y, y', \ldots, y^{(k-1)}) = \sum_{\nu=0}^{k-1} a_\nu(t)y^{(\nu)} + g(t), \quad (3.1.15)$$

$$k(t, s, y, y', \ldots, y^{(k)}) = \sum_{\nu=0}^{k} K_\nu(t, s)y^{(\nu)}, \quad (3.1.16)$$

with continuous functions $g,\ a_\nu$ and K_ν, to render the subsequent convergence analysis more transparent. Note that we allow the derivative of order k of y to occur as argument in the kernel of the VIDE, thus generalising the neutral VIDE (3.1.13).

We shall now show briefly that, under the above continuity assumptions, the linear VIDE possesses a unique solution $y \in C^k(I)$ which assumes the prescribed initial values. An analogous (generally only local) existence and uniqueness result can be obtained for the nonlinear VIDE (3.1.10), by suitably adapting the arguments presented below.

Let $\mathbf{w}(t) := (w_0(t), w_1(t), \ldots, w_k(t))^T := (y(t), y'(t), \ldots, y^{(k)}(t))^T \in \mathbb{R}^{k+1}$: the components of $\mathbf{w}(t)$ are coupled by

$$w_\nu(t) = y_0^{(\nu)} + \int_0^t w_{\nu+1}(s)ds, \quad t \in I \quad (\nu = 0, 1, \ldots, k-1). \quad (3.1.17)$$

The above (linear) VIDE can then be rewritten as an equivalent *system of VIEs* of the second kind, namely,

$$w_k(t) = g(t) + \sum_{\nu=0}^{k-1} a_\nu(t)\left(y_0^{(\nu)} + \int_0^t w_{\nu+1}(s)ds\right) + \int_0^t \sum_{\nu=0}^{k} K_\nu(t, s)w_\nu(s)ds, \quad (3.1.18)$$

which, by means of the vector function

$$\gamma(t) := \left(y_0^{(0)}, y_0^{(1)}, \ldots, y_0^{(k-1)}, g(t) + \sum_{\nu=0}^{k-1} a_\nu(t)y_0^{(\nu)}\right)^T \in \mathbb{R}^{k+1}$$

and the kernel matrix $\mathbf{K}(\cdot,\cdot) \in L(\mathbb{R}^{k+1})$,

$$\mathbf{K}(t,s) := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & & 1 \\ K_0(t,s) & a_0(t) + K_1(t,s) & \cdots & \cdots & a_{k-1}(t) + K_k(t,s) \end{bmatrix},$$

assumes the compact form

$$\mathbf{w}(t) = \gamma(t) + \int_0^t \mathbf{K}(t,s)\mathbf{w}(s)ds, \quad t \in I. \tag{3.1.19}$$

We have seen in Section 2.1.3 that because γ and $\mathbf{K}$ are continuous, this system possesses a unique solution $\mathbf{y} \in C(I)$ whose representation,

$$\mathbf{w}(t) = \gamma(t) + \int_0^t \mathbf{R}(t,s)\gamma(s)ds, \quad t \in I, \tag{3.1.20}$$

is based on the (matrix) resolvent kernel $\mathbf{R} \in L(\mathbb{R}^{k+1})$ of $\mathbf{K}$. If we write this matrix resolvent kernel as

$$\mathbf{R}(t,s) := \begin{pmatrix} R_{0,0}(t,s) & \dots & R_{0,k}(t,s) \\ \vdots & & \vdots \\ R_{k,0}(t,s) & \dots & R_{k,k}(t,s) \end{pmatrix},$$

then the representation (3.1.20) permits the explicit derivation of the expressions for the $k+1$ components, e.g. for $w_0(t) = y(t)$, of the solution vector $\mathbf{w}(t)$.

The above equivalence between the initial-value problem for the kth-order neutral VIDE given by (3.1.14)–(3.1.16), and the system of $k+1$ linear Volterra integral equations (3.1.19) allows us, by appealing to Theorem 2.1.7, to obtain the following regularity result.

Theorem 3.1.6 *Assume that* a_ν $(\nu = 0, 1, \dots, k-1)$ *and* g *are in* $C(I)$, *and* $K_\nu \in C(D)$ $(\nu = 0, 1, \dots, k)$. *Then for any initial values* $y_0^{(\nu)}$ $(\nu = 0, 1, \dots, k-1)$ *the linear kth-order VIDE (3.1.14) corresponding to (3.1.15), (3.1.16) possesses a unique solution* $y \in C^k(I)$ *satisfying the given initial conditions.*

If the given functions have continuous derivatives of order m *on their respective domains* I *and* D, *then the solution* y *lies in the space* $C^{k+m}(I)$.

We will return to this result, and the equivalence property underlying it, in Section 3.2.6 when we analyse the attainable order of global and local superconvergence of the collocation solution in $S_{m+d}^{(d)}(I_h)$ $(d = k-1)$ for (3.1.14).

3.1.3 Nonlinear and non-standard VIDEs

Consider first the initial-value problem for the general nonlinear first-order VIDE

$$y'(t) = f(t, y(t)) + \int_0^t k(t, s, y(s))ds, \quad t \in I, \quad y(0) = y_0. \tag{3.1.21}$$

Since it is equivalent to a nonlinear VIE of the second kind, namely,

$$y(t) = y_0 + \int_0^t \left(f(s, y(s)) + \int_s^t k(v, s, y(s)) \right) ds, \quad t \in I,$$

Theorem 2.1.10 is readily adapted to establish the (local) existence and uniqueness of a solution.

We observe that in certain applications, in particular in the spatial semidiscretisation of parabolic VIDEs (see for example Thomée (1988), Thomée and Zhang (1989), Zhang (1990), Kauthen (1989b, 1992), Larsson, Thomée and Wahlbin (1998), Lin (1998), and Chen and Shih (1998), as well as the the references in Brunner (1989b)), nonlinear VIDEs typically have a more *structured* form. A representative example is

$$y'(t) = f(y(t)) + (\mathcal{H}y)(t), \quad t \in I \tag{3.1.22}$$

(compare also the end of Section 3.1.1), where $\mathcal{H}$ denotes the *Hammerstein operator* introduced before,

$$(\mathcal{H}y)(t) := \int_0^t K(t, s)G(s, y(s))ds,$$

corresponding to a smooth function $G : I \times \mathbb{R} \to \mathbb{R}$ (or, in the case of a spatially semidiscretised partial IDE, $G : I \times \mathbb{R}^M \to \mathbb{R}^M$, for some $M \gg 1$). The integrated form of this VIDE is given by

$$y(t) = y_0 + \int_0^t (f(y(s)) + K_2(t, s)G(s, y(s)))ds, \tag{3.1.23}$$

with

$$K_2(t, s) := \int_s^t K(v, s)dv.$$

Thus, the resulting nonlinear VIE is a special case ($r = 2$, $K_1(t, s) \equiv 1$, $G_1(s, y) = f(y)$) of the more general Hammerstein equation

$$y(t) = g(t) + \int_0^t \left(\sum_{i=1}^r K_i(t, s)G_i(s, y(s)) \right) ds. \tag{3.1.24}$$

Results on the existence and uniqueness of solutions to initial-value problems for *higher-order (neutral) VIDEs* of the form (3.1.14) can be proved by rewriting the VIDE as a *system of* $k+1$ *nonlinear second-kind VIEs*, analogous to (3.1.19).

Other mathematical modelling processes (we mention those in *population growth* and *viscoelasticity* as two important examples; cf. Volterra (1927, 1928, 1931, 1959), Cushing (1977), Lodge, McLeod and Nohel (1978), Markowich and Renardy (1983), and the Notes at the end of the chapter) lead to more general, *non-standard VIDEs* in which the integrand depends both on $y(s)$ and $y(t)$. A typical generic form is given by

$$y'(t) = f(t, y(t)) + \int_0^t k(t-s)G(y(t), y(s))ds, \tag{3.1.25}$$

and the best known example is probably the logistic equation with memory term,

$$y'(t) = \left(\varepsilon - ay(t) - \int_0^t k(t-s)y(s)ds\right) y(t), \quad t \geq 0, \tag{3.1.26}$$

which corresponds to $f(t, y) = f(y) = (\varepsilon - ay)y$ and $G(y, z) = -yz$ (see, for example, Volterra's papers of 1928 and 1934); an existence result can be found in Miller (1966)). The memory kernel k is usually of the form

$$k(t) = (\gamma_0 b^{-1} + \gamma_1 b^{-2} t)\exp(-t/b),$$

with $\gamma_1 > \gamma_0 \geq 0,\ \gamma_0 + \gamma_1 = 1$, and $b > 0$ (see also Cushing (1977) and Aves, Davies and Higham (1996, 2000)).

The dynamics of two interacting species was first modelled by Volterra (1927) (and preocccupied him until his death in 1940 – see Volterra (1939)): if we denote by $N_1(t)$ and $N_2(t)$ the size of the two populations at time $t \geq 0$, then the resulting system of non-standard VIDEs has the form

$$N_1'(t) = \left(\varepsilon_1 - a_1 N_2(t) - \int_\theta^t k_1(t-s)N_2(s)ds\right) N_1(t),$$

$$N_2'(t) = \left(-\varepsilon_2 + a_2 N_1(t) + \int_\theta^t K_2(t-s)N_1(s)ds\right) N_2(t).$$

Here, the ε_i and a_i denote given positive constants, and the lower limit of integration is either $\theta = 0,\ \theta = -\infty$, or $\theta = \theta(t) = t - \tau\ (\tau > 0)$ (VIDEs with constant and variable delays $\tau > 0$ will be discussed in Chapter 4). In his 1927 paper Volterra presents a detailed analysis of the quantitative and qualitative properties of the solutions to the above system of VIDEs.

3.2 Collocation for linear VIDEs

3.2.1 The exact collocation equations

Consider the first-order Volterra integro-differential equation

$$y'(t) = f(t, y(t)) + (\mathcal{V}y)(t), \quad t \in I := [0, T], \tag{3.2.1}$$

with initial condition $y(0) = y_0$. The operator $\mathcal{V} : C(I) \to C(I)$ for now denotes again the linear Volterra integral operator defined by

$$(\mathcal{V}\phi)(t) := \int_0^t K(t, s)\phi(s)ds, \quad t \in I,$$

where $K \in C(D)$. Since the VIDE (3.2.1) can be viewed as being a 'perturbation' of the ODE (1.1.1), with perturbation term given by the memory term $(\mathcal{V}y)(t)$, it will be interesting to see how this perturbation affects the order results we derived for ODEs, and how the usually necessary quadrature approximations for the memory term influence these results for the exact collocation equation. Here, the insights we obtained in Chapter 2 will of course be helpful.

As in the case of ODEs we approximate the solution y by collocation in the piecewise polynomial space $S_m^{(0)}(I)$: the collocation solution u_h is the element in this space that satisfies the collocation equation

$$u_h'(t) = f(t, u_h(t)) + (\mathcal{V}u_h)(t), \quad t \in X_h, \tag{3.2.2}$$

together with the initial condition $u_h(0) = y(0) = y_0$. Since the dimension of this collocation space is dim $S_m^{(0)}(I_h) = Nm + 1$, the set X_h of collocation points will be as in Sections 1.1.1 and 2.2.1, namely,

$$X_h := \{t_{n,i} := t_n + c_i h_n : \ 0 \leq c_1 < \ldots < c_m \leq 1 \ (0 \leq n \leq N-1)\}.$$

If we admit sets X_h with $c_1 = 0$ and $c_m = 1$ $(m \geq 2)$, then the collocation solution lies again in the smoother space $S_m^{(0)}(I_h) \cap C^1(I) =: S_m^{(1)}(I_h)$, provided the given functions in (3.2.1) are continuous. Its dimension is dim $S_m^{(1)}(I_h) = N(m-1) + 2$, implying that we need a second ('artificial') initial condition, $u_h'(0) = y'(0) = f(0, y_0)$, in order to start the recursive process for solving (3.2.2).

We have already encountered the local (Lagrange) representation of $u_h \in S_m^{(0)}(I_h)$ on $\bar{\sigma}_n$ in Section 1.1.1: setting $Y_{n,j} := u_h'(t_n + c_j h_n)$, and

$$u_h'(t_n + vh_n) = \sum_{j=1}^{m} L_j(v)Y_{n,j}, \quad v \in (0, 1],$$

it is given by

$$u_h(t_n + vh_n) = y_n + h_n \sum_{j=1}^{m} \beta_j(v) Y_{n,j}, \quad v \in [0, 1], \tag{3.2.3}$$

where $y_n := u_h(t_n)$ and $\beta_j(v) := \int_0^v L_j(s)ds$. Since $(\mathcal{V}u_h)(t_{n,i})$ may be written as

$$(\mathcal{V}u_h)(t_{n,i}) = F_n(t_{n,i}) + h_n \int_0^{c_i} K(t_{n,i}, t_n + sh_n) u_h(t_n + sh_n) ds,$$

with lag term $F_n(t)$ as in (2.2.8) (see also (3.2.7) below), the computational form of the collocation equation (3.2.2) on $\bar{\sigma}_n$ becomes

$$Y_{n,i} = f(t_{n,i}, y_n + h_n \sum_{j=1}^{m} a_{i,j} Y_{n,j}) + h_n^2 \sum_{j=1}^{m} \left(\int_0^{c_i} K(t_{n,i}, t_n + sh_n) \beta_j(s) ds \right) Y_{n,j}$$
$$+ F_n(t_{n,i}) + h_n \left(\int_0^{c_i} K(t_{n,i}, t_n + sh_n) ds \right) y_n \quad (i = 1, \ldots, m). \tag{3.2.4}$$

Note that since u_h is continuous on I, the value y_n is given by

$$y_n = u_h(t_n) = u_h(t_{n-1} + h_{n-1}) = y_{n-1} + h_{n-1} \sum_{j=1}^{m} b_j Y_{n-1,j} \quad (n = 1, \ldots, N),$$

where $b_j := \beta_j(1)$. We note in passing that it will again occasionally be convenient to write

$$\int_0^{c_i} K(t_{n,i}, t_n + sh_n) \beta_j(s) ds = c_i \int_0^1 K(t_{n,i}, t_n + sc_i h_n) \beta_j(sc_i) ds$$

when $c_i < 1$, especially when deriving the fully discretised collocation equation (Section 3.2.2).

In the remainder of this section we will assume, for ease of exposition, that f in (3.2.1) is *linear*,

$$f(t, y) = a(t)y + g(t), \quad \text{with} \quad a, g \in C(I). \tag{3.2.5}$$

(Various nonlinear versions of (3.2.1) will be considered in Section 3.3.) The collocation equation corresponding to (3.2.4) then assumes the linear form

$$Y_{n,i} - h_n a(t_{n,i}) \sum_{j=1}^{m} a_{i,j} Y_{n,j} - h_n^2 \sum_{j=1}^{m} \left(\int_0^{c_i} K(t_{n,i}, t_n + sh_n) \beta_j(s) ds \right) Y_{n,j}$$
$$= g(t_{n,i}) + F_n(t_{n,i}) + \left(a(t_{n,i}) + h_n \int_0^{c_i} K(t_{n,i}, t_n + sh_n ds \right) y_n \tag{3.2.6}$$

$(i = 1, \dots, m)$, where the lag term,

$$F_n(t_{n,i}) := \int_0^{t_n} K(t_{n,i}, s)u_h(s)ds = \sum_{\ell=0}^{n-1} h_\ell \int_0^1 K(t_{n,i}, t_\ell + sh_\ell)u_h(t_\ell + sh_\ell)ds, \tag{3.2.7}$$

may be written as

$$F_n(t_{n,i}) = \sum_{\ell=0}^{n-1} h_\ell \int_0^1 K(t_{n,i}, t_\ell + sh_\ell)\{y_\ell + h_\ell \sum_{j=1}^{m} \beta_j(s) Y_{\ell,j}\}ds. \tag{3.2.8}$$

Let us introduce the vectors in $\mathbb{R}^m$,

$$\mathbf{Y}_n := (Y_{n,1}, \dots, Y_{n,m})^T, \quad \mathbf{a}_n := (a(t_{n,1}), \dots, a(t_{n,m}))^T$$
$$\mathbf{g}_n := (g(t_{n,1}), \dots, g(t_{n,m}))^T, \quad \mathbf{G}_n := (F_n(t_{n,1}), \dots, F_n(t_{n,m}))^T,$$

and the matrices in $L(\mathbb{R}^m)$,

$$A := \begin{pmatrix} a_{i,j} \\ (i, j = 1, \dots, m) \end{pmatrix}, \quad A_n := \text{diag}(a(t_{n,i}))A \ ,$$

$$C_n := \begin{pmatrix} \int_0^{c_i} K(t_{n,i}, t_n + sh_n)\beta_j(s)ds \\ (i, j = 1, \dots, m) \end{pmatrix}$$

$$C_n^{(\ell)} := \begin{pmatrix} \int_0^{1} K(t_{n,i}, t_\ell + sh_\ell)\beta_j(s)ds \\ (i, j = 1, \dots, m) \end{pmatrix} \quad (\ell < n).$$

with $a_{i,j} = \beta_j(c_i)$. Moreover, set

$$\boldsymbol{\kappa}_n := \mathbf{a}_n + h_n \left(\int_0^{c_i} K(t_{n,i}, t_n + sh_n)ds \quad (i = 1, \dots, m) \right)^T \in \mathbb{R}^m$$

and, for $0 \le \ell < n \le N - 1$,

$$\boldsymbol{\kappa}_n^{(\ell)} := \left(\int_0^{1} K(t_{n,i}, t_\ell + sh_\ell)ds \quad (i = 1, \dots, m) \right)^T \in \mathbb{R}^m.$$

The system of linear algebraic equations (3.2.6) then becomes

$$[\mathcal{I}_m - h_n(A_n + h_n C_n)]\mathbf{Y}_n = \mathbf{g}_n + \mathbf{G}_n + \boldsymbol{\kappa}_n y_n \quad (n = 0, 1, \dots, N-1), \tag{3.2.9}$$

where

$$\mathbf{G}_n := (F_n(t_{n,1}), \dots, F_n(t_{n,m}))^T = \sum_{\ell=0}^{n-1} h_\ell^2 C_n^{(\ell)} \mathbf{Y}_\ell + \sum_{\ell=0}^{n-1} h_\ell \boldsymbol{\kappa}_n^{(\ell)} y_\ell.$$

When the solution $\mathbf{Y}_n$ of (3.2.9) has been found, the collocation solution on the interval $\bar{\sigma}_n$ is determined by

$$u_h(t_n + vh_n) = y_n + h_n \boldsymbol{\beta}^T(v)\mathbf{Y}_n, \quad v \in [0,1], \tag{3.2.10}$$

where $\boldsymbol{\beta}(v) := (\beta_1(v), \ldots, \beta_m(v))^T \in \mathbb{R}^m$.

Theorem 3.2.1 *Assume that the functions a, g and K in the VIDE (3.2.1), with f given by (3.2.5), are continuous on their respective domains I and D. Then there exists an $\bar{h} > 0$ so that for any mesh I_h with mesh diameter $h > 0$ satisfying $h < \bar{h}$, each of the linear algebraic systems (3.2.9) has a unique solution $\mathbf{Y}_n \in \mathbb{R}^m$. Hence the collocation equation (3.2.2) defines a unique collocation solution $u_h \in S_m^{(0)}(I_h)$ for the initial-value problem (3.2.1), (3.2.5), and its representation on the subinterval $\bar{\sigma}_n$ is given by (3.2.10).*

Proof It follows from the assumptions on a and K that the matrices

$$\mathcal{C}_n := A_n + h_n C_n \in L(\mathbb{R}^m) \quad (0 \le n \le N-1)$$

have bounded elements for any mesh I_h. Thus, the argument in the proof of Theorem 2.2.1 can again be used to deduce that the inverses $(\mathcal{I}_m - h_n\mathcal{C}_n)^{-1}$ exist and are bounded whenever $h_n \in (0, \bar{h})$, for some sufficiently small $\bar{h} > 0$, implying that each of the systems $(\mathcal{I}_m - h_n\mathcal{C}_n)\mathbf{Y}_n = \mathbf{g}_n + \mathbf{G}_n + \boldsymbol{\kappa}_n y_n$ is uniquely solvable in $\mathbb{R}^m$ when $h = \max_{(n)} h_n < \bar{h}$. This proves Theorem 3.2.1. For $\mathcal{V} = 0$ (i.e. $C_n = 0$ for all n) we obtain the uniqueness of the collocation solution $u_h \in S_m^{(0)}(I_h)$ for ODEs.

Example 3.2.1 $u_h \in S_1^{(0)}(I_h)$ $(m = 1)$, $\quad 0 < c_1 =: \theta \le 1,\ t_{n,1} = t_n + \theta h_n$: Here we have, as in Example 1.1.1, $\beta_1(v) = v,\ A = a_{1,1} = \theta$, and

$$u_h(t_n + vh_n) = (1 - v)y_n + vy_{n+1}, \quad v \in [0,1], \quad y_n = u_h(t_n) \tag{3.2.11}$$

(since $u_h(t_n + vh_n) = y_n + vh_nY_{n,1}$ yields, for $v = 1,\ h_nY_{n,1} = y_{n+1} - y_n$). It thus follows from (3.2.6) that y_{n+1} is given by the solution of the linear algebraic equation

$$\begin{aligned}\left(1 - \theta h_n a(t_{n,1}) - h_n^2 \int_0^\theta K(t_{n,1}, t_n + sh_n)s\,ds\right) y_{n+1} \\ = h_n(g(t_{n,1}) + F_n(t_{n,1})) + \Big(1 + (1-\theta)h_n a(t_{n,1}) \\ + h_n^2 \int_0^\theta K(t_{n,1}, t_n + sh_n)(1-s)ds\Big) y_n,\end{aligned} \tag{3.2.12}$$

with lag term

$$F_n(t_{n,1}) = \sum_{\ell=0}^{n-1} h_\ell \int_0^1 K(t_{n,1}, t_\ell + sh_\ell)[(1-s)y_\ell + sy_{\ell+1}]ds. \qquad (3.2.13)$$

The collocation method defined by (3.2.11)–(3.2.13) will be referred to as the *(exact) continuous θ-method* for the linear VIDE (3.2.1),(3.2.5). Its *nonlinear* counterpart is given by (3.2.11) and by

$$\begin{aligned} y_{n+1} &= y_n + h_n f(t_{n,1}, (1-\theta)y_n + \theta y_{n+1}) + F_n(t_{n,1}) \\ &\quad + h_n^2 \theta \int_0^1 k(t_{n,1}, t_n + s\theta h_n, (1-s\theta)y_n + s\theta y_{n+1})ds, \end{aligned}$$

with lag term

$$F_n(t_{n,1}) := \sum_{\ell=0}^{n-1} h_\ell \int_0^1 k(t_{n,1}, t_\ell + sh_\ell, (1-s)y_\ell + sy_{\ell+1})ds.$$

Example 3.2.2 $u_h \in S_2^{(0)}(I_h)$ $(m = 2)$, $\quad 0 < c_1 < c_2 \le 1$:
Here,

$$\beta_1(v) = \int_0^v L_1(s)ds = \frac{v(2c_2 - v)}{2(c_2 - c_1)},$$

$$\beta_2(v) = \int_0^v L_2)ds = \frac{v(v - 2c_1)}{2(c_2 - c_1)},$$

which permits the computation of the elements of the matrix $A \in L(\mathbb{R}^2)$, $a_{i,j} = \beta_j(c_i)$ $(i, j = 1, 2)$ (compare also Example 1.1.2). The elements of the matrix $C_n \in L(\mathbb{R}^2)$ in (3.2.9) are

$$(C_n)_{i,1} = \frac{1}{2(c_2 - c_1)} \int_0^{c_i} K(t_{n,i}, t_n + sh_n)s(2c_2 - s)ds \quad (i = 1, 2),$$

and

$$(C_n)_{i,2} = \frac{1}{2(c_2 - c_1)} \int_0^{c_i} K(t_{n,i}, t_n + sh_n)s(s - 2c_1)ds \quad (i = 1, 2).$$

See also Example 1.1.2 (and Brunner (1984b)) for further details, including collocation at the Gauss and Radau II points.

3.2.2 The fully discretised collocation equations

The (exact) collocation equation (3.2.6) for the linear VIDE (3.2.1),(3.2.5) can only be used for the numerical computation of u_h if the integrals in the equation (and the lag term (3.2.7)) can be found analytically. Since this will in general not be possible (compare, however, Section 3.3) these integrals will have to be approximated by feasible numerical quadrature processes which, as in Section 2.2.1, will be given, or are based on, interpolatory m-point quadrature formulas whose abscissas are determined by the collocation parameters $\{c_i\}$. Hence, using the notation introduced in (2.2.19), (2.2.20), the fully discretised version of (3.2.6) assumes the form

$$\hat{Y}_{n,i} - h_n a(t_{n,i}) \sum_{j=1}^{m} a_{i,j} \hat{Y}_{n,j} - h_n^2 (\hat{Q}_n \hat{u}_h)(t_{n,i})$$
$$= g(t_{n,i}) + \hat{F}_n(t_{n,i}) + [a(t_{n,i}) + h_n \sum_{j=1}^{m} c_i b_j K(t_{n,i}, t_n + c_i c_j h_n)]\hat{y}_n$$
$$(i = 1, \ldots, m). \tag{3.2.14}$$

The discretised lag term is

$$\hat{F}_n(t_{n,i}) := \sum_{\ell=0}^{n-1} h_\ell (\hat{Q}_n^{(\ell)} \hat{u}_h)(t_{n,i}), \tag{3.2.15}$$

with $\hat{u}_h(t_\ell + s h_\ell) = \hat{y}_\ell + h_\ell \sum_{j=1}^{m} \beta_j(s) \hat{Y}_{\ell,j}$. We recall for convenience that the quadrature approximations introduced in Section 2.2.1 are defined by

$$(\hat{Q}_n \hat{u}_h)(t_{n,i}) := c_i \sum_{j=1}^{m} b_j K(t_{n,i}, t_n + c_i c_j h_n) \hat{u}_h(t_n + c_i c_j h_n)$$
$$= c_i \sum_{j=1}^{m} b_j K(t_{n,i}, t_n + c_i c_j h_n) \hat{y}_n$$
$$+ c_i h_n \sum_{j=1}^{m} \left(\sum_{k=1}^{m} b_k K(t_{n,i}, t_n + c_i c_k h_n) \beta_j(c_i c_k) \right) \hat{Y}_{n,j}, \tag{3.2.16}$$

and, for $\ell < n$, by

$$(\hat{Q}_n^{(\ell)} \hat{u}_h)(t_{n,i}) := \sum_{\ell=0}^{n-1} \sum_{j=1}^{m} b_j K(t_{n,i}, t_\ell + c_j h_\ell) \hat{u}_h(t_\ell + c_j h_\ell)$$
$$= \sum_{\ell=0}^{n-1} \sum_{j=1}^{m} b_j K(t_{n,i}, t_\ell + c_j h_\ell) \hat{y}_\ell$$
$$+ \sum_{\ell=0}^{n-1} h_\ell \sum_{j=1}^{m} \left(\sum_{k=1}^{m} b_k K(t_{n,i}, t_\ell + c_k h_\ell) \beta_j(c_k) \right) \hat{Y}_{\ell,j} \quad (\ell < n). \tag{3.2.17}$$

Here, $b_j = \beta_j(1)$ and $\beta_j(c_k) = a_{k,j}$. The solution $\hat{\mathbf{Y}}_n := (\hat{Y}_{n,1}, \ldots, \hat{Y}_{n,m})^T \in \mathbb{R}^m$ of the linear algebraic system (3.2.14) determines the discretised collocation solution on the subinterval $\bar{\sigma}_n$:

$$\hat{u}_h(t_n + vh_n) = \hat{y}_n + h_n \sum_{j=1}^{m} \beta_j(v)\hat{Y}_{n,j}, \quad v \in [0, 1], \tag{3.2.18}$$

with

$$\hat{y}_n := \hat{u}_h(t_n) = \hat{y}_{n-1} + h_{n-1} \sum_{j=1}^{m} b_j \hat{Y}_{n-1,j}.$$

In order to state and prove the result on the existence and uniqueness of the discretised collocation solution on I, we write (3.2.14) in a more concise form representing the discrete analogue of (3.2.9), namely

$$[\mathcal{I}_m - h_n(A_n + h_n\hat{C}_n)]\hat{\mathbf{Y}}_n = \mathbf{g}_n + \hat{\mathbf{G}}_n + \hat{\boldsymbol{\kappa}}_n\hat{y}_n \quad (n = 0, 1, \ldots, N-1), \tag{3.2.19}$$

with

$$\hat{\mathbf{G}}_n := (\hat{F}_n(t_{n,1}), \ldots, \hat{F}_n(t_{n,m}))^T = \sum_{\ell=0}^{n-1} h_\ell \hat{\boldsymbol{\kappa}}_n^{(\ell)} \hat{y}_\ell + \sum_{\ell=0}^{n-1} h_\ell \hat{C}_n^{(\ell)} \hat{\mathbf{Y}}_\ell.$$

Here,

$$\hat{C}_n := \begin{pmatrix} c_i \sum_{k=1}^{m} b_k K(t_{n,i}, t_n + c_i c_k h_n)\beta_j(c_i c_k) \\ (i, j = 1, \ldots, m) \end{pmatrix}$$

(cf. (3.2.16)) and

$$\hat{C}_n^{(\ell)} := \begin{pmatrix} \sum_{k=1}^{m} b_k K(t_{n,i}, t_\ell + c_k h_\ell)\beta_j(c_k) \\ (i, j = 1, \ldots, m) \end{pmatrix} \quad (\ell < n)$$

are the discretised versions of the matrices C_n and $C_n^{(\ell)}$ in (3.2.9), while

$$\hat{\boldsymbol{\kappa}}_n := \mathbf{a}_n + h_n \left(c_i \sum_{k=1}^{m} b_k K(t_{n,i}, t_n + c_i c_k h_n) \;\; (i = 1, \ldots, m) \right)^T, \tag{3.2.20}$$

and

$$\hat{\boldsymbol{\kappa}}_n^{(\ell)} := \left(\sum_{k=1}^{m} b_k K(t_{n,i}, t_\ell + c_k h_\ell) \;\; (i = 1, \ldots, m) \right)^T \quad (\ell < n). \tag{3.2.21}$$

We rewrite the local representation (3.2.18) of the discretised collocation solution $\hat{u}_h$ on $\bar{\sigma}_n$ as

$$\hat{u}_h(t_n + vh_n) = \hat{y}_n + h_n \boldsymbol{\beta}^T(v)\hat{\mathbf{Y}}_n, \quad v \in [0, 1], \tag{3.2.22}$$

with $\boldsymbol{\beta}(v) := (\beta_1(v), \ldots, \beta_m(v))^T$.

Theorem 3.2.2 *Assume that the given functions a, g and K in the linear VIDE (3.2.1), (3.2.5) satisfy the conditions of Theorem 3.2.1. If the exact collocation equation (3.2.6) is discretised by interpolatory m-point quadrature formulas based on the collocation parameters $\{c_i\}$ and given by (3.2.16) and (3.2.17), then there exists an $\hat{h} > 0$ so that for any mesh I_h with mesh diameter $h \in (0, \hat{h})$, each of the linear systems (3.2.19) has a unique solution $\hat{\mathbf{Y}}_n \in \mathbb{R}^m$. Hence the discretised collocation equation (3.2.14) defines a unique collocation solution $\hat{u}_h \in S_m^{(0)}(I_h)$ which on $\bar{\sigma}_n$ is given by (3.2.22).*

The **proof** is a straightforward adaptation of the proof for Theorem 3.2.1: for fixed $m \geq 1$ the weights of the above interpolatory m-point quadrature formulas are bounded for all $h > 0$, and hence, by the assumed continuity of a and K, the matrices $\hat{C}_n \in L(\mathbb{R}^m)$ have bounded elements for any h_n. This implies that the inverses of the matrices characterising the systems (3.2.18), $\hat{\mathcal{C}}_n := \mathcal{I}_m - h_n(A_n + h_n\hat{C}_n)$ $(n = 0, 1, \ldots, N-1)$, exist and are uniformly bounded for $h_n \in (0, \hat{h})$ for some $\hat{h} > 0$ which will in general be different from $\bar{h}$ defined in Theorem 3.2.1.

Example 3.2.3 m $= 1$ (*discretised θ-method*): It follows from Example 3.2.1 that this method is given by

$$\hat{u}_h(t_n + vh_n) = \hat{y}_n + (1 - v)\hat{y}_n + v\hat{y}_{n+1}, \quad v \in [0, 1],$$

and

$$[1 - \theta h_n a(t_{n,1}) - \theta^3 h_n^2 K(t_{n,1}, t_n + \theta^2 h_n)]\hat{y}_{n+1} = h_n[g(t_{n,1}) + \hat{F}_n(t_{n,1})]$$
$$+ [1 + (1-\theta)h_n a(t_{n,1}) + \theta(1-\theta^2)h_n^2 K(t_{n,1}, t_n + \theta^2 h_n)]\hat{y}_n,$$

with

$$\hat{F}_n(t_{n,1}) := \sum_{\ell=0}^{n-1} h_\ell K(t_{n,1}, t_\ell + \theta h_\ell)[(1-\theta)\hat{y}_\ell + \theta\hat{y}_{\ell+1}].$$

For the nonlinear VIDE we have

$$\hat{y}_{n+1} = \hat{y}_n + h_n f(t_{n,1}, (1-\theta)\hat{y}_n + \theta\hat{y}_{n+1}) + h_n\hat{F}_n(t_{n,1})$$
$$+ \theta^2 h_n^2 k(t_{n,1}, t_n + \theta^2 h_n, (1-\theta^2)\hat{y}_n + \theta^2\hat{y}_{n+1}),$$

with discretised lag term

$$\hat{F}_n(t_{n,1}) := \sum_{\ell=0}^{n-1} h_\ell k(t_{n,1}, t_\ell + \theta h_\ell, (1-\theta)\hat{y}_\ell + \theta\hat{y}_{\ell+1}).$$

3.2.3 Global convergence results

We start by deriving global error estimates for the collocation solution $u_h \in S_m^{(0)}(I_h)$ to the linear VIDE

$$y'(t) = a(t)y(t) + g(t) + (\mathcal{V}y)(t), \quad t \in I, \quad y(0) = y_0, \tag{3.2.23}$$

where

$$(\mathcal{V}y)(t) := \int_0^t K(t,s)y(s)ds.$$

Theorem 3.2.3 *Assume:*

(a) The given functions in (3.2.23) satisfy $a, g \in C^m(I), \; K \in C^m(D)$.
(b) $u_h \in S_m^{(0)}(I_h)$ *is the collocation solution to (3.2.23) defined by (3.2.3), (3.2.4) with* $h \in (0, \bar{h})$.

Then the estimates

$$||y^{(\nu)} - u_h^{(\nu)}||_\infty \le C_\nu ||y^{(m+1)}||_\infty h^m \quad (\nu = 0, 1) \tag{3.2.24}$$

hold for any set X_h *of collocation points with* $0 \le c_1 < \ldots < c_m \le 1$. *The constants* C_ν *depend on the collocation parameters* $\{c_i\}$ *and but not on* h.

Proof The collocation error $e_h := y - u_h$ satisfies the equation

$$e_h'(t) = a(t)e_h(t) + (\mathcal{V}e_h)(t), \quad t \in X_h, \tag{3.2.25}$$

with $e_h(0) = 0$. Recall now the analogous error equations for ODEs, (1.1.25), and for second-kind VIEs, (2.2.32), as well as the local representations (1.1.22) and (1.1.23) for e_h and e_h'; they are, respectively,

$$e_h(t_n + vh_n) = e_h(t_n) + h_n \sum_{j=1}^m \beta_j(v)\mathcal{E}_{n,j} + h_n^{m+1} R_{m+1,n}(v), \tag{3.2.26}$$

$$e_h'(t_n + vh_n) = \sum_{j=1}^m L_j(v)\mathcal{E}_{n,j} + h_n^m R_{m+1,n}^{(1)}(v), \quad v \in (0,1], \tag{3.2.27}$$

with $\mathcal{E}_{n,j} := Z_{n,j} - Y_{n,j}$. These representations are based on the fact that, by assumption (a), the solution y of (3.2.23) is in $C^{m+1}(I)$. Hence, not surprisingly if we recall the proofs of Theorem 1.1.2 and Theorem 2.2.3, all the essential ingredients for proving Theorem 3.2.3 are in place, and therefore we will just focus on the main steps, leaving most of the details to the reader. Consider first the expression for $(\mathcal{V}e_h)(t_{n,i})$ in (2.2.32): the only changes necessary to adapt it to the present situation consist in replacing the Lagrange polynomials $L_j(s)$ by their integrals $\beta_j(s)$, and the Peano remainder terms $R_{m,\ell}(s)$ by $R_{m+1,\ell}(s)$ $(\ell \le n)$.

Secondly, since e_h is continuous at the mesh points $t = t_n$ we have again the recurrence relation (1.1.27),

$$e_h(t_n) = \sum_{\ell=0}^{n-1} h_\ell \sum_{j=1}^{m} b_j \mathcal{E}_{\ell,j} + \sum_{\ell=0}^{n-1} h_\ell^{m+1} R_{m+1,\ell}(1) \quad (n = 1, \ldots, N-1). \tag{3.2.28}$$

Consider now the error equation (3.2.25) at $t = t_{n,i} = t_n + c_i h_n$. Observe first that, by (3.2.27), its left-hand side reduces to

$$e_h'(t_{n,i}) = \mathcal{E}_{n,i} + h_n^m R_{m+1,n}^{(1)}(c_i).$$

The contribution of the first term on its right-hand side is known from Section 1.1.2; it reads

$$a(t_{n,i})[e_h(t_n) + h_n \sum_{j=1}^{m} a_{i,j} \mathcal{E}_{n,j} + h_n^{m+1} R_{m+1,n}(c_i)].$$

By (3.2.26) the explicit expression for the Volterra term $(\mathcal{V}e_h)(t)$ at $t = t_{n,i}$ is

$$\begin{aligned}
(\mathcal{V}e_h)(t) = & \sum_{\ell=0}^{n-1} h_\ell \left(\int_0^1 K(t, t_\ell + s h_\ell) ds \right) e_h(t_\ell) \\
& + h_n \left(\int_0^{c_i} K(t, t_n + s h_n) ds \right) e_h(t_n) \\
& + \sum_{\ell=0}^{n-1} h_\ell^2 \sum_{j=1}^{m} \left(\int_0^1 K(t_{n,i}, t_\ell + h_\ell) \beta_j(s) ds \right) \mathcal{E}_{\ell,j} \\
& + h_n^2 \sum_{j=1}^{m} \left(\int_0^{c_i} K(t, t_n + s h_n) \beta_j(s) ds \right) \mathcal{E}_{n,j} \\
& + \sum_{\ell=0}^{n-1} h_\ell^{m+2} \int_0^1 K(t, t_\ell + s h_\ell) R_{m+1,\ell}(s) ds \\
& + h_n^{m+2} \int_0^{c_i} K(t_{n,i}, t_n + s h_n) R_{m+1,n}(s) ds.
\end{aligned}$$

Therefore, letting $t = t_{n,i} \in X_h$, (3.2.25) can be written as

$$\begin{aligned}
\mathcal{E}_{n,i} = & \; h_n a(t_{n,i}) \sum_{j=1}^{m} a_{i,j} \mathcal{E}_{n,j} + h_n^2 \sum_{j=1}^{m} \left(\int_0^{c_i} K(t_{n,i}, t_n + s h_n) \beta_j(s) ds \right) \mathcal{E}_{n,j} \\
& + \sum_{\ell=0}^{n-1} h_\ell^2 \sum_{j=1}^{m} \left(\int_0^1 K(t_{n,i}, t_\ell + s h_\ell) \beta_j(s) ds \right) \mathcal{E}_{\ell,j} \\
& + \sum_{\ell=0}^{n-1} h_\ell \kappa_{n,i}^{(\ell)} e_h(t_\ell) + \kappa_{n,i} e_h(t_n) + \sum_{\ell=0}^{n-1} h_\ell^{m+2} \rho_{n,i}^{(\ell)} + h_n^m \rho_{n,i}.
\end{aligned} \tag{3.2.29}$$

Here, the components of the vectors κ_n and $\kappa_n^{(\ell)}$ $(\ell < n)$, introduced in Section 3.2.1 (preceding Theorem 3.2.1), are given by

$$\kappa_{n,i}^{(\ell)} := \int_0^1 K(t_{n,i}, t_\ell + sh_\ell)ds \quad (\ell < n),$$

$$\kappa_{n,i} := a(t_{n,i}) + h_n \int_0^{c_i} K(t_{n,i}, t_n + sh_n)ds,$$

and we have set (in analogy to the proof of Theorem 2.2.3)

$$\rho_{n,i} := h_n[a(t_{n,i})R_{m+1,n}(c_i) + h_n \int_0^{c_i} K(t_{n,i}, t_n + sh_n)R_{m+1,n}(s)ds] - R^{(1)}_{m+1,n}(c_i)\,,$$

$$\rho_{n,i}^{(\ell)} := \int_0^1 K(t_{n,i}, t_\ell + sh_\ell)R_{m+1,\ell}(s)ds \quad (\ell < n).$$

It follows that $\boldsymbol{\mathcal{E}}_n := (\,\mathcal{E}_{n,1}, \ldots, \mathcal{E}_{n,m}\,)^T$ is given by the unique solution of the linear algebraic system

$$\begin{aligned}
&[\mathcal{I}_m - h_n(A_n + h_n C_n)]\boldsymbol{\mathcal{E}}_n \\
&= \sum_{\ell=0}^{n-1} h_\ell^2 C_n^{(\ell)}\boldsymbol{\mathcal{E}}_\ell + \sum_{\ell=0}^{n-1} h_\ell \boldsymbol{\kappa}_n^{(\ell)} e_h(t_\ell) + \boldsymbol{\kappa}_n e_h(t_n) \\
&\quad + \sum_{\ell=0}^{n-1} h_\ell^{m+2}\boldsymbol{\rho}_n^{(\ell)} + h_n^m \boldsymbol{\rho}_n \quad (n = 0, 1, \ldots, N-1). \qquad (3.2.30)
\end{aligned}$$

Its left-hand side matrix, $\mathcal{I}_m - h_n\mathcal{C}_n$, of course coincides with the one in (3.2.9). Also, the nodal errors $e_h(t_\ell)$ $(\ell \le n)$ can be expressed in terms of the components of $\boldsymbol{\mathcal{E}}_\ell$, as shown by (3.2.28).

According to Theorem 3.2.1 this linear system has a unique solution whenever $h_n \in (0, \bar{h})$, and hence there exists a constant $D_0 < \infty$ so that $||(\mathcal{I}_m - h_n\mathcal{B}_n)^{-1}||_1 \le D_0$ uniformly for $0 \le n \le N-1$. Equation (3.2.30) now leads to the estimate

$$\begin{aligned}
||\boldsymbol{\mathcal{E}}_n||_1 \le D_0\,[&\sum_{\ell=0}^{n-1} h_\ell^2 ||\boldsymbol{\mathcal{E}}_\ell||_1 + m\bar{K}\sum_{\ell=0}^{n-1} h_\ell |e_h(t_n)| + m(A_0 + h\bar{K})|e_h(t_n)| \\
&+ h^{m+1} m\bar{K} k_m M_{m+1} \sum_{\ell=0}^{n-1} h_\ell + h^{m+1} m(A_0 + h\bar{K})k_m M_{m+1}]. \qquad (3.2.31)
\end{aligned}$$

Here we have used the notation introduced in Sections 1.1.2 and 2.2.2. It follows from the continuity relation (3.2.28) that

$$|e_h(t_n)| \le \bar{b}\sum_{\ell=0}^{n-1} h_\ell ||\boldsymbol{\mathcal{E}}_\ell||_1 + h^m k_m M_{m+1} T \quad (n = 1, \ldots, N-1). \qquad (3.2.32)$$

Moreover, bounds for the error terms $\boldsymbol{\rho}_n^{(\ell)}$ ($\ell < n$) and ρ_n are readily found from the definition of their components given above; they are, respectively,

$$||\boldsymbol{\rho}_n^{(\ell)}||_1 \le m\bar{K}k_m M_{m+1} \quad (\ell < n) \quad \text{and} \quad ||\boldsymbol{\rho}_n||_1 \le m(A_0 + h\bar{K})k_m M_{m+1}.$$

Thus, observing that sums of the form $\sum_{\ell=0}^{n-1} h_\ell$ are bounded by T uniformly for $0 \le n \le N-1$, the above inequality for $||\boldsymbol{\mathcal{E}}_n||_1$ reduces to a generalized discrete Gronwall inequality,

$$||\boldsymbol{\mathcal{E}}_n||_1 \le \gamma_0 \sum_{\ell=0}^{n-1} h_\ell ||\boldsymbol{\mathcal{E}}_\ell||_1 + \gamma_1 M_{m+1} h^m, \quad 0 \le n \le N-1.$$

Hence, as in the proofs of Theorems 1.1.2 and 2.2.3, this leads to the uniform estimate

$$||\boldsymbol{\mathcal{E}}_n||_1 \le \gamma_1 M_{m+1} h^m \exp(\gamma_0 T) =: B M_{m+1} h^m, \quad 0 \le n \le N-1,$$

and so (3.2.32) yields

$$|e_h(t_n)| \le (\bar{b}B + k_m) T M_{m+1} h^m \quad (1 \le n \le N-1).$$

Recalling the error representations (3.2.26) and (3.2.27) and employing by now familiar notation, we find the estimates

$$\begin{aligned} |e_h(t_n + vh_n)| &\le |e_h(t_n)| + h\bar{\beta}||\boldsymbol{\mathcal{E}}_n||_1 + h^{m+1} k_m M_{m+1} \\ &\le [(\bar{b}B + k_m)T + h(\bar{\beta}B + hk_m] M_{m+1} h^m =: C_0 M_{m+1} h^m \end{aligned}$$

and

$$\begin{aligned} |e_h'(t_n + vh_n)| &\le \Lambda_m ||\boldsymbol{\mathcal{E}}_n||_1 + h^m k_m M_{m+1} \\ &\le (\Lambda_m B + k_m) M_{m+1} h^m =: C_1 M_{m+1} h^m, \end{aligned}$$

uniformly for $v \in [0, 1]$ and $0 \le n \le N-1$ ($h \in (0, \bar{h})$). Since the constants C_0 and C_1 depend (via the bound B for $||\boldsymbol{\mathcal{E}}_n||_1$ and the bound M_{m+1}) on $||y^{(m+1)}||_\infty$, this concludes the proof of Theorem 3.2.3.

As we know from the error analyses in the two preceding chapters, less than full regularity ($y \in C^d(I)$ with $d < m+1$) will imply a lower order of global convergence for u_h. The above proof, with appropriate form of Peano's Theorem, is thus readily modified to furnish the following convergence result.

Theorem 3.2.4 *Suppose that the regularity assumption (a) in Theorem 3.2.3 is replaced by:*

(a′) The given functions in (3.2.3) satisfy $a,\ g \in C^d(I)$ and $K \in C^d(D)$, for some d with $1 \le d < m$.

If (b) of Theorem 3.2.3 holds, then the estimates

$$||y^{(\nu)} - u_h^{(\nu)}||_\infty \leq C_\nu ||y^{(d+1)}||_\infty h^d \quad (\nu = 0, 1)$$

are best possible and hold for any set X_h of collocation points with $0 \leq c_1 < \ldots < c_m \leq 1$.

On the other hand, a judicious choice of the $\{c_i\}$ and a higher degree of regularity for y than in Theorem 3.2.3 will lead to *global superconvergence on I*, in complete analogy to Theorem 1.1.3 for ODEs.

Theorem 3.2.5 *Assume that the given functions in the linear VIDE (3.2.23) satisfy $a, g \in C^{m+1}(I)$ and $K \in C^{m+1}(D)$, and let $u_h \in S_m^{(0)}(I_h)$ be the collocation solution to (3.2.23) corresponding to the collocation points X_h. If the m collocation parameters $\{c_i\}$ defining X_h are chosen so that the orthogonality condition*

$$J_0 := \int_0^1 \prod_{i=1}^m (s - c_i) ds = 0$$

holds (cf. (1.1.34)), then for all meshes I_h with mesh diameter $h \in (0, \bar{h})$ the global order of the collocation solution u_h on I_h exceeds m:

$$||y - u_h||_\infty \leq Ch^{m+1}, \tag{3.2.33}$$

with C depending on the $\{c_i\}$ and on $||y^{(m+2)}||_\infty$ but not on h.

Proof Starting with the defect δ_h induced by u_h,

$$\delta_h(t) := -u_h'(t) + f(t, u_h(t)) + (\mathcal{V}u_h)(t), \quad t \in I,$$

with $f(t, y) = a(t)y + g(t)$, we have

$$\delta_h(t) = e_h'(t) - a(t)e_h(t) - (\mathcal{V}e_h)(t), \quad t \in I,$$

and thus, by the estimates in Theorem 3.2.3,

$$||\delta_h||_\infty \leq C_1 h^m + A_0 C_0 h^m + \bar{K} C_0 h^m =: Dh^m,$$

with $A_0 := ||a||_\infty$ and $\bar{K} := ||\mathcal{V}||_\infty$. Hence, since the collocation error is the solution of the initial-value problem

$$e_h'(t) = a(t)e_h(t) + \delta_h(t) + (\mathcal{V}e_h)(t), \quad t \in I, \quad e_h(0) = 0, \tag{3.2.34}$$

it may be written in the form

$$e_h(t) = r(t, 0)e_h(0) + \int_0^t r(t, s)\delta_h(s)ds = \int_0^t r(t, s)\delta_h(s)ds, \quad t \in I, \tag{3.2.35}$$

(cf. Theorem 3.1.1). Except for the definition of the resolven kernel $r(t, s)$, which is now given by the solution of the resolvent equations (3.1.8) or (3.1.9), the above error representation is formally identical with (1.1.38) for linear ODEs. Thus, taking into account the regularity of $r(t, s)$ and of the defect δ_h in the subintervals σ_n, we are able to complete the proof of Theorem 3.2.3 exactly along the lines of the one for Theorem 1.1.3 (or Theorem 2.2.5), to arrive at the desired global estimate (3.2.33).

3.2.4 Local superconvergence results

When we proved the result on *global* superconvergence for VIDEs (Theorem 3.2.5) we pointed out that the formal analysis was identical with the one we used in Section 1.1.3 for establishing such results for ODEs. In addition, we saw that the key to establishing local superconvergence results for u_h and u_h' on I_h was the resolvent representation of the collocation error e_h and its derivative e_h' in terms of the defect δ_h, namely

$$e_h(t) = \int_0^t r(t, s)\delta_h(s)ds, \quad t \in I \tag{3.2.36}$$

and

$$e_h'(t) = r(t, t)\delta_h(t) + \int_0^t \frac{\partial r(t, s)}{\partial t}\delta_h(s)ds, \quad t \in I, \tag{3.2.37}$$

where $r(t, t) = 1$ on I. For linear VIDEs an analogous (and formally identical) resolvent representation holds (Theorem 3.1.1), with the differential resolvent $r(t, s)$ of the linear VIDE (3.2.23) defined by

$$\frac{\partial r(t, s)}{\partial s} = -r(t, s)a(s) - \int_s^t r(t, v)K(v, s)dv, \quad (s, t) \in D \tag{3.2.38}$$

(cf. (3.1.8)), again with $r(t, t) = 1$ for $t \in I$.

Thus, in view of these observations it is not surprising that Theorem 1.1.4 and its corollaries remain valid for first-order VIDEs.

Theorem 3.2.6 *Assume:*

(a) The given functions in (3.2.23) satisfy $a, g \in C^{m+\kappa}(I)$ and $K \in C^{m+\kappa}(D)$, for some integer $\kappa \geq 1$ specified in (c) below.

(b) $u \in S_m^{(0)}(I_h)$ is the collocation solution to (3.2.23) with respect to the collocation points X_h.

(c) The parameters $\{c_i\}$ defining X_h are chosen so that the generalised orthogonality condition

$$J_\nu := \int_0^1 s^\nu \prod_{i=1}^{m}(s - c_i)ds = 0, \quad \nu = 0, \ldots, \kappa - 1, \tag{3.2.39}$$

with $J_\kappa \neq 0$ (and $\kappa \leq m$), is fulfilled.

Then, for $h \in (0, \bar{h})$, with $\bar{h} > 0$ defined in Theorem 3.2.1, the collocation error satisfies

$$\max_{t \in I_h} |e_h(t)| \leq C_0 h^{m+\kappa}, \tag{3.2.40}$$

If $c_m = 1$, then u_h' exhibits the same order of local superconvergence as u_h:

$$\max_{t \in I_h \setminus \{0\}} |e_h'(t)| \leq C h_1^{m+\kappa}, \tag{3.2.41}$$

while for $c_m < 1$ we only obtain $|e_h'(t_n)| = \mathcal{O}(h^m)$ $(1 \leq n \leq N)$. The constants C_0 and C_1 depend on the $\{c_i\}$ and on $||y^{(m+\kappa+1)}||_\infty$ but not on h.

Proof Consider (3.2.36) with $t = t_n$:

$$\begin{aligned} e_h(t_n) &= \int_0^{t_n} r(t_n, s)\delta_h(s)ds \\ &= \sum_{\ell=0}^{n-1} h_\ell \int_0^1 r(t_n, t_\ell + sh_\ell)\delta_h(t_\ell + sh_\ell)ds \quad (n = 1, \ldots, N). \end{aligned}$$

The assertion (3.2.40) now follows immediately along the lines of the proof of Theorem 1.1.4, by observing the regularity of the differential resolvent $r = r(t, s)$ (which is governed by the assumed regularity of a and K; i.e. $r \in C^{m+\kappa+1}(D)$) and the piecewise smoothness of the defect $\delta = \delta_h(t)$ on each subinterval σ_n (depending on the regularity of g).

If $c_m = 1$ we have $\delta_h(t_n) = 0$ (since now $t_n \in X_h$), and hence (3.2.37) yields

$$e_h'(t_n) = \int_0^{t_n} \frac{\partial r(t_n, s)}{\partial t} \delta_h(s)ds, \quad n = 1, \ldots, N.$$

Thus, the by now familiar quadrature argument carries over, with the role of $r(t_n, s)$ in the above expression for $e_h(t_n)$ assumed by $(\partial/\partial t)r(t_n, s)$.

Theorem 3.2.6 yields the following obvious corollaries generalising the results of Corollaries 1.1.5–1.1.7:

Corollary 3.2.7 *Let the $\{c_i\}$ be so that the orthogonality condition (3.2.39) holds with $\kappa = m$, that is, collocation is at the Gauss points. Then*

$$\max_{t \in I_h} |e_h(t)| \leq C h^{2m},$$

while $\max_{t \in I_h} |e_h'((t)| = \mathcal{O}(h^m)$ *only.*

Corollary 3.2.8 *If the* $\{c_i\}$ *are the Radau II points (corresponding to (3.2.39) with* $\kappa = m - 1$ *and* $c_m = 1$*), then*

$$\max_{t \in I_h \setminus \{0\}} |e_h^{(\nu)}(t)| \le C_\nu h^{2m-1} \quad (\nu = 0, 1).$$

Corollary 3.2.9 *For the continuously differentiable collocation solution corresponding to the Lobatto points* $\{c_i\}$, $0 = c_1 < \ldots < c_m = 1$ *(*$\kappa = m - 2$*, with* $m \ge 2$ *in (3.2.39)), the attainable order of* $u_h \in S_m^{(1)}(I_h)$ *and* u_h' *is described by*

$$\max_{t \in I_h} |e_h^{(\nu)}(t)| \le C_\nu h^{2(m-1)} \quad (\nu = 0, 1).$$

Remark The above order estimates remain valid for the *discretised collocation solution* $\hat{u}_h$: the 'perturbation argument' of Section 2.2.6 can again be used to show that the order of $||u_h - \hat{u}_h||_\infty$, as well as the one for $|u_h(t_n) - \hat{u}_h(t_n)|$ $(n = 1, \ldots, N)$, match the orders of the exact collocation error on I and on I_h, respectively.

Theorem 3.2.10 *Suppose that the collocation equation defining the exact collocation solution* $u_h \in S_m^{(0)}(I_h)$ *for the VIDE (3.2.23) is discretised by interpolatory* m*-point quadrature formulas based on the collocation parameters* $\{c_i\}$*. Then the resulting discretised collocation solution* $\hat{u}_h \in S_m^{(0)}(I_h)$ *has the same global and local (super-) convergence properties as* u_h *itself.*

3.2.5 Neutral and higher-order VIDEs

In Section 3.1.2 we introduced the first-order VIDE (3.1.13),

$$y'(t) = f(t, y(t)) + \int_0^t k(t, s, y(s), y'(s))ds, \quad t \in I, \quad y(0) = y_0, \quad (3.2.42)$$

and its linear version,

$$y'(t) = a(t)y(t) + g(t) + (\mathcal{V}_0 y)(t) + (\mathcal{V}_1 y')(t), \quad (3.2.43)$$

with $\mathcal{V}_i : \; C(I) \to C(I)$ given by

$$(\mathcal{V}_0\phi)(t) := \int_0^t K_0(t, s)\phi(s)ds,$$

and

$$(\mathcal{V}_1\phi)(t) := \int_0^t K_1(t, s)\phi(s)ds,$$

as special cases of higher-order 'neutral' VIDEs. In this section we will derive the collocation equations and corresponding convergence results for the latter.

The proofs of the analogous theorems for kth-order ODEs follow then as special cases, as announced in Section 1.4.

Let $k \geq 2$ be a given integer and consider the initial-value problem

$$y^{(k)}(t) = f(t, y(t), y'(t), \ldots, y^{(k-1)}(t)) + (\mathcal{V}y)(t), \quad t \in I := [0, T], \quad (3.2.44)$$
$$y^{(\nu)}(0) = y_0^{(\nu)} \quad (\nu = 0, 1, \ldots, k-1),$$

where

$$(\mathcal{V}y)(t) := \int_0^t k(t, s, y(s), y'(s), \ldots, y^{(k)}(s))ds.$$

As before we will focus on its linear counterpart, described by

$$f(t, y, y', \ldots, y^{(k-1)}) = \sum_{\nu=0}^{k-1} a_\nu(t) y^{(\nu)}, \quad (3.2.45)$$

$$k(t, s, y, y', \ldots, y^{(k)}) = \sum_{\nu=0}^{k} K_\nu(t, s) y^{(\nu)}, \quad (3.2.46)$$

where the given functions a_ν and K_ν are assumed to be continuous on I and D, respectively.

We will seek the collocation solution for (3.2.44) in the smooth piecewise polynomial space

$$S_{m+d}^{(d)}(I_h) := \{v \in C^d(I) : \; v|_{\bar{\sigma}_n} \in \pi_{m+d} \; (0 \leq n \leq N-1)\}$$

with $d = k - 1 \geq 1$. We know that the dimension of this linear space is

$$\dim S_{m+d}^{(d)}(I_h) = Nm + d + 1 = Nm + k$$

(see Section 2.2.1). Let X_h, the set of collocation points in I, be as in Section 3.2.1. The collocation solution u_h in this space for (3.2.44) is thus defined by

$$u_h^{(k)}(t) = f(t, u_h(t), u_h'(t), \ldots, u_h^{(k-1)}(t)) + (\mathcal{V}u_h)(t), \quad t \in X_h, \quad (3.2.47)$$
$$u_h^{(\nu)}(0) = y_0^{(\nu)} \quad (\nu = 0, 1, \ldots, k-1).$$

Setting $y_n^{(\nu)} := u_h^{(\nu)}(t_n)$ $(y_n := y_n^{(0)})$, $\; Y_{n,j} := u_h^{(k)}(t_{n,j})$, and

$$u_h^{(k)}(t_n + vh_n) = \sum_{j=1}^{m} L_j(v) Y_{n,j}, \quad v \in (0, 1],$$

the local Lagrange representation of $u_h^{(\nu)}$ ($\nu = k-1, \ldots, 0$) on $\bar{\sigma}_n$ is that of Section 1.4,

$$u_h^{(\nu)}(t_n + vh_n) = \sum_{\ell=0}^{k-\nu-1} \frac{y_n^{(\nu+\ell)}}{\ell!}(h_n v)^\ell + h_n^{k-\nu} \sum_{j=1}^{m} \beta_{\nu,j}(v) Y_{n,j}, \quad v \in [0,1], \tag{3.2.48}$$

where we defined

$$\beta_{\nu,j}(v) := \int_0^v \frac{(v-s)^{k-\nu-1}}{(k-\nu-1)!} L_j(s)ds. \tag{3.2.49}$$

For $\nu = 0$, (3.2.48) yields

$$u_h(t_n + vh_n) = \sum_{\ell=0}^{k-1} \frac{y_n^{(\ell)}}{\ell!}(h_n v)^\ell + h_n^k \sum_{j=1}^{m} \beta_{0.j}(v) Y_{n,j}, \quad v \in [0,1]. \tag{3.2.50}$$

This permits us to write down the computational form of the collocation equation (3.2.47) for $t = t_{n,i} \in \bar{\sigma}_n$. We will do this in detail only when $k = 2$, that is, for the VIDE

$$y''(t) = \sum_{\nu=0}^{1} a_\nu(t) y^{(\nu)}(t) + g(t) + \int_0^t \sum_{\nu=0}^{2} K_\nu(t,s) y^{(\nu)}(s)ds, \quad t \in I. \tag{3.2.51}$$

The general case is treated in Brunner (1988a, 1988b).

Illustration 3.2.1 *The continuous m-stage Volterra–Runge–Kutta–Nyström method:*
Consider (3.2.44) with $k = 2$. It follows from

$$Y_{n,i} = f(t_{n,i}, u_h(t_{n,i}), u_h'(t_{n,i})) + (\mathcal{V}u_h)(t_{n,i}), \quad i = 1, \ldots, m, \tag{3.2.52}$$

that the components of the vector $Y_n := (\,Y_{n,1}, \ldots, Y_{n,m}\,)^T$ are given by the solution of the nonlinear algebraic system

$$\begin{aligned} Y_{n,i} = f\Bigg(& t_{n,i}, y_n + h_n v y_n^{(1)} + h_n^2 \sum_{j=1}^{m} \beta_{0,j}(c_i) Y_{n,j}, y_n^{(1)} + h_n \sum_{j=1}^{m} \beta_{1,j}(v) Y_{n,j} \Bigg) \\ & + F_n(t_{n,i}) + h_n \int_0^{c_i} k\left(t_{n,i}, t_n + sh_n, u_h(t_n + sh_n), u_h'(t_n + sh_n),\right. \\ & \left. u_h''(t_n + sh_n)\right) ds \end{aligned} \tag{3.2.53}$$

($i = 1, \ldots, m$), with lag term approximation

$$F_n(t_n) := \int_0^{t_n} k\left(t_{n,i}, s, u_h(s), u_h'(s), u_h''(s)\right) ds. \tag{3.2.54}$$

Once the solution $\mathbf{Y}_n := (\ Y_{n,1}, \ldots, Y_{n,m}\ )^T$ has been computed, the values of u_h and u'_h on $\bar{\sigma}_n$ are determined by the interpolation formulas

$$u_h(t_n + vh_n) = y_n + h_n v y_n^{(1)} + h_n^2 \sum_{j=1}^{m} \beta_{0,j}(v) Y_{n,j}, \quad v \in [0, 1], \qquad (3.2.55)$$

and

$$u'_h(t_n + vh_n) = y_n^{(1)} + h_n \sum_{j=1}^{m} \beta_{1,j}(v) Y_{n,j}, \quad v \in [0, 1], \qquad (3.2.56)$$

where

$$\beta_{1,j}(v) := \int_0^v L_j(s)ds \quad \text{and} \quad \beta_{0,j}(v) := \int_0^v (v - s) L_j(s)ds.$$

For the linear version of this VIDE, corresponding to

$$f(t, y, y') = a_0(t)y + a_1(t)y' + g(t) \quad \text{and}$$
$$k(t, s, y, y', y'') = \sum_{\nu=0}^{2} K_\nu(t, s) y^{(\nu)}(s), \quad t \in I,$$

the linear algebraic system corresponding to (3.3.53) reduces to

$$[\mathcal{I}_m - h_n(\mathcal{A}_n + \mathcal{C}_n)]\mathbf{Y}_n = \mathbf{g}_n + \mathbf{G}_n + \boldsymbol{\kappa}_{n,0} y_n + \boldsymbol{\kappa}_{n,1} y_n^{(1)}, \qquad (3.2.57)$$

with

$$\mathcal{A}_n := A_{n,1} + h_n A_{n,0}, \qquad \mathcal{C}_n := C_{n,2} + h_n C_{n,1} + h_n^2 C_{n,0}.$$

The five matrices in $L(\mathbb{R}^m)$ determining $\mathcal{A}_n$ and $\mathcal{C}_n$ are defined by

$$A_{n,0} := \mathrm{diag}(a_0(t_{n,i})) \begin{pmatrix} \beta_{0,j}(c_i) \\ (i, j = 1, \ldots, m) \end{pmatrix},$$

$$A_{n,1} := \mathrm{diag}(a_1(t_{n,i})) \begin{pmatrix} \beta_{1,j}(c_i) \\ (i, j = 1, \ldots, m) \end{pmatrix},$$

$$C_{n,0} := \begin{pmatrix} \int_0^{c_i} K_0(t_{n,i}, t_n + sh_n) \beta_{0,j}(s)ds \\ (i, j = 1, \ldots, m) \end{pmatrix},$$

$$C_{n,1} := \begin{pmatrix} \int_0^{c_i} K_1(t_{n,i}, t_n + sh_n) \beta_{1,j}(s)ds \\ (i, j = 1, \ldots, m) \end{pmatrix},$$

$$C_{n,2} := \begin{pmatrix} \int_0^{c_i} K_2(t_{n,i}, t_n + sh_n) L_j(s)ds \\ (i, j = 1, \ldots, m) \end{pmatrix}.$$

The right-hand side vectors $\mathbf{g}_n$ and $\mathbf{G}_n$ are as before, and the terms reflecting the C^0 behaviour of the collocation solution u_h at the mesh points $t = t_n$ have the forms

$$\boldsymbol{\kappa}_{n,0} := \left((a_0(t_{n,i}) + h_n \int_0^{c_i} K_0(t_{n,i}, t_n + sh_n) ds \quad (i = 1, \dots, m) \right)^T$$

$$\boldsymbol{\kappa}_{n,1} := \left(a_1(t_{n,i}) + h_n c_i a_0(t_{n,i}) + h_n \int_0^{c_i} K_1(t_{n,i}, t_n + sh_n) ds \right.$$
$$\left. + h_n^2 \int_0^{c_i} K_0(t_{n,i}, t_n + sh_n) s \, ds \quad (i = 1, \dots, m) \right)^T .$$

The equations (3.2.55)–(3.2.57) describe the continuous implicit m-stage VRKN method for (3.2.44)–(3.2.46) with $k = 2$.

Example 3.2.3 $m = 1$ (see also Example 3.2.1)
Setting $\theta := c_1 \in (0, 1]$, $t_{n,1} := t_n + \theta h_n$, and observing that we have $\beta_{1,1}(v) = v$, $\beta_{0,1}(v) = v^2/2$, the continuous one-stage VRKN θ-method is described by the collocation equation

$$Y_{n,1} = f(t_{n,1}, u_h(t_{n,1}), u_h'(t_{n,1})) + F_n(t_{n,1})$$
$$+ h_n \int_0^{\theta} k(t_{n,1}, t_n + sh_n, u_h(t_n + sh_n), u_h'(t_n + sh_n), Y_{n,1}) ds.$$

Here,

$$Y_{n,1} := u_h''(t_n + vh_n) = \frac{1}{h_n}[y_{n+1}^{(1)} - y_n^{(1)}], \quad v \in (0, 1],$$

and this can be employed to express the local representations of u_h, u_h',

$$u_h(t_n + vh_n) = y_n + h_n v y_n^{(1)} + \frac{h_n^2}{2} v^2 Y_{n,1},$$
$$u_h'(t_n + vh_n) = y_n^{(1)} + h_n v Y_{n,1}, \quad v \in [0, 1],$$

in the forms

$$u_h(t_n + vh_n) = y_n + \frac{h_n v}{2} \left((1 - v) y_n^{(1)} + v y_{n+1}^{(1)} \right),$$
$$u_h'(t_n + vh_n) = (1 - v) y_n^{(1)} + v y_{n+1}^{(1)}, \quad v \in [0, 1].$$

It will be seen below (Theorem 3.2.12) that for $\theta = 1/2$ the order of (local) superconvergence on I_h is $p^* = 2m = 2$.

We now state the global and local (super-) convergence theorems for the collocation solution $u_h \in S^{(d)}_{m+d}(I_h)$ $(d = k-1)$ for (3.2.44)–(3.2.46),

$$y^{(k)}(t) = \sum_{\nu=0}^{k-1} a_\nu(t) y^{(\nu)}(t) + g(t) + \sum_{\nu=0}^{k} (\mathcal{V}_\nu y)(t), \quad t \in I, \tag{3.2.58}$$

where

$$(\mathcal{V}_\nu y)(t) := \int_0^t K_\nu(t,s) y^{(\nu)}(s) ds. \tag{3.2.59}$$

In analogy to the case $k = 1$ these results are readily extended to the nonlinear neutral VIDE (3.2.44); the proofs given below for the linear neutral VIDE suggest how to adapt the key ideas to the nonlinear problem.

Theorem 3.2.11 *Assume that the given functions a_ν, g and K_ν in the linear VIDE (3.2.44)–(3.2.46) are m times continuously differentiable on their respective domains I and D. Then for all sufficiently small $h > 0$ and any $\{c_i\}$ with $0 \le c_1 < \ldots < c_m \le 1$ we have the estimates*

$$||y^{(\nu)} - u_h^{(\nu)}||_\infty \le C_\nu h^m \quad (\nu = 0, 1, \ldots, k-1)$$

and

$$||y^{(k)} - u_h^{(k)}||_\infty := \sup_{t \in I_h \setminus \{0\}} |y^{(k)}(t) - u_h^{(k)}(t)| \le C_k h^m.$$

The constants C_ν depend on the $\{c_i\}$ but not on h.

Proof We leave the proof as an exercise: it is a straightforward (but notationwise somewhat tedious) generalisation of the proof of the global convergence result in Theorem 3.2.3 for first-order VIDEs.

As we saw in Section 1.4 when we studied the question of *global superconvergence* in collocation solutions for higher-order ODEs, a judicious choice of the collocation parameters $\{c_i\}$ leads to $\mathcal{O}(h^{m+1})$-convergence on I for $u_h^{(\nu)}$ $(\nu = 0, 1, \ldots, k-1)$. The following theorem shows that this remains true for higher-order VIDEs.

Theorem 3.2.12 *Assume that the given functions a_ν, g and K_ν in (3.2.58) and (3.2.59) are in $C^d(I)$ and $C^d(D)$, respectively, with $d \ge m+1$, and let the $\{c_i\}$ be chosen such that the orthogonality condition*

$$J_0 := \int_0^1 \prod_{i=1}^{m} (s - c_i) ds = 0$$

is satisfied. Then, for all sufficiently small $h > 0$,

$$||y^{(\nu)} - u_h^{(\nu)}||_\infty \le C_\nu h^{m+1} \quad (\nu = 0, 1, \ldots, k-1):$$

the collocation solution $u_h \in S_{m+d}^{(d)}(I_h)$ $(d = k-1)$ *and its derivatives* $u_h', \ldots, u_h^{(k-1)}$ *are globally superconvergent on* I, *with (optimal) order* $p^* = m+1$. *The error* $||y^{(k)} - u_h^{(k)}||_\infty$ *will, in general, be only* $\mathcal{O}(h^m)$.

Proof By definition, the collocation error $e_h := y - u_h$ associated with the collocation solution $u_h \in S_{m+d}^{(d)}(I_h)$ $(d = k-1)$ is the solution of the initial-value problem

$$e_h^{(k)}(t) = \sum_{\nu=0}^{k-1} a_\nu(t) e_h^{(\nu)}(t) + \delta_h(t) + \sum_{\nu=0}^{k} (\mathcal{V}_\nu e_h)(t), \quad t \in I, \qquad (3.2.60)$$

$$e_h^{(\nu)}(0) = 0 \quad (\nu = 0, 1, \ldots, k-1),$$

with $\delta_h(t) = 0$ for $t \in X_h$. If we introduce the vectors

$$\boldsymbol{\epsilon}_h(t) := (e_h(t), \ldots, e_h^{(k)}(t))^T \quad \text{and} \quad \mathbf{d}_h(t) := (0, \ldots, 0, \delta_h(t))^T$$

in $\mathbb{R}^{k+1}$, and recall the matrix kernel $\mathbf{K}(t,s) \in L(\mathbb{R}^{k+1})$ introduced in Section 3.1.3 (cf. (3.1.18)), we can write the VIDE for the collocation error as a system of $k+1$ first-order VIDEs, in analogy to Section 3.1.3:

$$\boldsymbol{\epsilon}_h(t) = \mathbf{d}_h(t) + \int_0^t \mathbf{K}(t,s) \boldsymbol{\epsilon}_h(s) ds, \quad t \in I. \qquad (3.2.61)$$

Hence, according to Theorem 2.1.7, its unique solution is given by

$$\boldsymbol{\epsilon}_h(t) = \mathbf{d}_h(t) + \int_0^t \mathbf{R}(t,s) \mathbf{d}_h(s) ds, \quad t \in I, \qquad (3.2.62)$$

where the (matrix) resolvent kernel $\mathbf{R}(t,s) \in L(\mathbb{R}^{k+1})$ (compare Sections 3.1.3 and 2.1.3) possesses the elements $R_{\nu,j}(t,s)$ $(\nu, j = 0, 1, \ldots, k)$. Due to the special structure of the non-homogeneous term $\mathbf{d}_h(t)$ this implies that, for $\nu = 0, 1, \ldots, k-1$,

$$e_h^{(\nu)}(t) = \int_0^t \sum_{j=0}^{k} R_{\nu,j}(t,s) (\mathbf{d}_h(s))_j ds$$

$$= \int_0^t R_{\nu,k}(t,s) \delta_h(s) ds (\nu = 0, 1, \ldots, k-1), \qquad (3.2.63)$$

while

$$e_h^{(k)}(t) = \delta_h(t) + \int_0^t \sum_{j=0}^{k} R_{k,j}(t,s)(\mathbf{d}_h(s))_j ds = \delta_h(t) + \int_0^t R_{k,k}(t,s) \delta_h(s) ds. \qquad (3.2.64)$$

Here, $(\mathbf{d}_h(s))_j$ denotes the jth component of the vector $\mathbf{d}_h(s)$.

Suppose now that $t = t_n + vh_n \in \sigma_n$. It follows from the representation (3.2.62) that, for $\nu = 0, 1, \ldots, k-1$,

$$e_h^{(\nu)}(t) = \sum_{\ell=0}^{n-1} \int_0^1 R_{\nu,k}(t, t_\ell + sh_\ell)\delta_h(t_\ell + sh_\ell)ds$$
$$+ h_n \int_0^v R_{\nu,k}(t, t_n + sh_n)\delta_h(t_n + sh_n)ds.$$

We see that we are back on familiar territory: the quadrature arguments we used in the previous chapters to prove global (and local) superconvergence results are clearly applicable here, too. Thus, the proof is brought to its end by observing that $||\delta_h||_\infty \le Dh^m$ (as a consequence of the global convergence result in Theorem 3.2.10 and the error equation (3.2.59)). We omit the details. Note, however, that $\mathcal{O}(h^{m+1})$-convergence does not hold for $u_h^{(k)}$ because of the presence of the term $\delta_h(t)$ in (3.2.63). For this term we have, according to Theorem 3.2.11 and (3.2.60), $||\delta_h||_\infty = \mathcal{O}(h^m)$ only.

Finally, and by now not surprisingly, the optimal *local superconvergence* results of Section 1.2.4 carry over to kth-order VIDEs, since the argument in the proof of the previous theorem on global superconvergence can be readily adapted, by setting $t = t_n$ $(n = 1, \ldots, N)$ in the error representation (3.2.61),(3.2.62). We deduce that for collocation parameters $\{c_i\}$ with $c_m < 1$ (e.g. the Gauss points) superconvergence of order $p^* = 2m$ at the mesh points can only be achieved for $u_h^{(\nu)}$ with $\nu < k$. If we have $c_m = 1$ (as for the Radau II points), then $u_h^{(k)}(t_n)$ will also exhibit (the same order of) local superconvergence as $u_h(t_n)$ itself, since $t_n \in X_h$ and hence $\delta_h(t_n) = 0$ in (3.2.63). We summarise this in

Theorem 3.2.13 *In the neutral VIDE (3.2.43) let* $a_\nu \in C^{m+\kappa}(I)$ $(\nu = 0, 1, \ldots, k-1)$, $g \in C^{m+\kappa}(I)$, $K_\nu \in C^{m+\kappa}(D)$ $(\nu = 0, 1, \ldots, k)$ *for some* κ *with* $1 \le \kappa \le m$. *Assume that the* $\{c_i\}$ *satisfy*

$$J_\ell := \int_0^1 s^\ell \prod_{i=1}^m (s - c_i)ds = 0, \quad \ell = 0, 1, \ldots, \kappa - 1,$$

with $J_\kappa \ne 0$. *Then for all sufficiently small mesh diameters* $h > 0$ *the collocation solution* $u_h \in S_{m+d}^{(d)}(I_h)$ $(d = k-1)$ *and its derivatives* $u_h^{(\nu)}$ $(\nu = 1, \ldots, k-1)$ *are superconvergent on the mesh* I_h:

$$\max_{t \in I_h} |y^{(\nu)}(t) - u_h^{(\nu)}(t)| \le C_\nu h^{m+\kappa} \quad (\nu = 0, 1, \ldots, k-1).$$

In particular, $\kappa = m$ *(for which the* $\{c_i\}$ *are the* m *Gauss points in* $(0, 1)$*) leads to*

$$\max_{t \in I_h} |y^{(\nu)}(t) - u_h^{(\nu)}(t)| \le C_\nu h^{2m} \quad (\nu = 0, 1, \ldots, k-1),$$

with $\max\{|e_h^{(k)}(t)| : t \in I_h \setminus \{0\}\} = \mathcal{O}(h^m)$ *only.*

If $\kappa = m - 1$ *and* $c_m = 1$ *(corresponding to the Radau II points in* $(0, 1]$*), then local superconvergence holds also for* $u_h^{(k)}$*: we now have*

$$\max_{t\in I_h} |y^{(\nu)}(t) - u_h^{\nu}(t)| \le C_\nu h^{2m-1} \quad (\nu = 0, 1, \ldots, k-1)$$

and

$$\max_{t\in I_h\setminus\{0\}} |y^{(k)}(t) - u_h^{(k)}(t)| \le C_k h^{2m-1}.$$

3.2.6 Collocation in smoother piecewise polynomial spaces

As a consequence of the results of Loscalzo and Talbot (1967) and Loscalzo (1968), it is clear that the collocation solution $u_h \in S_4^{(3)}(I_h)$ (classical quartic splines of degree four) for the VIDE (3.1.1) will be divergent. Hung (1970) showed that collocation solutions $u_h \in S_\mu^{(\mu-1)}(I_h)$ are convergent when $\mu = 2, 3$ and $c_1 = 1$. He also established the convergence of Hermite-type collocation in the space $S_4^{(2)}(I_h)$ (whose dimension is $2N + 3$), with collocation at the points $t = t_{n+1}$ $(n = 0, 1, \ldots)$ corresponding to $c_1 = c_2 = c_3 = 1$. More recently, Oja and Tarang (2001) and Oja and Saveljeva (2002) have obtained a number of significant results on the dependence of the convergence of smooth collocation solutions on the location of the collocation parameters $\{c_i\}$, using techniques different from those of Mülthei (1979, 1980a).

The general divergence results by Mülthei on collocation in $S_\mu^{(d)}(I_h)$ for ODEs remain of course valid for general VIDEs of the form (3.1.1). However, it is not clear – and remains an open problem – if they can be refined (and are possibly different) for the *special* VIDE

$$y'(t) = g(t) + (\mathcal{V}y)(t),$$

with $\mathcal{V}$ as in (3.2.23).

3.3 Collocation for nonlinear VIDEs

3.3.1 Local superconvergence results

The analysis of global convergence of the collocation solution $u_h \in S_m^{(0)}(I_h)$ to the initial-value problem

$$y'(t) = f(t, y(t)) + \int_0^t k(t, s, y(s))ds, \quad t \in I, \quad y(0) = y_0, \qquad (3.3.1)$$

proceeds along the lines of the one for ODEs in Section 1.1.4, and VIEs in Section 2.3.2, using in addition the classical linearisation argument for

$k(t, s, y(s) - e_h(s))$. We leave the details of the proof to the reader and focus instead on the analysis of local superconvergence on I_h. Since the collocation error $u_h = y - e_h$ solves the initial-value problem

$$e_h'(t) = f(t, y(t)) - f(t, y(t) - e_h(t)) + \delta_h(t) + \int_0^t (k(t, s, y(s)) - k(t, s, y(s) - e_h(s)))ds, \quad (3.3.2)$$

with $e_h(0) = 0$, we may rewrite this error equation in the form

$$e_h'(t) = a_1(t)e_h(t) + \int_0^t H_1(t, s)e_h(s)ds + \delta_h(t) + a_2(t)e_h^2(t) + (\mathcal{W}_2 e_h)(t), \quad t \in I, \quad (3.3.3)$$

assuming that f and k possess continuous (and bounded) second-order partial derivatives with respect to y. The functions $a_1(t) := f_y(t, y(t))$, $H_1(t, s) := k_y(t, s, y(s))$ assume the roles of $a(t)$ and $K(t, s)$ in the linear VIDE (3.2.23), and we have set

$$a_2(t) := -\frac{1}{2} f_{yy}(t, w(t)),$$

$$(\mathcal{W}_2 e_h)(t) := -\frac{1}{2} \int_0^t k_{yy}(t, s, z(s))e_h^2(s)ds,$$

with suitable intermediate 'Taylor arguments' $w(t)$ and $z(s)$. If $r_1 = r_1(t, s)$ denotes the differential resolvent of the kernel $H_1 = H_1(t, s)$, then the collocation error can be expressed in a form reflecting the perturbed error equation (3.3.3), namely,

$$e_h(t) = \int_0^t r_1(t, s)\big(\delta_h(s) + a_2(s)e_h^2(s) + (\mathcal{W}_2 e_h)(s)\big)ds, \quad t \in I. \quad (3.3.4)$$

Comparing this with the error representation (3.2.36) we observe that we now have two additional nonlinear terms which depend on e_h^2. Thus, assuming that $f_{yy}(t, \cdot)$ and $k_{yy}(t, s, \cdot)$ are bounded, the previous quadrature argument combined with the global estimate in Theorem 3.2.3 (cf. the remark at the end of its proof) and the estimates for these nonlinear terms, e.g.,

$$|(\mathcal{W}_2 e_h)(t)| \le \text{const} \cdot ||e_h||_\infty^2 \le D_2(h^m)^2, \quad t \in I,$$

allow us readily to arrive at

$$|e_h(t_n)| \le h^{m+\kappa} Q \sum_{\ell=0}^{n-1} h_\ell + C_2 h^{2m} \le (QT + C_2 h^{m-\kappa})h^{m+\kappa} =: C_0 h^{m+\kappa} + C_1 h^{2m}, \quad 1 \le n \le N,$$

where $0 \le \kappa \le m$.

Therefore, the global and local superconvergence results of Theorems 3.2.5 and 3.2.6, and Corollaries 3.2.7–3.2.9 carry over to nonlinear VIDEs, under the appropriate smoothness and boundedness conditions. Compare also Brunner (1992a) for additional details.

Remark Alternatively, these superconvergence results can be proved by employing the nonlinear variation-of-constant formula for (3.3.1). We refer the reader to the paper by Burgstaller (2000) in which a corrected version of the nonlinear variation-of-constants formulas contained in Brauer (1972) and Bernfeld and Lord (1978) has been used. Nonlinear variation-of-constants formulas for VIDEs can also be found in Hu, Lakshmikantham and Rao (1988).

3.3.2 Kernels of 'non-standard' form $a(t-s)G(y(t), y(s))$

Consider the VIDE

$$y'(t) = f(t, y(t)) + \int_0^t k(t-s)G(y(t), y(s))ds, \quad t \in I, \quad y(0) = y_0, \tag{3.3.5}$$

and assume for ease of exposition, and without loss of generality, that $f(t, y) = g(t)$. The collocation error $e_h := y - u_h$ solves, as we have already seen above in a more particular situation, the initial-value problem

$$e_h'(t) = \delta_h(t) + \int_0^t k(t-s)\,(G(y(t), y(s)) - G(u_h(t), u_h(s)))ds, \quad t \in I,$$
$$e_h(0) = 0.$$

Under appropriate regularity assumptions on $G = G(y, z)$ we may write

$$G(y(t), y(s)) - G(y(t) - e_h(t), y(s) - e_h(s))$$
$$= G_y(y(t), y(s))e_h(t) + G_z(y(t), y(s))e_h(s) + T_2(t, s),$$

where the Taylor remainder term reads

$$T_2(t, s) : = -\frac{1}{2}\big(G_{yy}(\eta(s), \eta(t))e_h^2(t) + 2G_{yz}(\eta(s), \eta(t))e_h(s)e_h(t)$$
$$+ G_{zz}(\eta(t), \eta(s))e_h^2(s)\big)$$

and $\eta(\cdot) := y(\cdot) - \theta e_h(\cdot)$ $(0 < \theta < 1)$. Hence, the linearised initial-value problem for the collocation error becomes

$$e_h'(t) = \delta_h(t) + \int_0^t k(t-s)\,\big(G_y(y(t), y(s))e_h(t) + G_z(y(t), y(s))e_h(s)\big)ds$$
$$+ \int_0^t k(t-s)T_2(t, s)ds, \quad t \in I,$$

with $e_h(0) = 0$. (Here, we have set $f(t, y) \equiv 0$ for simplicity: we recall that this term has been dealt with in Section 1.1.4.) The above error equation is of the form

$$e_h'(t) = A_0(t)e_h(t) + \delta_h(t) + W_2(t) + \int_0^t K_0(t, s)e_h(s)ds, \quad t \in I, \quad (3.3.6)$$

where we have set

$$A_0(t) := \int_0^t k(t-s)G_y(y(t), y(s))ds, \quad W_2(t) := \int_0^t k(t-s)T_2(t, s)ds,$$

and $K_0(t, s) := k(t-s)G_z(y(t), y(s))$. Hence, the solution of the linearised initial-value problem (3.3.6) for e_h is given by

$$e_h(t) = \int_0^t r_0(t, s)[\delta_h(s) + W_2(t)]ds, \quad t \in I. \quad (3.3.7)$$

As in the case of standard nonlinear VIDEs the global order of the collocation error e_h corresponding to the collocation solution $u_h \in S_m^{(0)}(I_h)$ for (3.3.5),

$$||e_h||_\infty \leq Ch^m, \quad \text{for any set } \{c_i\},$$

can be derived in a straighforward way. It thus follows from the definition of $W_2(t)$ and $T_2(t, s)$ that, under appropriate smoothness assumptions on the given functions,

$$|W_2(t)| \leq \text{const.} \int_0^t |k(t-s)|ds \cdot ||e_h||_\infty^2 = \mathcal{O}(h^{2m}),$$

for all $t \in I$. If we now combine this result with the familiar quadrature argument used to derive the earlier 'classical' global (on I) and local (on I_h) superconvergence results we see that they carry over to non-standard VIDEs; that is, Theorems 3.2.5 and 3.2.6, and Corollaries 3.2.7–3.3.9 remain valid for (3.3.5).

3.4 Partial VIDEs: time-stepping

The *spatial semidiscretisation* of initial-boundary-value problems for (parabolic) partial integro-differential equations is the main source of (high-dimensional) systems of semilinear VIDEs. Two representative examples are given below.

Example 3.4.1 Consider the parabolic problem with memory term,

$$u_t(t, x) - \Delta u(t, x) = f(t, x) + \int_0^t k(t-s)\mathcal{B}u(s, x)ds, \quad t \geq 0, \quad x \in \Omega,$$

$$u(0, x) = u_0(x), \quad x \in \Omega; \quad u(t, x) = 0, \quad t \geq 0, \quad x \in \partial\Omega,$$

where $\Omega \subset \mathbb{R}^d$ is bounded, with piecewise smooth boundary $\partial\Omega$, and $\mathcal{B}$ denotes a (linear or nonlinear) spatial partial differential operator of order not exceeding two.

Spatial discretisation (e.g. by finite difference or finite element techniques) leads to a (large) system of ordinary VIDEs for $\mathbf{U}(t) \in \mathbb{R}^M$, with $M \gg 1$,

$$\frac{\mathbf{U}_h(t)}{dt} = \mathcal{A}_h \mathbf{U}_h(t) + \int_0^t k(t-s)(\mathcal{B}_h \mathbf{U}_h)(s)ds, \quad t \geq 0,$$

corresponding to discrete versions $\mathcal{A}_h$ and $\mathcal{B}_h$ of the Laplace operator Δ and the operator $\mathcal{B}$. For the common spatial discretisations the dimension M corresponds to the number of interior mesh points resulting from the triangulation of Ω. Note that $\mathcal{A}_h \in L(\mathbb{R}^M)$ is an (unboundedly) stiff matrix (Dekker and Verwer (1984), Kauthen (1989a,b, 1992)).

Example 3.4.2 In the mathematical modelling of population dynamics involving spatial dependencies one encounters PVIDEs of the form

$$u_t(t,x) - \Delta u(t,x) = g(t,x) + (\mathcal{T}u)(t,x), \quad x \in \Omega \subset \mathbb{R}^d, \ t \in I \quad (3.4.1)$$

(cf. Zhao (2003) and its references), where $\mathcal{T}$ denotes the *Volterra–Fredholm* integral operator given by

$$(\mathcal{T}u)(t,x) := \int_0^t \int_\Omega K(t,\tau,x,\xi)u(\tau,\xi)d\xi \ d\tau. \quad (3.4.2)$$

The integral operator may even contain a delay argument.

Spatial semidiscretisation, in which Ω is replaced by a suitable triangulation Ω_h, and the use of corresponding appropriate quadrature approximations for the Fredholm part of the integral operator $\mathcal{T}$, leads to a (large) system of ordinary VIDEs, similar to the one in Example 3.4.1.

Example 3.4.3 It is well known that, depending on the geometry of the (bounded or unbounded) spatial domain $\Omega \subset \mathbb{R}^d$ and the 'size' of the initial function $u_0(x) \geq 0$, solutions of

$$u_t(t,x) - \Delta u(t,x) = u^p(t,x), \quad t > 0, \quad x \in \Omega \quad (p > 1),$$

with $u(0,x) = u_0(x), \ x \in \Omega$ and $u(t,x) = 0, \ x\partial\Omega, \ t \geq 0$, will blow up in finite time (see, e.g., the survey by Bandle and Brunner (1998) and its references). Bellout (1987) showed that the same is true for solutions of parabolic equations in which the local reaction term u^p is replaced by a memory term, e.g.

$$u_t(t,x) - \Delta u(t,x) = \int_0^t k(t-s)G(u(s,x))ds, \quad t > 0, \quad x \in \Omega, \quad (3.4.3)$$

with $G(u) = u^p$ $(p > 1)$, with initial and boundary conditions as before, and bounded Ω (the analogous problem for unbounded Ω remains open).

If the VIDE (3.4.3) is semidiscretised in space, with respect to a mesh Ω_h, then we obtain a (generally large) system of nonlinear VIDEs of the form

$$\frac{\mathbf{U}_h(t)}{dt} = \mathcal{A}_h \mathbf{U}_h(t) + \int_0^t k(t-s)\mathbf{G}_h(\mathbf{U}(s))ds, \quad t > 0. \tag{3.4.4}$$

It is clear that since the dynamics of the system (3.4.4) depends both on the geometry of Ω_h and on the approximating (finite element or collocation) space and will thus be different from that of (3.4.3), blow-up of the solution of (3.4.3) will not necessarily imply blow-up for $\mathbf{U}_h$.

Remarks

1. If time-stepping in the above semidiscretised systems of VIDEs is based on collocation in the space $S_m^{(0)}(I_h)$ with $m \geq 2$, the approximation of the time integrals (when deriving the fully discretised time-stepping scheme) will become prohibitively expensive. There are a number of ways to 'economise' these quadrature approximations; see, for example, Sloan and Thomée (1986).
2. The numerical detection of blow-up remains remains essentially open, in particular for the classes of (parabolic) partial VIDEs described in Bellout (1987) and Souplet (1998a, 1998b). The same is true for the accurate computation of the blow-up time and corresponding realistic (a posteriori) error estimates, especially in two- and three-dimensional spatial domains. Compare also Bandle and Brunner (1994, 1998), especially for references.

3.5 Exercises and research problems

Exercise 3.5.1 Derive the adjoint resolvent equation (3.1.9). In particular, show that $\partial r(t,s)/\partial t \in C(D)$.

Exercise 3.5.2 Consider the linear VIDEs with convolution kernel,

$$y'(t) = a(t)y(t) + g(t) + \int_0^t k(t-s)y(s)ds, \quad t \in I := [0,T].$$

Assuming that the given functions $a,\ g,\ k$ are in $C(I)$, does the resolvent kernel $r(t,s)$ inherit the convolution structure of k? Derive the resolvent equations for this case.

Exercise 3.5.3 For certain classes of linear VIEs and VIDEs, the Laplace transform provides a powerful tool for analysing quantitative and qualitative properties of their solutions. An important result is given by a generalisation of the *Paley–Wiener Theorem* (Paley and Wiener (1934)). We state these two results for scalar VIEs and VIDEs; analogous results hold for systems of such Volterra equations with convolution kernels (see Miller (1971a, Appendix I), also for background material on the Laplace transform).

1. The Paley–Wiener Theorem deals with the question of integrable resolvent kernels for convolution-type VIEs of the form
$$y(t) = g(t) + \int_0^t k(t-s)y(s)ds, \quad t \geq 0.$$
If $k \in L^1(\mathbb{R}^+)$ and if $K(s) := \int_0^\infty \exp(-st)k(t)dt$ denotes the Laplace transform of k, then the resolvent kernel $R = R(t-s)$ corresponding to the convolution kernel $k = k(t-s)$ is in $L^1(\mathbb{R}^+)$ if, and only if, $A(s) \neq 1$ for all s with $\mathrm{Re}(s) \geq s$.
Prove this result. An analogous condition holds for matrix kernels $k(\cdot) \in \mathbb{R}^m$: $\det(\mathcal{I} - K(s)) \neq 0$ whenever $\mathrm{Re}(s) \geq 0$.
2. Grossman and Miller (1973) extended this result to the VIDE
$$y'(t) = ay(t) + g(t) + \int_0^t k(t-s)y(s)ds, \quad t \geq 0:$$
The resolvent kernel $r = r(t-s)$ associated with the given convolution kernel k satisfies $r \in L^1(\mathbb{R}^+)$ if, and only if, $s - a - K(s) \neq 0$ for all s with $\mathrm{Re}(s) \geq 0$.
Prove this theorem. The corresponding matrix condition reads
$$\det[s\mathcal{I} - A - K(s)] \neq 0 \quad \text{whenever} \quad \mathrm{Re}(s) \geq 0.$$
Here, $A \in \mathbb{R}^m$ is the matrix replacing a in the VIDE.

(Compare also Lubich (1983b), pp. 461-463.)

Exercise 3.5.4 Assume that the memory kernel in the logistic ('non-standard') VIDE (3.1.26) is of the form
$$k(t-s) = \sum_{i=1}^{r} \gamma_i \exp(\lambda_i(t-s)),$$
with distinct (non-positive) constants λ_i. Show that the VIDE can then be reduced to a system of $r+1$ ODEs, and use this result to establish an existence and uniqueness result for the given VIDE.

Exercise 3.5.5 Analyse the optimal superconvergence properties of the *discretised* collocation solution $\hat{u}_h \in S_m^{(0)}(I_h)$ for the VIDE (3.2.1),(3.2.5), by establishing the orders of $||u_h^{(\nu)} - \hat{u}_h^{(\nu)}||_\infty$ and $|u_h^{(\nu)}(t_n - \hat{u}_h^{(\nu)}(t_n)|$ $(\nu = 0, 1;\ 1 \leq n \leq N)$. (Recall Section 2.2.6 and Theorem 2.2.10.)

Exercise 3.5.6 Assume that the collocation parameters $\{c_i\}$ are the m Lobatto points in [0, 1] (i.e., $0 = c_1 < c_1 < \ldots < c_m = 1$). State and prove the local superconvergence result for the collocation solution $u_h \in S_{m+d}^{(d+1)}(I_h)$ $(d = k - 1)$ to the linear version of the neutral kth-order VIDE (3.2.44).
Illustrate this by deriving the method for $k = 2$ and $m = 3$.

Exercise 3.5.7 Suppose that the VIDE (3.1.10) is rewritten as a system of two nonlinear second-kind VIEs,

$$y(t) = y_0 + \int_0^t F(s, y(s), z(s)),$$

$$z(t) = \int_0^t k(t, s, y(s))ds, \quad t \in I.$$

Here, we have set

$$F(t, y, z) := f(t, y) + z \qquad \text{and} \qquad z(t) := \int_0^t k(t, s, y(s))ds.$$

Derive superconvergence results for the collocation solutions $u_h,\ v_h$ in $S_{m-1}^{(-1)}(I_h)$ (and their iterates) approximating y and z. Compare the results with those correponding to 'direct' collocation of the VIDE in $S_m^{(0)}(I_h)$: do the two approaches yield identical approximations?

Exercise 3.5.8 Extend the proofs of the global and local superconvergence results (Theorems 3.2.5 and 3.2.6) to nonlinear VIDEs

$$y'(t) = f(y(t)) + \int_0^t k(t - s)G(y(s), y'(s))ds.$$

Exercise 3.5.9 Formulate and prove the result on the global order of the collocation solution $u_h \in S_m^{(0)}(I_h)$ for the non-standard VIDE (3.3.5).

Exercise 3.5.10 Consider the VIDE (personal communication by J.H. Gordis, September 1993),

$$y(t) = g(t) + \int_0^t k(t - s)y''(s)ds, \quad t \in I := [0, T], \tag{3.5.1}$$

with continuous data g and k and appropriate initial conditions.

(a) Discuss the existence and uniqueness of a solution when $k \in C^2(I)$, by rewriting (3.5.1) as a first-order VIDE.
(b) Assuming that $k \in C(I)$, apply Laplace transform techniques to (3.5.1) to obtain an existence and uniqueness result.

(c) Under the assumption in (b) write the original Volterra equation (3.5.1) as a VIE of the first kind, and discuss its solvability.
(d) Suppose (3.5.1) is solved directly, by collocation in $S_{m+1}^{(1)}(I_h)$. Determine the attainable orders of global and local superconvergence of the collocation solution u_h.

Exercise 3.5.11 (Research problem)
Nonlinear VIDEs with blow-up solutions: Extend the approach and the results of Stuart and Floater (1990) to

$$y'(t) = \lambda y(t) + \int_0^t k(t-s)y^p(s)ds, \quad y(0) = y_0 > 0,$$

with $\lambda \leq 0,\ p > 1$, and $k \in C[0, \infty)$ positive and non-increasing. (See also: partial VIDEs with blow-up solutions, as analysed by Bellout (1987) and Souplet (1998a, 1998b); in addition; consult the list of references in the survey paper by Bandle and Brunner (1998).) Here, one is above all interested in computing very accurate approximations for the *blow-up time*. However, the more challenging problem is the *numerical detection of blow-up*, especially in partial VIDEs: since numerical time-stepping is usually based on some spatially semidiscretised version of the given PVIDE, the dynamics of the resulting (high-dimensional) system of ordinary VIDEs will be different from the one of the original problem.

Exercise 3.5.12 (Research problem)
Discuss the solvability of the system of nonlinear algebraic equations arising in the (exact and discretised) collocation equation for (stiff) nonlinear VIDEs (including 'non-standard' VIDEs). Analyse the effect of the stopping error in iterative methods used to solve these systems, e.g. in Newton's method, on the attainable order of the method. This will generalise analogous investigations for ODEs, as given for example in Liu and Kraaijevanger (1988), Spijker (1994), Jackson, Kvaernø and Nørsett (1996); see also Hairer and Wanner (1996, pp. 215–224) (and compare with Exercise 2.7.30).

Exercise 3.5.13 (Research problem)
Extend the perturbed collocation method for ODEs (Section 1.2) to VIDEs. In particular, discuss the difference between the exact and the (fully) discretised perturbed collocation methods.

Exercise 3.5.14 (Research problem)
Derive and analyse the (super-) convergence properties of two-step collocation for linear first-order VIDEs.

Exercise 3.5.15 (Research problem)
Derive and analyse the discontinuous Galerkin method for VIDEs (recall Section 1.6 where the dG method was introduced for ODEs). Can the dG method be viewed as a (non-trivial) perturbed collocation method?

3.6 Notes

3.1: Review of basic Volterra theory (II)
An excellent account of the early theory of VIDEs, and especially of Volterra's contributions, can be found in Hellinger and Toeplitz (1927, pp. 1494–1498). The monograph by Gripenberg, Londen and Staffans (1990) contains many results on linear and nonlinear VIDEs, as does Corduneanu (1991). See also the important paper by Nohel and Shea (1976) on global existence of solutions to nonlinear (Hammerstein-type) VIDEs. A variation-of-constant formula for linear neutral VIDEs is presented in Wang, Wu and Li (1986); the result is used to prove the existence of periodic solutions.

Extensions of the Paley–Wiener theorem (Paley and Wiener (1934, pp. 58–63)) to linear VIDEs with convolution kernels can be found in Grossman and Miller (1973), Shea and Wainger (1975), and in Lubich (1983b); see also Gripenberg, Londen and Staffans (1990) and B. Zhang (1997) on the integrability of resolvent kernels.

Ordinary and partial VIDEs with blow-up solutions are studied in Bellout (1987), Hattori and Lightbourne (1990), and Souplet (1998a, 1998b). The last paper contains a good bibliography. See also Chadam, Pierce and Yin (1992) for related problems involving nonlinear Fredholm operators.

Good references on nonlinear Gronwall-type inequalities and comparison theorems for VIDEs are the books by Gripenberg, Londen and Staffans (1990, Chapter 10 (Lemma 3.10)) and Györi and Ladas (1991, Section 9.2).

Applications of VIDEs
The following monographs and conference proceedings contain numerous applications of VIDEs, as well as extensive lists of references: Cushing (1977), Brunner (1982a), Burton (1983), Lakshmikantham (1987), Yanik and Fairweather (1988), Brunner (1989b), Corduneanu (1991), Prüss (1993), Wu (1996), Agarwal and O'Regan (2000), and Zhao (2003). As in the Notes to the previous chapter we list a brief selection of more specific references; most of these papers and book feature detailed bibliographies.

- *Modelling of heredity effects:* Volterra (1913, pp. 139–141) considered a 'problème dynamique de la torsion héréditaire' modelled by the VIDE

$$\omega(t) = k\{P(t) - \mu\omega''(t)\} + \int_0^t \{P(\tau) - \mu\omega''(\tau)\}\Phi(t,\tau)d\tau.$$

A survey of such problems was given in Volterra (1912); see also Volterra (1928). His book of 1959 contains a review of VIDEs arising as models of hysteresis effects. The book by Visintin (1994) treats the modern theory of hysteresis. As indicated there, it appears that in spite of Volterra's pioneering work of around 1910 many challenging problems remain in the theory and, especially, the numerical analysis of Volterra integral and integro-differential equations with hysteresis.

- *Population dynamics:* Volterra (1927, 1931, 1934, 1939). The classical reference is Cushing (1977). More recent papers are by Ruan and Wu (1994) (with extensive references), Aves, Davies and Higham (1996, 2000), Thieme and Zhao (2003), and Zhao (2003). Consult also Brunner, van der Houwen and Sommeijer (2003) on relevant references.
- *Identification problems in partial VIDEs:* A. Lorenzi and his numerous collaborators have made extensive contributions to this topic; here, we mention only the papers by Favaron and Lorenzi (2003) and by Grasselli and Lorenzi (1991) (and their references). See also v. Wolfersdorf (1994), Janno and v. Wolfersdorf (1997a, 1997b), and the dissertation by Kiss (1999). These papers also contain extensive bibliographies.
- *Financial mathematics:* Chukwu (1999), Makroglou (2000, 2003).
- *Rheology / viscoelasticity:* Lodge, McLeod and Nohel (1978), Jordan (1978), Markowich and Renardy (1983), Angell and Olmstead (1985), Renardy, Hrusa and Nohel (1988), Hrusa, Nohel and Renardy (1988), Shaw, Warby and Whiteman (1994, 1996, 1997), Shaw and Whiteman (1997, 2000b, 2001).
- *Turbulent diffusion:* Tang and Yuan (1987), Yuan and Tang (1990), Bui Doan Khanh (1994).
- *Wave-power hydrolics:* Elliott and McKee (1981).
- *Capillary theory:* A. Corduneanu and Morosanu (1996).
- *Medicine:* Clements and Smith (1996).

Information on the physical origin of the Volterra equation (3.5.1) in Exercise 3.5.9 can be found in Arfken and Weber (2001, Chapter 16).

3.2 / 3.3: Collocation for linear and nonlinear VIDEs

The paper by Brunner and Lambert (1974) contains a detailed quantitative and qualitative analysis of various one-step methods (based on the explicit and implicit Euler methods) that may be viewed as fully discretised collocation methods. The A-stability of such methods was also studied in Matthys (1976). The general convergence and local superconvergence analysis of piecewise polynomial collocation for linear VIDEs was given in Brunner (1984b, 1988a, 1988b), Brunner and van der Houwen (1986, Chapter 5), and Aguilar and Brunner (1988). Analogous results for nonlinear VIDEs are the subject of Brunner (1989a, 1989b), while local superconvergence results for higher-order nonlinear

VIDEs can be found in Brunner (1992a). The paper by Brunner (1992b) deals with collocation methods for VIDEs with Hammerstein nonlinearities. See also the doctoral dissertation by Burgstaller (1993) for a good treatment of many aspects of collocation methods for VIDEs.

A survey of spline collocation methods for (partial) differential and integro-differential equations was given by Fairweather and Meade (1989); it contains a comprehensive list of references. See also the more recent papers by Fairweather (1994), Ganesh and Spence (1998), Ganesh and Sloan (1999), and Bialecki and Fairweather (2001) on orthogonal spline collocation methods.

In his doctoral dissertation, Wahr (1977) studies the convergence and numerical implementation of collocation methods for boundary-value problems in mth-order VIDEs. See also the related paper by Hangelbroek, Kaper and Leaf (1977) (extension of superconvergence results of de Boor and Swartz (1973)).

The analysis of *pseudo-spectral methods* for VIDEs is still incomplete. The reader may wish to consult the paper by Akyüz and Sezer (1999) (which deals with Chebyshev collocation) for additional references.

The theory and numerical solution of *Fredholm integro-differential equations* is presented in, e.g., Karpilovskaya (1965), Hangelbrook et al. (1977) (extension of superconvergence results by de Boor and Swartz (1973)), Volk (1985), Fairweather and Meade (1989) (survey paper), Micula and Micula (1992), Micula and Fairweather (1993), Ngyuen and Nguyen (1997) (Volterra–Fredholm IDEs), Hu (1998a), Ganesh and Spence (1998), Ganesh and Sloan (1999); see also the book by Appell, Kalitvin and Zabrejko (2000).

3.4: Partial VIDEs: time-stepping

Of the numerous literature we cite the papers by Sloan and Thomée (1986), Yanik and Fairweather (1988), Thomée (1988) (survey paper), Kauthen (1989b, 1992), Lin, Thomée and Wahlbin (1991), Thomée and Wahlbin (1994), Fairweather (1994) (hyperbolic VIDEs), Brunner, Kauthen and Ostermann (1995) (parabolic VIDEs as abstract ODEs), van der Houwen and Sommeijer (1997) (splitting methods), Larsson, Thomée and Wahlbin (1998), Kolobov and Molorodov (1999) (choice of the collocation parameters), and Brunner, van der Houwen and Sommeijer (2003). The monograph by Chen and Shih (1998) contains a comprehensive treatment of spatial (finite element) and time-discretisation techniques, as well as a good bibliography.

Post-processing methods

The post-processing (by extrapolation or multilevel iteration correction techniques) of collocation solutions to VIDEs was studied by Hu (1996b) and Hu and Peng (2000); see also the related analysis of Hu (1998a) for Fredholm integro-differential equations. We also mention the papers by Zhang, T. Lin,

Y. Lin and Rao (2001) (Galerkin methods), T. Lin, Y. Lin, Rao and Zhang (2000) and T. Lin, Y. Lin, Luo, Rao and Zhang (2001) (Petrov–Galerkin methods).

Q. Lin and his collaborators have done extensive research on Richardson extrapolation and defect correction methods for improving the accuracy of collocation and Galekin finite-element methods for parabolic and hyperbolic partial VIDEs. The paper by Q. Lin, Zhang and Yan (1998a) lists many of their papers. In addition, see Q. Lin and Lü (1984), Q. Lin and Zhang (1997), Q. Lin, Zhang and Yan (1997, 1998a, 1998b), and Q. Lin and Zhou (1997a,b).

The discontinuous Galerkin method for VIDEs

The doctoral dissertation by Ma (2004) presents a detailed convergence analysis of the discontinuous Galerkin methods, especially for non-standard (nonlinear) VIDEs. Compare also Ma and Brunner (2003).

WR and TR methods

Due to limitation of space we have not dealt with the question on how best to solve the (large) linear or nonlinear systems of algebraic equations resulting from the computational form of the collocation equations for systems of VIDEs. As we have briefly seen in Section 3.4, such systems are typically encountered in time-stepping for semi-discretised semi-linear partial VIDEs. Waveform relaxation (WR) methods and their discrete analogues, time-point relaxation methods, have recently received considerable attention. The reader will find information on theoretical and computational aspects of such methods in, e.g., Crisci, Ferraro and Russo (1996), Crisci, Russo and Vecchio (1997, 1998); see also the doctoral thesis by Parsons (1999) and the paper by Brunner, Crisci, Russo and Vecchio (2003) (on weakly singular VIDEs). Jackiewicz and Kwapisz (1997) and Zubik-Kowal and Vandewalle (1999) study WR methods for general functional differential equations.

4

Initial-value problems with non-vanishing delays

The functional equations considered in the previous three chapters had the common feature that smooth data led to smooth solutions. This will in general no longer be true if the equation contains a non-vanishing delay: such delays induce so-called primary discontinuity points at which the regularity of the solution will be lower, at least initially, than that of the given functions. Thus, superconvergence can only occur if the meshes underlying the collocation spaces are chosen so as to reflect this behaviour of the analytic solutions.

4.1 Basic theory of Volterra equations with delays

4.1.1 Definitions and notation

The initial-value problem for a first-order delay differential equation (DDE) is described by

$$\begin{aligned} y'(t) &= f(t, y(t), y(\theta(t))), \quad t \in I := [t_0, T], \\ y(t) &= \phi(t), \quad t \le t_0. \end{aligned} \tag{4.1.1}$$

The DDE (4.1.1) is also referred to as a *retarded* differential equation. We will assume that the delay function (or: *lag function*) $\theta(t) := t - \tau(t)$ is continuous and strictly increasing on I, and that the delay $\tau(t)$ is strictly positive on $I: \ \tau(t) \ge \tau_0 > 0$ for all $t \in I$.

A DDE containing also the derivative of the unknown solution at the points $\theta(t)$,

$$y'(t) = f(t, y(t), y(\theta(t)), y'(\theta(t))), \quad t \in I, \tag{4.1.2}$$

is called a *neutral* DDE. Neutral DDEs often occur in a somewhat different, but related form – often called *Hale's form* – namely,

$$\frac{d}{dt}[y(t) - G(t, y(\theta(t))] = f(t, y(t), y(\theta(t))), \quad t \in I. \tag{4.1.3}$$

The most complex situation arises if the lag function θ depends also on the unknown solution, $\theta = \theta(t, y(t)) = t - \tau(t, y(t))$: we then speak of a DDE with *state-dependent delay*.

The following definition, together with examples, may be found in Bellen and Zennaro (2003, Section 2.2).

Definition 4.1.1 The points $\{\xi_\mu : \mu \geq 0\}$ generated by the recursion

$$\theta(\xi_{\mu+1}) = \xi_{\mu+1} - \tau(\xi_{\mu+1}) = \xi_\mu, \quad \mu = 0, 1, \ldots; \; \xi_0 := t_0, \tag{4.1.4}$$

are called the *primary discontinuity points* associated with the lag function $\theta(t) = t - \tau(t)$.

As the name indicates, at these points the solution of a DDE, regardless of how regular the given functions are, will in general exhibit a low degree of regularity: for example, at $t = \xi_0 = t_0$ the solution will be continuous but will have a discontinuous derivative. We note that additional, so-called *secondary* discontinuity points may arise if the given initial function ϕ is only piecewise continuous, that is, if it contains one ore more finite jump discontinuities.

Illustration

1. If $\tau(t) = \tau > 0$ is constant, then the primary discontinuity points induced by $\theta(t) = t - \tau$ are

$$\xi_\mu = t_0 + \mu\tau, \quad \mu = 0, 1, \ldots.$$

2. Let $I := [t_0, T]$ be such that $t_0 > 0$. Then the lag function $\theta(t) = qt = t - (1-q)t$ $(0 < q < 1)$ corresponds to the *non-vanishing proportional delay* $\tau(t) = (1-q)t$. The corresponding primary discontinuity points are given by

$$\xi_\mu = q^{-\mu} t_0, \quad \mu = 0, 1, \ldots.$$

We note that for $t_0 = 0$ there are no primary discontinuity points: as we will see in Chapter 5, in this case smooth data lead to smooth solutions on I.

Definition 4.1.2

(i) The lag function $\theta(t) = t - \tau(t)$ describes a *fading memory* if there exists a $\tau_1 > 0$ so that $\tau(t) \leq \tau_1$ for all $t \geq t_0$.

(ii) The delay τ is said to be *bounded* if $\sup\{\tau(t) : t \geq t_0\} < \infty$.
(iii) The delay τ is called *unbounded* if $\tau(t) \to \infty$ as $t \to \infty$.

Note that for the delay function $\theta(t) := qt = t - (1-q)t$ $(0 < q < 1)$, with $t_0 > 0$, we have $\tau(t) = (1-q)t$, and hence τ is unbounded, while the *constant delay* $\tau(t) = \tau > 0$ is bounded.

A detailed discussion of DDEs with more general (e.g. non-monotonic) θ can be found in Chapters 1 and 2 of the monograph by Bellen and Zennaro (2003). In addition, see the papers by de Gee (1985), Willé and Baker (1992), Baker, Paul and Willé (1995a), and the survey Baker (2000) on the implications of delays of various types on their solutions.

Standard introductions to the general theory of DDEs are Bellman and Cooke (1964), Halanay (1966), El'sgol'ts and Norkin (1973), Driver (1977), Hale (1977), and Hale and Verduyn Lunel (1993). Compare also Diekmann et al. (1995) and Wu (1996) for more advanced treatments.

4.1.2 Second-kind Volterra integral equations with non-vanishing delays

The general linear Volterra integral equation with delay (or lag) function $\theta(t)$ has the form

$$y(t) = g(t) + (\mathcal{V}y)(t) + (\mathcal{V}_\theta y)(t), \quad t \in (t_0, T]. \tag{4.1.5}$$

Here, $\mathcal{V} : C(I) \to C(I)$ denotes the classical Volterra integral operator introduced in Chapter 2,

$$(\mathcal{V}y)(t) := \int_{t_0}^{t} K_1(t,s)y(s)ds, \tag{4.1.6}$$

with kernel $K_1 \in C(D)$, $D := \{(t,s) : t_0 \leq s \leq t \leq T\}$. The kernel K_2 of the delay integral operator

$$(\mathcal{V}_\theta y)(t) := \int_{t_0}^{\theta(t)} K_2(t,s)y(s)ds, \tag{4.1.7}$$

is assumed to be continuous in $D_\theta := \{(t,s) : \theta(t_0) \leq s \leq \theta(t),\ t \in I\}$, with $I := [t_0, T]$. Throughout this chapter the lag function θ will be subject to the following conditions (D1)–(D3):

(D1) $\theta(t) = t - \tau(t)$, $\tau \in C^d(I)$ for some $d \geq 0$;
(D2) $\tau(t) \geq \tau_0 > 0$ for $t \in I$;
(D3) θ is strictly increasing on I.

Remark The subsequent discussion will reveal that condition (D3) has been introduced mainly for technical reasons. However, the reader is encouraged to consult Section 2.1 in Bellen and Zennaro (2003) for many illuminating examples and remarks on the complications arising if (D3) does not hold. While the vast majority of delay Volterra equations (including of course those with constant delay $\tau > 0$) satisfy (D1)–(D3), severe complication will usually arise when the lag function θ depends on the solution y.

In applications (for example, in mathematical models for population growth; see Section 4.1.5 below) one often encounters delay integral equations of the type

$$y(t) = g(t) + (\mathcal{W}_\theta y)(t), \quad t \in (t_0, T], \tag{4.1.8}$$

corresponding to the delay Volterra integral operator

$$(\mathcal{W}_\theta y)(t) := \int_{t_0}^{t} K(t, s)y(s)ds \tag{4.1.9}$$

(or to its nonlinear version, see Section 4.1.5). This delay equation may be viewed as a particular case of (4.1.5), obtained formally by setting $K_2 = -K_1 =: -K$.

As for DDEs, the given delay integral equation will have to be complemented by an initial condition,

$$y(t) = \phi(t), \quad t \in [\theta(t_0), t_0].$$

We observe that, in contrast to initial-value problem for DDEs and DVIDEs with non-vanishing delays (compare Section 4.1.3), the interval in which (4.1.5) is considered is the left-open interval $(t_0, T]$: we shall see below (Theorem 4.1.1) that solutions to Volterra integral equations with non-vanishing delays typically possess a finite (jump) discontinuity at $t = t_0$, while for first-order DDEs (and DVIDEs) the solution y is continuous at this point, with the discontinuity occurring in y'.

However, in complete analogy to DDEs the non-vanishing delay θ gives rise to the *primary discontinuity points* $\{\xi_\mu\}$ for the solution y of (4.1.5): they are determined by the recursion

$$\theta(\xi_\mu) = \xi_{\mu-1}, \quad \mu \geq 1 \quad (\xi_\mu = t_0).$$

Condition (D2) ensures that these discontinuity points have the (uniform) separation property

$$\xi_\mu - \xi_{\mu-1} = \tau(\xi_\mu) \geq \tau_0 > 0 \quad \text{for all} \quad \mu \geq 1.$$

Theorem 4.1.1 *Assume that the given functions in (4.1.5) are continuous on their respective domains and that the lag function θ satisfies the above conditions (D1)–(D3). Then for any initial function $\phi \in C[\theta(t_0), t_0]$ there exists a unique (bounded) $y \in C(t_0, T]$ solving the delay integral equation (4.1.5) and coinciding with ϕ on $[\theta(t_0), t_0]$. In general, this solution has a finite (jump) discontinuity at $t = t_0$:*

$$\lim_{t \to t_0^+} y(t) \neq \lim_{t \to t_0^-} y(t) = \phi(t_0).$$

The solution is continuous at $t = t_0$ only if the initial function is such that

$$g(t_0) - \int_{\theta(t_0)}^{t_0} K_2(t_0, s)\phi(s)ds = \phi(t_0).$$

Proof For $t \in I^{(\mu)} := [\xi_\mu, \xi_{\mu+1}]$ the initial-value problem for (4.1.5) may be written as a Volterra integral equation of the second kind,

$$y(t) = g_\mu(t) + \int_{\xi_\mu}^{t} K_1(t, s)y(s)ds, \tag{4.1.10}$$

with $g_\mu(t) := g(t) + \Phi_\mu(t)$ and

$$\Phi_\mu(t) := \int_{t_0}^{\xi_\mu} K_1(t, s)y(s)ds + \int_{t_0}^{\theta(t)} K_2(t, s)y(s)ds.$$

For $\mu = 0$ this function is known and given by

$$\Phi_0(t) = -\int_{\theta(t)}^{t_0} K_2(t, s)\phi(s)ds;$$

by our assumptions we have $\Phi_0 \in C(I^{(0)})$. It follows from the classical Volterra theory of Chapter 2 that for each $\mu \geq 0$ (so that $I^{(\mu)} \subset I$) the integral equation (4.1.10) possesses a unique continuous solution in $I^{(\mu)}$.

As for its regularity, we first observe that for $\mu = 0$ (with $\xi_0 = t_0$),

$$\lim_{t \to t_0^+} y(t) = g(t_0) + \Phi_0(t_0) = g(t_0) - \int_{\theta(t_0)}^{t_0} K_2(t_0, s)\phi(s)ds$$

which, for arbitrary (continuous) data $g,\ K_2,\ \phi$, will not coincide with the value $\phi(t_0)$. For $\mu \geq 1$ we derive

$$y(\xi_\mu^-) = g(\xi_\mu) + \int_{t_0}^{\xi_\mu} K_1(\xi_\mu, s)y(s)ds + \int_{t_0}^{\theta(\xi_\mu)} K_2(\xi_\mu, s)y(s)ds$$

and

$$y(\xi_\mu^+) = g(\xi_\mu) + \int_{t_0}^{\xi_\mu} K_1(\xi_\mu, s)y(s)ds + \int_{t_0}^{\theta(\xi_\mu)} K_2(\xi_\mu, s)y(s)ds.$$

Hence,

$$y(\xi_\mu^+) - y(\xi_\mu^-) = 0,$$

whenever $g,\ K_1,\ K_2$ and θ are continuous functions. This completes the proof of Theorem 4.1.1.

We have seen in Chapter 2 that the solution of a linear Volterra integral equation of the second kind can be expressed in terms of the resolvent kernel and the non-homogeneous term g (recall Theorem 2.1.2); this 'variation-of-constants' formula proved to be the key to the establishing of (global and local) superconvergence results for collocation solutions to such equations. As the above proof implicitly shows, an analogous representation can be derived for the solution of the delay Volterra integral equation (4.1.5), since by (D2) the delay $\tau = \tau(t)$ in $\theta(t) = t - \tau(t)$ does not vanish in I. Suppose, for ease of notation and without loss of generality, that T in $I = [t_0, T]$ is such that $\xi_{M+1} = T$ (or, alternatively, $T \in (\xi_M, \xi_{M+1}]$) for some $M \geq 1$.

Theorem 4.1.2 *Suppose that (D1)–(D3) and the assumptions of Theorem 4.1.1 hold, and set*

$$g_0(t) := g(t) - \int_{\theta(t)}^{t_0} K_2(t,s)\phi(s)ds \quad \textit{for} \quad t \in I^{(0)}.$$

Then for $t \in I^{(\mu)} := [\xi_\mu, \xi_{\mu+1}]$ ($\mu \geq 1$) the unique solution y of (4.1.5) corresponding to the initial function ϕ can be expressed in the form

$$y(t) = g(t) + \int_{\xi_\mu}^{t} R_1(t,s)g(s)ds + F_\mu(t) + \Phi_\mu(t), \tag{4.1.11}$$

with

$$F_\mu(t) := \int_{t_0}^{\xi_1} R_{\mu,0}(t,s)g_0(s)ds + \sum_{\nu=1}^{\mu-1} \int_{\xi_\nu}^{\xi_{\nu+1}} R_{\mu,\nu}(t,s)g(s)ds,$$

$$\Phi_\mu(t) := \int_{t_0}^{\theta^\mu(t)} Q_{\mu,0}(t,s)g_0(s)ds + \sum_{\nu=1}^{\mu-1} \int_{\xi_\nu}^{\theta^{\mu-\nu}(t)} Q_{\mu,\nu}(t,s)g(s)ds.$$

On the initial interval $(\xi_0, \xi_1]$ (with $\xi_0 = t_0$) the solution y is given by

$$y(t) = g_0(t) + \int_{t_0}^{t} R_1(t,s)g_0(s)ds. \tag{4.1.12}$$

Here, R_1 is the resolvent kernel associated with the given kernel K_1 of the Volterra integral operator (4.1.6), $R_{\mu,\nu}$ and $Q_{\mu,\nu}$ denote functions which are continuous on their respective domains and depend on $K_1,\ K_2,\ R_1$ and θ, and $\theta^k := \underbrace{\theta \circ \cdots \circ \theta}_{k}$.

Remarks

1. The structure of the above variation-of-constants formula (4.1.11) clearly reveals the interaction between the classical lag term $F_\mu(t)$ (governed by the classical Volterra operator $\mathcal{V}$) and the delay term $\Phi_\mu(t)$ (which reflects the action of the non-vanishing delay function θ). The insight obtained from the latter will play a crucial role in the selection of appropriate ('θ-invariant') meshes underlying local superconvergence results (Section 4.2.1).
2. Cerha (1976) showed that the solution of a delay VIE whose delay occurs in the integrand,

$$y(t) = g(t) + \int_0^t K(t,s)y(\theta(s))ds, \quad t \in I,$$

with $g \in C(I)$ and $K \in C(D)$, admits a simpler 'resolvent representation', namely

$$y(t) = g(t) + \int_0^t R(t,s)g(\theta(s))ds, \quad t \in I.$$

The resolvent kernel R associated with the given kernel K satisfies the resolvent equations

$$R(t,s) = K(t,s) + \int_s^t K(t,v)R(\theta(v),s)dv, \quad (t,s) \in D),$$

and

$$R(t,s) = K(t,s) + \int_s^t R(t,v)K(\theta(v),s)dv, \quad (t,s) \in D.$$

Proof The solution of the integral equation (4.1.10),

$$y(t) = g_\mu(t) + \int_{\xi_\mu}^t K_1(t,s)y(s)ds, \quad t \in I^{(\mu)},$$

is given by

$$y(t) = g_\mu(t) + \int_{\xi_\mu}^t R_1(t,s)g_\mu(s)ds, \quad t \in I^{(\mu)}, \tag{4.1.13}$$

with R_1 defined by the resolvent equation

$$R_1(t,s) = K_1(t,s) + \int_s^t R_1(t,v)K_1(v,s)dv, \quad (t,s) \in D^{(\mu)}$$

(cf. (2.1.9)), where $D^{(\mu)} := \{(t,s) : \xi_\mu \le s \le t \le \xi_{\mu+1}\}$. The expression (4.1.12) for the solution on the interval $I^{(0)}$ thus follows immediately.

On $I^{(1)}$ we thus have, using Dirichlet's formula and (4.1.12),

$$
\begin{aligned}
g_1(t) &= g(t) + \int_{t_0}^{\xi_1} K_1(t,s)y(s)ds + \int_{t_0}^{\theta(t)} K_2(t,s)y(s)ds \\
&= g(t) + \int_{t_0}^{\xi_1} K_1(t,s)g_0(s)ds + \int_{t_0}^{\theta(t)} K_2(t,s)g_0(s)ds \\
&\quad + \int_{t_0}^{\xi_1} \left(\int_{v}^{\xi_1} K_1(t,s)R_1(s,v)ds \right) g_0(v)dv \\
&\quad + \int_{t_0}^{\theta(t)} \left(\int_{v}^{\theta(t)} K_2(t,s)R_1(s,v)ds \right) g_0(v)dv,
\end{aligned}
$$

and hence

$$
\begin{aligned}
g_1(t) \; &= \; g(t) + \int_{t_0}^{\xi_1} \left(K_1(t,s) + \int_{s}^{\xi_1} K_1(t,v)R_1(v,s)dv \right) g_0(s)ds \\
&\quad + \int_{t_0}^{\theta(t)} \left(K_2(t,s) + \int_{s}^{\theta(t)} K_2(t,v)R_1(v,s)dv \right) g_0(s)ds \\
&=: g(t) + \int_{t_0}^{\xi_1} Q_{1,1}^{(1)}(t,s)g_0(s)ds + \int_{t_0}^{\theta(t)} Q_{1,0}^{(1)}(t,s)g_0(s)ds,
\end{aligned}
$$

with obvious meaning of the (continuous) functions $Q_{1,0}^{(1)}$ and $Q_{1,1}^{(1)}$.

Recall now the representation (4.1.13) of the solution y on $I^{(1)}$: after trivial algebraic manipulation it can be written as

$$
\begin{aligned}
y(t) = g(t) &+ \int_{t_0}^{t} R_1(t,s)g(s)ds + \int_{t_0}^{\xi_1} \left(Q_{1,1}^{(1)}(t,s) + \hat{Q}_{1,1}^{(1)}(t,s) \right) g_0(s)ds \\
&+ \int_{t_0}^{\theta(t)} \left(Q_{1,0}^{(1)}(t,s) + \hat{Q}_{1,0}^{(1)}(t,s) \right) g_0(s)ds.
\end{aligned}
$$

This yields (4.1.11) for $\mu = 1$, by setting

$$
R_{1,0}(t,s) := Q_{1,1}^{(1)}(t,s) + \hat{Q}_{1,1}^{(1)}(t,s), \quad Q_{1,0}(t,s) := Q_{1,0}^{(1)}(t,s) + \hat{Q}_{1,0}^{(1)}(t,s).
$$

Clearly, the functions describing this expression for y are continuous in the region where they are defined.

The proof is now concluded by a simple but (notationwise) tedious induction argument. This argument reveals that in the variation-of-constants formula (4.1.11) the integrals over $[\xi_\mu, \xi_{\mu+1}]$ with $\mu \geq 1$ will contribute terms involving only $g(t)$, while the integrals over $[\xi_0, \xi_1]$ and $[\xi_0, \theta^\mu(t)]$ contain the 'entire' initial function $g_0(t)$.

The result of Theorem 4.1.2 and its proof lead to the following result on the regularity of solutions of (4.1.5).

Theorem 4.1.3 *Assume that (D1)–(D3) are satisfied and that the functions describing the delay Volterra integral equation (4.1.5) all possess continuous derivatives of at least order $m \geq 1$ on their respective domains. Then:*

(a) The (unique) solution y of (4.1.5) is in $C^m(\xi_\mu, \xi_{\mu+1}]$ for each $\mu = 0, 1, \ldots, M$ and is bounded on $Z_M := \{\xi_\mu : \mu = 0, 1, \ldots, M\}$, and hence on I.

(b) At $t = \xi_\mu$ $(\mu = 1, \ldots, \min\{m, M\})$ we have

$$\lim_{t \to \xi_\mu^-} y^{(\mu-1)}(t) = \lim_{t \to \xi_\mu^+} y^{(\mu-1)}(t),$$

while the μ-th derivative of y is in general not continuous at ξ_μ. In addition, if $\min\{m, M\} = m < M$, the solution also lies in $C^m[\xi_{m+1}, T]$.

The **proof** is left as an exercise (Exercise 4.7.3).

In Section 4.4.2 we shall meet a second-kind delay VIE that is somewhat more general than (4.1.8), namely,

$$y(t) = g(t) + b(t)y(\theta(t)) + (\mathcal{W}_\theta y)(t), \quad t \in (\theta(t_0), t_0]. \tag{4.1.14}$$

Since the delay τ in $\theta(t) = t - \tau(t)$ does not vanish on I the above result on the existence and uniqueness of a solution of the corresponding initial-value problem (Theorem 4.1.1), the variation-of-constant formula (Theorem 4.1.2), and the regularity properties (Theorem 4.1.3) can be generalised to encompass (4.1.14). We leave the proofs of these generalisations as an exercise (Exercise 4.7.4).

Turning to the *regularity* and smoothing properties of solutions of delay VIEs, it is not diffult, on the basis of Theorem 4.1.2, to establish results that are close analogues of those for DDEs with non-vanishing delays. Due to limitation of space we will simply summarise some of these results; the proofs of some of these can be found in Brunner and Zhang (1999). The extension of the results to delay VIEs with weakly singular kernels can be found in Section 6.1.7 (Table 6.1).

4.1.3 First-kind VIEs with non-vanishing delays

Consider the linear first-kind Volterra integral equation with lag function θ satisfying (D1)–(D3),

$$(\mathcal{V}y)(t) + (\mathcal{V}_\theta y)(t) = g(t), \quad t \in (t_0, T], \tag{4.1.15}$$

subject to the initial condition $y(t) = \phi(t),\ t \in [\theta(t_0), t_0]$. The (linear) Volterra integral operators are those of (4.1.6) and (4.1.7). Using the notation of the

Table 4.1. *Regularity and smoothing of solutions to delay VIEs*

Delay VIE (C^m–data)	Regularity at $t = \xi_\mu$ ($\mu = 0, 1, \ldots, M$)
• $y(t) = g(t) + (\mathcal{V}_\theta y)(t)$	$C^{\mu-1}$ (finite jump at $t = t_0$)
• $y(t) = g(t) + (\mathcal{W}_\theta y)(t)$	$C^{\mu-1}$ (finite jump at $t = t_0$)
• $y(t) = g(t) + b(t)y(\theta(t)) + (\mathcal{V}_\theta y)(t)$	C^{-1} (finite jump at $t = t_0$; no smoothing at $t = \xi_\mu$)

previous section we can write (4.1.15) in the local form

$$\int_{\xi_\mu}^{t} K_1(t,s)y(s)ds = g_\mu(t), \quad t \in (\xi_\mu, \xi_{\mu+1}], \tag{4.1.16}$$

with

$$g_\mu(t) := g(t) - \int_{t_0}^{\xi_\mu} K_1(t,s)y(s)ds - \int_{t_0}^{\theta(t)} K_2(t,s)y(s)ds \tag{4.1.17}$$

($\mu \geq 1$). For $t \in (\xi_0, \xi_1]$ this becomes

$$g_0(t) := g(t) + \int_{\theta(t)}^{t_0} K_2(t,s)\phi(s)ds. \tag{4.1.18}$$

This reveals that for arbitrary continuous $K_2,\ g,\ \phi,\ \theta$, we have

$$g_0(t_0) = g(t_0) + \int_{\theta(t_0)}^{t_0} K_2(t_0,s)\phi(s)ds \neq 0.$$

Hence, according to the classical Volterra theory of 1896, it follows that typically the solution of (4.1.16) (with $\mu = 0$) will be unbounded at $t = t_0^+$:

$$\lim_{t \to t_0^-} y(t) = \phi(t_0) \neq \lim_{t \to t_0^+} y(t) = \pm\infty.$$

For the solution to be bounded at $t = t_0^+$ the initial function must be such that

$$\int_{\theta(t_0)}^{t_0} K_2(t_0,s)\phi(s)ds = -g(t_0) \tag{4.1.19}$$

holds.

Theorem 4.1.4 *Assume:*

(a) $K_1 \in C^1(D)$, *with* $|K_1(t,t)| \geq \kappa_0 > 0,\ t \in I := [t_0, T]$;
(b) $K_2 \in C^1(D_\theta)$;

(c) $g \in C^1(I)$;
(d) θ *is subject to (D1)–(D3) of Section 4.1.2, with* $d = 1$ *in (D1).*

Then for any initial function $\phi \in C[\theta(t_0), t_0]$ *there exists a unique* y *with* $y \in C(\xi_\mu, \xi_{\mu+1}]$ $(\mu = 0, 1, \ldots, M)$ *which solves (4.1.15) and coincides with* ϕ *on* $[\theta(t_0), t_0]$. *This solution* y *remains bounded at* $t = t_0 = \xi_0$ *if, and only if, (4.1.19) holds.*

Proof We know from Section 2.1.3 that under the assumptions (a)–(d) the first-kind Volterra integral equation (4.1.16) possesses, for each $\mu = 0, 1, \ldots, M$, a unique solution $y \in C(\xi_\mu, \xi_{\mu+1}]$. At $t = \xi_0^+ = t_0^+$ the solution is bounded if, and only if, $g_0(\xi_0) = 0$ which, according to (4.1.18), is equivalent to the condition (4.1.19).

Is the smoothing property we encountered in solutions of delay Volterra integral equations of the second kind (Theorem 4.1.3) also present for solutions of the first-kind delay equation (4.1.15)? Let us obtain some insight into the general answer by looking at a representative example.

Illustration

For the smooth kernels $K_1(t, s) \equiv 1$, $K_2(t, s) \equiv \lambda_2 \neq 0$, (4.1.15) reads

$$\int_{t_0}^{t} y(s)ds + \int_{t_0}^{\theta(t)} \lambda_2 y(s)ds = g(t), \quad t \in (t_0, T], \tag{4.1.20}$$

with $y(t) = \phi(t) = \phi_0$ for $t \in [\theta(t_0), t_0]$. On $(\xi_\mu, \xi_{\mu+1}]$ this delay equation is given by

$$\int_{\xi_\mu}^{t} y(s)ds = g_\mu(t),$$

where

$$g_\mu(t) := g(t) - \int_{t_0}^{\xi_\mu} y(s)ds - \int_{t_0}^{\theta(t)} \lambda_2 y(s)ds \quad (\mu \geq 1),$$

and

$$g_0(t) := g(t) + \int_{\theta(t)}^{t_0} \lambda_2 \phi(s)ds = g(t) + \lambda_2 \phi_0 \cdot (t_0 - \theta(t)).$$

On $(t_0, \xi_1]$ we find the solution to be

$$y(t) = g_0'(t) = g'(t) - \lambda_2 \phi_0 \theta'(t).$$

It is bounded at $t = t_0^+$ if, and only if, the initial function is such that $g(t_0) = 0$, implying that

$$\phi_0 = -\frac{g(t_0)}{\lambda_2 \tau(t_0)}$$

holds (recall that $t - \theta(t) = \tau(t)$, with strictly positive delay $\tau(t)$). We observe also that under this hypothesis,

$$y(t_0^+) = g'(t_0) - \lambda_2 \phi_0 \theta'(t_0) = g'(t_0) + g(t_0)\theta'(t_0)/\tau(t_0);$$

thus, $y(t_0^+) \neq \phi(t_0)$ for general data.

Let now $\mu = 1$: using the above results and definitions we find

$$g_1(t) = g(t) - \int_{\xi_0}^{\xi_1} \left(g'(s) - \lambda_2 \phi_0 \theta'(s)\right) ds - \int_{\xi_0}^{\theta(t)} \lambda_2 \left(g'(s) - \lambda_2 \phi_0 \theta'(s)\right) ds$$

and hence

$$g_1(\xi_1) = g(\xi_0) + \lambda_2 \phi_0 (\xi_0 - \theta(\xi_0)) = g(t_0) + \lambda_2 \phi_0 \tau(t_0).$$

If the boundedness condition (4.1.19) is true, then it follows that $g_1(\xi_1) = g_0(t_0) = 0$. In other words, boundedness of y at $t = t_0 = \xi_0$ implies boundedness at $t = \xi_1$; an analogous argument yields then boundedness at the remaining points of Z_M. Moreover, we find from the expressions

$$y(\xi_1^-) = g'(\xi_1) + g(\xi_0)\theta'(\xi_0)/\tau(\xi_0)$$

and

$$y(\xi_1^+) = g'(\xi_1) - \lambda_2 \theta'(\xi_1)\{g'(\xi_0) + g(\xi_0)\theta'(\xi_0)/\tau(\xi_0)\}.$$

that the jump at $t = \xi_1$ is given by

$$y(\xi_1^+) - y(\xi_1^-) = -\lambda_2 g'(t_0)\theta'(\xi_1) - g(t_0)\theta'(t_0)[1 + \lambda_2 \theta'(t_0)]/\tau(t_0).$$

Clearly, the jump discontinuity of y at $t = t_0$ will lead to such a discontinuity at the next primary discontinuity point $t = \xi_1$: even if the boundedness condition (4.1.19) is fulfilled, there is no smoothing in the solution of the first-kind delay Volterra integral equation (4.1.20).

The analysis for the particular first-kind delay integral equation (4.1.20) is readily generalised to encompass (4.1.15). We summarise the result in the following theorem but leave the proof of the result to the reader.

Theorem 4.1.5 *Let the assumptions of Theorem 4.1.4 for the given functions in (4.1.15) hold, and assume that the initial function $\phi \in C[\theta(t_0), t_0]$ is such that the solution y of the initial-value problem for (4.1.15) is bounded at $t = t_0^+$.*

If y possesses a finite discontinuity at $t = t_0$, then the discontinuity persists at the other points of Z_M.

4.1.4 VIDEs with non-vanishing delays

In this section we study the properties of solutions of the linear first-order delay VIDE

$$y'(t) = a(t)y(t) + b(t)y(\theta(t)) + g(t) + (\mathcal{V}y)(t) + (\mathcal{V}_\theta y)(t), \quad t \in I := [t_0, T], \tag{4.1.21}$$

corresponding to the Volterra integral operators $\mathcal{V}$ and $\mathcal{V}_\theta$ introduced in (4.1.6) and (4.1.7). It includes the analogue of the particular delay VIE (4.1.8), namely

$$y'(t) = a(t)y(t) + b(t)y(\theta(t)) + g(t) + (\mathcal{W}_\theta y)(t), \quad t \in I. \tag{4.1.22}$$

The solutions y of the delay VIDE (4.1.21) (and hence those of (4.1.22)) will in general again have lower regularity at the *primary discontinuity points* $\{\xi_\mu\}$ defined by the recursion

$$\theta(\xi_\mu) = \xi_{\mu-1}, \quad \mu = 1, \ldots \; (\xi_0 = t_0)$$

(cf. Section 4.1.1).

We start with a basic result on the existence and uniqueness of solutions of the initial-value problem for (4.1.21).

Theorem 4.1.6 *Assume:*

(a) $a,\ b,\ g,\ \theta \in C(I)$, $K_1 \in C(D)$, $K_2 \in C(D_\theta)$;
(b) $\theta(t) = t - \tau(t)$ satisfies the conditions (D1)–(D3) of Section 4.1.2.

Then for any initial function $\phi \in C[\theta(t_0), t_0]$ there exists a unique function $y \in C(I) \cap C^1(t_0, T]$ which satisfies the delay VIDE (4.1.21) on I and coincides with ϕ on $[\theta(t_0), t_0]$. At $t = t_0$ its derivative is, in general, discontinuous (but bounded):

$$\lim_{t \to t_0^+} y'(t) \neq \lim_{t \to t_0^-} y'(t) = \phi'(t_0)$$

(assuming that $\theta'(t_0)$ exists).

The **proof** is left as an exercise (for which the lines preceding Theorem 4.1.9 below may be helpful).

The (unique) solution y of the initial-value problem for (4.1.21) can be expressed by a variation-of-constant formula, analogous to the one in Theorem

4.1.2 for the delay VIE (4.1.5). This result is based on the 'local' form of the above delay VIDE, that is, on the initial-value problem with respect to the interval $I^{(\mu)} := [\xi_\mu, \xi_{\mu+1}]$ ($\mu = 1, \ldots, M$):

$$y'(t) = a(t)y(t) + g_\mu(t) + \int_{\xi_\mu}^{t} K_1(t,s)y(s)ds, \quad t \in I^{(\mu)}, \tag{4.1.23}$$

where $y(\xi_\mu)$ is known and g_u is defined by

$$g_\mu(t) := g(t) + b(t)y(\theta(t)) + \int_{t_0}^{\xi_\mu} K_1(t,s)y(s)ds$$
$$+ \int_{t_0}^{\theta(t)} K_2(t,s)y(s)ds, \quad t \in I^{(\mu)}. \tag{4.1.24}$$

For $\mu = 0$ the above lag term reduces to

$$g_0(t) := g(t) + b(t)\phi(\theta(t)) - \int_{\theta(t)}^{t_0} K_2(t,s)\phi(s)ds, \quad t \in I^{(0)}. \tag{4.1.25}$$

According to Theorem 3.1.1, the solution of the (local) VIDE (4.1.23) has the form

$$y(t) = r_1(t,\xi_\mu)y(\xi_\mu) + \int_{\xi_\mu}^{t} r_1(t,s)g_\mu(s)ds, \quad t \in I^{(\mu)}, \tag{4.1.26}$$

with the resolvent kernel r_1 given by the solution of the resolvent equation

$$\frac{\partial r_1(t,s)}{\partial s} = -r_1(t,s)a(s) - \int_{s}^{t} r_1(t,v)K_1(v,s)dv, \quad (t,s) \in D^{(\mu)}, \tag{4.1.27}$$

subject to the initial condition $r_1(t,t) = 1$ for $t \in I^{(\mu)}$.

Theorem 4.1.7 *Let the given functions $a,\ b,\ g,\ K_1,\ K_2,\ \phi$ be continuous, and assume that the delay function θ is subject to (D1)–(D3). Then on the interval $I^{(\mu)} := [\xi_\mu, \xi_{\mu+1}]$ ($\mu \geq 1$) the solution of the initial-value problem for (4.1.21) can be written as*

$$y(t) = r_1(t,\xi_\mu)y(\xi_\mu) + + \int_{\xi_\mu}^{t} r_1(t,s)g(s)ds + F_\mu(t) + \Phi_\mu(t), \tag{4.1.28}$$

with

$$F_\mu(t) := \sum_{\nu=1}^{\mu-1} \rho_{\mu,\nu}(t)y(\xi_\nu) + \int_{\xi_0}^{\xi_1} r_{\mu,0}(t,s)g_0(s)ds + \sum_{\nu=1}^{\mu-1} \int_{\xi_\nu}^{\xi_{\nu+1}} r_{\mu,\nu}(t,s)g(s)ds,$$

$$\Phi_\mu(t) := \int_{\xi_0}^{\theta^\mu(t)} q_{\mu,0}(t,s)g_0(s)ds + \sum_{\nu=1}^{\mu-1} \int_{\xi_\nu}^{\theta^{\mu-\nu}(t)} q_{\mu,\nu}(t,s)g(s)ds.$$

On the first interval $I^{(0)}$ *this representation reduces to*

$$y(t) = r_1(t, t_0)y(t_0) + \int_{t_0}^{t} r_1(t, s)g_0(s)ds, \qquad (4.1.29)$$

where $y(t_0) = \phi(t_0)$*. The functions* $\rho_{\mu,\nu}$, $r_{\mu,\nu}$, *and* $q_{\mu,\nu}$ *depend on* a, b, K_1, K_2, r_1 *and* θ *and are continuous on their respective domains;* $r_1 = r_1(t, s)$ *denotes the resolvent kernel for* $K_1 = K_1(t, s)$ *defined by the resolvent equation (4.1.27).*

Remark As in Theorem 4.1.2 we see again how the presence of the delay term $(\mathcal{V}_\theta y)(t)$ in (4.1.21) influences the resolvent representation of the classical (non-delay) VIDE on the macro-interval $I^{(\mu)}$. In addition, we now have terms reflecting the initial values $y(\xi_\nu)$ $(0 \le \nu \le \mu)$.

Proof The basic idea governing the proof of the above result is essentially the one used to establish Theorem 4.1.2, except that now the variation-of-constant formula is based on the resolvent representation of the solution of the 'local' VIDE (4.1.26) and will thus reflect the initial values $y(\xi_\mu)$. Due to this similarity, we just sketch the first steps of the proof. For $\mu = 0$ $(t \in I^{(0)})$ the solution of (4.1.26) is

$$y(t) = r_1(t, t_0)y(t_0) + \int_{t_0}^{t} r_1(t, s)g_0(s)ds, \quad t \in I^{(0)},$$

with g_0 defined in (4.1.25). For $t \in I^{(\mu)}$ $(\mu \ge 1)$ we obtain, according to (4.1.26) and Theorem 3.1.1,

$$\begin{aligned} y(t) &= r_1(t, \xi_\mu)y(\xi_\mu) + \int_{\xi_\mu}^{t} r_1(t, s)g_\mu(s)ds \\ &= r_1(t, \xi_\mu)y(\xi_\mu) + \int_{\xi_\mu}^{t} r_1(t, s)g(s)ds + \Psi_\mu(t), \end{aligned}$$

with

$$\begin{aligned} \Psi_\mu(t) := \int_{\xi_\mu}^{t} r_1(t, s)b(s)y(\theta(s))ds + \int_{\xi_\mu}^{t} r_1(t, s)\left(\int_{\xi_0}^{\xi_\mu} K_1(s, v)y(v)dv\right)ds \\ + \int_{\xi_\mu}^{t} r_1(t, s)\left(\int_{\xi_0}^{\theta(s)} K_2(s, v)y(v)dv\right)ds. \end{aligned}$$

Thus, starting with $\mu = 1$, and noting that the double integrals in the corresponding above equation have already been encountered in the proof of Theorem 4.1.2, the use of Dirichlet's formula and a simple induction argument yield the main proposition of Theorem 4.1.8 in a straightforward way.

For the sake of completeness, and since in Section 4.5.4 on neutral functional integro-differential equations we shall have to resort to the result, we add the following corollary on solutions to DDEs with non-vanishing delays.

Corollary 4.1.8 *Consider the delay differential equation*

$$y'(t) = a(t)y(t) + b(t)y(\theta(t)) + g(t), \quad t \in I := [t_0, T], \tag{4.1.30}$$

with $y(t) = \phi(t)$ *on* $[\theta(t_0), t_0]$. *If the given functions are continuous, with* θ *subject to the conditions (D1)–(D3) of Section 4.1.2, then the (unique) solution* y *of this initial-value problem is given on* $I^{(\mu)}$ $(\mu \geq 1)$ *by*

$$\begin{aligned} y(t) = r(t, \xi_\mu)y(\xi_\mu) + \sum_{\nu=1}^{\mu-1} r_{\mu,\nu}(t)y(\xi_\nu) + \int_{\xi_\mu}^{t} r(t,s)g(s)ds \\ + \int_{\xi_0}^{\theta^\mu(t)} q_{\mu,0}(t,s)g_0(s)ds + \sum_{\nu=1}^{\mu-1} \int_{\xi_\nu}^{\theta^{\mu-\nu}(t)} q_{\mu,\nu}(t,s)g(s)ds \end{aligned} \tag{4.1.31}$$

Here,

$$g_0(t) := g(t) + b(t)\phi(\theta(t)), \quad t \in I^{(0)}, \tag{4.1.32}$$

and

$$r(t,s) := \exp\left(\int_s^t a(v)dv\right), \quad (t,s) \in D.$$

The continuous functions $r_{\mu,\nu}$ *and* $q_{\mu,\nu}$ *depend on* $a,\ b,\ \theta$ *and* r.

If the data in the delay VIDE (4.1.21) are smooth functions, the corresponding solution will essentially inherit this smoothness, except – similar to delay VIEs of the second kind – at the primary discontinuity points $\{\xi_\mu\}$. This is made precise in

Theorem 4.1.9 *Let* $a,\ b,\ g,\ K_1,\ K_2$ *and* ϕ *in (4.1.21) be* C^m*-functions on their respective domains, and assume that the delay* θ *is subject to the conditions (D1)–(D3) of Section 4.1.2, with* $d \geq m$. *Then:*

(a) The (unique) solution of the initial-value problem for (4.1.21) is $(m+1)$*-times continuously differentiable on each left-open macro-interval* $(\xi_\mu, \xi_{\mu+1}]$ *and has a bounded first derivative on* I.

(b) At $t = \xi_\mu$ $(\mu = 0, 1\ldots, \min\{m, M\})$ *we have*

$$\lim_{t\to\xi_\mu^-} y^{(\mu)}(t) = \lim_{t\to\xi_\mu^+} y^{(\mu)}(t),$$

while the $(\mu + 1)$*st derivative of* y *is in general not continuous at* $t = \xi_\mu$. *If* $\min\{m, M\} = m < M$, *the solution possesses a continuous* $(m + 1)$*st derivative on* $[\xi_m, T]$.

The **proof** can be found in, e.g. Brunner and Zhang (1999). Compare also El'sgol'ts and Norkin (1973), Neves and Feldstein (1976), de Gee (1985), and Bellen and Zennaro (2003) for related ideas in proofs for DDEs.

We will again summarise a number of regularity and smoothing results in a table; proofs (which can be based on Theorem 4.1.7) are left to the reader. See also Brunner and Zhang (1999).

Table 4.2. *Regularity and smoothing of solutions to delay VIDEs*

Delay VIDE (C^m–data)	Regularity at $t = \xi_\mu$ ($\mu = 0, 1, \ldots, M$)
• $y'(t) = f(t, y(t)) + (\mathcal{V}_\theta y)(t)$	$C^{2\mu}$ ('super-smoothing')
• $y'(t) = f(t, y(t), y(\theta(t))) + (\mathcal{V}_\theta y)(t)$	C^μ
• $y'(t) = f(t, y(t), y(\theta(t))) + (\mathcal{W}_\theta y)(t)$	C^μ
• $y'(t) = f(t, y(t), y(\theta(t)), y'(\theta(t))) + (\mathcal{V}_\theta y)(t)$	C^0 (no smoothing at $t = \xi_\mu$)

The reader may wish to complete the table by adding smoothing results for the delay VIDEs in which $\mathcal{V}_\theta$ has been replaced by $\mathcal{W}_\theta$ or by

$$(\mathcal{W}_\theta^1 y)(t) := \int_{\theta(t)}^t k(t, s, y(s), y'(s))ds.$$

4.1.5 Nonlinear delay problems

Nonlinear Volterra integral and integro-differential equations with non-vanishing delays have been used since the 1920s as mathematical models of population growth and related phenomena in biology. In this section we will briefly describe two such classes of Volterra functional equations; comments pointing to additional sources of nonlinear delay Volterra equations will be added at the end.

Example 4.1.1
In Part IV ('Studio delle azioni ereditarie') of his 1927 paper Volterra refined his earlier celebrated (ODE) 'predator–prey' model to include situations where 'historical actions cease after a certain interval of time' (see also Volterra (1939),

p. 8). This leads to a system of nonlinear Volterra integro-differential equations with constant delay $T_0 > 0$ (using again Volterra's notation),

$$\frac{dN_1}{dt} = N_1(t)\left(\varepsilon_1 - \gamma_1 N_2(t) - \int_{t-T_0}^{t} F_1(t-\tau)N_1(\tau)d\tau\right), \quad (4.1.33)$$
$$\frac{dN_2}{dt} = N_2(t)\left(-\varepsilon_2 + \gamma_2 N_1(t) + \int_{t-T_0}^{t} F_2(t-\tau)N_2(\tau)d\tau\right),$$

with $\varepsilon_i > 0,\ \gamma_i \geq 0$, and continuous $F_i(t) \geq 0$. Volterra later extended this model and its analysis to n interacting populations (see also his survey paper of 1939). Cushing (1977) is an excellent source on the further development of such population models based on VIDEs with delays; see also Bocharov and Rihan (2000) and its bibliography.

Example 4.1.2
Many basic mathematical models in epidemiology and population growth (see, e.g. Cooke and Yorke (1973), Waltman (1974), Cooke (1976), and Smith (1977)) are described by nonlinear Volterra integral equations of the second kind with (constant) delay $\tau > 0$, namely,

$$y(t) = \int_{t-\tau}^{t} P(t-s)G(s, y(s))ds + g(t), \quad t > t_0, \quad (4.1.34)$$

or

$$y(t) = \int_{t-\tau}^{t} P(t-s)G(y(s) + g(s))ds, \quad t > t_0. \quad (4.1.35)$$

Here, g is usually assumed to be such that $\lim_{t\to\infty} g(t) =: g(\infty)$ exists. These delay integral equations model the deterministic growth of a population $y = y(t)$ (e.g. of animals, or cells) or the spread of an epidemic with *immigration* into the population; it also has applications in economics.

Example 4.1.3
A generalisation of the above model is discussed in Bélair (1991): here, the delay τ in the delay (or: lag) function $\theta(t) := t - \tau(y(t))$ (life span) is no longer constant but depends on the size $y(t)$ of the population at time t (reflecting, e.g. crowding effects). Bélair's model corresponds to the delay VIE with *state-dependent delay*,

$$y(t) = \int_{t-\tau(y(t))}^{t} P(t-s)G(y(s))ds, \quad t > 0, \quad (4.1.36)$$

with $P(t) \equiv 1$. Here it is assumed that the number of births is a function of the population size only (that is, the birth rate is density dependent but not age dependent). For this choice of the kernel P it is tempting to 'simplify' the delay

VIE, by differentiating it with respect to t, to obtain the state-dependent (but 'local') DDE

$$y'(t) = \frac{G(y(t)) - G(y(t-\tau(y(t))))}{1-\tau'(y(t))G(y(t-\tau(y(t))))}\,. \tag{4.1.37}$$

While *any* constant $y(t) = y_c$ solves the above DDE, this is *not* true in the original DVIE (2.1.36): it is easily verified that $y(t) = y_c$ is a solution if, and only if, $y_c = G(y_c)\tau(y_c)$. This simple example also contains a warning: the use of the the DDE (2.1.37) as the basis for the ('indirect') numerical solution of the delay VIE (2.1.36) may lead to approximations for $y(t)$ that do not correctly reflect the dynamics of the original (highly nonlinear) delay integral equation.

4.1.6 Volterra functional equations of neutral type

The delay differential and integro-differential equations we have studied so far are functional equations of *retarded type*: the derivative $y'(t)$ of the unknown function depends on $y(t)$ and $y(\theta(t))$ but not on $y'(\theta(t))$. As we have seen, one of the consequences of this is that the solutions become smoother at the primary discontinuity points $\{\xi_\mu\}$. This is in general no longer true if the delay equation of of *neutral type*, as the simple linear model problem

$$y'(t) - cy'(\theta(t)) = ay(t) + by(\theta(t)), \quad t \geq t_0, \quad y(t) = \phi(t) \not\equiv 0,\ t \leq t_0, \tag{4.1.38}$$

with $c \neq 0$ and $\theta(t) = t - \tau$, readily shows. The following theorem is representative of neutral differential equations more general than (4.1.38). Its proof, and a general discussion of DDEs of neutral type, can be found in, e.g. the books by El'sgol'ts and Norkin (1973) and, especially, Hale (1977) and Hale and Verduyn Lunel (1993).

Theorem 4.1.10 *Assume that $\phi \in C^1[\theta(t_0, t_0]$. Then there exists a unique function y that coincides with ϕ on $[\theta(t_0, t_0]$, is in C^1 and satisfies (4.1.38) for $t \geq 0$, except possibly at the points $\xi_\mu = \mu\tau$ $(\mu \geq 0)$. This solution cannot have higher regularity than the initial function ϕ, and it is a C^1-function for $t \geq 0$ if, and only if, ϕ is such that*

$$\phi'(t_0 - c\phi'(\theta(t_0) = a\phi(t_0) + b\phi(\theta(t_0) + g(t_0).$$

Will the result of Theorem 4.1.10 substantially remain valid if the right-hand side of (4.1.38) also contains a memory term, for example

$$(\mathcal{W}_\theta^1 y)(t) := \int_{\theta(t)}^{t} \big(K_1(t,s)y(s) + K_2(t,s)y'(s)\big)\,ds?$$

To study this question consider the general (nonlinear) first-order delay VIDE of neutral type,

$$y'(t) = f(t, y(t), y(\theta(t)), y'(\theta(t))) + (\mathcal{V}^1 y)(t) + (\mathcal{V}^1_\theta y)(t), \quad t \in I, \quad (4.1.39)$$

where the kernels of the Volterra operators $\mathcal{V}^1$ and $\mathcal{V}^1_\theta$ also depend on $y'(s)$:

$$(\mathcal{V}^1 y)(t) := \int_{t_0}^{t} k_1(t, s, y(s), y'(s))ds, \quad (\mathcal{V}^1_\theta y)(t) := \int_{t_0}^{\theta(t)} k_2(t, s, y(s), y'(s))ds.$$

We will also consider the important case

$$y'(t) = f(t, y(t), y(\theta(t)), y'(\theta(t))) + (\mathcal{W}^1_\theta y)(t), \quad t \in I, \quad (4.1.40)$$

corresponding to the (nonlinear) Volterra operator

$$(\mathcal{W}^1_\theta y)(t) := \int_{\theta(t)}^{t} k(t, s, y(s), y'(s))ds.$$

As we have seen in Section 4.1.1, neutral delay differential equations often occur in what we called ‘Hale’s form’, which for (4.1.38) with $\theta(t) = t - \tau$ is given by

$$\frac{d}{dt}[y(t) - cy(\theta(t))] = ay(t) + by(\theta(t)), \quad t \geq t_0, \quad (4.1.41)$$

with $y(t) = \phi(t)$ when $t \in [\theta(t_0), t_0]$ (see also Liu (1999a, 1999b)). Note that here the initial function need only satisfy $\phi \in C[\theta(t_0), t_0]$. An obvious generalisation of this simple neutral functional equation is given by

$$\frac{d}{dt}[y(t) - (\mathcal{V}_\theta y)(t)] = F(t, y(t), y(\theta(t))), \quad t \in I, \quad (4.1.42)$$

with $y(t) = \phi(t), \ t \in [\theta(t_0, t_0]$ and

$$(\mathcal{V}_\theta y)(t) := \int_{0}^{\theta(t)} k_2(t, s, y(s))ds$$

(compare also Brunner and Vermiglio (2003)). Its ‘local’ counterpart is the neutral DDE

$$\frac{d}{dt}[y(t) - G(t, y(\theta(t)))] = F(t, y(t), y(\theta(t))), \quad (4.1.43)$$

corresponding to a smooth function G.

In view of applications, and to prepare for the collocation analysis to be presented in Section 4.5.4 we will focus on NFIDEs in the class (4.1.42). Results on the existence and uniqueness of solutions to the above initial-value problems can be obtained by considering the integrated forms of (4.1.42) and (4.1.43):

they are, respectively,

$$y(t) = \Phi_0 + (\mathcal{V}_\theta y)(t) + \int_{t_0}^{t} F(s, y(s), y(\theta(s)))ds, \quad t \in I, \tag{4.1.44}$$

with $y(t) = \phi(t),\ t \le t_0$, and $\Phi_0 := \phi(t_0) - (\mathcal{V}_\theta \phi)(t_0)$; and

$$y(t) = \Phi_0 + G(t, y(\theta(t)) + \int_{t_0}^{t} F(s, y(s), y(\theta(s)))ds, \quad t \in I, \tag{4.1.45}$$

with $\Phi_0 := \phi(t_0) - G(t_0, \phi(t_0))$. These NFIDEs are thus equivalent to initial-value problems for nonlinear second-kind Volterra integral equations with delay function θ. Their existence and uniqueness theory is a straightforward consequence of Theorems 4.1.1.

Alternatively, setting

$$z(t) := y(t) - (\mathcal{V}_\theta y)(t), \quad t \in I, \tag{4.1.46}$$

and

$$H(t, z, w) := F(t, z + \mathcal{V}_\theta y, w), \tag{4.1.47}$$

the initial-value problem (4.1.42) can be reformulated as an initial-value problem for z,

$$\begin{aligned} z'(t) &= H(t, z(t), y(\theta(t))), \quad t \in I, \\ z(t_0) &= \phi(t_0) - (\mathcal{V}_\theta \phi)(t_0)\ (= \Phi_0), \end{aligned} \tag{4.1.48}$$

whose solution then determines the solution y of the orginal problem (4.1.42) via the recursion

$$y(t) = z(t) + (\mathcal{V}_\theta y)(t), \quad t \in I, \tag{4.1.49}$$

with $y(t) = \phi(t)$ when $t \le t_0$. Clearly, the DDE (4.1.48) may be viewed as a sequence of initial-value problems on $I^{(\mu)} := [\xi_\mu, \xi_{\mu+1}]$ for a nonlinear ODE, with the explicit recursion (4.1.49) furnishing the expression for $y(\theta(t))$. This reformulation can also be used to obtain insight into the regularity properties of the solution y of the original NFIDE (4.1.42).

We note in passing the the analogous reformulation of the NDE (4.1.43) is

$$\begin{aligned} z'(t) &= H(t, z(t), y(\theta(t))), \quad t \in I, \\ z(t_0) &= \phi(t_0) - G(t_0, \phi(t_0))\ (= \Phi_0\ ), \end{aligned} \tag{4.1.50}$$

with

$$z(t) := y(t) - G(t, y(\theta(t))) \quad \text{and} \quad H(t, z, w) := F(t, z + G(t, y), w),$$

and with

$$y(t) = z(t) + G(t, y(\theta(t))), \quad t \in I.$$

(compare Liu (1999a), Vermiglio and Torelli (1998), Torelli and Vermiglio (2002) and, especially, Bellen and Zennaro (2003, Section 3.2.3)).

4.2 Collocation methods for DDEs: a brief review

A comprehensive treatment of continuous (explicit and implicit) Runge–Kutta methods for various classes of delay differential equations, including their convergence and asymptotic stability properties, can be found in the 2003 monograph by Bellen and Zennaro. We therefore restrict our discussion to the presentation of the basic definitions and to the description of piecewise polynomial collocation methods for DDEs. The corresponding convergence results will be obtained as particular cases of theorems for delay VIDEs (Section 4.5).

4.2.1 Constrained and θ-invariant meshes

Assume that the given delay function $\theta(t) = t - \tau(t)$ satisfies the assumptions (D1)–(D3) of Section 4.1.2 which we recall for the convenience of the reader:

(D1) $\theta \in C^d(I)$ for some $d \geq 0$, with $I := [t_0, T]$;
(D2) $\tau(t) \geq \tau_0 > 0$ for $t \in I$;
(D3) θ is strictly increasing on I.

This implies that the primary discontinuity points $\{\xi_\mu\}$ induced by θ and given by

$$\theta(\xi_\mu) = \xi_\mu - \tau(\xi_\mu) = \xi_{\mu-1}, \quad \mu = 1, \ldots, \quad \text{with } \xi_0 := t_0,$$

have the (uniform) separation property

$$\xi_\mu - \xi_{\mu-1} \geq \tau_0 > 0 \quad \text{for all } \mu \geq 1.$$

For ease of notation we will again assume that T defining $I = [t_0, T]$ is such that

$$T = \xi_{M+1} \quad \text{for some } M \geq 1,$$

and we recall the set $Z_M := \{\xi_\mu : \mu = 0, 1, \ldots, M\}$ introduced in Theorem 4.1.3.

Since, as we have already seen in Section 4.1.1, solutions of delay problems with non-vanishing delays generally suffer from a loss of reguarity at the primary discontinuity points $\{\xi_\mu\}$, the mesh I_h underlying the collocation space will have to include these points if the collocation solution is to attain its optimal global (or local) order (of superconvergence). Thus, we shall employ meshes of the form

$$I_h := \bigcup_{\mu=0}^{M} I_h^{(\mu)}, \quad I_h^{(\mu)} := \{t_n^{(\mu)} : \ \xi_\mu = t_0^{(\mu)} < t_1^{(\mu)} < \ldots < t_{N_\mu}^{(\mu)} = \xi_{\mu+1}\}. \tag{4.2.1}$$

Such a mesh is called a *constrained mesh* (with respect to θ) for I. We will refer to I_h as the *macro-mesh* and call the $I_h^{(\mu)}$ the underlying *local meshes*.

Definition 4.2.1

A mesh I_h for $I := [t_0, T]$ is said to be θ*-invariant* if it is constrained (that is, given by (4.2.1)) and if

$$\theta(I_h^{(\mu)}) = I_h^{(\mu-1)} \quad (\mu = 1, \ldots, M) \tag{4.2.2}$$

holds. We then have $N_\mu = N$ for all $\mu \geq 0$.

Observe that if I_h is θ-invariant then

$$t \in I_h^{(\mu)} \quad \Longrightarrow \quad \theta^{\mu-\nu}(t) \in I_h^{(\nu)} \quad (\nu = 0, 1, \ldots, \mu). \tag{4.2.3}$$

In analogy to the previous chapters we will use the following notation:

$$\sigma_n^{(\mu)} := (t_n^{(\mu)}, t_{n+1}^{(\mu)}], \quad h_n^{(\mu)} := t_{n+1}^{(\mu)} - t_n^{(\mu)}, \quad h^{(\mu)} := \max_{(n)} h_n^{(\mu)}, \quad h := \max_{(\mu)} h^{(\mu)},$$

and $\bar{\sigma}_n^{(\mu)} := [t_n^{(\mu)}, t_{n+1}^{(\mu)}]$.

For a given θ-invariant mesh I_h the collocation solution u_h will be an element of a piecewise polynomial space

$$S_{m+d}^{(d)}(I_h) := \{v \in C^d(I_h) : \ v|_{\sigma_n^{(\mu)}} \in \pi_{m+d} \ (0 \leq n < N; \ 0 \leq \mu \leq M)\}. \tag{4.2.4}$$

It follows from Section 2.1.1 (Definition 2.2.1) that this linear space has the dimension

$$\dim S_{m+d}^{(d)}(I_h) = (M+1)Nm + d + 1.$$

This suggests we choose the set of collocation points as

$$X_h := \bigcup_{\mu=0}^{M} X_h^{(\mu)} : \tag{4.2.5}$$

it is based on the local sets

$$X_h^{(\mu)} := \{t_n^{(\mu)} + c_i h_n^{(\mu)} : \ 0 \le c_1 < \ldots < c_m \le 1 \ (0 \le n \le N-1)\}.$$

In the collocation equation for a given delay equation with *non-vanishing* delay $\tau(t)$ we shall encounter the mapping $\theta(X_h^{(\mu)})$ (see, for example, (4.3.2)). It is clear that for *linear lag functions* θ and a given θ-invariant mesh I_h the set X_h defined in (4.2.5) is also θ-invariant. However, for *nonlinear* delays this will no longer be true. We record this important fact – which will affect the computational form of the collocation equation – in the following lemma. Its proof is straightforward and is left as an exercise.

Lemma 4.2.1 *Assume that the lag function θ satisfies (D1)–(D3), and let I_h be a θ-invariant mesh on $I = [t_0, T]$.*

(a) If θ is linear, then

$$\theta(X_h^{(\mu)}) = X_h^{(\mu-1)}, \quad \mu = 1, \ldots, M :$$

the set X_h of collocation points is also θ-invariant.

(b) For nonlinear θ this is no longer true: setting

$$\theta(t_n^{(\mu)} + c_i h_n^{(\mu)}) = t_n^{(\mu-1)} + \tilde{c}_i h_n^{(\mu-1)} =: \tilde{t}_{n,i}^{(\mu-1)} \quad (i = 1, \ldots, m),$$

the images $\{\tilde{c}_i\}$ of the $\{c_i\}$ satisfy

$$0 \le \tilde{c}_1 < \ldots < \tilde{c}_m \le 1 \quad (\textit{with} \quad \tilde{c}_i \ne c_i \quad \textit{in general}),$$

and they depend on the micro-interval $\sigma_n^{(\mu)}$ containing the collocation point $t_{n,i}^{(\mu)}$; that is, we have

$$\tilde{c}_i = \tilde{c}_i(n; \mu) \quad (i = 1, \ldots, m).$$

4.2.2 Collocation and continuous implicit Runge–Kutta methods

As will become apparent in more detail in the following section on delay VIEs and VIDEs, collocation solutions in $S_m^{(0)}(I_h)$ to DDEs with non-vanishing delays satisfying (D1)–(D3) will have the same global and local (super-) convergence properties as those for ODEs, provided the underlying mesh is constrained and (for superconvergence) θ-invariant. In the case of ODEs this was first observed by Bellen (1984). Thus, we will not state these convergence results explicitly here since we shall obtain them in Section 4.1.4 as particular cases of results for VIDEs with non-vanishing delays. However, we briefly describe the collocation equations for DDEs (the reader may wish to consult the monograph by Bellen and Zennaro (2003) for a more detailed treatment).

Consider the general (neutral) DDE

$$\begin{aligned} y'(t) &= f(t, y(t), y(\theta(t)), y'(\theta(t))), \quad t \in I := [t_0, T], \\ y(t) &= \phi(t), \quad t \le t_0, \end{aligned} \tag{4.2.6}$$

and assume that the delay function θ satisfies (D1)–(D3). For a θ-invariant mesh I_h given by (4.2.1) and (4.2.2), let $u_h \in S_m^{(0)}(I_h)$ be the collocation solution to (4.2.6):

$$\begin{aligned} u_h'(t) &= f(t, u_h(t), u_h(\theta(t)), u_h'(\theta(t))), \quad t \in X_h, \\ u_h(t) &= \phi(t), \quad t \le t_0, \end{aligned} \tag{4.2.7}$$

with initial function $\phi \in C^1[\theta(t_0), t_0]$. On the subinterval $\bar{\sigma}_n^{(\mu)}$ we use the local representations

$$\begin{aligned} u_h'(t_n^{(\mu)} + vh_n^{(\mu)}) &= \sum_{j=1}^{m} L_j(v) Y_{n,j}^{(\mu)}, \quad v \in (0, 1], \\ u_h(t_n^{(\mu)} + vh_n^{(\mu)}) &= y_n^{(\mu)} + h_n^{(\mu)} \sum_{j=1}^{m} \beta_j(v) Y_{n,j}^{(\mu)}, \quad v \in [0, 1], \end{aligned} \tag{4.2.8}$$

where $y_n^{(\mu)} := u_h(t_n^{(\mu)},\ Y_{n,j}^{(\mu)} := u_h'(t_{n,j}^{(\mu)}$. Note that the assumed θ-invariance of I_h implies, by Lemma 4.2.1, that

$$u_h(\theta(t_n^{(\mu)} + vh_n^{(\mu)}) = y_n^{(\mu-1)} + h_n^{(\mu-1)} \sum_{j=1}^{m} \beta_j(\tilde{v}) Y_{n,j}^{(\mu-1)}, \quad v \in [0, 1],$$

since $\theta(t_n^{(\mu)} + vh_n^{(\mu)}) = t_n^{(\mu-1)} + \tilde{v} h_n^{(\mu)}$ for appropriate $\tilde{v} \in [0, 1]$. If $\mu = 0$ then

$$u_h^{(\nu)}(\theta(t_n^{(0)} + vh_n^{(0)}) = \phi(t_n^{(-1)} + \tilde{v} h_n^{(-1)}) \quad (\nu = 0, 1),$$

where we have set $\theta(t_n^{(0)} + vh_n^{(0)}) =: t_n^{(-1)} + \tilde{v} h_n^{(-1)}$. The computational form of the collocation equation (4.2.7) at $t = t_{n,i}^{(\mu)}$ then becomes

$$Y_{n,i}^{(\mu)} = f\left(t_{n,i}^{(\mu)}, y_n^{(\mu)} + h_n^{(\mu)} \sum_{j=1}^{m} a_{i,j} Y_{n,j}^{(\mu)}, \Phi_{n,i}^{(\mu-1)}\right) \quad (i = 1, \ldots, m), \tag{4.2.9}$$

with

$$\Phi_{n,i}^{(\mu-1)} := \sum_{j=1}^{m} L_j(\tilde{c}_i) Y_{n,j}^{(\mu-1)}.$$

Recall that if the lag function θ is *nonlinear* then $\tilde{c}_i \neq c_i$ in general; for *linear* θ we have $\tilde{c}_i = c_i$, and hence $\sum_{j=1}^{m} L_j(\tilde{c}_i) Y_{n,j}^{(\mu-1)} = Y_{n,i}^{(\mu-1)}$. Equations (4.2.8) and (4.2.9) describe an m-stage *continuous implicit Runge–Kutta method* for the DDE (4.2.6). We observe that, in contrast to 'classical' (discrete) Runge–Kutta methods, the collocation scheme automatically furnishes the 'natural' continuous (local) interpolant on each $\bar{\sigma}_n^{(\mu)}$.

Since the results on the attainable orders of global and local (super-) convergence of collocation solutions $u_h \in S_m^{(0)}(I_h)$ to DDEs will be obtained as special cases of such results for VIDEs with non-vanishing delays (Sections 4.4.2 and 4.5.3), we will not state them explicitly here. The first comprehensive convergence analysis of collocation solutions to DDEs was given by Bellen (1984).

4.3 Collocation for second-kind VIEs with delays

4.3.1 The exact collocation equations

The collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ for the delay integral equation

$$y(t) = g(t) + (\mathcal{V}y)(t) + (\mathcal{V}_\theta y)(t), \quad t \in (t_0, T], \tag{4.3.1}$$

with

$$(\mathcal{V}y)(t) := \int_{t_0}^{t} K_1(t,s)y(s)ds, \qquad (\mathcal{V}_\theta y)(t) := \int_{t_0}^{\theta(t)} K_2(t,s)y(s)ds,$$

and with initial condition $y(t) = \phi(t),\ t \le t_0$, is defined by the collocation equation

$$u_h(t) = g(t) + (\mathcal{V}u_h)(t) + (\mathcal{V}_\theta u_h)(t), \quad t \in X_h. \tag{4.3.2}$$

The values of u_h at $t \in [\theta(t_0), t_0]$ are determined by the given initial function for (4.2.1), $u_h(t) = \phi(t)$. As for classical second-kind Volterra integral equations we will also consider the *iterated collocation solution* corresponding to u_h:

$$u_h^{it}(t) := g(t) + (\mathcal{V}u_h)(t) + (\mathcal{V}_\theta u_h)(t), \quad t \in (t_0, T]. \tag{4.3.3}$$

The lag function $\theta = \theta(t) = t - \tau(t)$ will be assumed to satisfy the conditions (D1)–(D3) of Section 4.1.2, and the mesh I_h on $I := [t_0, T]$ will be assumed to be the θ-invariant mesh defined by (4.2.2). As we indicated in Section 4.2.1 (cf. (4.2.5)) the collocation points X_h are given by

$$X_h := \bigcup_{\mu=0}^{M} X_h^{(\mu)}, \quad X_h^{(\mu)} := \{t_{n,i}^{(\mu)} := t_n^{(\mu)} + c_i h_n^{(\mu)} :$$
$$0 \le c_1 < \ldots < c_m \le 1\ (0 \le n \le N-1)\}.$$

On $\sigma_n^{(\mu)} := (t_n^{(\mu)}, t_{n+1}^{(\mu)}]$ the collocation solution will have the usual local Lagrange representation,

$$u_h(t_n^{(\mu)} + vh_n^{(\mu)}) = \sum_{j=1}^{m} L_j(v)U_{n,j}^{(\mu)}, \quad v \in (0,1], \quad \text{with} \quad U_{n,j}^{(\mu)} := u_h(t_{n,j}^{(\mu)}). \tag{4.3.4}$$

Since the contribution of the classical Volterra term $\mathcal{V}u_h$ to the computational form of the collocation was analysed in detail in Chapter 2, we will focus here on the terms induced by the delay part $(\mathcal{V}_\theta u_h)(t)$ with $t = t_{n,i}^{(\mu)}$.

Assume first that the lag function θ is *linear*. Since, as we have seen in Lemma 4.2.1, the θ-invariance of the mesh I_h implies the θ-invariance of the set X_h of collocation points, we may write, using the fact that $\theta(t_{n,i}^{(\mu)}) = t_{n,i}^{(\mu-1)}$,

$$(\mathcal{V}_\theta u_h)(t_{n,i}^{(\mu)}) = \int_{t_0}^{\theta(t_{n,i}^{(\mu)})} K_2(t_{n,i}^{(\mu)}, s)u_h(s)ds = \int_{t_0}^{t_{n,i}^{(\mu-1)}} K_2(t_{n,i}^{(\mu)}, s)u_h(s)ds, \tag{4.3.5}$$

and hence, recalling the local representation (4.2.4) of u_h,

$$\begin{aligned}(\mathcal{V}_\theta u_h)(t_{n,i}^{(\mu)}) &= \Psi_n^{(\mu-1)}(t_{n,i}^{(\mu)}) \\ &\quad + h_n^{(\mu-1)} \sum_{j=1}^{m} \left(\int_0^{c_i} K_2(t_{n,i}^{(\mu)}, t_n^{(\mu-1)} + sh_n^{(\mu-1)})L_j(s)ds \right) U_{n,j}^{(\mu-1)},\end{aligned} \tag{4.3.6}$$

with lag term

$$\Psi_n^{(\mu-1)}(t) := \int_{t_0}^{\xi_{\mu-1}} K_2(t,s)u_h(s)ds + \int_{\xi_{\mu-1}}^{t_n^{(\mu-1)}} K_2(t,s)u_h(s)ds \quad (t \in \sigma_n^{(\mu)}). \tag{4.3.7}$$

If θ is *nonlinear*, then the above terms have to be modified: by the (strict) monotonicity assumption (D3) for θ (cf. Section 4.2.1), the image of $t_{n,i}^{(\mu)} \in \sigma_n^{(\mu)}$ under θ lies in $\sigma_n^{(\mu-1)}$ (but will be different from the collocation point $t_{n,i}^{(\mu-1)}$); that is,

$$\theta(t_{n,i}^{(\mu)}) = t_n^{(\mu-1)} + \tilde{c}_i h_n^{(\mu-1)} =: \tilde{t}_{n,i}^{(\mu-1)} \quad (i = 1, \ldots, m), \tag{4.3.8}$$

with

$$0 \le \tilde{c}_1 < \ldots < \tilde{c}_m \le 1 \quad \text{and} \quad \tilde{c}_i = \tilde{c}_i(n;\mu)$$

(cf. Lemma 4.2.1). Accordingly, the expression (4.2.6) for $(\mathcal{V}_\theta u_h)(t_{n,i}^{(\mu)})$ now reads

$$(\mathcal{V}_\theta u_h)(t_{n,i}^{(\mu)}) = \Psi_n^{(\mu-1)}(t_{n,i}^{(\mu)}) + h_n^{(\mu-1)} \sum_{j=1}^{m} \left(\int_0^{\tilde{c}_i} K_2(t_{n,i}^{(\mu)}, t_n^{(\mu-1)} + s h_n^{(\mu-1)}) L_j(s) ds \right) U_{n,j}^{(\mu-1)}. \tag{4.3.9}$$

Hence, adapting the notation of Section 2.2.2 for the classical Volterra part $\mathcal{V}u_h$, in the collocation equation, (4.3.2) at $t = t_{n,i}^{(\mu)}$ $(i = 1, \dots, m)$ can now be written as

$$U_{n,i}^{(\mu)} = g(t_{n,i}^{(\mu)}) + F_n^{(\mu)}(t_{n,i}^{(\mu)}) + (\mathcal{V}_\theta u_h)(t_{n,i}^{(\mu)}) + h_n^{(\mu)} \sum_{j=1}^{m} \left(\int_0^{c_i} K_1(t_{n,i}^{(\mu)}, t_n^{(\mu)} + s h_n^{(\mu)}) L_j(s) ds \right) U_{n,j}^{(\mu)}. \tag{4.3.10}$$

Let $\mathbf{U}_n^{(\mu)} := (U_{n,1}^{(\mu)}, \dots, U_{n,m}^{(\mu)})^T$ and, in analogy to Section 2.2.2 (cf (2.2.12)), define the matrices

$$B_n^{(\mu)} := \left(\begin{array}{c} \int_0^{c_i} K_1(t_{n,i}^{(\mu)}, t_n^{(\mu)} + s h_n^{(\mu)}) L_j(s) ds \\ (i, j = 1, \dots, m) \end{array} \right),$$

$$\tilde{B}_n^{(\mu-1)} := \left(\begin{array}{c} \int_0^{\tilde{c}_i} K_2(t_{n,i}^{(\mu)}, t_n^{(\mu-1)} + s h_n^{(\mu-1)}) L_j(s) ds \\ (i, j = 1, \dots, m) \end{array} \right).$$

Finally, set $\mathbf{g}_n^{(\mu)} := (g(t_{n,1}^{(\mu)}), \dots, g(t_{n,m}^{(\mu)}))^T$, $\quad \mathbf{G}_n^{(\mu)} := (F(t_{n,1}^{(\mu)}), \dots, F_n^{(\mu)}(t_{n,m}^{(\mu)})^T$, and

$$\mathbf{Q}_n^{(\mu-1)} := (\Psi_n^{(\mu-1)}(t_{n,1}^{(\mu)}), \dots, \Psi_n^{(\mu-1)}(t_{n,m}^{(\mu)}))^T.$$

Thus, the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ to (4.3.1) on $\sigma_n^{(\mu)}$ is described by (4.3.4) in which the $\mathbf{U}_n^{(\mu)}$ is the solution of the linear algebraic system (4.3.10) which we now write in the form

$$[\mathcal{I}_m - h_n^{(\mu)} B_n^{(\mu)}]\mathbf{U}_n^{(\mu)} = \mathbf{g}_n^{(\mu)} + \mathbf{G}_n^{(\mu)} + \mathbf{Q}_n^{(\mu-1)} + h_n^{(\mu-1)} \tilde{B}_n^{(\mu)} \mathbf{U}_n^{(\mu-1)} \tag{4.3.11}$$

$(n = 0, 1, \dots, m;\ \mu = 0, 1, \dots, M)$.

The following theorem on the existence of a unique collocation solution is a natural, and obvious, extension of Theorem 2.2.1.

Theorem 4.3.1 *Assume that $g,\ \theta,\ K_1$ and K_2 are continuous on their respective domains $I,\ D$ and D_θ, with the delay θ satisfying (D1)–(D3) of Section 4.2.1.*

Then there exists an $\bar{h} > 0$ so that for any θ-invariant mesh I_h with $h \in (0, \bar{h})$ and any initial function $\phi \in [\theta(t_0), t_0]$ each of the linear algebraic systems (4.3.11) possesses a unique solution $\mathbf{U}_n^{(\mu)} \in \mathbb{R}^m$. Hence, the collocation equation (4.3.2) defines a unique collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ for (4.3.1) whose local representation on the subinterval $\sigma_n^{(\mu)}$ is given by (4.3.4).

The computational form of the *iterated collocation solution* (4.3.3) at $t = t_n^{(\mu)} + vh_n^{(\mu)} \in \bar{\sigma}_n^{(\mu)}$ can be written as

$$\begin{aligned} u_h^{it}(t) = {} & g(t) + F_n^{(\mu)}(t) + \Psi_n^{(\mu-1)}(t) \\ & + h_n^{(\mu)} \sum_{j=1}^{m} \left(\int_0^v K_1(t, t_n^{(\mu)} + sh_n^{(\mu)}) L_j(s) ds \right) U_{n,j}^{(\mu)} \\ & + h_n^{(\mu-1)} \sum_{j=1}^{m} \left(\int_0^{\tilde{v}} K_2(t, t_n^{(\mu-1)} + sh_n^{(\mu-1)}) L_j(s) ds \right) U_{n,j}^{(\mu-1)}, \quad v \in [0, 1]. \end{aligned} \tag{4.3.12}$$

The classical lag term (cf. (2.2.8)) has, for $t \in \sigma_n^{(\mu)}$, the form

$$F_n^{(\mu)}(t) := \int_{t_0}^{\xi_\mu} K_1(t, s) u_h(s) ds + \int_{\xi_\mu}^{t_n^{(\mu)}} K_1(t, s) u_h(s) ds \tag{4.3.13}$$

while the lag term $\Psi_n^{(\mu-1)}(t)$ corresponding to the delay operator $\mathcal{V}_\theta$ is given above by (4.3.7). The image $\tilde{t} := t_n^{(\mu-1)} + \tilde{v}h_n^{(\mu-1)}$ of $t = t_n^{(\mu)} + vh_n^{(\mu)}$ under θ depends on the nature of the delay function θ: if θ is *linear* then we have $\tilde{v} = v$; for *nonlinear* θ the value of $\tilde{v} \in [0, 1]$ must be obtained from

$$\theta(t_n^{(\mu)} + vh_n^{(\mu)}) =: t_n^{(\mu-1)} + \tilde{v}h_n^{(\mu-1)}, \quad v \in (0, 1]. \tag{4.3.14}$$

Observe that $u_h^{it} \in C(t_0, T]$ whenever the given data defining the initial-value problem for (4.3.1) are continuous functions and if

$$u_h^{it}(t_0) = g(t_0) - \int_{\theta(t_0)}^{t_0} K_2(t_0, s)\phi(s)) ds$$

holds. Moreover, if the right-hand side of the above equation coincides with $\phi(t_0)$ (cf. Theorem 4.1.1), then u_h^{it} is also continuous at $t = t_0$.

$$u_h^{it}(t) = u_h(t) \quad \text{for all} \quad t \in X_h.$$

Since second-kind Volterra integral equations with non-vanishing delays often arise in the particular form (4.1.8),

$$y(t) = g(t) + (\mathcal{W}_\theta y)(t), \quad t \in (t_0, T], \tag{4.3.15}$$

where

$$(\mathcal{W}_\theta y)(t) := \int_{\theta(t)}^{t} K(t,s)y(s)ds,$$

we present the corresponding computational form of the collocation equation defining $u_h \in S_{m-1}^{(-1)}(I_h)$ in some detail (although it could of course be formally obtained by setting $K_2 = -K_1$ in (4.3.9) and (4.3.10)).

We first note that for $t = t_{n,i}^{(\mu)}$ we have

$$\begin{aligned}(\mathcal{W}_\theta u_h)(t) = &\int_{\theta(t)}^{t_{n+1}^{(\mu-1)}} K(t,s)u_h(s)ds \\ &+ \int_{t_{n+1}^{(\mu-1)}}^{\xi_\mu} K(t,s)u_h(s)ds + \int_{\xi_\mu}^{t_n^{(\mu)}} K(t,s)u_h(s)ds \\ &+ h_n^{(\mu)} \int_0^{c_i} K(t, t_n^{(\mu)} + sh_n^{(\mu)})u_h(t_n^{(\mu)} + sh_n^{(\mu)})ds,\end{aligned} \tag{4.3.16}$$

where

$$\theta(t) = \theta(t_{n,i}^{(\mu)}) = \begin{cases} t_{n,i}^{(\mu-1)} = t_n^{(\mu-1)} + c_i h_n^{(\mu-1)} & \text{if } \theta \text{ is } \textit{linear}, \\ \tilde{t}_{n,i}^{(\mu-1)} := t_n^{(\mu-1)} + \tilde{c}_i h_n^{(\mu-1)} & \text{if } \theta \text{ is } \textit{nonlinear}. \end{cases}$$

Define, for $t = t_n^{(\mu)} + c_i h_n^{(\mu)}$,

$$\begin{aligned}\bar{\Psi}_n^{(\mu-1)}(t) := &h_n^{(\mu-1)} \int_{\tilde{c}_i}^{1} K(t, t_n^{(\mu-1)} + sh_n^{(\mu-1)})u_h(t_n^{(\mu-1)} + sh_n^{(\mu-1)})ds \\ &+ \int_{t_{n+1}^{(\mu-1)}}^{\xi_\mu} K(t,s)u_h(s)ds + \int_{\xi_\mu}^{t_n^{(\mu)}} K(t,s)u_h(s)ds.\end{aligned} \tag{4.3.17}$$

The collocation equation for (4.3.15) on $\sigma_n^{(\mu)}$ then becomes

$$\begin{aligned}U_{n,i}^{(\mu)} = &g(t_{n,i}^{(\mu)}) + \bar{\Psi}_n^{(\mu-1)}(t_{n,i}^{(\mu)}) \\ &+ h_n^{(\mu)} \sum_{j=1}^{m} \left(\int_0^{c_i} K(t_{n,i}^{(\mu)}, t_n^{(\mu)} + sh_n^{(\mu)})L_j(s)ds \right) U_{n,j}^{(\mu)} \quad (i = 1, \dots, m).\end{aligned} \tag{4.3.18}$$

Hence, the resulting linear algebraic system for $\mathbf{U}_n^{(\mu)} \in \mathbb{R}^m$ defining the local representation of u_h on $\sigma_n^{(\mu)}$ (cf. (4.3.4)) has the form

$$[\mathcal{I}_m - h_n^{(\mu)} B_n^{(\mu)}]\mathbf{U}_n^{(\mu)} = \mathbf{g}_n^{(\mu)} + \bar{\mathbf{G}}_n^{(\mu-1)}, \tag{4.3.19}$$

with $\mathbf{g}_n^{(\mu)} := (g(t_{n,1}^{(\mu)}), \dots, g(t_{n,m}^{(\mu)}))^T$ and $\bar{\mathbf{G}}_n^{(\mu-1)} := (\bar{\Psi}_n^{(\mu-1)}(t_{n,1}^{(\mu)}), \dots, \bar{\Psi}_n^{(\mu-1)}(t_{n,m}^{(\mu)}))^T$.

The corresponding *iterated collocation solution* at $t = t_n^{(\mu)} + vh_n^{(\mu)} \in \bar{\sigma}_n^{(\mu)}$

can be then computed via

$$u_h^{it}(t) = g(t) + \bar{\Psi}_n^{(\mu-1)}(t) + h_n^{(\mu)} \sum_{j=1}^{m} \left(\int_0^{c_i} K(t, t_n^{(\mu)} + sh_n^{(\mu)}) L_j(s) ds \right) U_{n,j}^{(\mu)}, \quad v \in [0, 1]. \tag{4.3.20}$$

4.3.2 Global convergence results

The collocation error $e_h := y - u_h$ associated with the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ for the delay integral equation (4.3.1) solves the initial-value problem

$$e_h(t) = \delta_h(t) + (\mathcal{V}e_h)(t) + (\mathcal{V}_\theta e_h)(t), \quad t \in (t_0, T], \tag{4.3.21}$$

with initial condition $e_h(t) = 0$ for $t \in [\theta(t_0), t_0]$. The defect δ_h vanishes on the set X_h. For $t \in \sigma_n^{(\mu)}$ $(\mu \geq 1)$ the above error equation can be written as

$$e_h(t) = E_\mu(t) + \delta_h(t) + \int_{\xi_\mu}^{t_0} K_1(t, s) e_h(s) ds, \tag{4.3.22}$$

where

$$E_\mu(t) := \sum_{\nu=0}^{\mu-1} \int_{\xi_\nu}^{\xi_{\nu+1}} K_1(t, s) e_h(s) ds + (\mathcal{V}_\theta e_h)(t). \tag{4.3.23}$$

On the first macro-interval $(t_0, \xi_1]$ we have

$$E_0(t) := (V_\theta e_h)(t) = -\int_{\theta(t)}^{t_0} K_2(t, s) e_h(s) ds = 0.$$

If the given functions in (4.3.1) have continuous derivatives of at least order m on their respective domains, the global convergence and order analysis can be based again on the (local) representation of the collocation error by means of the Peano Kernel Theorem: in analogy to the approach in Section 2.2.4 we now have

$$e_h(t_n^{(\mu)} + vh_n^{(\mu)}) = \sum_{j=1}^{m} L_j(v) \mathcal{E}_{n,j}^{(\mu)} + (h_n^{(\mu)})^m R_{m,n}^{(\mu)}(v), \quad v \in (0, 1], \tag{4.3.24}$$

with $\mathcal{E}_{n,j}^{(\mu)} := e_h(t_{n,j}^{(\mu)})$. The definition of the Peano remainder terms $R_{m,n}^{(\mu)}(v)$ is obvious from (2.2.30), (2.2.31) in the proof of Theorem 2.2.3.

In order to obtain an estimate for e_h on $(t_0, \xi_1]$ we can resort directly to the proof of Theorem 2.2.3: a trivial change in the notation yields

$||\mathcal{E}_n^{(0)}||_1 = \mathcal{O}((h^{(0)})^m)$ (we have set $\mathcal{E}_n^{(\mu)} := (\mathcal{E}_{n,1}^{(\mu)}, \dots, \mathcal{E}_{n,m}^{(\mu)})^T$), and hence it follows that

$$||e_h||_{0,\infty} := \sup_{t \in I^{(0)}} |e_h(t)| \le C_0(h^{(0)})^m \quad (n = 0, 1, \dots, N-1).$$

A simple induction argument, employing the estimates for the terms $E_\mu(t)$ on $I^{(\mu)}$ in (4.3.22), together with the observation that by the conditions (D1)–(D3) for the lag function θ the number $(M+1)$ of macro-intervals $I^{(\mu)} := [\xi_\mu, \xi_{\mu+1}]$ is finite, yields the results summarised in the following theorem.

Theorem 4.3.2 *Assume*

(a) The given functions g, K_1, K_2 and ϕ in (4.3.1) all possess continuous derivatives of order m on their respective domains.
(b) The lag function $\theta(t) = t - \tau(t)$ is subject to the conditions (D1)–(D3) of Section 4.2.1, with $d \ge m$ in (D1).
(c) $u_h \in S_{m-1}^{(-1)}(I_h)$ is the collocation solution to (4.3.1) corresponding to a θ-invariant mesh I_h with $h \in (0, \bar{h})$, with $\bar{h}$ defined in Theorem 4.3.1.

Then for any set of collocation parameters $\{c_i : 0 \le c_1 < \dots < c_m \le 1\}$ the collocation error $e_h := y - u_h$ has the property that

$$||e_h||_\infty := \sup_{t \in (t_0, T]} |e_h(t)| \le Ch^m. \tag{4.3.25}$$

The constant C depends on the $\{c_i\}$ but not on $h := \max_{(n,\mu)} h_n^{(\mu)}$.

Not surprisingly, the global superconvergence result of Theorem 2.2.5 for the *iterated collocation solution* u_h^{it} remains valid in the case of second-kind Volterra integral equations with non-vanishing delays.

Theorem 4.3.3 *Suppose that the assumptions (a)–(c) of Theorem 4.3.2 hold, but with $m+1$ replacing m in (a) and (b). If the collocation parameters $\{c_i\}$ are chosen so that the orthogonality condition*

$$J_0 := \int_0^1 \prod_{i=1}^m (s - c_i) ds = 0 \tag{4.3.26}$$

is satisfied, then the iterated collocation solution corresponding to the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ for (4.3.1) is globally superconvergent on I_h:

$$||y - u_h^{it}||_\infty \le Ch^{m+1},$$

with C depending on the $\{c_i\}$ but not on h.

Proof The key to the proof of Theorem 2.2.5 (and of Theorem 2.2.6) on superconvergence of iterated collocation solutions for classical second-kind VIEs was the variation-of-constants formula (or 'resolvent representation') of

$$e_h^{it} := y - u_h^{it} = e_h - \delta_h,$$

together with the general global convergence result of Theorem 2.2.3. It is clear that the analogous approach, based on Theorem 4.1.2, works here, too: for $t = t_n^{(\mu)} + vh_n^{(\mu)} \in \bar{\sigma}_n^{(\mu)}$ Theorem 4.1.2 yields, with e_h and δ_h replacing $y,\ g$ and $g_0 = g$, respectively,

$$e_h^{it}(t) = \int_{\xi_\mu}^{t} R_1(t,s)\delta_h(s)ds + \sum_{\nu=0}^{\mu-1}\int_{\xi_\nu}^{\xi_{\nu+1}} R_{\mu,\nu}(t,s)\delta_h(s)ds$$
$$+\sum_{\nu=0}^{\mu-1}\int_{\xi_\nu}^{\theta^{\mu-\nu}(t)} Q_{\mu,\nu}(t,s)\delta_h(s)ds. \tag{4.3.27}$$

We now adapt the techniques employed in the proofs of Theorem 1.1.3 (cf. (1.1.39) and (1.1.40)) and Theorem 2.2.5 ((2.2.40), (2.2.41)) to (4.3.27). The integrals over subintervals $[t^{(\nu)}, t_{l+1}^{(\nu)}]$ can then be replaced by the sum of an interpolatory m-point quadrature formula with respect to the collocation points in that interval and the corresponding quadrature error. The expression given by the quadrature formula has value zero, since $\delta_h(t) = 0$ for $t \in X_h$. Due to the assumed regularity of the data (which is inherited on D by the resolvent R_1 and piecewise on I and D, respectively, by the defect δ_h and the functions $R_{\mu,\nu},\ Q_{\mu,\nu}$), the orthogonality condition (4.3.26) implies that all quadrature errors are $\mathcal{O}(h^{m+1})$.

It remains to deal with the integrals

$$\int_{t_n^{(\mu)}}^{t} R_1(t,s)\delta_h(s)ds \qquad \text{and} \qquad \int_{t_n^{(\nu)}}^{\theta^{\mu-\nu}(t)} Q_{\mu,\nu}(t,s)\delta_h(s)ds$$

(recall from (4.2.3), following the definition of a θ-invariant mesh, that $\theta^{\mu-\nu}(t) \in \sigma_n^{(\nu)}$ if $t \in \sigma_n^{(\mu)}$). It is easily verified (using the global convergence result of Theorem 4.3.2 and (4.3.21)) that $||\delta_h||_\infty = \mathcal{O}(h^m)$. Thus, in the estimation of the above integrals (via the usual scaling) the uniform estimate for δ_h is multiplied by h, leading to the required $\mathcal{O}(h^{m+1})$-term in Theorem 4.3.3.

Corollary 4.3.4 *In the particular delay integral equation (4.3.15) assume that $g \in C^{m+1}(I)$ and $K \in C^{m+1}(\bar{D}_\theta)$, with $\bar{D}_\theta := \{(t,s) :\ \theta(t) \le s \le t,\ t \in I\}$, and let the delay function θ satisfy (D1)–(D3) with $d \ge m+1$. Then the iterated collocation solution based on $u_h \in S_{m-1}^{(-1)}(I_h)$ and defined by (4.3.20) has the global superconvergence property*

$$||y - u_h^{it}||_\infty \le Ch^{m+1}$$

provided the mesh I_h is θ-invariant, the $\{c_i\}$ underlying the set X_h of collocation points satisfy $J_0 = 0$ (cf. (4.3.26)), and $\phi \in C^{m+1}[\theta(t_0), t_0]$.

4.3.3 Local superconvergence results

The proof of the global superconvergence result in Theorem 4.3.3 indicates that we can readily modify it – as we have already seen in Section 2.2.5 when we established local superconvergence results for classical Volterra integral equations of the second kind – to obtain the 'non-vanishing delay analogues' of Theorem 2.2.6 and Corollaries 2.2.7–2.2.9. The key to the proofs of these results is once more the variation-of-constants formula of Theorem 4.1.2, evaluated at $t = t_n^{(\mu)} \in I_h$.

Theorem 4.3.5 *Let the given functions $g,\ K_1,\ K_2$ and ϕ in the delay integral equation (4.3.1) have continuous derivatives of order $m+\kappa$ in their respective domains $I,\ D,\ D_\theta$ and $[\theta(t_0), t_0]$, and assume that the delay function θ is subject to the conditions (D1)–(D3) of Section 4.2.1, with $d \geq m+\kappa$ in (D1). If $u_h \in S_{m-1}^{(-1)}(I_h)$ denotes the collocation solution, for a θ-invariant mesh I_h, with corresponding iterated collocation solution u_h^{it}, and if the collocation parameters are so that the orthogonality conditions (2.2.42),*

$$J_\nu := \int_0^1 s^\nu \prod_{i=1}^{m}(s-c_i)ds \quad (0 \leq \nu \leq \kappa - 1),$$

hold, with $J_\kappa \neq 0$, then

$$\max_{t \in I_h \setminus \{t_0\}} |y(t) - u_h^{it}| \leq C h^{m+\kappa}$$

is true whenever $h \in (0, \bar{h})$.

If, in addition, we have $c_m = 1$ (implying $\kappa < m$), then u_h itself exhibits local superconvergence at the mesh points:

$$\max_{t \in I_h \setminus \{t_0\}} |y(t) - u_h(t)| \leq C h^{m+\kappa}.$$

Proof Our starting point is (4.3.27) in the proof of Theorem 4.3.3 where we now set $t = t_n^{(\mu)}$. Hence,

$$e_h^{it}(t_n^{(\mu)}) = \int_{\xi_\mu}^{t_n^{(\mu)}} R_1(t_n^{(\mu)}, s)\delta_h(s)ds + \sum_{\nu=0}^{\mu-1} \int_{\xi_\nu}^{\xi_{\mu+1}} R_{\mu,\nu}(t_n^{(\mu)}, s)\delta_h(s)ds$$
$$+ \sum_{\nu=0}^{\mu-1} \int_{\xi_\nu}^{\theta^{\mu-\nu}(t_n^{(\mu)})} Q_{\mu,\nu}(t_n^{(\mu)}, s)\delta_h(s)ds$$

($0 \leq n < N;\ 0 \leq \mu \leq M$), with $\theta^{\mu-\nu}(t_n^{(\mu)}) = t_n^{(\nu)}$ (cf. (4.2.3)). Hence, the by now rather familiar quadrature argument is applicable: since the defect δ_h vanishes on X_h, and since the orthogonality and regularity conditions imply that the quadrature errors induced by the interpolatory m-point quadrature formulas based on the $\{c_i\}$ are all of order $\mathcal{O}(h^{m+\kappa})$, with the number $M+1$ of macro-intervals $I^{(\mu)}$ being finite, the first assertion in Theorem 4.3.4 follows immediately.

The second assertion is based on the fact that when $c_m = 1$, each mesh point $t_n^{(\mu)}$ $(1 \leq n \leq N)$ is a collocation point and thus $u_h^{it}(t_n^{(\mu)}) = u_h(t_n^{(\mu)})$, since $\delta_h(t_n^{(\mu)}) = 0$. Note also that $e_h^{it}(t_0) = 0$ because $u_h^{it}(t_0) = y(t_0^+)$.

Corollary 4.3.6 *Assume $\kappa = m$ in Theorem 4.3.4. Then collocation in $S_{m-1}^{(-1)}(I_h)$ at the Gauss points leads to an iterated collocation solution with the property that*

$$\max_{t \in I_h} |y(t) - u_h^{it}(t)| \leq Ch^{2m},$$

while

$$\max_{t \in I_h \setminus \{t_0\}} |y(t) - u_h(t)| \leq Ch^m \quad \textit{only.}$$

Corollary 4.3.7 *Suppose that $\kappa = m - 1$ and $c_m = 1$. The optimal order of convergence of the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ corresponding to the Radau II points is then given by*

$$\max_{t \in I_h \setminus \{t_0\}} |y(t) - u_h(t)| \leq Ch^{2m-1}.$$

Recall that we have $u_h^{it}(t) = u_h(t)$ for $t \in I_h \setminus \{t_0\}$ whenever $c_m = 1$ (i.e. when $t_n \in X_h,\ n = 1, \ldots, N$).

We illustrate these results by an example:

Example 4.3.1 *Non-vanishing proportional delay*

On $I = [t_0, T]$ with $t_0 > 0$, the delay function $\theta(t) = qt$ $(0 < q < 1)$ corresponds to a non-vanishing delay $\tau(t)$ since

$$\theta(t) = qt = t - (1 - q)t =: t - \tau(t),$$

with $\tau(t) \geq (1 - q)t_0 > 0$ for $t \in I$. Hence, the primary discontinuity points $\{\xi_\mu\}$ are given by

$$\xi_\mu = q^{-\mu} t_0 \quad (\mu \geq 0).$$

We will assume, for ease of exposition and without loss of generality, that T is such that $\xi_{M+1} = T$ for some $M > 1$. Hence, we may write

$$\xi_\mu = q^{M+1-\mu} T, \quad \mu = 0, 1, \ldots, M + 1.$$

Suppose that the mesh I_h is constrained, and let each local mesh $I_h^{(\mu)}$ be *uniform*:

$$I_h^{(\mu)} := \{t_n^{(\mu)} := \xi_\mu + nh^{(\mu)} :\ n = 0, 1, \ldots, N\ (h^{(\mu)} = q^{-(\mu+1)}(1 - q)t_0/N)\}.$$

A mesh of this type is often called a *quasi-geometric mesh* (see also Section 5.5.3). The linearity of θ then implies that I_h is θ-invariant, and the same is true for the set X_h of collocation points.

This choice of the local meshes defining I_h implies that

$$h = h^{(M)} = \frac{1}{N}(\xi_{M+1} - \xi_M) = (1-q)\frac{T}{N},$$

and

$$h^{(\mu)} = \frac{1}{N}(\xi_{\mu+1} - \xi_\mu) = q^{M+1-\mu-1}(1-q)\frac{T}{N} \quad (\mu = 0, 1, \ldots, M).$$

The result of, e.g. Theorem 4.3.5 then becomes

$$\max_{t \in I_h \setminus \{t_0\}} |y(t) - u_h^{it}(t)| \le C(q) N^{-(m+\kappa)}.$$

Note that this result also holds for the delay VIE (4.3.15),

$$y(t) = g(t) + (\mathcal{W}_\theta y)(t), \quad t \in I := [t_0, T] \ (t_0 > 0),$$

with $\theta(t) = qt$ $(0 < q < 1)$.

We shall return to this example in Section 5.5.3 when we describe collocation on quasi-geometric meshes for VIDEs with vanishing delays.

4.3.4 Nonlinear delay VIEs

We turn to the nonlinear version of (4.3.1),

$$y(t) = g(t) + (\mathcal{V}y)(t) + (\mathcal{V}_\theta y)(t), \quad t \in (t_0, T], \tag{4.3.28}$$

where now

$$(\mathcal{V}y)(t) := \int_{t_0}^{t} k_1(t, s, y(s))ds, \qquad (\mathcal{V}_\theta y)(t) := \int_{t_0}^{\theta(t)} k_2(t, s, y(s))ds. \tag{4.3.29}$$

The computational form of the collocation equation for $u_h \in S_{m-1}^{(-1)}(I_h)$ is readily obtained by adapting (4.3.10), and thus we will not write it down in detail. Instead, we focus on the particular nonlinear delay VIE

$$y(t) = g(t) + (\mathcal{W}_\theta y)(t), \quad t \in (t_0, T], \tag{4.3.30}$$

where we assume that the Volterra operator W_θ is now of *Hammerstein type*,

$$(\mathcal{W}_\theta y)(t) := \int_{\theta(t)}^{t} k(t-s) G(s, y(s)) ds. \tag{4.3.31}$$

As we have already seen in Section 2.3.3, there are two ways of generating collocation approximations to solutions of Volterra–Hammerstein integral equations of the second kind. In the 'direct' approach we approximate y by $u_h \in S_{m-1}^{(-1)}(I_h)$, followed by the iterated collocation solution u_h^{it} based on u_h. The equations defining these approximations are the nonlinear analogues of those in (4.3.18), namely,

$$U_{n,i}^{(\mu)} = g(t_{n,i}^{(\mu)}) + \bar{\Psi}_n^{(\mu-1)}(t_{n,i}^{(\mu)}) + h_n^{(\mu)} \int_0^{c_i} k((c_i - s)h_n^{(\mu)}) G(t_n^{(\mu)} + sh_n^{(\mu)}, \sum_{j=1}^m L_j(s) U_{n,j}^{(\mu)}) ds \quad (4.3.32)$$

($i = 1, \ldots, m$), with lag term approximation at $t = t_{n,i}^{(\mu)}$ as in (4.3.17),

$$\bar{\Psi}_n^{(\mu-1)}(t) := \int_{t_{n+1}^{(\mu-1)}}^{\xi_\mu} G(s, u_h(s)) ds + \int_{\xi_\mu}^{t_n^{(\mu)}} k(t-s) G(s, u_h(s)) ds + h_n^{(\mu-1)} \int_{\tilde{c}_i}^1 k(t - t_n^{(\mu-1)} - sh_n^{(\mu-1)}) G\left(t_n^{(\mu-1)} + sh_n^{(\mu-1)}, u_h(t_n^{(\mu-1)} + sh_n^{(\mu-1)})\right) ds. \quad (4.3.33)$$

The local representation of u_h on $\sigma_n^{(\mu)}$ is again described by (4.3.4).

The *iterated collocation solution* at $t = t_n^{(\mu)} + vh_n^{(\mu)} \in \sigma_n^{(\mu)}$ is then determined by

$$u_h^{it}(t) = g(t) + \bar{\Psi}_n^{(\mu-1)}(t) + h_n^{(\mu)} \int_0^v k((v-s)h_n^{(\mu)}) G(t_n^{(\mu)} + sh_n^{(\mu)}, u_h(t_n^{(\mu)} + sh_n^{(\mu)})) ds, \; v \in [0, 1]. \quad (4.3.34)$$

Here, $\bar{\Psi}_n^{(\mu-1)}(t)$ is the nonlinear counterpart of (4.3.17).

Alternatively, we can resort to what we called *implicitly linear collocation* in Section 2.3.3. Setting $z(t) := G(t, y(t))$ (recall the *Niemytzki operator* introduced at the end of Section 2.1.5), the nonlinear delay VIE (4.3.30) becomes an *implicitly linear delay VIE* for z,

$$z(t) = G\left(t, g(t) + \int_{\theta(t)}^t k(t-s) z(s) ds\right), \quad t \in (t_0, T], \quad (4.3.35)$$

with initial condition $z(t) = G(t, \phi(t)),\ t \in [\theta(t_0), t_0]$. The solution of the original DVIE is then obtained via the recursion

$$y(t) = g(t) + (\mathcal{L}_\theta z)(t), \quad t \in (t_0, T], \quad (4.3.36)$$

where $\mathcal{L}_\theta$ denotes the linear delay Volterra operator

$$(\mathcal{L}_\theta y)(t) := \int_{\theta(t)}^t k(t-s) z(s) ds.$$

The solution z of (4.3.35) will be approximated by $z_h \in S_{m-1}^{(-1)}(I_h)$, using the same collocation points X_h as in the direct approach: it is defined by the *implicit linear collocation equation*

$$z_h(t) = G\left(t, g(t) + \int_{\theta(t)}^{t} k(t-s)z_h(s))ds\right), \quad t \in X_h, \tag{4.3.37}$$

with initial values $z_h(t) = G(t, \phi(t)),\ t \in [\theta(t_0), t_0]$. This leads to the approximation y_h for the solution y of the original DVIE,

$$y_h(t) := g(t) + (\mathcal{L}_\theta z_h)(t), \quad t \in [t_0, T]. \tag{4.3.38}$$

Setting

$$z_h(t_n^{(\mu)} + vh_n^{(\mu)}) = \sum_{j=1}^{m} L_j(v) Z_{n,j}^{(\mu)}, \quad v \in (0, 1], \quad \text{with} \quad Z_{n,i}^{(\mu)} := z_h(t_{n,j}^{(\mu)}), \tag{4.3.39}$$

the computational forms of these equations at $t = t_{n,i}^{(\mu)}$ and at $t = t_n^{(\mu)} + vh_n^{(\mu)}$, respectively, are

$$Z_{n,i}^{(\mu)} = G\Bigg(t_{n,i}^{(\mu)}, g(t_{n,i}^{(\mu)}) + \bar{\Psi}_n^{(\mu-1)}(t_{n,i}^{(\mu)}) + h_n^{(\mu)} \times \sum_{j=1}^{m}\left(\int_0^{c_i} k((c_i - s)h_n^{(\mu)})L_j(s)ds\right) Z_{n,j}^{(\mu)}\Bigg) \tag{4.3.40}$$

$(i = 1, \ldots, m)$, where for $t = t_n^{(\mu)} + vh_n^{(\mu)} \in \bar{\sigma}_n^{(\mu)}$ we have

$$\bar{\Psi}_n^{(\mu-1)}(t) := h_n^{(\mu-1)} \int_{\tilde{v}}^{1} k(t - t_n^{(\mu-1)} - sh_n^{(\mu-1)}) z_h(t_n^{(\mu-1)} + sh_n^{(\mu-1)})ds + \int_{t_{n+1}^{(\mu-1)}}^{\xi_\mu} k(t-s)z_h(s)ds + \int_{\xi_\mu}^{t_n^{(\mu)}} k(t-s)z_h(s)ds,$$

and

$$y_h(t) = g(t) + \bar{\Psi}_n^{(\mu-1)}(t) + h_n^{(\mu)} \sum_{j=1}^{m}\left(\int_0^{v} k((v-s)h_n^{(\mu)})L_j(s)ds\right) Z_{n,j}^{(\mu)}, \quad v \in [0, 1]. \tag{4.3.41}$$

Recall that the number $\tilde{v} \in [0, 1]$ is obtained from

$$\theta(t_n^{(\mu)} + vh_n^{(\mu)}) =: t_n^{(\mu-1)} + \tilde{v}h_n^{(\mu-1)}, \quad v \in [0, 1],$$

with $\tilde{v} = v$ if the lag function θ is linear.

4.4 Collocation for first-kind VIEs with delays

4.4.1 The collocation space $S_{m-1}^{(-1)}(I_h)$

In this section we will study the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ for the (linear) first-kind Volterra integral equation with non-vanishing delay θ,

$$(\mathcal{V}y)(t) + (\mathcal{V}_\theta y)(t) = g(t), \quad t \in I := (t_0, T], \tag{4.4.1}$$

subject to the initial condition $y(t) = \phi(t)$ when $t \leq t_0$.

Employing the notation introduced in Section 4.3.1, the computational form of the corresponding collocation equation defining the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ for (4.4.1),

$$(\mathcal{V}u_h)(t) + (\mathcal{V}_\theta u_h)(t) = g(t), \quad t \in X_h, \tag{4.4.2}$$

with initial condition $u_h(t) = \phi(t),\ t \leq t_0$, is

$$\begin{aligned} h_n^{(\mu)} \sum_{j=1}^{m} &\left(\int_0^{c_i} K_1(t_{n,i}^{(\mu)}, t_n^{(\mu)} + s h_n^{(\mu)}) L_j(s) ds \right) U_{n,j}^{(\mu)} \\ &= g(t_{n,i}^{(\mu)}) - F_{n,i}^{(\mu)}(t_{n,i}^{(\mu)}) - (\mathcal{V}_\theta u_h)(t_{n,i}^{(\mu)}), \quad (i = 1, \ldots, m). \end{aligned} \tag{4.4.3}$$

For $t = t_n^{(\mu)} + v h_n^{(\mu)} \in \sigma_n^{(\mu)}$ we have

$$\begin{aligned} F_n^{(\mu)}(t) &:= \int_{t_0}^{t_n^{(\mu)}} K_1(t, s) u_h(s) ds \\ &= \int_{t_0}^{\xi_\mu} K_1(t, s) u_h(s) ds + \int_{\xi_\mu}^{t_n^{(\mu)}} K_1(t, s) u_h(s) ds, \end{aligned}$$

and

$$\begin{aligned} (\mathcal{V}_\theta u_h)(t) &= \Psi_n^{(\mu-1)}(t) \\ &\quad + h_n^{(\mu-1)} \sum_{j=1}^{m} \left(\int_0^{\tilde{v}} K_2(t, t_n^{(\mu-1)} + s h_n^{(\mu-1)}) L_j(s) ds \right) U_{n,j}^{(\mu-1)} \end{aligned}$$

(recall (4.3.9) and (4.3.13)), with u_h on $\sigma_n^{(\mu)}$ given by

$$u_h(t_n^{(\mu)} + v h_n^{(\mu)}) = \sum_{j=1}^{m} L_j(v) U_{n,j}^{(\mu)}, \quad v \in (0, 1], \quad \text{with} \quad U_{n,j}^{(\mu)} := u_h(t_{n,j}^{(\mu)}). \tag{4.4.4}$$

The vector $\mathbf{U}_n := (U_{n,1}^{(\mu)}, \ldots, U_{n,m}^{(\mu)})^T$ is determined by the solution of the linear algebraic system in $\mathbb{R}^m$,

$$B_n^{(\mu)} \mathbf{U}_n^{(\mu)} = (h_n^{(\mu)})^{-1} [\mathbf{g}_n^{(\mu)} - \mathbf{G}_n^{(\mu)} - \mathbf{Q}_n^{(\mu-1)} - h_n^{(\mu-1)} \tilde{B}_n^{(\mu)} \mathbf{U}_n^{(\mu-1)}], \tag{4.4.5}$$

in complete analogy to (4.3.11).

As we have observed before, an important special case of (4.4.1) is

$$(\mathcal{W}_\theta y)(t) = g(t), \quad t \in (t_0, T], \tag{4.4.6}$$

with given $y(t) = \phi(t)$ on the initial interval $[\theta(t_0), t_0]$. The kernel K of the integral operator

$$(\mathcal{W}_\theta y)(t) := \int_{\theta(t)}^{t} K(t,s)y(s)ds$$

is assumed to satisfy the hypotheses stated in Theorem 4.1.4.

Since we have already done our homework in Section 4.3.1, the computational form of the collocation equation

$$(\mathcal{W}_\theta u_h)(t) = g(t), \quad t \in X_h, \tag{4.4.7}$$

with initial values given by $u_h(t) = \phi(t)$, $t \in [\theta(t_0), t_0]$, derives immediately from (4.3.18) and reads, for $t = t_{n,i}^{(\mu)} \in \sigma_n^{(\mu)}$,

$$\begin{aligned} h_n^{(\mu)} \sum_{j=1}^{m} &\left(\int_0^{c_i} K(t, t_n^{(\mu)} + s h_n^{(\mu)}) L_j(s) ds \right) U_{n,j}^{(\mu)} \\ &= g(t_{n,i}^{(\mu)}) - \bar{\Psi}_n^{(\mu-1)}(t) - h_n^{(\mu-1)} \\ &\times \sum_{j=1}^{m} \left(\int_{\tilde{c}_i}^{1} K(t, t_n^{(\mu-1)} + s h_n^{(\mu-1)}) L_j(s) ds \right) U_{n,j}^{(\mu-1)}. \end{aligned} \tag{4.4.8}$$

With the notation of Section 4.3.1 (see (4.3.19)) this leads to the linear algebraic system for $\mathbf{U}_n^{(\mu)}$,

$$B_n^{(\mu)} \mathbf{U}_n^{(\mu)} = (h_n^{(\mu)})^{-1} [\mathbf{g}_n^{(\mu)} - \bar{\mathbf{G}}_n^{(\mu-1)}] \tag{4.4.9}$$

(compare (4.3.19) where this notation was introduced). For known $\mathbf{U}_n^{(\mu)}$, the collocation solution on $\sigma_n^{(\mu)}$ is thus given by

$$u_h(t_n^{(\mu)} + v h_n^{(\mu)}) = \sum_{j=1}^{m} L_j(s) U_{n,j}^{(\mu)}, \quad v \in (0, 1].$$

Remark In Example 4.3.1 we considered the lag function $\theta(t) = qt$ $(0 < q < 1)$ on $I = [t_0, T]$ with $t_0 > 0$. The corresponding delay $\tau(t) = (1-q)t$ does not vanish on I. Consider the delay equation

$$(\mathcal{W}_\theta y)(t) = g(t), \quad t \in I \quad (g(0) = 0),$$

where the kernel K in $\mathcal{W}$ and the function g are such that it possesses a unique solution $y \in C^d(I)$ for some $d \geq 1$. Suppose that y is approximated by the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$, with I_h being the θ-invariant (quasi-geometric) mesh of Example 4.3.1.

Does the 'stability condition' in Theorem 2.4.2 for the collocation parameters,

$$-1 \leq \rho_m := (-1)^m \prod_{i=1}^{m} \frac{1-c_i}{c_i} \leq 1,$$

imply the uniform convergence of u_h to y on I, as $N \to \infty$? The answer to this question remains to be found (see Exercise 4.7.13).

4.4.2 Direct versus indirect collocation

We know from Section 2.4.4 that local superconvergence at the mesh points is not possible in collocation solutions for first-kind Volterra integral equations, and that hence it is often advantageous to use its differentiated form (a VIE of the second kind) as the basis for generating high-order solutions. The same is true for first-kind VIEs with non-vanishing delays θ. Thus, if the given functions in (4.4.1) satisfy the conditions in Theorem 4.1.4, differentiation of both sides of the given equation yields the delay integral equation

$$y(t) = f(t) + b(t)y(\theta(t)) + (\tilde{\mathcal{W}}_\theta y)(t), \quad t \in (t_0, T], \tag{4.4.10}$$

where we have introduced the functions $f(t) := g'(t)/K(t,t)$,

$$b(t) := -K(t,\theta(t))\theta'(t)/K(t,t), \qquad H(t,s) := -[\partial K(t,s)/\partial t]/K(t,t),$$

and the Volterra integral operator

$$(\tilde{\mathcal{W}}_\theta y)(t) := -\int_{\theta(t)}^{t} H(t,s)y(s)ds.$$

Thus, instead of the given first-kind integral equation (5.4.1) we solve (4.4.5) by collocation in $S_{m-1}^{(-1)}(I_h)$, with θ-invariant mesh I_h:

$$u_h(t) = f(t) + b(t)u_h(\theta(t)) + (\bar{\mathcal{W}}_\theta u_h)(t), \quad t \in X_h, \tag{4.4.11}$$

with $u_h(t) = \phi(t)$ when $t \in [\theta(t_0), t_0]$. Since the delay function θ does not vanish on I, the result of Theorem 4.3.1 on the existence of a unique collocation solution u_h carries over to the more general equation (4.4.11). Hence, the iterated collocation solution for (4.4.1) corresponding to the unique 'indirect' collocation solution u_h is obtained from

$$u_h^{it}(t) := f(t) + b(t)u_h(\theta(t)) + (\bar{\mathcal{W}}_\theta u_h(t), \quad t \in (t_0, T].$$

Theorem 4.4.1 *Suppose that the functions defining the first-kind delay Volterra integral equation (4.4.1) satisfy, for $d \geq m + \kappa$:*

(a) $K \in C^{d+1}(\bar{D}_\theta)$;
(b) $g \in C^{d+1}(I)$;
(c) $\theta \in C^{d+1}(I)$, *and θ subject to the conditions (D1)–(D3) of Section 4.2.1;*
(d) $\phi \in C^{d+1}[\theta(t_0), t_0]$.

If the collocation points X_h are defined by the Gauss points $\{c_i\}$, then the 'indirect' iterated collocation solution u_h^{it} corresponding to $u_h \in S_{m-1}^{(-1)}(I_h)$, with θ-invariant mesh I_h, and defined by (4.4.11), has the superconvergence properties

$$||y - u_h^{it}||_\infty \leq C h^{m+1} \quad (\textit{if } \kappa = 1), \tag{4.4.12}$$

and

$$\max_{t \in I_h \setminus \{t_0\}} |y(t) - u_h^{it}(t)| \leq C h^{2m} \quad (\textit{if } \kappa = m). \tag{4.4.13}$$

If the collocation parameters are the Radau II points ($\kappa = m - 1$) then u_h itself is superconvergent on $I_h \setminus \{t_0\}$:

$$\max_{t \in I_h \setminus \{t_0\}} |y(t) - u_h(t)| \leq C h^{2m-1}.$$

Proof We leave it as Exercise 4.7.12.

4.5 Collocation for VIDEs with delays

4.5.1 The exact collocation equations

The description and analysis of collocation methods in Chapters 1 and 3, and in the previous sections of the present chapter, have introduced all the ideas required to deal with collocation solutions for the initial-value problem

$$y'(t) = f(t, y(t), y(\theta(t))) + (\mathcal{V}y)(t) + (\mathcal{V}_\theta y)(t), \quad t \in I := [t_0, T], \tag{4.5.1}$$
$$y(t) = \phi(t), \quad t \in [\theta(t_0), t_0],$$

with Volterra integral operators $\mathcal{V}$ and $\mathcal{V}_\theta$ given by (4.3.29) or by their linear counterparts in (4.3.1). The delay function θ will again be assumed to satisfy conditions (D1)–(D3) of Section 4.1.1. Therefore, the collocation equation defining $u_h \in S_m^{(0)}(I_h)$ in the subinterval $\bar{\sigma}_n^{(\mu)}$ of the θ-invariant mesh I_h is

$$u_h(t) = f(t, u_h(t), u_h(\theta(t))) + (\mathcal{V}u_h(t) + (\mathcal{V}_\theta u_h)(t), \quad t \in X_h, \tag{4.5.2}$$

with $u_h(t) := \phi(t)$ if $t \le t_0$. For $t \in \sigma_n^{(\mu)}$ we define the lag term approximations

$$F_n^{(\mu)}(t) := \int_{t_0}^{\xi_\mu} k_1(t, s, u_h(s)))ds + \int_{\xi_\mu}^{t_n^{(\mu)}} k_1(t, s, u_h(s)))ds, \qquad (4.5.3)$$

and

$$(\mathcal{V}_\theta u_h)(t) = \Psi_n^{(\mu-1)}(t) + \int_{t_n^{(\mu-1)}}^{\theta(t)} k_2(t, s, u_h(s)))ds. \qquad (4.5.4)$$

In analogy to (4.3.6) we have

$$\Psi_n^{(\mu-1)}(t) = \int_{t_0}^{\xi_{\mu-1}} k_2(t, s, u_h(s)))ds + \int_{\xi_{\mu-1}}^{t_n^{(\mu-1)}} k_2(t, s, u_h(s)))ds.$$

Recall from Section 4.2.1 (Lemma 4.2.1) that $\theta(t_{n,i}^{(\mu)}) = t_n^{(\mu-1)} + \tilde{c}_i h_n^{(\mu-1)}$ which coincides with the collocation point $t_{n,i}^{(\mu-1)}$ $(i = 1, \ldots, m)$ only if θ is linear.

With the usual local Lagrange representation of u_h on $\bar{\sigma}_n^{(\mu)}$,

$$u_h(t_n^{(\mu)} + sh_n^{(\mu)}) = y_n^{(\mu)} + h_n^{(\mu)} \sum_{j=1}^{m} \beta_j(v) Y_{n,j}^{(\mu)}, \quad v \in [0, 1],$$
$$\text{with } \; Y_{n,j}^{(\mu)} := u_h'(t_{n,j}^{(\mu)}), \qquad (4.5.5)$$

the computational form of (4.5.2) becomes

$$Y_{n,i}^{(\mu)} = f\left(t_{n,i}^{(\mu)}, y_n^{(\mu)} + h_n^{(\mu)} \sum_{j=1}^{m} a_{i,j} Y_{n,j}^{(\mu)}, u_h(\theta(t_{n,i}^{(\mu)}))\right)$$
$$+ h_n^{(\mu)} \int_0^{c_i} k_1\left(t_{n,i}^{(\mu)}, t_n^{(\mu)} + sh_n^{(\mu)}, y_n^{(\mu)} + h_n^{(\mu)} \sum_{j=1}^{m} \beta_j(s) Y_{n,j}^{(\mu)})\right) ds$$
$$+ F_n^{(\mu)}(t_{n,i}^{(\mu)}) + (\mathcal{V}_\theta u_h)(t_{n,i}^{(\mu)}) \quad (i = 1, \ldots, m). \qquad (4.5.6)$$

For the *linear* version of (4.5.1),

$$y'(t) = a(t)y(t) + b(t)y(\theta(t)) + g(t) + (\mathcal{V}y)(t) + (\mathcal{V}_\theta y)(t), \quad t \in I, \quad (4.5.7)$$

with $\mathcal{V}$ and $\mathcal{V}_\theta$ given by (4.1.2) and (4.1.3), the collocation solution $u_h \in S_m^{(0)}(I_h)$ on the subinterval $\bar{\sigma}_n^{(\mu)}$ is defined by the local representation (4.5.5) and the solution $\mathbf{Y}_n^{(\mu)} \in \mathbb{R}^m$ of the linear algebraic system

$$[\mathcal{I}_m - h_n^{(\mu)}(A_n^{(\mu)} + h_n^{(\mu)} C_n^{(\mu)})]\mathbf{Y}_n^{(\mu)} = \mathbf{g}_n^{(\mu)} + \mathbf{G}_n^{(\mu)} + \boldsymbol{\kappa}_n^{(\mu)} y_n^{(\mu)}$$
$$+ \mathbf{Q}_n^{(\mu-1)} + \tilde{\boldsymbol{\kappa}}_n^{(\mu-1)} y_n^{(\mu-1)} + (h_n^{(\mu-1)})^2 \tilde{C}_n^{(\mu-1)} \mathbf{Y}_n^{(\mu-1)} \qquad (4.5.8)$$

($n = 0, 1, \ldots, N-1$; $\mu = 0, 1, \ldots, M$). The matrices in $L(\mathbb{R}^m)$ defining (4.5.8) are

$$A_n^{(\mu)} := \mathrm{diag}(a(t_{n,i}^{(\mu)}))A, \quad \text{with } A := (a_{i,j});$$
$$\tilde{A}_n^{(\mu)} := \mathrm{diag}(b(t_{n,i}^{(\mu)}))\tilde{A}, \quad \text{with } \tilde{A} := (\beta_j(\tilde{c}_i));$$
$$C_n^{(\mu)} := \begin{pmatrix} \int_0^{c_i} K_1(t_{n,i}^{(\mu)}, t_n^{(\mu)} + sh_n^{(\mu)})\beta_j(s)ds \\ (i, j = 1, \ldots, m) \end{pmatrix};$$
$$\tilde{C}_n^{(\mu-1)} := \begin{pmatrix} \int_0^{\tilde{c}_i} K_2(t_{n,i}^{(\mu)}, t_n^{(\mu-1)} + sh_n^{(\mu-1)})\beta_j(s)ds \\ (i, j = 1, \ldots, m) \end{pmatrix},$$

and we have set

$$\boldsymbol{\kappa}_n^{(\mu)} := \boldsymbol{a}_n^{(\mu)} + h_n^{(\mu)} \left(\int_0^{c_i} K_1(t_{n,i}^{(\mu)}, t_n^{(\mu)} + sh_n^{(\mu)})ds \;\; (i = 1, \ldots, m) \right)^T,$$
$$\tilde{\boldsymbol{\kappa}}_n^{(\mu-1)} := \boldsymbol{b}_n^{(\mu)} + h_n^{(\mu-1)} \left(\int_0^{\tilde{c}_i} K_2(t_{n,i}^{(\mu)}, t_n^{(\mu-1)} + sh_n^{(\mu-1)})ds \;\; (i = 1, \ldots, m) \right)^T,$$

with

$$\mathbf{a}_n^{(\mu)} := (a(t_{n,i}^{(\mu)}) \;\; (i = 1, \ldots, m))^T, \quad \mathbf{b}_n^{(\mu)} := (b(t_{n,i}^{(\mu)}) \;\; (i = 1, \ldots, m))^T.$$

The vectors $\mathbf{G}_n^{(\mu)}$ and $\mathbf{Q}_n^{(\mu-1)}$ are defined by

$$\mathbf{G}_n^{(\mu)} := (F_n^{(\mu)}(t_{n,1}^{(\mu)}), \ldots, F_n^{(\mu)}(t_{n,m}^{(\mu)}))^T,$$
$$\mathbf{Q}_n^{(\mu-1)} := (\Psi_n^{(\mu-1)}(t_{n,1}^{(\mu)}), \ldots, \Psi_n^{(\mu-1)}(t_{n,m}^{(\mu)}))^T;$$

for $t = t_{n,i}^{(\mu)} \in \bar{\sigma}_n^{(\mu)}$ their components are given respectively by

$$F_n^{(\mu)}(t) := \int_{t_0}^{\xi_\mu} K_1(t, s)u_h(s)ds + \int_{\xi_\mu}^{t_n^{(\mu)}} K_1(t, s)u_h(s)ds,$$
$$\Psi_n^{(\mu-1)}(t) := \int_{t_0}^{\xi_{\mu-1}} K_2(t, s)u_h(s)ds + \int_{\xi_{\mu-1}}^{t_n^{(\mu-1)}} K_2(t, s)u_h(s)ds$$

(see also (4.3.7)).

Theorem 4.5.1 *Assume that the given functions describing the linear delay VIDE (4.5.7) are continuous on their respective domains, and let the delay functions θ be subject to the hypotheses (D1)–(D3) in Section 4.1.2. Then there exists a $\bar{h} > 0$ so that for any θ-invariant mesh I_h with $h \in (0, \bar{h})$ and any initial function $\phi \in C[\theta(t_0), t_0]$ each of the linear algebraic systems (4.5.8) possesses a unique solution $Y_n^{(\mu)} \in \mathbb{R}^m$. Therefore, the collocation equation (4.5.2) defines a* *unique collocation solution $u_h \in S_m^{(0)}(I_h)$ for (4.5.7) whose local representation on $\bar{\sigma}_n^{(\mu)}$ is given by (4.5.5).*

4.5.2 Global convergence results

Let $e_h := y - u_h$ denote the collocation error for the collocation solution $u_h \in S_m^{(0)}(I_h)$ to the linear version (4.5.7) of the delay VIDE (4.5.1), where I_h is the θ-invariant mesh defined in (4.2.1), (4.2.2). It obviously solves the initial-value problem

$$e_h'(t) = a(t)e_h(t) + b(t)e_h(\theta(t)) + \delta_h(t) + (\mathcal{V}e_h)(t) + (\mathcal{V}_\theta e_h)(t), \quad t \in I, \tag{4.5.9}$$

$e_h(t) = 0$ on $[\theta(t_0), t_0]$, where the defect δ_h vanishes on X_h, the set of collocation points. For $t \in I^{(\mu)} := [\xi_\mu, \xi_{\mu+1}]$ we write the above error equation in the form

$$e_h'(t) = a(t)e_h(t) + \delta_h(t) + G_\mu(t) + \int_{\xi_\mu}^t K_1(t,s)e_h(s)ds, \quad t \in I^{(\mu)}, \tag{4.5.10}$$

with given initial value $e_h(\xi_\mu)$ and lag term

$$G_\mu(t) := b(t)e_h(\theta(t)) + \int_{t_0}^{\xi_\mu} K_1(t,s)e_h(s)ds + (\mathcal{V}_\theta e_h)(t).$$

When $\mu = 0$ we have

$$e_h'(t) = a(t)e_h(t) + \delta_h(t) + \int_{t_0}^t K_1(t,s)e_h(s)ds, \quad t \in I^{(0)}, \tag{4.5.11}$$

since the initial condition $e_h(t) = 0,\ t \le 0$ implies $G_0(t) = 0$ in $[\theta(t_0), t_0]$.

Hence, on the first macro-interval $I^{(0)}$ the gobal convergence result of Theorem 3.2.1 for classical VIDEs holds: under appropriate assumptions on the regularity of the solution (see Theorem 4.5.1 below) the collocation error can be estimated by

$$||e_h^{(\nu)}||_{0,\infty} := \sup_{t\in I^{(0)}} |e_h^{(\nu)}(t)| \le C_\nu (h^{(0)})^m \quad (\nu = 0, 1).$$

This implies in particular that $e_h^{(\nu)}(\xi_1) = \mathcal{O}((h^{(0)})^m)$.

An analogous global error estimate can now be derived on each subsequent macro-interval $I^{(\mu)}$ $(1 \le \mu \le M)$, by applying the global convergence estimates of Section 3.2.3 to the VIDE (4.5.10) on $I^{(\mu)}$. We leave these obvious details to the reader and simply summarise the result in

Theorem 4.5.2 *Assume:*

(a) $a,\ b,\ g \in C^m(I)$, *and* $\phi \in C^{m+1}[\theta(t_0), t_0]$*;*

(b) $K_1 \in C^m(D),\ K_2 \in C^m(D_\theta)$*;*
(c) θ *satisfies the conditions (D1)–(D3) of Section 4.1.2, with* $d \geq m$ *in (D1);*
(d) $u_h \in S_m^{(0)}(I_h)$ *is the collocation solution to the delay VIDE (4.5.7), where* I_h *is* θ*-invariant and* $h \in (0, \bar{h})$ *so that the linear algebraic systems (4.5.8) all have unique solutions.*

Then the estimates

$$||y^{(\nu)} - u_h^{(\nu)}||_\infty \leq C_\nu h^m \quad (\nu = 0, 1) \tag{4.5.12}$$

hold for any set $\{c_i\}$ *of distinct collocation parameters in* $[0, 1]$*. The constants* C_ν *depend on these parameters but are independent of* h*.*

Since the delay τ in the VIDE (4.5.1) does not vanish on I, a gain of one can be achieved in the global order of convergence of u_h by a judicious choice of the $\{c_i\}$, thus extending the global superconvergence result of Theorem 3.2.5 for classical VIDEs.

Theorem 4.5.3 *Let the assumed degree of regularity for the given functions in the initial-value problem for the linear delay VIDE (4.5.7) be raised by one (to* $m + 1$ *and* $m + 2$*, respectively) in Theorem 4.5.1. If the collocation parameters satisfy the orthogonality condition*

$$J_0 := \int_0^1 \prod_{i=1}^m (s - c_i) ds = 0$$

then for all θ*-invariant meshes* I_h *with* $h \in (0, \bar{h})$*, the collocation solution* $u_h \in S_m^{(0)}(I_h)$ *is globally superconvergent on* I*:*

$$||y - u_h||_\infty \leq C h^{m+1}, \tag{4.5.13}$$

with C *depending on the* $\{c_i\}$ *but not on* h*.*

Proof The key to establishing this global superconvergence result (and the local superconvergence results in the next section) is the variation-of-constants formula of Theorem 4.1.7, where y and g are replaced, respectively, by e_h and δ_h, and where the initial condition is given by $e_h(t) = 0$ $(t \leq t_0)$. It is then easy to show that $e_h(\xi_\mu) = \mathcal{O}((h^{(\mu)})^{m+1})$ $(1 \leq \mu \leq M;\ h^{(\mu)} \leq h)$, by applying Theorem 3.2.5 on each of the macro-intervals $I^{(\mu)}$. Note that, as discussed in detail in Sections 4.3.1 and 4.3.2 (proof of Theorem 4.3.3), the image of a point $t = t_n^{(\mu)} + v h_n^{(\mu)} \in \sigma_n^{(\mu)}$ under $\theta^{\mu - \nu}$ $(0 \leq \nu \leq \mu - 1)$ is given either by $t_n^{(\nu)} + v h_n^{(\nu)}$ $(v \in [0, 1])$ if θ is linear, or by $t_n^{(\nu)} + \tilde{v} h_n^{(\nu)}$ (for some $\tilde{v} \in [0, 1]$, with $\tilde{v} \neq v$) if θ is *nonlinear*.

The remaining details of the proof are left as an exercise.

Remark The convergence results of Theorems 4.5.2 and 4.5.3 contain, as special cases, global convergence and superconvergence results for DDEs (corresponding to $K_i = 0$ on D and D_θ, respectively).

4.5.3 Local superconvergence results

In the previous section we have briefly described the foundation for proving optimal superconvergence results on I_h for the collocation solution $u_h \in S_m(^{(0)}(I_h)$ to the linear delay VIDE (4.5.7): it is given by the variation-of-constants formula (or 'resolvent representation') for the collocation error $e_h := y - u_h$ derived from Theorem 4.1.7 The essential ingredients of the proof of the local superconvergence result are thus all in place: the θ-invariance of the mesh I_h and the resulting mapping (4.2.2) of mesh points $t_n^{(\mu)}$ into corresponding previous mesh points $t_n^{(\nu)}$ (which is of course true regardless of whether the delay function θ is linear or nonlinear) and the order of the quadrature errors corresponding to the interpolatory m-point quadrature formulas based on the collocation points and depending on the familiar orthogonality conditions for the collocation parameters $\{c_i\}$. Thus, without any more ado we state

Theorem 4.5.4 *Assume:*

(a) The given functions $a,\ b,\ g$ and $K_1,\ K_2$ in the DVIDE (4.5.7) are in $C^{m+\kappa}$ on their respective domains, for some κ with $1 \le \kappa \le m$, as specified in (d).
(b) The lag function θ is subject to (D1)–(D3) in Section 4.2.1, with $d \ge m + \kappa + 1$ in (D1).
(c) $u_h \in S_m^{(0)}(I_h)$ is the collocation solution, with θ-invariant mesh I_h, for the delay VIDE (4.5.7).
(d) The collocation parameters $\{c_i\}$ are such that the orthogonality conditions (3.2.39),

$$J_\nu := \int_0^1 s^\nu \prod_{i=1}^{m}(s - c_i)ds = 0, \quad \nu = 0, 1, \ldots, \kappa - 1,$$

with $J_\kappa \neq 0$, hold.

Then, for all sufficiently small h (i.e. $h \in (0, \bar{h})$) the collocation error $e_h := y - u_h$ satisfies

$$\max_{t \in I_h} |e_h(t)| \le C h^{m+\kappa} \tag{4.5.14}$$

for some constant C which depends on the $\{c_i\}$ but not on h.

If, in addition, $c_m = 1$ (implying $\kappa \leq m-1$), then we also have

$$\max_{t \in I_h \setminus \{t_0\}} |e_h'(t)| \leq C_1 h^{m+\kappa}. \tag{4.5.15}$$

4.5.4 Neutral VFIDEs

In Section 4.1.4 we introduced two classes of neutral Volterra functional integro-differential equations, namely

$$y'(t) = f(t, y(t), y(\theta(t)), y'(\theta(t))) + (\mathcal{V}'y)(t) + (\mathcal{V}'_\theta y)(t) \tag{4.5.16}$$

where the kernels of $\mathcal{V}'$ and $\mathcal{V}'_\theta$ depend also on $y'(s)$, and

$$\frac{d}{dt}\left[y(t) - (\mathcal{V}_\theta y)(t)\right] = F(t, y(t), y(\theta(t))) \tag{4.5.17}$$

with

$$(\mathcal{V}_\theta y)(t) := \int_{t_0}^{\theta(t)} k(t, s, y(s))ds.$$

Since the global and local (super-) convergence properties of collocation solutions $u_h \in S_m^{(0)}(I_h)$ to (4.5.16) can be derived along the lines of the ones for the 'classical' neutral VIDEs we considered in Section 3.2.6 (see also Brunner (1994b) for the case of constant delay $\tau > 0$), we leave their derivation as an exercise (Exercise 4.7.14) and focus instead on collocation methods for (4.5.17) and the corresponding initial-value problem introduced in (4.1.48),

$$\begin{aligned} z'(t) &= H(t, z(t), y(\theta(t)), \quad t \in I := [t_0, T], \\ z(t_0) &= \Phi_0 := \phi(t_0) - (\mathcal{V}_\theta \phi)(t_0), \end{aligned} \tag{4.5.18}$$

where

$$z(t) := y(t) - (\mathcal{V}_\theta y)(t) \quad \text{and} \quad H(t, z, w) := F(t, z + \mathcal{V}_\theta y, w).$$

The solution to (4.5.17) is then obtained from

$$y(t) = z(t) + (\mathcal{V}_\theta z)(t), \quad t \in I$$

(see also Brunner and Vermiglio (2003)).

Suppose that the solution z of the initial-value problem (4.5.18) is approximated by the collocation solution $z_h \in S_m^{(0)}(I_h)$ where I_h is a θ-invariant mesh on I:

$$z_h'(t) = H(t, z_h(t), y_h(\theta(t))), \quad t \in X_h; \quad z_h(t_0) = \Phi_0. \tag{4.5.19}$$

The induced approximation y_h to the solution y of the original problem (4.5.17) is then determined by

$$y_h(t) := z_h(t) + (\mathcal{V}_\theta y_h)(s)ds, \quad t \in I. \tag{4.5.20}$$

On the first macro-interval $I^{(0)} := [\xi_0, \xi_1]$, with $\xi_0 = t_0$, the above equations reduce to

$$z_h'(t) = H(t, z(t), \phi(\theta(t))), \quad t \in X_h^{(0)}; \quad z_h(t_0) = \Phi_0,$$

and

$$y_h(t) = z_h(t) + (\mathcal{V}_\theta \phi)(t) = z_h(t) - \int_{\theta(t)}^{t_0} k(t, s, \phi(s))ds, \quad t \in I^{(0)}.$$

On the subsequent macro-intervals $I^{(\mu)}$ ($\mu = 1, \ldots, M$) the approximations z_h and y_h are then generated recursively: for $t = t_n^{(\mu)} + vh_n^{(\mu)}$ ($v \in [0, 1]$) we have

$$y_h(t) = z_h(t) + \int_{t_0}^{\theta(t)} k(t, s, y_h(s))ds, \quad t \in I^{(\mu)}, \tag{4.5.21}$$

where z_h is determined by

$$z_h'(t) = H(t, z_h(t), y_h(\theta(t))), \quad t \in X_h^{(\mu)}, \tag{4.5.22}$$

with

$$z_h(t_n^{(\mu)} + sh_n^{(\mu)}) = z_n^{(\mu)} + h_n^{(\mu)} \sum_{j=1}^{m} \beta_j(v) W_{n,j}^{(\mu)}, \quad v \in [0, 1],$$

and $z_n^{(\mu)} := z_h(t_n^{(\mu)})$, $W_{n,j}^{(\mu)} := z_h'(t_{n,j}^{(\mu)})$. The explicit computational forms of (4.5.22) and (4.5.21) are

$$W_{n,i}^{(\mu)} = H\left(t_{n,i}^{(\mu)}, z_n^{(\mu)} + h_n^{(\mu)} \sum_{j=1}^{m} a_{i,j} W_{n,j}^{(\mu)}, y_h(t_n^{(\mu-1)} + \tilde{c}_i h_n^{(\mu-1)})\right) \tag{4.5.23}$$

($i = 1, \ldots, m$), and

$$\begin{aligned} y_h(t) = z_h(t) + \Psi_n^{(\mu-1)} + h_n^{(\mu-1)} \int_0^{\tilde{v}} k(t, t_n^{(\mu-1)} \\ + sh_n^{(\mu-1)}, y_h(t_n^{(\mu-1)} + sh_n^{(\mu-1)}))ds, \quad v \in [0, 1] \end{aligned} \tag{4.5.24}$$

with $\theta(t_n^{(\mu)} + vh_n^{(\mu)}) =: t_n^{(\mu-1)} + \tilde{v}h_n^{(\mu-1)}$ ($v \in [0, 1]$). The images $\theta(t_{n,i}^{(\mu)}) =: t_n^{(\mu-1)} + \tilde{c}_i h_n^{(\mu-1)}$ of $t_{n,i}^{(\mu)} := t_n^{(\mu)} + c_i h_n^{(\mu)}$ coincide with $t_{n,i}^{(\mu-1)}$ ($i = 1, \ldots, m$) only if θ is linear.

The convergence analysis is straightforward because it can be based on our previous techniques. Consider first the error $z(t) - z_h(t)$ associated with

(4.5.19): since $z_h \in S_m^{(0)}(I_h)$ approximates the solution of a delay differential equation, we know that

$$||z - z_h||_\infty \leq Ch^{m+1}$$

holds if the $\{c_i\}$ satisfy the orthogonality condition $J_0 = 0$ (cf. Theorem 4.5.3). Also, we have

$$\max_{t \in I_h} |z(t) - z_h(t)| \leq Ch^{2m},$$

provided $J_\nu = 0$ ($\nu = 0, 1, \ldots, m-1$) (Theorem 4.5.4). It is then easy to show that the same order results are true for the approximation y_h to the solution y of (4.5.17): this follows from

$$|(\mathcal{V}_\theta y)(t) - (\mathcal{V}_\theta y_h)(t)| \leq \int_0^{\theta(t)} |K_2(t,s)| \cdot |y(s) - y_h(s)| ds, \quad t \in I,$$

and the estimate

$$|y(t) - y_h(t)| \leq |z(t) - z_h(t)|, \quad t \in I.$$

Here, we have used the fact that $y(t) = y_h(t) = \phi(t),\ t \leq t_0$, where ϕ is the given initial function.

Remark When solving neutral delay VIDEs (or neutral DDEs), for example the analogue of problem (4.5.17)),

$$\frac{d}{dt}[y(t) - (\mathcal{V}_\theta y)(t)] = F(t, y(t), t(\theta(t), y'(\theta(t))),$$

it is desirable to have high-order approximations to $y(t)$ *and* its derivative $y'(t)$ on I_h. In this case the use of the *Radau II points* as the collocation parameters defining X_h is to be preferred over the Gauss points. The resulting order of local superconvergence on I_h is then $p^* = 2m - 1$, both for y_h and y_h'.

4.6 Functional equations with state-dependent delays

4.6.1 DDEs with state-dependent delays

The numerical analysis of DDEs with state-dependent delays is now quite well understood. The pioneering papers by Neves and Feldstein (1976) and Feldstein and Neves (1984), as well as those by, e.g. Neves (1975a, 1975b), Neves and Thompson (1992) (with many examples and additional references), Willé and Baker (1994), Karoui and Vaillancourt (1994), Hartung and Turi (1995), Györi, Hartung and Turi (1995), Hartung, Herdman and Turi (1997), Györi, Hartung and Turi (1998), and the monograph by Bellen and Zennaro (2003, pp. 30–32) convey a good picture of its development and current state of the art.

4.6.2 Collocation for VIEs and VIDEs with state-dependent delays

For Volterra functional differential systems with state-dependent delays (which include integro-differential equations with such delays) we have the substantial work by Tavernini (1978) on general one-step methods. It also contains some superconvergence results. The numerical solution of second-kind Volterra integral equations with state-dependent delays was studied by Cahlon and Nachman (1985) and Cahlon (1992).

However, except for the results in Cryer and Tavernini (1972) (Euler's method may be viewed as a simple collocation method) the general (super-) convergence analysis for piecewise collocation methods is still outstanding. For example, we do not know if the collocation solution $u_h \in S_1^{(0)}(I_h)$ for DVIDEs of the form

$$y'(t) = g(t) + \int_{t-\tau(y(t))}^{t} k(t-s)G(y(s))ds$$

(i.e. the VIDE analogue of (4.1.36)) exhibits $\mathcal{O}(h^2)$-superconvergence if collocation is based on the Gauss point $c_1 = 1/2$. We are similarly ignorant about the optimal order of convergence on I_h for u_h^{it} corresponding to the collocation solution $u_h \in S_0^{(-1)}(I_h)$ for Bélair's state-dependent delay integral equation (4.1.36).

The major obstacle in these still missing analyses is of course the fact that now the location of the primary discontinuity points $\{\xi_\mu\}$ is not known a priori since these points depend on the unknown solution of the functional equation. The problem of tracking the $\{\xi_\mu\}$ is addressed in Neves and Feldstein (1976), Feldstein and Neves (1984), Neves and Thompson (1992), and Willé and Baker (1994). See also Bellen and Zennaro (2003) for a good exposition of these results for DDEs. The same problem is discussed in Brunner and Zhang (1999); they also analyse the regularity of solutions to VIEs and VIDEs with state-dependent delays (extending the techniques due to Feldstein and Neves). The exploitation of these results in collocation methods remains open.

4.7 Exercises and research problems

Exercise 4.7.1 Suppose that g, K_1 K_2 and θ in (4.1.5) are continuous on their respective domains. For which continuous initial functions ϕ is the solution y in $C(I)$?

Exercise 4.7.2 Derive the analogue of the variation-of-constants formula of Theorem 4.1.2 for the special delay VIE (4.1.14) corresponding to the delay integral operator $\mathcal{W}_\theta$.

Exercise 4.7.3 Prove Theorem 4.1.3. State and prove Theorem 4.1.3 directly for (4.1.8).

Exercise 4.7.4 Extend the variation-of-constants formula in Theorem 4.1.2 to the more general second-kind delay VIE (4.1.14), and to the modified version in which $\mathcal{W}_\theta$ has been replaced by $\mathcal{V}_\theta$.

Exercise 4.7.5 Prove Theorem 4.1.5. What can be said about the 'size' of the jumps at the points $\{\xi_\mu\}$ as μ increases from $\mu = 0$?

Exercise 4.7.6 Prove Theorem 4.1.6. For which initial functions $\phi \in C^1[\theta(t_0), t_0]$ does the solution of the DVIDE possess a continuous first derivative at $t = t_0$?

Exercise 4.7.7 Let $\theta(t) = t - \tau$ $(\tau > 0)$ and consider the DVIDEs

$$y'(t) = ay(t) + \int_0^{\theta(t)} [\lambda_1 y(s) + \lambda_2 y'(s)]ds$$

and

$$y'(t) = cy'(\theta(t)) + \int_0^{\theta(t)} [\lambda_1 y(s) + \lambda_2 y'(s)]ds,$$

with $\lambda_2 \neq 0$. Does smoothing occur at the points $\xi_\mu = \mu\tau$ as μ increases from $\mu = 0$?

Exercise 4.7.8 Prove Lemma 4.2.1.

Exercise 4.7.9 A mesh $I^{(\mu)}$ is called *quasi-uniform* if

$$q^{\mu)} := \max_{0 \le n < N} h_n^{(\mu)} / \min_{0 \le n < N} h_n^{(\mu)} \le \gamma_n < \infty$$

for all $N \in \mathbf{N}$.

(a) Show that if I_h is θ-invariant (cf. (4.2.1), (4.2.2)) and $I_h^{(0)}$ (or $I_h^{(M)}$) is chosen to be quasi-uniform, then each $I_h^{(\mu)}$, $\mu = 1, \ldots, M$ $[\mu = 0, \ldots, M-1]$ is also quasi-uniform, provided θ is linear.
(b) Does (a) remain true for nonlinear θ?

Exercise 4.7.10 Suppose the DV2 is given in the form

$$y(t) = g(t) + (\mathcal{V}y)(t) + \int_0^t K_2(t, s)y(\theta(s))ds.$$

Compare the resulting collocation equations (for $u_h \in S_{m-1}^{(-1)}(I_h)$ and u_h^{it}) with those obtained for (4.3.2) in which the lag function θ occurs as the upper limit in $\mathcal{V}_\theta$. Compare the computational implementations, and discuss their relative merits.

Exercise 4.7.11 Describe and analyse collocation in $S_{m-1}^{(-1)}(I_h)$ and associated iterated collocation for DVIEs with *multiple* (constant) delays in $\theta_\nu(t) := t - \tau_\nu$ $(0 = \tau_0 < \tau_1 < \ldots < \tau_r)$,

$$y(t) = g(t) + \sum_{\nu=0}^{r} \int_0^{\theta_\nu} (t) K_\nu(t, s) y(s) ds, \quad t \in I.$$

(Compare also Torelli and Vermiglio (1993) for a similar analysis for related DDEs.)

Exercise 4.7.12 Prove Theorem 4.4.1 on the convergence of the 'indirect' collocation solution for a first-kind VIE with non-vanishing delay.

Exercise 4.7.13 (Research problem)
Recall the Remark at the end of Section 4.4.1: if the solution of $\mathcal{W}_\theta y = g$ is approximated by collocation in $S_{m-1}^{(-1)}(I_h)$, and if the underlying mesh I_h is quasi-geometric, find a necessary and sufficient condition on the collocation parameters so that u_h converges uniformly to y on I. (Assume that the given functions $K,\ g,\ \phi$ are such that y is bounded on I.)

Exercise 4.7.14 State and prove global and local superconvergence results for the neutral DVIDEs (4.5.16) and (4.5.7).

Exercise 4.7.15 (Research problem)
Suppose that Volterra's system of 'non-standard' delay VIDEs of Section 4.1.5 is solved numerically by approximating the unknown solutions N_1 and N_2 by collocation in $S_m^{(0)}(I_h)$. Derive optimal global and local superconvergence estimates for the collocation solutions.

Exercise 4.7.16 Do the superconvergence results described at the end of Section 4.5.4 remain valid if the delay VIDE (4.5.17) is replaced by

$$\frac{d}{dt}\left(y(t) - (\mathcal{W}_\theta y)(t)\right) = F(t, y'(\theta(t))),$$

with

$$(\mathcal{W}_\theta y)(t) := \int_{\theta(t)}^{t} k(t-s) G(y(s)) ds?$$

4.8 Notes

4.1: Basic theory of Volterra equations with delays (I)
Variation-of-constants formulas for DDEs (including neutral equations) can be found for example in the books by Hale (1977), Hale and Verduyn Lunel (1993), and Diekmann et al. (1995). The paper by Cerha (1976) presents various variation-of-constants formulas and related solution representations for second-kind VIEs with variable delays; while Corduneau (1989) establishes analogous results for abstract Volterra differential equations.

An important early paper on nonlinear delay VIEs is Nohel and Levin (1964) (see also Ford, Baker and Roberts (1998)). The analysis of such functional equations received considerable momentum from the study of mathematical models of biological growth processes: see, e.g. Cooke (1976), Cooke and Kaplan (1976), Hethcote, Lewis and van den Driessche (1989), and Hethcote and van den Driessche (2000) (also for additional references).

The papers by Cahlon, Nachman and Schmidt (1984), Cahlon (1990, 1995b) and Cahlon and Schmidt (1997) deal, in addition to numerical solutions, with various aspects of the theory of delay VIEs of the second kind. Compare also Cahlon and Dentz (2000) for related results. A more general stability analysis can be found in Luzyanina, Roose and Engelborghs (2003). DVIEs with state-dependent delays are discussed in Cahlon and Nachman (1985) and Cahlon (1992).

Results on the existence and uniqueness of solutions to (nonlinear) delay VIEs of the first kind can be found in Meis (1978). See also Esser (1976, 1978) for related results for second-kind DVIEs.

The regularity of solutions to functional differential and integral equations was studied in Neves and Feldstein (1976) (DDEs with state-dependent delays), de Gee (1985), Willé and Baker (1992), Baker and Paul (1997), and in Brunner and Zhang (1999) and Ma (2004) (DVIDEs). The papers by Tavernini (1971) and by Cryer and Tavernini (1972) contain results on the solvability of general Volterra functional differential equations. We also point out the important contribution by Kappel and Kunisch (1987) on invariance results for delay and Volterra functional equations.

Applications of Volterra functional equations
As we have already seen briefly in Section 4.1.5, one of the principal sources of DDEs and, especially, Volterra integral and integro-differential equations with constant or more general non-vanishing delays is the mathematical modelling in population dynamics, with Volterra's pioneering work of the late 1920s marking its beginning. Many basic (early) mathematical models in epidemiology and population growth are described in, e.g. Waltman (1974), Cooke

(1976), Smith (1977), and Busenberg and Cooke (1980). The monographs by Volterra (1931), Volterra and d'Ancona (1935), Cushing (1977), Webb (1985), Kuang (1993), Wu (1996) (partial functional differential equations), Brauer and Castillo-Chávez (2001), and Zhao (2003) contain a wealth of material on the theory and application of population models, as do the proceedings edited by Schmitt (1972), Metz and Diekmann (1986) (especially Chapter IV), and Ruan, Wolkowicz and Wu (2003), and the survey papers by Cooke and Yorke (1973), Busenberg and Cooke (1980), Ruan and Wu (1994), Bocharov and Rihan (2000), and Brauer and van den Driessche (2003) (the last two papers feature extensive bibliographies). Among the milestone papers on this subject are the papers by Volterra (1927, 1928, 1934, 1939), Cooke (1976), Cooke and Kaplan (1976), Smith (1977), Hethcote and Tudor (1980), Hethcote et al. (1989), Cañada and Zertiti (1994), Hethcote and van den Driessche (1995, 2000). In addition, the reader may find it worthwhile to look at Tychonoff (1938) (for early applications of Volterra functional equations), Corduneanu and Lakshmikantham (1980) (on functional equations with unbounded delays), Ruan and Wu (1994) (on non-standard Volterra integro-differential equations), and Thieme and Zhao (2003), not least because of the numerous additional references contained in these papers.

Detailed treatments (and numerous additional applications) of nonlinear delay VIEs and VIDEs can be found in Marshall (1979), Lakshmikantham (1987), Györi and Ladas (1991), Yoshizawa and Kato (1991), Kolmanovskii and Myshkis (1992), Yatsenko (1995), Hritonenko and Yatsenko (1996), Piila (1996), Ruan and Wolkowicz (1996), and Corduneanu and Sandberg (2000). Compare also the papers by Tavernini (1978) and Cahlon and Nachman (1985), and their lists of references, on Volterra equations with state-dependent delays. The second chapter in Vogel (1965) contains an illuminating survey of the historical development of Volterra equations with delays and corresponding detailed references. Finally, the recent monograph by Ito and Kappel (2002) is the authoritative source for information on the mathematical framework for, and applications of, neutral functional integro-differential equations of the type (1.10).

4.2: Collocation for DDEs: a review

The monograph by Bellen and Zennaro (2003) gives a comprehensive account of numerical methods for DDEs, with the focus being on (classical and continuous) Runge–Kutta methods and their asymptotic stability properties. Early papers on the subject are by Bellman (1961), Bellman and Cooke (1965). The papers by Torelli (1989) and Zennaro (1993) are landmarks in the analysis of contractivity of Rung–Kutta approximations to DDEs (but see also Reverdy

(1981, 1990) for closely related results). Related stability results were derived by numerous authors; we cite Bellen and Zennaro (1992), Zennaro (1993, 1997), Bellen (1997), Bellen, Guglielmi and Zennaro (1999), and Torelli and Vermiglio (2003).

The important question of delay-dependent stability was answered by Guglielmi (1998, 2000, 2001); see also Guglielmi and Hairer (2001a, 2001b). In order to obtain an impression of the development of the numerical analysis of DDEs and related functional equations, the reader may wish to consult the surveys by Cryer (1972), Bellen (1985), Jackiewicz and Kwapisz (1991), Zennaro (1995) and Baker (1997).

Bellen (1984) established the local superconvergence results of collocation methods (for the Gauss points) for nonlinear DDEs with non-vanishing delays; see also Vermiglio (1985) and Zennaro (1985, 1986, 1988).

Collocation methods for functional differential equations with periodic solutions were studied in Bellen (1979) and, more recently, in Engelborghs, Luzyanina, in 't Hout and Roose (2000) and Engelborghs and Doedel (2002).

4.3: Collocation for second-kind VIEs with delays
The superconvergence analysis of collocation and iterated collocation solutions for linear second-kind VIEs with constant delays is due to Brunner (1994a). It was extended to nonlinear equations in Brunner (1992a). Related convergence results may be found in Hu (1997c, 1999).

The asymptotic stability of collocation solutions for delay VIEs (with constant delays) was studied by Vermiglio (1992). As we mentioned in the Preface, the stability properties of numerical methods for Volterra integral and more general functional equations are not yet well understood. Stability analyses for some classes of numerical methods for (special) delay VIEs can be found Cahlon (1990, 1995a, 1995b), Cahlon and Dentz (1992), and Cahlon and Schmidt (1997, 2000); see also Tian and Kuang (1995). A more general approach is given in Luzyanina, Roose and Engelborghs (2003).

4.4: Collocation for first-kind VIEs with delays:
Brunner (1999b) uses the integrated form of the neutral functional equation $(d/dt)[(\mathcal{W}_\theta y)(t)] = f(t)$ (with $\theta(t) = t - \tau$) to generate piecewise polynomial solutions and prove corresponding superconvergence results.

4.5: Collocation for VIDEs with delays
Collocation methods for VIDEs with constant delays were studied by Brunner (1994b). These superconvergence results were extended to VIDEs with non-vanishing proportional (and more general) delays in Brunner, Bellen, Maset and Torelli (2002).

Brunner and Vermiglio (2003) analyse continuous Runge-Kutta and collocation methods for delay VIDEs in 'Hale's form'; the focus of the analysis is on contractivity properties of the aproximate solutions. A different approach (based on the integrated form of the neutral Volterra integro-differential equation) is studied in Brunner (1999b).

The papers by Kappel and Kunisch (1982) and Ito and Kappel (1989) are concerned with spline approximation methods to neutral functional differential equations and Volterra functional equations with infinite delay, respectively. The basis for these methods is the semigroup framework generated by the given functional equations (see also the Notes to Chapter 7, and the monograph by Ito and Kappel (2002)).

5

Initial-value problems with proportional (vanishing) delays

Delay differential equations and Volterra functional equations with smooth data and proportional delays that vanish at the left endpoint of their interval of integration $I = [0, T]$ possess smooth solutions on I. However, the superconvergence analysis of collocation solutions to functional equations with these seemingly 'innocent' delays is much more complex, not least due to the fact that variation-of-constants formulas for the representation of their solutions do no longer exist. A thorough understanding of the numerical analysis of these functional equations will be crucial when dealing with more general problems including vanishing delays.

5.1 Basic theory of functional equations with proportional delays

5.1.1 Volterra's 1897 paper and some early history

In the first Nota of his 1896 papers Volterra had studied and solved the problem of 'inverting' definite integrals of the form

$$(\mathcal{V}y)(t) := \int_0^t K(t, s)y(s)ds = g(t), \quad t \in I := [0, T], \quad g(0) = 0,$$

where $K \in C(D)$. He then turned his attention to the more general inversion problem in which the lower limit of integration in the Volterra integral operator is also variable. In particular he considered the delay equation

$$(\mathcal{W}_\theta y)(t) = g(t), \quad t \in I, \quad g(0) = 0, \qquad (5.1.1)$$

where the Volterra integral operator $\mathcal{W}_\theta$ is defined by

$$(\mathcal{W}_\theta \phi)(t) := \int_{\theta(t)}^{t} K(t,s)\phi(s)ds, \quad \text{with} \quad \theta(t) := qt \ (0 < q < 1). \tag{5.1.2}$$

He gave a complete answer to this question in Volterra (1897), by adapting the techniques he used in *Nota I* (1896a) to this new, and very different, situation. Under suitable conditions on K and g (similar to those in Theorem 2.1.8) this equation can be transformed into the equivalent second-kind equation

$$K(t,t)y(t) - qK(t,qt)y(qt) + \int_{qt}^{t} \frac{\partial K(t,s)}{\partial t} y(s)ds = g'(t), \quad t \in I. \tag{5.1.3}$$

(see also Brunner (1997b)). This reformulation is the basis for Volterra's 1897 result which we state below. We set $\bar{D}_\theta := \{(t,s) : \ 0 \leq \theta(t) \leq s \leq t \leq T\}$.

Theorem 5.1.1 *Assume:*

(a) $g \in C^1(I)$, *with* $g(0) = 0$;
(b) $K \in C^(\bar{D}_\theta)$, $\partial K/\partial t \in C(\bar{D}_\theta)$, *with* $|K(t,t)| \geq k_0 > 0$ *for all* $t \in I$.

Then for each $\theta(t) = qt$ *with* $q \in (0,1)$ *the first-kind delay integral equation (5.1.1) possesses a unique solution* $y \in C(I)$.

Proof Volterra starts the proof by the following observation (Volterra (1897, pp. 156–157). Suppose that the given (real-valued) functions λ and φ are continuous on I, with $|\lambda(0)| \leq 1$, and consider the infinite series

$$\theta(t) := \varphi(t) + \sum_{j=1}^{\infty} \alpha^j \left(\prod_{l=0}^{j-1} \lambda(\alpha^l t) \right) \varphi(\alpha^j t), \quad t \in I.$$

This series converges uniformly on I, and hence its limit θ lies in $C(I)$. On the other hand, if $\theta \in C[0,T]$ is given, then replacing t in the above equation by αt and then multiplying by $\alpha\lambda(x)$ readily leads to an expression for the unknown function φ,

$$\theta(t) - \alpha\lambda(t)\theta(\alpha t) = \varphi(t), \quad t \in I.$$

In other words, these two equations are reciprocal to each other. This observation was then used by Volterra to establish the desired result for the delay integral equation (5.1.2) in a rather elegant way. We shall encounter the second functional equation again later, in Section 5.3.4; see also Liu (1995b).

Volterra's analysis – which relies on Picard iteration techniques – was extended by Lalesco (1908, 1911) (see also Volterra (1913, pp. 92–101) and Fenyö and Stolle (1984, pp. 324–327)) to first-kind integral equations with more general vanishing delays.

We note in passing that the above result was generalised by, among others, Lalesco (1908, 1911), and more recently by Denisov and Korovin (1992) and by Denisov and Lorenzi (1995). From the latter paper we cite the following result.

Theorem 5.1.2 *Assume the lag function θ in $\mathcal{W}_\theta$ satisfies*

(a) $\theta \in C^3(I)$, *with* $\theta(0) = 0,\ \theta'(0) = 1,\ \theta''(0) < 0,\ \theta(t) < t\ (t \in (0, T])$, $\theta'(t) > 0$ *for* $t \in I$,

and let

(b) $g \in C^2(I)$, *with* $g(0) = g'(0) = 0$;
(c) $K \in C^3(\bar{D}_\theta)$, *with* $|K(t, t)| \geq k_0 > 0\ (t \in I)$.

Then the first-kind delay integral equation $(\mathcal{W}_\theta y)(t) = g(t)$ *has a unique solution* $y \in C(I)$.

Remark A similar result was proved in Denisov and Korovin (1992), but under the hypothesis that $\theta'(0) < 1$. If, as in the above theorem, $\theta'(0) = 1$, the domain $\bar{D}_\theta$ has a *cusp* at the point $(t, s) = (0, 0)$, and new techniques are needed to deal with this situation. We note that the case $\theta'(0) = 1$ was already treated, albeit in a somewhat sketchy way, by Lalesco (1911).

In his papers of 1913 and 1914 Andreoli studied the class of 'pure' delay integral equations of the second kind described by

$$y(t) = g(t) + (\mathcal{V}_\theta y)(t), \quad t \in I,$$

with

$$(\mathcal{V}_\theta \phi)(t) := \int_0^{\theta(t)} K(t, s)\phi(s)ds,$$

and he illustrated his observation (Andreoli (1914, p. 77)) that

> '... *la $\theta(t)$ avrà un'enorme influenza sulle formole di soluzione* ...'

by two examples, namely

$$y(t) = g(t) + \int_0^{qt} K(t, s)y(s)ds, \quad t \in I = [0, T] \quad (0 < q < 1) \tag{5.1.4}$$

(which is also analysed in Chambers (1990)), and

$$y(t) = g(t) + \int_0^{t^r} K(t, s)y(s)ds, \quad t \in [0, 1] \quad (r > 0).$$

The use of Picard iteration, extending Volterra's approach of 1896, reveals immediately that the representation of the solution is now much more complex (see Theorem 5.1.4 below). Andreoli's statement is even more true in the numerical analysis of such proportional delay VIEs, as we shall see in Sections 5.2.5 and 5.3.6!

If the kernel in (5.1.4) is constant, $K(t, s) = b/q$, and $g \in C^1(I)$, then the delay VIE is equivalent to the proportional delay differential equation

$$y'(t) = g'(t) + by(qt), \quad t \in I, \quad y(0) = g(0).$$

We shall have a closer look at 'innocent' DDEs of this type in the following section (cf. (5.1.6)).

5.1.2 Linear differential equations with proportional delays

The linear DDE with constant coefficients,

$$y'(t) = ay(t) + by(qt), \quad t \geq 0 \quad (0 < q < 1), \tag{5.1.5}$$

arose in the mathematical modelling of the wave motion in the supply line to an overhead current collector (*pantograph*) of an electric locomotive (see Ockendon and Tayler (1971) and Fox et al. (1971); also Tayler (1986, pp. 40–45, 50–53)): the resulting *pantograph equation* is a (seemingly!) very simple example of a DDE with vanishing delay on any interval $I := [0, T]$: here, we have $\theta(t) = t - \tau(t)$, with $\tau(t) = (1 - q)t \geq 0$.

A special case of (5.1.5) is the 'pure delay' equation

$$y'(t) = by(qt), \quad t \geq 0, \quad y(0) = y_0 \quad (b \neq 0) \tag{5.1.6}$$

(which we have already met at the end of the previous section). Its (unique) solution is given by

$$y(t) = \sum_{j=0}^{\infty} \frac{q^{j(j-1)/2}}{j!}(bt)^j \cdot y_0, \quad t \geq 0. \tag{5.1.7}$$

Iserles (1993) presents an illuminating introduction into the beautifully complex world of solutions to (5.1.5) and its generalisations.

The following result can be found in Kato and McLeod (1971); compare also Frederickson (1971), Morris, Feldstein and Bowen (1972), Derfel (1990), Iserles (1993), and Terjéki (1995).

Theorem 5.1.3 *For any* $q \in (0, 1)$ *and any* y_0 *the delay differential equation* (5.1.5) *possesses a unique solution* $y \in C^1(I)$ *with* $y(0) = y_0$*, regardless of the* *choice of* $a,\ b \neq 0$, *and* $T > 0$. *It is given by*

$$y(t) = \sum_{n=0}^{\infty} \gamma_n(q) t^n,$$

where

$$\gamma_n(q) := \frac{1}{n!} \prod_{j=1}^{n} (a + bq^{j-1}).$$

Proof We apply Picard iteration to the equivalent Volterra integral equation,

$$y(t) = y_0 + \int_0^t (ay(s) + by(qs))ds, \quad t \in I.$$

It can be shown that the resulting sequence $\{y_n(t)\}$ $(n \geq 0,\ y_0(t) := y_0)$ converges uniformly on any interval I. Moreover, setting

$$y(t) := \sum_{n=0}^{\infty} \gamma_n(q) t^n,$$

one verifies the power series has infinite radius of convergence, since its coefficients satisfy

$$\frac{\gamma_n}{\gamma_{n-1}} = \frac{1}{n}[a + bq^{n-1}], \quad n \geq 1.$$

Remarks

1. The above result (except for the last statement) remains true for (5.1.5) with variable coefficients $a,\ b \in C(I)$. More precisely, if $a,\ b \in C^m(I)$ then, for any $q \in (0, 1)$ and any y_0, the solution y lies in $C^{m+1}(I)$. See also Terjéki (1995) on various representations of solutions to linear pantograph DDEs. Properties of solutions of nonlinear versions of these equations (e.g. Riccati-type equations) can be found in the papers by Iserles (1994b) and Iserles and Terjéki (1995).
2. These results reveal a crucial difference between the regularity of solutions of DDEs with non-vanishing delays and DDEs of pantograph-type DDEs: for the latter, smooth data lead to solutions that are smooth on the *entire* interval $[0, T]$. In particular, solutions to (5.1.6) are *entire functions* of *order zero*. It follows from classical complex function theory (Ahlfors's Theorem) that an entire function of order zero cannot have finite asymptotes. This implies that, for $b < 0$, non-trivial solutions of (5.1.6) are not bounded on $\mathbb{R}^+$; also, the number of sign changes (zeros) is infinite. (See also Iserles (1993, 1997b), Liu (1997), and Exercise 5.6.2.) The properties of solutions to the second-order analogue of (5.1.6) were analysed in Bélair (1981).

Table 5.1. *Zeros of* $y(t)$ *for* $b = -1$

$q = 0.05$	$q = 0.5$	$q = 0.95$
$z_1 = 1.02631$	$z_1 = 1.48808$	$z_1 = 8.96684$
$z_2 = 40.3651$	$z_2 = 4.88114$	$z_2 = 10.8942$
	$\vdots$	$\vdots$
$z_3 = 1205.57$	$z_{10} = 5223.38$	$z_{46} = 5258.99$

To give the reader an idea of how these zeros depend on q, Table 5.1 exhibits a representative sample of zeros of y. Additional information (for $q = 1/4,\ q = 3/4$) can be found in Iserles (1993, p. 5).

The values of $|y(t)|$ in the interval given by the last listed zero and the following one exceed 10^{15}.

The reader interested in details on the asymptotic distribution of the zeros of such solutions may wish to consult the 1992 paper by Elbert (which includes a reference to the first study of this subject, a 1967 report by Feldstein and Kolb).

5.1.3 Linear Volterra integral equations with proportional delays

We now return to the delay VIE (5.1.4) considered by Andreoli (1913, 1914) (and by Chambers (1990)), and to his remark about the effect the (vanishing) proportional delay has on the representation of the solution.

Theorem 5.1.4 *Let* g *and* K *in (5.1.4) satisfy* $g \in C(I)$ *and* $K \in C(D_\theta)$*, where* $D_\theta := \{(t,s):\ 0 \le s \le \theta(t),\ t \in I\}$*. Then for any* $\theta(t) := qt$ *with* $q \in (0,1)$ *the delay integral equation (5.1.4) possesses a unique solution* $y \in C(I)$*. This solution is given by*

$$y(t) = g(t) + \sum_{n=1}^{\infty} \int_0^{q^n t} K_n(t,s) g(s) ds$$
$$= g(t) + \int_0^t \left(\sum_{n=1}^{\infty} q^n K_n(t, q^n s) g(q^n s) \right) ds, \quad t \in I. \tag{5.1.8}$$

The iterated kernels $K_n(t,s)$ $(= K_n(t,s;q))$ *are obtained recursively by*

$$K_{n+1}(t,s) := \int_{q^{-n}s}^{qt} K(t,v) K_n(v,s) dv, \quad (t,s) \in D_\theta^{(n+1)} \quad (n \ge 1),$$

with $K_1(t,s) := K(t,s)$ *and*

$$D_\theta^{(k)} := \{(t,s):\ 0 \le s \le q^k t,\ t \in I\}, \quad k \ge 1 \ \ (D_\theta^{(1)} = D_\theta).$$

Proof The Picard iteration process we applied to the integrated form (5.1.8) of the pantograph DDE can of course be used for the delay VIE (5.1.4), with suitably adapted Dirichlet's formula when changing the order of integration in the double integrals: here again, the resulting limits of integration depend on the iteration number n. To see this in some more detail, we have, setting $y_0(t) := g(t)$,

$$y_1(t) := g(t) + \int_0^{qt} K_1(t,s)g(s)ds,$$

and hence

$$\begin{aligned} y_2(t) &:= g(t) + \int_0^{qt} K_1(t,s)\left(g(s) + \int_0^{qs} K_1(s,v)g(v)dv\right)ds \\ &= g(t) + \int_0^{qt} K_1(t,s)g(s)ds + \int_0^{qt}\left(\int_{q^{-1}v}^{qt} K_1(t,s)K_1(s,v)ds\right)y(v)dv. \end{aligned}$$

It is now easily verified by induction that the iterated kernels $K_n(t,s)$ of the given kernel $K(t,s) =: K_1(t,s)$ are generated recursively by

$$K_{n+1}(t,s) = \int_{q^{-n}s}^{qt} K(t,v)K_n(v,s)dv, \quad (t,s) \in D_\theta^{(n+1)} \quad (n \ge 1)$$

(see also Chambers (1990)).

We leave the detailed steps of the proof as an exercise but state the uniform bounds for the iterated kernels. This (readily verified) result will play a role in the analysis of global superconvergence of collocation solutions for (5.1.9) below (compare the proof of Theorem 5.3.4).

Lemma 5.1.5 *Uniform bounds on* $I = [0,T]$ *for the iterated kernels* $K_n(t,s)$ *defined in Theorem 5.1.5 are given by*

$$|K_n(t,s)| \le \frac{q^{n(n-1)/2}}{(n-1)!}T^{n-1}\bar{K}_\theta^n, \quad (t,s) \in D_\theta^{(n)} \quad (n \ge 1),$$

where we have set $\bar{K}_\theta := \max\{|K(t,s)|:\ (t,s) \in D_\theta\}$.

The existence, uniqueness and regularity properties hold also for the more general linear delay VIE with proportional delay,

$$y(t) = g(t) + (\mathcal{V}y)(t) + (\mathcal{V}_\theta y)(t), \quad t \in I, \tag{5.1.9}$$

corresponding to the Volterra integral operators

$$(\mathcal{V}y)(t) := \int_0^t K_1(t,s)y(s)ds, \quad (\mathcal{V}_\theta y)(t) := \int_0^{\theta(t)} K_2(t,s)y(s)ds,$$

with $\theta(t) := qt$ $(0 < q < 1)$, $K_1 \in C(D)$ and $K_2 \in C(D_\theta)$.

Theorem 5.1.6 *Assume that $K_1 \in C^d(D)$ and $K_2 \in C^d(D_\theta)$, for some $d \geq 0$. Then the delay integral equation (5.1.9) with $\theta(t) = qt$ $(0 < q < 1)$ has a unique solution $y \in C^d(I)$ for any g with $g \in C^d(I)$.*

Proof Theorem 5.1.4 shows that the iterated kernels $K_n(t,s;q)$ associated with the kernel K of the special delay integral equation (5.1.4) inherit the regularity of K. Since the additional term $(\mathcal{V}y)(t)$ in the general linear delay VIE (5.1.9) will not lead to lower regularity in the Picard iteration process, the assertion of Theorem 5.1.5 follows from the uniform convergence of the Picard iterates on I, for any $q \in (0,1)$.

We shall see in Section 5.1.5 that this regularity result can also be derived by means of embedding techniques.

Remark The paper by Morris, Feldstein and Bowen (1972, pp. 518–523) contains an illuminating discussion of the connection between general pantograph DDEs and certain Volterra integral and integro-differential equations with (multiple) proportional delays. Compare also Iserles and Liu (1994).

5.1.4 Volterra integro-differential equations with proportional delays

In order to obtain some first insight into the properties of solutions of linear VIDEs with proportional delays we will consider the analogue of Andreoli's 'pure delay' problem (5.1.4), namely

$$y'(t) = g(t) + \int_0^{qt} K(t,s)y(s)ds, \quad t \in I := [0,T], \quad y(0) = y_0, \quad (5.1.10)$$

assuming that $g \in C(I)$, $K \in C(D_\theta)$, and $0 < q < 1$. This initial-value problem is equivalent to the delay VIE

$$y(t) = g_0(t) + \int_0^{qt} H(t,s;q)y(s)ds, \quad t \in I, \quad (5.1.11)$$

where

$$g_0(t) := y_0 + \int_0^t g(s)ds, \quad H(t,s;q) := \int_{q^{-1}s}^t K(v,s)dv.$$

We now apply Theorem 5.1.4: setting $H_1(t,s) := H(t,s;q)$, and denoting by $H_n(t,s)$ the corresponding iterated kernels, the unique solution y of (5.1.11) (which, since g_0 and $H(\cdot,\cdot;q)$ are continuously differentiable functions, lies in $C^1(I)$) can be expressed in the form*

$$y(t) = g_0(t) + \sum_{n=1}^{\infty} \int_0^{q^n t} H_n(t,s) g_0(s) ds, \quad t \in I,$$

where the infinite series converges absolutely and uniformly. If we now substitute the expressions for $g_0(t)$, an obvious rearrangement (using Dirichlet's formula) leads to the following result.

Theorem 5.1.7 *For any $g \in C(I)$ and $K \in C(D_\theta)$, the unique solution $y \in C^1(I)$ to the initial-value problem (5.1.10) has the representation*

$$y(t) = \left(1 + \sum_{n=1}^{\infty} \tilde{H}_n(t,0)\right) y_0 + \sum_{n=0}^{\infty} \int_0^{q^n t} \tilde{H}_n(t,s) g(s) ds, \quad t \in I.$$

Here, we have set

$$\tilde{H}_n(t,s) := \int_s^{q^n t} H_n(t,v) dv, \quad (t,s) \in D_\theta^{(n)} \quad (n \geq 1),$$

with $\tilde{H}_0(t,s) := 1$ on D, and we note that

$$\tilde{H}_n(t,0) = \int_0^{q^n t} H_n(t,v) dv \quad t \in I.$$

The initial-value problem for the *general* linear VIDE with proportional delay,

$$y'(t) = a(t)y(t) + b(t)y(qt) + g(t) + (\mathcal{V}y)(t) + (\mathcal{V}_\theta y)(t), \quad t \in I, \quad (5.1.12)$$

with $\theta(t) = qt$ $(0 < q < 1)$, is equivalent to the delay VIE

$$\begin{aligned} y(t) &= g_0(t) + \int_0^t \left(a(s) + \int_s^t K_1(v,s) dv\right) y(s) ds \\ &\quad + \int_0^{qt} \left((1/q) b(s/q) + \int_{q^{-1}s}^t K_2(v,s) dv\right) y(s) ds \\ &=: g_0(t) + \int_0^t G_1(t,s) y(s) ds + \int_0^{qt} G_2(t,s;q) y(s) ds, \end{aligned}$$

where

$$g_0(t) := y_0 + \int_0^t g(s) ds.$$

The regularity of the kernels G_1 and $G_2(\cdot;\cdot;q)$ is determined by that of the original data $a,\ b$ and $K_1,\ K_2$. Thus, Theorem 5.1.6 implies

Theorem 5.1.8 *Assume:*
(a) $a,\ b,\ g \in C^d(I)$ *for some* $d \geq 0$*;*
(b) $K_1 \in C^d(D)$ *and* $K_2 \in C^d(D_\theta)$.

Then for each initial value y_0 *the delay VIDE (5.1.12) possesses a unique solution* $y \in C^{d+1}(I)$.

5.1.5 Embedding techniques

The embedding of a (proportional) delay differential equation into an infinite system of ordinary differential equations was studied in detail in the 1995 paper by Feldstein, Iserles and Levin (1995). The motivation behind this approach was to explore another way of obtaining results on the asymptotic stability (or the boundedness) of solutions of such DDEs, and for constructing feasible methods for their numerical solution. It also permits the derivation of regularity results for the exact solutions.

Here, we extend these embedding techniques to the delay Volterra integral equation (5.1.9) and to the delay Volterra integro-differential equation (5.1.12). Note that these delay VEs contain the important special cases characterised by $K_2(t,s) = -K_1(t,s) =: -K(t,s)$:

$$y(t) = g(t) + (\mathcal{W}_\theta y)(t), \quad t \in I, \tag{5.1.13}$$

and

$$y'(t) = a(t)y(t) + b(t)y(qt) + (\mathcal{W}_\theta y)(t), \quad t \in I; \quad y(0) = y_0. \tag{5.1.14}$$

corresponding to the delay Volterra operator $\mathcal{W}_\theta$ introduced in (5.1.2). The following embedding results (which can be extended to the nonlinear counterparts of the above pantograph-type Volterra equations) contain the key not only to establishing results on the existence, uniqueness, and regularity of solutions but possibly also to the analysis of the local superconvergence properties of collocation solutions to such functional equations.

Embedding results for the DVIE (5.1.9)

Lemma 5.1.9 *The delay VIE (5.1.9) can be embedded into an infinite-dimensional system of 'classical' VIEs of the second kind,*

$$z_\nu(t) = g_\nu(t) + \int_0^t \big(K_{1,\nu}(t,s)z_\nu(s) + K_{2,\nu}(t,s)z_{\nu+1}(s)\big)ds \quad (\nu \in \mathbb{N}_0), \tag{5.1.15}$$

where

$$z_\nu(t) := y(q^\nu t), \quad g_\nu(t) := g(q^\nu t)$$

and

$$K_{1,\nu}(t,s) := q^\nu K_1(q^\nu t, q^\nu s), \quad K_{2,\nu}(t,s) := q^{\nu+1} K_2(q^\nu t, q^{\nu+1} s).$$

The **proof** of this embedding result is left as an exercise.

Consider now the *truncated* (finite) system corresponding to (5.1.15),

$$z_{M,\nu}(t) = g_\nu(t) + \int_0^t \big(K_{1,\nu}(t,s) z_{M,\nu}(s) + K_{2,\nu}(t,s) z_{M,\nu+1}(s)\big) ds$$
$$(\nu = 0, 1, \dots, M-1), \tag{5.1.16}$$

$$z_{M,M}(t) = g_M(t) + \int_0^t K_{1,M}(t,s) z_{M,M}(s) ds, \quad t \in I. \tag{5.1.17}$$

Lemma 5.1.10 *Assume that* $g \in C(I)$, $K_1 \in C(D)$, $K_2 \in C(D_\theta)$. *Then for* $\nu = M, M-1, \dots, 0$, *the (unique) solution of (5.1.16), (5.1.17) satisfies*

$$||z_\nu - z_{M,\nu}||_\infty \le C q^{\tilde{M}}, \quad \textit{with} \ \ \tilde{M} \ge M+1.$$

Proof Setting $\varepsilon_{M,\nu} := z_\nu - z_{M,\nu}$, it follows from (5.1.16) and (5.1.17) that

$$\varepsilon_{M,\nu}(t) = \int_0^t K_{1,\nu}(t,s) \varepsilon_{M,\nu}(s) ds + \Phi_{M,\nu}(t), \quad t \in I \tag{5.1.18}$$

$(\nu = 0, 1, \dots, M)$ with

$$\Phi_{m,\nu}(t) := \begin{cases} \displaystyle\int_0^t K_{2,M}(t,s) z_{M+1}(s) ds & \text{if } \ \nu = M \\ \displaystyle\int_0^t K_{2,\nu}(t,s) \varepsilon_{M,\nu+1}(s) ds & \text{if } \ M-1 \ge \nu \ge 0. \end{cases}$$

Let $R_{1,\nu} = R_{1,\nu}(t,s)$ denote the resolvent kernel associated with the kernel $K_{1,\nu}$ in (5.1.15); we know from Section 2.1.1 that $K_1 \in C(D)$ implies $R_{1,\nu} \in C(D)$ for all $\nu \ge 0$. The (unique) solution of the finite system (5.1.18) may thus be written as

$$\varepsilon_{M,\nu}(t) = \int_0^t R_{1,\nu}(t,s) \Phi_{M,\nu}(s) ds + \Phi_{M,\nu}(t), \quad t \in I \tag{5.1.19}$$

$(\nu = M, M-1, \dots, 0)$. Since $|K_{2,\nu}(t,s)| \le \bar{K}_2 q^{\nu+1}$, $(t,s) \in D_\theta$, where $\bar{K}_2 := \max_{D_\theta} |K_2(t,s)|$, setting $\nu = M$ in (5.1.19) leads to

$$|\varepsilon_{M,M}(t)| \le C q^{M+1}, \quad t \in I.$$

Thus, assuming that $||\varepsilon||_\infty \le Cq^{\tilde{M}}$ $(\tilde{M} \ge M)$ for $\nu = M, M-1, \dots, M_0+1$, we find

$$|\Phi_{M_0,\nu}(t)| \le \bar{K}_2 T q^{\nu+1} C_0 q^{\tilde{M}} =: C q^{\tilde{M}+\nu+1}, \quad \nu \ge \tilde{M}+1,$$

and hence,

$$|\varepsilon_{M,M_0}(t)| \le Cq^{\tilde{M}},\ t \in I, \quad \text{with} \ \ \tilde{M} \ge M+1.$$

This establishes the uniform estimates in Lemma 5.1.10.

Embedding results for the DVIDE (5.1.12)

Lemma 5.1.11 *The delay VIDE (5.1.12) can be embedded into an infinite-dimensional system of 'classical' VIDEs, namely,*

$$z_\nu'(t) = \tilde{a}_\nu(t) z_\nu(t) + \tilde{b}_\nu(t) z_{\nu+1}(t) + \int_0^t \big(\tilde{K}_{1,\nu}(t,s) z_\nu(s) + \tilde{K}_{2,\nu}(t,s) z_{\nu+1}(s)\big) ds \tag{5.1.20}$$

($\nu \in \mathbb{N}_0$), with

$$\tilde{a}_\nu(t) := q^\nu a(q^\nu t), \quad \tilde{b}_\nu(t) := q^\nu b(q^\nu t),$$

and

$$\tilde{K}_{i,\nu}(t,s) := q^\nu K_{i,\nu}(t,s) \quad (i = 1, 2).$$

The kernels $K_{i,\nu}$ are those defined in Lemma 5.1.9.

This easily verified result leads to the VIDE analogue of Lemma 5.1.10:

Lemma 5.1.12 *Assume that $a, b \in C(I)$, $K_1 \in C(D)$, and $K_2 \in C(D_\theta)$. Then the (unique) solution of the truncated (finite) system of VIDEs corresponding to (5.1.20),*

$$\begin{aligned} z_{M,\nu}'(t) &= \tilde{a}_\nu(t) z_{M,\nu}(t) + \tilde{b}_\nu(t) z_{M,\nu+1}(t) \\ &\quad + \int_0^t \big(\tilde{K}_{1,\nu}(t,s) z_{M,\nu}(s) + \tilde{K}_{2,\nu}(t,s) z_{M,\nu+1}(s)\big) ds \qquad (5.1.21) \\ &\quad (\nu = 0, 1, \dots, M-1), \\ z_{M,M}'(t) &= \tilde{a}_\nu(t) z_{M,M}(t) + \int_0^t \tilde{K}_{1,M}(t,s) z_{M,M}(s) ds, \quad t \in I, \end{aligned} \tag{5.1.22}$$

with $z_{M,\nu}(0) = y_0$, satisfies

$$||z_\nu(t) - z_{M,\nu}(t)||_\infty \le Cq^{\tilde{M}} \quad (\nu = 0, 1, \dots, M), \quad \textit{with} \quad \tilde{M} \ge M.$$

Proof Setting $\varepsilon_{M,\nu} := z_\nu - z_{M,\nu}$, we have

$$\varepsilon'_{M,\nu}(t) = \tilde{a}_\nu(t)\varepsilon_{M,\nu}(t) + \int_0^t \tilde{K}_{1,\nu}(t,s)\varepsilon_{M,\nu}(s)ds + \Psi_{M,\nu}(t), \quad t \in I, \tag{5.1.23}$$

with $\varepsilon_{M,\nu}(0) = 0$ ($\nu = M, M-1, \dots, 0$). Here,

$$\Psi_{M,\nu}(t) := \begin{cases} \tilde{b}_M(t)z_{M+1}(t) + \int_0^t \tilde{K}_{2,M}(t,s)z_{M+1}(s)ds & \text{if } \nu = M \\ \tilde{b}_\nu(t)\varepsilon_{M,\nu+1}(t) + \int_0^t \tilde{K}_{2,\nu}(t,s)\varepsilon_{M,\nu+1}(s)ds & \text{if } \nu < M. \end{cases}$$

Let $r_{1,\nu} = r_{1,\nu}(t,s)$ denote the (differential) resolvent kernel corresponding to the functions $\tilde{a}_\nu$ and $\tilde{K}_{1,\nu}$ in (5.1.20); that is, $r_{1,\nu}$ is defined by the (unique) solution of the (differential) resolvent equation

$$\frac{\partial r_{1,\nu}(t,s)}{\partial s} = -r_{1,\nu}(t,s)\tilde{a}_\nu(s) - \int_s^t r_{1,\nu}(t,z)\tilde{K}_{1,\nu}(z,s)\,dz, \quad (t,s) \in D,$$

with $r_{1,\nu}(t,t) = 1,\ t \in I$ (recall Theorem 3.1.2). The solution of the initial-value problem (5.1.21),(5.1.22) can then be written in the form

$$\varepsilon_{M,\nu}(t) = r_{1,\nu}(t,0)\varepsilon_{M,\nu}(0) + \int_0^t r_{1,\nu}(t,s)\Psi_{M,\nu}(s)ds, \quad t \in I, \tag{5.1.24}$$

($\nu = M, M-1, \dots, 0$), where $\varepsilon_{M,\nu}(0) = 0$ for all ν. Since

$$|\Psi_{M,M}(t)| \le \gamma_0 q^M + \gamma_1 q^{2M+1}, \quad t \in I,$$

for some finite constants γ_1 (recall that $\tilde{K}_{2,\nu}(t,s) = q^\nu K_{2,\nu}(t,s)$ and $|K_{2,\nu}(t,s)| \le \bar{K}_2 q^{\nu+1}$), we readily derive the uniform estimate

$$|\varepsilon_{M,M}(t)| \le C_0 q^M + C_1 q^{2M+1} =: Cq^M, \quad t \in I \quad (M \in \mathbb{N}_0),$$

where $C = C(q,M) < \infty$ and $q \in (0,1)$.

For $\nu < M$ the argument for bringing the proof to its conclusion is analogous to the one in the proof of Lemma 5.1.10: we employ the representation (5.1.24) and the estimate for $||\varepsilon_{M,M}||_\infty$. Details are left to the reader.

Remark The (uniform) convergence results in Lemmas 5.1.10 and 5.1.12 allow us not only to deduce the existence of unique solutions to the delay problems (5.1.9) and (5.1.12) but also to establish the global regularity results already alluded to: C^m-data imply that the solutions of the DVIE and the DVIDE lie, respectively, in $C^m(I)$ and $C^{m+1}(I)$.

5.1.6 Nonlinear pantograph-type functional equations

Results on the existence, uniqueness and qualitative behaviour of solutions to various classes of nonlinear DDEs with vanishing proportional delays can be found for example in Iserles (1994a), Iserles and Tejéki (1995), Feldstein, Iserles and Levin (1995) (embedding techniques), and Feldstein and Liu (1998) (see also for additional references).

Three typical examples are described below; the first two were studied in detail by Iserles (1994a) and Feldstein and Liu (1998).

Example 5.1.1 The nonlinear DDE with proportional delay,

$$y'(t) = ay(t) + by(qt)(1 - y(qt)), \quad t \geq 0 \ \ (0 < q < 1). \tag{5.1.25}$$

is a Riccati-type equation which may be viewed as the proportional delay analogue of the classical logistic equation.

Example 5.1.2 The rational version of (5.2.5) is

$$y'(t) = \frac{ay(t)}{1 + by(qt)}, \quad t \geq 0 \ \ (0 < q < 1). \tag{5.1.26}$$

Example 5.1.3 Nonlinear second-kind Volterra integral equations with proportional delays were studied by Chambers (1990). He showed that the equation

$$y(t) = g(t) + \int_0^{qt} k(t, s, y(s))ds, \quad t \in I := [0, T],$$

possesses a unique solution $y \in C(I)$ if $g \in C(I)$ and k satisfies

$$|k(t, s, y) - k(t, s, z)| \leq P(t)Q(s)|y - z| \quad \text{for} \quad (t, s) \in D_\theta, \ \ y, z \in \mathbb{R},$$

for some functions P and Q with $P(t) = pt^\alpha, \ Q(s) = s^\beta \ (\alpha, \ \beta \geq 0)$.

Results on the existence and uniqueness of solutions of more general nonlinear VIEs (and VIDEs) with vanishing proportional delays can be established by using the embedding techniques described in the previous section. We leave the details to the reader.

5.2 Collocation for DDEs with proportional delays

Since a complete understanding of the effects of the (seemingly simple) vanishing proportional delay $\theta(t) = qt$ on the structure of the collocation equations and hence on the convergence properties of collocation solutions to functional equations with vanishing proportional delays is essential for tackling analogous questions for problems with state-dependent delays, we will present a rather detailed description of the collocation equations and the corresponding error analysis, perhaps more so than in the previous chapters. Although Runge–Kutta methods for pantograph-type DDEs are studied in some detail in Bellen and Zennaro (2003), our approach in Section 5.2.1 may yield some additional insight into collocation-based continuous RK methods.

5.2.1 Collocation and continuous Runge–Kutta methods

Assume that the initial-value problem

$$y'(t) = f(t, y(t), y(qt)), \quad t \in I := [0, T], \quad y(0) = y_0 \ (0 < q < 1), \tag{5.2.1}$$

possesses a unique solution $y \in C^1(I)$. As we have seen in Section 5.1.2, solutions of equations with proportional delays on any interval $[0, T]$ essentially inherit the regularity of the given data on that interval; that is, there are no primary discontinuity points, in sharp contrast to DDEs with *non-vanishing* delays (e.g. (5.2.1) on $I = [t_0, T]$ with $t_0 > 0$). Hence, the meshes I_h underlying the collocation space $S_m^{(0)}(I_h)$ need not be constrained ones, and we may choose, as in Section 1.1 for ODEs,

$$I_h := \{t_n : \ 0 = t_0 < t_1 < \ldots < t_N = T\},$$

with

$$\sigma_n := (t_n, t_{n+1}], \quad h_n := t_{n+1} - t_n, \quad h := \max_{(n)} h_n,$$

and $\bar{\sigma}_n := [t_n, t_{n+1}]$. The resulting computational form of the collocation equation for $u_h \in S_m^{(0)}(I_h)$,

$$u_h'(t) = f(t, u_h(t), u_h(qt)), \quad t \in X_h, \quad u_h(0) = y_0, \tag{5.2.2}$$

with collocation points

$$X_h := \{t_{n,i} := t_n + c_i h_n : \ 0 \leq c_1 < \ldots < c_m \leq 1 \ (0 \leq n \leq N-1)\},$$

is, however, much more complex than the one for ODEs or for DDEs with non-vanishing delays. This is due to the presence of the terms $q(t_n + c_i h_n)$: since $t_0 = 0$, these points $qt_{n,i}$ will initially lie in the same subinterval σ_n as the collocation points $t_{n,i}$ themselves, and this will be followed in general by 'partial overlap'. The collocation equations assume a structure similar to the one of the collocation equation corresponding to a DDE with non-vanishing delay only after some 'transition phase', when we have reached the subintervals σ_n for which $qt_{n,i} \leq t_n$ for all $i = 1, \ldots, m$.

We shall limit our analysis to *uniform meshes* and to certain *(quasi-) geometric meshes*. The (non-trivial!) case of more general meshes – important when designing collocation methods on *adaptive meshes* I_h – will be left as a research problem (Exercise 5.6.24).

Assume first that $I_h := \{t_n := nh : \ n = 0, 1, \ldots, N; \ ; h_n = h = T/N\}$ is a *uniform mesh*. Set

$$q_{n,i} := \lfloor q(n + c_i)\rfloor, \quad \gamma_{n,i} := q(n + c_i) - q_{n,i} \in [0, 1), \tag{5.2.3}$$

where, for $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the largest integer not exceeding x. Hence,

$$qt_{n,i} = q(t_n + c_i h) = h \cdot (q_{n,i} + \gamma_{n,i}) = t_{q_{n,i}} + \gamma_{n,i} h \in [t_{q_{n,i}}, t_{q_{n,i}+1}]. \tag{5.2.4}$$

We denote by $\lceil x \rceil$ the least upper integer bound of $x \in \mathbb{R}$.

Lemma 5.2.1 *Let $q \in (0, 1)$ and $0 < c_1 < \ldots < c_m \leq 1$ be given, and assume that I_h is a uniform mesh with mesh diameter $h = T/N$. Then:*

(i) For $n = 0$ we have $qt_{n,i} \in (t_n, t_{n+1})$ for $i = 1, \ldots, m$.
(ii) If $n \geq 1$, then $qt_{n,i} \in (t_n, t_{n+1})$ for $i = 1, \ldots, m$ if, and only if, $n < \lceil \frac{q}{1-q} c_1 \rceil =: q^I$.
(iii) $qt_{n,i} \leq t_n$ for $i = 1, \ldots, m$ if, and only if, $\lceil \frac{q}{1-q} c_m \rceil =: q^{II} \leq n \leq N - 1$.

Proof

(i) Assuming that $c_1 > 0$ we clearly have $q(t_0 + c_i h) = h \cdot qc_i \in (0, h)$ for all i; i.e., 'complete overlap' occurs always at least for $n = 0$, for any $q \in (0, 1)$.
(ii) Since $qt_{n,i} \geq qt_{n,1}$, $qt_{n,1} \in (t_n, t_{n+1})$ if, and only if, $q(n + c_1) > n$. This holds if, and only if, $n < qc_1/(1 - q)$.
(iii) We have $qt_{n,i} \leq t_n$ for $i = 1, \ldots, m$ if, and only if, $qt_{n,m} \leq t_n$. This leads to the condition that $n \geq qc_m/(1 - q)$, and hence to the final assertion in Lemma 5.2.1.

The above results and their proofs are readily modified to cover sets $\{c_i\}$ where $c_1 = 0$ (e.g. the Lobatto points). We leave this as an exercise.

Lemma 5.2.1 shows that the recursive computation of the collocation solution for the DDE (5.2.1) with vanishing proportional delay qt (or, as we shall see in subsequent sections, for analogous DV2s and DVIDEs with vanishing proportional delays) consists in general of *three phases*:

- *Phase I:* This 'initial phase' (*complete overlap*) is described by the values n satisfying

$$0 \leq n < \left\lceil \frac{q}{1-q} c_1 \right\rceil =: q^I$$

(assuming again that $c_1 > 0$). For these values of n we have $q(t_n + c_i h) > t_n$ for $i = 1, \ldots, m$. As already metioned in Lemma 5.2.1 this is always true when $n = 0$, for any $q \in (0, 1)$.

- *Phase II:* The 'transition phase' (*partial overlap*) is characterised by the values of n with

$$q^I \leq n < \left\lceil \frac{q}{1-q} c_m \right\rceil =: q^{II}.$$

In this phase there exists, for given n, a $\nu_n \in \{1, \ldots, m-1\}$ so that

$$q(t_n + c_i h) \leq t_n \quad \text{for} \quad i = 1, \ldots, \nu_n,$$

while

$$q(t_n + c_i h) > t_n \quad \text{for} \quad i = \nu_n + 1, \ldots, m.$$

Note that this phase may be *empty* (recall Example 5.2.1, and see Exercise 5.6.11).

- *Phase III:* The 'pure delay phase' (*no overlap*) consists of those values of n for which

$$q^{II} \leq n \leq N - 1.$$

Here,

$$q(t_n + c_i h) \leq t_n \quad \text{for all} \quad i = 1, \ldots, m,$$

that is, $q_{n,i} \leq n - 1$ for all $i = 1, \ldots, m$.

Example 5.2.1
For $q = 1/2$ and $n = 1$,

$$q(t_n + c_i h) = hq(1 + c_i) \leq h(1 + c_i)/2 \leq h, \;\; i = 1, \ldots, m.$$

Thus, we have either (assuming $c_1 > 0$)

$$qt_{n,i} > n \; (i = 1, \ldots, m) \implies n = 0,$$

or

$$qt_{n,i} \leq t_n \; (i = 1, \ldots, m) \quad \text{if} \quad 1 \leq n \leq N - 1.$$

Hence, $q^I = q^{II} = 1$: Phase II is empty for all values of m and any set $\{c_i\}$ with $c_1 > 0$.
Compare also Example 5.4.1 ($m = 1$) and the values for $q_{n,1}$ and $\gamma_{n,1}$ given in Tables 5.4–5.7.

Example 5.2.2
Let $q = 1/2$ and $u_h \in S_2^{(0)}(I_h)$ $(m = 2)$. Hence,

$$q_{n,i} = \lfloor (n + c_i)/2 \rfloor = \lfloor c_i/2 + n/2 \rfloor \ (i = 1, 2).$$

- Let the $\{c_i\}$ be the *Gauss points*: $c_1 = (3 - \sqrt{3})/6, \ c_2 = (3 + \sqrt{3})/6$. It is easily seen that

  $$q_{n,i} = \lfloor n/2 \rfloor \ (i = 1, 2),$$

  and

  $$\gamma_{n,1} = \begin{cases} (3 - \sqrt{3})/12 & \text{if } n \text{ is even} \\ (9 - \sqrt{3})/12 & \text{if } n \text{ is odd} \end{cases}$$

  $$\gamma_{n,2} = \begin{cases} (3 + \sqrt{3})/12 & \text{if } n \text{ is even} \\ (9 + \sqrt{3})/12 & \text{if } n \text{ is odd.} \end{cases}$$

  Here, we have $q^I = \lceil c_1 \rceil = 1, \quad q^{II} = \lceil c_2 \rceil = 1$, a particular case of the previous example.
- For the *Radau II points*, $c_1 = 1/3, \ c_2 = 1$, we find

  $$q_{n,1} = \lfloor 1/6 + n/2 \rfloor = \lfloor n/2 \rfloor, \quad q_{n,2} = \lfloor (n+1)/2 \rfloor,$$

  and

  $$\gamma_{n,1} = \begin{cases} 1/6 & \text{if } n \text{ is even} \\ 2/3 & \text{if } n \text{ is odd} \end{cases}$$

  $$\gamma_{n,2} = \begin{cases} 1/2 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

  The values of q^I and q^{II} are again given by $q^I = q^{II} = 1$.

Example 5.2.3
Assume that $q = 0.9$ and $u_h \in S_2^{(0)}(I_h)$ $(m = 2)$.

- For the *Gauss points*, Lemma 5.2.1 yields

  $$qt_{n,i} > t_n \ (i = 1, 2) \ \text{ if, and only if } \ n < \lceil 9c_1 \rceil = 2,$$

  and

  $$qt_{n,i} \le t_n \ (i = 1, 2) \ \text{ if, and only if } \ n \ge \lceil 9c_2 \rceil = 8.$$

  Thus, for $n = 2, \ldots, 7$ we have

  $$qt_{n,1} \in (t_{n-1}, t_n] \quad \text{and} \quad qt_{n,2} \in (t_n, t_{n+1}).$$

Table 5.2. $m = 2$

	Gauss points				Radau II points			
$q =$	1/2	2/3	0.9	0.99	1/2	2/3	0.9	0.99
$q^I =$	1	1	2	21	1	1	3	33
$q^{II} =$	1	2	8	79	1	2	9	99

It follows that $q^I = \lceil 9c_1 \rceil = 2, \quad q^{II} = \lceil 9c_2 \rceil = 8$.

- For the *Radau II points*,

$$qt_{n,i} > t_n \ (i = 1, 2) \ \text{ if, and only if } \ n < \lceil 9c_1 \rceil = 3,$$

and

$$qt_{n,i} \le t_n \ (i = 1, 2) \ \text{ if, and only if } \ n \ge \lceil 9c_2 \rceil = 9,$$

implying that for $n = 3, \ldots, 8$ we have

$$qt_{n,1} \in (t_{n-1}, t_n] \quad \text{and} \quad qt_{n,2} \in (t_n, t_{n+1}).$$

Hence, $q^I = \lceil 9c_1 \rceil = 3, q^{II} = \lceil 9c_2 \rceil = 9$.

For later reference we add a brief summary of values of q^I and q^{II} corresponding to $m = 2, \ m = 3$ and to two prominent sets of collocation parameters.

- *Gauss points:*

  $m = 2: \ \ c_1 = (3 - \sqrt{3})/6, \qquad c_2 = (3 + \sqrt{3})/6,$

  $m = 3: \ \ c_1 = (5 - \sqrt{15})/10, \quad c_2 = 1/2, \quad c_3 = (5 + \sqrt{15})/10;$

- *Radau II points:*

  $m = 2: \ \ c_1 = 1/3, \qquad c_2 = 1,$

  $m = 3: \ \ c_1 = (4 - \sqrt{6})/10, \quad c_2 = (4 + \sqrt{6})/10, \quad c_3 = 1$

We complement these illustrations by a more general result: it deals with a class of values of q for which we have $q^I = q^{II}$ and for which in Phase III ($q^{II} \le n \le N - 1$) all values $q(t_n + c_i h \ (i = 1, \ldots, m)$ lie in the same subinterval.

Lemma 5.2.2 *Let* $0 < c_1 < \ldots < c_m \le 1$, *and assume that* q *is of the form* $q = 1/r$, *with* $r \in \mathbb{N}, \ r \ge 2$. *Then:*

(a) $q^I = q^{II} = 1$: *Phase I consists of* $n = 0$ *only, Phase II is empty, and hence Phase III is described by* $1 \le n \le N - 1$.

(b) *In Phase III, the images* $\theta(t_{n,i}) := q(t_n + c_i h)$ $(i = 1, \ldots, m)$ *all lie in the same subinterval* $(t_{q_n}, t_{q_n+1}]$ *for some* $q_n < n - 1$.
(c) *For* $kr \le n < (k+1)r$ *we have* $q_n = k$.

Proof See Exercise 5.6.13.

Table 5.3. $m = 3$

	Gauss points				Radau II points			
$q =$	1/2	2/3	0.9	0.99	1/2	2/3	0.9	0.99
$q^I =$	1	1	2	12	1	1	2	16
$q^{II} =$	1	2	8	88	1	2	9	99

We now return to the collocation equation (5.2.2) (for uniform mesh I_h). Let the local representation of $u_h \in S_m^{(0)}(I_h)$ be given by

$$u_h(t_n + vh) = y_n + h \sum_{j=1}^{m} \beta_j(v) Y_{n,j}, \quad v \in [0, 1], \tag{5.2.5}$$

with $y_n := u_h(t_n)$ and $Y_{n,j} := u_h(t_{n,j})$. For a given collocation point $t_{n,i} \in \sigma_n$, equation (5.2.2) becomes

$$Y_{n,i} = f(t_{n,i}, y_n + h \sum_{j=1}^{m} a_{i,j} Y_{n,j}, y_{q_{n,i},i} + h \sum_{j=1}^{m} \beta_j(\gamma_{n,i}) Y_{q_{n,i},j}) \tag{5.2.6}$$

(because, by (5.2.4), $qt_{n,i} \in [t_{q_{n,i}}, t_{q_{n,i}+1}]$), with $a_{i,j} := \beta_j(c_i)$. Thus, by Lemma 5.2.1, as n increases from 0 to $N - 1$ the above systems of nonlinear algebraic equations for $\mathbf{Y}_n := (Y_{n,1}, \ldots, Y_{n,m})^T \in \mathbb{R}^m$ assume the following forms:

(I) *Initial phase (complete overlap)* $0 \le n < q^I$.
For these values of n we have, according to Lemma 5.2.1, $qt_{n,i} > t_n$ $(i = 1, \ldots, m)$; this is always true at least for $n = 0$. Since now $q_{n,i} = n$ $(i = 1, \ldots, m)$, the above system of algebraic equations (5.2.6) is

$$Y_{n,i} = f\left(t_{n,i}, y_n + h \sum_{j=1}^{m} a_{i,j} Y_{n,j}, y_n + h \sum_{j=1}^{m} \beta_j(\gamma_{n,i}) Y_{n,j}\right) \quad (i = 1, \ldots, m), \tag{5.2.7}$$

with $\gamma_{n,i} > 0$ for all i.

If the given DDE (5.2.1) is *linear*,

$$y'(t) = a(t)y(t) + b(t)y(qt) + g(t), \quad t \in I, \tag{5.2.8}$$

with $a, b \in C(I)$ (and, for simplicity, $g(t) \equiv 0$), then the linear algebraic system corresponding to (5.2.7) has the form

$$Y_{n,i} = a(t_{n,i})[y_n + h \sum_{j=1}^{m} a_{i,j} Y_{n,j}] + b(t_{n,i})$$
$$[y_n + h \sum_{j=1}^{m} \beta_j(\gamma_{n,i}) Y_{n,j}] \ (i = 1, \dots, m).$$

Recalling the $m \times m$ matrices introduced in Section 1.1.1,

$$A := (a_{i,j}), \quad A_n := \mathrm{diag}(a(t_{n,i}))A,$$

and

$$A_n^I(q) := \mathrm{diag}(b(t_{n,i}))(\beta_j(\gamma_{n,i})),$$

and defining

$$\mathbf{r}_n := \mathrm{diag}(a(t_{n,i}))\mathbf{e}, \quad \mathbf{r}_n^I(q) := \mathrm{diag}(b(t_{n,i}))\mathbf{e},$$

with $\mathbf{e} := (1, \dots, 1)^T \in \mathbb{R}^m$, we obtain

$$[\mathcal{I}_m - h(A_n + A_n^I(q))]\mathbf{Y}_n = (\mathbf{r}_n + \mathbf{r}_n^I(q))y_n. \tag{5.2.9}$$

(II) *Transition phase (partial overlap)* $\quad q^I \le n < q^{II}$.
If this set of values n is not empty (recall the remark following Lemma 5.2.1), let $\nu_n \in \mathbb{N}$ with $1 \le \nu_n < m$ be such that

$$q_{n,i} = n - 1 \ (i = 1, \dots, \nu_n) \quad \text{and} \quad q_{n,i} = n, \quad \gamma_{n,i} > 0 \ (i = \nu_n + 1, \dots, m).$$

Thus, the collocation equation (5.2.6) separates into

$$Y_{n,i} = f\left(t_{n,i}, y_n + h\sum_{j=1}^{m} a_{i,j} Y_{n,j}, y_{n-1} + h\sum_{j=1}^{m} \beta_j(\gamma_{n,i}) Y_{n-1,j})\right)$$
$$(i = 1, \dots, \nu_n), \tag{5.2.10}$$

and

$$Y_{n,i} = f\left(t_{n,i}, y_n + h \sum_{j=1}^{m} a_{i,j} Y_{n,j}, y_n + h \sum_{j=1}^{m} \beta_j(\gamma_{n,i}) Y_{n,j}\right),$$
$$(i = \nu_n + 1, \dots, m). \tag{5.2.11}$$

In the linear case (5.2.8) we employ the notation

$$A_n^{II}(q) := \mathrm{diag}(\underbrace{0, \dots, 0}_{\nu_n}, 1, \dots, 1) A_n^I(q),$$

$$\mathbf{r}_n^{II}(q) := \mathrm{diag}(b(t_{n,1}), \dots, b(t_{n,\nu_n}), 0, \dots, 0)\mathbf{e}, \qquad \hat{\mathbf{r}}_n^{II}(q) := A_n^{II}(q)\mathbf{e},$$

and

$$S_n^{II}(q) := \mathrm{diag}(\underbrace{1,\ldots,1}_{\nu_n},0,\ldots,0)A_n^{I}(q),$$

to express the linear algebraic system corresponding to (5.2.10),(5.2.11) as

$$[\mathcal{I}_m - h(A_n + A_n^{II}(q))]\mathbf{Y}_n = hS_n^{II}(q)\mathbf{Y}_{n-1} + (\mathbf{r}_n + \mathbf{r}_n^{II}(q))y_n + \hat{\mathbf{r}}_n^{II}(q)y_{n-1}. \tag{5.2.12}$$

(III) *Pure delay phase (no overlap)* $\quad q^{II} \le n \le N-1$.
Here, $qt_{n,i} \le t_n$ $(i = 1,\ldots,m)$, and $q_{n,i} < n$ for all i. Depending on the value of q, the indices $q_{n,i}$ and $q_{n,j}$ $(i \ne j)$ are either equal or differ by one. Thus, for such an n there is an integer $\nu_n \in \{1,\ldots,m\}$ so that

$$q_{n,i} = q_n \ (i = 1,\ldots,\nu_n) \ \text{ and } \ q_{n,i} = q_n + 1, \quad \gamma_{n,i} > 0 \ (i = \nu_n + 1,\ldots,m),$$

with $q_n + 1 < n$. The algebraic system (5.2.6) now decomposes into

$$Y_{n,i} = f(t_{n,i}, y_n + h\sum_{j=1}^{m} a_{i,j}Y_{n,j}, y_{q_n} + h\sum_{j=1}^{m}\beta_j(\gamma_{n,i})Y_{q_n,j}) \ (i = 1,\ldots,\nu_n), \tag{5.2.13}$$

and

$$\begin{aligned} Y_{n,i} = f(t_{n,i}, y_n + h\sum_{j=1}^{m} a_{i,j}y_{n,j}, y_{q_n+1} \\ + h\sum_{j=1}^{m}\beta_j(\gamma_{n,i})Y_{q_n+1,j}) \ (i = \nu_n + 1,\ldots,m). \end{aligned} \tag{5.2.14}$$

Setting $\mathbf{r}_n^{III} := \mathrm{diag}(a(t_{n,i}))\mathbf{e}$,

$$\hat{\mathbf{r}}_n^{III}(q) := \mathrm{diag}(b(t_{n,1}),\ldots,b(t_{n,\nu_n},0,\ldots,0)\mathbf{e},$$

$$\mathbf{r}_n^{III}(q) := \mathrm{diag}(\underbrace{0,\ldots,0}_{\nu_n}, b(t_{n,\nu_n+1}),\ldots,b(t_{n,m}))\mathbf{e},$$

$$\hat{S}_n^{III}(q) := \mathrm{diag}(\underbrace{1,\ldots,1}_{\nu_n},0,\ldots,0)A_n^{I}(q),$$

and

$$S_n^{III}(q) := \mathrm{diag}(\underbrace{0,\ldots,0}_{\nu_n},1,\ldots,1)A_n^{I}(q),$$

we can write the linear algebraic system for (5.2.8) corresponding to (5.2.13),(5.2.14) as

$$\begin{aligned}[\mathcal{I}_m - hA_n]\mathbf{Y}_n = h[S_n^{III}(q)\mathbf{Y}_{q_n+1} + \hat{S}_n^{III}(q)\mathbf{Y}_{q_n}] \\ + \mathbf{r}_n y_n + \mathbf{r}_n^{III}(q)y_{q_n+1} + \hat{\mathbf{r}}_n^{III}(q)y_{q_n}. \end{aligned} \tag{5.2.15}$$

Remark We will employ the above notation also in subsequent sections: if a matrix or vector carries the argument q, it is to indicate a contribution from a delay term; a hat over a matrix or vector suggests that this quantity originates from an index $i \in \{1, \dots, \nu_n\}$ with $\nu_n < m$.

Summary The collocation solution $u_h \in S_m^{(0)}(I_h)$ for uniform mesh I_h and collocation parameters $\{0 < c_1 < \dots < c_m \leq 1\}$ has, on the subinterval $\sigma_n := [t_n, t_{n+1}]$, the local representation (5.2.5) in which $\mathbf{Y}_n = (Y_{n,1}, \dots, Y_{n,m})^T$ is determined by a system of algebraic equations in $\mathbb{R}^m$, as follows:

Phase I (complete overlap): $0 \leq n < \lceil qc_1/(1-q) \rceil =: q^I$:

(5.2.7) (nonlinear DDE (5.2.1)), or (5.2.9) (linear DDE (5.2.8));

Phase II (partial overlap): $q^I \leq n < \lceil qc_m/(1-q) \rceil =: q^{II}$:

(5.2.10),(5.2.11) (nonlinear DDE (5.2.1)), or (5.2.12) (linear DE (5.2.8));

Phase III (no overlap): $q^{II} \leq n \leq N-1$:

(5.2.13), (5.2.14) (nonlinear DDE (5.2.1)), or (5.2.15) (linear DDE (5.2.8)).

Although we have based the above discussion on the assumption that $c_1 > 0$, these arguments are – as we have already briefly indicated – readily modified to include the case where $c_1 = 0$ (occurring, for example, when we choose $c_1 = 0$ and $c_m = 1$, thus generating a *continuous* collocation solution u_h). We leave this as an exercise (Exercise 5.6.15).

The existence and uniqueness of the collocation solution $u_h \in S_m^{(0)}(I_h)$, i.e. the unique solvability for sufficiently small $h > 0$ of the nonlinear or linear algebraic systems mentioned in the above summary, follows from arguments essentially identical with those in Section 1.1 (Theorem 1.1.2) and Section 2.1 (Theorem 2.2.1). Note in particular that the matrix of the linear system (5.2.15) (pure delay phase) coincides of course with the ones for the linear ODEs in (1.1.16).

Theorem 5.2.3 *Assume that the given functions a and b in (5.2.8) are continuous on I. Then there exists an $\bar{h} > 0$ so that the linear algebraic systems (5.2.9), (5.2.12) and (5.2.15) are uniquely solvable whenever the mesh diameter of the underlying uniform mesh I_h satisfies $h \in (0, \bar{h})$. Thus, for such meshes the collocation equation (5.2.2), with $f(t, y, z) := a(t)y + b(t)z$, defines a unique collocation solution $u_h \in S_m^{(0)}(I_h)$ whose local representation on $\bar{\sigma}_n$ is given by (5.2.5).*

We shall see in Section 5.2.3 that the analysis of the attainable order of local superconvergence (on I_h) is very complex (and is not yet fully understood) if the mesh I_h is uniform. This is due to the fact that such meshes are not θ-invariant and, as we have seen above, lead to initial 'overlap'. Moreover, collocation on uniform meshes will lead to severe storage problems if it is used for *long-time integration* (compare the papers by Iserles (1997b) and by Liu (1997) for illuminating comments and illustrations in the case of DDEs with proportional delays). Hence, it seems natural to ask if (non-uniform) meshes can be constructed for which the 'non-vanishing delay techniques' of Chapter 4 can be employed.

Two such approaches have recently been analysed: Brunner, Hu and Lin (2001) consider collocation solutions $u_h \in S_m^{(0)}(I_h)$ where I_h is a *geometric mesh* defined by

$$t_n = t_n^{(N)} := q^{1/\kappa} T \quad (n = 1, \ldots, N), \tag{5.2.16}$$

where $\kappa = \kappa(q; N)$ depends not only on $q \in (0, 1)$ but also on the number N of subintervals σ_n corresponding to the mesh I_h.

Collocation (or, more generally, continuous implicit Runge–Kutta methods) on *quasi-geometric* meshes requires the computation of a sufficiently accurate approximation y_0 to y on some (small) initial interval $[0, t_0]$, with $t_0 = q^M T$ (Bellen (2002)). Once a feasible (small) $t_0 > 0$ has been chosen, we define the points $\{\xi_\mu\}$ by setting

$$\xi_\mu := q^{M-\mu} T \quad (\mu = 0, 1, \ldots, M)$$

(these points may be viewed as the primary discontinuiy points generated by $\xi_0 := t_0 > 0$), each of the subintervals $[\xi_\mu, \xi_{\mu+1}]$ is endowed with a (usually uniform) mesh defined by

$$I_h^{(\mu)} := \{t_n^{(\mu)} := \xi_\mu + (n/N)[\xi_{\mu+1} - \xi_\mu] : \; n = 0, 1, \ldots, N-1\}.$$

We shall describe the details in Sections 5.5.3 and 5.5.4 when studying collocation solutions for DVIDEs with proportional delays. Consult also Example 4.3.1.

5.2.2 Global convergence results: uniform I_h

We first study the convergence of the collocation solution $u_h \in S_m^{(0)}(I_h)$ on uniform meshes I_h. In order to exhibit the basic principles underlying the global convergence and error analysis more clearly and without additional technicalities we will deal first with the *linear* proportional delay equation (5.2.8).

Theorem 5.2.4 *Consider the linear DDE (5.2.8),*

$$y'(t) = a(t)y(t) + b(t)y(qt) + g(t), \quad t \in I = [0, T] \quad (0 < q < 1),$$

with initial condition $y(0) = y_0$, and assume that

(a) $a, b, g \in C^m(I)$;
(b) $u_h \in S_m^{(0)}(I_h)$ *is the (unique) collocation solution to (5.2.8) corresponding to uniform* I_h *and collocation parameters* $\{c_i\}$ *with* $0 \le c_1 < \ldots < c_m \le 1$.

Then for all uniform meshes I_h *with diameter* $h \in (0, \bar{h})$ *(cf. Theorem 5.2.3) we have*

$$||y^{(\nu)} - u_h^{(\nu)}||_\infty := \sup_{t \in I} |y^{(\nu)}(t) - u_h^{(\nu)}(t)| \le C_\nu ||y^{(m+1)}||_\infty h^m \quad (\nu = 0, 1), \tag{5.2.17}$$

and this optimal order estimate holds for any set $\{c_i\}$ *defining the set of collocation points* X_h. *The constants* C_ν *depend on the* $\{c_i\}$ *but not on* h.

Proof Theorem 5.1.3 shows that assumption (a) implies $y \in C^{m+1}(I)$. Thus, the local representations (1.1.22), (1.1.24) of the collocation error $e_h := y - u_h$ carry over to the present situation, except that now collocation is based on uniform meshes I_h. To be more precise, we have

$$e_h(t_n + vh) = e_h(t_n) + h \sum_{j=1}^{m} \beta_j(v) \mathcal{E}_{n,j} + h^{m+1} R_{m+1,n}(v), \quad v \in [0, 1], \tag{5.2.18}$$

with $\mathcal{E}_{n,j} := e_h'(t_{n,j})$ and with $R_{m+1,n}(v)$ denoting the Peano remainder term (see (1.1.22) and (1.1.24) with $h_n = h$). The continuity constraints of e_h at the interior mesh points furnish the recurrence relation (cf. (1.1.27))

$$e_h(t_n) = h \sum_{\ell=0}^{n-1} \sum_{j=1}^{m} b_j \mathcal{E}_{\ell,j} + h^{m+1} \sum_{\ell=0}^{n-1} R_{m+1,\ell}(1), \quad n = 1, \ldots, N-1, \tag{5.2.19}$$

with $e_h(0) = 0$. By definition of the collocation solution for (5.2.8), e_h satisfies

$$e_h'(t_{n,i}) = a(t_{n,i}) e_h(t_{n,i}) + b(t_{n,i}) e_h(q t_{n,i}), \quad i = 1, \ldots, m \ (0 \le n \le N-1),$$

and so, using the recursion (5.2.19) for $e_h(t_n)$ we obtain

$$\begin{aligned} \mathcal{E}_{n,i} = a(t_{n,i}) &\left(h \sum_{\ell=0}^{n-1} \sum_{j=1}^{m} b_j \mathcal{E}_{\ell,j} + h \sum_{j=1}^{m} a_{i,j} \mathcal{E}_{n,j} \right) + h^m \rho_{n,i} \\ &+ b(t_{n,i}) \left(h \sum_{\ell=0}^{q_{n,i}-1} \sum_{j=1}^{m} b_j \mathcal{E}_{\ell,j} + h \sum_{j=1}^{m} \beta_j(\gamma_{n,i}) \mathcal{E}_{q_{n,i},j} \right) + h^m \rho_{n,i}(q) \end{aligned} \tag{5.2.20}$$

where (cf. (1.1.30))

$$\rho_{n,i} := a(t_{n,i}) \left(h \sum_{\ell=0}^{n-1} R_{m+1,\ell}(1) + h R_{m+1,n}(c_i) \right) - R_{m+1,n}^{(1)}(c_i),$$

and

$$\rho_{n,i}^{I}(q) := b(t_{n,i})\left(h\sum_{\ell=0}^{q_{n,i}-1} R_{m+1,\ell}(1) + hR_{m+1,n}(\gamma_{n,i})\right). \tag{5.2.21}$$

We set $\boldsymbol{\rho}_n := (\rho_{n,i}, \ldots, \rho_{n,m})^T$ and $\boldsymbol{\rho}_n^I(q) := (\rho_{n,1}^I(q), \ldots, \rho_{n,m}^I(q))^T$. A glimpse at Phases I, II and III in Section 5.2.1 (cf. (5.2.9), (5.2.12), and (5.2.15)) will help in making the following analysis obvious:

(I): $0 \le n < q^I$: Since $q_{n,i} = n$ for all values of i, the vector $\boldsymbol{\mathcal{E}}_n := (\mathcal{E}_{n,1}, \ldots, \mathcal{E}_{n,m})^T$ is defined by the linear algebraic system

$$\begin{aligned}
&[\mathcal{I}_m - h(A_n + A_n^I(q))]\boldsymbol{\mathcal{E}}_n \\
&\quad = h\cdot \mathrm{diag}(a(t_{n,i}))\mathbf{e}\sum_{\ell=0}^{n-1}\mathbf{b}^T\boldsymbol{\mathcal{E}}_\ell + h\cdot \mathrm{diag}(b(t_{n,i}))\mathbf{e}\sum_{\ell=0}^{n-1}\mathbf{b}^T\boldsymbol{\mathcal{E}}_\ell \\
&\qquad + h^m[\boldsymbol{\rho}_n + \boldsymbol{\rho}_n^I(q)].
\end{aligned} \tag{5.2.22}$$

Here, the matrix $A_n^I(q) \in L(\mathbb{R}^m)$ coincides with the one in (5.2.9), and the components of $\boldsymbol{\rho}_n^I(q)$ are given by (5.2.24) with $q_{n,i} = n$ $(i = 1, \ldots, m)$.

(II): $q^I \le n < q^{II}$: As before, let ν_n be such that $q_{n,i} = n - 1$ for $i = 1, \ldots, \nu_n$ and $q_{n,i} = n$ when $i = \nu_n + 1, \ldots, m$ (with $\gamma_{n,i} > 0$). Setting

$$\rho_{n,i}^{II}(q) := b(t_{n,i})\left(h\sum_{\ell=0}^{n-2} R_{m+1,\ell}(1) + hR_{m+1,n-1}(\gamma_{n,i})\right), \quad (i = 1, \ldots, \nu_n),$$

and

$$\rho_{n,i}^{II}(q) := b(t_{n,i})\left(h\sum_{\ell=0}^{n-1} R_{m+1,\ell}(1) + hR_{m+1,n}(\gamma_{n,i})\right), \quad (i = \nu_n + 1, \ldots, m),$$

the equations defining the components of $\mathcal{E}_n$ read

$$\begin{aligned}
\mathcal{E}_{n,i} &= a(t_{n,i})\left(h\sum_{\ell=0}^{n-1}\sum_{j=1}^{m} b_j\mathcal{E}_{\ell,j} + h\sum_{j=1}^{m} a_{i,j}\mathcal{E}_{n,j}\right) + h^m\rho_{n,i} \\
&\quad + b(t_{n,i})\left(h\sum_{\ell=0}^{n-2} b_j\mathcal{E}_{\ell,j} + h\sum_{j=1}^{m}\beta_j(\gamma_{n,i})\mathcal{E}_{n-1,j}\right) + h^m\rho_{n,i}^{II}(q)
\end{aligned}$$

when $i = 1, \ldots, \nu_n$, and

$$\begin{aligned}
\mathcal{E}_{n,i} &= a(t_{n,i})\left(h\sum_{\ell=0}^{n-1}\sum_{j=1}^{m} b_j\mathcal{E}_{\ell,j} + h\sum_{j=1}^{m} a_{i,j}\mathcal{E}_{n,j}\right) + h^m\rho_{n,i} \\
&\quad + b(t_{n,i})\left(h\sum_{\ell=0}^{n-1}\sum_{j=1}^{m} b_j\mathcal{E}_{\ell,j} + h\sum_{j=1}^{m}\beta_j(\gamma_{n,i})\mathcal{E}_{n,j}\right) + h^m\rho_{n,i}^{II}(q)
\end{aligned}$$

for $i = \nu_n + 1, \ldots, m$. Thus, the linear algebraic system defining $\mathcal{E}_n$ is

$$[\mathcal{I}_m - h(A_n + A_n^{II}(q))]\boldsymbol{\mathcal{E}}_n = h \cdot \mathrm{diag}(a(t_{n,i}))\mathbf{e}\sum_{\ell=0}^{n-1}\mathbf{b}^T\boldsymbol{\mathcal{E}}_\ell + h \cdot \mathrm{diag}(b(t_{n,i}))\mathbf{e}\sum_{\ell=0}^{n-2}\mathbf{b}^T\boldsymbol{\mathcal{E}}_\ell + hS_n^{II}(q)\boldsymbol{\mathcal{E}}_{n-1} + h^m[\boldsymbol{\rho}_n + \boldsymbol{\rho}_n^{II}(q)], \tag{5.2.23}$$

where the matrices $A_n^{II}(q),\ S_n^{II}(q) \in L(\mathbb{R}^m)$ are as in (5.2.12), and the vector $\boldsymbol{\rho}_n^{II}(q)$ is described by the components

$$\rho_{n,i}^{II}(q) := b(t_{n,i}) \cdot \begin{cases} h\sum_{\ell=0}^{n-2} R_{m+1,\ell}(1) + hR_{m+1,n-1}(\gamma_{n,i}) & \text{for } i = 1, \ldots, \nu_n \\ h\sum_{\ell=0}^{n-1} R_{m+1,\ell}(1) + hR_{m+1,n}(\gamma_{n,i}) & \text{for } i = \nu_n + 1, \ldots, m. \end{cases}$$

(III) $q^{II} \le n \le N-1$ For given n let ν_n $(1 \le \nu_n \le m)$ be the integer for which $q_{n,i} = q_n$ $(i = 1, \ldots, \nu_n)$, and $q_{n,i} = q_n + 1,\ \gamma_{n,i} > 0$ $(i = \nu_n + 1, \ldots, m)$, with $q_n < n-1$. The algebraic equations determining $\mathcal{E}_n$ are then

$$\mathcal{E}_{n,i} = a(t_{n,i})\left(h\sum_{\ell=0}^{n-1}\sum_{j=1}^{m} b_j\mathcal{E}_{\ell,j} + h\sum_{j=1}^{m} a_{i,j}\mathcal{E}_{n,j}\right) + h^m\rho_{n,i} + b(t_{n,i})\left(h\sum_{\ell=0}^{q_n-1}\sum_{j=1}^{m} b_j\mathcal{E}_{\ell,j} + h\sum_{j=1}^{m}\beta_j(\gamma_{n,i})\mathcal{E}_{q_n,j}\right) + h^m\rho_{n,i}^{III}(q)$$

when $i \le \nu_n$, and

$$\mathcal{E}_{n,i} = a(t_{n,i})\left(h\sum_{\ell=0}^{n-1}\sum_{j=1}^{m} b_j\mathcal{E}_{\ell,j} + h\sum_{j=1}^{m} a_{i,j}\mathcal{E}_{n,j}\right) + h^m\rho_{n,i} + b(t_{n,i})\left(h\sum_{\ell=0}^{q_n}\sum_{j=1}^{m} b_j\mathcal{E}_{\ell,j} + h\sum_{j=1}^{m}\beta_j(\gamma_{n,i})\mathcal{E}_{q_n+1,j}\right) + h^m\rho_{n,i}^{III}(q)$$

when $\nu_n + 1 \le i \le m$. Here,

$$\rho_{n,i}^{III}(q) := b(t_{n,i}) \cdot \begin{cases} h\sum_{\ell=0}^{q_n-1} R_{m+1,\ell}(1) + hR_{m+1,q_n}(\gamma_{n,i}), & 1 \le n \le \nu_n \\ h\sum_{\ell=0}^{q_n} R_{m+1,\ell}(1) + hR_{m+1,q_n+1}(\gamma_{n,i}), & \nu_n < n \le m. \end{cases}$$

The corresponding linear algebraic system for $\mathcal{E}_n$ can be written concisely as

$$[\mathcal{I}_m - hA_n]\boldsymbol{\mathcal{E}}_n = h \cdot \mathrm{diag}(a(t_{n,i}))\boldsymbol{e} \sum_{\ell=0}^{n-1} \mathbf{b}^T \boldsymbol{\mathcal{E}}_\ell + h \cdot \mathrm{diag}(b(t_{n,i}))\mathbf{e} \sum_{\ell=0}^{q_n-1} \mathbf{b}^T \boldsymbol{\mathcal{E}}_\ell + h[S_n^{III}(q)\boldsymbol{\mathcal{E}}_{q_n+1} + \hat{S}_n^{III}(q)\boldsymbol{\mathcal{E}}_{q_n}] + h^m[\boldsymbol{\rho}_n + \boldsymbol{\rho}_n^{III}(q)]. \tag{5.2.24}$$

By Theorem 5.2.3, each of the linear algebraic systems (5.2.22), (5.2.23), (5.5.24) possesses a unique solution $\boldsymbol{\mathcal{E}}_n$ whenever $h \in (0, \bar{h})$. Thus, the arguments we used to pass from the linear system (1.1.31) to a generalised discrete Gronwall inequality for $||\boldsymbol{\mathcal{E}}_n||_1$ are readily adapted to the present situation: denoting by D_0^I, D_0^{II}, D_0^{III} ($= D_0$ for (1.1.31)) (uniform) upper bounds for the ℓ_1-norms of the inverses of the matrices $\mathcal{I}_m - h\mathcal{A}_n \in L(\mathbb{R}^m)$ on the left-hand side of these three linear systems, with

$$\mathcal{A}_n := \begin{cases} A_n + A_n^I(q) & \text{if} \quad 0 \le n < q^I \\ A_n + A_n^{II}(q) & \text{if} \quad q^I \le n < q^{II} \\ A_n & \text{if} \quad q^{II} \le n \le N-1\,, \end{cases}$$

and recalling that the integers q^I and q^{II} characterising Phase I and Phase II do not depend on h (or N), we obtain the discrete Gronwall inequalities

$$||\boldsymbol{\mathcal{E}}_n||_1 \le \begin{cases} \gamma_0^I h \sum_{\ell=0}^{n-1} ||\boldsymbol{\mathcal{E}}_\ell||_1 + \gamma_1^I M_{m+1} h^m & \text{if} \quad 0 \le n < q^I \\ \gamma_0^{II} h \sum_{\ell=0}^{n-1} ||\boldsymbol{\mathcal{E}}_\ell||_1 + \gamma_1^{II} M_{m+1} h^m & \text{if} \quad q^I \le n < q^{II} \\ \gamma_0^{III} h \sum_{\ell=0}^{n-1} ||\boldsymbol{\mathcal{E}}_\ell||_1 + \gamma_1^{III} M_{m+1} h^m & \text{if} \quad q^{II} \le n \le N-1. \end{cases} \tag{5.2.25}$$

Thus, the standard argument of Section 1.1.1 and the local representations for e_h and e_h' yield, respectively, $||\boldsymbol{\mathcal{E}}_n||_1 \le BM_{m+1}h^m$ ($n = 0, 1, \ldots, N-1$), and hence the asserted estimates (5.2.17) follow.

Remark If the regularity assumption $y \in C^{m+1}(I)$ replaced by $y \in C^{d+1}(I)$ with $1 \le d < m$, then a trivial modification (employing the Peano Kernel Theorem with remainder terms $R_{d+1,n}(v)$) leads to

Theorem 5.2.5 *Suppose that assumption (a) of Theorem 5.2.4 is replaced by* $a, b, g \in C^d(I)$, $1 \le d \le m$. *Then the collocation error* $e_h := y - u_h$

corresponding to $u_h \in S_m(I_h)$ *is governed by*

$$||e_h^{(\nu)}||_\infty \le C_\nu ||y^{(d+1)}||_\infty h^d \quad (\nu = 0, 1), \tag{5.2.26}$$

for all uniform I_h *with* $h \in (0, \bar{h})$.

We now enter what may be called 'new territory' – as predicted by Andreoli (1914) (cf. Section 5.1.1): we shall see that while the *global* superconvergence results for DDEs and VIEs with non-vanishing delays (e.g. Theorems 4.5.3 and 4.3.3) remain valid for pantograph-type functional equations (but with much less obvious proofs!), this is no longer true for *local* superconvergence statements. The following result can be proved by adapting the analysis employed in Brunner and Hu (2003).

Theorem 5.2.6 *Assume that the given functions* $a,\ b,\ g$ *in the linear pantograph equation*

$$y'(t) = a(t)y(t) + b(t)y(qt) + g(t), \quad t \in I,$$

are in $C^{m+1}(I)$. *If the collocation parameters satisfy the orthogonality condition*

$$J_0 := \int_0^1 \prod_{i=1}^m (s - c_i)ds = 0,$$

then the collocation solution $u_h \in S_m^{(0)}(I_h)$ *on uniform* I_h *can be estimated by*

$$||y - u_h||_\infty \le Ch^{m+1}. \tag{5.2.27}$$

This holds for any $q \in (0, 1)$, *and the exponent* $m + 1$ *is best possible.*

We shall obtain this result as a special case of the more general convergence theorem for delay VIDEs with vanishing proportional delays (Theorem 5.5.4).

5.2.3 Attainable order at $t = t_1 = h$

To obtain some first insight into the optimal local superconvergence properties of the collocation solution at the points of a *uniform mesh* I_h, suppose that the solution of the initial-value problem

$$y'(t) = by(qt), \quad t \in I := [0, T], \quad y(0) = 1, \quad (0 < q < 1), \tag{5.2.28}$$

is approximated by $u_h \in S_m^{(0)}(I_h)$. What can be said about the order of $y(t) - u_h(t)$ at $t = t_1 = h$? Since $u_h \in \pi_m$ in $\bar{\sigma}_0 := [0, h]$, the collocation equation $u_h'(t) = bu_h(qt),\ t \in X_h \cap \bar{\sigma}_0$, may be written in the form

$$u_h'(t) - bu_h(qt) = K \cdot M_m((t - t_n)/h), \quad t \in \bar{\sigma}_0, \quad v_h(0) = 1. \tag{5.2.29}$$

The polynomial

$$M_m(s) := \frac{1}{m!}\prod_{i=1}^{m}(s - c_i) \quad (s = (t - t_n)/h)$$

(which for brevity we will often denote just by $M(s)$) is the collocation polynomial introduced in Lemma 1.1.11, and K is a constant to be determined. (Compare also Nørsett (1974, 1984) and the monograph by Iserles and Nørsett (1991, pp. 29–32).) Note that $KM((t - t_n)/h)$ is of course closely related to the *defect* $\delta_h(t)$ induced by the collocation solution u_h: it is defined by

$$\delta_h(t) := -u_h'(t) + bu_h(qt) = -KM((t - t_n)/h), \quad t \in \bar{\sigma}_0.$$

The following result may also be found in Brunner (1997a). Observe that for $q = 1$ we obtain the result of Lemma 1.1.12.

Theorem 5.2.7 *The value of the collocation solution $u_h \in S_m^{(0)}(I_h)$ for (5.2.28) at $t = t_1 = h$ is given by*

$$u_h(h) = v_h(h) = P_{m,m}(z; q)/Q_{m,m}(z; q),$$

with

$$P_{m,m}(z; q) := \sum_{j=0}^{m} q^{j(2m-j+1)/2} M^{(m-j)}(1/q^{m-j+1}) z^j,$$

$$Q_{m,m}(z; q) := \sum_{j=0}^{m} q^{j(2m-j+1)/2} M^{(m-j)}(0) z^j.$$

Remark Observe that the term in $P_{m,m}(z; q)$ corresponding to $j = m$ contains the factor $M(1/q)$ which, for $q \in (0, 1)$, does not vanish. Hence, the numerator polynomial in the expression for $u_h(h)$ is of exact degree m for any set $\{c_i\}$ with $c_i \in [0, 1]$.

Proof The proof is a straightforward adaptation of ideas due to Nørsett (see, e.g. Nørsett (1975) or Iserles and Nørsett (1991, pp. 29–32)). It follows from equation (5.2.29) that

$$u_h''(t) - bqu_h'(t) = KM'((t - t_0)/h), \quad t \in \bar{\sigma}_0,$$

and hence,

$$u_h''(t) - bq\{bu_h(q^2 t) + KM(q(t - t_0)/h)\} - KM'((t - t_0)/h) = 0, \quad t \in \bar{\sigma}_0,$$

where we have set, without loss of generality, $h = 1$. This leads to

$$u_h^{(m+1)}(t) - z^{m+1} q^{m(m+1)/2} u_h(q^{m+1} t) - K \sum_{j=0}^{m} z^{m-j} q^{(m-j)(m+j+1)/2} M^{(j)}(q^{m-j} t)$$
$$= 0.$$

Here, $z := bh$, and we have $u_h^{(m+1)}(t) \equiv 0$ on $\bar{\sigma}_0$. Setting $t = 0$ and $t = 1/q^{m+1}$, respectively, and observing that $u_h(0) = 0$, we readily find

$$u_h(h) = \frac{\sum_{j=0}^{m} q^{(m-j)(m+j+1)/2} M^{(j)}(1/q^{j+1}) z^{m-j}}{\sum_{j=0}^{m} q^{(m-j)(m+j+1)/2} M^{(j)}(0) z^{m-j}}.$$

An obvious change in the order of summation leads to the desired result.

Example 5.2.1 $m = 1$
Here, the collocation polynomial is $M(s) = s - c_1$, and the expressions for $u_h(h)$ and the [1/1]-Padé approximant are given respectively by

$$u_h(h) = \frac{1 + (1 - qc_1)z}{1 - qc_1 z} \quad \text{and} \quad R_{1/1}(z; q) = \frac{1 + (1 - q/2)z}{1 - (q/2)z} \quad (z := bh).$$

Note that these expressions coincide for any $q \in (0, 1)$ if, and only if, $c_1 = 1/2$ (collocation at the Gauss points).

Example 5.2.2 $m = 2$
The collocation polynomial has the form $M(s) = (s - c_1)(s - c_2)/2$, and we readily obtain

$$u_h(h) = \frac{1 + [1 - (1/2)q^2(c_1 + c_2)]z + (q/2)[1 - q(c_1 + c_2) + q^2 c_1 c_2]z^2}{1 - (1/2)q^2(c_1 + c_2)z + (1/2)q^3 c_1 c_2 z^2}$$

(see also [195]), while the [2, 2]-Padé approximant is found to be

$$R_{2/2}(z; q) = \frac{1 + [(6 - 4q - 2q^2 + q^4)/(2(3 - 2q))]z + [q(18 - 24q + 10q^3 - 3q^4)/(12(3 - 2q))]z^2}{1 - [q^2(2 - q^2)/(2(3 - 2q))]z + [q^4(4 - 3q)/(12(3 - 2q))]z^2}.$$

For the *Gauss points*, $c_1 = (3 - \sqrt{3})/6,\ c_2 = (3 + \sqrt{3})/6$, and $q = 1/2$ the above expressions become, respectively,

$$v_h(h) = \frac{1 + (7/8)z + (13/96)z^2}{1 - (1/8)z + (1/96)z^2} \quad \text{and}$$

$$R_{2/2}(z; 1/2) = \frac{1 + (57/64)z + (113/768)z^2}{1 - (7/64)z + (5/768)z^2}.$$

It follows that they are identical if, and only if, $q = 1$.

For the Radau II points $c_1 = 1/3,\ c_2 = 1$ we obtain

$$u_h(h) = \frac{1 + [1 - (2/3)q^2]z + (q/2)[1 - (4/3)q + (1/3)q^2]z^2}{1 - (2q^2/3)z + (q^3/6)z^2}.$$

Note that this rational function is not the [1/2]-Padé approximant to $y(h)$.

Do there exist (distinct) collocation parameters $\hat{c}_i = \hat{c}_i(q) \in [0, 1]$ so that $v_h(h) = R_{m/m}(z; q)$ for all $q \in (0, 1)$? This question was answered by Brunner (1997a) for $m = 2$ and by Takama, Muroya and Ishiwata (2000) for arbitrary $m \geq 3$. See also Ishiwata (2000).

5.2.4 Local superconvergence on uniform meshes

The optimal order estimates for the collocation solution at the first mesh point $t = t_1 = h$ of the previous section might suggest that, for collocation at the Gauss points, the optimal order of convergence of $uh(t)$ at $t \in I_h$ is again $p^* = 2m$, or at least $p^* = 2m - 1$. Numerical examples suggest that we have $p^* = 2m$ when $m = 2$ (see Brunner (1997a); however, this is no longer true for $m > 2$. Instead the following result (whose proof is still elusive) appears to hold when $m \geq 3$. It is a special case of Conjecture 5.5.5.

Conjecture 5.2.8 *Assume that the assumptions on a, b, g of Theorem 5.2.6 hold, but with $C^{m+1}(I)$ replaced by $C^d(I)$ $(d \geq m+2)$. If the collocation solution $u_h \in S_m^{(0)}(I_h)$ corresponds to the collocation parameters given by the Gauss points $\{c_i\}$ and if $m \geq 2$, then*

$$\max_{t \in I_h} |y(t) - u_h(t)| \leq C h^{m+2},$$

where $m+2$ cannot be replaced by $m+3$. This estimate is true for all $q \in (0, 1)$.

Remark We shall see in Section 5.5.4 that the classical local superconvergence results (e.g. $p^* = 2m$ for collocation at the Gauss points) can be restored if we use *quasi-geometric meshes*. For DDEs with proportional delays this was shown by Bellen (2001).

5.3 Second-kind VIEs with proportional delays

As we mentioned at the beginning of Section 5.1.1, second-kind Volterra integral equations with proportional delays,

$$y(t) = g_1(t) + b(t)y(qt) + \int_{qt}^{t} K(t,s)y(s)ds, \quad t \in I := [0, T], \quad 0 < q < 1, \tag{5.3.1}$$

and

$$y(t) = g(t) + \int_{0}^{qt} K(t,s)y(s)ds, \quad t \in I, \quad 0 < q < 1, \tag{5.3.2}$$

were studied by Volterra in (1897) and by Andreoli (1914). The first of these delay equations arose in the analysis of the 'invertibility' of the delay integral equation of the first kind,

$$\int_{qt}^{t} H(t,s)y(s)ds = g(t), \quad t \in I, \quad 0 < q < 1. \tag{5.3.3}$$

We shall return to (5.3.3) in more detail in Section 5.4. We will first focus on general second-kind delay VIEs of which (5.3.1) with $b(t) \equiv 0$ is a particular case. It will be seen later that the analysis of collocation methods for (5.3.1) with $b \not\equiv 0$ is much harder.

5.3.1 The collocation equations for uniform meshes

Set $\theta(t) := qt,\ 0 < q < 1$, and recall the delay integral equation (5.1.9),

$$y(t) = g(t) + (\mathcal{V}y)(t) + (\mathcal{V}_\theta y)(t), \quad t \in I := [0, T], \tag{5.3.4}$$

with Volterra operators $\mathcal{V}$ and $\mathcal{V}_\theta$ given by

$$(\mathcal{V}y)(t) := \int_0^t K_1(t,s)y(s)ds, \quad t \in I,$$

and

$$(\mathcal{V}_\theta y)(t) := \int_0^{\theta(t)} K_2(t,s)y(s)ds, \quad t \in I.$$

Their kernels are assumed to be continuous on their respective domains $D := \{(t,s):\ 0 \le s \le t \le T\}$ and $D_\theta := \{(t,s):\ 0 \le s \le \theta(t),\ t \in I\}$.

As for DDEs with (vanishing) proportional delay, smooth data $g,\ K_1$ and K_2 in (5.3.4) yield correspondingly smooth solutions y (Theorem 5.1.5), and hence we may choose the same collocation space as for classical (non-delay) second-kind VIEs, namely $S_{m-1}^{(-1)}(I_h)$, with unconstrained mesh I_h and with collocation points again given by

$$X_h := \{t_{n,i} := t_n + c_i h_n :\ 0 \le c_1 < \ldots < c_m \le 1\ (0 \le n \le N-1)\}.$$

The equations

$$u_h(t) = g(t) + (\mathcal{V}u_h)(t) + (\mathcal{V}_\theta u_h)(t), \quad t \in X_h, \tag{5.3.5}$$

and

$$u_h^{it}(t) := g(t) + (\mathcal{V}u_h)(t) + (\mathcal{V}_\theta u_h)(t), \quad t \in I, \tag{5.3.6}$$

determine, respectively, the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ and the corresponding iterated collocation solution $u_h^{it} \in C(I)$ for (5.3.4). As in Section 5.2.1

we shall first study the computational form of the collocation equation (5.3.5) on *uniform meshes*; collocation on *geometric meshes* will be considered in Section 5.3.7.

For *uniform meshes* I_h the general framework is the one introduced in Section 5.2.1. Recall in particular that, for $t = t_{n,i} := t_n + c_i h \in X_h$, we set

$$qt_{n,i} = t_{q_{n,i}} + \gamma_{n,i} h \in [t_{q_{n,i}}, t_{q_{n,i}+1}], \quad \text{with} \quad q_{n,i} := \lfloor q(n + c_i) \rfloor,$$
$$\gamma_{n,i} := q(n + c_i) - q_{n,i}.$$

Hence, the collocation equation (5.3.5) at $t = t_{n,i}$ $(i = 1, \ldots, m)$ becomes

$$U_{n,i} = g(t_{n,i}) + F_n(t_{n,i}) + h \int_0^{c_i} K_1(t_{n,i}, s) u_h(s) ds + (\mathcal{V}_\theta u_h)(t_{n,i}). \quad (5.3.7)$$

The lag term corresponding to the operator $\mathcal{V}$ is

$$\begin{aligned} F_n(t_{n,i}) &:= \int_0^{t_n} K_1(t_{n,i}, s) u_h(s) ds \\ &= h \sum_{\ell=0}^{n-1} \sum_{j=1}^{m} \left(\int_0^1 K_1(t_{n,i}, t_\ell + sh) L_j(s) ds \right) U_{\ell,j}, \end{aligned} \quad (5.3.8)$$

while the one corresponding to $\mathcal{V}_\theta$ can be expressed in the form

$$\begin{aligned} (\mathcal{V}_\theta u_h)(t_{n,i}) = h &\sum_{\ell=0}^{q_{n,i}-1} \sum_{j=1}^{m} \left(\int_0^1 K_2(t_{n,i}, t_\ell + sh) L_j(s) ds \right) U_{\ell,j} \\ &+ h \sum_{j=1}^{m} \left(\int_0^{\gamma_{n,i}} K_2(t_{n,i}, t_{q_{n,i}} + sh) L_j(s) ds \right) U_{q_{n,i},j}. \end{aligned} \quad (5.3.9)$$

Here, we have employed again the local representation of u_h on the subintervals σ_n,

$$u_h(t_n + vh) = \sum_{j=1}^{m} L_j(v) U_{n,j}, \quad v \in (0, 1], \quad \text{with} \quad U_{n,j} := u_h(t_{n,j}). \quad (5.3.10)$$

The computational form of the collocation equation (5.3.5) on σ_n is thus given by

$$\begin{aligned} U_{n,i} = h &\sum_{j=1}^{m} \left(\int_0^{c_i} K_1(t_{n,i}, t_n + sh) L_j(s) ds \right) U_{n,j} \\ &+ g(t_{n,i}) + F_n(t_{n,i}) + (V_\theta u_h)(t_{n,i}) \qquad (i = 1, \ldots, m). \end{aligned} \quad (5.3.11)$$

Recall from Section 5.2.1 that the integer $q_{n,i}$ is not necessarily the same for all $i \in \{1, \ldots, m\}$: it is possible that $q_{n,i} = q_{n,j} - 1$ for some $i < j$.

(I) *Initial phase (complete overlap)* $\quad 0 \le n < \lceil qc_1/(1-q) \rceil =: q^I$.
We know from Lemma 5.2.1 that for this (finite) set of values of n (which always includes $n = 0$), we have $q_{n,i} = n$ and $\gamma_{n,i} > 0$ $(i = 1, \dots, m)$, provided $c_1 > 0$. Hence, setting (recall also Section 2.2.2)

$$B_n^{(\ell)} := \begin{pmatrix} \int_0^1 K_1(t_{n,i}, t_\ell + sh) L_j ds \\ (i, j = 1, \dots, m) \end{pmatrix} \quad (\ell < n),$$

$$B_n := \begin{pmatrix} \int_0^{c_i} K_1(t_{n,i}, t_n + sh) L_j(s) ds \\ (i, j = 1, \dots, m) \end{pmatrix},$$

$$B_n^{(\ell)}(q) := \begin{pmatrix} \int_0^1 K_2(t_{n,i}, t_\ell + sh) L_j(s) ds \\ (i, j = 1, \dots, m) \end{pmatrix} \quad (\ell < n),$$

$$B_n^I(q) := \begin{pmatrix} \int_0^{\gamma_{n,i}} K_2(t_{n,i}, t_n + sh) L_j(s) ds \\ (i, j = 1, \dots, m) \end{pmatrix},$$

where, following the convention introduced in Section 5.2.1, the argument q attached to a matrix (or a vector below) indicates that it originates with the delay integral operator V_θ, we may write the collocation equation (5.3.11) as

$$[\mathcal{I}_m - h(B_n + B_n^I(q))]\mathbf{U}_n = \mathbf{g}_n + h \sum_{\ell=0}^{n-1} (B_n^{(\ell)} + B_n^{(\ell)}(q))\mathbf{U}_\ell, \qquad (5.3.12)$$

with $\mathbf{U}_n := (U_{n,1}, \dots, U_{n,m})^T$ and $\mathbf{g}_n := (g(t_{n,1}), \dots, g(t_{n,m}))^T$.

(II) *Transition phase (partial overlap)* $\quad q^I \le n < \lceil qc_m/(1-q) \rceil =: q^{II}$.
If this set of values n is not empty, there exists, for given n, an integer $\nu_n \in \{1, \dots, m-1\}$ so that

$$q_{n,i} = n - 1 \ (i = 1, \dots, \nu_n) \quad \text{and} \quad q_{n,i} = n, \ \gamma_{n,i} > 0 \ (i = \nu_n + 1, \dots, m);$$

that is, we have $t_{q_{n,i}} \le t_n$ for $i = 1, \dots, \nu_n$, and $t_{q_{n,i}} > t_n$ when $i > \nu_n$. Accordingly, we define the matrices

$$B_n^{II}(q) := \text{diag}(\underbrace{0, \dots, 0}_{\nu_n}, 1, \dots, 1) B_n^I(q),$$

$$S_{n-1}^{II}(q) := \text{diag}(\underbrace{0, \dots, 0}_{\nu_n}, 1, \dots, 1) B_n^{(n-1)}(q),$$

$$\hat{S}_{n-1}^{II}(q) := \text{diag}(\underbrace{1, \dots, 1}_{\nu_n}, 0, \dots, 0) B_{n-1}^{II}(q),$$

where

$$B_{n-1}^{II}(q) := \begin{pmatrix} \int_0^{\gamma_{n,i}} K_2(t_{n,i}, t_{n-1} + sh)L_j(s)ds \\ (i, j = 1, \ldots, m) \end{pmatrix}.$$

The linear algebraic system describing Phase II then becomes

$$[\mathcal{I}_m - h(B_n + B_n^{II}(q))]\mathbf{U}_n = \mathbf{g}_n + h\sum_{\ell=0}^{n-1} B_n^{(\ell)}\mathbf{U}_\ell + h\sum_{\ell=0}^{n-2} B_n^{(\ell)}(q)\mathbf{U}_\ell$$
$$+ h(\hat{S}_{n-1}^{II}(q) + S_{n-1}^{II}(q))\mathbf{U}_{n-1}\,. \qquad (5.3.13)$$

(III) *Pure delay phase (no overlap)* $\quad q^{II} \le n \le N-1$.
According to Lemma 5.2.1, the points $qt_{n,i}$ now all satisfy $qt_{n,i} \le t_n$. Assume that, for given n,

$$q_{n,i} = q_n \ (i = 1, \ldots, \nu_n) \ \text{ and } \ q_{n,i} = q_n + 1, \ \gamma_{n,i} > 0, (i = \nu_n + 1, \ldots, m),$$

for some $\nu_n \in \{1, \ldots, m\}$, where $q_n + 1 < n$. Hence, using (5.3.11) and (5.3.9) with the above values of $q_{n,i}$, and defining the matrices

$$\hat{S}_{q_n}^{III)}(q) := \text{diag}(\underbrace{1, \ldots, 1}_{\nu_n}, 0, \ldots, 0)B_{q_n}^{III}(q),$$

$$S_{q_n+1}^{III}(q) := \text{diag}(\underbrace{0, \ldots, 0}_{\nu_n}, 1, \ldots, 1)B_{q_n+1}^{III}(q),$$

with

$$B_{q_n}^{III}(q) := \begin{pmatrix} \int_0^{\gamma_{n,i}} K_2(t_{n,i}, t_{q_n} + sh)L_j(s)ds \\ (i, j = 1, \ldots, m) \end{pmatrix},$$

the linear algebraic system for Phase III assumes the form

$$[\mathcal{I}_m - hB_n]\mathbf{U}_n = \mathbf{g}_n + h\sum_{\ell=0}^{n-1} B_n^{(\ell)}\mathbf{U}_\ell + h\sum_{\ell=0}^{q_n-1} B_n^{(\ell)}(q)\mathbf{U}_\ell$$
$$+ h(\hat{S}_{q_n}^{III}(q) + B_n^{(q_n)}(q))\mathbf{U}_{q_n} + hS_{q_n+1}^{(III)}(q)\mathbf{U}_{q_n+1}. \qquad (5.3.14)$$

This confirms of course that once we have reached Phase III the matrix characterising the linear algebraic system (5.3.14) coincides with the one in (2.2.14), the linear algebraic system for second-kind Volterra integral equations without delay argument.

The existence of a unique collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ is guaranteed by

Theorem 5.3.1 *Assume that g, K_1, K_2 are continuous on their domains I, D, D_q, and let the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ to the delay VIE*

(5.3.4) be defined by (5.3.5) and (5.3.10). Then there exists an $\bar{h} > 0$ (depending on q) so that for all $h \in (0, \bar{h})$ each of the linear algebraic systems (5.3.12), (5.3.13), (5.3.14) possesses a unique solution $\mathbf{U}_n$. Thus, for such a mesh I_h the collocation solution u_h is unique for all $q \in (0, 1)$, and it is given locally, on σ_n, by (5.3.10).

The **proof** is completely analogous to the one for Theorem 2.2.1 and is readily carried out by applying the Neumann Lemma to each of the linear algebraic systems (5.3.12)–(5.3.14). We leave the details to the reader.

Once the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ has been computed, the corresponding *iterated collocation solution* u_h^{it} at $t = t_n + vh$ ($v \in [0, 1]$ can be obtained from

$$u_h^{it}(t_n + vh) = g(t_n + vh) + F_n(t_n + vh) + (\mathcal{V}_\theta u_h)(t_n + vh) + h \sum_{j=1}^{m} \left(\int_0^v K_1(t_n + vh, t_n + sh) L_j(s) ds \right) U_{n,j}. \tag{5.3.15}$$

Here,

$$q_n(v) := \lfloor q(n + v) \rfloor, \quad \gamma_n(v) := q(n + v) - q_n(v) \in [0, 1),$$

and hence $q(t_n + vh) = t_{q_n(v)} + \gamma_n(v)h \in [t_{q_n(v)}, t_{q_n(v)+1}]$. The lag term corresponding to $\mathcal{V}$ in (5.3.6) is, for $t = t_n + vh_n \in \bar{\sigma}_n$,

$$F_n(t) = h \sum_{\ell=0}^{n-1} \int_0^1 K_1(t, t_\ell + sh) u_h(t_\ell + sh) ds, \tag{5.3.16}$$

and we have

$$(\mathcal{V}_\theta u_h)(t) = h \sum_{\ell=0}^{q_n(v)-1} \int_0^1 K_2(t, t_\ell + sh) u_h(t_\ell + sh) ds + h \sum_{j=1}^{m} \left(\int_0^{\gamma_n(v)} K_2(t, t_{q_n(v)}) + sh) L_j(s) ds \right) U_{q_n(v),j}. \tag{5.3.17}$$

We observe once more that, in contrast to u_h, the iterated collocation solution u_h^{it} is continuous in I whenever the given functions g, K_1 and K_2 are continuous.

5.3.2 Two prominent DVIEs with proportional delay

We will now briefly illustrate the foregoing analysis by looking at two particular cases of the general delay Volterra integral equation (5.3.1) which we have met before. These delay integral equations have both historical and practical significance. The first is the 'pure delay' Volterra integral equation corresponding to $K_1 = 0$,

$$y(t) = g(t) + (\mathcal{V}_\theta y)(t), \quad t \in I = [0, T], \tag{5.3.18}$$

where

$$(\mathcal{V}_\theta y)(t) := \int_0^{\theta(t)} K(t, s)y(s)ds, \tag{5.3.19}$$

with $\theta(t) := qt$ $(0 < q < 1)$. Its collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ is computed by solving the linear algebraic systems (5.3.12)–(5.3.14) in which $B_n = 0$ and $B_n^{(\ell)}$ for all n and $\ell < n$. See also Exercise 5.6.16.

The more interesting equation corresponds formally to $K_2 = -K_1$ in (5.3.4),

$$y(t) = g(t) + (\mathcal{W}_\theta y)(t), \quad t \in I, \tag{5.3.20}$$

with $\mathcal{W}_\theta : C(I) \to C(I)$ given by

$$(\mathcal{W}_\theta y)(t) := \int_{\theta(t)}^{t} K(t, s)y(s)ds. \tag{5.3.21}$$

Let $t = t_{n,i} := t_n + c_i h \in X_h$ be given. If n is such that $qt_{n,i} \le t_n$, we may write

$$\begin{aligned}
(\mathcal{W}_\theta u_h)(t_{n,i}) &= \int_{qt_{n,i}}^{t_{n,i}} K(t_{n,i}, s)u_s ds \\
&= h \int_{\gamma_{n,i}}^{1} K(t_{n,i}, t_{q_{n,i}} + sh)u_h(t_{q_{n,i}} + sh)ds \\
&\quad + h \sum_{\ell=q_n+1}^{n-1} \int_0^1 K(t_{n,i}, t_\ell + sh)u_h(t_\ell + sh)ds \\
&\quad + h \int_0^{c_i} K(t_{n,i}, t_n + sh)u_h(t_n + sh)ds.
\end{aligned} \tag{5.3.22}$$

We recall that $q_{n,i}$ and $\gamma_{n,i}$ were defined in (5.2.4).

If $qt_{n,i} > t_n$ we have $q_{n,i} = n$ and $\gamma_{n,i} > 0$ (which is true for all $i = 1, \ldots, m$ in Phase I, and at least for some i in Phase II, unless it is empty). Hence, the above equation reduces to

$$(\mathcal{W}_\theta u_h)(t_{n,i}) = h \int_{\gamma_{n,i}}^{c_i} K(t_{n,i}, t_n + sh)u_h(t_n + sh)ds. \tag{5.3.23}$$

We now readily derive the systems of algebraic equations resulting from the collocation equation

$$u_h(t_{n,i}) = g(t_{n,i}) + (\mathcal{W}_\theta u_h)(t_{n,i}) \quad (i = 1, \ldots, m),$$

and which define the vector $\mathbf{U}_n := (U_{n,1}, \ldots, U_{n,m})^T$ in (5.3.10):

Phase I $0 \le n < q^I$ $(q_{n,i} = n,\ \gamma_{n,i} > 0$ for all $i)$:

$$[\mathcal{I}_m - h\bar{B}_n^I(q)]\mathbf{U}_n = \mathbf{g}_n, \tag{5.3.24}$$

where

$$\bar{B}_n^I(q) := \left(\int_{\gamma_{n,i}}^{c_i} K(t_{n,i}, t_n + sh) L_j(s) ds \right) \in L(\mathbb{R}^m),$$

which is of course formally equivalent to $B_n + B_n^I(q)$ with $K_2 = -K_1 =: -K$ (recall (5.3.12)).

Phase II $q^I \le n < q^{II}$ (where $q_{n,i} = n - 1 = q_n,\ i = 1, \ldots, \nu_n;\ q_{n,i} = n,\ \gamma_{n,i} > 0,\ i = \nu_n + 1, \ldots, m)$:

Here, we obtain

$$[\mathcal{I}_m - h\bar{B}_n^{II}(q)]\mathbf{U}_n = \mathbf{g}_n + h\bar{S}_{n-1}^{II}(q)\mathbf{U}_{n-1}, \tag{5.3.25}$$

with

$$\bar{B}_n^{II}(q) := \mathrm{diag}(\underbrace{1, \ldots, 1}_{\nu_n}, 0, \ldots, 0)B_n + \mathrm{diag}(\underbrace{0, \ldots, 0}_{\nu_n}, 1, \ldots, 1)\bar{B}_n^I(q),$$

$$\bar{S}_{n-1}^{II}(q) := \mathrm{diag}(\underbrace{1, \ldots, 1}_{\nu_n}, 0, \ldots, 0) \left(\int_{\gamma_{n,i}}^{1} K(t_{n,i}, t_{n-1} + sh) L_j(s) ds \right).$$

Phase III $q^{II} \le n \le N - 1$ (with $q_{n,i} = q_n < n - 1,\ i = 1, \ldots, \nu_n;\ q_{n,i} = q_n + 1,\ i = \nu_n + 1, \ldots, m)$:

The system of linear equations describing the final, pure delay phase is given by

$$[\mathcal{I}_m - hB_n]\mathbf{U}_n = \mathbf{g}_n + h[\bar{S}_{q_n}^{III}(q)\mathbf{U}_{q_n} + \sum_{\ell=q_n+1}^{n-1} B_n^{(\ell)}\mathbf{U}_\ell + S_{q_n+1}^{III}(q)\mathbf{U}_{q_n+1}], \tag{5.3.26}$$

where

$$\bar{S}_{q_n}^{III}(q) := \mathrm{diag}(\underbrace{1, \ldots, 1}_{\nu_n}, 0, \ldots, 0\,) \left(\int_{\gamma_{n,i}}^{1} K(t_{n,i}, t_{q_n} + sh) L_j(s) ds \right),$$

$$S_{q_n+1}^{III}(q) := \mathrm{diag}(\underbrace{1, \ldots, 1}_{\nu_n}, 0, \ldots, 0)B_n^{(q_n+1)}$$
$$+ \mathrm{diag}(\underbrace{0, \ldots, 0}_{\nu_n}, 1, \ldots, 1) \left(\int_{\gamma_{n,i}}^{1} K(t_{n,i}, t_{q_n+1} + sh) L_j(s) ds \right).$$

The matrices $B_n^{(\ell)} \in L(\mathbb{R}^m)$ $(\ell < n)$ coincide with those in (5.3.14).

5.3.3 Global convergence results: uniform I_h

Consider the linear delay integral equation introduced in Section 5.3.1,

$$y(t) = g(t) + (\mathcal{V}y)(t) + (\mathcal{V}_\theta y)(t), \quad t \in I := [0, T], \tag{5.3.27}$$

with $\mathcal{V}$ and $\mathcal{V}_\theta$ as in (5.3.4), and $\theta(t) := qt$ $(0 < q < 1)$.

Theorem 5.3.2 *Assume:*

(a) The given functions describing (5.3.27) satisfy the regularity conditions $g \in C^m(I)$, $K_1 \in C^m(D)$, and $K_2 \in C^m(D_\theta)$.
(b) For given uniform mesh I_h and collocation points X_h, $u_h \in S_{m-1}^{(-1)}(I_h)$ is the collocation solution to (5.3.27).

Then for any uniform mesh I_h with mesh diameter $h \in (0, \bar{h})$, with $\bar{h}$ as in Theorem 5.3.1, and any set $\{c_i\}$ of m distinct collocation parameters in $[0, 1]$, *the collocation error $e_h := y - u_h$ can be estimated by*

$$||e_h||_\infty \leq C||y^{(m)}||_\infty h^m. \tag{5.3.28}$$

The constant C depends on the $\{c_i\}$ but not on h.

Proof We have seen in Theorem 5.1.5 that assumption (a) implies that the (unique) solution y of (5.3.27) lies in $C^m(I)$. Thus, we may again resort to the local representation (2.2.31) for the collocation error $e_h := y - u_h$ on σ_n,

$$e_h(t_n + vh) = \sum_{j=1}^{m} L_j(v)\mathcal{E}_{n,j} + h^m R_{m,n}(v), \quad v \in (0, 1], \quad \mathcal{E}_{n,j} := e_h(t_{n,j}). \tag{5.3.29}$$

On X_h it satisfies the error equation

$$e_h(t) = (\mathcal{V}e_h)(t) + (\mathcal{V}_\theta e_h)(t).$$

For $t = t_{n,i}$ this equation becomes

$$\mathcal{E}_{n,i} = h \int_0^{c_i} K_1(t_{n,i}, t_n + sh)e_h(t_n + sh)ds + F_n(t_{n,i}) + (\mathcal{V}_\theta e_h)(t_{n,i}), \tag{5.3.30}$$

with lag term corresponding to $\mathcal{V}$ given by

$$F_n(t_{n,i}) = h \sum_{\ell=0}^{n-1} \int_0^1 K_1(t_{n,i}, t_\ell + sh) \left(\sum_{j=1}^{m} L_j(s)\mathcal{E}_{\ell,j} + h^m R_{m,\ell}(s) \right) ds. \tag{5.3.31}$$

The contribution of the delay operator $\mathcal{V}_\theta$ in the above error equation is described by

$$(\mathcal{V}_\theta e_h)(t) := h \sum_{\ell=0}^{q_{n,i}-1} \int_0^1 K_2(t, t_\ell + sh) \left(\sum_{j=1}^{m} L_j(s)\mathcal{E}_{\ell,j} + h^m R_{m,\ell}(s) \right) ds$$
$$+ h \int_0^{q_{n,i}} K_2(t, t_{q_{n,i}} + sh) \left(\sum_{j=1}^{m} L_j \mathcal{E}_{q_{n,i},j} + h^m R_{m,q_{n,i}}(s) \right) ds, \tag{5.3.32}$$

where $t = t_{n,i}$.

The description in the previous section of the structure of the recursive process underlying the collocation method for the proportional delay VIE (5.3.27) contains all the essential ingredients for proving Theorem 5.3.2: since Phase I and Phase II, corresponding to the values of n for which, respectively, $0 \le n < q^I := \lceil c_1 q/(1-q) \rceil$ and $q^I \le n < q^{II} := \lceil c_m q/(1-q) \rceil$ holds, involve only finitely many time steps, regardless of the choice of h, the order of convergence is governed by a generalised discrete Gronwall inequality arising from the pure delay Phase III. We observe that these systems of linear algebraic equations for $\mathcal{E}_n$ closely resemble the ones for $\mathbf{U}_n$, namely (5.3.12)–(5.3.14): the role of $\mathbf{g}_n$ is now assumed by terms reflecting the (Peano) error terms in the local representation (5.3.29) of the collocation error. Thus, depending on the value of n and the corresponding $q_{n,i}$ the details are as follows:

(I): $0 \le n < q^I := \lceil q c_1/(1-q) \rceil$

Here, $q_{n,i} = n$ for all values of $i = 1, \ldots, m$, with $\gamma_{n,i} > 0$. Thus, proceeding along familiar lines and using the notation introduced in (5.3.13), the vector $\boldsymbol{\mathcal{E}}_n := (\mathcal{E}_{n,1}, \ldots, \mathcal{E}_{n,m})^T$ is defined by the solution of the linear algebraic system

$$[\mathcal{I}_m - h(B_n + B_n^I(q))]\boldsymbol{\mathcal{E}}_n = h \sum_{\ell=0}^{n-1} (B_n^{(\ell)} + B_n^{(\ell)}(q))\boldsymbol{\mathcal{E}}_\ell + h^m [h \sum_{\ell=0}^{n-1} \boldsymbol{\rho}_n^{(\ell)} + h\boldsymbol{\rho}_n]$$
$$+ h^m [h \sum_{\ell=0}^{n-1} \boldsymbol{\rho}_n^{(\ell)}(q) + h\boldsymbol{\rho}_n^I(q)], \tag{5.3.33}$$

where the matrices $B_n,\ B_n^I(q),\ B_n^{(\ell)},\ B_n^{(\ell)}(q)$ $(\ell < n)$ are those of (5.3.13), and where we have set

$$\boldsymbol{\rho}_n^I(q) := \left(\int_0^{\gamma_{n,i}} K_2(t_{n,i}, t_n + sh) R_{m,n}(s) ds \quad (i = 1, \ldots, m) \right)^T,$$

$$\boldsymbol{\rho}_n^{(\ell)}(q) := \left(\int_0^1 K_2(t_{n,i}, t_\ell + sh) R_{m,\ell}(s) ds \quad (i = 1, \ldots, m) \right)^T \quad (\ell < n).$$

(II): $q^{I} \leq n < q^{II} := \lceil qc_m/(1-q) \rceil$
For given n let ν_n, $1 \leq \nu_n < m$, be such that

$$q_{n,i} = n-1 \;\text{ for }\; i = 1, \ldots, \nu_n; \;\; q_{n,i} = n, \;\gamma_{n,i} > 0 \;\text{ for }\; i = \nu_n + 1, \ldots, m.$$

It then follows readily from (5.3.30)–(5.3.32) and the analysis of Phase II in Section 5.3.1 that the algebraic system for $\mathcal{E}_n$ has the form

$$\begin{aligned}
&[\mathcal{I}_m - h(B_n + B_n^{II}(q))]\boldsymbol{\mathcal{E}}_n \\
&\quad = h\sum_{\ell=0}^{n-1} B_n^{(\ell)}\boldsymbol{\mathcal{E}}_\ell + h\sum_{\ell=0}^{n-2} B_n^{(\ell)}(q)\boldsymbol{\mathcal{E}}_\ell + h(\hat{S}_n^{II}(q) + S_n^{II}(q))\boldsymbol{\mathcal{E}}_{n-1} \\
&\qquad + h^m[h\sum_{\ell=0}^{n-1} \boldsymbol{\rho}_n^{(\ell)} + h\boldsymbol{\rho}_n] + h^m[h\sum_{\ell=0}^{n-2} \boldsymbol{\rho}_n^{(\ell)}(q) + h(\hat{\boldsymbol{\rho}}_{n-1}^{II}(q) + \boldsymbol{\rho}_n^{II}(q))],
\end{aligned} \tag{5.3.34}$$

with matrices $B_n^{II}(q)$, $B_n^{(\ell)}$, $\hat{B}_n^{(\ell)}(q)$, $\hat{S}_n^{II}(q)$, $S_n^{II}(q)$ as in (5.3.13), and

$$\begin{aligned}
\hat{\boldsymbol{\rho}}_{n-1}^{II}(q) &:= \mathrm{diag}(\underbrace{1, \ldots, 1}_{\nu_n}, 0, \ldots, 0) \\
&\qquad \left(\int_0^{\gamma_{n,i}} K_2(t_{n,i}, t_{n-1} + sh) R_{m,n-1}(s)ds \;\; (i = 1, \ldots, m)\right), \\
\boldsymbol{\rho}_n^{II}(q) &:= \mathrm{diag}(\underbrace{0, \ldots, 0}_{\nu_n}, 1, \ldots, 1)[\boldsymbol{\rho}_n^{(n-1)}(q) + \boldsymbol{\rho}_n^{I}(q)].
\end{aligned}$$

(III): $q^{II} \leq n \leq N-1$
Let now ν_n with $1 \leq \nu_n \leq m$ be such that

$$\begin{aligned}
&q_{n,i} = q_n \;\text{ for }\; i = 1, \ldots, \nu_n, \quad \text{and } \; q_{n,i} = q_n + 1, \; \gamma_{n,i} > 0 \\
&\quad \text{for }\; i = \nu_n + 1, \ldots, m,
\end{aligned}$$

with $q_n < n-1$. Using these values for $q_{n,i}$ in (5.3.32) we are led to the linear algebraic system

$$\begin{aligned}
[\mathcal{I}_m - hB_n]\boldsymbol{\mathcal{E}}_n &= h\sum_{\ell=0}^{n-1} B_n^{(\ell)}\boldsymbol{\mathcal{E}}_\ell + h\sum_{\ell=0}^{q_n-1} B_n^{(\ell)}(q)\boldsymbol{\mathcal{E}}_\ell \\
&\quad + h(\hat{S}_{q_n}^{III}(q) + B_n^{(q_n)}(q))\boldsymbol{\mathcal{E}}_{q_n} + hS_{q_n+1}^{III}(q)\boldsymbol{\mathcal{E}}_{q_n+1} \\
&\quad + h^m[h\sum_{\ell=0}^{n-1} \boldsymbol{\rho}_n^{(\ell)} + h\boldsymbol{\rho}_n] + h^m[h\sum_{\ell=0}^{q_n-1} \boldsymbol{\rho}_n^{(\ell)}(q) \\
&\quad + h(\hat{\boldsymbol{\rho}}_{q_n}^{III}(q) + \boldsymbol{\rho}_{q_n+1}^{III}(q))],
\end{aligned} \tag{5.3.35}$$

with $\hat{S}^{III}_{q_n}(q)$ and $S^{III}_{q_n+1}(q)$ as in (5.3.14) and with

$$\hat{\boldsymbol{\rho}}^{III}_{q_n}(q) := \operatorname{diag}(\underbrace{1, \ldots, 1}_{\nu_n}, 0, \ldots, 0)$$
$$\times \left(\int_0^{\gamma_{n,i}} K_2(t_{n,i}, t_{q_n} + sh) R_{m,q_n}(s) ds \;\; (i = 1, \ldots, m) \right),$$
$$\boldsymbol{\rho}^{III}_{q_n+1}(q) := \operatorname{diag}(\underbrace{0, \ldots, 0}_{\nu_n}, 1, \ldots, 1)[\boldsymbol{\rho}^{(q_n-1)}_n(q) + \boldsymbol{\rho}_{q_n+1}(q)].$$

According to Theorem 5.3.1 each of the above linear algebraic systems (5.3.33)–(5.3.35) possesses a unique solution for (uniform) meshes I_h with $h \in (0, \bar{h})$. Thus, we may proceed as in the proof of Theorem 2.2.3: denoting by D_0 the constant for which we have

$$||(\mathcal{I}_m - h\mathcal{B}_n)^{-1}||_1 \leq D_0, \quad h \in (0, \bar{h}) \;\; (n = 0, 1, \ldots, N-1),$$

with

$$\mathcal{B}_n := \begin{cases} B_n + B^I_n(q) & \text{if } 0 \leq n < q^I \\ B_n + B^{II}_n(q) & \text{if } q^I \leq n < q^{II} \\ B_n & \text{if } q^{II} \leq n \leq N-1, \end{cases}$$

we are led to a generalised discrete Gronwall inequality for $||\boldsymbol{\mathcal{E}}_n||_1$ of the type (5.2.25), except that now the last term reads $\gamma_1 M_m h^m$, because of the lower-order bounds for the ρ-terms in (5.3.33)–(5.3.35). Hence, in complete analogy to the final argument in the proof of Theorem 2.2.3 we obtain $||\boldsymbol{\mathcal{E}}_n||_1 \leq BM_m h^m$, leading via the local error representation (5.3.29) to $||e_h||_\infty \leq CM_m h^m$.

The following corollary to Theorem 5.3.2 addresses again the case where the solution y of (5.3.27) does not have full regularity; that is, if instead of $y \in C^m(I)$ we only have $y \in C^d(I),\ 1 \leq d < m$ (which corresponds to the assumption that the given functions lie only in $C^d(I)$).

Corollary 5.3.3 *If assumption (a) in Theorem 5.3.2 is replaced by* $g \in C^d(I),\ K_1 \in C^d(D),\ K_2 \in C^d(D_\theta)$, *for some* d *with* $1 \leq d < m$, *then the estimate*

$$||e_h||_\infty \leq C||y^{(d)}||_\infty h^d \tag{5.3.36}$$

for the collocation error e_h *corresponding to* $u_h \in S^{(-1)}_{m-1}(I_h)$ *is true for any set* X_h *of collocation points defined by distinct* $\{c_i\}$ *in* $[0, 1]$.

In Theorem 2.2.4 we showed that the iterated collocation solution for a classical Volterra integral equation of the second kind can be globally superconvergent if the collocation parameters are chosen judiciously. An analogous result holds for second-kind VIEs with proportional delays. However, the proof of this result (due to Brunner and Hu (2003) and based on interpolatory projection techniques) is much more complex.

Theorem 5.3.4 *If the collocation parameters* $\{c_i\}$ *are chosen so that*

$$J_0 := \int_0^1 \prod_{i=1}^{m} (s - c_i)ds = 0,$$

then

$$||y - u_h^{it}||_\infty \le Ch^{m+1},$$

provided we have $g \in C^{d+1}(I)$ *and* $K_1 \in C^{d+1}(D)$, $K_2 \in C^{d+1}(D_\theta)$, *with* $d \ge m$. *This global superconvergence result is true for any* $q \in (0, 1)$, *and the exponent* $m + 1$ *can in general not be replaced by* $m + 2$.

Proof Since the crucial term in the delay VIE (5.3.1) is $\mathcal{V}_\theta y$, we will prove Theorem 5.3.4 for the 'pure delay' delay VIE (5.3.18), employing an approach that is different from the one in Brunner and Hu (2003). This will exhibit more clearly how the 'overlapping effect' seen in the solution representation of Theorem 5.1.4 affects the superconvergence analysis.

The collocation error $e_h := y - u_h$ for $u_h \in S_{m-1}^{(-1)}(I_h)$ satisfies the equation

$$e_h(t) = \delta_h(t) + \int_0^{qt} K(t, s)e_h(s)ds, \quad t \in I,$$

with $\delta_h = 0$ on X_h. Hence, by Theorem 5.1.4 we may write (because of $e_h^{it}(t) = e_h(t) - \delta_h(t)$)

$$e_h^{it}(t) = \sum_{k=1}^{\infty} \int_0^{q^k t} K_k(t, s)\delta_h(s)ds, \quad t \in I, \tag{5.3.37}$$

with iterated kernels $K_k(t, s)$ as defined in (5.1.9). Suppose that $t = t_n + vh$ ($v \in [0, 1]$). We define

$$q_{k,n}(v) := \lfloor q^k(n + v)\rfloor, \qquad \gamma_{k,n}(v) := q^k(n + v) - q_{k,n}(v) \in [0, 1).$$

Thus, the representation of $e_h^{it}(t)$ assumes the form

$$\begin{aligned} e_h^{it}(t) &= \sum_{k=1}^{\infty} \Big(\int_0^{t_{q_{k,n}(v)}} K_k(t, s)\delta_h(s)ds \\ &\quad + h \int_0^{\gamma_{k,n}(v)} K_k(t, t_{q_{k,n}(v)} + sh)\delta_h(t_{q_{k,n}(v)} + sh)ds \Big) \\ &=: S_n^I(v) + S_n^{II}(v). \end{aligned} \tag{5.3.38}$$

For fixed n, consider first the individual terms of $S_n^I(v)$, written as

$$\int_0^{t_{q_{k,n}(v)}} K_k(t,s)\delta_h(s)ds = h\sum_{\ell=0}^{q_{k,n}(v)-1}\int_0^1 K_k(t,t_\ell+sh)\delta_h(t_\ell+sh)ds,$$

for all integers $q_{k,n}(v)$ with $q_{k,n}(v)\geq 1$. This holds as long as $q^k(n+1)\geq 1$, or

$$k \leq \lfloor -\log(n)/\log(q)\rfloor =: k_n^*(q).$$

In any case, we have $q_{k,n}(v) < N$ for $v\in[0,1]$ and all $q\in(0,1)$. Hence, by the standard quadrature argument employed in our earlier superconvergence analyses,

$$\begin{aligned} S_n^I(v) &= h\sum_{k=1}^{k_n^*(q)}\sum_{\ell=0}^{q_{k,n}(v)-1}\int_0^1 K_k(t,t_\ell+sh)\delta_h(t_\ell+sh)ds \\ &= h\sum_{k=1}^{k_n^*(q)}\sum_{\ell=0}^{q_{k,n}(v)-1} E_{k,n}^{(\ell)}(v), \quad v\in[0,1], \end{aligned}$$

where the terms $E_{k,n}^{(\ell)}(v)$ denote the quadrature errors induced by the interpolatory m-point quadrature formulas with abscissas $\{t_\ell + c_j h\}$. Since the orthogonality condition $J_0 = 0$ implies that these formulas possess degree of precision of (at least) m, it follows that $|E_{k,n}^{(\ell)}(v)|\leq Q_m h^{m+1}$ uniformly for $v\in[0,1]$. Thus,

$$\begin{aligned} |S_n^I(v)| &\leq hQ_m h^{m+1}\sum_{k=1}^{k_n^*(q)}\sum_{\ell=0}^{q_{k,n}(v)-1} 1 \leq hQ_m h^{m+1}\sum_{k=1}^{k_n^*(q)} q^k N \\ &\leq Nh\cdot Q_m h^{m+1}\sum_{k=1}^{k_n^*(q)} q^k \leq Q_m T h^{m+1} q/(1-q) \quad (0\leq n\leq N-1). \end{aligned}$$

In order to derive an upper bound for $S_n^{II}(v)$ in (5.3.38), recall first that the iterated kernels $K_k(t,s)$ are bounded by

$$|K_k(t,s)| \leq \frac{q^{k(k-1)/2}}{(k-1)!}T^{k-1}\bar{K}_\theta^k, \quad (t,s)\in D_\theta^{(k)} \quad (k\geq 1)$$

(cf. Lemma 5.1.5). Moreover, by Theorem 5.3.2 we have $||\delta_h||_\infty \leq C_\delta h^m$ for any choice of $\{c_i\}$. These observations lead to

$$\begin{aligned} |S_n^{II}(v)| &\leq h\sum_{k=1}^{\infty}\int_0^{\gamma_{k,n}(v)} |K_k(t,t_{q_{k,n}(v)}+sh)||\delta_h(t_{q_{k,n}(v)}+sh)|ds \\ &\leq C_\delta h^{m+1}\sum_{k=1}^{\infty}\gamma_{k,n}(v)\frac{q^{k(k-1)/2}}{(k-1)!}T^{k-1}\bar{K}_\theta^k \\ &\leq C_\delta h^{m+1}\bar{K}_\theta\sum_{k=1}^{\infty}\frac{q^{k(k-1)/2}}{(k-1)!}(T\bar{K}_\theta)^{k-1}. \end{aligned}$$

For any finite T and $q \in (0, 1)$, the infinite series is convergent (note its similarity to the expression for the solution of the initial-value problem

$$y'(t) = \bar{K}_\theta y(qt), \quad t \in [0, T], \quad y(0) = y_0,$$

at $t = T$, as seen in (5.1.7)!). We therefore conclude that there exist constants $C^I(q)$ and $C^{II}(q)$ so that

$$|e_h^{it}(t)| \leq (C^I(q) + C^{II}(q))h^{m+1}, \quad t \in I.$$

5.3.4 A more general VIE with proportional delay

As we have seen at the beginning of Section 5.1.1, the first-kind delay integral equation (5.1.1) can often be recast as an equation of the second kind,

$$y(t) = g(t) + b(t)y(qt) + (\mathcal{W}_\theta y)(t), \quad t \in I := [0, T] \tag{5.3.39}$$

(cf. (5.1.3))), with $b(t) \not\equiv 0$. This more general delay equation is a particular case of

$$y(t) = g(t) + b(t)y(qt) + (\mathcal{V}y)(t) + (\mathcal{V}_\theta y)(t), \quad t \in I. \tag{5.3.40}$$

It is immediately clear that the analysis of existence and uniqueness of the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$, defined by

$$u_h(t) = g(t) + b(t)u_h(qt) + (\mathcal{V}u_h)(t) + (\mathcal{V}_\theta u_h)(t), \quad t \in X_h,$$

and that of its attainable order of convergence on I and I_h is much more complex, due to the presence of the term $b(t)u_h(qt)$ on the right-hand side of the collocation equation. The additional matrix representing the contribution of $b(t_{n,i})u_h(qt_{n,i})$, for example to Phase I,

$$\mathcal{D}_n^I := \text{diag}(b(t_{n,i}), \ldots, b(t_{n,m})) \begin{pmatrix} L_j(\gamma_{n,i}) \\ (i, j = 1, \ldots, m) \end{pmatrix},$$

implies that the matrix $\mathcal{I}_m - h[B_n + B_n^I(q)]$ characterising the linear algebraic system (5.3.12) is now replaced by $\mathcal{I}_m - \mathcal{D}_n^I - h[B_n + B_n^I(q)]$. Thus, since $\mathcal{D}_n^I$ does not carry the factor h, the statement Theorem 5.3.1, guaranteeing the existence of a (unique) solution $\mathbf{U}_n$ for all sufficiently small h, will in general no longer remain valid, unless we have $||\mathcal{D}_n^I|| < 1$ for all n. An analogous remark applies to Phase II, while Phase III is no longer affected by the additional delay term $b(t)u_h(qt)$.

A very particular case was studied by Y. Liu (1995b): it essentially corresponds to the choice $m = 1,\ c_1 = 1$, and it already exhibits the different, much more difficult nature of the analysis.

5.3.5 Attainable order at $t = t_1 = h$

Consider now the *integrated form* of the DDE (5.2.28),

$$y(t) = 1 + \int_0^{qt} (b/q)y(s)ds, \quad t \in I, \tag{5.3.41}$$

and suppose that its solution is approximated by $u_h \in S_{m-1}^{(-1)}(I_h)$, using the same collocation parameters $\{c_i\}$ as for $v_h \in S_m^{(0)}(I_h)$, the collocation solution to

$$y'(t) = by(qt), \quad t \in I, \quad y(0) = 1.$$

Will the results of Section 1.1.5 (Corollary 1.1.10, corresponding to $q = 1$: no delay) remain valid when $0 < q < 1$?

Theorem 5.3.5 *The collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ and the corresponding iterated collocation solution u_h^{it} for (5.2.41) at $t = t_1 = h$ (with $z := bh$) have the values*

$$u_h(h) = \frac{\sum_{j=0}^{m-1} q^{j(2m-j-1)/2} M^{(m-j)}(1/q^{m-j})z^j}{\sum_{j=0}^{m} q^{j(2m-j-1)/2} M^{(m-j)}(0)z^j}$$

and

$$u_h^{it}(h) = \frac{\sum_{j=0}^{m} q^{j(2m-j-1)/2} M^{(m-j)}(1/q^{m-j})z^j}{\sum_{j=0}^{m} q^{j(2m-j-1)/2} M^{(m-j)}(0)z^j},$$

where $z := bh$.

Remark We observe that the two rational approximants describing $u_h(h)$ and $u_h^{it}(h)$ are very closely related: they essentially coincide except that in the numerator of $u_h(h)$ the upper limit of the sum is $m - 1$, compared to m in $u_h^{it}(h)$. The result remains of course true for $q = 1$; see Sections 1.1.5 and 1.1.6 (Corollary 1.1.10 and Lemma 1.1.12).

Proof In analogy to the proof of the previous theorem the collocation equation for $u_h \in S_{m-1}^{(-1)}(I_h)$ on the first subinterval $\bar{\sigma}_0 = [0, h]$ may be written as

$$u_h(t) = 1 + \int_0^{qt} (b/q)u_h(s)ds + \tilde{K}M(t), \quad t \in \sigma_0, \tag{5.3.42}$$

where $M(t)$ denotes the collocation polynomial with respect to the points $\{c_i\}$. There is, however, one major difference: since we are now in the discontinuous space $S_{m-1}^{(-1)}(I_h)$ we have, in general, $u_h(0) \neq y_0 = 1$; that is,

$$u_h(0) = 1 + \tilde{K} M(0) \neq 1 \quad (\text{unless } c_1 = 0).$$

Applying m-fold differentiation to the collocation equation (5.2.40) on $\bar{\sigma}_0$ (and setting again $h = 1$ for simplicity) we find

$$0 \equiv b^m q^{m(m-1)/2} u_h(q^m t) + \tilde{K} \sum_{j=0}^{m-1} q^{j(2m-j-1)/2} M^{(m-j)}(q^j t) z^j.$$

It thus follows from the above value of $u_h(0)$ and by setting, respectively, $t = 0$ and $t = 1/q^m$ in the differentiated collocation equation that the first assertion of Theorem 5.3.5 is true.

To prove the second statement we first note that, by definition,

$$u_h^{it}(t) = 1 + \int_0^{qt} (b/q) u_h(s) ds, \quad t \in \bar{\sigma}_0.$$

Hence,

$$(d/dt) u_h^{it}(t) = b u_h(qt), \quad t \in \bar{\sigma}_0.$$

Since on $\bar{\sigma}_0$ the iterated collocation solution u_h^{it} for (5.3.39) reduces to a polynomial of degree m, we find that

$$0 \equiv (d^m/dt^m) u_h^{it}(t) = b^{m+1} q^{m(m+1)/2} u_h^{it}(q^{m+1} t) + b q^2 \tilde{K} \sum_{j=0}^{m} q^{j(2m-j-1)/2} M^{(mj)}(q^{j+1} t).$$

The proof is brought to its conclusion by setting $t = 0$ and $t = 1/q^{m+1}$ and by observing that $u_h(0) = 1$, in complete analogy to the proof of Theorem 5.2.7.

Example 5.3.1
For $m = 1$ we obtain

$$u_h^{it}(h) = \frac{1 + (1 - c_1) z}{1 - c_1 z} \quad (z := bh)$$

for all values of $q \in (0, 1]$. Thus, for $c_1 = 1/2$ (Gauss point) this coincides with the [1, 1]-Padé approximant for $\exp(z)$, regardless of q.

Example 5.3.2
For $m = 2$ Theorem 5.3.5 yields

$$u_h^{it}(h) = \frac{1 + (1 - [q(c_1 + c_2)/2]z + [q(1 - c_1)(1 - c_2)/2]z^2}{1 - [q(c_1 + c_2)/2]z + [q c_1 c_2/2]z^2}.$$

If c_1 and c_2 are the *Gauss points* then

$$u_h^{it}(h) = \frac{1 + (1 - q/2)z + (q/12)z^2}{1 - (q/2)z + (q/12)z^2}.$$

This rational function is different from the one for $v_h(h)$ (recall Example 5.2.2), and it also differs from the [2, 2]-Padé approximant for $y(h)$, whenever $q \in (0, 1)$.

If we compare the expressions for $v_h(h)$ and $u_h^{it}(h)$ given, respectively, in Theorem 5.2.7 and Theorem 5.3.5, we see that the following result (answering one of the questions raised above) is now obvious. To state it, assume that $v_h \in S_m^{(0)}(I_h)$ is based on the collocation parameters $\{c_i\}$ while $u_h \in S_{m-1}^{(-1)}(I_h)$ corresponds to the m collocation parameters $\{\hat{c}_i\}$.

Theorem 5.3.6 *For $q \in (0, 1)$, we obtain*

$$u_h^{it}(h) = v_h(h) \quad \textit{if, and only if, } \hat{c}_i = qc_i \ (i = 1, \ldots, m).$$

The **proof** of this result can be found in Takama, Muroya and Ishiwata (2000); for $m = 2$ it was given in Brunner (1997a).

Remark For $q = 1$ the result of Theorem 5.3.6 reduces to the one in Corollary 1.1.10 (Section 1.1.5).

5.3.6 Local superconvergence analysis on uniform meshes

We have already seen that for DDEs with proportional delay the classical local superconvergence order of $p^* = 2m$ for collocation at the Gauss points can no longer be attained if $m > 2$ (Conjecture 5.2.8). For pantograph-type delay Volterra integral equations of the second kind the situation is even worse, as the following theorem shows (Brunner and Hu (2003)). This is not really too surprising in view of Theorem 5.3.6 on the relationship between the collocation solutions for a special case of the pantograph equation and its integrated form.

To be more precise, we will now show that, in contrast to the global superconvergence result of Theorem 5.3.2, the attainable order of *local superconvergence* on the uniform mesh I_h differs rather substantially from earlier classical $\mathcal{O}(h^{m+\kappa})$-estimates ($\kappa \leq m$) when $m \geq 3$.

Theorem 5.3.7 *Let $u_h \in S_{m-1}^{(-1)}(I_h)$ be the collocation solution to the DVIE (5.3.27), and let u_h^{it} be the corresponding iterated collocation solution. If the collocation parameters $\{c_i\}$ are the Gauss points, then the order p^* in the estimate*

$$\max_{t \in I_h \setminus \{0\}} |y(t) - u_h^{it}(t)| \leq C h^{p^*}$$

cannot exceed $m+2$*. More precisely, the following is true:*

(i) If $q = 1/2$*, then*

$$p^* = \begin{cases} m+2 & \textit{if } m \textit{ is even}, \\ m+1 & \textit{if } m \textit{ is odd}. \end{cases}$$

(ii) For $q \in (0,1) \setminus \{1/2\}$ *we attain only* $p* = m+1$*.*

Proof See Brunner and Hu (2003).

We note in passing that the superconvergence results presented in Theorems 5.3.4 and 5.3.7 are of course also true for the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ to the particular delay VIEs (5.3.20) and (5.3.18).

5.3.7 Local superconvergence on geometric meshes

The special form of the delay function $\theta(t) = qt$ $(0 < q < 1)$ suggests that u_h^{it} might possibly attain the classical optimal order of superconvergence $p^* = 2m$ on a suitable *geometric mesh*, if collocation is at the Gauss points. That this is (almost) so was verified in Brunner, Hu and Lin (2001). We briefly describe this result and sketch its proof.

Assume that I_h is a *geometric mesh* defined by

$$I_h := \{t_n : t_n = \gamma^{N-n}T, \quad n = 0, 1, \ldots, N; \quad \gamma \in (0,1)\}. \tag{5.3.43}$$

As we shall see below, the mesh parameter γ will depend on N (but not on n), on q, and on m. The mesh (5.3.43) possesses the following obvious properties:

(i) $h_n := t_{n+1} - t_n = \gamma^{N-n-1}(1-\gamma)T \quad (n = 0, 1, \ldots, N-1)$;
(ii) $\max_{(n)} h_n = h_{N-1} = (1-\gamma)T$ (for *any* $N \in \mathbb{N}$). Hence, $\gamma = \gamma(N)$ will have to be chosen so that $\gamma \to 1$, as $N \to \infty$, for all $q \in (0,1)$.

Let $\rho \in \mathbb{N}$ be defined by

$$\rho := \left\lfloor \frac{\ln(q)}{\ln(1 - \frac{2m\ln(N)}{(m+1)N})} \right\rfloor : \tag{5.3.44}$$

it is the largest integer for which

$$q^{1/\rho} \leq 1 - \frac{2m \cdot \ln(N)}{(m+1)N}.$$

Theorem 5.3.8 will reveal the motivation for introducing this integer ρ. Observe that for given (fixed) $q \in (0,1)$ and $m \geq 1$, we have $\rho > 1$ for all sufficiently large N. This is true because

$$1 - \frac{2m \cdot \ln(N)}{(m+1)N} \longrightarrow 1^-, \quad \text{as } N \to \infty,$$

for any $m \in \mathbb{N}$. The following is the result of Brunner, Hu and Lin (2001).

Theorem 5.3.8 *Assume:*

(a) $g \in C^{2m}(I),\ K_1 \in C^{2m}(D),\ K_2 \in C^{2m}(D_\theta)$;
(b) I_h *is the geometric mesh described by (5.3.43) and (5.3.44), with* $\gamma = q^{1/\rho}$;
(c) $u_h \in S_{m-1}^{(-1)}(I_h)$ *is the collocation solution to the delay VIE (5.3.5), with the* $\{c_i\}$ *given by the Gauss points, and* u_h^{it} *denotes the corresponding iterated collocation solution.*

Then for all sufficiently large N *the resulting local order of convergence of* u_h^{it} *is given by*

$$\max_{t \in I_h \setminus \{0\}} |y(t) - u_h^{it}(t)| \le C(q) N^{-(2m-\varepsilon_N)},$$

where

$$\varepsilon_N := \log_N \left(\frac{(2m \cdot \ln(N))^{2m}}{(2m+1)(m+1)^{2m}} \right)$$

satisfies

$$\lim_{N \to \infty} \varepsilon_N = 0.$$

Proof Since the proof is technically quite complex (using interpolatory projection techniques), we will only point to one of the key ingredients.

Lemma 5.3.9 *Let* I_h *be the geometric mesh defined by (5.3.43) and (5.3.44), with* $\gamma = q^{1/\rho}$. *Then:*

(i) $h_0 \le C N^{-2m/(m+1)}$;
(ii) $\sum_{n=1}^{N-1} h_n^{2m+1} \le C N^{-(2m-\varepsilon_N)}$;
(iii) For $\rho + 1 \le n \le N$ *we have* $q t_n = t_{n-\rho} \in I_h \setminus \{0\}$.

Note that (iii) may be viewed as generalised θ-invariance of this geometric mesh I_h.

Remarks

1. Geometric meshes similar to the ones employed here were introduced by Hu (1998c) for piecewise polynomial collocation methods applied to VIDEs with weakly singular kernels, to obtain local superconvergence of the collocation solution on I_h.
2. The analysis in Brunner, Hu and Lin (2001) suggests that analogous superconvergence results can be derived for collocation solutions in $S_m^{(0)}(I_h)$, with suitable geometric mesh I_h, for Volterra integro-differential equations with

vanishing proportional delays. This has not yet been worked out in detail, and hence the reader is invited to take up the challenge.

3. As Bellen (2001) has shown (see also the remark at the end of Section 5.2.5), the classical local superconvergence results on the mesh points I_h can be revovered if one switches from uniform to *quasi-geometric* meshes. We will not write down the details of this approach for second-kind VIEs with proportional delays; the reader should be able to derive them from the presentation in Sections 5.5.4 and 5.5.5.

5.4 Collocation for first-kind VIEs with proportional delays

We have seen at the beginning of the present chapter that the analysis of the existence and uniqueness of solutions becomes significantly more difficult when we move from the 'classical' first-kind integral equation in Volterra (1896a),

$$(\mathcal{V}y)(t) = g(t), \quad t \in I := [0, T], \tag{5.4.1}$$

with $K \in C^1(D),\ |K(t,t)| \geq \kappa_0 > 0,\ g \in C^1(I),\ g(0) = 0$, to the related delay integral equation

$$(\mathcal{W}_\theta y)(t) := \int_{\theta(t)}^{t} K(t,s)y(s)ds = g(t), \quad t \in I, \tag{5.4.2}$$

with lag function $\theta(t) = qt$ $(0 < q < 1)$. This is, as Volterra (1897) has described (cf. (5.1.3)), closely related to the problem of analysing the solution of the functional equation

$$y(t) - qy(qt) = f(t), \quad t \in I.$$

This increase in complexity is even more pronounced in the convergence analysis of collocation solutions for (5.4.2). Therefore, it will not come as a surprise to the reader that the convergence analysis on *uniform* I_h is not yet understood since, as we have already seen in Sections 2.4.2 and 2.4.3, even for 'classical' first-kind VIEs (5.4.1) we cannot expect uniform convergence of u_h for arbitrary $\{c_i\}$.

5.4.1 Collocation in $S_{m-1}^{(-1)}(I_h)$: uniform I_h

The collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ to (5.4.2) with $\theta(t) = qt$ $(0 < q < 1)$ is determined by

$$(\mathcal{W}_\theta u_h)(t) = g(t), \quad t \in X_h, \tag{5.4.3}$$

where, as in Section 5.3.2, the set X_h of collocation points now corresponds to collocation parameters satisfying $0 < c_1 < \ldots < c_m \leq 1$. Since we have already done our homework at the end of Section 5.3.2, the precise form of the collocation equations is already available: for $t = t_{n,i} \in X_h$ we see that the linear algebraic systems for $\mathbf{U}_n$ in the local representation (5.3.10) have the following forms:

Phase I $\quad 0 \leq n < q^I$ $(q_{n,i} = n,\ \gamma_{n,i} > 0)$
Here, (5.4.3) reduces to

$$\bar{B}_n^I(q)\mathbf{U}_n = h^{-1}\mathbf{g}_n, \tag{5.4.4}$$

where the matrix $\bar{B}_n^I(q)$ was introduced in (5.3.24); the vector $\mathbf{g}_n$ has the components $g(t_{n,i})$.

Phase II $\quad q^I \leq n < q^{II}$ $(q_{n,i} = n-1,\ \ i = 1, \ldots, \nu_n;\ \ q_{n,i} = n,\ \gamma_{n,i} > 0$ when $i > \nu_n)$
A glimpse at (5.3.25) reveals that $\mathbf{U}_n$ is now given by the system

$$\bar{B}_n^{II}(q)\mathbf{U}_n = h^{-1}\mathbf{g}_n - \bar{S}_{n-1}^{II}(q)\mathbf{U}_{n-1}, \tag{5.4.5}$$

with the matrices $\bar{B}_n^{II}(q)$ and $\bar{S}_{n-1}^{II}(q)$ as in (5.3.25).

Phase III $\quad q^{II} \leq n \leq N-1$ $(q_{n,i} = q_n < n-1,\ i = 1, \ldots, \nu_n;\ q_{n,i} = q_n + 1,\ \gamma_{n,i} > 0$ when $i > \nu_n)$
Since we have now reached the pure delay stage in the recursion, the left-hand side matrix in these linear systems coincides with the one for the 'classical' first-kind equation (5.4.1) and we obtain

$$B_n\mathbf{U}_n = h^{-1}\mathbf{g}_n - [\bar{S}_n^{III}(q)\mathbf{U}_{q_n} + \sum_{\ell=q_n+1}^{n-1} B_n^{(\ell)}\mathbf{U}_\ell + S_{q_n+1}^{III}(q)\mathbf{U}_{q_n+1}], \tag{5.4.6}$$

in complete analogy to (5.3.26) for the second-kind delay VIE of Section 5.3.3.

Example 5.4.1 $u_h \in S_0^{(-1)}(I_h)$ $(m = 1)$
Here, we have $q^I = q^{II}$. According to (5.4.3) and (5.3.24), the collocation equation of *Phase I* $(0 \leq n < q^I)$ assumes the form

$$\left(\int_{\gamma_{n,1}}^{c_1} K(t_{n,1}, t_n + sh)ds\right) y_{n+1} = h^{-1}g(t_{n,1}).$$

Since Phase II is empty, the collocation equation for *Phase III* $(n \geq q^{II} = q^I)$ is given by

$$(\mathcal{W}_\theta u_h)(t_{n,1}) = \int_{qt_{n,1}}^{t_n} K(t_{n,1}, s)u_h(s)ds + \int_{t_n}^{t_{n,1}} K(t_{n,1}, s)u_h(s)ds,$$

and this can be written as

$$\left(\int_0^{c_1} K(t_{n,1}, t_n + sh)ds\right) y_{n+1} = h^{-1}g(t_{n,1}) - \left(\int_{\gamma_{n,1}}^1 K(t_{n,1}, t_{q_{n,1}} + sh)ds\right) y_{q_{n,1}+1}$$
$$- \sum_{\ell=q_{n,1}+1}^{n-1} \left(\int_0^1 K(t_{n,1}, t_\ell + sh)ds\right) y_{\ell+1}.$$

Setting

$$B_n := \int_0^{c_1} K(t_{n,1}, t_n + sh)ds,$$
$$B_n^{(\ell)} := \int_0^1 K(t_{n,1}, t_\ell + sh)ds \quad (q_{n,1} + 1 \le \ell \le n - 1),$$

and

$$\bar{B}_{q_n}^{III}(q) := \int_{\gamma_{n,1}}^1 K(t_{n,1}, t_{q_{n,1}} + sh)ds,$$

with $q_{n,1} := \lfloor q(n + c_1) \rfloor$ and $\gamma_{n,1} := q(n + c_1) - q_{n,1}$, the above difference equation defining the values $\{y_{n+1}\}$ becomes

$$B_n y_{n+1} + \sum_{\ell=q_{n,1}+1}^{n-1} B_n^{(\ell)} y_{\ell+1} + \bar{B}_{q_n}^{III}(q) y_{q_{n,1}+1} = h^{-1}g(t_{n,1}). \tag{5.4.7}$$

If $K(t, s) \equiv 1$ the delay integral equation (5.4.3) reduces to

$$\int_{qt}^t y(s)ds = g(t), \quad t \in I \quad (g(0) = 0),$$

and this is equivalent to the functional equation

$$y(t) - qy(qt) = g'(t), \quad t \in I.$$

The corresponding collocation solution $u_h \in S_0^{(-1)}(I_h)$ is thus determined by the solution of the difference equations

$$(c_1 - \gamma_{n,1})y_{n+1} = h^{-1}g(t_{n,1}) \quad (0 \le n < q^I) \tag{5.4.8}$$

(for *Phase I*), and

$$c_1 y_{n+1} + \sum_{\ell=q_{n,1}+1}^{n-1} y_{\ell+1} + (1 - \gamma_{n,1})y_{q_{n,1}+1} = h^{-1}g(t_n + c_1 h) \quad (n \ge q^I = q^{II}) \tag{5.4.9}$$

(for Phase III; Phase II is empty). We will briefly return to this in Section 5.4.3. Note that Liu (1995b) analysed similar difference equations in the special case where $c_1 = 1$.

Table 5.4. $q = 1/2,\ c_1 = 1/2\ (q^I = q^{II} = 1)$

n	0	1	2	3	4	5	6
$q_{n,1}$	0	0	1	1	2	2	3
$\gamma_{n,1}$	1/4	3/4	1/4	3/4	1/4	3/4	1/4

Table 5.5. $q = 0.9,\ c_1 = 1/2\ (q^I = q^{II} = 5)$

n	0	1	2	3	4	5	6
$q_{n,1}$	0	1	2	3	4	4	5
$\gamma_{n,1}$	0.45	0.35	0.25	0.15	0.05	0.95	0.85

Table 5.6. $q = 1/2,\ c_1 = 1\ (q^I = q^{II} = 1)$

n	0	1	2	3	4	5	6
$q_{n,1}$	0	1	1	2	2	3	3
$\gamma_{n,1}$	1/2	0	1/2	0	1/2	0	1/2

Table 5.7. $q = 0.9,\ c_1 = 1\ (q^I = q^{II} = 9)$

n	0	1	2	3	...	8	9	10
$q_{n,1}$	0	1	2	3	...	8	9	9
$\gamma_{n,1}$	0.9	0.8	0.7	0.6	...	0.1	0.0	0.9

In order to illustrate the structure of the above difference equations corresponding to collocation at the Gauss points ($c_1 = 1/2$) we give a sample of values of $q_{n,1}$ and $\gamma_{n,1}$ (Tables 5.4 and 5.5). For comparison we also show a sample of values of $q_{n,1}$ and $\gamma_{n,1}$ for $c_1 = 1$ (Tables 5.6 and 5.7).

If the given functions K and g in (5.4.2) satisfy the hypotheses stated in Theorem 5.1.1, the existence of a unique collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ to (5.4.2), for all meshes with sufficiently small $h > 0$, can be established along the lines of the analysis in Section 2.4.1, by proceeding from Phases I and II to the 'pure delay' Phase III.

5.4.2 Convergence results for $S_{m-1}^{(-1)}(I_h)$ on uniform meshes

What can be said about the (order of) global convergence of the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ to (5.4.2)? Perhaps not surprisingly, we do not even know sufficient conditions on the $\{c_i\}$ for which u_h converges uniformly to y on I. Numerical evidence suggests that the condition

$$-1 \leq \rho_m := (-1)^m \prod_{i=1}^{m} \frac{1-c_i}{c_i} \leq 1$$

(which guarantees uniform convergence when $q = 0$ in $\mathcal{W}_\theta$ with $\theta(t) = qt$; see Section 2.4.2) is necessary but certainly no longer sufficient for uniform convergence. In particular, it is not even known for which values of $c_1 \in (0, 1]$ the solution of the simple difference equations (5.4.8) and (5.4.9) remains uniformly bounded as $N \to \infty$ $(h \to 0,\ Nh = T)$ when $q \in (0, 1)$.

5.5 VIDEs with proportional delays

5.5.1 The collocation equations and their discretisations

We now study the convergence of collocation solutions for the delay VIDE with proportional delay $\theta(t) := qt$ $(0 < q < 1)$,

$$y'(t) = f(t, y(t), y(qt)) + (\mathcal{V}y)(t) + (\mathcal{V}_\theta y)(t), \quad t \in I, \quad y(0) = y_0, \tag{5.5.1}$$

with $\mathcal{V}$ and $\mathcal{V}_\theta$ denoting the nonlinear Volterra integral operators from $C(I)$ to $C(I)$,

$$(\mathcal{V}y)(t) := \int_0^t k_1(t, s, y(s))ds, \quad t \in I,$$

and

$$(\mathcal{V}_\theta y)(t) := \int_0^{qt} k_2(t, s, y(s))ds, \quad t \in I. \tag{5.5.2}$$

The kernel functions k_i $(i = 1, 2)$ are supposed to be (Lipschitz-) continuous. Hence, the derivation and the analysis of the collocation equation defining $u_h \in S_m^{(0)}(I_h)$ for (5.5.1),

$$u_h'(t) = f(t, u_h(t), u_h(qt)) + (\mathcal{V}u_h)(t) + (\mathcal{V}_\theta u_h)(t), \quad t \in I, \quad u_h(0) = y_0, \tag{5.5.3}$$

are by now straightforward, since they will be based on the machinery introduced in Sections 5.2. and 5.3. For ease of exposition we will often resort to the linear version of (5.5.1), because it captures most of the essential features.

This delay VIDE is

$$y'(t) = a(t)y(t) + b(t)y(qt) + (\mathcal{V}y)(t) + (\mathcal{V}_\theta y)(t), \quad t \in I. \tag{5.5.4}$$

where $a, b \in C(I)$. The Volterra integral operators $\mathcal{V}$ and $\mathcal{V}_\theta$ are the linear counterparts of the above nonlinear operators, as defined at the beginning of Section 5.3. It will again be assumed that the kernels K_1 and K_2 defining these linear integral operators $\mathcal{V}$ and $\mathcal{V}_\theta$ are continuous on their respective domains D and D_θ.

Suppose now that the mesh I_h is *uniform* and that the local representation of $u_h \in S_m^{(0)}(I_h)$ on the subinterval $\bar{\sigma}_n$ is

$$u_h(t_n + vh_n) = y_n + h_n \sum_{j=1}^{m} \beta_j(v) Y_{n,j}, \quad v \in [0, 1], \tag{5.5.5}$$

where $y_n := u_h(t_n)$ and $Y_{n,j} := u_h'(t_{n,j})$ (cf. (1.1.5)). Recall the notation introduced in Sections 5.2.1 and 5.3.1.:

$$q_{n,i} := \lfloor q(n + c_i) \rfloor \in \mathbb{N}_0, \quad \gamma_{n,i} := q(n + c_i) - q_{n,i} \in [0, 1).$$

Thus, at the collocation points $t_{n,i} := t_n + c_i h$ $(i = 1, \ldots, m)$ the collocation equation (5.5.3) assumes the form

$$\begin{aligned} Y_{n,i} = \; & f(t_{n,i}, y_n + h \sum_{j=1}^{m} a_{i,j} Y_{n,j}, y_{q_{n,i}} + h \sum_{j=1}^{m} \beta_j(\gamma_{n,i}) Y_{q_{n,i},j}) \\ & + F_n(t_{n,i}) + h \int_0^{c_i} k_1(t_{n,i}, t_n + sh, y_n + h \sum_{j=1}^{m} \beta_j(s) Y_{n,j}) ds \\ & + (V_\theta u_h)(t_{n,i}) \quad (i = 1, \ldots, m). \end{aligned} \tag{5.5.6}$$

The lag term associated with the Volterra operator $\mathcal{V}$ is

$$\begin{aligned} F_n(t_{n,i}) &:= \int_0^{t_n} k_1(t_{n,i}, s, u_h(s)) ds \\ &= h \sum_{\ell=0}^{n-1} \int_0^1 k_1(t_{n,i}, t_\ell + sh, y_\ell + h \sum_{j=1}^{m} \beta_j(s) Y_{\ell,j}) ds, \end{aligned}$$

and the term $(\mathcal{V}_\theta u_h)(t_{n,i})$ has the form

$$\begin{aligned} (\mathcal{V}_\theta u_h)(t_{n,i}) &= \int_0^{q t_{n,i}} k_2(t_{n,i}, s, u_h(s)) ds \\ &= Q_{q_{n,i}}(t_{n,i}) \\ &\quad + h \int_0^{\gamma_{n,i}} k_2(t_{n,i}, t_{q_{n,i}} + sh, y_{q_{n,i}} + h \sum_{j=1}^{m} \beta_j(s) Y_{q_{n,i},j}) ds, \end{aligned} \tag{5.5.7}$$

with

$$Q_{q_{n,i}}(t_{n,i}) := h \sum_{\ell=0}^{q_{n,i}-1} \int_0^1 k_2(t_{n,i}, t_\ell + sh, y_\ell + h \sum_{j=1}^{m} \beta_j(s) Y_{\ell,j}) ds. \quad (5.5.8)$$

The description of the three phases in the computation of u_h for the nonlinear VIDE (5.5.1) of course closely resembles the one in Section 5.3.1.

(I) *Initial phase (complete overlap)* $0 \le n < \lceil qc_1/(1-q) \rceil =: q^I$.
Employing the local representation (5.5.5) and letting $t = t_{n,i} := t_n + c_i h$ in (5.5.3), we find that the system of nonlinear algebraic equations (5.5.6) for $\mathbf{Y}_n := (Y_{n,1}, \dots, Y_{n,m})^T$ has the form

$$\begin{aligned} Y_{n,i} = {} & f(t_{n,i}, y_n + h \sum_{j=1}^{m} a_{i,j} Y_{n,j}, y_n + h \sum_{j=1}^{m} \beta_j(\gamma_{n,i}) Y_{n,j}) \\ & + F_n(t_{n,i}) + h \int_0^{c_i} k_1(t_{n,i}, t_n + sh, y_n + h \sum_{j=1}^{m} \beta_j(s) Y_{n,j}) ds \\ & + Q_n(t_{n,i}) + h \int_0^{\gamma_{n,i}} k_2(t_{n,i}, t_n + sh, y_n + h \sum_{j=1}^{m} \beta_j(s) Y_{n,j}) ds \end{aligned} \quad (5.5.9)$$

($i = 1, \dots, m$), since $q_{n,i} = n$ for $i = 1, \dots, m$. The lag terms are defined above, where $q_{n,i}$ in $Q_{q_{n,i}}(t_{n,i})$ now assumes the value n for all i.

(II) *Transition phase (partial overlap)* $q^I \le n < \lceil qc_m/(1-q) \rceil =: q^{II}$.
If this set of values of n is not empty there is an integer $\nu_n \in \{1, \dots, m-1\}$ so that

$$q_{n,i} = n - 1 \ \text{ for } i = 1, \dots, \nu_n \quad \text{and} \quad q_{n,i} = n, \ \gamma_{n,i} > 0$$
$$\text{for } i = \nu_n + 1, \dots, m.$$

The collocation equation (5.5.6) assumes the separated form described by

$$\begin{aligned} Y_{n,i} = {} & f(t_{n,i}, y_n + h \sum_{j=1}^{m} a_{i,j} Y_{n,i}, y_{n-1} + h \sum_{j=1}^{m} \beta_j(\gamma_{n,i}) Y_{n-1,j}) \\ & + F_n(t_{n,i}) + h \int_0^{c_i} k_1(t_{n,i}, t_n + sh, y_n + h \sum_{j=1}^{m} \beta_j(s) Y_{n,j}) ds \\ & + Q_{n-1}(t_{n,i}) + h \int_0^{\gamma_{n,i}} k_2(t_{n,i}, t_{n-1} + sh, y_{n-1} + h \sum_{j=1}^{m} \beta_j(s) Y_{n-1,j}) ds \\ & (i = 1, \dots, \nu_n), \end{aligned} \quad (5.5.10)$$

and

$$\begin{aligned} Y_{n,j} &= f(t_{n,i}, y_n + h\sum_{j=1}^{m} a_{i,j}Y_{n,j}, y_n + h\sum_{j=1}^{m} \beta_j(\gamma_{n,i})Y_{n,j}) \\ &\quad + F_n(t_{n,i}) + h\int_0^{c_i} k_1(t_{n,i}, t_n + sh, y_n + h\sum_{j=1}^{m} \beta_j(s)Y_{n,j})ds \\ &\quad + Q_n(t_{n,i}) + h\int_0^{\gamma_{n,i}} k_2(t_{n,i}, t_n + sh, y_n + h\sum_{j=1}^{m} \beta_j(s)Y_{n,j})ds \\ &\qquad (i = \nu_n + 1, \ldots, m). \end{aligned} \tag{5.5.11}$$

(III) *Pure delay phase (no overlap)* $q^{II} \leq n \leq N-1$.
Assume that for given n we have

$$q_{n,i} = q_n \ (i = 1, \ldots, \nu_n) \quad \text{and} \quad q_{n,i} = q_n + 1 < n, \ \gamma_{n,i} > 0,$$
$$(i = \nu_n + 1, \ldots, m)$$

for some integer ν_n with $1 \leq \nu_n \leq m$. The resulting system of nonlinear algebraic equations corresponding to (5.5.6) is now given by the sets of equations

$$\begin{aligned} Y_{n,i} &= f(t_{n,i}, y_n + h\sum_{j=1}^{m} a_{i,j}Y_{n,j}, y_{q_n} + h\sum_{j=1}^{m} \beta_j(\gamma_{n,i})Y_{q_n,j}) \\ &\quad + F_n(t_{n,i}) + h\int_0^{c_i} k_1(t_{n,i}, t_n + sh, y_n + h\sum_{j=1}^{m} \beta_j(s)Y_{n,j})ds \\ &\quad + Q_{q_n}(t_{n,i}) + h\int_0^{\gamma_{n,i}} k_2(t_{n,i}, t_{q_n} + sh, y_{q_n} + h\sum_{j=1}^{m} \beta_j(s)Y_{q_n,j})ds, \\ &\qquad (i = 1, \ldots, \nu_n), \end{aligned} \tag{5.5.12}$$

and

$$\begin{aligned} Y_{n,i} &= f(t_{n,i}, y_n + h\sum_{j=1}^{m} a_{i,j}Y_{n,j}, y_{q_n+1} + h\sum_{j=1}^{m} \beta_j(\gamma_{n,i})Y_{q_n+1,j}) \\ &\quad + F_n(t_{n,i}) + h\int_0^{c_i} k_1(t_{n,i}, t_n + sh, y_n + h\sum_{j=1}^{m} \beta_j(s)Y_{n,j})ds \\ &\quad + Q_{q_n+1}(t_{n,i}) + h\int_0^{\gamma_{n,i}} k_2(t_{n,i}, t_{q_n+1} + sh, y_{q_n+1} + h\sum_{j=1}^{m} \beta_j(s)Y_{q_n+1,j})ds \\ &\qquad (i = \nu_n + 1, \ldots, m). \end{aligned} \tag{5.5.13}$$

In order to understand the precise structure of these (seemingly) rather complex algebraic systems, and to prepare the ground for the analysis of the collocation error, we will use the *linear* VIDE with proportional delay (5.5.4) to make the above collocation equations more transparent. The corresponding systems of linear algebraic equations describing the three phases of the computational form of the collocation equation are presented below. The reader may find it instructive to compare these systems with those corresponding to 'classical' linear Volterra integro-differential equations (Section 3.2.1) and the ones encountered in Section 5.3.1 (cf. (5.3.12)–(5.3.14)).

We first study the contributions arising from the classical (non-delay) VIDE part, $a(t)u_h(t) + (\mathcal{V}u_h)(t)$, and the delay part, $b(t)u_h(t) + (\mathcal{V}_\theta u_h)(t)$, separately. Since the former was studied in Section 3.2, we can be brief: recall that, for all values of $n = 0, 1, \ldots, N-1$,

$$\begin{aligned} &a(t_{n,i})u_h(t_{n,i}) + (\mathcal{V}u_h)(t_{n,i}) \\ &= ha(t_{n,i})\sum_{j=1}^{m} a_{i,j}Y_{n,j} + h^2\sum_{j=1}^{m}\left(\int_0^{c_i} K_1(t_{n,i}, t_n + sh)\beta_j(s)ds\right)Y_{n,j} \\ &\quad + h^2\sum_{\ell=0}^{n-1}\sum_{j=1}^{m}\left(\int_0^1 K_1(t_{n,i}, t_\ell + sh)\beta_j(s)ds\right)Y_{\ell,j} \\ &\quad + \left(a(t_{n,i}) + h\int_0^{c_i} K_1(t_{n,i}, t_n + sh)ids\right)y_n \\ &\quad + h\sum_{\ell=0}^{n-1}\left(\int_0^1 K_1(t_{n,i}, t_\ell + sh)ds\right)y_\ell. \end{aligned} \tag{5.5.14}$$

Thus, the resulting contribution to the linear system for $\mathbf{Y}_n$ coming from the non-delay terms in the collocation equation (5.5.6) for the linear VIDE (5.5.4) is given by

$$h(A_n + hC_n)\mathbf{Y}_n + h^2\sum_{\ell=0}^{n-1} C_n^{(\ell)}\mathbf{Y}_\ell + (\mathbf{r}_n + h\kappa_n)y_n + h\sum_{\ell=0}^{n-1}\kappa_n^{(\ell)}y_\ell, \tag{5.5.15}$$

where the matrices $A_n,\ C_n,\ C_n^{(\ell)}\ (\ell < n)$ and the vectors $\mathbf{r}_n,\ \kappa_n,\ \kappa_n^{(\ell)}\ (\ell < n)$ were introduced in Section 3.2.1 (cf. (3.2.9)).

Consider now the contribution due to the delay terms $D_{n,i} := b(t_{n,i})u_h(qt_{n,i}) + (\mathcal{V}_\theta u_h)(t_{n,i})$: We first note that

$$\begin{aligned} (\mathcal{V}_\theta e_h)(t_{n,i}) &= h\sum_{\ell=0}^{q_{n,i}-1}\int_0^1 K_2(t_{n,i}, t_\ell + sh)\left(y_\ell + h\sum_{j=1}^{m}\beta_j(s)Y_{\ell,j}\right)ds \\ &\quad + h\int_0^{\gamma_{n,i}} K_2(t_{n,i}, t_{q_{n,i}} + sh)\left(y_{q_{n,i}} + h\sum_{j=1}^{m}\beta_j(s)Y_{q_{n,i},j}\right)ds. \end{aligned} \tag{5.5.16}$$

Hence,

$$
\begin{aligned}
D_{n,i} &= hb(t_{n,i})\sum_{j=1}^{m}\beta_j(\gamma_{n,i})Y_{q_{n,i}} + h^2\sum_{j=1}^{m}\left(\int_0^{\gamma_{n,i}} K_2(t_{n,i}, t_{q_{n,i}} + sh)\beta_j(s)ds\right)Y_{q_{n,i},j} \\
&\quad + h^2\sum_{\ell=0}^{q_{n,i}-1}\sum_{j=1}^{m}\left(\int_0^1 K_2(t_{n,i}, t_\ell + sh)\beta_j(s)ds\right)Y_{\ell,j} \\
&\quad + \left(b(t_{n,i}) + h\int_0^{\gamma_{n,i}} K_2(t_{n,i}, t_{q_{n,i}} + sh)ds\right)y_{q_{n,i}} \\
&\quad + h\sum_{\ell=0}^{q_{n,i}-1}\left(\int_0^1 K_2(t_{n,i}, t_\ell + sh)ds\right)y_\ell. \qquad (5.5.17)
\end{aligned}
$$

The precise structure of the corresponding matrices and vectors will now of course depend on the value of $q_{n,i}$: We have $q_{n,i} = n$ in Phase I; $q_{n,i} \in \{n-1, n\}$ (Phase II); and $q_{n,i} \in \{q_n, q_n + 1\}$, with $q_n < n - 1$ (Phase III). Thus, let $\nu_n \in \{0, 1, \ldots, m\}$ be such that, for given n,

$$
q_{n,i} = \begin{cases} q_n & \text{for } \; i = 1, \ldots, \nu_n \\ q_n + 1 \; (\gamma_{n,i} > 0) & \text{for } \; i = \nu_n + 1, \ldots, m. \end{cases}
$$

The three phases I–III are then characterised by

$$
q_{n,i} = \begin{cases} q_n + 1 = n & (\nu_n = 0): \quad \text{Phase I} \\ q_n = n - 1 & (1 \le \nu_n < m): \quad \text{Phase II} \\ q_n < n - 1 & (1 \le \nu_n \le m): \quad \text{Phase III.} \end{cases} \qquad (5.5.18)
$$

If we are in Phase II or Phase III, the equations (5.5.17) will in general split into two separated forms: for $q_{n,i} = q_n$ $(i = 1, \ldots, \nu_n)$ we obtain

$$
\begin{aligned}
D_{n,i} &= hb(t_{n,i})\sum_{j=1}^{m}\beta_j(\gamma_{n,i})Y_{q_n,j} + h^2\sum_{j=1}^{m}\left(\int_0^{\gamma_{n,i}} K_2(t_{n,i}, t_{q_n} + sh)\beta_j(s)ds\right)Y_{q_n,j} \\
&\quad + h^2\sum_{\ell=0}^{q_n-1}\sum_{j=1}^{m}\left(\int_0^1 K_2(t_{n,i}, t_\ell + sh)\beta_j(s)ds\right)Y_{\ell,j} \\
&\quad + \left(b(t_{n,i}) + h\int_0^{\gamma_{n,i}} K_2(t_{n,i}, t_{q_n} + sh)ds\right)y_{q_n} \\
&\quad + h\sum_{\ell=0}^{q_n-1}\left(\int_0^1 K_2(t_{n,i}, t_\ell + sh)ds\right)y_\ell \quad (i = 1, \ldots, \nu_n), \qquad (5.5.19)
\end{aligned}
$$

while for $q_{n,i} = q_n + 1$ $(i = \nu_n + 1, \ldots, m$, with $\gamma_{n,i} > 0)$, we have

$$
D_{n,i} = hb(t_{n,i})\sum_{j=1}^{m}\beta_j(\gamma_{n,i})Y_{q_n+1,j}
$$

$$+h^2 \sum_{j=1}^{m} \left(\int_0^{\gamma_{n,i}} K_2(t_{n,i}, t_{q_n+1} + sh)\beta_j(s)ds \right) Y_{q_n+1,j}$$
$$+h^2 \sum_{\ell=0}^{q_n-1} \sum_{j=1}^{m} \left(\int_0^1 K_2(t_{n,i}, t_\ell + sh)\beta_j(s)ds \right) Y_{\ell,j}$$
$$+h^2 \sum_{j=1}^{m} \left(\int_0^1 K_2(t_{n,i}, t_{q_n} + sh)\beta_j(s)ds \right) Y_{q_n,j}$$
$$+ \left(b(t_{n,i}) + h \int_0^{\gamma_{n,i}} K_2(t_{n,i}, t_{q_n+1} + sh)ds \right) y_{q_n+1}$$
$$+h \sum_{\ell=0}^{q_n-1} \left(\int_0^1 K_2(t_{n,i}, t_\ell + sh)ds \right) y_\ell + h \int_0^1 i \left(K_2(t_{n,i}, t_{q_n} + sh)ds \right) y_{q_n}. \tag{5.5.20}$$

Hence, the right-hand sides of these two equations (5.5.19) and (5.5.20) may be written concisely as

$$(D_{n,i}) = h\hat{A}_n^{\{*\}}(q)\mathbf{Y}_{q_n} + hA_n^{\{*\}}(q)\mathbf{Y}_{q_n+1} + h^2\hat{S}_{q_n}^{\{*\}}(q)\mathbf{Y}_{q_n} + S_{q_n+1}^{\{*\}})(q)\mathbf{Y}_{q_n+1}$$
$$+ \hat{\mathbf{r}}_n^{\{*\}}(q)y_{q_n} + \mathbf{r}_n^{\{*\}}(q)y_{q_n+1} + h[\hat{\boldsymbol{\kappa}}_{q_n}^{\{*\}}(q)y_{q_n} + \boldsymbol{\kappa}_{q_n+1}^{\{*\}}(q)y_{q_n+1}],$$

where $\{*\}$ stands for I, II or III, depending on the value of n. We are now ready to describe the linear algebraic systems corresponding to the three phases; the definitions of the above matrices and vectors will then also become clear.

(I) *Initial phase* $\quad 0 \le n < q^I$.

Combining (5.5.13) and (5.5.19) (with $q_n + 1 = n$ ($\nu_n = 0$) and $\{*\} = \{I\}$), the linear algebraic system corresponding to (5.5.6) may then be written as

$$[\mathcal{I}_m - h(A_n + A_n^I(q)) - h^2(C_n + C_n^I(q))]\mathbf{Y}_n$$
$$= h^2 \sum_{\ell=0}^{n-1} (C_n^{(\ell)} + C_n^{(\ell)}(q))\mathbf{Y}_\ell + (\mathbf{r}_n + \mathbf{r}_n^I(q))y_n$$
$$+h \sum_{\ell=0}^{n-1} (\boldsymbol{\kappa}_n^{(\ell)} + \boldsymbol{\kappa}_n^{(\ell)}(q))y_\ell + h(\boldsymbol{\kappa}_n + \boldsymbol{\kappa}_n^I(q)). \tag{5.5.21}$$

where

$$A_n^I(q) := \mathrm{diag}(b(t_{n,i}))(\beta_j(\gamma_{n,i})),$$
$$C_n^I(q) := \begin{pmatrix} \int_0^{\gamma_{n,i}} K_2(t_{n,i}, t_n + sh)\beta_j(s)ds \\ (i, j = 1, \ldots, m) \end{pmatrix},$$
$$C_n^{(\ell)}(q) := \begin{pmatrix} \int_0^1 K_2(t_{n,i}, t_\ell + sh)\beta_j(s)ds \\ (i, j = 1, \ldots, m) \end{pmatrix} \quad (\ell < n)$$

(the latter matrices are the analogues of the matrices $B_n^I(q)$ and $B_n^{(\ell)}(q)$ of Section 5.3.1), and

$$\kappa_{n,i}^I(q) := \int_0^{\gamma_{n,i}} K_2(t_{n,i}, t_n + sh)ds,$$
$$\kappa_{n,i}^{(\ell)}(q) := \int_0^1 K_2(t_{n,i}, t_\ell + sh)ds \ (\ell < n).$$

(II) *Transition phase* $q^I \leq n < q^{II}$.
If this set is not empty we now have $q_n = n - 1$ $(i = 1, \ldots, \nu_n)$ and $q_n + 1 = n$ $(i = \nu_n + 1, \ldots, m)$. The analogue of the above linear system (5.5.21) for Phase II is

$$\begin{aligned}
&[\mathcal{I}_m - h(A_n + A_n^{II}(q)) - h^2(C_n + C_n^{II}(q))]\mathbf{Y}_n \\
&\quad = h^2 \sum_{\ell=0}^{n-1} C_n^{(\ell)}\mathbf{Y}_\ell + h^2 \sum_{\ell=0}^{n-2} C_n^{(\ell)}(q)\mathbf{Y}_\ell \\
&\qquad + h\hat{A}_n^{II}(q)\mathbf{Y}_{n-1} + h^2]\hat{S}_{n-1}^{II}(q) + S_n^{II}(q)]\mathbf{Y}_{n-1} \\
&\qquad + \mathbf{r}_n y_n + \hat{\mathbf{r}}_n^{II}(q)y_{n-1} + \mathbf{r}_n^{II}(q)y_n) \\
&\qquad + h\sum_{\ell=0}^{n-1} \boldsymbol{\kappa}_n^{(\ell)} y_\ell + h\boldsymbol{\kappa}_n y_n + h\sum_{\ell=0}^{n-2} \boldsymbol{\kappa}_n^{(\ell)}(q)y_\ell + h(\hat{\boldsymbol{\kappa}}_n^{II}(q)y_{n-1} + \boldsymbol{\kappa}_n^{II}(q)y_n.
\end{aligned} \tag{5.5.22}$$

Here, we have introduced the matrices in $L(\mathbb{R}^m)$,

$$A_n^{II}(q) := \text{diag}(\underbrace{0, \ldots, 0}_{\nu_n}, 1, \ldots, 1)A_n^I(q),$$
$$\hat{A}_n^{II}(q) := \text{diag}(\underbrace{1, \ldots, 1}_{\nu_n}, 0, \ldots, 0)A_n^I(q),$$
$$C_n^{II}(q) := \text{diag}(\underbrace{0, \ldots, 0}_{\nu_n}, 1, \ldots, 1)C_n^I(q),$$

as well as

$$\begin{aligned}
\hat{S}_{n-1}^{II}(q) &:= \text{diag}(\underbrace{1, \ldots, 1}_{\nu_n}, 0, \ldots, 0)C_n^I(q) \\
&\quad + \text{diag}(\underbrace{0, \ldots, 0}_{\nu_n}, 1, \ldots, 1)C_n^{(n-1)}(q),
\end{aligned}$$
$$S_n^{II}(q) := \text{diag}(\underbrace{0, \ldots, 0}_{\nu_n}, 1, \ldots, 1)C_n^{(n-1)}(q),$$

and the vectors

$$\hat{\mathbf{r}}_n^{II}(q) := \mathrm{diag}(\underbrace{1,\ldots,1}_{\nu_n},0,\ldots,0)\mathbf{r}_n^I(q),$$

$$\mathbf{r}_n^{II}(q) := \mathrm{diag}(\underbrace{0,\ldots,0}_{\nu_n},1,\ldots,1)\mathbf{r}_n^I(q),$$

$$\hat{\boldsymbol{\kappa}}_{n-1}^{II}(q) := \mathrm{diag}(\underbrace{1,\ldots,1}_{\nu_n},0,\ldots,0)\Big(\int_0^{\gamma_{n,i}} K_2(t_{n,i},t_{n-1}+sh)ds$$

$$+\int_0^1 K_2(t_{n,i},t_{n-1}+sh)ds \;\; (i=1,\ldots,m)\Big)^T,$$

$$\boldsymbol{\kappa}_n^{II}(q) := \mathrm{diag}(\underbrace{0,\ldots,0}_{\nu_n},1,\ldots,1)\Big(\int_0^{\gamma_{n,i}} K_2(t_{n,i},t_n+sh)ds \; (i=1,\ldots,m)\Big)^T.$$

(III) *Pure delay phase* $q^{II} \le n \le N-1$.
We have now reached the stage where $q_{n,i} = q_n < n-1$ $(i = 1,\ldots,\nu_n)$ and $q_{n,i} = q_n + 1,\ \gamma_{n,i} > 0$ $(i = \nu_n + 1,\ldots,m)$. Hence, the resulting system of linear algebraic equations for $\mathbf{Y}_n$ is given by

$$\begin{aligned}
&[\mathcal{I}_m - hA_n - h^2C_n]\mathbf{Y}_n \\
&= h^2\sum_{\ell=0}^{n-1} C_n^{(\ell)}\mathbf{Y}_\ell + h^2\sum_{\ell=0}^{q_n-1} C_n^{(\ell)}(q)\mathbf{Y}_\ell \\
&\quad + h[\hat{A}_n^{III}(q)\mathbf{Y}_{q_n} + A_n^{III}(q)\mathbf{Y}_{q_n+1}] + h^2[\hat{S}_{q_n}^{III}(q)\mathbf{Y}_{q_n} + S_{q_n+1}^{III}(q)\mathbf{Y}_{q_n+1} \\
&\quad + \mathbf{r}_n y_n + \hat{\mathbf{r}}_n^{III}(q)y_{q_n} + \mathbf{r}_n^{III}(q)y_{q_n+1} + h\sum_{\ell=0}^{n-1}\boldsymbol{\kappa}_n^{(\ell)}y_\ell + h\boldsymbol{\kappa}_n y_n \\
&\quad + h\sum_{\ell=0}^{q_n-1}\boldsymbol{\kappa}_n^{(\ell)}(q)y_\ell + h[(\hat{\boldsymbol{\kappa}}_{q_n}^{III}(q)y_{q_n} + \boldsymbol{\kappa}_{q_n+1}^{III}(q)y_{q_n+1}].
\end{aligned} \tag{5.5.23}$$

We refrain from writing down the by now self-explanatory meanings of the matrices and vectors describing the above linear algebraic system (5.5.23): a brief look at (5.2.15) and the explicit equations (5.5.14) and (5.5.20) will help the reader readily to do this.

Theorem 5.5.1 *Assume that a, b and K_1, K_2 in (5.5.4) are continuous on their respective domains I, D and D_θ. Then there exists an $\bar{h} > 0$ (depending on q) so that for every uniform mesh I_h with $h \in (0,\bar{h})$ the linear algebraic systems (5.5.21)–(5.5.23) have unique solutions $\mathbf{Y}_n$ for any $q \in (0,1)$; that is, the collocation equation (5.5.3) corresponding to the linear DVIDE (5.5.4) defines a unique collocation solution $u_h \in S_m^{(0)}(I_h)$ which on $\bar{\sigma}_n$ is described by (5.5.5).*

Illustration

If $K_2 = -K_1 =: -K$, the (linear) delay VIDE (5.5.4) becomes

$$y'(t) = a(t)y(t) + b(t)y(qt) + (\mathcal{W}_\theta y)(t), \quad t \in I := [0, T], \tag{5.5.24}$$

where the delay operator $\mathcal{W}_\theta$ is as in (5.4.2). In the corresponding collocation equation for $u_h \in S_m^{(0)}(I_h)$,

$$u_h'(t) = a(t)u_h(t) + b(t)u_h(qt) + (\mathcal{W}_\theta u_h)(t), \quad t \in X_h, \tag{5.5.25}$$

the term $(\mathcal{W}_\theta u_h)(t)$ for $t = t_{n,i}$ assumes the forms

$$\begin{aligned}
(\mathcal{W}_\theta u_h)(t_{n,i}) &= \int_{qt_{n,i}}^{t_{n,i}} K(t_{n,i}, s)u_h(s)ds \\
&= h^2 \sum_{j=1}^{m} \left(\int_{\gamma_{n,i}}^{1} K(t_{n,i}, t_{q_n} + sh)\beta_j(s)ds \right) Y_{q_n,j} \\
&\quad + h^2 \sum_{\ell=q_n+1}^{n-1} \sum_{j=1}^{m} \left(\int_0^1 K(t_{n,i}, t_\ell + sh)\beta_j(s)ds \right) Y_{\ell,j} \\
&\quad + h^2 \sum_{j=1}^{m} \left(\int_0^{c_i} K(t_{n,i}, t_n + sh)\beta_j(s)ds \right) Y_{n,j} \\
&\quad + h \int_{\gamma_{n,i}}^{1} \big(K(t_{n,i}, t_{q_n} + sh)ds\big)\, y_{q_n} \\
&\quad + h \sum_{\ell=q_n+1}^{n-1} \int_0^1 \big(K(t_{n,i}, t_\ell + sh)ds\big)\, y_\ell \\
&\quad + h \int_0^{c_i} \big(K(t_{n,i}, t_n + sh)ds\big)\, y_n,
\end{aligned}$$

when we have $qt_{n,i} \le t_n$ (that is, $q_{n,i} = q_n < n$). If $qt_{n,i} > t_n$ (which is true during Phase I and part of Phase II), the above expression reduces to

$$\begin{aligned}
(\mathcal{W}_\theta u_h)(t_{n,i}) &= h^2 \sum_{j=1}^{m} \left(\int_{\gamma_{n,i}}^{c_i} K(t_{n,i}, t_n + sh)\beta_j(s)ds \right) Y_{n,j} \\
&\quad + h \int_{\gamma_{n,i}}^{c_i} \big(K(t_{n,i}, t_n + sh)ds\big)\, y_n.
\end{aligned}$$

This allows us to write down the linear algebraic systems for Y_n corresponding to Phase I, II, and III (cf. (5.5.22)–(5.5.24) with $K_2 = -K_1 =: -K$), and the collocation solution u_h on any subinterval σ_n is then given by the local representation (5.5.5). The extension of the above equation and remarks to the nonlinear analogue of (5.5.25),

$$y'(t) = f(t, y(t), y(qt)) + \int_{qt}^{t} k(t, s, y(s)ds, \quad t \in I := [0, T], \tag{5.5.26}$$

which formally corresponds to (5.5.1) with $k_2 = -k_1$, is obvious.

5.5.2 Convergence results on uniform meshes

Consider first the linear VIDE

$$\begin{aligned} y'(t) &= a(t)y(t) + b(t)y(qt) + (\mathcal{V}y)(t) + (\mathcal{V}_\theta y)(t), \quad t \in I = [0, T], \\ y(0) &= y_0, \end{aligned} \tag{5.5.27}$$

with the linear Volterra integral operators given by

$$(\mathcal{V}y)(t) := \int_0^t K_1(t, s)y(s)ds, \quad (\mathcal{V}_\theta y)(t) := \int_0^{\theta(t)} K_2(t, s)y(s)ds,$$

and $\theta(t) := qt,\ 0 < q < 1$.

Theorem 5.5.2 *Assume:*

(a) The given functions in (5.5.27) are sufficiently regular: $a, b \in C^m(I),\ K_1 \in C^m(D)$, and $K_2 \in C^m(D_\theta)$.
(b) For given uniform mesh I_h, $u_h \in S_m^{(0)}(I_h)$ is the collocation solution to (5.5.27).

Then for all $h \in (0, \bar{h})$ and any set $\{c_i\}$ of distinct collocation parameters in $[0, 1]$ *the collocation error $e_h := y - u_h$ satisfies*

$$||e_h^{(\nu)}||_\infty \leq C_\nu ||y^{(m+1)}||_\infty h^m \quad (\nu = 0, 1), \tag{5.5.28}$$

with constants C_ν depending on the $\{c_i\}$ but not on h. This estimate is true for any $q \in (0, 1)$.

Proof Consider the equation satisfied by the collocation error $e_h := y - u_h$,

$$e_h'(t) = a(t)e_h(t) + (\mathcal{V}e_h)(t) + (\mathcal{V}_\theta e_h)(t), \quad t \in X_h, \tag{5.5.29}$$

and set $t = t_{n,i} := t_n + c_i h$: using the local Peano representation of e_h,

$$e_h(t_n + vh) = e_h(t_n) + h \sum_{j=1}^{m} \beta_j(v)\mathcal{E}_{n,j} + h^{m+1} R_{m+1,n}(v), \quad v \in [0, 1] \tag{5.5.30}$$

(cf. (5.2.18)), with $\mathcal{E}_{n,j} := e_h'(t_{n,i})$, the contribution of the delay term,

$$\Delta_{n,i} := b(t_{n,i})e_h(t_{n,i}) + (\mathcal{V}_\theta e_h)(t_{n,i}),$$

to the error equation is given by

$$\begin{aligned}\Delta_{n,i} =& hb(t_{n,i})\sum_{j=1}^{m}\beta_j(\gamma_{n,i})\mathcal{E}_{n,i}\\ &+h^2\sum_{j=1}^{m}\left(\int_0^{\gamma_{n,i}} K_2(t_{n,i}, t_{q_{n,i}}+sh)\beta_j(s)ds\right)\mathcal{E}_{q_{n,i},j}\\ &+h^2\sum_{\ell=0}^{q_{n,i}-1}\sum_{j=1}^{m}\left(\int_0^1 K_2(t_{n,i}, t_\ell+sh)\beta_j(s)ds\right)\mathcal{E}_{\ell,j}\\ &+b(t_{n,i})e_h(t_{q_{n,i}}) + h\int_0^{\gamma_{n,i}}\left(K_2(t_{n,i}, t_{q_{n,i}}+sh)ds\right)e_h(t_{q_{n,i}})\\ &+h\sum_{\ell=0}^{q_{n,i}-1}\left(\int_0^1 K_2(t_{n,i}, t_\ell+sh)ds\right)e_h(t_\ell)\\ &+b(t_{n,i})h^{m+1}R_{m+1,q_{n,i}}(\gamma_{n,i}) + h^{m+2}\int_0^{\gamma_{n,i}} K_2(t_{n,i}, t_{q_{n,i}}+sh)R_{m+1,q_{n,i}}(s)ds\\ &+h^{m+2}\sum_{\ell=0}^{q_{n,i}-1}\int_0^1 K_2(t_{n,i}, t_\ell+sh)R_{m+1,\ell}(s)ds \qquad (5.5.31)\end{aligned}$$

(compare with (5.5.17)). As described in detail in Section 5.5.1, the value of $q_{n,i}$ depends crucially on that of n and the corresponding phase: it is $q_{n,i} = n$ ($\gamma_{n,i} > 0$) in Phase I; $q_{n,i} \in \{n-1, n\}$ in Phase II (which may be empty); and $q_{n,i} \in \{q_n, q_n+1\}$, with $q_n < n-1$, in Phase III. This leads to the three sets of linear algebraic systems, in complete analogy to Section 5.2.2 and Section 5.3.2. The proof of Theorem 5.5.2 is then achieved in a by now familar way. In order not to become overly repetitive, we leave the details of the precise structure of these algebraic systems and the derivation of the corresponding generalised discrete Gronwall inequalities for $||\mathcal{E}_n||_1$, and hence the completion of the proof of Theorem 5.5.2, to the reader.

The proof of Theorem 5.2.2 shows that Theorem 5.5.3 can be modified in a straightforward way to derive optimal global error estimates for solutions of (5.5.4) possessing a lower degree of regularity:

Theorem 5.5.3 *If $y \in C^{d+1}(I)$ with $1 \le d < m$ (corresponding to the assumption that $a, b \in C^d(I)$, $K_1 \in C^d(D)$, $K_2 \in C^d(D_\theta)$), then the optimal estimates (5.5.28) have to be modified to read*

$$||y^{(\nu)} - u_h^{(\nu)}||_\infty \le C_d||y^{(d+1)}||_\infty h^d \quad (\nu = 0, 1). \qquad (5.5.32)$$

The key to the proof is again Peano's Kernel Theorem and the corresponding local error representations (5.5.32) and (5.5.31), with $d + 1$ replacing $m + 1$.

Remark Both Theorem 5.5.2 and the above observation regarding the order of global convergence in the case of lower regularity in y include the special delay VIDE (5.5.26),

$$y'(t) = a(t)y(t) + b(t)y(qt) + (\mathcal{W}_\theta y)(t),$$

as well as its nonlinear counterpart

$$y'(t) = f(t, y(t)), y(qt)) + \int_{qt}^{t} k(t, s, y(s))ds. \qquad (5.5.33)$$

Do the global and local superconvergence results we derived for classical VIDEs (Sections 3.2.3 and 3.2.4) and VIDEs with *non-vanishing delays* (Sections 4.5.2 and 4.5.3) remain valid for VIDEs with vanishing proportional delay? From what we have seen in this chapter, the answer will likely be in the affirmative for global convergence (as shown in Theorem 5.5.4). Regarding local superconvergence on I_h the answer appears to be no if $m > 2$ (Conjecture 5.5.5), and this is supported by numerical evidence.

Theorem 5.5.4 *Assume that the orthogonality condition $J_0 = 0$ holds. If the given data in the DVIDE (5.5.27) are in C^d with $d \geq m + 1$, then the attainable order of global superconvergence of the collocation solution $u_h \in S_m^{(0)}(I_h)$ is, for all $q \in (0, 1)$, described by*

$$||y - u_h||_\infty \leq Ch^{m+1}.$$

Proof In order to exhibit the crucial steps leading to the above global superconvergence result we will prove Theorem 5.5.4 for the delay VIDE

$$y'(t) = g(t) + (\mathcal{V}_\theta y)(t).$$

(cf. Section 5.1.4). The collocation error $e_h := y - u_h$ solves the initial-value problem

$$e_h'(t) = \delta_h(t) + (\mathcal{V}_\theta e_h)(t), \quad t \in I, \quad e_h(0) = 0,$$

and hence, according to Theorem 5.1.6, it can be written as

$$e_h(t) = \int_0^t \delta_h(s)ds + \sum_{k=1}^{\infty} \int_0^{q^k t} \bar{H}_k(t, s)\delta_h(s)ds, \quad t \in I. \qquad (5.5.34)$$

If we now compare the error representation (5.5.34) with (5.3.37), the one for the iterated collocation error for second-kind VIEs with proportional delay, we see that they possess an essentially identical structure, except that (5.5.34) contains the additional term $\int_0^t \delta_h(s)ds$. For $t = t_n + vh$ with $v \in [0, 1]$ we may write this term as

$$\int_t^t \delta_h(s)ds = h \sum_{\ell=0}^{n-1} \int_0^1 \delta_h(s)ds + h \int_0^v \delta_h(s)ds.$$

Hence, if we now approximate each integral over $[0, 1]$ by interpolatory m-point quadrature formulas based on the $\{c_i\}$, the orthogonality condition $J_0 = 0$ implies that the induced quadrature errors are $\mathcal{O}(h^{m+1})$. Furthermore, it follows from

$$\sup\{|\delta_h(t_n + vh)| : \; v \in [0, 1]\} = \mathcal{O}(h^m)$$

that the argument used in the proof of Theorem 5.3.4 is now easily modified, to yield the $\mathcal{O}(h^{m+1})$-estimate of Theorem 5.5.4.

Conjecture 5.5.5 *If $d \geq m + 2$ in Theorem 5.5.4, with $m \geq 2$, and if the collocation points X_h correspond to the Gauss points $\{c_i\}$, then the optimal value of p^* in the local estimate for the collocation solution $u_h \in S_m^{(0)}(I_h)$ to (5.5.27),*

$$\max_{t \in I_h} |y(t) - u_h(t)| \leq Ch^{p^*},$$

is given by $p^* = m + 2$. This holds for all $q \in (0, 1)$*, and the value $m + 2$ is best possible.*

5.5.3 Collocation on quasi-geometric meshes

Suppose that on some small initial subinterval $[0, t_0]$ of $I := [0, T]$ we have computed, by some continuous method, an approximation $y_0 = y_0(t)$ to the solution $y = y(t)$ of the initial-value problem for (5.5.1) with $\theta(t) = qt$ $(0 < q < 1)$, so that

$$||y - y_0||_{0,\infty} := \max_{t \in [0,t_0]} |y(t) - y_0(t)| \leq C_0 t_0^{p_0} \tag{5.5.35}$$

for some $p_0 \geq 1$ to be specified later. We will assume that this initial interval is defined by setting

$$t_0 = \theta^M(T) \quad \text{for some} \quad M \in \mathbb{N}, \tag{5.5.36}$$

where $\theta^M(T) := (\underbrace{\theta \circ \theta \circ \ldots \circ \theta}_{M})(T) = q^M T$.

High-order approximations to the solution y of (5.5.1) on $[0, t_0]$ can be generated either by computing the collocation solution $v_h \in S_{m+r}^{(0)}([0, t_0])$, using the single subinterval $[0, t_0]$ and with the choice of r depending on the desired order, or by resorting to the appropriate Taylor series for y (since on I the solution y of (5.5.1) is smooth when the data are smooth).

The original initial-value problem (5.5.1) is now replaced by

$$z'(t) = f(t, z(t), z(\theta(t))) + (\mathcal{V}z)(t) + (\mathcal{V}_\theta z)(t), \quad t \in \bar{I} := [t_0, T], \qquad (5.5.37)$$
$$z(t) = y_0(t), \quad t \in [\theta(t_0), t_0] = [qt_0, t_0] \subset (0, t_0].$$

Since, by assumption, the delay θ does not vanish on the interval $\bar{I}$ (recall condition (D1) in Section 4.1.2) we introduce on $\bar{I}$ the graded *macro-mesh* $\bar{\Omega}$ by

$$\bar{\Omega} := \{\xi_\mu : \; t_0 = \xi_0 < \xi_1 < \ldots < \xi_M = T, \quad \xi_\mu := \theta^{M-\mu}(T) \; (0 \le \mu \le M)\}, \qquad (5.5.38)$$

with $H^{(\mu)} := \xi_{\mu+1} - \xi_\mu$ $(\mu = 0, 1, \ldots, M-1)$ denoting the macro-steps. The local meshes $I_h^{(\mu)}$ on the subintervals $I^{(\mu)} := [\xi_\mu, \xi_{\mu+1}]$ $(\mu = 0, 1, \ldots, M-1)$ are defined by

$$I_h^{(\mu)} := \{t_n^{(\mu)} : \; \xi_\mu = t_0^{(\mu)} < t_1^{(\mu)} < \ldots < t_N^{(\mu)} = \xi_{\mu+1}\},$$

and we set

$$\sigma_n^{(\mu)} := [t_n^{(\mu)}, t_{n+1}^{(\mu)}], \quad h_n^{(\mu)} := t_{n+1}^{(\mu)} - t_n^{(\mu)}, \quad h^{(\mu)} := \max\{h_n^{(\mu)} : \; 0 \le n < N\}.$$

For *linear* delays θ we will assume, without loss of generality, that the submeshes $I_h^{(\mu)}$ are all uniform; that is, $h_n^{(\mu)} = h^{(\mu)} = H_\mu/N$ $(0 \le n < N)$. The corresponding mesh on $\bar{I}$,

$$\bar{I}_h := \bigcup_{\mu=0}^{M-1} I_h^{(\mu)}, \qquad (5.5.39)$$

is then both *constrained* and θ-*invariant*, that is,

$$\theta(I_h^{(\mu+1)}) = I_h^{(\mu)} \quad (\mu = 0, 1, \ldots, M-1). \qquad (5.5.40)$$

The collocation solution u_h for the delay VIDE (5.5.1) will be an element of the continuous piecewise polynomial space $S_m^{(0)}(\bar{I}_h)$. It is defined by the collocation equation

$$u_h'(t) = f(t, u_h(t), u_h(\theta(t))) + (\mathcal{V}u_h)(t) + (\mathcal{V}_\theta u_h)(t), \quad t \in \bar{X}_h, \qquad (5.5.41)$$
$$u_h(t) = y_0(t) \quad \text{for} \quad t \in [\theta(t_0), t_0].$$

Here, the set

$$\bar{X}_h := \bigcup_{\mu=0}^{M-1} X_h^{(\mu)},$$
$$X_h^{(\mu)} := \{t_n^{(\mu)} + c_i h_n^{(\mu)} : \; 0 \le c_1 < \ldots < c_m \le 1 \; (0 \le n < N)\}, \qquad (5.5.42)$$

denotes the set of collocation points for the underlying θ-invariant mesh I_h. Hence, the basic setting is the one for VIDEs with non-vanishing delays introduced in Section 4.5.1, and we can employ that notation to put (5.5.41) into a form that is feasible both for the subsequent convergence analysis and for the numerical computation of u_h. We employ the local Lagrange basis representation for u_h on $\sigma_n^{(\mu)} = [t_n^{(\mu)}, t_{n+1}^{(\mu)}]$, namely,

$$u_h(t_n^{(\mu)} + vh_n^{(\mu)}) = y_n^{(\mu)} + h_n^{(\mu)} \sum_{j=1}^{m} \beta_j(v) Y_{n,j}^{(\mu)}, \quad v \in [0, 1], \tag{5.5.43}$$

where

$$y_n^{(\mu)} := u_h(t_n^{(\mu)}), \quad Y_{n,j}^{(\mu)} := u_h'(t_{n,j}) \quad \text{and} \quad t_{n,j}^{(\mu)} := t_n^{(\mu)} + c_j h_n^{(\mu)}.$$

Recall from Section 4.2.1 that if the delay $\theta(t)$ is *linear*, the set $\bar{X}_h$ of collocation points is θ-invariant, too:

$$\theta(X_h^{(\mu+1)}) = X_h^{(\mu)} \qquad (\mu = 0, 1, \ldots, M-1). \tag{5.5.44}$$

As we have already seen in Lemma 4.2.1, this is of course no longer true for *nonlinear* delay functions $\theta(t)$.

The collocation equation (5.5.41) at $t = t_{n,i}^{(\mu)} \in \sigma_n^{(\mu)}$ can be written in the form

$$Y_{n,i}^{(\mu)} = f(t_{n,i}^{(\mu)}, u_h(t_{n,i}^{(\mu)}), u_h(\theta(t_{n,i}^{(\mu)})) + (\mathcal{V}u_h)(t_{n,i}^{(\mu)}) + (\mathcal{V}_\theta u_h)(t_{n,i}^{(\mu)}), \tag{5.5.45}$$

where $\theta(t_{n,i}^{(\mu)}) = t_{n,i}^{(\mu-1)}$. In (5.5.45)

$$(\mathcal{V}u_h)(t_{n,i}^{(\mu)}) = F_n^{(\mu)}(t_{n,i}^{(\mu)}) + h_n^{(\mu)} \int_0^{c_i} k_1(t_{n,i}^{(\mu)}, t_n^{(\mu)} + sh_n^{(\mu)}, u_h(t_n^{(\mu)} + sh_n^{(\mu)}))ds, \tag{5.5.46}$$

where the lag term $F_n^{(\mu)}(t)$ has the form

$$F_n^{(\mu)}(t) := \int_0^{\xi_0} k_1(t, s, y_0(s))ds + \int_{\xi_0}^{\xi_\mu} k_1(t, s, u_h(s))ds + \int_{\xi_\mu}^{t_n^{(\mu)}} k_1(t, s, u_h(s))ds, \tag{5.5.47}$$

with $t \in \sigma_n^{(\mu)}$. Moreover,

$$\begin{aligned}(\mathcal{V}_\theta u_h)(t_{n,i}^{(\mu)}) = & \int_0^{\theta(t_{n,i}^{(\mu)})} k_2(t_{n,i}^{(\mu)}, s, u_h(s))ds + \Psi_n^{(\mu-1)}(t_{n,i}^{(\mu)}) \\ & + h_n^{(\mu-1)} \int_0^{c_i} k_2(t_{n,i}^{(\mu)}, t_n^{(\mu-1)} + sh_n^{(\mu-1)}, u_h(t_n^{(\mu-1)} + sh_n^{(\mu-1)}))ds,\end{aligned} \tag{5.5.48}$$

where we have set

$$\begin{aligned}\Psi_n^{(\mu-1)}(t_{n,i}^{(\mu)}) := & \int_0^{\xi_0} k_2(t_{n,i}^{(\mu)}, s, y_0(s))ds + \int_{\xi_{\mu-1}}^{t_n^{(\mu-1)}} k_2(t_{n,i}^{(\mu)}, s, u_h(s))ds \\ & + \int_{\xi_{\mu-1}}^{t_n^{(\mu-1)}} k_2(t_{n,i}^{(\mu)}, s, u_h(s))ds\end{aligned} \tag{5.5.49}$$

$(\xi_0 = t_0 = t_0^{(0)})$. For $t = t_{n,i}^{(0)} \in X_h^{(0)}$ we obtain, defining $\theta(t_{n,i}^{(0)}) = \theta(t_0 + c_i h_n^{(0)}) =: t_{n,i}^{(-1)}$,

$$(\mathcal{V}_\theta u_h)(t_{n,i}^{(0)}) = \int_0^{t_{n,i}^{(-1)}} k_2(t_{n,i}^{(0)}, s, y_0(s))ds, \tag{5.5.50}$$

where $y_0(t)$ denotes the already computed initial approximation to $y(t)$ on the 'small' interval $[0, t_0]$ described by (5.5.36).

The (exact) collocation method in $S_m^{(0)}(\bar{I}_h)$ is described by (5.5.43) and (5.5.45)–(5.5.49). It involves the computation of the solution $\mathbf{Y}_n^{(\mu)} := (Y_{n,1}^{(\mu)}, \dots, Y_{n,m}^{(\mu)})^T \in \mathbb{R}^n$ of each nonlinear algebraic system (5.5.45). In order to make this discussion more transparent we will derive these algebraic systems in the case where the given VIDE (5.5.37) is *linear:*

$$y'(t) = a(t)y(t) + b(t)y(\theta(t)) + (\mathcal{V}y)(t) + (\mathcal{V}_\theta y)(t), \quad t \in I, \tag{5.5.51}$$

with

$$(\mathcal{V}y)(t) := \int_0^t K_1(t, s)y(s)ds, \quad (\mathcal{V}_\theta y)(t) := \int_0^{\theta(t)} K_2(t, s)y(s)ds.$$

The given kernel functions K_1 and K_2 are assumed to be continuous on their respective domains D and D_θ, respectively. The resulting linear algebraic systems can be described in concise form if we introduce matrices in $L(\mathbb{R}^m)$ given by

$$A := (a_{i,j}), \quad A_n^{(\mu)} := \text{diag}(a(t_{n,i}^{(\mu)}))A, \quad B_n^{(\mu)} := \text{diag}(b(t_{n,i}^{(\mu)})),$$

and

$$C_n^{(\mu)} := \begin{pmatrix} \int_0^{c_i} K_1(t_{n,i}^{(\mu)}, t_n^{(\mu)} + sh_n^{(\mu)})\beta_j(s)ds \\ (i, j = 1, \dots, m) \end{pmatrix},$$

$$D_n^{(\mu)} := \begin{pmatrix} \int_0^{c_i} K_2(t_{n,i}^{(\mu)}, t_n^{(\mu-1)} + sh_n^{(\mu-1)})\beta_i(s)ds \\ (i, j = 1, \dots, m) \end{pmatrix}.$$

In addition, we define the m-vectors

$$\boldsymbol{\kappa}_n^{(\mu)} := \left(a(t_{n,i}^{(\mu)}) + h_n^{(\mu)} \int_0^{c_i} K_1(t_{n,i}^{(\mu)}, t_n^{(\mu-1)} + sh_n^{(\mu-1)})ds \right)^T,$$

$$\tilde{\boldsymbol{\kappa}}_n^{(\mu-1)} := \left(\tilde{\kappa}_{n,1}^{(\mu-1)}, \dots, \tilde{\kappa}_{n,m}^{(\mu-1)} \right)^T,$$

where

$$\tilde{\kappa}_{n,i}^{(\mu-1)} := b(t_{n,i}^{(\mu)}) + h_n^{(\mu-1)} \int_0^{c_i} K_2(t_{n,i}^{(\mu)}, t_n^{(\mu-1)} + sh_n^{\mu-1)})ds.$$

Finally, set

$$\mathbf{G}_n^{(\mu)} := (F_n^{(\mu)}(t_{n,1}^{(\mu)}), \ldots, F_n^{(\mu)}(t_{n,m}^{(\mu)}))^T$$

and

$$\mathbf{Q}_n^{(\mu-1)} := (\Psi_n^{(\mu-1)}(t_{n,1}^{(\mu)}), \ldots, \Psi_n^{(\mu-1)}(t_{n,m}^{(\mu)}))^T,$$

with $F_n^{(\mu)}(t)$ and $\Psi_n^{(\mu-1)}(t)$ given by (5.5.47) and (5.5.49). The linear counterpart of (5.5.45) then assumes the form

$$\begin{aligned}[]
[\mathcal{I}_m - h_n^{(\mu)}(A_n^{(\mu)} + h_n^{(\mu)}C_n^{(\mu)})]\mathbf{Y}_n^{\mu)} &= h_n^{(\mu)}[B_n^{(\mu)} + h_n^{(\mu-1)}D_n^{(\mu)}]\mathbf{Y}_n^{(\mu-1)} \\
&\quad + \boldsymbol{\kappa}_n^{(\mu)}y_n^{(\mu)} + \tilde{\boldsymbol{\kappa}}_n^{(\mu-1)}y_n^{(\mu-1)} + \mathbf{G}_n^{(\mu)} + \mathbf{Q}_n^{(\mu-1)}. \qquad (5.5.52)
\end{aligned}$$

On the first interval $[\xi_0, \xi_1]$ of the macro-mesh ($\mu = 0$) the above algebraic system (5.5.53) reduces to

$$\begin{aligned}
Y_{n,i}^{(0)} &= a(t_{n,i}^{(0)})y_n^{(0)} + h_n^{(0)}a(t_{n,i}^{(0)})\sum_{j=1}^{m} a_{i,j}Y_{n,j}^{(0)} + b(t_{n,i}^{(0)})y_0(t_{n,i}^{(0)}) + F_n^{(0)}(t_{n,i}^{(0)}) \\
&\quad + h_n^{(0)}\int_0^{c_i} K_1(t_{n,i}^{(0)}, t_n^{(0)} + sh_n^{(0)})\left(y_n^{(0)} + h_n^{(0)}\sum_{j=1}^{m}\beta_j(s)Y_{n,j}^{(0)}\right)ds \\
&\quad + \int_0^{\theta(t_{n,i}^{(0)})} K_2(t_{n,i}^{(0)}, s)y_0(s)ds. \qquad (5.5.53)
\end{aligned}$$

In more compact notation (5.5.53) reads

$$[\mathcal{I}_m - h_n^{(0)}(A_n^{(0)} + h_n^{(0)}C_n^{(0)})]\mathbf{Y}_n^{(0)} = \boldsymbol{\kappa}_n^{(0)}y_n^{(0)} + \mathbf{G}_n^{(0)} + \mathbf{Q}_n^{(-1)} \qquad (5.5.54)$$

($n = 0, 1, \ldots, N-1$), where the components of $\mathbf{Q}_n^{(-1)} \in \mathbb{R}^m$ are given by

$$\Phi_{n,i}^{(-1)} := b(t_{n,i}^{(0)})y_0(\theta(t_{n,i}^{(0)})) + \int_0^{\theta(t_{n,i}^{(0)})} K_2(t_{n,i}^{(0)}, s)y_0(s)ds.$$

The exact collocation method in $S_m^{(0)}(\bar{I}_h)$ for the VIDE (5.5.1) with delay function $\theta(t) = qt$ ($0 < q < 0$) can now be summarised as follows:

1. Choose a small initial interval $[0, t_0]$ defined by $t_0 := \theta^M(T)$ (cf. (5.5.36)) where M denotes an appropriate 'large' integer which will be specified in Section 5.5.4.
2. The points $\xi_\mu := \theta^{M-\mu}(T)$ ($\mu = 0, 1, \ldots, M$), with $\xi_0 = t_0$, define the macro-intervals $I^{(\mu)} := [\xi_\mu, \xi_{\mu+1}]$ and the macro-steps $H^{(\mu)} := \xi_{\mu+1} - \xi_\mu$.
3. Introduce the local meshes $I_h^{(\mu)} := \{t_n^{(\mu)} : \xi_\mu = t_0^{(\mu)} < t_1^{(\mu)} < \ldots < t_n^{(\mu)} = \xi_{\mu+1}\}$ ($\mu = 0, 1, \ldots, M-1$) on the intervals $I^{(\mu)}$ and define the (constrained and θ-invariant) mesh on $\bar{I} := [t_0, T]$ by (5.5.39). These local

meshes will often be chosen to be uniform: $h_n^{(\mu)} = h^{(\mu)} := H^{(\mu)}/N$ ($n = 0, 1, \ldots, N-1$).

4. On the subinterval $\bar{\sigma}_n^{(\mu)} := [t_n^{(\mu)}, t_{n+1}^{(\mu)}]$ the collocation solution $u_h \in S_m^{(0)}(\bar{I}_h)$ to the delay VIDE (5.5.1) is determined by (5.5.43) and by the solutions $\mathbf{Y}_n^{(\mu)}$ of the algebraic systems (5.5.45) (or by (5.5.52) and (5.5.54) if the delay VIDE is the *linear* equation (5.5.51)).

The existence of a unique collocation solution u_h for (5.5.45) is guaranteed for any sufficiently small h, say $h \in (0, \bar{h})$, provided the given delay VIDE has a unique solution $y \in C^1(I)$. In the linear case this is obvious from the form of the linear algebraic systems (5.5.52) and (5.5.54) (recall also Theorem 4.5.1). In the nonlinear situation the assertion follows by the usual classical fixed-point argument.

5.5.4 Superconvergence results on quasi-geometric meshes

Suppose that the initial approximation $y_0(t)$ in $[0, t_0]$ satisfies (5.5.35) with some feasible order $p_0 \geq m$. How should t_0 be determined? It is suggestive that it be chosen so that

$$t_0 = \theta^M(T) = q^M T \leq \max_{(n,\mu)} h_n^{(\mu)} =: h. \tag{5.5.55}$$

If the *local meshes* $I_h^{(\mu)}$ are all *uniform* then

$$h = h^{(M-1)} = H^{(M-1)}/N = (T - \theta(T))/N = (1-q)T/N. \tag{5.5.56}$$

It then follows that (5.5.55) holds if M in (5.5.36) is such that

$$\theta^M(T) = q^M T \leq (T - \theta(T))/N = (1-q)T/N. \tag{5.5.57}$$

For the (linear) proportional delay $\theta(t) = qt$ $(0 < q < 1)$ we have $\theta(T) = qT$, and this leads to

$$M = M(q; N) \geq \frac{\log(1-q) - \log(N)}{\log(q)} \tag{5.5.58}$$

Hence, we choose

$$M = M(q; N) := \left\lceil \frac{\log(1-q) - \log(N)}{\log(q)} \right\rceil \tag{5.5.59}$$

(see Bellen (2001)). In order to obtain an idea on how large these values of M defining the number of macro-intervals are for specific values of $q \in (0, 1)$ and N, the number of subintervals corresponding to each local mesh $I_h^{(\mu)}$, it may be instructive to list a sample of such values (Table 5.8).

Table 5.8. *Values of* $M = M(q; N)$

q	$N = 100$	$N = 1000$	$N = 10000$
0.1	3	4	5
0.5	8	11	15
0.9	66	88	110
0.99	917	1146	1375

We will assume in the following that $\theta(t) = qt$ $(0 < q < 1)$ and that the local meshes $I_h^{(\mu)}$ $(0 \leq \mu \leq M-1)$ are all uniform, with M in (5.5.36) satisfying (5.5.58) (implying that $t_0 = \mathcal{O}(h)$). It then follows from the classical convergence analysis of collocation methods for VIDEs with non-vanishing delays (Section 4.5) that the collocation solution $u_h \in S_m^{(0)}(\bar{I}_h)$ for (5.5.1) induces the estimates

$$\max_{t_0 \leq t \leq \xi_{\mu+1}} |z^{(\nu)}(t) - u_h^{(\nu)}| \leq C_\nu (h^{(\mu)})^m \tag{5.5.60}$$

for $\mu = 0, 1, \ldots, M-1$, and hence

$$||z^{(\nu)} - u_h^{(\nu)}||_\infty := \max_{t \in \bar{I}} |z^{(\nu)}(t) - u_h^{(\nu)}(t)| \leq C_\nu h^m \quad (\nu = 0, 1), \tag{5.5.61}$$

for any choice of the m (distinct) parameters $\{c_i\}$, provided the mesh I_h is given by (5.5.39), and the exact solution z is in $C^{m+1}(I^{(\mu)})$ for $\mu = 0, 1, \ldots, M-1$.

These results are not particularly exciting since we obtained the same order of global convergence on I by collocation on uniform meshes. However, since the order of *local superconvergence* can apparently not exceed $m + 2$ on uniform I_h (Conjecture 5.5.5), the use of these quasi-geometric meshes leads to the optimal local superconvergence results we derived for VIDEs with non-vanishing delays.

Theorem 5.5.6 *Assume*

(a) $a,\ b \in C^{m+1}(I)$; $K_1 \in C^{m+1}(D)$, $K_2 \in C^{m+1}(D_\theta)$; $\theta(t) = qt$ $(0 < q < 1)$*;*
(b) $u_h \in S_m^{(0)}(\bar{I}_h)$ *is the collocation solution defined by (5.5.39), (5.5.54), (5.5.52);*
(c) $\bar{I}_h$ *is the (constrained and* θ*-invariant) mesh defined by (5.5.39) and (5.5.40), with uniform local meshes* $I_h^{(\mu)}$ *so that* $h \in (0, \bar{h})$*;*
(d) $p_0 \geq m + 1$ *in (5.5.35), with* M *as in (5.5.59).*

If the collocation parameters $\{c_i\}$ *satisfy the orthogonality condition*

$$J_0 := \int_0^1 \prod_{i=1}^{m}(s - c_i)ds = 0, \tag{5.5.62}$$

then

$$||z - u_h||_\infty := \max_{t \in \bar{I}} |z(t) - u_h(t)| \leq Ch^{m+1}. \tag{5.5.63}$$

As we have indicated earlier, the principal motivation for employing appropriate quasi-geometric meshes for the computation of the collocation solution to proportional delay VIDEs, with given initial approximation y_0, is that we can then resort to the results of Chapter 4 to allow us to generate collocation solutions with high-order local superconvergence.

Theorem 5.5.7 *Let the assumptions (b), (c) of Theorem 5.5.6 hold, and assume that (a) is replaced by*

$$a,\ b \in C^{m+\kappa}(I);\quad K_1 \in C^{m+\kappa}(D),\quad K_2 \in C^{m+\kappa}(D_\theta),$$

for some integer κ *with* $1 \leq \kappa \leq m$ *described in (5.5.64) below. Finally, in (5.5.35) let* $p_0 \geq m + \kappa$ *and choose* M *by (5.5.59).*

If the collocation parameters $\{c_i\}$ *are chosen so that the orthogonality conditions*

$$J_\nu := \int_0^1 s^\nu \prod_{i=1}^{m}(s - c_i)ds \quad (\nu = 0, 1, \ldots, \kappa - 1), \tag{5.5.64}$$

with $J_\kappa \neq 0$*, hold then, for all* $q \in (0, 1)$,

$$\max_{t \in \bar{I}_h} |z(t) - u_h(t)| \leq Ch^{m+\kappa}. \tag{5.5.65}$$

If, in addition, $c_m = 1$*, then also*

$$\max_{t \in \bar{I}_h \setminus \{\xi_\mu\}} |z'(t) - u_h'(t)| \leq Ch^{m+\kappa}. \tag{5.5.66}$$

In this case κ *cannot exceed* $m - 1$.

Corollary 5.5.8 *Let* $\kappa = m$ *in Theorem 5.5.7: the (unique)* $\{c_i\}$ *are the* m *Gauss points in* $(0, 1)$. *Then for all* $q \in (0, 1)$ *the local estimate*

$$\max_{t \in \bar{I}_h} |z(t) - u_h(t)| \leq Ch^{2m} \tag{5.5.67}$$

holds, while we only obtain

$$\max_{t \in \bar{I}_h \setminus \{\xi_\mu\}} |z'(t) - u_h'(t)| = \mathcal{O}(h^m).$$

If $\kappa = m - 1$, then collocation at the corresponding Radau II points (for which $c_m = 1$) yields

$$\max_{t \in \bar{I}_h \setminus \{\xi_\mu\}} |z^{(\nu)}(t) - u_h^{(\nu)}(t)| \leq C_\nu h^{2m-1} \quad (\nu = 0, 1). \tag{5.5.68}$$

Proof We proceed as in Section 4.5.3. The only difference is that now the initial condition $e_h(0) = 0$ is replaced by $e_h(t_0) \neq 0$, and thus the initial-value problem for $e_h := z - u_h$ is

$$e_h'(t) = a(t)e_h(t) + b(t)e_h(\theta(t)) + \delta_h(t) + (\mathcal{V}e_h)(t) + (\mathcal{V}_\theta e_h)(t), \quad t \in \bar{I},$$
$$e_h(t) = \varepsilon_0(t) \quad \text{for} \quad t \in [\theta(t_0), t_0]. \tag{5.5.69}$$

Here,

$$|\varepsilon_0(t)| \leq C_0 t_0^{p_0}, \quad t \in [0, t_0], \tag{5.5.70}$$

by our assumption (5.5.34). Note that $\varepsilon_0(t) \equiv 0$ if $y_0(t)$ coincides with the exact solution $y(t)$ of (5.5.1) for $t \in [0, t_0]$.

The key to the proofs of the superconvergence results is again the variation-of-constants formula for the representation of the (unique) solution of the initial-value problem (5.5.69), in analogy to delay VIDEs with non-vanishing delays treated in Chapter 4 (cf. Theorem 4.1.7). We now have

$$\begin{aligned} e_h(t) = r_1(t, \xi_\mu)e_h(\xi_\mu) &+ \int_{\xi_\mu}^t r_1(t, s)\delta_h(s)ds \\ &+ \sum_{\nu=1}^{\mu-1} r_{\mu,\nu}(t)e_h(\xi_\nu) + r_{\mu,0}(t)\varepsilon_0(\xi_0) \\ &+ \sum_{\nu=0}^{\mu-1} \int_{\xi_\nu}^{\xi_{\nu+1}} R_{\mu,\nu}(t, s)\delta_h(s)ds \\ &+ \sum_{\nu=0}^{\mu-1} \int_{\xi_\nu}^{\theta^{\mu-\nu}(t)} Q_{\mu,\nu}(t, s)\delta_h(s)ds + E_0^{(\mu)}(t). \end{aligned} \tag{5.5.71}$$

Here, $r_1(t, s)$ denotes the resolvent kernel associated with the data a and K_1 of the homogeneous VIDE

$$y'(t) = a(t)y(t) + (\mathcal{V}y)(t).$$

It satisfies $r_1(t, t) = 1$ on I, and $r_{\mu,\nu},\ R_{\mu,\nu},\ Q_{\mu,\nu}$ denote continuous (piecewise smooth) functions depending on the given functions $a,\ b,\ K_i,\ \theta$. Moreover,

$$\begin{aligned} E_0^{(\mu)}(t) := &\int_{\xi_0}^{\xi_1} R_{\mu,0}(t, s)\{b(s)\varepsilon_0(\theta(s)) - \int_{\theta(s)}^{\xi_0} K_2(s, v)\varepsilon_0(v)dv\}ds \\ &+ \int_{\xi_0}^{\theta^\mu(t)} Q_{\mu,0}(t, s)\{b(s)\varepsilon_0(\theta(s)) - \int_{\theta(s)}^{\xi_0} K_2(s, v)\varepsilon_0(v)dv\}ds, \end{aligned} \tag{5.5.72}$$

and $\theta^\nu := \underbrace{\theta \circ \cdots \circ \theta}_{\nu}$.

The superconvergence result of Theorems 5.5.7 will be obtained from the above representation of e_h by setting, respectively, $t = t_n^{(\mu)} + vh_n^{(\mu)}$ $(v \in [0, 1])$ and $t = t_n^{(\mu)} \in I_h^{(\mu)}$ in (5.5.71). Thus, we have to show that, in spite of the fact that $M = M(N; q) = \mathcal{O}(\log(N))$ and $N \to \infty$, the sums with upper limits equal to $\mu - 1$ remain uniformly bounded. This central basic ingredient in the subsequent convergence analysis is summarised in the following lemma (whose proof is obvious).

Lemma 5.5.9 *Assume that $\theta(t) = qt$ $(0 < q < 1)$ and let $\bar{I}_h$ be the constrained and θ-invariant mesh defined by (5.5.39) and (5.5.40), with M in $t_0 = \xi_0 = q^M T$ satisfying (5.5.59). Then for any $p \in \mathbb{N}$, $p \geq 1$,*

$$\sum_{\nu=0}^{\mu-1}\sum_{\ell=0}^{N-1}(h_\ell^{(\nu)})^{p+1} \leq h^p \sum_{\nu=0}^{\kappa-1}\left(\sum_{\ell=0}^{N-1} h_\ell^{(\nu)}\right) = (T - t_0)h^p, \tag{5.5.73}$$

uniformly in M and N.

We are now ready to prove the two main theorems. Consider first Theorem 5.5.6. It follows from the order m of global convergence $(p = m)$ that the defect δ_h can be bounded by

$$||\delta_h(t)||_{\mu,\infty} \leq A_0 C_0 (h^{(\mu)})^m + B_0 C_0 (h^{(\mu-1)})^m + K_{0,1}||e_h||_{\mu,\infty} + K_{0,2}||e_h||_{\mu-1,\infty},$$

where

$$A_0 := ||a||_\infty, \quad B_0 := ||b||_\infty, \quad K_{0,1} := \max_{t\in I} \int_{t_0}^{t} |K_1(t,s)|ds,$$
$$K_{0,2} := \max_{t\in I} \int_{t_0}^{\theta(t)} |K_2(t,s)|ds.$$

Hence,

$$||\delta_h||_\infty \leq D_0 (h^{(\mu)})^m \quad (0 \leq \mu \leq M - 1).$$

Let now $t = t_n^{(\mu)} + vh_n^{(\mu)}$ $(v \in [0, 1])$ in (5.5.71) and write

$$\int_{\xi_\mu}^{t} ds = h^{(\mu)} \left(\sum_{\ell=0}^{\mu-1}\int_0^1 ds + \int_0^v ds\right),$$

and

$$\int_{\xi_\mu}^{\theta^{\mu-\nu}(t)} ds = h^{(\nu)} \left(\sum_{\ell=0}^{\nu-1}\int_0^1 ds + \int_0^v ds\right).$$

In the remaining terms of (5.5.71) the integrals over the macro-intervals $I^{(\nu)}$ are broken down similarly into sums of (scaled) integrals over [0, 1] with factors $h^{(\nu)}$. Each of the integrals over [0, 1] is replaced by the sum consisting of the m-point interpolatory quadrature approximation using the collocation points as abscissas and the corresponding quadrature error $E_{\mu,\nu}(t)$. Due to the orthogonality assumption (5.5.62) all these quadrature errors are of order $\mathcal{O}(h^{m+1})$ since the integrands possess, by assumption on the given functions in (5.5.1), the required (piecewise) regularity. The integral over $[0, v]$ can be bounded by $h^{(\mu)} \cdot \text{const}||\delta_h||_{\mu,\infty} = \mathcal{O}((h^{(\mu)})^{m+1})$. Collecting all these estimates and invoking Lemma 5.5.9 we readily establish the desired global $\mathcal{O}(h^{m+1})$-estimate of Theorem 5.5.6.

We now turn to the proof of Theorem 5.5.7. Here, we set $t = t_n^{(\mu)}$ in the representation (5.5.71) of the collocation error $e_h(t_n^{(\mu)})$, and we employ again m-point interpolatory quadrature formulas based on the collocation parameters $\{c_i\}$ plus the corresponding error terms $E_{\mu,\nu}(t)$, to replace the (scaled) integrals over [0, 1]. To illustrate this, consider the sum in the third line of (5.5.71): it is replaced by

$$\sum_{\nu=0}^{\mu-1}\sum_{\ell=0}^{N-1} h_\ell^{(\nu)} \int_0^1 R_{\mu,\nu}(t, t_\ell^{(\nu)} + sh_\ell^{(\nu)})\delta_h(t_\ell^{(\nu)} + sh_\ell^{(\nu)})ds$$
$$= \sum_{\nu=0}^{\mu-1}\sum_{\ell=0}^{N-1} h_\ell^{(\nu)} \left(\sum_{j=1}^{m} w_j R_{\mu,\nu}(t, t_\ell^{(\nu)} + c_j h_\ell^{(\nu)})\delta_h(t_\ell^{(\nu)} + c_j h_\ell^{(\nu)}) + E_{\mu,\nu}(t)\right)$$
$$= \sum_{\nu=0}^{\mu-1}\sum_{\ell=0}^{N-1} h_\ell^{(\nu)} E_{\mu,\nu}(t).$$

It follows from the orthogonality conditions (5.5.64) (and the resulting degree of precision of these quadrature formulas) and from the assumed regularity of the given data that all quadrature errors are of order $\mathcal{O}(h^{m+r})$. Moreover, the sums with upper limit $\mu - 1$ can again be bounded, uniformly in N and M, using Lemma 5.5.9. The final estimate (5.5.65) is now obtained in a straightforward way, as is (5.5.66).

5.5.5 More general vanishing delays

The approach and the convergence results described in Sections 5.5.3 and 5.5.4 are not confined to vanishing linear delay functions $\theta(t) = qt$ $(0 < q < 1)$, but they remain valid, with obvious modifications, if $\theta(t)$ is *nonlinear* and is such that

(N1) $\theta \in C^1(I)$, with $\theta(0) = 0$ and $\theta(t) < t$ for $t > 0$;
(N2) $\min_{t \in I} \theta'(t) =: q_0 > 0$.

Then – as we already briefly indicated in Section 4.2.1 (Lemma 4.2.1) – (5.5.43) is no longer valid: the set $\bar{X}_h$ of collocation points in no longer θ-invariant. Hence the expression (5.5.48) for $(\mathcal{V}_\theta u_h)(t_{n,i}^{(\mu)})$ in the collocation equation (5.5.45) has to be modified. Since θ is strictly increasing on I we have

$$\theta(t_{n,i}^{(\mu)}) = \tilde{t}_{n,i}^{(\mu-1)} := t_n^{(\mu-1)} + \tilde{c}_i h_n^{(\mu-1)} \in \sigma_n^{(\mu-1)} \quad (i = 1, \ldots, m), \qquad (5.5.74)$$

for some $\{\tilde{c}_i\}$ with $0 \le \tilde{c}_1 < \ldots < \tilde{c}_m \le 1$. (It is understood that the value of each $\tilde{c}_i$ depends on both n *and* μ: $\tilde{c}_i = \tilde{c}_i(n; \mu)$. For the sake of ease of notation we will usually supress these arguments.) The θ-invariant mesh $\bar{I}_h$ on $\bar{I}$ is still given by (5.5.39) and (5.5.40). Since θ is *nonlinear*, this mesh $\bar{I}_h$ is no longer quasi-geometric: we will call it *quasi-graded*.

Using again the local Lagrange representation (5.5.43) for $u_h \in S_m^{(0)}(\bar{I}_h)$ on $\sigma_n^{(\mu-1)}$,

$$u_h(t_n^{(\mu-1)} + v h_n^{(\mu-1)}) = y_n^{(\mu-1)} + h_n^{(\mu-1)} \sum_{j=1}^{m} \beta_j(v) Y_{n,j}^{(\mu-1)}, \quad v \in [0, 1],$$

we see that the computational form of the collocation equation (5.5.41) remains essentially the same, except that now we have

$$(\mathcal{V}_\theta u_h)(t_{n,i}^{(\mu)}) = \Psi_n^{(\mu-1)}(t_{n,i}^{(\mu)}) + h_n^{(\mu-1)} \int_0^{\tilde{c}_i} k_2(t_{n,i}^{(\mu)}, t_n^{(\mu-1)} + s h_n^{(\mu-1)}, u_h(t_n^{(\mu-1)} + s h_n^{(\mu-1)}))ds, \qquad (5.5.75)$$

with lag term $\Psi_n^{(\mu-1)}(t)$ as in (5.5.49), and

$$b(t_n^{(\mu)}) u_h(\theta(t_{n,i}^{(\mu)}) = b(t_{n,i}^{(\mu)}) \left(y_n^{(\mu-1)} + h_n^{(\mu-1)} \sum_{j=1}^{m} \beta_j(\tilde{c}_i) Y_{n,j}^{(\mu-1)} \right). \qquad (5.5.76)$$

In a fixed subinterval $\sigma_n^{(\mu)}$ where we consider the collocation equation the (given) collocation points are $t_{n,i}^{(\mu)} = t_n^{(\mu)} + c_i h_n^{(\mu)}$ $(i = 1, \ldots, m)$. Thus, the parameters $\tilde{c}_i$ used in the above delay terms have to be computed from the images of these collocation points under θ.

The validity of the global and local superconvergence results of Theorems 5.5.6 and 5.5.7 hinges on the fact that, for such nonlinear delays θ the corresponding θ-invariant mesh has, as for linear delays θ, the property that

$$h := \max_{(n,\mu)} h_n^{(\mu)} \to 0 \quad \text{as} \quad N \to \infty,$$

provided the grading exponent $M = M(N; \mu)$ satisfies a nonlinear analogue of (5.5.58). In order to make this precise, let $\theta_0(t) := q_0 t$ denote the proportional delay corresponding to the value of q_0 in condition (N2) for θ, and let

$$\theta_0(\eta_\mu) =: \eta_{\mu-1}, \quad H_0^{(\mu)} := \eta_{\mu+1} - \eta_\mu, \quad H_0 := \max_{(\mu)} H_0^{(\mu)},$$
$$h_0 := H_0/N = (1 - q_0)T/N.$$

Assume in the following that one of the local meshes $I_h^{(\mu)}$, e.g. $I_h^{(0)}$ or $I_h^{(M-1)}$, has been prescribed; without loss of generality we will assume that this prescribed mesh is uniform. Hence,

$$h_n^{(0)} = h^{(0)} := H^{(0)}/N, \quad \text{or} \quad h_n^{(M-1)} = h^{(M-1)} := H^{(M-1)}/N.$$

Lemma 5.5.10 *Let θ satisfy (N1) and (N2), and let $\theta_0(t) := q_0 t$. Then, under the above hypotheses on the choice of $I_h^{(0)}$ (or $I_h^{(M-1)}$), we have*

$$H^{(\mu)} \le H_0 = (1 - q_0)T \quad (\mu = 0, 1, \dots, \kappa - 1),$$

and

$$h^{(\mu)} \le h_0 = (1 - q_0)T/N \quad (\mu = 0, 1, \dots, M - 1).$$

Proof The assertion is geometrically evident. Its analytical verification is left to the reader.

The above observations, including Lemma 5.5.10 and the (super-) convergence arguments based on it, show that *the linear pantograph VIDE (5.5.1), with q_0 replacing q, is a representative test equation for a large class of VIDEs with vanishing nonlinear delay functions θ.*

5.6 Exercises and research problems

Exercise 5.6.1

(a) Prove Theorem 5.1.3.
(b) Show that if the coefficients a and b in the more general pantograph equation,

$$y'(t) = a(t)y(t) + b(t)y(qt), \quad t \in I, \quad y(0) = y_0 \neq 0,$$

are in $C^m(I)$, then its unique solution y lies in $C^{m+1}(I)$ for all $q \in (0, 1)$.
(c) Does the uniqueness part of Theorem 5.1.3 remain valid if (5.1.5) is replaced by

$$y'(t) = ay(t) + by(qt) + cy'(qt) \quad (0 < q < 1),$$

with $a, b, c \in \mathbb{R}$ and $c \neq 0$?

Exercise 5.6.2 Use Theorem 5.1.4 to derive the expression for the solution of the DDE

$$y'(t) = by(qt), \quad 0 < q < 1.$$

Assuming $b < 0$, use both this expression and collocation in $S_m^{(0)}(I_h)$ $(m = 1, 2)$ to compute the zeros of the solution for $q = 0.5,\ q = 0.9,\ q = 0.99$ in the interval (0, 150).

Exercise 5.6.3 Consider the pantograph equation (5.1.5) with $q > 1$. What can be said about the existence or uniquenss of the solution corresponding to an initial value $y_0 \neq 0$?

Exercise 5.6.4 Compute the iterated kernels for the delay VIE (5.1.4) when $K(t, s) = \lambda$.

Exercise 5.6.5 Discuss the existence and uniqueness of the solution to the delay VIE

$$y(t) = 1 + by(qt) + \int_0^t k(t - s)y(s)ds, \quad t \in I := [0, T],$$

where $k \in C(I)$ is given and $0 < q < 1$.

Exercise 5.6.6 Use the embedding approach deribed in Section 5.1.5 and Lemma 5.1.12 to prove the regularity result of Theorem 5.1.8.

Exercise 5.6.7 Extend the embedding approach of Section 5.1.5 to the nonlinear delay VIE

$$y(t) = g(t) + \int_0^{qt} k(t - s)G(y(s))ds, \quad t \in I \quad (0 < q < 1).$$

Assume that $k \in C(I)$ and G is appropriately smooth. Discuss the application of embedding to the implicitly linear form of this DVIE, given by

$$z(t) = G\left(g(t) + \int_0^{qt} k(t - s)z(s)ds\right),$$

$$y(t) = g(t) + \int_0^{qt} k(t - s)z(s)ds, \quad t \in I.$$

Exercise 5.6.8 Let the lag function θ satisfy the conditions (N1) and (N2) of Section 5.5.5, with $\theta \in C^d(I),\ d \geq 1$. Establish regularity results for the solutions of the corresponding delay VEs (5.1.9) and (5.1.12).

Exercise 5.6.9 Analyse the solvability of the first-kind VIE with two proportional delays,

$$\int_{qt}^{rt} K(t,s)y(s)ds = g(t), \quad t \in I := [0,T],$$

with $0 < q < r < 1$ and $g(0) = 0$.

Exercise 5.6.10 Consider the pantograph VIDE

$$y^{(k)}(t) = \sum_{j=0}^{k} b_j(t)y^{(j)}(qt) + \int_0^{qt} \sum_{j=0}^{k} K_{2,j}(t,s)y^{(j)}(s)ds,$$

with $k \geq 2$ and continuous given functions. Show that for any prescribed set of inital values $\{y_0^{(\nu)} : \nu = 0, 1, \ldots, k-1\}$ the initial-value problem for the above delay VIDE has a unqiue solution $y \in C^k(I)$.

Use the result for $k = 2$, to establish the existence, uniqueness, and regularity properties of the generalisation of the second-order pantograph equation studied by Bélair (1981),

$$y''(t) = b_0(t)y(qt) + \int_0^{qt} \sum_{\nu=0}^{2} K_{2,\nu}(t,s)y^{(\nu)}(s)ds.$$

Exercise 5.6.11 (Section 5.1.2) For which values of $q \in (0,1)$ is Phase II *non-empty*? Consider both the Gauss points and the Radau II points.

Exercise 5.6.12 Prove Lemma 5.2.2.

Exercise 5.6.13 Assume that $q = 1/r$ where $r \in \mathbb{N},\ r \geq 2$. Determine q^I and q^{II}, and discuss the 'periodicity' of the corresponding values of $\gamma_{n,i}$. Illustrate your result by choosing $m = 2,\ m = 3$ and $r = 3, \ldots, 6$.

What happens if $q = \ell/r$ ($\ell \in \mathbb{N},\ 2 \leq \ell < r$)?

Exercise 5.6.14 Extend Theorem 5.2.3 on the existence of a unique collocation solution $u_h \in S_m^{(0)}(I_h)$ to the neutral pantograph equation given in part (c) of Exercise 5.6.1.

Exercise 5.6.15 State and prove Lemmas 5.2.1 and 5.2.2 when $0 = c_1 < c_1 < \ldots < c_m \leq 1$.

Exercise 5.6.16 Formulate the collocation equations defining $u_h \in S_{m-1}^{(-1)}(I_h)$ and u_h^{it} for the special DV2 (5.3.18). In particular, consider the case where $m = 2,\ m = 3$ and the $\{c_i\}$ are the Gauss points. Choose an example with known exact solution and compute the errors induced by u_h and u_h^{it}, and use the numerical results to deduce the orders of convergence, both on I and on $I_h \setminus \{0\}$.

Exercise 5.6.17 In Section 5.5.3 we described collocation on quasi-geometric meshes for delay VIDEs. Adapt this approach to the second-kind delay VIE

(5.3.4). In particular, state and prove the analogue of the superconvergence result of Theorem 5.5.7 and Corollary 5.5.8.

Exercise 5.6.18 In Section 5.2 we only considered the collocation solution in $S_{m-1}^{(-1)}(I_h)$ defined by the *exact* collocation equation. Discuss the choice of suitable quadrature formulas for obtaining the discretised collocation equation, and carry out the perturbation analysis for $u_h - \hat{u}_h$ and $u_h^{it} - \hat{u}_h^{it}$.

Exercise 5.6.19 Extend the results of Section 5.3.6 on the quasi-optimal order of local superconvergence ($p^* = 2m - \varepsilon_N$) for geometric meshes to the linear delay VIDE (5.5.50) and the general delay VIDE (5.5.1). As a corollary we obtain the result for the pantograph equation (5.1.5).

Exercise 5.6.20 Formulate the collocation equations for $u_h \in S_{m-1}^{(-1)}(I_h)$ approximating

$$\int_{qt}^{t} K(t,s)y(s)ds = g(t), \quad t \in I := [0,T] \quad (0 < q < 1),$$

when an approximation $y_0(t)$ to $y(t)$ has been found on $[0, t_0]$ and $\bar{I}_h$ is a *quasi-geometric* mesh for $\bar{I} := [t_0, T]$ ($t_0 > 0$). Compare the beginning of Section 5.5.3 for notation and assumptions on $\bar{I}_h$ and t_0.

Exercise 5.6.21 (Research problem)
Delay VIEs of the form

$$y(t) = g(t) + \int_{-t}^{t} K(t,s)y(s)ds, \quad t \in [0,T],$$

were analysed by Ghermanesco (1959, 1961) (see also Volterra (1913), pp. 92–94). Discuss the existence and uniqueness of solutions in $C(I)$, and analyse the (super-) convergence properties of collocation solutions, either for the given problem itself, or for the equivalent pair of integral equations (see the original papers for details).

Exercise 5.6.22 (Research problem)
Assume that $K(t,s) \equiv 1$ in the Volterra operator $\mathcal{W}_\theta$ characterising the proportional delay VIE of the first kind (5.4.2).

(i) If $m = 1$ find a sufficient condition for $c_1 \in (0,1]$ so that the collocation solution $u_h \in S_0^{(-1)}(I_h)$ defined by (5.4.9) converges uniformly to y on I.
(ii) Extend the result of (i) to $u_h \in S_{m-1}^{(-1)}(I_h)$ with $m \geq 2$, assuming $0 < c_1 < \ldots < c_m = 1$.

Exercise 5.6.23 (Research problem)
Analyse the convergence and superconvergence properties of the collocation solution $u_h \in S_m^{(0)}(I_h)$ to the Riccati–Hammerstein delay VIDE

$$y'(t) = [a - by(qt)]y(t) + \int_{qt}^{t} k(t-s)G(y(s), y(t))ds.$$

Exercise 5.6.24 (Research problem)
A posteriori error estimates and adaptive mesh selection for pantograph-type functional equations: extend the approaches in Eriksson et al. (1995a, 1995b, 1996) and Shaw and Whiteman (1996, 2000a) to discontinuous Galerkin methods for (i) the pantograph equation; (ii) the delay VIE (5.1.13); and (iii) to the delay VIDE (5.1.14).

5.7 Notes

5.1: Basic theory of functional equations with proportional delays
The book by Volterra (1913, pp. 85–88, 92–100) gives a detailed analysis of the solvability of integral equations with proportional (and more general vanishing) delays. This review is based on his own paper of 1997, as well as on work by Picard (1907) (on functional equations of the form $y(t) = g(t) + b(t)y(qt) + (\mathcal{V}y)(t)$), and Lalesco (1908, 1911). Hellinger and Toeplitz (1927) contains a concise overview of this development. Volume 3 of Fenyö and Stolle (1984) is, to my knowledge, the only 'modern' book that deals with pantograph-type VIEs. Compare also the papers by Chambers (1990), Pukhnacheva (1990), Denisov and Lorenzi (1997) and Mureşan (1999) for additional results for linear and nonlinear second-kind VIEs with proportional delays.

First-kind VIEs with variable upper and lower limits of integration are the subject of the monograph by Apartsin (2003).

The systematic study of the theory of the pantograph DDE and its various generalisations began with the papers by Ockendon and Tayler (1971), Fox, Mayers, Ockendon and Tayler (1971), and Kato and MacLeod (1971). These DDEs almost immediately received much attention by researchers in analysis; see, for example, the papers by Frederickson (1971), Kato (1972), Nussbaum (1972), Carr and Dyson (1976), Bélair (1981), Derfel (1990, 1991), Kuang and Feldstein (1990), Derfel and Molchanov (1990), Iserles (1993) (survey with extensive list of references), Iserles (1994b), Terjéki (1995), Iserles and Terjéki (1995), Derfel and Vogl (1996), Liu (1996a), Iserles (1997a), Iserles and Liu (1997), Feldstein and Liu (1998). However, the reader may also wish to look at the ‘early’ papers cited in Frederickson (1971), including the one by de Bruijn (1953).

5.2: Collocation for DDEs with proportional delays
Numerical analysts remained singularly inattentive to the challenges of the numerical analysis of pantograp-type DDEs: the fundamental paper by Fox et al. (1971) on the numerical solution of the pantograph DDE (and its formulation as Volterra functional equation) stood alone until the early 1990s, when Buhmann and Iserles (1991, 1992, 1993), Iserles (1993), and Buhmann, Iserles and Nørsett (1993) understood that this class of functional differential equations represents a rich source of deep mathematical problems, both for the (theoretical and computational) numerical analyst.

In the contributions just mentioned the focus was on the asymptotic properties of numerical approximations, by linear multistep and simple collocation methods, for the pantograph equation (5.1.5). The survey by Iserles (1994a) and the papers by Iserles (1994c, 1997a, 1997b), Y. Liu (1995a, 1995b, 1996a, 1996b, 1997), Liang and Liu (1996), Liang, Qiu and Liu (1996), Bellen, Guglielmi and Torelli (1997), Carvalho and Cooke (1998), Koto (1999), Liang and Liu (1999), Bellen (2001), Liu and Clements (2002), and Guglielmi and Zennaro (2003) describe various extensions of these early stability results, both on uniform and (quasi-) geometric meshes. Compare also the monograph by Bellen and Zennaro (2003) for a survey of many of these results, and Brunner (2003) for additional references.

Collocation methods and their (super-) convergence properties are considered in Buhmann, Iserles and Nørsett (1993) (for $u_h \in S_1^{(0)}(I_h)$ and $q = 1/2$), Brunner (1997a), Zhang (1998), Zhang and Brunner (1998), and Takama, Muroya and Ishiwata (2000). While these properties are now reasonably well understood, this is not true for the qualitative aspects of piecewise polynomial (and continuous Runge–Kutta) methods: as shown in, e.g. Buhmann, Iserles and Nørsett (1993) the present understanding is still at a very primitive level (except possibly when $q = 1/2$).

5.3: Second-kind VIEs with proportional delays
Fox et al. (1971, pp. 292–295) used the integrated form of the pantograph equation, i.e. a Volterra functional integral equation, to analyse the error induced by a variant of the classical Lanczos τ-method. Collocation methods for Volterra integral and integro-differential equations with proportional delays were studied in detail in Brunner (1997a), Zhang (1998), Brunner and Zhang (1998) (for second-order Volterra functional integro-differential equations), Takama, Muroya and Ishiwata (2000), Ishiwata (2000), Muroya, Ishiwata and Brunner

(2002), and Bellen et al. (2002). In these papers the focus is on the attainable orders of global and local (super-) convergence in collocation solutions. See also the survey by Brunner (2003).

As we mentioned before, the analysis of the asymptotic behaviour of collocation solutions to pantograph-type Volterra integral (and integro-differential) equations is completely open.

5.4: Collocation for first-kind VIEs with proportional delays
As we have already indicated in Section 5.4.2, the convergence analysis for collocation solutions to pantograph-type VIEs of the first kind is completely open (see also Brunner (1997b). The same is of course true for the more general first-kind VIEs with vanishing delays (Denisov and Korovin (1992), Denisov and Lorenzi (1995).

5.5: VIDEs with proportional delays
Piecewise polynomial collocation methods on uniform meshes for a rather general class of VIDEs with proportional delays are studied in Ishiwata (2000); her analysis (which focuses on the attainable order of the collocation solution at $t = h$) generalises the ones in Brunner (1997a) and Takama, Muroya and Ishiwata (2000). See also the sequel to this paper, Muroya, Ishiwata and Brunner (2003).

6

Volterra integral equations with weakly singular kernels

Volterra integral equations with weakly singular kernels (of algebraic or logarithmic type) typically have solutions whose derivatives are unbounded at the left endpoint of the interval of integration. Due to this singular behaviour the optimal global and local (super-) convergence results of Chapter 2 for collocation solutions in piecewise polynomial spaces on uniform meshes will no longer be valid. The use of appropriately graded meshes, or of non-polynomial collocation spaces on uniform meshes, are two of the possible alternative approaches for dealing with this order reduction problem.

6.1 Review of basic Volterra theory (III)

6.1.1 The Mittag-Leffler function

In Chapter 1 we encountered the special linear initial-value problem

$$y'(t) = \lambda y(t), \quad t \geq 0, \quad y(0) = y_0, \tag{6.1.1}$$

which is equivalent to the second-kind Volterra integral equation

$$y(t) = y_0 + \int_0^t \lambda y(s)ds, \quad t \geq 0, \tag{6.1.2}$$

and whose solution is given by $y(t) = \exp(\lambda t)y_0$. We then studied, in Chapter 5, a delay variant of this problem,

$$y'(t) = \lambda y(qt), \quad t \geq 0, \quad y(0) = y_0 \quad (0 < q < 1), \tag{6.1.3}$$

or, equivalently,

$$y(t) = y_0 + \int_0^{qt} (\lambda/q)y(s)ds, \quad t \geq 0 : \tag{6.1.4}$$

while its solution is also smooth for all $t \geq 0$, the analysis of the local superconvergence properties of the corresponding collocation solutions is vastly more complex.

Another variant of (6.1.2) and (6.1.4) arose in Chapter 4: for $\tau > 0$ the solution of

$$y(t) = y_0 + \int_0^{t-\tau} \lambda y(s)ds, \quad t > 0,$$

corresponding to an arbitrarily smooth initial function ϕ in the initial condition $y(t) = \phi(t),\ t \in [-\tau, 0]$, has low regularity at the points $\xi_\mu := \mu\tau$ $(\mu \geq 0)$. Smoothing occurs at ξ_μ as μ increases.

In the present chapter we meet an entirely different non-smooth behaviour of solutions. To illustrate this we choose as our starting point the VIE generalising (6.1.2), i.e.

$$y(t) = g(t) + \lambda \int_0^t (t-s)^{-\alpha} y(s)ds, \quad t \geq 0, \quad 0 < \alpha < 1, \tag{6.1.5}$$

with $g(t) = y_0$. The (unique) solution can be found in explicit form generalising the expression for the solution of (6.1.2), as shown in the following theorem (due to Hille and Tamarkin (1930); see also Friedman (1963)).

Theorem 6.1.1 *For any interval $I := [0, T]$ the unique solution $y \in C(I)$ of the VIE (6.1.5) with $0 < \alpha < 1$ is given by*

$$y(t) = E_{1-\alpha}(\lambda\Gamma(1-\alpha)t^{1-\alpha})y_0, \quad t \in I, \tag{6.1.6}$$

where

$$E_\beta(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k\beta)} \quad (\beta > 0) \tag{6.1.7}$$

denotes the Mittag-Leffler function.

Remark The *Mittag-Leffler function* was introduced early in the 20th century by the Swedish mathematician whose name it bears (see, e.g. his paper of 1903). It is an entire function of order $p = 1/\beta$ for any $\beta > 0$. For $\beta = 1/2$ we have

$$E_{1/2}(\pm z^{1/2}) = \exp(z)[1 + \operatorname{erf}(\pm z^{1/2})] = \exp(z)\operatorname{erfc}(\pm z^{1/2}),$$

with

$$\operatorname{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^z \exp(-v^2)dv \quad \text{and} \quad \operatorname{erfc}(z) := 1 - \operatorname{erf}(z)$$

denoting, respectively, the *error function* and the *complementary error function*.

For $\beta = 1$ we obtain of course $E_1(z) = \exp(z)$. Additional properties and applications can be found, for example, in Erdélyi (1955) and, especially, in the survey paper by Mainardi and Gorenflo (2000).

We do not prove Theorem 6.1.1 here since the result will be obtained as a special case of Theorem 6.1.2 (see Corollary 6.1.4). Note that for $\alpha = 0$ we recover the solution of (6.1.2), $y(t) = E_1(\lambda t) = \exp(\lambda t)$. If $\alpha \in (0, 1)$ the solution of (6.1.5) is no longer smooth on I: according to (6.1.6) near $t = 0^+$ its first derivative behaves like

$$y'(t) = \lambda y_0 t^{-\alpha} + \frac{(\lambda\Gamma(1-\alpha))^2}{\Gamma(2(1-\alpha))} y_0 t^{1-2\alpha} + \cdots .$$

As we shall see in the next section this representation also reflects the general situation: the solutions of general linear (and nonlinear) second-kind VIEs with algebraic kernel singularity $p_\alpha(t-s)$ $(0 < \alpha < 1)$, but otherwise smooth data, are smooth on $(0, T]$ but have an unbounded first derivative at $t = 0$; in the terminology of Section 6.2.3, y lies in the *Hölder space* $C^{1-\alpha}(I)$.

6.1.2 Linear VIEs of the second kind

The linear Volterra integral operators $\mathcal{V}_\alpha : C(I) \to C(I)$ we will consider in this and the next chapter have as part of their kernels the weakly singular (integrable) convolution factor

$$p_\alpha(t-s) := \begin{cases} (t-s)^{-\alpha} & \text{if} \ \ 0 < \alpha < 1, \\ \log(t-s) & \text{if} \ \ \alpha = 1. \end{cases} \qquad (6.1.8)$$

(We note in passing that writing $(t-s)^{\alpha-1}$ instead of $(t-s)^{-\alpha}$ is a seemingly more obvious choice of notation. However, it will become clear that the one chosen here will have certain advantages in our analysis.) Hence, $\mathcal{V}_\alpha$ has the form

$$(\mathcal{V}_\alpha \phi)(t) := \int_0^t p_\alpha(t-s)K(t,s)\phi(s)ds, \quad t \in I := [0, T]; \qquad (6.1.9)$$

we will assume that $K \in C(D)$, with $K(t,t) \neq 0$ for $t \in I$. The nonlinear case will be treated in Section 6.1.4.

Since the kernel $H_\alpha(t,s) := p_\alpha(t-s)K(t,s)$ in the corresponding *linear* VIE,

$$y(t) = g(t) + (\mathcal{V}_\alpha y)(t), \quad t \in I, \qquad (6.1.10)$$

is integrable on D, Picard iteration will lead to a uniformly and absolutely convergent Neumann series with limit $R_\alpha(t,s)$, in analogy to Theorem 2.1.2.

Hence, the solution of (6.1.10) will possess a representation similar to (2.1.11), namely

$$y(t) = g(t) + \int_0^t R_\alpha(t, s)g(s)ds, \quad t \in I. \tag{6.1.11}$$

This is made more precise, first for $0 < \alpha < 1$, in the following theorem.

Theorem 6.1.2 *Assume that $K \in C(D)$, and let $0 < \alpha < 1$. Then for any $g \in C(I)$ the linear, weakly singular Volterra integral equation (6.1.10) possesses a unique solution $y \in C(I)$. This solution is given by (6.1.11): here, the resolvent kernel R_α corresponding to the kernel H_α inherits the weak singularity $(t-s)^{-\alpha}$ and has the form*

$$R_\alpha(t, s) = (t-s)^{-\alpha} Q(t, s; \alpha), \quad 0 \le s < t \le T, \tag{6.1.12}$$

where

$$Q(t, s; \alpha) := \sum_{n=1}^{\infty} (t-s)^{(n-1)(1-\alpha)} \Phi_n(t, s; \alpha). \tag{6.1.13}$$

The functions Φ_n are defined recursively by

$$\Phi_n(t, s; \alpha) := \int_0^1 (1-z)^{-\alpha} z^{(n-1)(1-\alpha)-1} K(t, s+(t-s)z) \\ \Phi_{n-1}(s+(t-s)z, s; \alpha)dz$$

($n \ge 2$), with $\Phi_1(t, s; \alpha) := K(t, s)$ and $\Phi_n(\cdot, \cdot; \alpha) \in C(D)$. Moreover, $Q(\cdot, \cdot; \alpha)$ solves the resolvent equations

$$Q(t, s; \alpha) = K(t, s) + (t-s)^{\alpha} \int_s^t (t-v)^{-\alpha}(v-s)^{-\alpha} K(t, v) Q(v, s; \alpha)dv,$$

$$Q(t, s; \alpha) = K(t, s) + (t-s)^{\alpha} \int_s^t (t-v)^{-\alpha}(v-s)^{-\alpha} Q(t, v; \alpha) K(v, s)dv$$

on D.

Proof The Picard iteration process for (6.1.10) defines an infinite sequence $\{y_n(t)\}$ by choosing $y_0(t) := g(t)$ and setting

$$y_n(t) := g(t) + (\mathcal{V}_\alpha y_{n-1})(t), \quad t \in I \quad (n \ge 1).$$

In complete analogy to Section 2.1.1 the resulting *iterated kernels* $H_n(t, s; \alpha)$ corresponding to $H_\alpha(t, s) := p_\alpha(t-s)K(t, s) =: H_1(t, s; \alpha)$ are obtained recursively by

$$H_n(t, s; \alpha) := \int_s^t H_1(t, v; \alpha) H_{n-1}(v, s; \alpha)dv, \quad (t, s) \in D \quad (n \ge 2).$$

Using the variable transformation $v = s + (t - s)z$ and an induction argument it is easy to prove the following result on the form of the iterated kernels.

Lemma 6.1.3 *Let* $0 < \alpha < 1$ *and* $K \in C(D)$, *with* $\bar{K} := \max\{|K(t,s)| : (t,s) \in D\}$. *Then the iterated kernels* $\{H_n(t,s;\alpha)\}$ *corresponding to the kernel* $H_1(t,s;\alpha) := H_\alpha(t,s)$ *in (6.1.9) can be written as*

$$H_n(t,s;\alpha) = (t-s)^{-\alpha}(t-s)^{(n-1)(1-\alpha)}\Phi_n(t,s;\alpha) \quad (n \geq 2),$$

with

$$\Phi_n(t,s;\alpha) := \int_0^1 (1-z)^{-\alpha} z^{(n-1)(1-\alpha)-1} K(t, s+(t-s)z) \, \Phi_{n-1}(s+(t-s)z, s;\alpha)dz.$$

Moreover, the terms $\Psi_n(t,s;\alpha) := (t-s)^{(n-1)(1-\alpha)}\Phi_n(t,s;\alpha)$ *can be bounded uniformly by*

$$|\Psi_n(t,s;\alpha)| \leq \bar{K}^n T^{(n-1)(1-\alpha)} \frac{(\Gamma(1-\alpha))^n}{\Gamma(n(1-\alpha))}.$$

We note in passing that the above uniform estimate for the iterated kernels corresponding to the weakly singular kernel $H_\alpha(t,s)$ was already given by Tychonoff (1938). The resulting uniform convergence of the Neumann series,

$$\sum_{n=1}^{\infty} \Psi_n(t,s;\alpha) =: Q(t,s;\alpha), \quad (t,s) \in D,$$

implies that $Q(\cdot,\cdot;\alpha) \in C(D)$ for all $\alpha \in (0,1)$. The representation (6.1.11) then follows, thus generalising the analogous result (2.1.11) of Theorem 2.1.2.

To show that this solution $y \in C(I)$ given by (6.1.11) is unique, we observe that the existence of another solution $z \in C(I)$ leads to

$$y(t) - z(t) = (\mathcal{V}_\alpha(y-z))(t) = \int_0^t H_\alpha(t,s)[y(s) - z(s)]ds, \quad t \in I.$$

Hence,

$$|y(t) - z(t)| \leq \bar{K} \int_0^t p_\alpha(t-s)|y(s) - z(s)|ds, \quad t \in I.$$

Since $0 < \alpha < 1$, the generalised Gronwall inequality dealt with in Theorem 6.1.17 yields, since $\gamma(t) \equiv 0$,

$$|y(t) - z(t)| \leq E_{1-\alpha}(\bar{K}\Gamma(1-\alpha)t^{1-\alpha}) \cdot 0 = 0 \quad \text{for all} \quad t \in I.$$

The assertion regarding uniqueness of y thus follows from the continuity of $|y - z|$.

Remark The first proposition in Lemma 6.1.3 shows that the iterated kernels $H_n(t, s; \alpha)$ become *bounded* (that is, continuous) on D when $n \geq N = N(\alpha)$: it is easily verified that this value of N is $N(\alpha) = \lceil 1/(1-\alpha) \rceil$.

Corollary 6.1.4 *Let $g \in C(I)$ and $0 < \alpha < 1$. Then the (unique) solution $y \in C(I)$ of the weakly singular VIE (6.1.5) can be written as*

$$\begin{aligned} y(t) &= \frac{d}{dt}\left(\int_0^t E_{1-\alpha}(\lambda\Gamma(1-\alpha)(t-s)^{1-\alpha})g(s)ds\right) \\ &= g(t) + \int_0^t \left(\frac{d}{dt}E_{1-\alpha}(\lambda\Gamma(1-\alpha)(t-s)^{1-\alpha})\right)g(s)ds, \quad t \in I. \end{aligned}$$

Hence, the resolvent kernel associated with the kernel $H_\alpha(t, s) := \lambda(t-s)^{-\alpha}$ of (6.1.5) is

$$R_\alpha(t, s) = \frac{d}{dt}E_{1-\alpha}(\lambda\Gamma(1-\alpha)(t-s)^{1-\alpha}), \quad (t, s) \in D;$$

it can be written as

$$R_\alpha(t, s) = (t-s)^{-\alpha}\sum_{n=1}^{\infty}\frac{(\lambda\Gamma(1-\alpha))^n}{\Gamma(n(1-\alpha))}(t-s)^{(n-1)(1-\alpha)}.$$

Proof For constant kernel, $K(t, s) = \lambda$, Lemma 6.1.3 yields

$$\Phi_n(t, s; \alpha) = \lambda^n\frac{(\Gamma(1-\alpha))^n}{\Gamma(n(1-\alpha))},$$

and hence

$$H_n(t, s; \alpha) = \Phi_n(t, s; \alpha)(t-s)^{n(1-\alpha)-1} \quad (n \geq 1).$$

It follows that the unique solution of (6.1.5) is given by

$$\begin{aligned} y(t) &= g(t) + \int_0^t \left(\sum_{n=1}^{\infty} H_n(t, s; \alpha)\right) g(s)ds \\ &= g(t) + \lambda\Gamma(1-\alpha)\int_0^t (t-s)^{-\alpha}\left(\sum_{n=1}^{\infty}\frac{[\lambda\Gamma(1-\alpha)(t-s)^{1-\alpha}]^{n-1}}{\Gamma(n(1-\alpha))}\right)g(s)ds, \end{aligned}$$

with uniformly convergent series, and this reduces to

$$y(t) = g(t) + \int_0^t \left(\sum_{n=1}^{\infty}\frac{[\lambda\Gamma(1-\alpha)]^n}{\Gamma(n(1-\alpha))}(t-s)^{n(1-\alpha)-1}\right) g(s)ds \quad t \in I. \tag{6.1.14}$$

Observe that

$$\sum_{n=1}^{\infty}\frac{(\lambda\Gamma(1-\alpha))^n}{\Gamma(n(1-\alpha))}(t-s)^{n(1-\alpha)-1} = \frac{d}{dt}E_{1-\alpha}(\lambda\Gamma(1-\alpha)(t-s)^{1-\alpha}),$$

since $\Gamma(1+n(1-\alpha)) = n(1-\alpha)\Gamma(n(1-\alpha))$. The last statement in Corollary 6.1.4 follows by recalling that $E_\beta(0) = 1$ for all $\beta > 0$.

For $g(t) \equiv 1$ we can explicitly compute the above integrals (note that the infinite series converges absolutely and uniformly on D), and this allows us to derive the expression (6.1.6) for the solution of the special VIE (6.1.5).

We have seen in Section 2.1.1 that there exists an alternative representation of the solution in terms of the 'integrated' resolvent kernel $U(t, s)$ (cf. Theorem 2.1.4). This result extends to VIEs with weakly singular kernels.

Theorem 6.1.5 *Assume that $g \in C^1(I)$ and $K \in C(D)$, and let $\alpha \in (0, 1)$. Then the solution $y \in C(I)$ of (6.1.10) can be written as*

$$y(t) = U_\alpha(t, 0)g(0) + \int_0^t U_\alpha(t, s)g'(s)ds, \quad t \in I. \tag{6.1.15}$$

The function $U_\alpha = U_\alpha(t, s)$ is related to the resolvent $R_\alpha = R_\alpha(t, s)$ by

$$-\frac{\partial U_\alpha(t, s)}{\partial s} = R_\alpha(t, s), \quad 0 \le s < t \le T. \tag{6.1.16}$$

Proof Using integration by parts to rewrite the integral on the right-hand side of (6.1.15) we derive

$$\begin{aligned} y(t) &= U_\alpha(t, 0)g(0) + \left(U_\alpha(t, s)g(s)|_0^t - \int_0^t \frac{\partial U_\alpha(t, s)}{\partial s} g(s)ds \right) \\ &= U_\alpha(t, t)g(t) - \int_0^t \frac{U_\alpha(t, s)}{\partial s} g(s)ds, \quad t \in I. \end{aligned}$$

We already know from Theorem 6.1.2 that the VIE has a unique solution $y \in C(I)$ for any $\alpha \in (0, 1)$. Hence, by comparing the above expression for y with the one in (6.1.11) we deduce that (6.1.16) must hold uniquely.

The proofs of the previous results on the resolvent representation of the solution y of (6.1.10) also contain information on the regularity of y: for $0 < \alpha < 1$ it confirms that Theorem 6.1.1 in fact reflects the general qualitative regularity behaviour of the solution of (6.1.10) near $t = 0^+$.

Theorem 6.1.6 *Assume that $g \in C^m(I)$ and $K \in C^m(D)$, with $K(t, t) \neq 0$ on I. Then:*

(i) For any $\alpha \in (0, 1)$ the functions $\Phi_n(t, s; \alpha)$ $(n \ge 1)$ in (6.1.13) defining $Q(t, s; \alpha)$ lie in the space $C^m(D)$, and the regularity of the unique solution of the weakly singular VIE (6.1.10) is described by

$$y \in C^m(0, T] \cap C(I), \quad \textit{with} \quad |y'(t)| \le C_\alpha t^{-\alpha} \quad \textit{for} \quad t \in (0, T].$$

(ii) The solution y can be written in the form

$$y(t) = \sum_{(j,k)_\alpha} \gamma_{j,k}(\alpha) t^{j+k(1-\alpha)} + Y_m(t;\alpha), \quad t \in I. \tag{6.1.17}$$

Here, $(j,k)_\alpha := \{(j,k) : \ j,k \in \mathbb{N}_0, \ j + k(1-\alpha) < m\}$ *and* $Y_m(\cdot\,;\alpha) \in C^m(I)$*. The coefficients* $\gamma_{j,k}(\alpha)$ *are defined in the proof below.*

Proof The assertion regarding the regularity of y follows straightforwardly from the proof of Theorem 6.1.2, since $K \in C^m(D)$ implies – by Lemma 6.1.3 – that $\Phi_n(\cdot,\cdot;\alpha)$ possesses the same regularity: $\Phi_n(\cdot,\cdot;\alpha) \in C^m(D)$ $(n \geq 1)$ for any $\alpha \in (0,1)$.

Consider now the solution representation described by (6.1.11) and Theorem 6.1.2. By the uniform convergence of the infinite series defining $Q(t,s;\alpha)$ we may write

$$\int_0^t R_\alpha(t,s)g(s)ds = \sum_{k=1}^{\infty} \int_0^t (t-s)^{k(1-\alpha)-1} G_k(t,s;\alpha)ds,$$

where $G_k(t,s;\alpha) := \Phi_k(t,s;\alpha)g(s)$. It follows from the assumed regularity of g and K that $G_k(\cdot,\cdot;\alpha) \in C^m(D)$ $(k \geq 1)$. Hence, by Taylor's formula and by employing the more convenient multi-index notation $d := (d_1,d_2)$ $(d_i \in \mathbb{N}_0)$, with

$$|d| := d_1 + d_2, \quad d! := d_1!d_2!, \quad \mathbf{t}^d := t^{d_1}s^{d_2}, \quad D^d := \frac{\partial^{|d|}}{\partial^{d_1}\partial^{d_2}},$$

we write

$$G_k(t,s;\alpha) = \sum_{|d|<m} \frac{1}{d!} D^d G(0,0;\alpha)\mathbf{t}^d + \sum_{|d|=m} \frac{1}{d!} G(\zeta_1,\zeta_2;\alpha)\mathbf{t}^d.$$

Note that

$$\begin{aligned}\int_0^t (t-s)^{k(1-\alpha)-1} s^j ds &= t^{j+k(1-\alpha)} \int_0^1 (1-v)^{k(1-\alpha)-1} v^j dv \\ &= B(k(1-\alpha), j+1) \cdot t^{j+k(1-\alpha)},\end{aligned}$$

with $B(\cdot,\cdot)$ denoting the *Euler beta function* (compare also the remark following Theorem 6.1.13 in Section 6.1.3). By suitably rearranging all these terms, and by adding the contribution due to g,

$$g(t) = \sum_{j=0}^{m-1} \frac{g^{(j)}(0)}{j!} t^{j-1} + \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} g^{(m)}(s)ds, \quad t \in I,$$

the solution representation (6.1.11) can be expressed in the form

$$y(t) = \sum_{(j,k)_\alpha} \gamma_{k,j}(\alpha) t^{j+k(1-\alpha)} + Y_m(t;\alpha), \quad t \in I,$$

where $Y_m(t;\alpha)$ comprises those terms containing $t^{j+k(1-\alpha)}$ with $j + k(1-\alpha) \geq m$, and all Taylor remainder terms. This completes the proof of Theorem 6.1.6. (Compare also Cerezo (1996) and Cao, Herdman and Xu (2003) for a representation very similar to (6.1.17) and for an alternative proof.)

Remark If the given functions g and K are (real) *analytic* in their domains, then it can be shown (see Lubich (1983a) that there is a function $Y = Y(z_1, z_2)$, real and *analytic* at (0, 0), so that solution of the VIE (6.1.10) $(0 < \alpha < 1)$ can be written as $y(t) = Y(t, t^{1-\alpha})$. Related regularity results can be found in the papers by Miller and Feldstein (1971) and de Hoog and Weiss (1974).

The existence and uniqueness of a solution $y \in C(I)$ of (6.1.10) is also guaranteed if the kernel singularity is of *logarithmic* type, $p_1(t-s) := \log(t-s)$. We summarise this in the following theorem (but leave its proof as an exercise).

Theorem 6.1.7 *Let $\alpha = 1$ and $K \in C(D)$ in (6.1.10). Then for any $g \in C(I)$ the VIE*

$$y(t) = g(t) + \int_0^t \log(t-s) K(t,s) y(s) ds, \quad t \in I,$$

possesses a unique solution $y \in C(I)$. If $g \in C^m$ and $K \in C^m(D)$ then

$$y \in C^m(0, T] \cap C(I), \quad \textit{with} \quad |y'(t)| \leq C|\log(t)|, \quad t \in (0, T].$$

We conclude this section with a generalisation of some of the above regularity results: they cover VIEs with bounded but non-smooth kernels, and equations with non-smooth right-hand sides g.

Consider first the VIE

$$y(t) = g(t) + (\mathcal{V}_\nu y)(t), \quad t \in I, \tag{6.1.18}$$

corresponding to the Volterra integral operator (with a slight abuse of our previous notation)

$$(\mathcal{V}_\nu y)(t) := \int_0^t (t-s)^\nu K(t,s) y(s) ds$$

with $\nu := \rho - \alpha$, $\rho \in \mathbb{N}$, $0 < \alpha < 1$, and $K \in C(D)$, $K(t,t) \neq 0$ $(t \in I)$. A look at Theorem 6.1.2 and its proof shows that they, and the result of Lemma 6.1.3, remain valid if the role of $-\alpha$ is now assumed by $\nu = \rho - \alpha$. Hence, we readily derive

Theorem 6.1.8 *Let $\nu := \rho - \alpha$, with $\rho \in \mathbb{N}$ and $0 < \alpha < 1$. Then the unique solution $y \in C(I)$ of (6.1.18) is given by*

$$y(t) = g(t) + \int_0^t R_\nu(t,s)g(s)ds, \quad t \in I,$$

with resolvent kernel

$$R_\nu(t,s) = (t-s)^\nu Q(t,s;\nu)$$

and

$$Q(t,s;\nu) := \sum_{n=1}^{\infty}(t-s)^{(n-1)(1+\nu)}\Phi_n(t,s;\nu).$$

The (continuous) functions $\Phi_n(\cdot,\cdot;\nu)$ correspond to the ones introduced in Lemma 6.1.3, with $-\alpha$ replaced by ν.

If in addition we assume that $g \in C^m(I)$, $K \in C^m(D)$ $(m \geq 1)$, with $K(t,t) \neq 0$ on I, then the solution lies in $C^\rho(I)$ whenever $1 \leq \rho < m$, while $y^{(\rho+1)}(t)$ near $t = 0^+$ behaves like $t^{-\alpha}$. If $\rho \geq m$ then $y \in C^m(I)$.

Remark In Section 6.2.3 we will adopt the more concise (classical) notation to describe the regularity properties of solutions to VIEs with weakly singular kernels of algebraic type, by introducing the notion of a *Hölder space*. In that terminology, the regularity result in the above theorem will read: $y \in C^{\rho,1-\alpha}(I)$, with $1 \leq \rho < m$.

Theorem 6.1.8 yields an obvious generalisation of the result we met in Corollary 6.1.4, namely:

Corollary 6.1.9 *Assume that $g \in C(I)$, and let $\nu := \rho - \alpha (\rho \in \mathbb{N}, 0 < \alpha < 1)$. Then the (unique) solution of the integral equation*

$$y(t) = g(t) + \lambda \int_0^t (t-s)^\nu y(s)ds, \quad t \in I,$$

is

$$\begin{aligned} y(t) &= \frac{d}{dt}\left(\int_0^t E_{1+\nu}(\lambda\Gamma(1+\nu)(t-s)^{1+\nu})g(s)ds\right) \\ &= g(t) + \int_0^t \left(\frac{d}{dt}E_{1+\nu}(\lambda\Gamma(1+\nu)(t-s)^{1+\nu})\right)g(s)ds, \quad t \in I. \end{aligned}$$

For $g(t) \equiv y_0$ we obtain the generalisation of (6.1.6),

$$y(t) = E_{1+\nu}(\lambda\Gamma(1+\nu)t^{1+\nu})y_0, \quad t \in I.$$

The last statement in this theorem will often form the basis for finding the solutions of somewhat more general VIEs with weakly singular kernels, in complete analogy to Theorem 2.1.6 for second-kind VIEs with regular convolution kernels. A particular case corresponds to the choice $k(t-s) = p_\alpha(t-s)$ $(0 < \alpha \leq 1)$.

The next result (extending Theorem 2.1.6) puts Corollary 6.1.9 into a somewhat more general context.

Theorem 6.1.10 *Consider the linear convolution equations*

$$y(t) = g(t) + \int_0^t k(t-s)y(s)ds, \quad t \in I, \tag{6.1.19}$$

and

$$w(t) = 1 + \int_0^t k(t-s)w(s)ds, \quad t \in I. \tag{6.1.20}$$

Assume that $g \in C^1(I)$, *and* $k \in L^1(I)$. *Then the (unique) solutions* $y \in C(I)$ *and* $w \in C(I)$ *of (6.1.19) and (6.1.20) are related by*

$$\begin{aligned} y(t) &= g(0)w(t) + \int_0^t w(t-s)g'(s)ds \\ &= w(0)g(t) + \int_0^t w'(t-s)g(s)ds, \quad t \in I. \end{aligned} \tag{6.1.21}$$

Proof We leave it as an exercise. The reader is also referred to Bellman and Cooke (1963).

The final result in this section forms the basis for analysing the effect of a non-smooth function g on the regularity of the solution of the weakly singular VIE (6.1.10).

Theorem 6.1.11 *Let* $g(t) = g_1(t) + t^\beta g_2(t)$, *with* $g_i \in C(I)$ $(i = 1, 2)$ *and* $\beta > 0$ $(\beta \notin \mathbb{N})$, *and assume that* $K \in C(D)$. *Then the (unique) solution* $y \in C(I)$ *of (6.1.10) with this function* g *can be written as*

$$\begin{aligned} y(t) = g_1(t) &+ \int_0^t R_\alpha(t,s)g_1(s)ds \\ &+ t^\beta g_2(t) + \int_0^t R_\alpha(t,s)g_2(s)s^\beta ds, \quad t \in I. \end{aligned}$$

Here, $R_\alpha(t,s)$ *is the resolvent kernel given by (6.1.12) in Theorem 6.1.2.*

The proof uses the *superposition principle* for solutions of linear VIEs with the same kernel but different non-homogeneous terms, and the result of Theorem 6.1.2.

Remarks

1. The statement of Theorem 6.1.11 yields a regularity result for the solution of (6.1.10) corresponding to $g_1(t) \equiv 0$. An obvious modification of Theorem 6.1.6 and its proof then leads to the analogue of the representation (6.1.17) of y. See also Exercise 6.6.6.
2. Results on the regularity of the solution of more general linear (and nonlinear) VIEs with weakly singular, or other types of bounded but non-smooth kernels involving both algebraic and logarithmic terms can be found in Brunner, Pedas and Vainikko (1999, pp. 1080–1082).

6.1.3 Nonlinear VIEs of the second kind

Since the singular term $p_\alpha(t-s)$ in the general nonlinear second-kind VIE,

$$y(t) = g(t) + \int_0^t p_\alpha(t-s)k(t,s,y(s))ds, \quad t \in I \quad (0 < \alpha \leq 1), \qquad (6.1.22)$$

is integrable, it can be shown in a straightforward way that the existence and uniqueness result of Theorem 2.1.10 remains valid: however, the number δ_0 defining the existence interval I_0 now depends on α. We leave the proof as an exercise (Exercise 6.6.5) and instead consider briefly the Hammerstein-type Volterra equation

$$y(t) = g(t) + (\mathcal{H}_\alpha y)(t), \quad t \in I, \qquad (6.1.23)$$

where the weakly singular Volterra–Hammerstein operator is

$$(\mathcal{H}_\alpha y)(t) := \int_0^t p_\alpha(t-s)K(t,s)G(s,y(s))ds.$$

The functions p_α and K are subject to the assumptions stated at the beginning of Section 6.1.2, and $G: \ I \times \Omega \subset \mathbb{R} \to \mathbb{R}$ is smooth.

The short discussion we presented at the end of Section 2.1.5 is easily adapted to cover the above VHIE with weakly singular kernel: with the Niemytzki operator $\mathcal{N}$ as in (2.1.44),

$$z(t) := (\mathcal{N}y)(t) = G(t, y(t)),$$

(6.1.23) yields an implicitly linear VIE for z,

$$z(t) = G(t, g(t) + (\mathcal{V}_\alpha z)(t)), \quad t \in I, \qquad (6.1.24)$$

and this is followed by the recursion

$$y(t) = g(t) + (\mathcal{V}_\alpha z)(t), \quad t \in I. \qquad (6.1.25)$$

The operator $\mathcal{V}_\alpha$ is our linear, weakly singular Volterra integral operator,

$$(\mathcal{V}_\alpha z)(t) := \int_0^t p_\alpha(t-s)K(t,s)z(s)ds.$$

We will return to this reformulation in Section 6.2.11.

The analysis of second-kind VIEs with weakly singular kernels and Hammerstein nonlinearities has its origin in the early 1950s. Such equations arise in the modelling of one-dimensional heat flow with radiation cooling at the boundary, and they typically have the form

$$y(t) = \int_0^t (t-s)^{-\alpha} G(y(s))ds,$$

with $G(y) = \gamma(1-y^\nu); (\gamma > 0,\ \nu = 4)$. Mann and Wolf (1951) showed that for $\alpha = 1/2$, the solution y is increasing and satisfies

$$0 < y(t) < 1 \quad (t > 0), \quad \text{and} \quad \lim_{t\to\infty} y(t) = 1.$$

These results were extended by Roberts and Mann (1951) to arbitrary $\alpha \in (0, 1)$, and by Padmavally (1958) to nonlinearities $G(s, y(s))$. The papers by Nohel (1964, 1976) and Miller (2000) survey this development and contain additional references.

6.1.4 Linear VIEs of the first kind

We should of course have started the presentation of the classical theory of Volterra integral equations with weakly singular kernels with Niels Henrik Abel's classical results of 1823 and 1826 on the solution of the first-kind integral equation

$$\int_0^t (t-s)^{-\alpha} y(s)ds = g(t), \quad t \in (0, T] \quad (0 < \alpha < 1), \tag{6.1.26}$$

now named after him. In these papers he derived the *inversion formula*

$$y(t) = \frac{1}{\gamma_\alpha}\frac{d}{dt}\left(\int_0^t (t-s)^{\alpha-1} g(s)ds\right), \quad t \in (0, T], \tag{6.1.27}$$

with $\gamma_\alpha := \pi/\sin(\alpha\pi) = \Gamma(\alpha)\Gamma(1-\alpha)$, provided the function

$$G_\alpha(t) := \int_0^t (t-s)^{\alpha-1} g(s)ds \tag{6.1.28}$$

has a continuous derivative on $(0, T]$. This is certainly true if $g \in C^1(I)$ $(I := [0, T])$; if, in addition, we have $g(0) = 0$ then the solution y lies in $C(I)$ and is given by

$$y(t) = \frac{1}{\gamma_\alpha} \int_0^t (t-s)^{\alpha-1} g'(s)ds, \quad t \in I.$$

We summarise Abel's result in the following theorem.

Theorem 6.1.12 *Let $g \in C^1(I)$. Then for any $\alpha \in (0,1)$ the Abel integral equation (6.1.26) possesses a unique continuous solution on $(0, T]$. This solution can be written in the form*

$$y(t) = \frac{1}{\gamma_\alpha} \left(g(0)t^{\alpha-1} + \int_0^t (t-s)^{\alpha-1} g'(s)ds \right), \quad t \in (0, T]. \tag{6.1.29}$$

In his Nota II of 1896, Vito Volterra extended both his approach of Nota I and Abel's result (and his key idea in the proof) of 1823/26 to the more general first-kind integral equation

$$(\mathcal{V}_\alpha y)(t) = g(t), \quad t \in I = [0, T] \;\; (0 < \alpha < 1), \tag{6.1.30}$$

with $\mathcal{V}_\alpha$ as in (6.1.9) and $g(0) = 0$. He showed, by multiplying the equation by $(z-t)^{\alpha-1}$ and then integrating with respect to t over $[0, z]$, that the given equation (6.1.31) can be written as a first-kind VIE with *regular* (bounded) kernel,

$$\int_0^t H(t, s; \alpha) y(s)ds = G_\alpha(t), \quad t \in I, \tag{6.1.31}$$

where G_α is the function defined in (6.1.28) and

$$H(t, s; \alpha) := \int_0^1 \frac{K(s + (t-s)v, s)}{v^\alpha (1-v)^{1-\alpha}} dv.$$

Observe that this kernel $H(\cdot, \cdot; \alpha)$ inherits the regularity of the original kernel K: if $K \in C^m(D)$ then $H(\cdot, \cdot; \alpha) \in C^m(D)$. Moreover, $H(t, t; \alpha) = K(t, t)/\gamma_\alpha$. We shall return to these facts in Theorems 6.1.13 and 6.1.14.

The following theorem contains Volterra's fundamental result (Volterra (1896a, Nota II)).

Theorem 6.1.13 *Assume that*

(a) $g \in C^1(I)$, with $g(0) = 0$;
(b) $K \in C(D)$, $\partial K/\partial t \in C(D)$, with $|K(t, t)| \geq k_0 > 0$ when $t \in I$.

Then for any $\alpha \in (0, 1)$ the following is true:

(i) The first-kind VIE (6.1.30) possessing the weakly singular kernel $H_\alpha(t, s) := (t-s)^{-\alpha} K(t, s)$ is equivalent to the first-kind VIE (6.1.31) with bounded kernel $H(t, s; \alpha)$ and with non-smooth right-hand side $G_\alpha(t)$.

(ii) *The kernel $H(t, s; \alpha)$ and the right-hand side $G_\alpha(t)$ satisfy the hypotheses for $K(t, s)$ and $g(t)$ in Theorem 2.1.8, and hence the given VIE (6.1.30) possesses a unique solution $y \in C(I)$.*

We leave the proof as a simple exercise. It makes use of the fact that

$$\int_0^1 v^{\mu-1}(1-v)^{\nu-1}dv = B(\mu, \nu) = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} \quad (\mu, \nu > -1),$$

where $B(\cdot, \cdot)$ denotes *Euler's beta function* (see, e.g. Henrici (1962, pp. 24–62)).

We shall see in Theorem 6.1.14 that while the solution y is continuous on the interval $[0, T]$, its derivative near $t = 0^+$ will behave like $t^{\alpha-1}$, for any non-trivial C^d-data with $d \geq 2$.

Due to this equivalence between the first-kind VIE with weakly singular kernel and smooth right-hand side, and a first-kind VIE with smooth kernel but non-smooth right-hand side, the proof of Theorem 2.1.9 can be adapted to yield an analogous regularity result for the weakly singular VIE (6.1.31). Note, however, that since G_α is smooth only on the left-open interval $(0, T]$, the same will be true of the solution y when g and K are smooth. In other words, as the previous theorem already suggests, the regularity of $G_\alpha(t)$ at $t = 0$ will depend on the values of $g^{(\nu)}(0)$ $(\nu \geq 0)$.

In order to make this more precise we recall the definition of the *Pochhammer symbol*,

$$(\alpha)_k := \alpha(\alpha+1)\cdots(\alpha+k-1), \quad k \geq 1 \ (k \in \mathbb{N}).$$

Theorem 6.1.14 *Assume:*

(a) $g \in C^{m+1}(I)$;
(b) $K \in C^{m+1}(D)$, *with* $|K(t, t)| \geq k_0 > 0$ *when* $t \in I$;
(c) $g^{(\nu)}(0) = 0$ *for* $\nu = 0, 1, \ldots, q$ $(q < m)$.

Then the unique solution of (6.1.30) lies in the space $C^q(I) \cap C^m(0, T]$ for all $\alpha \in (0, 1)$, and $|y^{(q+1)}(t)| \leq Ct^{\alpha-1}$ on $(0, T]$.

For $q = 0$ the solution of (6.1.30) has a representation similar to (6.1.17) in Theorem 6.1.6, with α replacing $1 - \alpha$.

In the terminology to be introduced in Section 6.2.3 the solution lies in the Hölder space $C^{q,\alpha}(I)$.

The **proof** of Theorem 6.1.14 is based on the observation that, by (c), the function G_α in (6.1.27) can be written as

$$G_\alpha(t) = \frac{1}{(\alpha)_{q+1}} \int_0^t (t-s)^{q+\alpha} g^{(q+1)}(s)ds, \quad t \in I. \tag{6.1.32}$$

Hence it follows that $G_\alpha \in C^{q+1}(I)$, with $G_\alpha^{(\nu)}(0) = 0$ $(\nu = 0, \ldots, q)$.

Remark In his 1916 paper (Chapter 4) Volterra analysed the solution of first-kind VIEs whose kernels contain weakly singular factors of both algebraic and *logarithmic* type. He showed in particular that the derivation of the solution of the VIE corresponding essentially to $p_\alpha(t-s)$ with $\alpha = 1$ (cf. (6.1.8)),

$$\int_0^t [\log(t-s)+\gamma]y(s)ds = g(t), \quad t \in I := [0,T],$$

with $g(0) = 0$, and with

$$\gamma := -\Gamma'(1)/\Gamma(1) = -\int_0^\infty \exp(-s)\log(s)ds \doteq 0.57721$$

denoting the *Euler constant*, is considerably more complex than in the case $0 < \alpha < 1$. The starting point of the analysis is the fact that this VIE is equivalent to a first-kind equation of the form

$$\int_0^t \left(\frac{d}{d\alpha} \left(\frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1} \right) \Big|_{\alpha=1} \right) y(s)ds = g(t).$$

Details and results on related VIEs can also be found in Krasnov et al. (1977), pp. 141–143, Srivastava and Buschman (1977, p. 87), and in volume 3 of Fenyö and Stolle (1984).

We continue by briefly touching upon the first-kind Abel-type VIE corresponding to the 'non-standard' Volterra–Abel integral operator

$$(\mathcal{A}_\alpha \phi)(t) := \int_0^t (h(t)-h(s))^{-\alpha} K(t,s)\phi(s)ds, \quad 0 < \alpha < 1, \qquad (6.1.33)$$

with $K \in C(D)$, $K(t,t) \neq 0$ $(t \in I$, and $h \in C^1(I)$, $h'(t) > 0$ $(t > 0)$. In many applications (see, e.g. Anderssen (1977) and its bibliography) we have $h(t) = t^p$, $p > 1$. In this case we will denote the corresponding integral operator (6.1.29) by $\mathcal{A}_{p,\alpha}$.

The following theorem is concerned with this particular case; it is readily extended to encompass the general case, and the reader may wish to consult Schmeidler (1950), Sneddon (1972), Anderssen (1976, 1977), and Hung (1979). The regularity result (ii) is due to Atkinson (1974a) (see also Lubich (1987)). In addition, compare Smarzewski and Malinowksi (1978, 1983) where Volterra–Abel integral equations corresponding to the adjoint operator $\mathcal{A}^*_{p,\alpha}$,

$$(\mathcal{A}^*_{p,\alpha}\phi)(t) := \int_t^T (s^p - t^p)^{-\alpha} K(t,s)\phi(s)ds,$$

are studied.

Theorem 6.1.15 *Consider the Abel-type integral equation*

$$(\mathcal{A}_{p,\alpha} y)(t) = g(t), \quad t \in I := [0, T] \quad (0 < \alpha < 1, \ p > 1).$$

Assume that $g \in C^1(I)$ *and* $K \in C^1(D)$, *with* $|K(t,t)| \geq k_0 > 0$ *for* $t \in I$.

(i) *If* $K(t,t) \equiv 1$ *then the (unique) solution* $y \in C(0, T]$ *of* $(\mathcal{A}_\alpha y)(t) = g(t)$ *is given by the inversion formula*

$$y(t) = \frac{p}{\gamma_\alpha}\left(g(0)t^{\alpha p - 1} + t^{p-1}\int_0^t (t^p - s^p)^{\alpha-1} g'(s)ds\right), \quad t \in (0, T].$$

(ii) *If* $g(t) = g_0(t)t^\beta$, *with* $g_0 \in C^{m+1}(I)$ *and* $\beta > -p\alpha$, *and* $K \in C^{m+1}(D)$, *then the solution is of the form*

$$y(t) = t^{p\alpha+\beta-1}[c_0 + t\phi_0(t)], \quad t \in (0, T],$$

where $\phi_0 \in C^m(I)$ *and* $c_0 = 0$ *if, and only if,* $g_0(0) = 0$.

We conclude with a remark on the fundamental difference between the integral operators $\mathcal{V}_\alpha$ and $\mathcal{A}_{p,\alpha}$ $(0 < \alpha < 1, \ p > 1)$; we illustrate this with the example $p = 2, \ \alpha = 1/2$ and $K(t,s) \equiv 1$ (see also Atkinson (1997a), p. 20). If we define $\phi_\beta(t) := t^\beta$, direct computation shows that, for any $\beta \geq 0, \ \beta \in \mathbb{R}$,

$$(\mathcal{A}_{2,1/2}\phi_\beta)(t) = \int_0^t (t^2 - s^2)^{-1/2}\phi_\beta(s)ds = \lambda_\beta \phi_\beta(t), \quad t \in I,$$

with $(\mathcal{A}_{2,1/2}\phi)(0) := 0$ and

$$\lambda_\beta := \int_0^1 (1 - s^2)^{-1/2} s^\beta ds \ \in \ (0, \pi/2].$$

In other words, the integral operator $\mathcal{A}_{2,\alpha}$ $(0 < \alpha < 1)$ has a *continuous spectrum* $\sigma(\mathcal{A}_{2,\alpha}) = (0, \pi/2]$. This is equivalent to the statement that $\mathcal{A}_{2,1/2}$ is not a compact operator from $C(I) \to C(I)$. Thus, not surprisingly, the analysis of piecewise collocation methods for first-kind and second-kind integral equations described by such Abel-type operators is considerably more complex; it remains essentially open (Exercise 6.6.17).

6.1.5 Nonlinear VIEs of the first kind

As we shall see in Chapter 8, nonlinear VIEs of the first kind occur for example in systems of integral-algebraic equations, replacing the algebraic constraints in a DAE (Chapter 8). Their kernel functions are usually of Hammerstein type, and hence we will restrict our present discussion to VIEs of the form

$$(\mathcal{H}_\alpha y)(t) := \int_0^t p_\alpha(t-s)K(t,s)G(s,y(s))ds = g(t),$$
$$t \in I := [0,T] \quad (0 \le \alpha \le 1), \tag{6.1.34}$$

with $g(0) = 0$ and $K(t,t) \neq 0$ on I.

The following result is due to Deimling (1995). Related results can be found in Gladwin and Jeltsch (1974), Branca (1976, 1978), and Dixon, McKee and Jeltsch (1986).

Theorem 6.1.16 *Assume:*

(a) $g \in C^1(I)$*, with* $g(0) = 0$*;*
(b) $K \in C^1(D)$*, with* $|K(t,t)| \ge k_0 > 0,\ t \in I$*;*
(c) $G:\ I \times \mathbb{R} \to \mathbb{R}$ *is continuous and satisfies*

$$(G(t,y) - G(t,z))(y-z) > 0, \quad t \in I,\ y,z \in \mathbb{R}\ (y \neq z);$$

(d) $\lim_{|y|\to\infty} \frac{G(t,y)y}{|y|} \to \infty \quad (t \in I)$.

Then the nonlinear Volterra–Hammerstein equation (6.1.34) possesses a unique solution $y \in C(I)$ *for any* $\alpha \in [0,1)$*. For* $0 < \alpha < 1$ *and sufficiently regular functions* g*,* K *and* G *its regularity properties coincide with those described in Theorem 6.1.14. In particular, if* $g'(0) \neq 0$ *then* $|y'(t)| \le Ct^{\alpha-1}$ *near* $t = 0^+$*.*

Remark The above result remains valid for *systems* of first-kind Volterra–Hammerstein integral equations when the products and absolute values are replaced, respectively, by the standard inner product in $\mathbb{R}^m$ and the induced Euclidian norm (see Deimling (1995) and Section 8.1.2).

Proof Since $0 < \alpha < 1$ we can adopt Volterra's idea of rewriting the given VIE as an equivalent, now nonlinear, first-kind equation with bounded kernel function. Recalling the remarks preceding Theorem 6.1.13 this VIE is

$$\int_0^t H(t,s;\alpha)G(s,y(s))ds = G_\alpha(t), \quad t \in I, \tag{6.1.35}$$

with $H(t,s;\alpha)$ and $G_\alpha(t)$ as in (6.1.27). Differentiation with respect to t leads to

$$H(t,t;\alpha)G(t,y(t)) + \int_0^t \frac{\partial H(t,s;\alpha)}{\partial t} G(s,y(s))ds = G'_\alpha(t). \tag{6.1.36}$$

Since by (b) (see also (ii) in Theorem 6.1.13)) we have $H(t,t;\alpha) \neq 0$ on I, we have to show that this *implicit* second-kind VIE has a unique continuous solution on I. To this end, set $z(t) := G(t, y(t))$ and consider the integral equation

$$z(t) = F_\alpha(t) + \int_0^t H_1(t, s; \alpha) z(s) ds, \quad t \in I, \tag{6.1.37}$$

where

$$F_\alpha(t) := G'_\alpha(t)/H(t, t; \alpha), \quad H_1(t, s; \alpha) := -[\partial H(t, s; \alpha)/\partial t]/H(t, t; \alpha).$$

It follows from the theory of linear second-kind VIEs and assumptions (a), (b) that there exists a unique $z \in C(I)$ solving (6.1.35). The unique solvability of

$$G(t, y(t)) = z(t), \quad t \in I,$$

is now a consequence of the assumptions (c) and (d).

6.1.6 Weakly singular Volterra equations with non-vanishing delays

Let θ be a delay function satisfying the conditions (D1)–(D3) of Section 4.1.2:

(D1) $\theta(t) = t - \tau(t), \;\; \tau \in C^d(I)$ for some $d \geq 0$;
(D2) $\tau(t) \geq \tau_0 > 0$ for $t \in I$;
(D3) θ is strictly increasing on I.

Here, we assume that $I := [t_0, T]$ for some $t_0 \geq 0$.

For given θ and $\alpha \in (0, 1]$ we define the Volterra integral operator $\mathcal{V}_{\theta,\alpha}$ by

$$(\mathcal{V}_{\theta,\alpha} y)(t) := \int_0^{\theta(t)} p_\alpha(t - s) K_2(t, s) y(s) ds, \quad t \in I := [0, T], \tag{6.1.38}$$

where $K_2 \in C^d(D_\theta)$ for some $d \geq 0$. We also introduce

$$(\mathcal{W}_{\theta,\alpha} y)(t) := \int_{\theta(t)}^t p_\alpha(t - s) K(t, s) ds, \quad t \in I.$$

How does a non-vanishing delay affect the regularity of the solutions of the second-kind VIEs

$$y(t) = g(t) + (\mathcal{V} y)(t) + (\mathcal{V}_{\theta,\alpha} y)(t), \quad t \in I, \tag{6.1.39}$$

and

$$y(t) = g(t) + (\mathcal{W}_{\theta,\alpha} y)(t), \quad t \in I,$$

with $y(t) := \phi(t)$ if $t \leq 0$, when all the given functions are smooth, e.g. have continuous derivatives of order m on their respective domains?

Table 6.1. *Regularity and smoothing of solutions to weakly singular delay VIEs*

Delay Volterra integral equation (with arbitrarily smooth data)	Regularity on $I^{(\mu)} = (\xi_\mu, \xi_{\mu+1}]$ ($\mu = 0, 1, \dots, M$)
• $y(t) = g(t) + (\mathcal{V}_{\theta,\alpha} y)(t)$	$\begin{cases} C^{\mu,1-\alpha} & \text{if } \mu = 0, 1, \dots, \min\{m, M\} \\ C^m & \text{if } \mu > \min\{m, M\} \end{cases}$ (finite jump at $t = t_0$)
• $y(t) = g(t) + (\mathcal{W}_{\theta,\alpha} y)(t)$	$\begin{cases} C^{\mu,1-\alpha} & \text{if } \mu = 0, 1, \dots, \min\{m, M\} \\ C^m & \text{if } \mu > \min\{m, M\} \end{cases}$ (finite jump at $t = t_0$)
• $y(t) = b(t)y(\theta(t)) + (\mathcal{V}_{\theta,\alpha} y)(t)$	$C^{1-\alpha}$ (finite jump at $t = t_0$, no smoothing at $t = \xi_\mu$)
• $y(t) = b(t)y(\theta(t)) + (\mathcal{W}_{\theta,\alpha} y)(t)$	$C^{1-\alpha}$ (finite jump at $t = t_0$, no smoothing at $t = \xi_\mu$)

We summarise a number of relevant regularity results in Table 6.1; they extend those described in Section 4.1.2 (Table 4.1). The proofs are left as an exercise.

6.1.7 Comparison theorems and Gronwall-type inequalities

We conclude this look at the theory of weakly singular Volterra integral equations by describing a number of generalisations of the continuous and discrete comparisons theorems of Sections 2.1.8 and 2.1.9. These results (as well as more general variants) are due to McKee (1982a), Beesack (1985a, 1985b), and Dixon and McKee (1986); see also McKee and Tang (1991).

We start with an extension of the classical result of Gronwall.

Theorem 6.1.17 *Let $I := [0, T]$ and assume that*

(a) $g \in C(I)$, $g(t) \geq 0$ on I, and g is non-decreasing on I.
(b) The continuous, non-negative function z satisfies the inequality

$$z(t) \leq g(t) + M \int_0^t (t-s)^{-\alpha} z(s)ds, \quad t \in I, \tag{6.1.40}$$

for some $M > 0$ *and* $0 < \alpha < 1$.

Then:

$$z(t) \le E_{1-\alpha}(M\Gamma(1-\alpha)t^{1-\alpha})g(t), \quad t \in I. \tag{6.1.41}$$

Here, E_β *denotes the Mittag-Leffler function introduced in Section 6.1.1.*

Remark An extension of this result to VIEs with more general weakly singular kernels,

$$z(t) \le \gamma(t) + M\int_0^t s^q(t^p - s^p)^{-\alpha}z(s)ds, \quad t \in I; \quad q \ge 0, \ 1 \le p \le q+1, \tag{6.1.42}$$

can be found in the papers by McKee (1982a) and Beesack (1985b); see also the survey by McKee and Tang (1991).

The *comparison theorems* of Section 2.1.8 can be extended to equations with weakly singular kernels. We state the analogue of Theorem 2.1.15.

Theorem 6.1.18 *Assume that* $g \in C(I)$, *with* $g(t) \ge 0$ *on* I, *and* $K \in C(D)$, *with* $K(t,s) \ge 0$ *on* D. *Let* $R_\alpha(t,s) = (t-s)^{-\alpha}Q(t,s;\alpha)$ *denote the resolvent kernel associated with*

$$H_\alpha(t,s) := (t-s)^{-\alpha}K(t,s) \ (0 < \alpha < 1).$$

If $z \in C(I)$ *satisfies the inequality*

$$z(t) \le g(t) + \int_0^t H_\alpha(t,s)z(s)ds, \quad t \in I,$$

then

$$z(t) \le g(t) + \int_0^t R_\alpha(t,s)g(s)ds, \quad t \in I,$$

and we have $Q(t,s;\alpha) \ge K(t,s)$ *on* D.

Proof Theorem 6.1.2 and its proof show that

$$Q(t,s;\alpha) = \sum_{n=1}^{\infty} \Psi_n(t,s;\alpha), \quad (t,s) \in D,$$

with $\Psi_1(t,s;\alpha) = \Phi_1(t,s;\alpha) = K(t,s)$ and $\Psi_n(t,s;\alpha) \ge 0$ since, by their recursive definition, all $\Phi_n(t,s;\alpha)$ are non-negative on D.

We conclude this section by briefly considering a generalised *discrete Gronwall inequality*,

$$z_n \leq \gamma_n + Mh^{1-\alpha} \sum_{\ell=0}^{n-1} (n-\ell)^{-\alpha} z_\ell, \quad 0 \leq n \leq N, \tag{6.1.43}$$

where the sequence $\{\gamma_n\}$ is non-negative and non-decreasing, $M > 0$, and $0 < \alpha < 1$.

Theorem 6.1.19 *If the non-negative sequence* $\{z_n\}$ *satisfies the inequality (6.1.45), then its elements can be bounded by*

$$z_n \leq E_{1-\alpha}(M\Gamma(1-\alpha)(nh)^{1-\alpha})\gamma_n, \quad 0 \leq n \leq N. \tag{6.1.44}$$

We refer the reader to McKee and Tang (1991) for a **proof** of this theorem. A somewhat more general version of this result, and its proof, can be found in Dixon (1985). See also Beesack (1985b).

6.2 Collocation for weakly singular VIEs of the second kind

6.2.1 The exact collocation equations

As in Section 6.1.2 let the linear Volterra integral operator $\mathcal{V}_\alpha : C(I) \to C(I)$ be given by

$$(\mathcal{V}_\alpha y)(t) := \int_0^t H_\alpha(t,s)y(s)ds, \quad t \in I := [0,T], \tag{6.2.1}$$

with

$$H_\alpha(t,s) := p_\alpha(t-s)K(t,s), \quad 0 < \alpha \leq 1. \tag{6.2.2}$$

The kernel function $K = K(t,s)$ is assumed to satisfy $K \in C(D)$ and $K(t,t) \neq 0$ on I, and the integrable (weak) singularity p_α has the form

$$p_\alpha(t-s) := \begin{cases} (t-s)^{-\alpha} & \text{if } 0 < \alpha < 1, \\ \log(t-s) & \text{if } \alpha = 1. \end{cases} \tag{6.2.3}$$

Given a function $g \in C(I)$ we shall approximate the solution of the weakly singular VIE

$$y(t) = g(t) + (\mathcal{V}_\alpha y)(t), \quad t \in I, \tag{6.2.4}$$

by collocation in the piecewise polynomial space

$$S_{m-1}^{(-1)}(I_h) := \{v : v|_{\sigma_n} \in \pi_{m-1} \ (0 \leq n \leq N-1)\}.$$

The desired collocation solution u_h is therefore defined by

$$u_h(t) = g(t) + (\mathcal{V}_\alpha u_h)(t), \quad t \in X_h, \tag{6.2.5}$$

where the set of collocation points,

$$X_h := \{t_n + c_i h_n : \; 0 \leq c_1 < \cdots < c_m \leq 1 \; (n = 0, 1, \ldots, N-1)\}, \tag{6.2.6}$$

is determined by the given mesh I_h and the (distinct) collocation parameters $\{c_i\}$. As we have already observed at the beginning of Section 2.2.2, the choice $c_1 = 0$ and $c_m = 1$ $(m \geq 2)$ implies, for continuous g and K, that

$$u_h \in S_{m-1}^{(-1)}(I_h) \cap C(I) = S_{m-1}^{(0)}(I_h),$$

with dim $S_{m-1}^{(0)}(I_h) = N(m-1) + 1$. In this case we require u_h to satisfy the the initial condition $u_h(t_{0,1}) = u_h(0) = g(0)$.

The *iterated collocation solution* u_h^{it} corresponding to the collocation solution u_h is then defined by

$$u_h^{it}(t) := g(t) + (\mathcal{V}_\alpha u_h)(t), \quad t \in I. \tag{6.2.7}$$

It trivially satisfies

$$u_h^{it}(t) = u_h(t) \qquad \text{for all} \quad t \in X_h$$

and for any $\alpha \in (0, 1]$.

As in Chapter 2, the computational form of the collocation equation (6.2.5) will again be based on the local representation employing the Lagrange basis functions with respect to the collocation parameters $\{c_i\}$ which we will recall for convenience: setting

$$L_j(v) := \prod_{k \neq j}^{m} \frac{v - c_k}{c_j - c_k} \qquad \text{and} \qquad U_{n,j} := u_h(t_n + c_j h_n) \;\; (j = 1, \ldots, m),$$

the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ on the subinterval $\sigma_n := (t_n, t_{n+1}]$ is described by

$$u_h(t) = u_h(t_n + v h_n) = \sum_{j=1}^{m} L_j(v) U_{n,j}, \quad v \in (0, 1]. \tag{6.2.8}$$

Thus, for $t = t_{n,i} := t_n + c_i h_n$ the collocation equation (6.2.5) assumes the form

$$u_h(t) = g(t) + \int_0^{t_n} H_\alpha(t, s) u_h(s) ds + h_n \int_0^{c_i} H_\alpha(t, t_n + s h_n) u_h(t_n + s h_n) ds.$$

We write this as

$$U_{n,i} = g(t_{n,i}) + F_n(t_{n,i};\alpha) + h_n \sum_{j=1}^{m} \left(\int_0^{c_i} H_\alpha(t_{n,i}, t_n + sh_n) L_j(s) ds \right) U_{n,j} \tag{6.2.9}$$

$(i = 1, \ldots, m)$. For $t \in \sigma_n$ the lag term is

$$F_n(t;\alpha) := \int_0^{t_n} H_\alpha(t,s) u_h(s) ds = \sum_{\ell=0}^{n-1} h_\ell \int_0^1 H_\alpha(t, t_\ell + sh_\ell) u_h(t_\ell + sh_\ell) ds. \tag{6.2.10}$$

If $t = t_{n,i}$ this becomes, by (6.2.8),

$$F_n(t_{n,i};\alpha) = \sum_{\ell=0}^{n-1} h_\ell \sum_{j=1}^{m} \left(\int_0^1 H_\alpha(t_{n,i}, t_\ell + sh_\ell) L_j(s) ds \right) U_{\ell,j}.$$

Let $\mathbf{U}_n := (U_{n,1}, \ldots, U_{n,m})^T$, $\mathbf{g}_n := (g(t_{n,1}), \ldots, g(t_{n,m}))^T$, and define the matrices in $L(\mathbb{R}^m)$,

$$B_n^{(\ell)}(\alpha) := \begin{pmatrix} \int_0^1 H_\alpha(t_{n,i}, t_\ell + sh_\ell) L_j(s) ds \\ (i, j = 1, \ldots, m) \end{pmatrix} \quad (\ell < n), \tag{6.2.11}$$

and

$$B_n(\alpha) := \begin{pmatrix} \int_0^{c_i} H_\alpha(t_{n,i}, t_n + sh_n) L_j(s) ds \\ (i, j = 1, \ldots, m) \end{pmatrix}. \tag{6.2.12}$$

The collocation equation (6.2.9) then assumes the form

$$[\mathcal{I}_m - h_n B_n(\alpha)] \mathbf{U}_n = \mathbf{g}_n + \mathbf{G}_n(\alpha) \quad (n = 0, 1, \ldots, N-1), \tag{6.2.13}$$

where

$$\mathbf{G}_n(\alpha) := (F_n(t_{n,1};\alpha), \ldots, F_n(t_{n,m};\alpha))^T = \sum_{\ell=0}^{n-1} h_\ell B_n^{(\ell)}(\alpha) \mathbf{U}_\ell.$$

Here, $\mathcal{I}_m$ denotes again the identity matrix in $L(\mathbb{R}^m)$.

We note for later reference that the integrands defining the elements of $B_n^{(\ell)}(\alpha)$ and $B_n(\alpha)$ are, respectively,

$$H_\alpha(t_{n,i}, t_\ell + sh_\ell) = p_\alpha(t_{n,i} - t_\ell - sh_\ell) K(t_{n,i}, t_\ell + sh_\ell) \quad (\ell < n), \tag{6.2.14}$$

$$H_\alpha(t_{n,i}, t_n + sh_n) = p_\alpha((c_i - s)h_n) K(t_{n,i}, t_n + sh_n). \tag{6.2.15}$$

We also observe that for $0 < \alpha < 1$ we may write

$$p_\alpha(t_{n,i} - t_\ell - sh_\ell) = h_\ell^{-\alpha} \left(\frac{t_n + c_i h_n - t_\ell}{h_\ell} - s \right)^{-\alpha} \quad (\ell < n)$$

and

$$p_\alpha((c_i - s)h_n) = h_n^{-\alpha}(c_i - s)^{-\alpha}.$$

The left-hand side matrix in the system (6.2.13) then becomes $\mathcal{I}_m - h_n^{1-\alpha} B_n(\alpha)$, where we now have

$$B_n(\alpha) := \begin{pmatrix} \int_0^{c_i} (c_i - s)^{-\alpha} K(t_{n,i}, t_n + sh_n) L_j(s) ds \\ (i, j = 1, \ldots, m) \end{pmatrix}.$$

Due to the integrability of the kernel H_α in the Volterra integral operator $\mathcal{V}_\alpha$ it is clear that the result of Theorem 2.2.1 on the existence and uniqueness of the collocation solution u_h remains valid for any $\alpha \in (0, 1]$.

Theorem 6.2.1 *Assume that g and K in $H_\alpha(t, s) = p_\alpha(t - s)K(t, s)$ are continuous on their respective domains I and D. Then there exists an $\bar{h} = \bar{h}(\alpha) > 0$ so that, for every $\alpha \in (0, 1]$ and any mesh I_h with mesh diameter h satisfying $h \in (0, \bar{h})$, each of the linear algebraic systems (6.2.13) has a unique solution $\mathbf{U}_n \in \mathbb{R}^m$ $(n = 0, 1, \ldots, N - 1)$. Hence the collocation equation (6.2.5) defines a unique collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ for the weakly singular VIE (6.2.4), with local representation given by (6.2.8).*

Proof By our assumptions on the factor K in the kernel H_α of the Volterra operator $\mathcal{V}_\alpha$, the elements of the matrices $B_n(\alpha)$ in (6.2.12) are bounded for all $\alpha \in (0, 1]$. As in the case $\alpha = 0$ this implies that the inverse of the matrix $\mathcal{B}_n(\alpha) := \mathcal{I}_m - h_n B_n(\alpha) \in L(\mathbb{R}^m)$ exists if $h_n||B_n(\alpha)|| < 1$ for some matrix norm. This clearly holds whenever h_n is sufficiently small. In other words, there is an $\bar{h} = \bar{h}(\alpha) > 0$ so that for any mesh I_h with $h := \max\{h_n : 0 \leq n \leq N - 1\} < \bar{h}$, each matrix $\mathcal{B}_n(\alpha)$ $(n = 0, 1, \ldots, N - 1)$ has a uniformly bounded inverse. The assertion of Theorem 6.2.1 now follows.

When the collocation solution on the subinterval σ_n has been computed, the *iterated collocation solution* for $t = t_n + vh_n \in \bar{\sigma}_n := [t_n, t_{n+1}]$ is given by

$$u_h^{it}(t) = g(t) + F_n(t; \alpha) + h_n \sum_{j=1}^{m} \left(\int_0^v H_\alpha(t, t_n + sh_n) L_j(s) ds \right) U_{n,j}, \tag{6.2.16}$$

with lag term $F_n(t; \alpha)$ as in (6.2.10). For $0 < \alpha < 1$, (6.2.16) can be written as

$$u_h^{it}(t) = g(t) + F_n(t; \alpha) + h_n^{1-\alpha} \sum_{j=1}^{m} \left(\int_0^v (v - s)^{-\alpha} K(t, t_n + sh_n) L_j(s) ds \right) U_{n,j},$$

$v \in [0, 1]$.

Example 6.2.1 $u_h \in S_0^{(-1)}(I_h)$ $(m = 1)$, $\quad 0 < c_1 =: \theta \le 1$:
Here, $u_h(t_n + vh_n) = U_{n,1}$ for all $v \in (0, 1]$. Setting $y_{n+1} := U_{n,1}$ the collocation solution is determined by the equation

$$\left(1 - h_n \int_0^\theta H_\alpha(t_{n,1}, t_n + sh_n)ds\right) y_{n+1} = g(t_{n,1}) + F_n(t_{n,1}; \alpha), \qquad (6.2.17)$$

$(n = 0, 1, \ldots, N - 1)$, with $t_{n,1} = t_n + \theta h_n$ and with lag term given by

$$F_n(t_{n,1}; \alpha) = \sum_{\ell=0}^{n-1} h_\ell \left(\int_0^1 H_\alpha(t_{n,1}, t_\ell + sh_\ell)ds\right) y_{\ell+1}.$$

For $t = t_n + vh_n$ $(v \in [0, 1])$ the corresponding *iterated collocation solution* is then

$$u_h^{it}(t) = g(t) + F_n(t; \alpha) + h_n \left(\int_0^v H_\alpha(t, t_n + sh_n)ds\right) y_{n+1}. \qquad (6.2.18)$$

We remind the reader that $H_\alpha(t, t_n + sh_n) = p_\alpha((v - s)h_n)K(t, t_n + vh_n)$ when $t = t_n + vh_n$; hence, for $0 < \alpha < 1$ we have

$$H_\alpha(t, t_n + sh_n) = h_n^{-\alpha}(v - s)^{-\alpha} K(t, t_n + sh_n).$$

Example 6.2.2 $u_h \in S_1^{(-1)}(I_h)$ $(m = 2)$, $\quad 0 < c_1 < c_2 \le 1$:
Since the Lagrange fundamental polynomials corresponding to the two collocation parameters are

$$L_1(s) = (c_2 - s)/(c_2 - c_1) \quad \text{and} \quad L_2(s) = (s - c_1)/(c_2 - c_1),$$

the matrix $B_n(\alpha) \in L(\mathbb{R}^2)$ in (2.2.14) has the elements

$$(B_n(\alpha))_{i,1} = \frac{1}{c_2 - c_1} \int_0^{c_i} p_\alpha((c_i - s)h_n)K(t_{n,i}, t_n + sh_n)(c_2 - s)ds \;\; (i = 1, 2)$$

and

$$(B_n(\alpha))_{i,2} = \frac{1}{c_2 - c_1} \int_0^{c_i} p_\alpha((c_i - s)h_n)K(t_{n,i}, t_n + sh_n)(s - c_1)ds \;\; (i = 1, 2).$$

Moreover,

$$(B_n^{(\ell)}(\alpha))_{i,1} = \frac{1}{c_2 - c_1} \int_0^1 H_\alpha(t_{n,i}, t_\ell + sh_\ell)(c_2 - s)ds \quad (i = 1, 2),$$

and

$$(B_n^{(\ell)}(\alpha))_{i,2} = \frac{1}{c_2 - c_1} \int_0^1 H_\alpha(t_{n,i}, t_\ell + sh_\ell)(s - c_1)ds \quad (i = 1, 2).$$

The collocation solution is now determined by the corresponding system (6.2.13) in $\mathbb{R}^2$ and the local Lagrange representation (6.2.8) with $m = 2$, and (6.2.16) yields the iterated collocation solution on $\bar{\sigma}_n$.

6.2.2 The fully discretised collocation equations

The integrals occurring in the collocation equation (6.2.9), and in (6.2.10) and (6.2.16), usually cannot be found analytically but have to be approximated by suitable *numerical quadrature formulas*, similar to the situation we have already encountered in Section 2.2.3. Now, due to the presence of the weak singularity $p_\alpha(t-s)$ it will be natural, as we shall see below, to base this further discretisation step on (interpolatory) *product quadrature formulas* whose weights depend on the weakly singular factor $p_\alpha(t-s)$ in the kernel $H_\alpha(t,s)$.

Thus, the *fully discretised version* of (6.2.5) will be

$$\hat{u}_h(t) = g(t) + (\hat{\mathcal{V}}_{\alpha,h}\hat{u}_h)(t), \quad t \in X_h, \tag{6.2.19}$$

where $\hat{\mathcal{V}}_{\alpha,h}$ denotes the discrete version of the original Volterra integral operator $\mathcal{V}_\alpha$ in (6.2.4). The *weighted* interpolatory m-point quadrature formulas (which we will refer to simply as m-point *product quadrature formulas*) whose abscissas are given by, or based on, the m collocation parameters $\{c_k\}$ and whose weights depend on p_α will be employed to generate the quadrature approximations defining $\hat{\mathcal{V}}_{\alpha,h}$; they are

$$(\hat{Q}_n(\alpha)u_h)(t) := \sum_{k=1}^{m} w_{n,k}(v;\alpha)K(t, t_n + vc_kh_n)u_h(t_n + vc_kh_n) \tag{6.2.20}$$

and

$$(\hat{Q}_n^{(\ell)}(\alpha)u_h)(t) := \sum_{k=1}^{m} w_{n,k}^{(\ell)}(v;\alpha)K(t, t_\ell + c_kh_\ell)u_h(t_\ell + c_kh_\ell) \quad (\ell < n) \tag{6.2.21}$$

for the integrals

$$\begin{aligned}(Q_n(\alpha)u_h)(t) &:= \int_0^v H_\alpha(t, t_n + sh_n)u_h(t_n + sh_n)ds \\ &= v\int_0^1 H_\alpha(t, t_n + svh_n)u_h(t_n + svh_n)ds,\end{aligned}$$

and

$$(Q_n^{(\ell)}(\alpha)u_h)(t) := \int_0^1 H_\alpha(t, t_\ell + sh_\ell)u_h(t_\ell + sh_\ell)ds \quad (\ell < n),$$

respectively, when $t = t_n + vh_n \in \sigma_n$. The *product quadrature weights* are

$$w_{n,k}(v;\alpha) := \int_0^v p_\alpha((v-s)h_n)L_k(s/v)ds$$
$$= v\int_0^1 p_\alpha((1-z)vh_n)L_j(z)dz, \quad (v \in (0,1]), \qquad (6.2.22)$$

and, for $\ell < n$,

$$w_{n,k}^{(\ell)}(v;\alpha) := \int_0^1 p_\alpha(((t_n + vh_n - t_\ell)/h_\ell - s)h_\ell)L_k(s)ds \;\; (v \in (0,1], \;\; \ell < n). \qquad (6.2.23)$$

Note that for $\alpha = 0$ these quadrature weights reduce to $w_{n,j}(v;0) = vb_j$ and $w_{n,j}^{(\ell)}(v;0) = \beta_j(1) = b_j$, respectively (cf. (2.2.19), (2.2.20)).

We observe in passing that the integrals $Q_n^{(\ell)}(\alpha)u_h)(t)$ $(\ell < n)$ in the lag term (6.2.10) could, of course, also be discretised by the 'classical' (i.e. non-product) interpolatory quadrature formulas used in Section 2.2.3, since $p_\alpha(t_n + vh_n - t_\ell - sh_\ell)$ is now bounded for $v \in (0,1]$ when $\ell < n$.

The *fully discretised collocation equation* is obtained from the *exact collocation equation* (6.2.9) by replacing the integrals by the above quadrature approximations, disregarding the quadrature errors induced by this secondary discretisation process. As in Section 2.2.3 we will denote the resulting *discretised collocation solution* by $\hat{u}_h$: it is, of course, still an element of our space $S_{m-1}^{(-1)}(I_h)$, but in general we have $\hat{u}_h \neq u_h$. The local representation of $\hat{u}_h$ on σ_n is thus

$$\hat{u}_h(t_n + vh_n) = \sum_{j=1}^m L_j(v)\hat{U}_{n,j} \quad v \in (0,1], \quad \text{with} \quad \hat{U}_{n,j} := \hat{u}_h(t_n + c_jh_n). \qquad (6.2.24)$$

Thus, the fully discretised version of the collocation equation (6.2.9) is

$$\hat{U}_{n,i} = g(t_{n,i}) + \hat{F}_n(t_{n,i};\alpha) + h_n(\hat{Q}_n(\alpha)\hat{u}_h)(t_{n,i}) \quad (i = 1,\ldots,m), \qquad (6.2.25)$$

where $(\hat{Q}_n(\alpha)\hat{u}_h)(t_{n,i})$ is defined in (6.2.20) and where the fully discretised lag term $\hat{F}_n(t;\alpha)$ at $t = t_n + vh_n$ has the form

$$\hat{F}_n(t;\alpha) := \sum_{\ell=0}^{n-1} h_\ell(\hat{Q}_n^{(\ell)}(\alpha)\hat{u}_h)(t)$$
$$= \sum_{\ell=0}^{n-1} h_\ell\left(\sum_{j=1}^m w_{n,j}^{(\ell)}(v;\alpha)K(t, t_\ell + c_jh_\ell)\hat{U}_{\ell,j}\right). \qquad (6.2.26)$$

In analogy to (6.2.13) we can write the corresponding discretised collocation equation (6.2.25) in the more concise form

$$[\mathcal{I}_m - h_n \hat{B}_n(\alpha)]\hat{\mathbf{U}}_n = \mathbf{g}_n + \hat{\mathbf{G}}_n(\alpha) \quad (n = 0, 1, \ldots, N-1), \tag{6.2.27}$$

with

$$\hat{\mathbf{G}}_n(\alpha) := (\hat{F}_n(t_{n,1};\alpha), \ldots, \hat{F}_n(t_{n,m};\alpha))^T = \sum_{\ell=0}^{n-1} h_\ell \hat{B}_n^{(\ell)}(\alpha)\hat{\mathbf{U}}_\ell.$$

Here, $\hat{\mathbf{U}}_n := (\hat{U}_{n,1}, \ldots, \hat{U}_{n,m})^T \in \mathbb{R}^m$; the matrices $\hat{B}_n(\alpha)$ and $\hat{B}_n^{(\ell)}(\alpha)$ in $L(\mathbb{R}^m)$ – representing the discretised versions of $B_n(\alpha)$ and $B_n^{(\ell)}(\alpha)$ in (6.2.12 and (6.2.11) – are defined respectively by

$$\hat{B}_n(\alpha) := \begin{pmatrix} \sum_{k=1}^{m} w_{n,k}(c_i;\alpha)K(t_{n,i}, t_n + c_i c_k h_n)L_j(c_i c_k) \\ (i, j = 1, \ldots, m) \end{pmatrix},$$

and

$$\hat{B}_n^{(\ell)}(\alpha) := \begin{pmatrix} w_{n,j}^{(\ell)}(c_i;\alpha)K(t_{n,i}, t_\ell + c_j h_\ell) \\ (i, j = 1, \ldots, m) \end{pmatrix} \quad (\ell < n).$$

Observe again that for $0 < \alpha < 1$ we obtain, by (6.2.22) and (6.2.23),

$$w_{n,k}(v;\alpha) = h_n^{-\alpha} \int_0^v (v-s)^{-\alpha} L_k(s/v)ds \quad (v > 0),$$

and

$$w_{n,k}^{(\ell)}(v;\alpha) = h_\ell^{-\alpha} \int_0^1 \left(\frac{t_n + vh_n - t_\ell}{h_\ell} - s \right)^{-\alpha} L_k(s)ds \quad (\ell < n).$$

In accordance with the notation we introduced for the exact matrices $B_n(\alpha)$ corresponding to $\alpha \in (0, 1)$ we will write the matrix describing the left-hand side of the discretised algebraic system (6.2.27) as $\mathcal{I}_m - h_n^{1-\alpha}\hat{B}_n(\alpha)$, with appropriately redefined matrix $\hat{B}_n(\alpha)$.

Theorem 6.2.2 *Let the assumptions of Theorem 6.2.1 hold. Then there exists an* $\hat{h} = \hat{h}(\alpha) > 0$ *so that for* $\alpha \in (0, 1]$ *and any mesh* I_h *with mesh diameter* h *satisfying* $h \in (0, \hat{h})$ *there exists a unique discretised collocation approximation* $\hat{u}_h \in S_{m-1}^{(-1)}(I_h)$ *defined by the unique solutions* $\hat{\mathbf{U}}_n$ *of the linear algebraic systems (6.2.27)* ($n = 0, 1, \ldots, N-1$) *and the local representations (6.2.24).*

The **proof** of Theorem 6.2.2 of course closely resembles the one for Theorem 6.2.1: since, for any fixed m and any $\alpha \in (0, 1]$, the weights $\{w_{n,k}(c_i;\alpha)\}$ characterising the elements of $\hat{B}_n(\alpha)$ are bounded, it follows from the continuity of K and the Neumann Lemma that each matrix $\mathcal{I}_m - h_n\hat{B}_n(\alpha)$

($n = 0, 1, \ldots, N-1$) in (2.2.27) possesses a uniformly bounded inverse whenever $h_n < \hat{h}$, for some suitable $\hat{h} > 0$ depending on α; in general we have $\hat{h} \neq \bar{h}$.

For $t = t_n + vh_n \in \sigma_n$ the corresponding *discretised iterated collocation solution* $\hat{u}_h^{it}$ corresponding to $\hat{u}_h \in S_{m-1}^{(-1)}(I_h)$ is defined by

$$\hat{u}_h^{it}(t) := g(t) + \hat{F}_n(t;\alpha) + h_n(\hat{Q}_n(\alpha)\hat{u}_h)(t) \tag{6.2.28}$$

(recall (6.2.16)), where $(\hat{Q}_n(\alpha)\hat{u}_h)(t)$ is as in (6.2.20). The discretised lag term $\hat{F}_n(t;\alpha)$ was introduced in (6.2.26).

The following two illustrations are the discrete counterparts of the exact collocation methods described in Examples 6.2.1 and 6.2.2.

Example 6.2.3 $\hat{u}_h \in S_0^{(-1)}(I_h),\ 0 < c_1 =: \theta \leq 1$:
Setting $\hat{y}_{n+1} := \hat{U}_{n,1}$ and $t_{n,1} := t_n + \theta h_n$, equation (6.2.17) yields

$$[1 - w_{n,1}(\theta;\alpha)h_n K(t_{n,1}, t_n + \theta^2 h_n)]\hat{y}_{n+1} = g(t_{n,1}) + \hat{F}_n(t_{n,1};\alpha) \tag{6.2.29}$$

($n = 0, 1, \ldots, N-1$), with

$$w_{n,1}(v;\alpha) := \int_0^v p_\alpha((v-s)h_n)ds = \begin{cases} h_n^{-\alpha}\dfrac{v^{1-\alpha}}{1-\alpha} & \text{if } 0 < \alpha < 1, \\ v[\log(vh_n) - 1] & \text{if } \alpha = 1. \end{cases}$$

The discretised lag term is

$$\hat{F}_n(t_{n,1};\alpha) = \sum_{\ell=0}^{n-1} h_\ell w_{n,1}^{(\ell)}(\theta;\alpha)K(t_{n,1}, t_\ell + \theta h_\ell)\hat{y}_{\ell+1},$$

with weights given by

$$w_{n,1}^{(\ell)}(v;\alpha) := \int_0^1 p_\alpha(t_n + vh_n - t_\ell - sh_\ell)ds \quad (\ell < n).$$

The corresponding discretised iterated collocation solution at $t = t_n + vh_n$ ($v \in [0, 1]$) is

$$\hat{u}_h^{it}(t) = g(t) + \hat{F}_n(t;\alpha) + w_{n,1}(v;\alpha)h_n K(t, t_n + v\theta h_n)\hat{y}_{n+1}. \tag{6.2.30}$$

Example 6.2.4 $\hat{u}_h \in S_1^{(-1)}(I_h)$ ($m = 2$), $0 < c_1 < c_2 \leq 1$:
We see from Example 6.2.2 that the elements of the discretised matrices $\hat{B}_n \in L(\mathbb{R}^2)$ in (6.2.27) are

$$\begin{aligned}(\hat{B}_n(\alpha))_{i,1} = {} & w_{n,1}(c_i;\alpha)K(t_{n,i}, t_n + c_i c_1 h_n)L_1(c_i c_1) \\ & + w_{n,2}(c_i;\alpha)K)t_{n,i}, t_n + c_i c_2 h_n)L_1(c_i c_2)\end{aligned}$$

and

$$(\hat{B}_n(\alpha))_{i,2} = w_{n,1}(c_i;\alpha)K(t_{n,i}, t_n + c_i c_1 h_n)L_2(c_i c_1) + w_{n,2}(c_i;\alpha)K(t_{n,i}, t_n + c_i c_2 h_n)L_2(c_i c_2)$$

($i = 1, 2$), with quadrature weights

$$w_{n,1}(c_i;\alpha) := \frac{1}{c_2 - c_1} \int_0^{c_i} p_\alpha((c_i - s)h_n)(c_2 - s/c_i)ds,$$
$$w_{n,2}(c_i,\alpha) := \frac{1}{c_2 - c_1} \int_0^{c_i} p_\alpha((c_i - s)h_n)(s/c_i - c_1)ds.$$

The elements of the matrices $\hat{B}_n^{(\ell)}(\alpha)$ ($\ell < n$) describing the discretised lag term have the forms

$$(\hat{B}_n^{(\ell)}(\alpha))_{i,1} = w_{n,1}^{(\ell)}(c_i;\alpha)K(t_{n,i}, t_\ell + c_1 h_\ell)$$

and

$$(\hat{B}_n^{(\ell)}(\alpha))_{i,2} = w_{n,2}^{(\ell)}(c_i;\alpha)K(t_{n,i}, t_\ell + c_2 h_\ell).$$

6.2.3 Approximation of functions in Hölder spaces and graded meshes

We briefly mentioned in Section 2.2.1 that graded meshes would play an important role in the computation and convergence analysis of collocation methods for Volterra equations with weakly singular kernels. The regularity results of Section 6.1 indicate why this will be so: typically, the first derivative of solutions of second-kind VIEs with smooth data behave like $t^{-\alpha}$ if $0 < \alpha < 1$, or like $t \cdot \log(t)$ if $\alpha = 1$, near $t = 0^+$. Thus, it is intuitively clear that collocation in piecewise polynomial spaces based on uniform meshes I_h will, due to the lack of regularity in y at $t = 0$, lead to low orders of (global or local) convergence.

We recall from Section 2.2.1 that for an interval $I := [t_0, T]$ a *graded mesh* with *grading exponent* $r > 1$ is defined by

$$I_h := \{t_n = t_n^{(N)} := t_0 + (n/N)^r (T - t_0) : \ n = 0, 1, \ldots, N\}. \tag{6.2.31}$$

The sequence $\{h_n := t_{n+1} - t_n \ (n = 0, 1, \ldots, N - 1)\}$ is strictly increasing, and its mesh diameter is given by $h = h_{N-1}$.

In this chapter we will assume that $t_0 = 0$; graded meshes corresponding to more general t_0 will be encountered in Sections 6.4 and 7.4.2. The following elementary lemma summarises the key properties of graded meshes.

Lemma 6.2.3 *Let I_h be a graded mesh of the form (6.2.31), with $t_0 = 0$. Then:*

(a) $t_n = n^r t_1 \quad (n = 1, \ldots, N), \quad$ *with* $t_1 = h_0 = TN^{-r}$.
(b) $h = h_{N-1} = rTN^{-1}(1 - \theta N^{-1})^{r-1} \leq rTN^{-1}$, *for some* $\theta \in (0, 1)$.
(c) $h/h_0 = rN^{r-1}(1 - \theta N^{-1})^{r-1}$ *for some* $\theta \in (0, 1)$.

The grading exponent $r = r(\alpha) > 1$ of the meshes that will be employed in this and the next chapter will depend on the real number $\alpha \in (0, 1]$ characterising the weakly singular factor $p_\alpha(t - s)$ of the kernel $H_\alpha(t, s)$: our aim is to choose $r(\alpha)$ so that the collocation solutions u_h and u_h^{it}, or their discretised counterparts, exhibit optimal orders of global and local (super-) convergence.

In order to obtain some first insight into why mesh grading is crucial when approximating functions with low regularity we recall a number of relevant definitions and results from classical aproximation theory.

Definition A function $f : I := [0, T] \to \mathbb{R}$ is said to be *Hölder continuous* on I, with *Hölder exponent* $\beta \in (0, 1]$, if there exists a constant $L_\beta > 0$ so that the *Hölder condition*,

$$|f(t) - f(\tau)| \leq L_\beta |t - \tau|^\beta \quad \text{for all } \; t, \tau \in I,$$

holds. The space of Hölder continuous functions on I will be denoted by $C^\beta(I)$.

If a function $f : I \to \mathbb{R}$ is in $C^k(I)$ and $y^{(k)} \in C^\beta(I)$, then we shall write $f \in C^{k,\beta}(I)$, with $C^{0,\beta}(I) := C^\beta(I)$. For $\beta = 1$ we obtain the space of *Lipschitz continuous* functions on I which we will denote by $C^{0,1}(I)$. Clearly, $C^{0,1}(I)$ is a proper subset of $C(I)$.

Illustration 6.2.1 It is easily verified that for $\rho \in \mathbb{N}_0$ the function $f(t) := t^{\rho+\beta}$ is in $C^{\rho,\beta}[0, T]$ for any $T > 0$.

How well can a function $f \in C^\beta(I)$ $(0 < \beta < 1)$ be approximated by (continuous) piecewise polynomials? The key to the answer in the case of polynomial approximation is given by the classical results of Jackson (see, e.g. Timan (1963) or Schumaker (1981)), and they can be used to obtain corresponding optimal orders of convergence for piecewise polynomials of (fixed) degree. We first cite the following simple but instructive result (which, together with their proofs, can be found in the books by de Boor (2000) and Powell (1981, pp. 254–255)).

Theorem 6.2.4 *Let* $f(t) = t^\beta \; (0 < \beta < 1), \; t \in [0, 1]$, *and assume that* p_h *is the unique interpolant in* $S_1^{(0)}(I_h)$ *for* f *with respect to the points* $I_h := \{t_n : 0 = t_0 < t_1 < \ldots < t_N = 1\}$.

(i) *If I_h is the uniform mesh then*

$$||f - p_h||_\infty \leq C_1(\beta)N^{-\beta}.$$

This estimate is also true if I_h is a quasi-uniform mesh.

(ii) *If I_h is the graded mesh given by*

$$t_n := \left(\frac{n}{N}\right)^r \quad (n = 0, 1, \ldots, N), \quad \textit{with } r = r(\beta) = \frac{2}{\beta},$$

then

$$||f - p_h||_\infty \leq C_1^*(\beta)N^{-2}.$$

Proof It is not difficult to show that the interpolation error on the subinterval $\bar{\sigma}_n := [t_n, t_{n+1}]$ is given by

$$e_h(t) := f(t) - p_h(t) = t^\beta - [(t_{n+1} - t)t_n^\beta + (t - t_n)t_{n+1}^\beta]/h_n,$$

with $h_n := t_{n+1} - t_n$. The maximum of $|e_h(t)|$ on $\bar{\sigma}_n$ is attained at the point

$$\xi_n := [\beta h_n/(t_{n+1}^\beta - t_n^\beta)]^{1/(1-\beta}.$$

It now follows from the geometry of f that for uniform I_h we have $||e_h||_\infty = |e_h(\xi_0)|$, and a simple calculation then yields the result of (i).

The assertion of (ii) is proved analogously, by using the given specially graded mesh and the above expression for ξ_n.

In order to acquire more insight into the nature of this problem, and as a step towards the general best approximation result of Theorem 6.2.7 below, we will indicate how the results of Theorem 6.2.4 can be derived in a different way, by using what is called an *equidistribution principle* for the points of the mesh I_h. The proof can be found in, e.g. de Boor (2000).

Theorem 6.2.5 *Assume that $f \in C^2(0, 1)$ has the property that it is monotone near $t = 0^+$ and $t = 1^-$, with $\int_0^1 |f''(t)|^{1/2}dt < \infty$. If $p_h \in S_1^{(0)}(I_h)$ is the interpolant for f on I_h and if the points of I_h satisfy the equidistribution condition*

$$\int_0^{t_n} |f''(t)|^{1/2}dt = \frac{n}{N}\int_0^1 |f''(t)|^{1/2}dt \quad (n = 1, \ldots, N-1),$$

then the order of the interpolation error $e_h := f - p_h$ is optimal:

$$||e_h||_\infty \leq C_1^* N^{-2}.$$

As we will see in Illustration 6.2.2 below, for $f(t) = t^\beta$ $(0 < \beta < 1)$ the corresponding optimally graded mesh of Theorem 6.2.4(ii) can also be obtained by the above equidistribution equation.

So far we have only looked at *interpolation* for Hölder continuous functions. What can be said about the optimal approximation order exhibited by the *best uniform approximation* $p_h^* \in S_m^{(0)}(I_h)$? It turns out, not surprisingly if we look at the Jackson theorems, that the optimal order on uniform I_h is again $\mathcal{O}(N^{-\beta})$, regardless of the degree $m \geq 1$. The optimal order is recovered by a judicious grading of the mesh I_h, as the following theorem reveals.

Theorem 6.2.6 *Assume:*

(a) $f \in C^{m+1}(0, 1)$, *with* $\int_0^1 |f^{(m+1)}(t)|^{1/(m+1)} dt < \infty$.

(b) The points defining the mesh I_h *satisfy the equidistribution equations*

$$\int_0^{t_n} |f^{(m+1)}(t)|^{1/(m+1)} dt = \frac{n}{N} \int_0^1 |f^{(m+1)}(t)|^{1/(m+1)} dt,$$
$$n = 1, \ldots, N-1.$$

If $p_h^* \in S_m^{(0)}(I_h)$ *denotes the best uniform approximation to* f *on* I, *that is, if*

$$||f - p_h^*||_\infty \leq ||f - p_h||_\infty \quad \text{for all} \quad p_h \in S_m^{(0)}(I_h),$$

then

$$||f - p_h^*||_\infty \leq C^* N^{-(m+1)}.$$

The *proof* of this theorem can be found in Schumaker (1981, pp. 286–294); see also de Boor (2000).

Illustration 6.2.2 Let $f(t) = t^\beta$ $(0 < \beta < 1)$ and $I = [0, 1]$. It follows from $f^{(m+1)}(t) = \text{const} \cdot t^{\beta-m-1}$ that

$$|f^{(m+1)}(t)|^{1/(m+1)} = \text{const} \cdot t^{\beta/(m+1)-1}.$$

Hence, the equidistribution condition reduces to

$$\int_0^{t_n} t^{\beta/(m+1)-1} dt = \frac{n}{N} \int_0^1 t^{\beta/(m+1)-1} dt \quad (n = 1, \ldots, N-1),$$

and this yields the mesh described by

$$t_n = \left(\frac{n}{N}\right)^{(m+1)/\beta} \qquad (n = 0, 1, \ldots, N).$$

We shall encounter this optimal *grading exponent*, $r = r(\beta) := (m+1)/\beta$, in the subsequent convergence analyses for VIEs and VIDEs with weakly singular kernels.

Illustration 6.2.3 Consider $f(t) = t \log(t)$ in $I = [0, 1]$. Since we now have

$$f^{(k+1)}(t) = (-1)^{k+1}(k-1)!t^{-k} \ (k \geq 1, \ t > 0),$$

and hence

$$|f^{(m+1)}(t)|^{1/(m+1)} = \text{const} \cdot t^{-m/(m+1)},$$

the equdistribution condition yields the optimal grading exponent $r = m + 1$, and the corresponding graded mesh is given by

$$t_n = \left(\frac{n}{N}\right)^{m+1} \quad (n = 0, 1, \ldots, N).$$

We observe that this agrees formally with $r(1)$ in Illustration 6.2.2. See also the result of Theorem 6.2.11.

Readers who are looking for a more general setting and additional details on the results in this section are directed to the papers by Krantz (1983) and Graham (1985).

6.2.4 The error in product quadrature on graded meshes

Suppose that $f \in C^d(I)$ $(d \geq 0)$, with $I := [0, T]$. On a given mesh I_h we approximate the integrals

$$\begin{aligned}(Q(\alpha)f)(t_n) &:= \int_0^{t_n} p_\alpha(t_n - s) f(s) ds \\ &= \sum_{\ell=0}^{n-1} h_\ell \int_0^1 p_\alpha((t_n - t_\ell)/h_\ell - s)h_\ell) f(t_\ell + sh_\ell) ds\end{aligned}$$

$(n = 1, \ldots, N)$ by the interpolatory m-point (composite) *product quadrature formulas*

$$(\hat{Q}(\alpha)f)(t_n) := \sum_{\ell=0}^{n-1} h_\ell \sum_{k=1}^{m} w_{n,k}^{(\ell)}(\alpha) f(t_{\ell,k}).$$

Here, the abscissas $t_{\ell,k} := t_\ell + c_k h_\ell$ correspond to prescribed points $\{c_k\}$ with $0 \leq c_1 < \cdots < c_m \leq 1$, and the product quadrature weights are defined by

$$w_{n,k}^{(\ell)}(\alpha) := \int_0^1 p_\alpha((t_n - t_\ell)/h_\ell - s)h_\ell) L_k(s) ds \quad (k = 1, \ldots, m)$$

(cf. (6.2.23) with $v = 0$). If $0 < \alpha < 1$ we may write

$$w_{n,k}^{(\ell)}(\alpha) = h_\ell^{-\alpha} \int_0^1 \left(\frac{t_n - t_\ell}{h_\ell} - s\right)^{-\alpha} L_k(s) ds.$$

Note that for $\ell = n - 1$ these quadrature weights assume the form

$$w_{n,k}^{(n-1)}(\alpha) = h_{n-1}^{-\alpha} \int_0^1 (1 - s)^{-\alpha} L_k(s) ds.$$

The first result (see de Hoog and Weiss (1973c)) deals with the optimal order of the product quadrature error on *uniform meshes* when $p_\alpha(t-s) = (t-s)^{-\alpha}$ $(0 < \alpha < 1)$.

Theorem 6.2.7 *Assume:*

(a) $f \in C^d(I)$ *with* $d \geq m$;
(b) $\kappa := \min \left\{ \nu \in \mathbb{N}_0 : \; J_\nu := \int_0^1 s^\nu \prod_{i=1}^m (s - c_i) ds \neq 0 \quad (\kappa \leq m) \right\}$;
(c) I_h *is uniform:* $t_n := nh, \; n = 0, 1, \ldots, N \; (Nh = T)$.

Then for any $\alpha \in (0, 1)$ *and any* $d \geq m$,

$$|(Q(\alpha)f)(t_n) - (\hat{Q}(\alpha)f)(t_n)| \leq C(\alpha) \begin{cases} h^m & \text{if } \kappa = 0 \\ h^{m+1-\alpha} & \text{if } \kappa > 0. \end{cases}$$

For suitably *graded meshes* it is possible to attain a higher order of convergence (Schneider (1980); compare also Kaneko and Xu (1994), Köhler (1995), and Tamme (2000)):

Theorem 6.2.8 *Let* κ *be as in Theorem 6.2.7 and assume that:*

(a) $f \in C^{m+\kappa}(I)$ *if* $\kappa > 0$; *else* $f \in C^{m+1}(I)$;
(b) I_h *is the graded mesh given by*

$$t_n := \left(\frac{n}{N}\right)^{\rho^*}, \quad \text{with} \quad \rho^* = \rho^*(\alpha) := \frac{m + \kappa + 1}{m + 1 - \alpha}.$$

Then for any $\alpha \in (0, 1)$ *we have*

$$|(Q(\alpha)f)(t_n) - (\hat{Q}(\alpha)f)(t_n)| = \mathcal{O}(n^{-(m+\kappa)}) \quad (n = 1, \ldots, N).$$

This estimate remains true for graded meshes with grading exponents $\rho > \rho^*$ *and sufficiently small mesh diameters* h.

The principal application of these results in the subsequent analysis will be in the convergence analysis of the discretised collocation solution (Section 6.2.7). As a preview we note here that the optimal grading exponent ρ^* given in (b) of the above theorem satisfies

$$\rho^* = \frac{m + \kappa - \alpha}{m + 1 - \alpha} < \frac{m}{1 - \alpha} =: r$$

for all $\alpha \in (0, 1]$: since r will be seen to be the optimal grading exponent for the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ to attain the global order $p = m$ on I, Theorem 6.2.8 will show that the *discretised* collocation solution will retain this order, due to sufficient (over-) grading of the mesh.

An analogous result is true for the logarithmic kernel singularity $p_1(t-s) = \log(t-s)$ (corresponding to $\alpha = 1$). This can be proved by using, for example, Theorem 2.3 in Kaneko and Xu (1994), with $r = m$ in the above inequality.

6.2.5 Global convergence results

The collocation error $e_h := y - u_h$ associated with the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ to the (linear) VIE

$$y(t) = g(t) + (\mathcal{V}_\alpha y)(t), \quad t \in I := [0, T],$$

satisfies

$$e_h(t) = (\mathcal{V}_\alpha e_h)(t), \quad t \in X_h. \tag{6.2.32}$$

Assume first that $0 < \alpha < 1$ (the case of the logarithmic kernel singularity will be considered later). We know from Section 2.2.4 and the regularity property of the exact solution y that much of the global convergence analysis carries over to VIEs with weakly singular kernels. The significant new element is the *representation of* e_h on the *first subinterval* $\bar{\sigma}_0 := [0, h_0]$: since y possesses an unbounded derivative at the left endpoint $t = 0^+$, we have to replace the local representation (2.2.31) (which was based on the Peano Theorem) by

$$e_h(t_0 + vh_0) = \sum_{j=1}^{m} L_j(v)\mathcal{E}_{0,j} + h_0^{\beta} R_{m,0}(v;\alpha), \quad v \in [0, 1], \tag{6.2.33}$$

with appropriate (fractional) exponent $\beta > 0$ and remainder term $R_{m,0}(v;\alpha)$ to be determined. On the other subintervals σ_n $(n = 1, \dots, N-1)$ the representation (2.2.31),

$$e_h(t_n + vh_n) = \sum_{j=1}^{m} L_j(v)\mathcal{E}_{n,j} + h_n^{m} R_{m,n}(v), \quad v \in (0, 1], \tag{6.2.34}$$

remains valid. Hence, a look at the proof of Theorem 2.2.3 (recall in particular (2.2.33) and the subsequent discrete Gronwall argument for (2.2.34)) reveals that the order of the collocation error will be governed by the first remainder term $h_0^{\beta} R_{m,0}(v;\alpha)$. The detailed proof of the following theorem will show that for uniform meshes the value of β will be $\beta = 1 - \alpha$, regardless of the degree $m - 1$ of the piecewise polynomials in u_h. This is of course not unexpected after our short excursion, in Section 6.2.3, into the problem of approximating non-smooth functions. That insight also suggests that if I_h is a graded mesh with grading exponent $r = r(\alpha) \geq m/(1-\alpha)$ then u_h converges again with optimal order $p = m$.

Graded meshes in collocation (and Galerkin) methods for *Fredholm integral equations* of the second kind were first employed in the late 1970s, by Chandler (1979), Graham (1980), Schneider (1981), and Vainikko and Uba (1981) (compare also the Notes in Section 6.7). For VIEs with $0 < \alpha < 1$ they were used by Brunner (1985a, 1985c) (see also the Brunner and van der Houwen (1986, Chapter 6) and the survey by Brunner (1987)). The definitive convergence (and superconvergence) analysis, including weakly singular kernels of logarithmic type, as well as bounded but non-smooth kernels, can be found in Brunner, Pedas and Vainikko (1999).

The basic result is the following.

Theorem 6.2.9 *Assume:*

(a) The given functions in the Volterra integral equation (6.2.4) satisfy $K \in C^m(D)$ *and* $g \in C^m(I)$*.*

(b) The kernel singularity in $\mathcal{V}_\alpha$ *is* $p_\alpha(t-s) = (t-s)^{-\alpha}$*, with* $0 < \alpha < 1$*.*

(c) $u_h \in S_m^{(-1)}(I_h)$ *is the (unique) collocation solution to (6.2.4) defined by (6.2.5), with* $h \in (0, \bar{h})$ *and corresponding to the collocation points* X_h*.*

(d) The grading exponent $r = r(\alpha) \geq 1$ *determining the mesh* I_h *is given by*

$$r(\alpha) = \frac{\mu}{1-\alpha}, \quad \mu \geq 1-\alpha.$$

Then we have, setting $h := T/N$*:*

$$||y - u_h||_\infty := \sup_{t \in I} |y(t) - u_h(t)| \leq C(r) \begin{cases} h^\mu & \text{if } 1-\alpha \leq \mu \leq m, \\ h^m & \text{if } \mu \geq m, \end{cases} \tag{6.2.35}$$

holds for any set X_h *of collocation points with* $0 \leq c_1 < \cdots < c_m \leq 1$*. The constant* $C(r)$ *depends on the* $\{c_i\}$ *and on the grading exponent* $r = r(\alpha)$*, but not on* h*.*

Proof As we have indicated at the beginning of this section, the proof will follow closely the one for Theorem 2.2.1, except that now the local (Peano) representation of the exact solution y on $\bar{\sigma}_n$ remains valid only if $n = 1, \ldots, N-1$. The collocation error $e_h := y - u_h$ satisfies the error equation

$$e_h(t_{n,i}) = (\mathcal{V}_\alpha e_h)(t_{n,i}), \quad i = 1, \ldots, m \quad (0 \leq n \leq N-1). \tag{6.2.36}$$

Its right-hand side is

$$(\mathcal{V}_\alpha e_h)(t_{n,i}) = \int_0^{t_1} H_\alpha(t_{n,i}, s) e_h(s) ds + \int_{t_1}^{t_n} H_\alpha(t_{n,i}, s) e_h(s) ds$$
$$+ h_n \int_0^{c_i} H_\alpha(t_{n,i}, t_n + s h_n) e_h(t_n + s h_n) ds.$$

It follows from Theorem 6.1.6 and Section 2.2 that for $n = 1, \dots, N-1$ the collocation error on the corresponding subintervals σ_n has the local Lagrange (-Peano) representation

$$e_h(t_n + vh_n) = \sum_{j=1}^{m} L_j(v)\mathcal{E}_{n,j} + h_n^m R_{m,n}(v), \quad v \in (0, 1], \qquad (6.2.37)$$

where $\mathcal{E}(t_{n,j}) := e_h(t_{n,j})$ and

$$R_{m,n}(v) := \int_0^1 K_m(v, z) y^{(m)}(t_n + zh_n)dz,$$

with

$$K_m(v, z) := \frac{1}{(m-1)!}\left\{(v-z)_+^{m-1} - \sum_{k=1}^{m} L_k(v)(c_k - z)_+^{m-1}\right\}, \quad z \in [0, 1].$$

For $n = 0$ we resort to Theorem 6.1.6: it implies that on $\bar{\sigma}_0 = [t_0, t_1] = [0, h_0]$ the exact solution of (6.2.4) can be written in the form

$$y(t_0 + vh_0) = \sum_{(j,k)_\alpha} \gamma_{j,k}(\alpha)(t_0 + vh_0)^{j+k(1-\alpha)} + h_0^m \bar{Y}_{m,0}(v; \alpha), \quad v \in [0, 1],$$

with

$$(j, k)_\alpha := \{(j, k) : \ j, k \in \mathbb{N}_0, \ j + k(1-\alpha) < m\},$$

and with obvious adaptation of the meaning of the definition of $Y_{m,0}(v; \alpha)$; recall (6.1.17). (The general initial point t_0, instead of $t_0 = 0$, is being used in view of later applications to weakly singular VIE with non-vanishing *delays*; see Sections 6.5 and 7.5.)

We rewrite this representation as

$$y(t_0 + vh_0) = \sum_{(j,k)'_\alpha} \gamma_{j,k}(\alpha)h_0^{j+k(1-\alpha)} v^{j+k(1-\alpha)} + \sum_{(j,k)''_\alpha} \gamma_{j,k}(\alpha)h_0^{j+k(1-\alpha)} v^{j+k(1-\alpha)}$$
$$+ h_0^m Y_{m,0}(v; \alpha), \quad v \in [0, 1],$$

where

$$(j, k)'_\alpha := \{(j, k) : \ j + k(1-\alpha) \in \mathbb{N}_0; \ j + k(1-\alpha) < m\}$$

and

$$(j, k)''_\alpha := \{(j, k) : \ j + k(1-\alpha) \notin \mathbb{N}_0; \ j + k(1-\alpha) < m\}.$$

With self-explanatory meaning of the coefficients $c_{j,k}(\alpha)$ we thus obtain the local representation

$$y(t_0 + vh_0) = \sum_{j=0}^{m-1} c_{j,0}(\alpha)v^j + h_0^{1-\alpha}\Phi_{m,0}(v;\alpha) + h_0^m Y_{m,0}(v;\alpha), \quad v \in [0,1], \tag{6.2.38}$$

with

$$\Phi_{m,0}(v;\alpha) := \sum_{(j,k)''_\alpha} c_{j,k}(\alpha)v^{j+k(1-\alpha)}.$$

Suppose now that on $\bar{\sigma}_0$ the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ is expressed in the form

$$u_h(t_0 + vh_0) = \sum_{j=0}^{m-1} d_{j,0}v^j, \quad v \in [0,1].$$

This allows us to write the collocation error on $\bar{\sigma}_0$ as

$$e_h(t_0 + vh_0) = \sum_{j=0}^{m-1} \beta_{j,0}(\alpha)v^j + h_0^{1-\alpha} \sum_{(j,k))''_\alpha} c_{j,k}(\alpha)v^{j+k(1-\alpha)} + h_0^m R_{m,0}(v;\alpha),$$
$$v \in [0,1], \tag{6.2.39}$$

having set $\beta_{j,0}(\alpha) := c_{j,0}(\alpha) - d_{j,0}$.

We now return to the error equation (6.2.36) corresponding to $n = 0$. It follows from

$$\begin{aligned} e_h(t_0 + c_i h_0) &= (\mathcal{V}_\alpha e_h)(t_0 + c_i h_0) \\ &= h_0^\alpha \int_0^{c_i} (c_i - s)^{-\alpha} K(t_0 + c_i h_0, t_0 + sh_0) e_h(t_0 + sh_0)ds \end{aligned}$$

that the unknown coefficients $\beta_{j,0}(\alpha)$ in (6.2.39) solve the linear algebraic system

$$\begin{aligned} &\sum_{j=0}^{m-1} \left(c_i^j - h_0^{1-\alpha} \int_0^{c_i} (c_i - s)^{-\alpha} K(t_{0,i}, t_0 + sh_0) s^j ds \right) \beta_{j,0}(\alpha) \\ &= -h_0^{1-\alpha} \sum_{(j,k)''_\alpha} \Big(c_i^{j+k(1-\alpha)} - h_0^{1-\alpha} \\ &\quad \times \int_0^{c_i} (c_i - s)^{-\alpha} K(t_{0,i}, t_0 + sh_0) s^{j+k(1-\alpha)} ds \Big) c_{j,k}(\alpha) \\ &\quad - h_0^m \left(R_{m,0}(c_i;\alpha) - h_0^{1-\alpha} \int_0^{c_i} (c_i - s)^{-\alpha} K(t_{0,i}, t_0 + sh_0) R_{m,0}(s;\alpha) ds \right) \end{aligned} \tag{6.2.40}$$

$(i = 1, \ldots, m)$. It can be written compactly as

$$[V_m - h_0^{1-\alpha} B_0(\alpha)]\beta_0(\alpha) = h_0^{1-\alpha}\mathbf{q}_0(\alpha) + h_0^m \rho_0(\alpha).$$

Here, $V_m \in L(\mathbb{R}^m)$ denotes the Vandermonde matrix based on the collocation parameters $\{c_i\}$, and the components of the vectors $\mathbf{q}_0(\alpha)$ and $\rho_0(\alpha)$ can be deduced from (6.2.40). Due to the continuity and boundedness of the kernel K and the remainder term $R_{m,0}(\cdot;\alpha)$ the inverse matrix $[V_m - h_0^{1-\alpha} B_0(\alpha)]^{-1}$ exists for all $\alpha \in (0, 1)$ and is uniformly bounded for sufficiently small h_0. This in turn implies that, since $m \geq 1$,

$$||\beta_0(\alpha)||_1 \leq Bh_0^{1-\alpha} \quad (\alpha \in (0, 1))$$

holds for some constant B, and thus, by (6.2.39),

$$|e_h(t_0 + vh_0)| \leq ||\beta_0(\alpha)||_1 + \gamma_0(\alpha)h_0^{1-\alpha} + \gamma_1(\alpha)h_0^m, \quad v \in [0, 1],$$

with appropriate constants $\gamma_0(\alpha),\ \gamma_1(\alpha)$ and $h_0 \in (0, \bar{h})$. If the grading exponent $r = r(\alpha)$ is chosen as $r = \mu/(1-\alpha)$, with $1 - \alpha \leq \mu \leq m$, then we have

$$h_0^{1-\alpha} = (TN^{-r})^{1-\alpha} = T^{1-\alpha}N^{-\mu} = \mathcal{O}(h^\mu) \quad (h := T/N),$$

by Lemma 6.2.3(a), and hence

$$||e_h||_{0,\infty} := \max_{v\in[0,1]} |e_h(t_0 + vh_0)| = \mathcal{O}(h^\mu). \tag{6.2.41}$$

Assume now that $1 \leq n \leq N-1$. It follows from the error equation (6.2.36) and the corresponding expression for $(\mathcal{V}_\alpha e_h)(t_{n,i})$ that

$$\begin{aligned}
&\mathcal{E}_{n,i} - h_n^{1-\alpha}\sum_{j=1}^{m}\left(\int_0^{c_i}(c_i - s)^{-\alpha}K(t_{n,i}, t_n + sh_n)L_j(s)ds\right)\mathcal{E}_{n,j}\\
&= \sum_{\ell=1}^{n-1} h_\ell^{1-\alpha}\sum_{j=1}^{m}\left(\int_0^1((t_{n,i} - t_\ell)/h_\ell - s)^{-\alpha}K(t_{n,i}, t_\ell + sh_\ell)L_j(s)ds\right)\mathcal{E}_{\ell,j}\\
&\quad + h_0^{1-\alpha}\int_0^1((t_{n,i} - t_0)/h_0 - s)^{-\alpha}K(t_{n,i}, t_0 + sh_0)e_h(t_0 + sh_0)ds\\
&\quad + h_n^{m+1-\alpha}\int_0^{c_i}(c_i - s)^{-\alpha}K(t_{n,i}, t_n + sh_n)R_{m,n}(s;\alpha)ds\\
&\quad + \sum_{\ell=1}^{n-1} h_\ell^{m+1-\alpha}\int_0^1((t_{n,i} - t_\ell)/h_\ell - s)^{-\alpha}K(t_{n,i}, t_\ell + sh_\ell)R_{m,\ell}(s;\alpha)ds
\end{aligned} \tag{6.2.42}$$

$(i = 1, \ldots, m)$. This represents a linear algebraic system,

$$[\mathcal{I}_m - h_n^{1-\alpha} B_n(\alpha)]\boldsymbol{\mathcal{E}}_n = \sum_{\ell=1}^{n-1} h_\ell^{1-\alpha} B_n^{(\ell)}(\alpha)\boldsymbol{\mathcal{E}}_\ell + h_0^{1-\alpha}\mathbf{q}_n^{(0)}(\alpha)$$
$$+ h_n^{m+1-\alpha}\boldsymbol{\rho}_n(\alpha) + \sum_{\ell=1}^{n-1} h_\ell^{m+1-\alpha}\boldsymbol{\rho}_n^{(\ell)}(\alpha), \quad (6.2.43)$$

described by the vectors

$$\mathbf{q}_0(\alpha) := \left(\int_0^1 ((t_{n,i} - t_0)/h_0 - s)^{-\alpha} K(t_{n,i}, t_0 + sh_0)e_h(t_0 + sh_0)ds \right.$$
$$\left. (i = 1, \ldots, m)\right)^T,$$

$$\boldsymbol{\rho}_n(\alpha) := \left(\int_0^{c_i} (c_i - s)^{-\alpha} K(t_{n,i}, t_n + sh_n)R_{m,n}(s;\alpha)ds \quad (i = 1, \ldots, m)\right)^T,$$

$$\boldsymbol{\rho}_n^{(\ell)}(\alpha) := \left(\int_0^1 ((t_{n,i} - t_\ell)/h_\ell - s)^{-\alpha} K(t_{n,i}, t_\ell + sh_\ell)R_{m,\ell}(s;\alpha)ds \right.$$
$$\left. (i = 1, \ldots, m)\right)^T,$$

and the matrices $B_n(\alpha)$ and $B_n^{(\ell)}(\alpha)$ $(\ell < n)$ whose meaning is clear from (6.2.42) (see also (6.2.11) and (6.2.12)). As Theorem 6.2.1 showed, $[\mathcal{I}_m - h_n^{1-\alpha} B_n(\alpha)]^{-1}$ exists and is uniformly bounded whenever $h_n \in (0, \bar{h})$: there is a constant $D_0(\alpha)$ so that

$$||(\mathcal{I}_m - h_n^{1-\alpha} B_n(\alpha))^{-1}||_1 \leq D_0(\alpha) \quad (n = 1, \ldots, N-1). \quad (6.2.44)$$

Thus, (6.2.43) yields a generalised discrete Gronwall inequality,

$$||\boldsymbol{\mathcal{E}}_n||_1 \leq D_0(\alpha)\left(\sum_{\ell=1}^{n-1} h_\ell^{1-\alpha}||B_n^{(\ell)}(\alpha)||_1 \cdot ||\boldsymbol{\mathcal{E}}_\ell||_1 + h_0^{1-\alpha}||\mathbf{q}_n^{(0)}(\alpha)||_1\right.$$
$$\left. + h_n^{m+1-\alpha}||\boldsymbol{\rho}_n(\alpha)||_1 + \sum_{\ell=1}^{n-1} h_\ell^{m+1-\alpha}||\boldsymbol{\rho}_n^{(\ell)}(\alpha)||_1\right) \quad (n = 1, \ldots, N-1).$$
$$(6.2.45)$$

In order to derive the desired ℓ^1-estimates for the above vectors and matrices (so as to transform (6.2.45) into a discrete Gronwall inequality of the form (6.1.45)), we have to appeal to Lemma 6.2.3 and the following

Lemma 6.2.10 *Let I_h be the graded mesh (6.2.31) on $I = [0, T]$, with grading exponent $r \geq 1$. If the $\{c_i\}$ satisfy $0 \leq c_1 < \cdots < c_m \leq 1$ then, for $1 \leq \ell < n \leq$*

$N-1$ and $\nu \in \mathbb{N}_0$,

$$\int_0^1 \left(\frac{t_{n,i}-t_\ell}{h_\ell}-s\right)^{-\alpha} s^\nu ds \le \gamma(\alpha)(n-\ell)^{-\alpha} \quad (i=1,\dots,m)$$

with $\gamma(\alpha) := 2^\alpha/(1-\alpha)$.

Proof Consider first the case $\ell = n-1$ for which

$$\begin{aligned}\int_0^1 \left(\frac{t_{n,i}-t_{n-1}}{h_{n-1}}-s\right)^{-\alpha} s^\nu ds &\le \int_0^1 (1+c_i h_n/h_{n-1}-s)^{-\alpha} ds \\ &\le 1/(1-\alpha) < 2^\alpha/(1-\alpha),\end{aligned}$$

$(i=1,\dots,m;\ 0<\alpha<1)$. Here, we have used the fact that $r \ge 1$ $(r>1)$ implies that $h_{n-1} \le h_n$ $(h_{n-1} < h_n)$ for $n = 1,\dots,N-1$.

Assume now that $\ell < n-1$. In this case we obtain

$$\begin{aligned}\int_0^1 \left(\frac{t_{n,i}-t_\ell}{h_\ell}-s\right)^{-\alpha} s^\nu ds &\le \int_0^1 \left(\frac{t_n-t_\ell}{h_\ell}-s\right)^{-\alpha} ds \\ &= \frac{1}{1-\alpha}\left\{\left(\frac{t_n-t_\ell}{h_\ell}\right)^{1-\alpha} - \left(\frac{t_n-t_\ell}{h_\ell}-1\right)^{1-\alpha}\right\} \\ &= \frac{1}{1-\alpha}\left(\frac{t_n-t_\ell}{h_\ell}\right)^{1-\alpha} \\ &\quad \times \left\{1-\left[1-\left(\frac{t_n-t_\ell}{h_\ell}\right)^{-1}\right]^{1-\alpha}\right\}.\end{aligned}$$

The application of the Mean-Value Theorem to the function $f(z) := (1-z)^{1-\alpha}$, with $z := [(t_n-t_\ell)/h_\ell]^{-1}$, leads without difficulty to

$$\int_0^1 \left(\frac{t_{n,i}-t_\ell}{h_\ell}-s\right)^{-\alpha} ds \le \left(\frac{t_n-t_\ell}{h_\ell}\right)^{-\alpha}\left(1-\theta_{n,\ell}\left(\frac{t_n-t_\ell}{h_\ell}\right)^{-1}\right)^{-\alpha},$$

where $\theta_{n,\ell}$ is some number between 0 and 1. Since, as pointed out above,

$$0 < h_0 < h_1 < \cdots < h_{n-1} < h_n < \cdots < h_{N-1} = h,$$

it follows that

$$\frac{t_n-t_\ell}{h_\ell} = \frac{h_{n-1}+\cdots+h_{\ell+1}+h_\ell}{h_\ell} \ge \frac{(n-\ell)h_\ell}{h_\ell} = n-\ell,$$

and so

$$1-\theta_{n,\ell}\left(\frac{t_n-t_\ell}{h_\ell}\right)^{-1} \ge 1-\left(\frac{t_n-t_\ell}{h_\ell}\right)^{-1} \ge 1-\left(\frac{h_{\ell+1}+h_\ell}{h_\ell}\right)^{-1} \ge \frac{1}{2},$$

whenever $\ell \le n-2$.

We have thus shown that

$$\int_0^1 \left(\frac{t_{n,i} - t_\ell}{h_\ell} - s\right)^{-\alpha} s^\nu ds \le 2^\alpha (n-\ell)^{-\alpha} < \frac{2^\alpha}{1-\alpha}(n-\ell)^{-\alpha}$$

for $i = 1, \dots, m$, $\ell \le n-2$, and $0 < \alpha < 1$. This completes the proof of Lemma 6.2.10.

Recall now the definition of the matrices $B_n^{(\ell)}(\alpha)$ and the vectors $\boldsymbol{\rho}_n^{(\ell)}(\alpha)$ ($\ell < n$) from Section 6.2.1. It is easy to verify, along the lines of the proof of Theorem 2.2.3, that

$$||B_n^{(\ell)}(\alpha)||_1 \le D_1(\alpha)(n-\ell)^{-\alpha} \quad (\ell < n)$$

and

$$||\boldsymbol{\rho}_n^{(\ell)}(\alpha)||_1 \le R_1(\alpha)(n-\ell)^{-\alpha} \quad (\ell < n),$$

with appropriate constants $D_1(\alpha)$ and $R_1(\alpha)$ depending on m and the bounds for K and the uniform norms of the Lagrange fundamental polynomials L_j. The inequality (6.2.45) now becomes

$$||\boldsymbol{\mathcal{E}}_n||_1 \le \gamma_0(\alpha)h^{1-\alpha}\sum_{\ell=1}^{n-1}(n-\ell)^{-\alpha}||\boldsymbol{\mathcal{E}}_\ell||_1 + \gamma_1(\alpha)h_0^{1-\alpha}$$
$$+ \gamma_2(\alpha)h_n^{m+1-\alpha} + \gamma_3(\alpha)\sum_{\ell=1}^{n-1} h_\ell^{m+1-\alpha}(n-\ell)^{-\alpha}, \qquad (6.2.46)$$

with $1 \le n \le N-1$ and appropriate constants $\gamma_i(\alpha)$ ($i = 1, 2, 3$). (It is instructive to compare this with (2.2.34).)

Recall now the generalised discrete Gronwall inequality (6.1.45) and Theorem 6.1.19: we now have $z_\ell := ||\boldsymbol{\mathcal{E}}_\ell||_1$, and the sequence $\{\gamma_n\}$ given by

$$\gamma_n := \gamma_1(\alpha)h_0^{1-\alpha} + \gamma_2(\alpha)h^{m+1-\alpha} + \gamma_3(\alpha)\sum_{\ell=1}^{n-1} h_\ell^{m+1-\alpha}(n-\ell)^{-\alpha} \quad (n \ge 1)$$

is clearly non-decreasing. Moreover, we have

$$\sum_{\ell=1}^{n-1} h_\ell^{1-\alpha}(n-\ell)^{-\alpha} \le \frac{T^{1-\alpha}}{1-\alpha}, \quad n = 1, \dots, N.$$

This is easily verified by observing that, for any uniform mesh,

$$\int_0^{t_n}(t_n - s)^{-\alpha}ds = h^{1-\alpha}\sum_{\ell=0}^{n-1}\int_0^1 (n-\ell-s)^{-\alpha}ds \ge h^{1-\alpha}\sum_{\ell=0}^{n-1}(n-\ell)^{-\alpha},$$

where the last expression represents the lower Riemann sum (left rectangular quadrature approximation) for the given integral whose integrand is convex on $[0, t_n)$.

Hence, we have found a uniform upper bound for γ_n, namely,

$$\begin{aligned}\gamma_n \leq \bar{\gamma} &:= \gamma_1(\alpha)h_0^{1-\alpha} + \gamma_2(\alpha)h^{m+1-\alpha} + \gamma_3(\alpha)h^m T^{1-\alpha}/(1-\alpha)\\ &= \gamma_1(\alpha)h_0^{1-\alpha} + [\gamma_2(\alpha)h^{1-\alpha} + \gamma_3(\alpha)T^{1-\alpha}/(1-\alpha)]h^m,\end{aligned}$$

and with this (6.2.46) leads to

$$||\mathcal{E}_n||_1 \leq E_{1-\alpha}(\gamma_0(\alpha)\Gamma(1-\alpha)(nh)^{1-\alpha}) \cdot h_0^{1-\alpha} \cdot \bar{\gamma}.$$

Lemma 6.2.3 shows that

$$nh \leq nrTN^{-1} = (n/N)rT \leq rT, \quad n = 1, \dots, N,$$

and we have

$$h_0^{1-\alpha} = (TN^{-r})^{1-\alpha} = T^{1-\alpha}N^{-r(1-\alpha)} = T^{1-\alpha}N^{-\mu}, \tag{6.2.47}$$

for any graded I_h with grading exponent $r = \mu/(1-\alpha)$ $(1-\alpha \leq \mu \leq m)$. Therefore, $||\mathcal{E}_n||_1 \leq Bh^\mu$ $(1 \leq n \leq N-1)$, and so, by (6.2.37) and (6.2.41), we arrive at the desired estimate for $||e_h||_\infty$.

If the weakly singular part of the kernel $H_\alpha(t, s)$ in (6.2.2) is of *logarithmic type* (corresponding to $\alpha = 1$) we recover the optimal (global) order of convergence if $r = m$, as the following theorem (due to Brunner, Pedas and Vainikko (1999)) shows.

Theorem 6.2.11 *Assume that in (6.2.2), (6.2.4) we have* $\alpha = 1$ *(that is,* $p_\alpha(t-s) = \log(t-s)$*) and* $g \in C^m(I)$, $K \in C^m(D)$. *Let* I_h *be the graded mesh (6.2.31) with grading exponent* $r \geq 1$, *and set* $h := T/N$. *Then the global order of convergence of the collocation solution* $u_h \in S_{m-1}^{(-1)}(I_h)$ *for (6.2.4) is described by*

$$||y - u_h||_\infty \leq C(r) \begin{cases} h(1 + |\log(h)|) & \text{if } r = 1 \text{ and } m = 1,\\ h & \text{if } r > 1 \text{ and } m = 1,\\ h^r & \text{if } 1 \leq r \leq m \text{ and } m \geq 2,\\ h^m & \text{if } r \geq m \text{ and } m \geq 2.\end{cases}$$

Proof The proof is based on the regularity result of Theorem 6.1.7 (recall also Illustration 6.2.3). Details may be found in the paper just mentioned: they include the embedding of the given VIE (6.2.4) into a second-kind Fredholm integral equation and the corresponding tools for the analysis of its collocation solution.

We conclude this section by complementing the above convergence results with one for VIEs whose kernels are bounded but have unbounded derivatives. As a typical example, consider

$$(\mathcal{V}_\nu y)(t) := \int_0^t (t-s)^\nu K(t,s)y(s)ds, \quad t \in I := [0,T], \tag{6.2.48}$$

with $\nu := \rho - \alpha$ ($\rho \in \mathbb{N}$ and $0 < \alpha < 1$). Assume also that $K(t,t) \neq 0$ for $t \in I$. (VIEs with more general non-smooth (but bounded) kernels have been studied in Brunner, Pedas and Vainikko (1999).)

Theorem 6.2.12 *Assume that the given functions g and K in*

$$y(t) = g(t) + (\mathcal{V}_\nu y)(t), \quad t \in I,$$

satisfy $g \in C^m(I)$, $K \in C^m(D)$. If I_h is the graded mesh (6.2.31) with grading exponent

$$r = r(\nu) = \frac{\mu}{1+\nu} \quad (\mu \geq 1+\nu),$$

then the corresponding collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ satisfies

$$||y - u_h||_\infty \leq C(r) \begin{cases} h^\mu & \text{if } 1+\nu \leq \mu \leq m, \\ h^m & \text{if } \mu \geq m, \end{cases}$$

where we have again defined $h := T/N$. These estimates hold for any $\{c_i\}$ with $0 \leq c_1 < \cdots < c_m \leq 1$.

The **proof** is left as an exercise: its starting point is the regularity result of Theorem 6.1.8, and it consists essentially in a straightforward adaption of the proof for Theorem 6.2.9 where $1-\alpha$ is now replaced by $1+\nu = \rho + 1 - \alpha$. The choice of the grading exponent then implies

$$h_0^{1+\nu} = (TN^{-r})^{1+\nu} = T^{1+\nu}N^{-\mu}.$$

Compare also Brunner (1985b), and see Brunner, Pedas and Vainikko (1999) for the extension of Theorem 6.2.12 to bounded but non-smooth kernels of the form $(t-s)^k \log(t-s)$ ($k \in \mathbb{N}$).

6.2.6 Global and local superconvergence results

So far we have only considered the attainable order of (global) convergence on I for the collocation solution u_h when $\{c_i\}$ is an arbitrary set of collocation parameters. If u_h^{it} is the corresponding iterated collocation solution,

$$u_h^{it}(t) := g(t) + (\mathcal{V}_\alpha u_h)(t), \quad t \in I,$$

can it exhibit global or local superconvergence (on I or $I_h \setminus \{0\}$, respectively), and what are the optimal orders? It is intuitively clear that on uniform meshes little can be gained. As the following theorems show, the possible orders of superconvergence for suitably graded meshes are only marginally higher than m.

Theorem 6.2.13 *Assume:*

(a) $g \in C^{m+1}(I)$, $K \in C^{m+1}(D)$, *with* $K(t,t) \neq 0$ *on* I, *and* $0 < \alpha < 1$;
(b) $u_h \in S_{m-1}^{(-1)}(I_h)$ *is the collocation solution to (6.2.4), with corresponding iterated collocation solution* u_h^{it};
(c) the collocation parameters satisfy $J_0 := \int_0^1 \prod_{i=1}^m (s - c_i)ds = 0$;
(d) I_h *is the graded mesh (6.2.31) with grading exponent* $r \geq 1$, *and* $h := T/N$.

Then:

$$||y - u_h^{it}||_\infty \leq C(r) \begin{cases} h^{2(1-\alpha)} & \text{if } r = 1, \\ h^{m+1-\alpha} & \text{if } r \geq \frac{m}{1-\alpha}. \end{cases}$$

Proof The relationship between $e_h^{it} := y - u_h^{it}$ and the defect δ_h, defined by

$$\delta_h(t) := -u_h(t) + g(t) + (\mathcal{V}_\alpha u_h)(t), \quad t \in I,$$

is – in complete analogy to the case $\alpha = 0$ of Section 2.2 – given by

$$e_h^{it}(t) = \int_0^t R_\alpha(t,s)\delta_h(s)ds, \quad t \in I, \tag{6.2.49}$$

where $R_\alpha(t,s)$ is the resolvent kernel introduced in Theorem 6.1.2,

$$R_\alpha(t,s) := (t-s)^{-\alpha} Q(t,s;\alpha) \quad (0 < \alpha < 1).$$

We first observe that

$$\delta_h(t) = e_h(t) - (\mathcal{V}_\alpha e_h)(t), \quad t \in I,$$

implies, by the global convergence result of Theorem 6.2.9,

$$||\delta_h||_\infty \leq (1 + ||\mathcal{V}_\alpha||_\infty)||e_h||_\infty \leq (1 + ||\mathcal{V}_\alpha||_\infty)C(r)h^\mu =: D(r)h^\mu,$$

provided the grading exponent has been chosen as $r = \mu/(1-\alpha)$ with $1 - \alpha \leq \mu \leq m$. Here, the norm of the Volterra integral operator $\mathcal{V}_\alpha$ is given by

$$||\mathcal{V}_\alpha||_\infty = \max_{t \in I} \int_0^t p_\alpha(t-s)|K(t,s)|ds.$$

Consider (6.2.49) for $t = t_n + vh_n \in \sigma_n$. If $n = 0$ there exist constants $D = D(r)$ and Q_α so that

$$
\begin{aligned}
|e_h^{it}(t)| &\leq \int_0^t (t-s)^{-\alpha}|Q(t,s;\alpha)||\delta_h(s)|ds \\
&\leq ||\delta_h||_\infty \int_0^t (t-s)^{-\alpha}|Q(t,s;\alpha)|ds \\
&\leq D(r)h^\mu \cdot Q_\alpha t^{1-\alpha}/(1-\alpha) \leq D(r)h^\mu Q_\alpha h_0^{1-\alpha}/(1-\alpha) = \mathcal{O}(h^{\mu+1-\alpha}), \\
&\quad v \in [0,1].
\end{aligned}
$$

Thus, on *uniform* I_h we obtain

$$|e_h^{it}(t_0 + vh)| = \mathcal{O}(h^{2(1-\alpha)} \quad (v \in [0,1].$$

If I_h is a *graded* mesh, with $r = m$, there follows

$$|e_h^{it}(t_0 + vh_0)| = \mathcal{O}(h^{m+1-\alpha}), \quad v \in [0,1],$$

with $h := T/N$

If $1 \leq n \leq N-1$ and $t = t_n + vh_n \in \sigma_n$, equation (6.2.49) yields

$$
\begin{aligned}
e_h^{it}(t) = &\int_0^{t_n} (t-s)^{-\alpha} Q(t,s;\alpha)\delta_h(s)ds + h_n^{1-\alpha} \\
&\int_0^v (v-s)^{-\alpha} Q(t, t_n + sh_n;\alpha)\delta_h(t_n + sh_n)ds.
\end{aligned}
$$

Consider the second ('local') term on the right-hand side: since $||\delta_h||_\infty = \mathcal{O}(h^\mu)$ an upper bound for its absolute value is given by

$$h_n^{1-\alpha}h^\mu Q_0(\alpha)/(1-\alpha) = \mathcal{O}(N^{-(\mu+1-\alpha)}) \quad (1-\alpha \leq \mu \leq m).$$

The sum in the first term can be written as

$$\sum_{\ell=0}^{n-1} h_\ell^{1-\alpha} \int_0^1 \left(\frac{t_n + vh_n - t_\ell}{h_\ell} - s\right)^{-\alpha} Q(t, t_\ell + sh_\ell;\alpha)\delta_h(t_\ell + sh_\ell)ds.$$

Since we now have $\ell < n$, the integrands of the individual integrals are no longer singular. Hence, the orders of the quadrature errors induced by (weighted) interpolatory quadrature based on the m collocation points in each σ_ℓ match that of the second term (see also Schneider (1980) and Kaneko and Xu (1994)), because the collocation parameters $\{c_i\}$ are assumed to satisfy the orthogonality condition $J_0 = 0$.

6.2.7 Fully discretised collocation and product integration methods

We have shown in Section 2.2.6 that a judicious choice of the quadrature formulas in the discretisation of the integrals occurring in the collocation equation will not reduce the order of convergence of the resulting discretised collocation solution $\hat{u}_h$. This remains true for VIEs with weakly singular kernels when appropriate product quadrature formulas are used. To be more precise, assume that u_h and $\hat{u}_h$ denote again the exact and the discretised collocation solution in $S_{m-1}^{(-1)}(I_h)$: they are determined, respectively, by the equations

$$u_h(t) = g(t) + (\mathcal{V}_\alpha u_h)(t), \quad t \in X_h$$

and

$$\hat{u}_h(t) = g(t) + (\hat{\mathcal{V}}_{\alpha,h}\hat{u}_h)(t), \quad t \in X_h.$$

The discretised Volterra operator $\hat{\mathcal{V}}_{\alpha,h}$ was introduced in Section 6.2.2: it is given by

$$(\hat{\mathcal{V}}_{\alpha,h}\hat{u}_h)(t_{n,i}) := \hat{F}_n(t_{n,i};\alpha) + h_n^{1-\alpha}(\hat{Q}_n(\alpha)\hat{u}_h)(t_{n,i}),$$

where

$$\hat{F}_n(t_{n,i};\alpha) := \sum_{\ell=0}^{n-1}(\hat{Q}_n^{(\ell)}(\alpha)\hat{u}_h)(t_{n,i})$$

and with product quadrature operators $\hat{Q}_n(\alpha)$ and $\hat{Q}_n^{(\ell)}(\alpha)$ defined in (6.2.20) and (6.2.21). The discretised iterated collocation solution $\hat{u}_h^{it}$ associated with $\hat{u}_h$ is defined by

$$\hat{u}_h^{it}(t) := g(t) + (\hat{\mathcal{V}}_{\alpha,h}\hat{u}_h)(t), \quad t \in I.$$

What can be said about the orders of the perturbations

$$z_h(t) := u_h(t) - \hat{u}_h(t),$$

and

$$z_h^{it}(t) := u_h^{it}(t) - \hat{u}_h^{it}(t), \quad t \in I \ ?$$

In order to show that its order agrees with that of the exact collocation solution itself, we introduce the product quadrature errors,

$$(\hat{Q}_n^{(\ell)}(\alpha)\hat{u}_h)(t_{n,i}) = (Q_n^{(\ell)}(\alpha)\hat{u}_h)(t_{n,i}) - E_n^{(\ell)}(t_{n,i};\alpha) \quad (\ell < n),$$
$$(\hat{Q}_n(\alpha)\hat{u}_h)(t_{n,i}) = (Q_n(\alpha)\hat{u}_h)(t_{n,i}) - E_n(t_{n,i};\alpha),$$

and we define

$$\epsilon_n(t_{n,i}) := \sum_{\ell=0}^{n-1} h_\ell^{1-\alpha} E_n^{(\ell)}(t_{n,i};\alpha) + h_n^{1-\alpha} E_n(t_{n,i};\alpha) \quad (i = 1, \ldots, m). \quad (6.2.50)$$

Let $\boldsymbol{\epsilon}_n := (\epsilon_n(t_{n,1}), \ldots, \epsilon_n(t_{n,m}))^T$, and set $Z_{n,i} := z_h(t_{n,i})$. Since we have

$$z_h(t_{n,i}) = (\mathcal{V}_\alpha u_h)(t_{n,i}) - (\hat{\mathcal{V}}_{\alpha,h}\hat{u}_h)(t_{n,i}),$$

it follows that $\mathbf{Z}_n := (Z_{n,1}, \ldots, Z_{n,m})^T$ solves the algebraic system

$$[\mathcal{I}_m - h_n^{1-\alpha} B_n(\alpha)]\mathbf{Z}_n = \sum_{\ell=0}^{n-1} h_\ell^{1-\alpha} B_n^{(\ell)}(\alpha)\mathbf{Z}_\ell + \boldsymbol{\epsilon}_n, \quad (6.2.51)$$

$n = 0, 1, \ldots, N-1;\ 0 < \alpha < 1$. (Before proceeding, the reader may wish to have another look at (6.2.43)–(6.2.45) and at Lemma 6.2.10.) Hence,

$$||\mathbf{Z}_n||_1 \le \gamma_0(\alpha)h^{1-\alpha}\sum_{\ell=0}^{n-1}(n-\ell)^{-\alpha}||\mathbf{Z}_\ell||_1 + D_0(\alpha)||\boldsymbol{\epsilon}_n||_1,$$

whenever $h \in (0, \bar{h})$. Since the integrands in $Q_n^{\ell)}(\alpha)\hat{u}_h$ and $Q_n(\alpha)\hat{u}_h$ are smooth on each subinterval $\bar{\sigma}_n$, the orders of the quadrature errors in (6.2.50) are governed by the results of Theorem 6.2.7 (if I_h is uniform) and Theorem 6.2.8 (if I_h is graded); see also Brunner (1984b) and Brunner and van der Houwen (1986, pp. 365–369). We summarise these order results in

Theorem 6.2.14 *Assume:*

(a) $g \in C^d(I),\ K \in C^d(D)$ *for some* $d \ge m$, *and* $0 < \alpha < 1$;
(b) $u_h \in S_{m-1}^{(-1)}(I_h)$ *is the exact collocation solution, with corresponding iterated collocation slution* u_h^{it}, *for the weakly singular VIE (6.2.4);*
(c) $\hat{u}_h \in S_{m-1}^{(-1)}(I_h)$ *denotes the discretised collocation solution, for the same collocation points* X_h, *given by (6.2.19), with discretised iterated collocation solution* $\hat{u}_h^{it}$. *The underlying quadrature formulas are the (interpolatory) product quadrature rules (6.2.20) and (6.2.21).*
(d) The collocation parameters c_i *satisfy the orthogonality condition of Theorem 6.2.7 (assumption (b)).*

Then:

(i) The estimates

$$||u_h - \hat{u}_h||_\infty \le C(\alpha)\begin{cases} h^m & \text{if } \kappa = 0, \\ h^{m+1-\alpha} & \text{if } \kappa \ge 1 \end{cases}$$

hold for uniform I_h. *The same orders are attained by* $||u_h^{it} - \hat{u}_h^{it}||_\infty$.

(ii) If I_h is graded, with grading exponent r satisfying $r \geq (m+\kappa+1)/(m+1-\alpha)$, then we obtain (setting $h := T/N$)

$$||u_h - \hat{u}_h||_\infty \leq C(r)h^{m+\kappa} \quad \textit{and} \quad ||u_h^{it} - \hat{u}_h^{it}||_\infty \leq C(r)h^{m+\kappa},$$

for all $\alpha \in (0, 1)$.

Remark The use of non-product (but interpolatory) quadrature formulas for the integrals $Q_n^{(\ell)}(\alpha)u_h$ ($\ell < n$) whose abscissas are the collocation points in the subintervals σ_ℓ lead to the same estimates, but possibly with larger error constants.

Proof In Theorem 6.2.7 (product quadrature on *uniform* I_h) and Theorem 6.2.8 (product quadrature on *graded meshes*) we presented the relevant information on the orders of the resulting product quadrature errors. Note that the smooth part of the integrand in Section 6.2.4 is now given by the product of the kernel K and the restriction of the collocation solution u_h (or $\hat{u}_h$) on the subintervals $\bar{\sigma}_n$. Moreover, as we observed after Theorem 6.2.8,

$$r = m/(1-\alpha) > (m+\kappa+1)/(m+1-\alpha)$$

for $\alpha \in (0, 1)$ and all $m \geq 1$.

6.2.8 Comparison with weakly singular Fredholm integral equations

Solutions of second-kind Fredholm integral equations with weakly singular kernels but otherwise smooth data g and K (with $K(t, t) \neq 0$),

$$y(t) = g(t) + \lambda(\mathcal{F}_\alpha y)(t), \quad t \in I := [0, T],$$

where the Fredholm operator $\mathcal{F}_\alpha : C(I) \to C(I)$ has the form

$$(\mathcal{F}_\alpha \phi)(t) := \int_0^T p_\alpha(|t-s|)K(t, s)\phi(s)ds, \quad 0 < \alpha \leq 1,$$

and $p_\alpha(t)$ as before, possess unbounded derivatives at both endpoints of I. To be more precise, assume that $\lambda^{-1} \notin \sigma(\mathcal{F}_\alpha)$ and that $g \in C^m(I)$, $K \in C^m(I \times I)$. The unique solution y then lies in $C^m(0, T)$, and the behaviour at $t = 0^+$ and $t = T^-$ is described by

$$|y^{\nu)}(t)| \leq C[t^{1-\alpha-\nu} + (T-t)^{1-\alpha-\nu}], \quad t \in (0, T), \quad (\nu = 1, \ldots, m).$$

Details and proofs of this and more general regularity results for weakly singular FIEs can be found in Richter (1976), Schneider (1979), Vainikko and Uba (1981) (see also for earlier papers, in Russian, by Vainikko and others), Graham

(1982a), and in the book by Vainikko, Pedas and Uba (1984). More recent papers are by Kaneko, Noren and Xu (1992) and by Pedas and Vainikko (1997) (nonlinear FIEs). The standard reference for regularity results of solutions of multidimensional FIEs with weakly singular kernels is Vainikko (1993).

It is clear from our earlier analysis that this singular behaviour of the solution at $t = 0$ and $t = T$ will in general again result in a reduction of the attainable order of piecewise polynomial collocation solutions. Assume that $u_h \in S_m^{(-1)}(I_h)$ satifies the collocation equation

$$u_h(t) = g(t) + \lambda(\mathcal{F}_\alpha u_h)(t), \quad t \in X_h,$$

with corresponding iterated collocation solution given by

$$u_h^{it}(t) := g(t) + \lambda(\mathcal{F}_\alpha u_h)(t), \quad t \in I.$$

The set X_h of collocation points is again based on m distinct collocation parameters $\{c_i\}$ in I. In order to reflect the symmetric location of the points where y has unbounded derivatives we choose the points of the mesh I_h by

$$t_n := \left(\frac{n}{N}\right)^r \frac{T}{2}, \quad n = 0, 1, \ldots, N; \qquad t_{N+n} := T - t_{N-n}, \quad n = 1, \ldots, N,$$

with $r \geq 1$ denoting the grading parameter. It can then be shown (see, e.g. Vainikko and Pedas (1981), Schneider (1981), Pedas and Vainikko (1997); also Kaneko, Noren and Xu (1992) and Kaneko, Noren and Padilla (1997)) that, for sufficiently regular g and K and $0 < \alpha < 1$,

$$||y - u_h||_\infty := \sup\{|y(t) - u_h(t)| : \ t \in I\} = \mathcal{O}(N^{-\mu})$$

if the grading exponent is given by $r = \mu/(1-\alpha)$ $(1 - \alpha \leq \mu \leq m)$, in complete analogy to the result in Theorem 6.2.9 for weakly singular VIEs. The iterated collocation solution exhibits (slight) *global superconvergence* on I, namely

$$||y - u_h^{it}||_\infty = \mathcal{O}(N^{-(m+1-\alpha)},$$

provided the collocation parameters satisfy the orthogonality condition $J_0 = 0$ (cf. Theorem 6.2.13).

Similar optimal estimates hold when $\alpha = 1$ $(p_\alpha(t) = \log(t))$; see Pedas and Vainikko (1997) for details.

6.2.9 Hammerstein-type VIEs: Implicitly linear collocation

Most nonlinear VIEs with weakly singular kernels arising in the mathematical modelling of physical or biological phenomena are of *Hammerstein type* (see, for example, Mann and Wolf (1951), Roberts and Mann (1951), Padmavally (1958), Levin (1960), Olmstead and Handelsman (1976), Groetsch (1989, 1991), and their references). A typical example arises in nonlinear heat condution and superfluidity: it is

$$y(t) = \gamma \int_0^t (t-s)^{-\alpha}[f(s) - (y(s))^k]ds \quad (\alpha = 1/2, \quad k > 1)$$

(Roberts and Mann (1951); see also Gorenflo and Kilbas (1995)). Therefore we will not extend the previous convergence analyses to completely general nonlinear VIEs but restrict our considerations to problems of the form

$$y(t) = g(t) + (\mathcal{H}_\alpha y)(t), \quad t \in I := [0, T], \tag{6.2.52}$$

with

$$(\mathcal{H}_\alpha y)(t) := \int_0^t H_\alpha(t,s)G(s, y(s))ds.$$

Here, $H_\alpha(t,s) := p_\alpha(t-s)K(t,s)$, with smooth K and G and with $K(t,t) \neq 0$ on I. The VHIE (6.2.53) can again be rewritten in a form that leads to a computationally more attractive version of the collocation method. Setting

$$z(t) := (\mathcal{N}y)(t) := G(t, y(t)), \quad t \in I, \tag{6.2.53}$$

where $\mathcal{N}$ denotes the Niemytzki operator, equation (6.2.52) becomes an implicitly linear integral equation for z, namely,

$$z(t) = (\mathcal{N}(g + \mathcal{V}_\alpha z))(t) = G(t, g(t) + (\mathcal{V}_\alpha z)(t)), \quad t \in I, \tag{6.2.54}$$

with *linear* Volterra operator

$$(\mathcal{V}_\alpha z)(t) := \int_0^t p_\alpha(t-s)K(t,s)z(s)ds.$$

The solution y is then found by the iteration

$$y(t) = g(t) + (\mathcal{V}_\alpha z)(t), \quad t \in I. \tag{6.2.55}$$

Hence, as in Section 2.33, we approximate z by the collocation solution $z_h \in S_{m-1}^{(-1)}(I_h)$,

$$z_h(t) = G(t, g(t) + (\mathcal{V}_\alpha z_h)(t)), \quad t \in X_h, \tag{6.2.56}$$

and define the approximation y_h to the solution y of the original VHIE (6.2.50) by

$$y_h(t) := g(t) + (\mathcal{V}_\alpha z_h)(t), \quad t \in I. \tag{6.2.57}$$

The computational form of the collocation equation (6.2.56) on σ_n uses the local representation

$$z_h(t_n + vh_n) = \sum_{j=1}^{m} L_j(v) Z_{n,j}, \quad v \in (0,1], \quad Z_{n,j} := z_h(t_{n,j}), \qquad (6.2.58)$$

and is thus given by

$$\begin{aligned} Z_{n,i} &= G(t_{n,i}, g(t_{n,i}) + F_n(t_{n,i}; \alpha) \\ &\quad + h_n \sum_{j=1}^{m} \left(\int_0^{c_i} H_\alpha(t_{n,i}, t_n + sh_n) L_j(s) ds \right) Z_{n,j}) \end{aligned} \qquad (6.2.59)$$

($i = 1, \ldots, m$), with lag term

$$F_n(t; \alpha) := \int_0^{t_n} H_\alpha(t,s) z_h(s) ds \quad (t \in \sigma_n).$$

When z_h is known we can compute the approximation to the solution y of the given VIE at $t = t_n + vh_n \in \bar{\sigma}_n$ by means of

$$\begin{aligned} y_h(t_n + vh_n) &:= g(t_n + vh_n) + F_n(t_n + vh_n; \alpha) \\ &\quad + h_n \sum_{j=1}^{m} \left(\int_0^{v} H_\alpha(t_n + vh_n, t_n + sh_n) L_j(s) ds \right) Z_{n,j}. \end{aligned} \qquad (6.2.60)$$

As we have already observed in Section 2.3.3 (see the remark preceding Theorem 2.3.4), the principal merit of implicitly linear collocation is that it eliminates the necessity of re-computing the integrals in the (nonlinear) collocation equation: since the integrals in (6.2.59) do not depend on the unknown $\mathbf{Z}_n := (Z_{n,1}, \ldots, Z_{n,m})^T$, they need to be evaluated only once, before the beginning of the iteration process chosen for the solution of the nonlinear algebraic system (6.2.59).

It turns out that the approximation y_h obtained by implicitly linear collocation and the iterated collocation solution u_h^{it} generated by 'direct' collocation are essentially identical when $K(t,s) \equiv 1$. Thus, for judiciously chosen collocation parameters $\{c_i\}$ both approaches yield superconvergent approximations of the same global and local order, in particular on optimally graded meshes (cf. Theorem 6.2.13). This is made precise in

Theorem 6.2.15 *Assume:*

(a) $g \in C(I)$, $K(t,s) \equiv 1$, *and* G *is smooth and such that the Volterra–Hammerstein equation (6.2.52) has a unique solution* $y \in C(I)$ *for given* $\alpha \in (0,1]$.

(b) y_h *is the approximation to the solution* y *of the VHIE equation (6.2.52) with* $0 < \alpha \leq 1$, *obtained by implicitly linear collocation (6.2.56), (6.2.57).*
(c) $\hat{u}_h^{it}$ *is the discretised iterated collocation solution corresponding to the direct collocation solution* $\hat{u}_h \in S_{m-1}^{(-1)}(I_h)$ *defined by the fully discretised collocation equation (6.2.61), (6.2.62) below, using the same collocation points* X_h *as for the computation of* $z_h \in S_{m-1}^{(-1)}(I_h)$ *in (6.2.56).*

Then for any $\alpha \in (0, 1]$ *we have*

$$\hat{u}_h^{it}(t) = y_h(t) \quad \textit{for all} \quad t \in I.$$

Proof If we solve the given Volterra–Hammerstein integral equation (6.2.52) by 'direct' collocation in $S_{m-1}^{(-1)}(I_h)$, the the exact collocation equation reads

$$u_h(t) = g(t) + (\mathcal{H}_\alpha u_h)(t), \quad t \in I,$$

and the corresponding exact iterated collocation solution is found from

$$u_h^{it}(t) := g(t) + (\mathcal{H}_\alpha u_h)(t), \quad t \in I.$$

Recall that by our assumption on K we now have $H_\alpha(t, s) = p_\alpha(t - s)$. For $t = t_{n,i}$ $(i = 1, \ldots, m)$ and $t = t_n + vh_n$ $(v \in [0, 1])$ these equations become, respectively,

$$\begin{aligned} U_{n,i} &= g(t_{n,i}) + \Phi_n(t_{n,i}; \alpha) \\ &\quad + h_n \int_0^{c_i} p_\alpha((c_i - s)h_n) G(t_n + sh_n, \sum_{j=1}^m L_j(s) U_{n,j}) ds, \end{aligned}$$

and

$$\begin{aligned} u_h^{it}(t_n + vh_n) &= g(t_n + vh_n) + \Phi_n(t_n + vh_n; \alpha) \\ &\quad + h_n \int_0^{v} p_\alpha((v - s)h_n) G(t_n + vh_n, \sum_{j=1}^m L_j(s) U_{n,j}) ds. \end{aligned}$$

Here, we have set

$$\Phi_n(t_n + vh_n; \alpha) := \int_0^{t_n} p_\alpha(t_n + vh_n - s) G(s, u_h(s)) ds, \quad v \in [0, 1].$$

Consider now their fully discretised versions based on interpolatory m-point product quadrature formulas with weight function $H_\alpha(\cdot, s) = p_\alpha(\cdot - s)$ (because $K(t, s) = 1$) and abscissas given by the collocation points X_h: in analogy to Section 6.2.2 they are given by

$$\begin{aligned} \hat{U}_{n,i} &= g(t_{n,i}) + \hat{\Phi}_n(t_{n,i}; \alpha) \\ &\quad + h_n \sum_{\nu=1}^m \left(\int_0^{c_i} p_\alpha((c_i - s)h_n) L_\nu(s) ds \right) G(t_{n,\nu}, \hat{U}_{n,\nu}) \quad (6.2.61) \end{aligned}$$

and

$$\hat{u}_h^{it}(t_n + vh_n) = g(t_n + vh_n) + \hat{\Phi}_n(t_n + vh_n; \alpha) + h_n \sum_{\nu=1}^{m} \left(\int_0^v p_\alpha((v-s)h_n) L_\nu(s) ds \right) G(t_{n,\nu}, \hat{U}_{n,\nu}). \tag{6.2.62}$$

Let $\hat{V}_{n,i} := G(t_{n,i}, \hat{U}_{n,i})$. From (6.2.61) and (6.2.62) we thus obtain the equations

$$\hat{V}_{n,i} = G\left(t_{n,i}, g(t_{n,i}) + \hat{\Phi}_n(t_{n,i}; \alpha) + h_n \sum_{\nu=1}^{m} \left(\int_0^{c_i} p_\alpha(c_i - s)h_n) L_\nu(s) ds \right) \hat{V}_{n,\nu}\right) \tag{6.2.63}$$

and

$$\hat{u}_h^{it}(t_n + vh_n) = g(t_n + vh_n) + \hat{\Phi}_n(t_n + vh_n; \alpha) + h_n \sum_{\nu=1}^{m} \left(\int_0^v p_\alpha((v-s)h_n) L_\nu(s) ds \right) \hat{V}_{n,\nu}. \tag{6.2.64}$$

It remains to show that $\hat{V}_{n,i} = Z_{n,i},\ i = 1, \ldots, m\ (n = 0, 1, \ldots, N-1)$, where the stage values $Z_{n,i}$ are defined by the solution of the nonlinear algebraic system (6.2.59). This is readily verified by induction, using the obvious fact that the assertion is true for $n = 0$. We summarise the convergence result in Theorem 6.2.16 and leave the remaining details of its proof as an exercise.

Theorem 6.2.16 *Assume that the given functions describing the VHIE (6.2.52) satisfy* $g \in C^m(I),\ K(t,s) \equiv 1$, *with* $G \in C^m(I \times \Omega)\ (\Omega \subset \mathbb{R})$ *such that the integral possesses a unique solution* $y \in C^m(I)$. *If (6.2.52) is solved by implicitly linear collocation (6.2.56), (6.2.57), and if the underlying mesh* I_h *is the graded mesh (6.2.31) with grading exponent* $r = \mu/(1-\alpha)$, *then*

$$||y - y_h||_\infty \le C(r) \begin{cases} h^\mu & \text{if } 1 - \alpha \le \mu \le m, \\ h^m & \text{if } \mu \ge m, \end{cases}$$

for any set $\{c_i\}$, *and with* $h := T/N$.

It is clear that the superconvergence results of Theorem 6.2.13 carry over to $\hat{u}_h^{it}$, because of Theorems 6.2.14 and 6.2.15. We omit the detailed statement of these by now expected results.

6.3 Collocation for weakly singular first-kind VIEs

We know from the analysis in Section 2.4 that collocation solutions in $S_{m+d}^{(d)}(I_h)\ (d = -1,\ d = 0)$ to first-kind VIEs with regular kernels (i.e. $\alpha = 0$)

do not converge to the exact solution for any choice of the collocation parameters $\{c_i\}$. Hence, it is intuitively clear that the same will be true when the Volterra integral operator contains the weakly singular factor $p_\alpha(t-s)$ $(0<\alpha\le 1)$. It turns out, however, that to describe this quantitatively poses a formidable challenge, and so far only a few partial results are known.

We will focus on the case where $u_h \in S^{(-1)}_{nm-1}(I_h)$; in view of the results of Theorem 2.4.5 (Kauthen and Brunner (1997)) the convergence analysis in the continuous collocation space $S^{(0)}_m(I_h)$ will be even more intractable at present.

6.3.1 Collocation in $S^{(-1)}_{m-1}(I_h)$

The collocation solution $u_h \in S^{(-1)}_{m-1}(I_h)$ to the linear weakly singular first-kind VIE

$$(\mathcal{V}_\alpha y)(t) := \int_0^t p_\alpha(t-s)K(t,s)y(s)ds = g(t), \quad t\in I := [0,T] \\ (0<\alpha\le 1), \tag{6.3.1}$$

is defined by the collocation equation

$$(\mathcal{V}_\alpha u_h)(t) = g(t), \quad t\in X_h, \tag{6.3.2}$$

and by the local representation

$$u_h(t_n+sh_n) = \sum_{j=1}^m L_j(v)U_{n,j}, \quad v\in(0,1], \quad \text{with} \quad U_{n,j} := u_h(t_{n,j}). \tag{6.3.3}$$

The vector $\mathbf{U}_n := (U_{n,1},\ldots,U_{n,m})^T \in \mathbb{R}^m$ is the solution of the linear algebraic system

$$B_n(\alpha)\mathbf{U}_n = h_n^{-1}[\mathbf{g}_n - \mathbf{G}_n(\alpha)] \;\; (n=0,1,\ldots,N-1), \tag{6.3.4}$$

in complete analogy to Section 2.4.2. Here $\mathbf{G}_n(\alpha)$ is the vector whose components are the lag term values $F_n(t_{n,i};\alpha)$ and which can be written as

$$\mathbf{G}_n(\alpha) := \sum_{\ell=0}^{n-1} h_\ell B_n^{(\ell)}(\alpha)\mathbf{U}_\ell$$

(see also (6.2.13) in Section 6.2.1). The matrices $B_n(\alpha)$ and $B_n^{(\ell)}(\alpha)$ in $L(\mathbb{R}^m)$ are the ones we introduced in (6.2.12) and (6.2.11), namely,

$$B_n(\alpha) := \left(\begin{array}{c}\int_0^{c_i} p_\alpha((c_i-s)h_n)K(t_{n,i},t_n+sh_n)L_j(s)ds \\ (i,j=1,\ldots,m)\end{array}\right), \tag{6.3.5}$$

and

$$B_n^{(\ell)}(\alpha) := \begin{pmatrix} \int_0^1 p_\alpha(t_{n,i} - t_\ell - sh_\ell)K(t_{n,i}, t_\ell + sh_\ell)L_j(s)ds \\ (i, j = 1, \dots, m) \end{pmatrix} \quad (\ell < n). \tag{6.3.6}$$

Under the assumptions of Theorem 6.1.13 the matrix $B_n(\alpha)$ is non-singular for all sufficiently small values of h_n. To see this, recall that the assumption $K \in C^1(D)$ allows us to write, by Taylor's Theorem,

$$\begin{aligned} K(t_{n,i}, t_n + sh_n) = K(t_n, t_n) + h_n[&c_i K_t(t_n + \theta_1 c_i h_n, t_n + \theta_2 sh_n) \\ &+ sK_s(t_n + \theta_1 c_i h_n, t_n + \theta_2 sh_n)], \end{aligned}$$

where $\theta_k \in (0, 1)$ $(k = 1, 2)$. Hence, the element of the matrix $B_n(\alpha)$ corresponding to the index pair (i, j) can be expressed in the form

$$\int_0^{c_i} p_\alpha((c_i - s)h_n)[K(t_n, t_n) + \mathcal{O}(h_n)]L_j(s)ds \quad (i, j = 1, \dots, m),$$

and this reveals that for sufficiently small $h_n > 0$,

$$B_n(\alpha) = \begin{pmatrix} K(t_n, t_n)\int_0^{c_i} p_\alpha((c_i - s)h_n)L_j(s)ds \\ (i, j = 1, \dots, m) \end{pmatrix} + \mathcal{O}(h_n),$$

is invertible for $n = 0, 1, \dots, N-1$ and all $\alpha \in (0, 1]$, under the assumption that $|K(t, t)| \geq k_0 > 0,\ t \in I$.

Theorem 6.3.1 *Assume that g and K in the first-kind Volterra integral equation (6.3.1) satisfy*

$$g \in C^1(I),\ g(0) = 0; \quad K \in C^1(D),\ |K(t, t)| \geq k_0 > 0,\ t \in I.$$

Then for any $\alpha \in (0, 1]$ *there exists an* $\bar{h} = \bar{h}(\alpha) > 0$ *so that for all meshes* I_h *with diameter* $h \in (0, \bar{h})$ *each of the linear algebraic systems (6.3.4) possesses a unique solution* $\mathbf{U}_n \in \mathbb{R}^m$. *Hence, for such meshes the collocation equation (6.3.2) defines a unique collocation solution* $u_h \in S_{m-1}^{(0)}(I_h)$ *which on* σ_n *is given by (6.3.3).*

Example 6.3.1 $u_h \in S_0^{(-1)}(I_h),\ 0 < c_1 \leq 1$:
Since u_h is constant on each σ_n we again set $y_{n+1} := u_h(t_n + vh_n$ $(v \in (0, 1])$. The collocation equation follows immediately from Example 6.2.1 and now reads (writing $\theta := c_1$)

$$\left(\int_0^\theta H_\alpha(t_{n,1}, t_n + sh_n)ds\right) y_{n+1} = h_n^{-1}[g(t_{n,1}) - F_n(t_{n,1}; \alpha)],$$

$(n = 0, 1, \ldots, N-1)$, with $t_{n,1} = t_n + \theta h_n$ and with lag term given by

$$F_n(t_{n,1};\alpha) = \sum_{\ell=0}^{n-1} h_\ell \left(\int_0^1 H_\alpha(t_{n,1}, t_\ell + sh_\ell)ds \right) y_{\ell+1}.$$

We recall that $H_\alpha(t, t_\ell + sh_\ell) = p_\alpha(t - t_\ell - s)h_\ell)K(t, t_\ell + vh_\ell)$ when $t = t_n + vh_n$.

We shall see below (Section 6.3.4) that this collocation solution converges uniformly on I only if

$$\theta \geq \theta^*(\alpha) := \frac{1}{2}(\alpha(1-\alpha)\gamma_\alpha)^{1/(1-\alpha)} \ (0 < \alpha < 1).$$

On uniform meshes the order then cannot exceed $p = \alpha$, while on suitably graded meshes (with grading exponent $r \geq m/\alpha$) we observe $\mathcal{O}(h)$- convergence (having set $h := T/N$).

Example 6.3.2 $u_h \in S_1^{(-1)}(I_h),\ 0 < c_1 < c_2 \leq 1$:
We know from Example 6.2.2 that the local representation of the collocation solution is

$$u_h(t_n + vh_n) = \frac{1}{c_2 - c_1}[(c_2 - v)U_{n,1} + (v - c_1)U_{n,2}], \quad v \in (0, 1].$$

The vector $\mathbf{U}_n := (U_{n,1}, U_{n,2})^T \in \mathbb{R}^2$ is the solution of the linear system

$$B_n(\alpha)U_n = h_n^{-1}[\mathbf{g}_n - \mathbf{G}_n(\alpha)]$$

(recall (6.3.4)) with the elements of the matrix $B_n(\alpha) \in L(\mathbb{R}^2)$ as in Example 6.2.2. The matrix $B_n(\alpha)$ and the ones describing the lag term $\mathbf{G}_n(\alpha)$ are given, resepectively, by

$$(B_n(\alpha))_{i,1} = \frac{1}{c_2 - c_1} \int_0^{c_i} p_\alpha((c_i - s)h_n)(c_2 - s)K(t_{n,i}, t_n + sh_n)ds \ \ (i = 1, 2),$$

$$(B_n(\alpha))_{i,2} = \frac{1}{c_2 - c_1} \int_0^{c_i} p_\alpha((c_i - s)h_n)(s - c_1)K(t_{n,i}, t_n + sh_n)ds \ \ (i = 1, 2),$$

and

$$(B_n^{(\ell)}(\alpha))_{i,1} = \frac{1}{c_2 - c_1} \int_0^1 H_\alpha(t_{n,i}, t_\ell + sh_\ell)(c_2 - s)ds \quad (i = 1, 2),$$

$$(B_n^{(\ell)}(\alpha))_{i,2} = \frac{1}{c_2 - c_1} \int_0^1 H_\alpha(t_{n,i}, t_\ell + sh_\ell)(s - c_1)ds \quad (i = 1, 2).$$

The collocation solution is now determined by the solution $(U_{n,1}, U_{n,2})^T$ of the linear system (6.3.4), the local Lagrange representation (6.3.3) with $m = 2$.

As we shall see in Conjecture 6.3.5 at the end of Section 6.3.3, no necessary *and* sufficient condition on the two collocation parameters is known yet under which u_h converges uniformly on I to y.

6.3.2 Collocation in $S_m^{(0)}(I_h)$

We have seen in Section 2.4.3 that the imposition of continuity at the mesh points on the collocation solution u_h for a first-kind VIE leads to a more severe constraint on the collocation parameters $\{c_i\}$ for which u_h is convergent, even in the case of smooth exact solutions. Since in a weakly singular first-kind VIE (6.3.1) with smooth g (satisfying $g(0) = 0$) and K the exact solution has an unbounded derivative at $t = 0^+$ (Theorem 6.1.14), such a continuity requirement will likely make the conditions on the $\{c_i\}$ more stringent (and will certainly lead to very challenging arguments in the convergence analysis!). At the time of writing this problem remains essentially open (except for some special results in the case $m = 1$ and $c_1 = 1$; see, e.g. Weiss (1972b), Benson (1973), Eggermont (1981), and Capobianco (1990a, 1990b)).

The collocation equation determining $u_h \in S_m^{(0)}(I_h)$ is

$$(\mathcal{V}_\alpha u_h)(t) = g(t), \quad t \in X_h, \quad \text{with} \;\; u_h(0) = y(0), \tag{6.3.7}$$

where

$$y(0) = \lim_{t \to 0^+} \frac{(1-\alpha)t^{\alpha-1} g(t)}{K(0,0)}$$

must be known. As in Section 2.4.3, let the local representation of u_h on σ_n be given by

$$u_h(t_n + v h_n) = \sum_{j=0}^{m} L_j(v) U_{n,j}, \quad v \in [0,1], \quad \text{with} \;\; U_{n,j} := u_h(t_n + c_j h_n); \tag{6.3.8}$$

here, we have introduced $c_0 := 0$ and

$$L_0(v) := (-1)^m \prod_{k=1}^{m} (v - c_k)/c_k,$$

$$L_j(v) := (v/c_j) \prod_{k=0, k \neq j}^{m} (v - c_k)/(c_j - c_k) \; (j = 1, \ldots, m).$$

Hence, (6.3.8) can be written in the form

$$u_h(t_n + v h_n) = L_0(v) y_n + \sum_{j=1}^{m} L_j(v) U_{n,j}, \quad v \in [0,1], \tag{6.3.9}$$

and this implies that

$$y_n := u_h(t_n) = u_h(t_{n-1} + h_{n-1}) \quad (n = 1, \dots, N-1),$$

because the collocation solution u_h is continuous at the mesh points.

The collocation equation on σ_n now becomes

$$\int_0^{t_n} H_\alpha(t_{n,i}, s)u_h(s)ds + h_n \int_0^{c_i} H_\alpha(t_{n,i}, t_n + sh_n)u_h(t_n + sh_n)ds$$
$$= g(t_{n,i}) \quad (i = 1, \dots, m),$$

or, using (6.3.9),

$$\sum_{j=1}^{m} \left(\int_0^{c_i} H_\alpha(t_{n,i}, t_n + sh_n)L_j(s)ds \right) U_{n,j}$$
$$= h_n^{-1}[g(t_{n,i}) - F_n(t_{n,i}; \alpha)] - \int_0^{c_i} H_\alpha(t_{n,i}, t_n + sh_n)L_0(s)ds \cdot y_n, \tag{6.3.10}$$

with obvious meaning of the lag term $F_n(t_{n,i}; \alpha)$. Setting

$$\boldsymbol{\rho}_n(\alpha) := -\left(\int_0^{c_i} H_\alpha(t_{n,i}, t_n + sh_n)L_0(s)ds \quad (i = 1, \dots, m) \right)^T$$

we are led to a linear algebraic system for $\mathbf{U}_n := (U_{n,1}, \dots, U_{n,m})^T$ which resembles (6.3.4) but which contains on its right-hand side an additional term reflecting the continuity of u_h at the mesh points:

$$B_n(\alpha)\mathbf{U}_n = h_n^{-1}[\mathbf{g}_n - \mathbf{G}_n(\alpha)] + \boldsymbol{\rho}_n(\alpha)y_n \quad (n = 0, 1, \dots, N-1). \tag{6.3.11}$$

The matrix $B_n(\alpha)$ and the vectors $\mathbf{g}_n$ and $\mathbf{G}_n(\alpha)$ are as in (6.3.4)–(6.3.6).

Note that the existence of a unique collocation solution $u_h \in S_m^{(0)}(I_h)$ is assured by Theorem 6.3.1 because in the systems of linear algebraic equations (6.3.11) we have the same coefficient matrices $B_n(\alpha)$ as in (6.3.4).

Example 6.3.3 $u_h \in S_1^{(0)}(I_h),\ 0 < c_1 =: \theta \le 1$:
Here, we have

$$L_0(v) = (\theta - v)/\theta, \qquad L_1(v) = v/\theta,$$

and

$$B_n(\alpha) = \left(\frac{1}{\theta} \int_0^{\theta} p_\alpha((\theta - s)h_n)K(t_{n,1}, t_n + sh_n)sds \right).$$

The collocation solution is thus determined by

$$u_h(t_n + vh_n) = L_0(v)y_n + L_1(v)U_{n,1}, \quad v \in (0, 1],$$

by the solution of

$$B_n(\alpha)U_{n,1} = h_n^{-1}[g_{n,1} - F_n(t_{n,1};\alpha)] \\ -\frac{1}{\theta}\left(\int_0^\theta p_\alpha((\theta - s)h_n)K(t_{n,1}, t_n + sh_n)(\theta - s)ds\right) y_n$$

($n = 0, 1, \ldots, N-1$). The 'artificial' initial value y_0 must be prescribed, as indicated in (6.3.7).

The *continuous (exact) product trapezoidal method* is obtained by choosing $\theta = 1$. Its fully discretised counterpart will be presented in Example 6.3.6.

6.3.3 Convergence analysis; conjectures

For the weakly singular first-kind Volterra integral equation (6.3.1) with $0 < \alpha < 1$, no necessary and sufficient conditions under which the collocation solutions in the spaces $S_{m+d}^{(d)}(I_h)$ ($d = -1, 0$) are convergent are yet known (compare, however, Eggermont (1984, 1988b) for an analysis of related questions in Galerkin methods for (6.3.1) and for a comparison of Galerkin and collocation methods).

Before describing a partial answer for the collocation space $S_0^{(-1)}(I_h)$ of piecewise constant functions we first show that we can answer the question on the global order of convergence of $u_h \in S_{m-1}^{(-1)}(I_h)$, provided we know that the collocation parameters $\{c_i\}$ are such that $||y - u_h||_\infty \to 0$, as $h \to 0$, is true.

Theorem 6.3.2 *Let $0 < \alpha < 1$ and assume that K and g in (6.3.1) are, respectively, in $C^{m+1}(D)$ and $C^{m+1}(I)$, with $K(t,t) \neq 0$ for all $t \in I$. In addition suppose that*

$$g^{(j)}(0) = 0, \qquad j = 0, 1, \ldots, q, \tag{6.3.12}$$

for some q with $0 \leq q < m$. If the collocation parameters $\{c_i : 0 < c_1 < \cdots < c_m \leq 1\}$ are such that the corresponding collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ to (6.3.1) converges uniformly to y on I then we have, for all $h \in (0, \bar{h})$,

$$\|y - u_h\|_\infty \leq Ch^{\alpha+q} \tag{6.3.13}$$

for any $\alpha \in (0, 1)$, regardless of the choice of m, if the mesh I_h is uniform.

For graded *meshes given by*

$$t_n = \left(\frac{n}{N}\right)^r T \quad (n = 0, 1, \ldots, N), \quad \text{with } r = \frac{\mu}{\alpha + q}, \tag{6.3.14}$$

we obtain the estimates

$$\|y - u_h\|_\infty \leq C(r)\begin{cases} h^\mu & \text{if } \alpha + q \leq \mu \leq m, \\ h^m & \text{if } \mu \geq m, \end{cases} \tag{6.3.15}$$

with $h := T/N$.

Proof We will sketch the main steps but leave the details to the reader. First, we recall that the collocation error, $e_h := y - u_h$, satisfies

$$(\mathcal{V}_\alpha e_h)(t) = 0 \qquad \text{for } \ t \in X_h.$$

Since $0 < \alpha < 1$, this equation can be written more explicitly as

$$\begin{aligned} &h_n^{1-\alpha} \int_0^{c_i} (c_i - s)^{-\alpha} K_{n,i}(t_n + sh_n) e_h(t_n + sh_n) ds \\ &\quad = - \sum_{\ell=0}^{n-1} h_\ell^{1-\alpha} \int_0^1 \left(\frac{t_n + c_i h_n - t_\ell}{h_\ell} - s \right)^{-\alpha} K(t_{n,i}, t_\ell + sh_\ell) e_h(t_\ell + sh_\ell) ds \end{aligned} \tag{6.3.16}$$

($i = 1, \ldots, m;\ n = 0, 1, \ldots, N-1$). While $u_h\ |_{(t_n, t_{n+1}]}$ is a polynomial of degree $m-1$, the exact solution y of (6.3.1) has, according to Theorem 6.1.14, lower regularity at $t = 0$, namely $y \in C^{q,\alpha}(I)$. Thus, the collocation error may be expressed in the form

$$e_h(t_n + vh_n) = \begin{cases} \displaystyle\sum_{j=1}^{m} \beta_{0,j}(\alpha) v^{j-1} + h_0^{\alpha+q} \rho_0(v;\alpha) + h_0^m R_{m,0}(v;\alpha), & \text{if } \ n = 0 \\ \displaystyle\sum_{j=1}^{m} L_j(v) \mathcal{E}_{n,j} + h_n^m R_{m,n}(v), & \text{if } \ 1 \le n \le N-1, \end{cases} \tag{6.3.17}$$

in analogy to (6.2.39) and (6.2.37). Here, $\rho_0(\cdot;\alpha)$, $R_{m,0}(\cdot;\alpha)$, and $R_{m,n}$ are bounded functions analogous to those in Theorems 6.1.6 and 6.1.14 (compare also the results in Section 6.5.2) below).

If the above expressions (6.3.17) for e_h are substituted in the error equation (with $t = t_{n,i}$) we obtain, as in the proof of Theorem 6.2.9 but with $\alpha + q$ replacing $1 - \alpha$, first an estimate of the form $||\beta_0(\alpha)||_1 \le Bh_0^{\alpha+q}$ and so

$$|e_h(t_0 + vh_0)| = \mathcal{O}(h_0^{\alpha+q}), \quad v \in [0, 1].$$

The subsequent systems of difference equations (for $1 \le n \le N-1$) are uniquely solvable for $h \in (0, \bar{h})$, thanks to the crucial assumption on the collocation parameters $\{c_i\}$ (which is equivalent to the statement that the matrices $B_n(\alpha)$ all possess uniformly bounded inverses for $\alpha \in (0, 1)$). A discrete Gronwall argument leads to uniform $\mathcal{O}(h_0^{\alpha+q})$-bounds for the quantities $||\boldsymbol{\mathcal{E}}_n||_1$ and hence, by the local representation (6.3.17) for e_h, for $||e_h||_\infty$ itself.

Note that for the graded meshes I_h with grading exponent $r = \mu/(\alpha + q)$ we obtain

$$h_0 = t_1 = (N^{-1})^r T = N^{-\mu/(\alpha+q)} T,$$

and hence $h_0^{\alpha+q} = \mathcal{O}(N^{-\mu}) = \mathcal{O}(h^\mu)$ ($\alpha + q \le \mu \le m$) similar to (6.2.47).

The order result of Theorem 6.3.2 hinges on the *assumed uniform convergence* of the collocation solution u_h. We will now investigate this assumption for the simple collocation space $S_0^{(-1)}(I_h)$ and derive a sufficient condition for $c_1 \in (0, 1]$ under which u_h is convergent. Some related convergence results were obtained in the early 1970s: In Weiss and Anderssen (1972) it was shown that uniform convergence holds for the (discretised) collocation solution u_h when $c_1 = 1$. It is thus natural to ask if this is true for all $c_1 \geq 1/2$, as in the nonsingular case $\alpha = 0$. The following theorem answers this question in the affirmative. The result shows in particular that the collocation equation corresponding to $m = 1$ has uniformly bounded solutions as $h \to 0$ even for certain values of $c_1 < 1/2$, depending on the given value of α. This is not entirely surprising because the first-kind VIE (6.3.1) becomes 'less ill-conditioned' as α moves from 0^+ to 1^-.

Theorem 6.3.3 *Let K and g in (6.3.1) satisfy the conditions stated in Theorem 6.3.2 (with $m = 1$), and let $u_h \in S_0^{(-1)}(I_h)$ be the collocation solution corresponding to the collocation parameter $c_1 \in (0, 1]$ and uniform mesh I_h. If*

$$c_1 \geq c_1^*(\alpha) := \frac{1}{2}\left(\alpha(1-\alpha)\gamma_\alpha\right)^{1/(1-\alpha)}, \qquad (6.3.18)$$

where $\gamma_\alpha := \pi/\sin(\alpha\pi)$ $(= \Gamma(\alpha)\Gamma(1-\alpha))$, then u_h converges uniformly on I to the solution y of (6.3.1).

We present a sample of values for $c_1^*(\alpha)$ in the following table, as an illustration of (6.3.18).

While numerical experiments indicate that the lower bound (6.3.18) for c_1 appears to be also necessary for u_h to remain uniformly bounded for *all* $\alpha \in (0, 1)$, a result analogous to that for $\alpha = 0$ (cf. Theorem 2.4.2), giving a necessary and sufficient condition, is yet to be established.

Proof We will outline a few of the key steps in the proof of the above result; details are left to the reader. The setting is as follows. We have seen in Section 6.1.3 (Theorem 6.1.13) that, for $0 < \alpha < 1$, (6.3.1) is equivalent to the regular

Table 6.2. *A selection of values for $c_1^*(\alpha)$*

α	0	0.10	0.25	0.5	0.90	0.99	1^-
$c_1^*(\alpha)$	0.5	0.3919	0.4520	0.3084	0.2056	0.1861	0.1839 $(= e^{-1}/2)$

first-kind integral equation

$$\int_0^t H(t,s;\alpha)y(s)ds = G_\alpha(t), \quad t \in I, \tag{6.3.19}$$

where

$$H(t,s;\alpha) := \int_0^1 v^{-\alpha}(1-v)^{\alpha-1}K(s+(t-s)v,s)\,dv, \quad (t,s) \in D,$$

and

$$G_\alpha(t) := \int_0^t (t-s)^{\alpha-1}g(s)ds = \frac{1}{(\alpha)_{q+1})}\int_0^t (t-s)^{\alpha+q}g^{(q+1)}(s)ds, \quad t \in I.$$

Suppose now that the collocation solution for (6.3.1) is $u_h \in S_0^{(-1)}(I_h)$, with collocation at $t_n + c_1 h,\ 0 < c_1 \le 1$, and the solution of the (equivalent) first-kind Volterra integral equation (6.3.19) is approximated by z_h in the same space, but with collocation at the points $t_n + d_1 h$, for some $d_1 \in (0, 1]$. Since $H(\cdot,\cdot;\alpha)$ and G_α in (6.3.19) satisfy the assumptions of Theorem 2.4.2, we know that z_h converges uniformly on I to y if, and only if, $d_1 \ge 1/2$.

The collocation equation determining u_h is given in Example 6.3.1. For ease of exposition we will assume that $K(t,s) \equiv 1$, implying that $H(t,s;\alpha) = \gamma_\alpha$ on D; suppose also that $q = 0$. Thus, setting $u_h(t_n + vh_n) =: y_{n+1}$ $(v \in (0,1])$, we find

$$\begin{aligned} y_{n+1} &= \frac{1-\alpha}{h^{1-\alpha}c_1^{1-\alpha}}\, g(t_n + c_1 h) \\ &\quad -\frac{1}{c_1^{1-\alpha}}\sum_{\ell=0}^{n-1}\left((n-\ell+c_1)^{1-\alpha} - (n-\ell+c_1-1)^{1-\alpha}\right) y_{\ell+1} \end{aligned} \tag{6.3.20}$$

$(n = 0, 1, \ldots, N-1)$. For (6.3.19) we obtain

$$\begin{aligned} z_{n+1} &+ \frac{1}{d_1}\sum_{\ell=0}^{n-1} z_{\ell+1} \\ &= \frac{1}{\gamma_\alpha h^{1-\alpha}d_1}\int_0^{d_1}(d_1 - s)^{\alpha-1}\, g(t_n + sh)\, ds \\ &\quad + \frac{1}{\gamma_\alpha h^{1-\alpha}d_1}\sum_{\ell=0}^{n-1}\int_0^1 (n-\ell+d_1-s)^{\alpha-1}g(t_\ell + sh)ds \end{aligned} \tag{6.3.21}$$

with $z_{n+1} := z_h(t_n + vh_n)$ $(v \in (0,1])$. If we approximate the integrals in (6.3.21) by one-point interpolatory product quadrature with abscissas

$\{t_\ell + \xi_1 h \ (\xi_1 \in (0, d_1])\}$, we are led to the Volterra difference equation

$$\begin{aligned} z_{n+1} &+ \frac{1}{d_1}\sum_{\ell=0}^{n-1} z_{\ell+1} \\ &= \frac{1}{\alpha\gamma_\alpha h^{1-\alpha} d_1^{1-\alpha}}\, g(t_n + \xi_1 h) \\ &\quad + \frac{1}{\alpha\gamma_\alpha h^{1-\alpha} d_1}\sum_{\ell=0}^{n-1}\left((n-\ell+d_1)^\alpha - (n-\ell+d_1-1)^\alpha\right) g(t_\ell + \xi_1 h). \end{aligned} \tag{6.3.22}$$

For $n = 0$ the equations (6.3.20) and (6.3.22) respectively reduce to

$$u_1 = u_h(t_1) = \frac{1-\alpha}{h^{1-\alpha} c_1^{1-\alpha}}\, g(t_0 + c_1 h)$$

and

$$v_1 = v_h(t_1) = \frac{1}{\alpha\gamma_\alpha h^{1-\alpha} d_1^{1-\alpha}}\, g(t_0 + \xi_1 h).$$

The following statement is now obvious.

Lemma 6.3.4 *We have $u_h(t_1) = z_h(t_1)$ for any $g \in C^1(I)$ with $g(0) = 0$ if, and only if, $\xi_1 = c_1$ and*

$$c_1 = \phi(\alpha) \cdot d_1,$$

where $\phi(\alpha) := (\alpha(1-\alpha)\gamma_\alpha)^{1/(1-\alpha)}$ and $\gamma_\alpha := \pi/\sin(\alpha\pi)$.

Note that the function ϕ is strictly decreasing on $(0, 1)$, with $\phi(0) = 1$ and $\phi(1^-) = \mathrm{e}^{-1}$.

For $n \geq 1$, $|z_{n+1}|$ $(n = 0, 1, \ldots, N-1)$ is uniformly bounded as $h \to 0$ and $Nh = T$ if, and only if, $d_1 \geq 1/2$ (Theorem 2.4.2)). In order to complete the proof of Theorem 6.3.3 we now have to show that, using (6.3.20) and (6.3.22), $|z_{n+1} - y_{n+1}|$ $(n = 0, 1, \ldots, N-1)$ remains uniformly bounded as $N \to \infty,\ t_N = T$, whenever $c_1 \in (0, 1]$ is chosen so that

$$c_1 \geq \phi(\alpha)/2 = c_1^*(\alpha) > (1/2)\mathrm{e}^{-1} \doteq 0.1839 \quad (0 < \alpha < 1).$$

The details are left to the reader.

As we observed at the beginning of the present section, an analogue of Theorem 6.3.3 for collocation in $S_{m-1}^{(-1)}(I_h)$ with $m \geq 2$ is not yet known. There appears to be a close connection between the answer to this open problem and the *asymptotic behaviour* of the collocation solution $v_h \in S_{m-1}^{(-1)}(I_h)$, with the same collocation parameters $\{c_i\}$, for the *second kind* VIE

$$y(t) = 1 + \int_0^t \lambda(t-s)^{-\alpha} y(s)ds, \quad 0 < \alpha < 1, \quad \lambda < 0. \tag{6.3.23}$$

We will say that the solution y of (6.3.23) is A_α-stable if, for $t_n := nh$ ($n = 0, 1, \ldots$), with fixed $h > 0$,

$$\lim_{n\to\infty} u_h(t_n) = 0, \quad \text{for all} \quad \lambda < 0.$$

Conjecture 6.3.5 *Assume that the collocation parameters* $\{c_i\}$ *satisfy* $0 < c_1 < \cdots < c_m \le 1$ *and let the mesh* I_h *be uniform. The collocation solution* $u_h \in S_{m-1}^{(-1)}(I_h)$ *for the weakly singular first-kind VIE (6.3.1) with* $0 < \alpha < 1$ *converges uniformly to the exact solution* y *as* $h \to 0$ *if, and only if, the collocation solution* $v_h \in S_{m-1}^{(-1)}(I_h)$*, using the same set* $\{c_i\}$*, for the weakly singular second-kind equation (6.3.23) is* A_α*-stable.*

However, it is at present not known for which $\{c_i\}$ the collocation solution v_h for (6.3.23) has the property of being A_α-stable. A partial answer (sufficient condition) was given in Brunner, Crisci, Russo and Vecchio (1991) for the case $m = 1$.

The final remark in this section concerns collocation for the *singularly perturbed* VIE

$$\varepsilon y(t) = g(t) + (\mathcal{V}_\alpha y)(t), \quad t \in I, \quad 0 < \alpha < 1,$$

with $0 < \varepsilon \ll 1$. We have seen that for $\varepsilon = 1$ the collocation $u_h \in S_{m-1}^{(-1)}(I_h)$ exhibits optimal order of convergence $p = m$ if the mesh I_h is graded and $r =: r_1 = m/(1-\alpha)$. If $\varepsilon = 0$ and the collocation parameters $\{c_i\}$ are such that the collocation solution in this space converges uniformly to y, then it attains the same optimal order p only if the grading exponent is $r =: r_0 = m/\alpha$. While we have $r_1 = r_0$ when $\alpha = 1/2$, numerical experiments show clearly that for $\alpha \ne 1/2$ the collocation solution corresponding to mesh grading with r_1 loses its optimal order as $\varepsilon \to 0^+$. Thus, it will be important to understand the dependence of the order, and hence that of the optimal grading exponent, on ε, as $\varepsilon \to 0$.

An excellent survey of singularly perturbed Volterra equations and the state of the art in their numerical analysis can be found in Kauthen (1997a).

6.3.4 Fully discretised collocation

The secondary discretisation step in the (exact) collocation equation (6.3.2) will, as in the case of second-kind VIEs with weakly singular kernels, be based on interpolatory m-point quadrature formulas whose abscissas are given by the collocation parameters. Since the generalities have been dicsussed in detail in Section 6.2.2, we will not repeat them here. Instead we present the fully discretised collocation equations for two important special cases: these product integration methods were analysed in the early papers Weiss and Anderssen (1972), Weiss (1972b), Benson (1973), and Eggermont (1981).

Example 6.3.4 $\hat{u}_h \in S_0^{(-1)}(I_h),\ 0 < c_1 =: \theta \le 1$
Setting $\hat{y}_{n+1} := \hat{u}_h(t_n + vh_n) = \hat{U}_{n,1}$ $(v \in (0,1])$ and $t_{n,1} := t_n + \theta h_n$, Example 6.2.1 yields

$$w_{n,1}(\theta;\alpha)K(t_{n,1}, t_n + \theta^2 h_n))\hat{y}_{n+1} = h_n^{-1}[g(t_{n,1}) - \hat{F}_n(t_{n,1};\alpha)] \qquad (6.3.24)$$

$(n = 0, 1, \ldots, N-1)$, with

$$w_{n,1}(v;\alpha) := \int_0^v p_\alpha((v-s)h_n)ds = \begin{cases} h_n^{-\alpha}v^{1-\alpha}/(1-\alpha) & \text{if } 0 < \alpha < 1, \\ v[\log(vh_n) - 1] & \text{if } \alpha = 1. \end{cases}$$

The discretised lag term is

$$\hat{F}_n(t_{n,1};\alpha) = \sum_{\ell=0}^{n-1} w_{n,1}^{(\ell)}(\theta;\alpha)K(t_{n,1}, t_\ell + \theta h_\ell)\hat{y}_{\ell+1},$$

where the weights are given by

$$w_{n,1}^{(\ell)}(v;\alpha) := \int_0^1 p_\alpha(t_n + vh_n - t_\ell - sh_\ell)ds \quad (\ell < n).$$

The order of convergence of the *discretised midpoint method* $(\theta = 1)$ was analysed by Weiss and Anderssen (1972).

Example 6.3.5 $\hat{u}_h \in S_1^{(-1)}(I_h),\ 0 < c_1 < c_2 \le 1$:
Using the interpolatory two-point quadrature formulas for the integrals in Example 6.3.2 we find that the matrices describing the fully discretised collocation equation have the elements (cf. Example 6.2.4)

$$\begin{aligned}(\hat{B}_n(\alpha))_{i,1} &= w_{n,1}(c_i;\alpha)K(t_{n,i}, t_n + c_ic_1h_n)L_1(c_ic_1) \\ &\quad + w_{n,2}(c_i;\alpha)K)t_{n,i}, t_n + c_ic_2h_n)L_1(c_ic_2), \\ (\hat{B}_n(\alpha))_{i,2} &= w_{n,1}(c_i;\alpha)K(t_{n,i}, t_n + c_ic_1h_n)L_2(c_ic_1) \\ &\quad + w_{n,2}(c_i;\alpha)K(t_{n,i}, t_n + c_ic_2h_n)L_2(c_ic_2),\end{aligned}$$

and, for $\ell < n$,

$$\begin{aligned}(\hat{B}_n^{(\ell)}(\alpha))_{i,1} &= w_{n,1}^{(\ell)}(c_i;\alpha)K(t_{n,i}, t_\ell + c_1h_\ell), \\ (\hat{B}_n^{(\ell)}(\alpha))_{i,2} &= w_{n,2}^{(\ell)}(c_i;\alpha)K(t_{n,i}, t_\ell + c_2h_\ell)\end{aligned}$$

$(i = 1, 2)$, with quadrature weights

$$w_{n,1}(c_i;\alpha) := \frac{1}{c_2 - c_1}\int_0^{c_i} p_\alpha((c_i - s)h_n)(c_2 - s/c_i)ds,$$

$$w_{n,2}(c_i,\alpha) := \frac{1}{c_2 - c_1}\int_0^{c_i} p_\alpha((c_i - s)h_n)(s/c_i - c_1)ds$$

and

$$w_{n,1}^{(\ell)}(c_i;\alpha := \frac{1}{c_2 - c_1}\int_0^{1} p_\alpha(t_{n,i} - t_\ell - sh_\ell)(c_2 - s)ds,$$

$$w_{n,2}^{(\ell)}(c_i;\alpha) := \frac{1}{c_2 - c_1}\int_0^{1} p_\alpha(t_{n,i} - t_\ell - sh_\ell)(s - c_1)ds.$$

Example 6.3.6 $\hat{u}_h \in S_1^{(0)}(I_h),\ 0 < c_1 =: \theta \le 1$:
Upon recalling that $L_0(v) := (\theta - v)/\theta,\ L_1(v) := v/\theta$, the quadrature weights of the interpolatory two-point product quadrature formulas for approximating the integrals in the exact collocation (Example 6.3.3) are

$$w_{n,1}(\theta;\alpha) = \frac{1}{\theta}\int_0^{\theta} p_\alpha((\theta - s)h_n)(\theta - s)ds,$$

$$w_{n,2}(\theta;\alpha) = \frac{1}{\theta}\int_0^{\theta} p_\alpha((\theta - s)h_n)sds,$$

and, for $\ell < n$,

$$w_{n,1}^{(\ell)}(\theta;\alpha) = \frac{1}{\theta}\int_0^{1} p_\alpha(t_{n,1} - t_\ell - sh_\ell)(\theta - s)ds,$$

$$w_{n,2}^{(\ell)}(\theta;\alpha) = \frac{1}{\theta}\int_0^{1} p_\alpha(t_{n,1} - t_\ell - sh_\ell)sds.$$

The *discretised (continuous) product trapezoidal method* corresponds to the choice $\theta = 1$: it defines the collocation solution $\hat{u}_h$ on $\bar{\sigma}_n$ by

$$\hat{u}_h(t_n + vh_n) = (1 - v)\hat{y}_n + v\hat{y}_{n+1}, \quad v \in [0, 1],$$

with $\hat{y}_{n+1}$ given by the solution of

$$w_{n,2}(\theta;\alpha)K(t_{n+1}, t_{n+1})\hat{y}_{n+1} = h_n^{-1}[g(t_{n+1}) - \hat{F}_n(t_{n+1};\alpha)] - w_{n,1}(\theta;\alpha)K(t_{n+1}, t_n)\hat{y}_n.$$

The discretised lag term has the form

$$\hat{F}_n(t_{n+1};\alpha) = \sum_{\ell=0}^{n-1}[w_{n,1}^{(\ell)}(\theta;\alpha)K(t_{n+1}, t_\ell)\hat{y}_\ell + w_{n,2}^{(\ell)}(\theta;\alpha)K(t_{n+1}, t_{\ell+1})\hat{y}_{\ell+1}.$$

The *discretised product trapezoidal method* was studied by Weiss (1972a, 1972b); see also Benson (1973) and, especially, Eggermont (1981).

Remark Convergence results for the product midpoint and product trapezoidal methods were proved in Weiss and Anderssen (1972) and Weiss (1972b) (and by Eggermont (1981)) under the assumption that the underlying exact solution of (6.3.1) has *Lipschitz-continuous derivatives*, namely $y \in C^{1,1}(I)$ and $y \in C^{2,1}(I)$, respectively. The resulting orders of convergence on *uniform meshes* are then given by $\mathcal{O}(h)$ and $\mathcal{O}(h^2)$. As we have seen in Theorem 6.1.14, this regularity is present only under special assumptions on $g^{(j)}(0)$ $(j \geq 1)$. In general, smooth K and g lead to $y \in C^{0,\alpha}(I)$ only. Hence, in order to attain $\mathcal{O}(h)$- and $\mathcal{O}(h^2)$-convergence the mesh I_h must be *graded*, with grading exponents $r = m/\alpha$ $(m = 1,\ m = 2)$, as shown in Theorem 6.3.2. On *uniform* meshes we have only $\mathcal{O}(h^\alpha)$-convergence, both for $m = 1$ and $m = 2$.

6.4 Non-polynomial spline collocation methods

6.4.1 Weakly singular VIEs of the second kind

In Section 6.2.5 (proof of Theorem 6.2.9) we observed that the reason behind the low ($\mathcal{O}(h^{1-\alpha})$-) convergence of the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ for *uniform meshes* lies in the fact that on the first subinterval σ_0 the polynomial $u_h \in \pi_{m-1}$ cannot match the fractional-power terms of the exact solution y (cf. Theorem 6.1.6). This suggests that, on uniform I_h, it may be more natural to seek the collocation solution in special *non-polynomial spline spaces* whose elements reflect the expansion (6.1.38) of y. In other words, if we choose a collocation space $Z_{m-1}^{(-1)}(I_h)$ with the property that $z_h \in Z_{m-1}^{(-1)}(I_h)$ on $\bar{\sigma}_0 = [t_0, t_1]$ $(t_0 = 0)$ reduces to

$$z_h(t_0 + vh) = \sum_{(j,k)_{1-\alpha}} b_{j,k} v^{j+k(1-\alpha)}, \quad v \in [0, 1], \tag{6.4.1}$$

where $(j, k)_{1-\alpha} := \{(j, k) : j + k(1 - \alpha) < m,\ j, k \in \mathbb{N}_0\}$ and $b_{j,k} = b_{j,k}(h)$, then this local representation exactly matches the terms in the first expression on the right-hand side of (6.1.38). The error analysis for the corresponding collocation solution z_h can then be carried out along familiar lines (using a standard Gronwall argument, as in the proof of Theorem 6.2.9) and it reveals that $\|y - z_h\|_\infty = \mathcal{O}(h^m)$ (see also Brunner and van der Houwen (1986, Section 6.2.5)).

The dimension of such a non-polynomial spline space depends on α and is of course much larger than that of $S_{m-1}^{(-1)}$. A detailed analysis of this dimension is given in Brunner (1983). In the practically important case where $\alpha = 1/2$ the number of basis functions used to represent z_h on σ_0 is $2m$. If $\alpha \to 1^-$, the number of basis functions tends to infinity. This observation is closely connected with a result of Lubich (1983a) which shows that the number of order conditions required for a Volterra–Runge–Kutta method for a weakly singular VIE to have order m tends to infinity as $\alpha \to 1^-$.

Variants of such non-polynomial collocation methods were studied by a number of authors. The paper by te Riele (1982) discusses hybrid methods (for $\alpha = 1/2$) that combine the above non-polynomial spline collocation technique (on a feasible small number of subintervals $\sigma_0, \dots, \sigma_{n_0}$) with subsequent piecewise polynomial collocation. The underlying mesh is uniform. See also Cao, Herdman and Xu (2003).

Another approach was investigated by Hu (1997a): here, the collocation solution is a so-called piecewise β-polynomial (employing integer powers of t^β, with suitable β, as basis functions), and I_h is a specially chosen *geometric mesh*. Compare also Hu (1998c) for postprocessing techniques based on this special non-polynomial spline approximation, and Hu and Luo (1997) for its use in Volterra–Hammerstein equations with weakly singular kernels.

6.5 Weakly singular Volterra functional equations with non-vanishing delays

6.5.1 Collocation for delay equations of the second kind

In Section 6.1.6 we introduced the delay Volterra operators $\mathcal{V}_{\theta,\alpha}$ and $\mathcal{W}_{\theta,\alpha}$: recall that they are respectively defined by

$$(\mathcal{V}_{\theta,\alpha} y)(t) := \int_0^{\theta(t)} p_\alpha(t-s) K_2(t,s) y(s)\,ds, \quad t \in I := [t_0, T]\ (t_0 \geq 0),$$

and

$$(\mathcal{W}_{\theta,\alpha} y)(t) := \int_{\theta(t)}^{t} p_\alpha(t-s) K(t,s)\,ds, \quad t \in I.$$

The lag function θ is assumed to be subject to the conditions (D1)–(D3) of Section 6.1.7, and the kernels K_2 and K are smooth on their domains D_θ and $\bar{D}_\theta$. Suppose we approximate the solutions of the corresponding VIEs

$$y(t) = g(t) + (\mathcal{V}_\alpha y)(t) + (\mathcal{V}_{\theta,\alpha} y)(t), \quad t \in (t_0, T],$$

and

$$y(t) = g(t) + (\mathcal{W}_{\theta,\alpha} y)(t), \quad t \in (t_0, T],$$

with $y(t) := \phi(t)$ if $t \leq 0$, by the collocation solutions $u_h \in S_{m-1}^{(-1)}(I_h)$ and the associated iterated collocation solutions u_h^{it}. What is the combined effect of the non-vanishing delay and the weakly singular kernel on the order of (super-) convergence of the collocation and iterated collocation solutions? Since optimal convergence orders are attained only if the mesh I_h is suitable graded (cf. Theorems 6.2.9 and 6.2.13), the answer to this question depends on the regularity of the solutions y at the primary discontinuity points $\{\xi_\mu\}$ (Table 6.1) and – as we shall see – on whether the lag function θ is linear or nonlinear. The following lemma provides the key to these results.

Lemma 6.5.1 *Assume that the mesh $I_h := \cup_{\mu=0}^{M} I_h^{(\mu)}$ is θ-invariant (Definition 4.2.1), and let the first local mesh $I_h^{(0)}$ be optimally graded:*

$$t_n^{(0)} := \xi_0 + \left(\frac{n}{N}\right)^{r_0} (\xi_1 - \xi_0) \ \ (n = 0, 1, \ldots, N;\ \xi_0 = t_0), \quad \textit{with} \ \ r_0 = \frac{m}{1-\alpha}.$$

(i) If the lag function θ is linear, then the other local meshes $I_h^{(\mu)}$ are also optimally graded, with grading exponents $r_\mu = r_0$ $(\mu = 1, \ldots, M)$.
(ii) If θ is nonlinear, then the grading is lost for $I_h^{(\mu)}$ $(\mu = 1, \ldots, M)$.

Proof The assertion in (i) is a direct consequence of the definitions of θ-invariance and the primary discontinuity points $\{\xi_\mu\}$, and the linearity of $\theta(t) = t - \tau(t)$. These reveal that the grading exponent r_0 remains invariant under θ. The validity of (ii) can be seen graphically, by constructing an example.

Table 6.1 shows that, depending on the type of the delay VIE, the solution at ξ_μ^+ is in $C^{\mu,1-\alpha}$ (smoothing), or in $C^{1-\alpha}$ for all $\mu = 0, 1, \ldots, M$ (no smoothing). Hence, Lemma 6.5.1 tells us that for linear lag functions the proofs of Theorems 6.2.9 and 6.2.13 carry over to second-kind VIEs with non-vanishing delays and weakly singular kernels, leading to the first assertion in the following theorem.

Theorem 6.5.2 *Assume:*

(a) The given functions $g,\ b,\ K_1,\ K_2,\ \theta,\ \phi$ are $d \geq m$-times continuously differentiable on their respective domains, with the lag function θ satisfying the conditions (D1)–(D3) and $\alpha \in (0, 1)$.
(b) $u_h \in S_{m-1}^{(-1)}(I_h)$, with θ-invariant mesh I_h.
(c) The first local mesh $I_h^{(0)}$ is optimally graded:

$$t_n^{(0)} := t_0 + \left(\frac{n}{N}\right)^{r_0} (\xi_0 - t_0) \ \ (n = 0, 1, \ldots, N), \quad \textit{with} \ \ r_0 = \frac{m}{1-\alpha}.$$

(I) *If θ is* linear, *the results of Theorems 6.2.9 and 6.2.13 remain valid on each subinterval $I^{(\mu)} := [\xi_\mu, \xi_{\mu+1}]$:*

$$\|y - u_h\|_{\mu,\infty} := \sup_{t \in I^{(\mu)}} |y(t) - u_h(t)| \le C(\alpha) N^{-m} \quad (0 \le \mu \le M),$$

and, for $d = m + 1$ and $\{c_i\}$ with $J_0 = 0$,

$$\|y - u_h^{it}\|_{\mu,\infty} \le C(\alpha) N^{-(m+1-\alpha)} \quad (0 \le \mu \le M).$$

(II) *If θ is* nonlinear, *the results of Theorems 6.2.9 and 6.2.13 are in general valid only on $I^{(0)}$. On the subsequent subintervals $I^{(\mu)}$ $(\mu \ge 1)$ the attainable orders of $\|y - u_h\|_{\mu,\infty}$ and $\|y - u_h^{it}\|_{\mu,\infty}$ will be less than m and lie between $1 - \alpha$ and m, except when $m = 1$ and we have smoothing in the exact solution.*

Remarks

1. For *linear* lag functions the (super-) convergence results of Brunner, Pedas and Vainikko (1999), in particular those involving logarithmic kernel singularities, remain true for second-kind delay VIEs with weakly singular, or bounded but non-smooth, kernels.
2. If θ is *nonlinear*, we can of course still obtain global convergence order $p = m$, by resorting to local meshes $I_h^{(\mu)}$ that are *individually graded*:
   - If the solution y has $C^{\mu,1-\alpha}$-regularity at $t = \xi_\mu^+$ (cf. Table 6.1) then – according to Theorem 6.2.9 – the optimal grading exponent for $I_h^{(\mu)}$ is

$$r_\mu = \begin{cases} \dfrac{m}{\mu + 1 - \alpha} & \text{for } \mu = 0, 1, \ldots, \min\{m, M\}, \\[2ex] 1 & \text{for } \mu = m + 1, \ldots, M. \end{cases}$$

   - If y has only $C^{1-\alpha}$-regularity at each ξ_μ (no smoothing), then we choose $r_\mu = m/(1 - \alpha$ for all $\mu = 0, 1, \ldots, M$.

   The corresponding global mesh I_h is of course no longer θ-invariant.

6.5.2 Collocation for weakly singular delay VIEs of the first kind

Since we do not yet understand for which sets of collocation parameters $\{c_i\}$ collocation solutions to first-kind VIEs with weakly singular kernels converge uniformly on I, the reader will not be surprised to read that the convergence analysis of collocation solutions $u_h \in S_{m-1}^{(-1)}(I_h)$ for the functional equation

$$(\mathcal{W}_{\theta,\alpha} y)(t) = g(t), \quad t \in (t_0, T] \quad (g(0) = 0), \tag{6.5.1}$$

is a completely open problem. The same is true for a closely related Volterra functional integro-differential equation, namely

$$\frac{d}{dt}[(\mathcal{W}_{\theta,\alpha}y)(t)] = f(t), \tag{6.5.2}$$

even when $\theta(t) = t - \tau$ ($\tau > 0$) (compare also the comments at the end of the paper by Ito and Turi (1991)). A new semigroup framework (different from the one in the paper just mentioned) for (6.5.2) has recently been established in Clément, Desch and Homan (2003): since, as shown in Ito and Turi (1991), the delay equation (6.5.1) can be recast in the form (6.5.2), this framework may yield the basis for the analysis of collocation solutions to (6.5.1).

6.6 Exercises and research problems

Exercise 6.6.1 Prove the resolvent equations for $Q(t, s; \alpha)$ in Theorem 6.1.2, and find $Q(t, s; \alpha)$ for the VIE (6.1.5).

Exercise 6.6.2 Prove Theorem 6.1.7: show that the VIE

$$y(t) = g(t) + (\mathcal{V}_1 y)(t), \quad t \in I,$$

corresponding to $p_1(t - s) = \log(t - s)$, has a unique solution $y \in C(I)$. Describe the regularity of y on $I = [0, T]$ when $g \in C^m(I)$ and $K \in C^m(D)$. What can be said about the structure of the resolvent kernel: does it inherit the factor $\log(t - s)$, in analogy to the case $0 < \alpha < 1$?

Exercise 6.6.3 Prove Theorem 6.1.10. Use the result to prove the statement in Corollary 6.1.4.

Exercise 6.6.4 Consider the second-kind VIE

$$y(t) = g(t) + \lambda \int_0^t (t - s)^{-\alpha} y(s)ds, \quad t \in I := [0, T] \quad (0 < \alpha < 1),$$

with $g \in C^m(I)$ and $\lambda \neq 0$. If $g^{(\nu)}(0) = 0$ for $\nu = 0, \ldots, q$ ($q < m$), does this affect the regularity of the solution y?

What happens if $(t - s)^{-\alpha}$ is replaced by $\log(t - s)$?

Exercise 6.6.5 Extend the result and the proof of Theorem 2.1.10 to the nonlinear VIE (6.1.22) in Section 6.1.3. Formulate this result for the Volterra–Hammerstein equation (6.1.23).

Exercise 6.6.6 Discuss the regularity of the solution of the VIE

$$y(t) = t^{\beta} + \lambda \int_0^t p_\alpha(t - s)y(s)ds, \quad t \in I,$$

when $\beta > 0$ ($\beta \notin \mathbb{N}$) and $0 \le \alpha \le 1$. Solve the problem when α is replaced by $\nu := \rho - \alpha$ ($\rho \in \mathbb{N},\ 0 < \alpha < 1$).

Exercise 6.6.7 Extend the result of Theorem 6.1.6 to the VIE

$$y(t) = g(t) + (\mathcal{V}_\nu y)(t), \quad \nu := \rho - \alpha\ (\rho \in \mathbb{N},\ 0 < \alpha < 1).$$

(Recall Theorem 6.1.8.)

Exercise 6.6.8 Derive the analogue of the regularity result in Theorem 6.1.6 for the first-kind VIE (6.1.26) (using as starting point Theorem 6.1.14).

Exercise 6.6.9 Prove the analogue of Theorem 6.1.17 for the kernel $p_\alpha(t - s) = \log(t - s)$ ($\alpha = 1$).

Exercise 6.6.10 Extend the Comparison Theorem 2.1.16 to linear second-kind VIEs with weakly singular kernels. What can be said if the weak singularities corresponding to $i = 1, 2$ are different, i.e. given by

$$p_{\alpha_i} := (t - s)^{-\alpha_i}, \quad 0 < \alpha_1 < \alpha_2 \le 1\ ?$$

Exercise 6.6.11 (Section 6.1.4) Show that the solution of the 'regular' V1 (6.1.31) solves the original VIE (6.1.30, $\mathcal{V}_\alpha y = g$.

Exercise 6.6.12 Assume that $h \in C^1(I)$ is strictly increasing on I. Derive the inversion formula giving the solution of

$$(\mathcal{A}_\alpha y)(t) := \int_0^t (h(t) - h(s))^{-\alpha} y(s)ds = g(t), \;\; t \in I \;\; (0 < \alpha < 1,\ g(0) = 0).$$

In particualr, let $h(t) = t^p,\ p > 1$.

Also:

$$(\mathcal{A}_\alpha^* y)(t) := \int_t^T (h(s) - h(t))^{-\alpha} y(s)ds = g(t), \quad t \in I:$$

Derive the inversion formula describing the solution. When is $y \in C(I)$?

Exercise 6.6.13 Establish the analogue of the regularity result in Theorem 6.1.14 for the nonlinear (Hammerstein) VIE of the first kind (6.1.34).

Exercise 6.6.14 Consider the VIE

$$y(t) = t^\beta g_0(t) + (\mathcal{V}y)(t), \quad t \in I := [0, T] \;\; (g_0(0) \ne 0),$$

with $\beta > 0,\ \beta \notin \mathbb{N}$; here, $\mathcal{V}$ denotes the classical Volterra integral operator with smooth kernel $K(t, s)$.

(a) Discuss the regularity of the solution y, and derive the analogue of the representation (6.1.17) (Theorem 6.1.6).

(b) Give a complete convergence analysis for $u_h \in S_{m-1}^{(-1)}(I_h)$ and the associated u_h^{it}. In particular, determine the optimal mesh grading.

Exercise 6.6.15 Solve the Volterra–Hammerstein integral equation

$$y(t) = g(t) + \lambda \int_0^t p_\alpha(t-s)G(y(s))ds, \quad t \in I := [0,T] \quad (0 < \alpha \le 1),$$

with $g(y) = \exp(-y)$ and $\lambda < 0$, by direct collocation in $S_{m-1}^{(-1)}(I_h)$ (followed by u_h^{it}), and by implicitly linear collocation. Compare the numerical results corresponding to uniform and optimally graded meshes, by selecting suitable 'test solutions' (determining the non-homogeneous term g).

Exercise 6.6.16 Nonlinear VIEs of the form

$$(y(t))^\beta = g(t) + (\mathcal{V}_\alpha y)(t), \;\; \beta > 1$$

(cf. Buckwar (1997, 2000) for the underlying existence and uniqueness theory), can be solved by a simple variant of *implicitly linear collocation*. Provide the computational details, and carry out the convergence analysis. In particular: is (global or local) superconvergence possible (when $\alpha = 0$, and when $0 < \alpha < 1$)?

Exercise 6.6.17 (Research problem)
Consider the more general (Abel–) Volterra integral equation

$$y(t) = g(t) + (\mathcal{V}_{\alpha,\beta} y)(t), \quad t \in I,$$

with

$$(\mathcal{V}_{\alpha,\beta} y)(t) := \int_0^t (t-s)^{-\alpha}(t+s)^{-\beta} K(t,s)y(s)ds, \text{ with}$$
$$0 < \alpha < 1, \;\; 0 < \beta \le 1-\alpha.$$

Assume that K is smooth, with $K(t,t) \ne 0$ on I.

(a) Analyse the attainable orders of convergence of the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ and the corresponding iterated collocation solution, for uniform and suitably graded meshes.
(b) Collocation in the same piecewise polynomial space for

$$(\mathcal{V}_{\alpha,\beta} y)(t) = g(t), \quad t \in I \quad (g(0) = 0).$$

(See also the remark at the end of Weiss and Anderssen (1972, p. 455).)

Exercise 6.6.18 (Research Problem)
Do Exercise 6.6.17 for the linear version of 'Lighthill's equation',

$$y(t) = g(t) + (\mathcal{A}_{p,\alpha} y)(t), \quad p = 3/2, \quad \alpha = 2/3$$

(cf. Franco, McKee and Dixon (1983)). See Section 6.1.5 for the definition of the Volterra operator $\mathcal{A}_{p,\alpha}$.

Exercise 6.6.19 Assume that the $\{c_i\}$ are such that the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ for the first-kind VIE $(\mathcal{V}_\alpha y)(t) = g(t)$ is convergent. Show that the corresponding *discretised* collocation solution $\hat{u}_h$ in the same collocation space is also convergent, with the same order of convergence. In other words, the result of Theorem 6.3.4 remains true for $\hat{u}_h$.

Exercise 6.6.20 (Research problem)
Extend the sequential collocation method/sequential future (constant/polynomial) regularisation methods of Lamm and Scofield (2000) and of Ring (2001) and Ring and Prix (2000) to weakly singular first-kind VIEs described by the operators $\mathcal{V}_\alpha$ and $\mathcal{V}_{p,\alpha}$ $(0 < \alpha < 1,\ p = 2)$.

Exercise 6.6.21 The result in Theorem 6.1.6 on the representation of the solution to the second-kind VIE $y(t) = g(t) + (\mathcal{V}_\alpha y)(t)$ $(0 < \alpha < 1)$ can be used, as already suggested in Section 6.4.1, as the basis for obtaining *non-polynomial* collocation solutions to this equation. Using the appropriate collocation space (which, on the first subinterval $\bar{\sigma}_0$, is spanned by the functions

$$\phi_{j,k}^{(0)}(t) := t^{j+k(1-\alpha)}, \quad j + k(1-\alpha) < m \quad (j, k \in \mathbb{N}_0\,)),$$

analyse the attainable orders of convergence of the corresponding collocation solution and its iterate. In particular, is superconvergence (globally, on I, and locally, on I_h) possible? (Compare also Brunner (1982b, 1983) and Brunner and van der Houwen (1986, Section 6.2.5).)

Exercise 6.6.22 Show that the starting methods used by Lubich for the *fractional linear multistep methods* for solving weakly singular VIEs of the second kind can be interpreted as non-polynomial spline collocation methods using equidistant collocation points (see also Exercise 6.6.23 below).

Exercise 6.6.23 (Research problem)
Give a convergence analysis of non-polynomial spline collocation for linear first-kind Volterra integral equations with weakly singular kernel $p_\alpha(t-s)$ $(0 < \alpha < 1)$ (see Section 6.4.2). In particular, find a necessary and sufficient condition for the collocation parameters which implies the uniform convergence of the collocation on I (assume that the mesh is uniform).

Exercise 6.6.24 (Research problem)
Solutions of certain nonlinear VIEs with weakly singular kernels can blow up for some finite value of t (recall also Section 2.1.5). Typical examples are encountered in the modelling of the formation of shear bands in steel, when subjected to very high strain rates (see Roberts, Lasseigne and Olmstead (1993) and the survey paper by Roberts (1998)). If $y = y(t)$ denotes the temperature at time t, a simple such model is given by

$$y(t) = \gamma \int_0^t (t-s)^{-\alpha}(1+s)^q[y(s)+1]^p ds, \quad t \geq 0,$$

where $\gamma > 0,\ 0 < \alpha < 1,\ q \geq 0,\ p > 1$. The numerical analysis (e.g. of collocation methods) for such problems is not yet understood. Thus, as a first step, consider the collocation solution $u_h \in S_0^{(-1)}(I_h)$, with $c_1 = 1/2$ and $c_1 = 1$, and the associated iterated collocation solution. The choice of the mesh I_h is initially governed by the non-smooth behaviour of y near $t = 0$ (graded mesh with $r = 1/(1-\alpha)$). As we approach the blow-up point $t = T_b$ (and u_h becomes large), the stepsize sequence $\{h_n\}$ must be such that the nonlinear algebraic equations remain uniquely solvable.

(a) Discuss the choice of I_h near $t = T_b^-$. (Compare also Bandle and Brunner (1994).)
(b) (Detection of blow-up.) Is it possible to generate collocation solutions $u_h,\ v_h$ (corresponding to two different values of c_1) so that, for a given mesh I_h,

$$v_h^{it}(t) \leq y(t) \leq u_h^{it}(t) \quad \text{for all} \quad t \in [0, T_b)\ ?$$

Exercise 6.6.25 Let $0 < \alpha < 1$ and assume that the lag function θ is nonlinear and satisfies (D1)–(D3). Let I_h be a θ-invariant whose first submesh $I_h^{(0)}$ is graded with grading exponent $r_0 = m/(1-\alpha)$. Are the submeshes $I_h^{(\mu)}$ ($\mu = 1, \ldots, M$) *quasi-uniform*? (See also Exercise 4.7.9.)

Exercise 6.6.26 Suppose that a weakly singular delay VIE with $\alpha \in (0, 1)$ and non-vanishing *linear* delay satisfying (D1)–(D3) is solved by collocation in $S_{m-1}^{(-1)}(I_h)$. If the submeshes $I_h^{(\mu)}$ ($\mu = 0, 1, \ldots, M$) are graded individually, each with optimal grading exponent r_μ (recall Remark 2 following Theorem 6.5.2), is local superconvergence of order $m + 1 - \alpha$ possible on $I_h \setminus \{t_0\}$?

Exercise 6.6.27 (Research problem)
Collocation analysis on graded meshes for

$$\frac{d}{dt}[(\mathcal{W}_{\theta,\alpha} y)(t)] = g(t), \quad t \in (0, T] \quad (0 < \alpha < 1),$$

with $y(t) = \phi(t),\ t \leq 0$: compare the convergence properties and the numerical implementation of direct collocation (based on setting $z(t) := (\mathcal{W}_{\theta,\alpha} y)(t)$ in the given VFDE) and indirect collocation (based on the integrated form of the equation). Compare also the remark on a related open problem at the end of the paper by Ito and Turi (1991).

6.7 Notes

6.1: Review of basic Volterra theory (III)

The two papers by Evans (1910, 1911) are based on his doctoral dissertation (written under the supervision of Bôcher); they represent the first detailed studies of second-kind VIEs with weakly singular kernels, as well as other types of singular VIEs.

The Swedish mathematician Mittag-Leffler introduced 'his' function in series of papers in the early 1900s; the one of 1903, listed in the References, is a good one to consult. It was used by Hille and Tamarkin (1930) to represent the solution of certain linear VIEs with weakly singular kernels. The survey paper by Mainardi and Gorenflo (2000) is a rich source of information on the history, theory, and applications of the Mittag-Leffler function; it also contains an extensive list of references. In addition see Erdélyi (1955), Wagner (1978) (Laplace transform techniques and asymptotic behaviour of solutions), Gorenflo (1987, 1996), W. Han (1994), and Kiryakova (2000).

Regularity results for weakly singular second-kind VIEs can be found for example in Evans (1910), Tychonoff (1938), Miller and Feldstein (1971), Lubich (1983a), Brunner (1983, 1985b, 1985c), Brunner and van der Houwen (1986, Ch. 6), Mydlarczyk (1990) and – especially – in Brunner, Pedas and Vainikko (1999). Compare also volume 2 of Fenyö and Stolle (1984). At the end of Section 6.1.5 we pointed out an example of a non-compact Volterra integral operator. The paper by Graham and Sloan (1979) establishes necessary and sufficient conditions for compactness of (Fredholm) integral operators, and introduces tests for deciding if these conditions are satisfied. Many of these results will of course be relevant for Volterra integral operators.

Blow-up for nonlinear weakly singular VIEs (possessing Hammerstein nonlinearities $G(s, y)$ with $G(s, 0) = 0$, implying the existence of nontrivial solutions, in addition to $y = 0$) is studied in Mydlarczyk (1994, 1996, 1999, 2003), Bushell and Okrasiński (1996), and Mydlarczyk and Okrasiński (2001, 2003). See also Constantin and Peszat (2000) on the generalisation of some results in Bushell and Okrasiński (1996).

Another important class of VIEs with finite-time blow-up is mentioned in the next paragraph on applications. The numerical analysis (e.g. of collocation methods) for blow-up problems in VIEs is not yet understood.

Existence results for nonlinear VIEs of the form $(y(t))^\beta = g(t) + (\mathcal{V}y)(t)$ $(\beta > 1)$, can be found in Bushell and Okrasiński (1992) and in Buckwar (1997, 2000). See also Kilbas and Saigo (1999).

Applications of VIEs with weakly singular kernels
Of the many papers listed (and annotated) in the bibliography we just mention a brief selection:

- *Heat transfer problems:* This is one of the major sources of (nonlinear) VIEs with weakly singular kernels. Beginning with the papers by Mann and Wolf (1951) and Roberts and Mann (1951), we have later contributions by Keller and Olmstead (1972), Olmstead and Handelsman (1976), Gorenflo (1987), Norbury and Stuart (1987), Groetsch (1989, 1991), Jumarhon (1994), Jumarhon and McKee (1996), Jumarhon et al. (1996), and Ibrahim and Alnasr (1997).

  An interesting VIE, due to Lighthill, is studied in Franco, McKee and Dixon (1983): its kernel is of form $(t^p - s^p)^{-\alpha}$, with $p = 3/2$ and $\alpha = 2/3$.
- *Gas absorption:* Olmstead (1977).
- *VIEs with blow-up solutions:* Solutions of certain nonlinear VIEs with weakly singular kernels can blow up for finite value of t (recall also Section 2.1.5). Typical examples are encountered in the modelling of the formation of shear bands in steel, when subjected to very high strain rates (see Roberts, Lasseigne and Olmstead (1993)). Related papers are by Olmstead and Roberts (1994, 1996), Olmstead, Roberts and Deng (1995), Roberts and Olmstead (1996), Olmstead (1997, 2000), Roberts (1997, 2000) and her survey paper of 1998. Compare also Exercise 6.6.23.
- Weakly singular VIEs of the first kind, either in Abel's original form, or with singularity $(t^p - s^p)^{-\alpha}$ $(p > 1)$, arise in many applications. The survey paper by Anderssen (1977) and the monograph by Gorenflo and Vessella (1991) list numerous sources and contain extensive bibliographies. See also Atkinson (1974a, 1974b), Brunner (1975), and Hung (1979). A related first-kind VIE (the generalised Tricomi equation, related to a boundary-value problem in PDEs) is studied in v. Wolfersdorf (1965); its numerical analysis is to my knowledge completely open.

  The reader should also consult the illuminating paper by Plato (1997a) in which a general resolvent theory for Abel-type integral operators of the above form is established; it also deals with Lavrentiev's iterated regularisation method.

- Systems of VIEs with non-smooth solutions arise for example in the spatial discretisation of partial VIEs of the form

$$u(t,x) = \phi(x) + \frac{t^{\alpha/2}}{\Gamma(1+\alpha/2)}\psi(x) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Delta u(s,x)ds,$$

with $1 \le \alpha \le 2$. Such PVIEs are studied in the two papers by Fujita (1990).

6.2: Collocation for weakly singular VIEs of the second kind
Many different aspects of the classical Hölder spaces (including approximation theory) can be found in Timan (1963), Kufner, John and Fačík (1977), Burchard (1977), Powell (1981), and Zeidler (1990). See also Rice (1969) (where a somewhat different terminology is employed), and the survey paper by Brunner (1987, p. 585).

Of the many papers dealing with product quadrature for weakly singular integrands we cite the theses by Benson (1973), Logan (1976) and Kutsche (1994), and the papers by de Hoog and Weiss (1973c, 1974), Schneider (1980), Palamara Orsi (1993), Kaneko and Xu (1994), Mastroianni and Monegato (1994), Köhler (1995), Monegato and Lyness (1998) and Tamme (2000). See also Schwab (1994) for an analysis of composite quadrature formulas of variable order, and Monegato and Sloan (1997) on quadrature approximations for Cauchy-type singular integrals.

Discretised collocation methods based on product integration (for weakly singular VIEs of the first and second kind) was suggested by Huber (1939); he used the space $S_1^{(0)}(I_h)$. His analysis was extended by Wagner (1954) (see also Mirkin and Nilov (1991) where non-uniform meshes were employed). The application of product integration to integral equations is the subject of Young (1954).

Of the subsequent papers we mention Oulès (1964), Linz (1969c), the doctoral theses of Benson (1973) and Logan (1976), de Hoog and Weiss (1974), Bownds (1979), Bownds and Wood (1976), Cameron and McKee (1984). Lubich (1983a) provided a comprehensive analysis (order conditions and their dependence on $\alpha \in (0,1)$) of Runge–Kutta methods for weakly singular VIEs.

Various aspects of piecewise collocation methods were analysed by Brunner and Nørsett (1981) (superconvergence), Kershaw (1982a, 1982b) (asymptotic stability in $S_1^{(0)}(I_h)$), Brunner (1985a, 1985c) (graded meshes), Brunner and van der Houwen (1986, Chapter 6) (see also the survey paper of Brunner (1987), Eggermont (1988a) (trapezoidal method on $\mathbb{R}^+$), Palamara Orsi (1996), Monegato and Scuderi (1998a,b), Kasemets and Pedas (1999) (discretisation of collocation equation by product quadrature), Tamme (1999), and Savchenko

(2003). Optimal superconvergence order estimates, also for L^p-norms and logarithmic kernel singularities, can be found in Brunner, Pedas and Vainikko (1999); this paper contains an extensive list of references.

Collocation methods for VIEs with other types of singular kernels are discussed in Diogo (1991), Diogo, McKee and Tang (1991, 1994), and Lima and Diogo (1997).

A numerical approach to weakly singular VIEs of the third kind (whose left-hand side is $p(t)y(t)$, with $p(t) = 0$ at a finite number of points in I) can be found in Pereverzev and Prössdorf (1997).

In order to avoid the use of graded meshes, it may be feasible to use a suitable transformation of the independent variable to obtain a VIE whose solution is smooth and which can be solved on a uniform mesh (with respect to the new variable). Such an approach were already described by Prasad (1924). See also Noble (1964), de Hoog and Weiss (1973c, p. 573), Norbury and Stuart (1987), Abdalkhani (1993), Diogo, McKee and Tang (1994), Monegato and Scuderi (1998b), Galperin et al. (2000), and Baratella and Orsi Palamara (2003).

Weakly singular Fredholm integral equations: The first analyses of collocation (and Galerkin) methods on graded meshes are those in Chandler (1979), Graham (1980, 1982b), Vainikko and Uba (1981), Schneider (1981), and Vainikko, Pedas and Uba (1984). More recent contributions are by Kaneko, Noren and Xu (1992), Kaneko and Xu (1994), Kaneko, Noren and Padilla (1997), Pedas and Vainikko (1997, 1999), and Tamme (1999). A comprehensive analysis of collocation methods for one- and multidimensional Fredholm integral equations of the second kind is contained in Vainikko (1993). See also the survey paper by Brunner (1987) and the monographs by Hackbusch (1995) and Atkinson (1997a).

Cauchy type singular integral equations: There is now an extensive literature on collocation methods for singular integral equations of, e.g. Cauchy type. A good (early) survey paper is by Elliott (1982); compare also Volume 4 of Fenyö and Stolle (1984). Elliott (1989) presents an illuminating comparison of Galerkin and collocation methods. The most comprehensive exposition of the numerical analysis of such IEs is Prössdorf and Silbermann (1991). The detailed survey by Junghanns and Silbermann (2000) and the paper by Junghanns and Müller (2000) contain numerous references to related contributions.

6.3: Collocation for weakly singular first-kind VIEs

Huber (1939) studied collocation in $S_1^{(0)}(I_h)$ for weakly singular VIEs. The convergence properties of the product midpoint and trapezoidal methods were analysed in detail in Weiss (1972a, 1972b), Weiss and Anderssen (1972), Benson (1973) and – especially – in Eggermont (1981). See also Cameron and McKee

(1984). Branca (1976, 1978) used a more general approach which is more in the spirit of collocation methods; it leads to higher-order spline methods. The paper by te Riele and Schroevers (1986) contains an illuminating numerical comparison of the performance of many discretisation methods, including collocation.

The papers by Brunner (1997b, 1999a, 1999c) convey a picture of the 'state of the art' in the analysis of collocation methods for such VIEs, including the open problems mentioned earlier in this chapter. In addition, see Eggermont (1984, 1988b) (also for a comparison of collocation and Galerkin methods), Capobianco (1988, 1990), and Capobianco and Formica (1998).

Numerical methods for first-kind VIEs with kernel singularities $(t^p - s^p)^{-\alpha}$ $p > 1,\ 0 < \alpha < 1$) can be found in, e.g. Atkinson (1974b), Brunner (1975), Anderssen (1976) (use of inversion formula and spectral differentiation), Anderssen (1977) (survey paper; see also the proceedings volume edited by Anderssen, de Hoog and Lukas (1980)), Smarzewski and Malinowski (1978), Hung (1979), and Smarzewski and Malinowski (1983) (singular kernels of the form $(h(t) - h(s))^{-\alpha}$).

The convergence analysis for collocation solutions in $S^{(d)}_{m+d}(I_h)$ ($d \in \{-1, 0\}$) remains to be established: as in the case $p = 1$ it is not known (except for $m = 1$ and $c_1 = 1$) under which conditions on the collocation parameters $\{c_i\}$ one obtains uniform convergence on I as $h \to 0$ (to my knowledge, there is not even a counterpart to Theorem 6.3.3 when $m = 1$).

Boundary integral equations

Collocation methods for BIEs have received considerable attention in the last dozen years or so. The surveys by Atkinson (1997b) and by Sloan (2000) convey a detailed picture of these developments. In addition, see also, e.g. Chandler (1984) (mesh grading), Elschner (1989), Chandler and Sloan (1990), Iso and Onishi (1991), Lubich and Schneider (1992) (time-discretisation), Sloan (1992, 1995), Hamina and Saranen (1994), McLean (1994) (comparison of exact and discretised collocation solutions), Sloan (1995), Berthold and Silbermann (1995), Elschner and Graham (1995), Saranen and Vainikko (1996) (trigonometric collocation), Tran and Sloan (1998), Hämäläinen (1998), Sloan and Tran (1998, 2001), Junghanns and Rathsfeld (2002), as well as the monographs by Saranen and Vainikko (2002) (and its bibliograpy) and Yu (2002).

Qualocation

The 'quadrature modified collocation method' (or qualocation method) combines the best properties of Galerkin methods (superconvergence, easy stability analysis) and collocation methods (no inner products, cheaper implementation); this is achieved by replacing the inner products by speciallly designed quadrature rules not based on the collocation parameters. Its principal application is in boundary integral equations. An excellent survey is Sloan (2000) (with numerous references); in addition, see Sloan (1998b, 1991), Hagen and Silbermann (1988), Wendland (1989), Yan (1990), Chandler and Sloan (1990), Tran and Sloan (1998), Sloan and Tran (2001), and the book by Hagen, Roch and Silbermann (1995). An alternative but related approach for periodic pseudo-differential equation, using 'corrected' collocation methods, described in Berthold and Silbermann (1995).

6.4: Non-polynomial spline collocation methods
Collocation methods in non-polynomial spline spaces (and on uniform meshes) reflecting the non-smooth behaviour of solutions of weakly singular VIEs were introduced in te Riele (1982) (for $\alpha = 1/2$) and Brunner (1983); see Brunner (1982b) and Brunner and van der Houwen (1986, Chapter 6). Hu (1997a, 1997) and Hu and Luo (1997) combined special non-polynomial splines ('β-polynomials'), geometric meshes, and interpolation postprocessing to obtain superconvergence results for such VIEs. An analogous approach to weakly singular VIDEs can be found in Hu (1996a, 1998b). Cao, Herdman and Xu (2003) describe a hybrid collocation method, combining non-polynomial spline collocation near the singular point $t = 0$ with polynomial spline collocation on suitably graded meshes in the rest of I. Riley (1989, 1992) applied Sinc methods to linear, weakly singular VIEs. Stenger (1993, 1995, 2000) should be consulted for a comprehensive treatment and applications of these functions. A different kind of non-polynomial spline approximation can be found in Horvath and Rogina (2002) (for singularly perturbed VIEs and VIDEs).

The reader may also wish to look at the survey paper by Unser and Blu (2000) on fractional splines and wavelets.

6.5: Weakly singular Volterra functional equations with non-vanishing delays
Numerical methods for the first-kind Volterra functional integro-differential equation (6.5.2) can be found in Herdman and Turi (1991a) and in Ito and Turi (1991). They use the semigroup framework of Burns, Herdman and Stech (1983) to rewrite the given equation as a hyperbolic PDE with non-local boundary conditions. As we mentioned before, the numerical exploitation of the alternative semi-group framework for (6.5.2) given in Clément, Desch and Homan (2002) remains to be studied. Collocation for the integrated form of this class of equations is discussed in Brunner (1999c).

7

VIDEs with weakly singular kernels

The order reduction we observed in Chapter 6 when approximating solutions of weakly singular Volterra integral equations by piecewise polynomial collocation on uniform meshes is also present in analogous Volterra integro-differential equations, although their solutions are slightly more regular. We shall see that the principal ideas underlying the convergence analysis in the previous chapter are readily adapted to derive analogous optimal convergence estimates for VIDEs with weakly singular kernels.

7.1 Review of basic Volterra theory (IV)

7.1.1 Linear weakly singular VIDEs

In this section we will focus on the regularity properties of solutions to initial-value problems for linear first-order VIDEs with weakly singular kernels,

$$y'(t) = a(t)y(t) + g(t) + (\mathcal{V}_\alpha y)(t), \quad t \in I := [0, T], \quad y(0) = y_0. \quad (7.1.1)$$

As in Section 6.1, $\mathcal{V}_\alpha : C(I) \to C(I)$ is defined by

$$(\mathcal{V}_\alpha \phi)(t) := \int_0^t p_\alpha(t-s)K(t,s)\phi(s)ds, \quad (7.1.2)$$

with p_α denoting either an algebraic or a logarithmic singularity,

$$p_\alpha(t-s) := \begin{cases} (t-s)^{-\alpha} & \text{if} \;\; 0 < \alpha < 1, \\ \log(t-s) & \text{if} \;\; \alpha = 1, \end{cases}$$

and with $K \in C(D)$, $K(t,t) \neq 0$ for $t \in I$. We will again set $H_\alpha(t,s) := p_\alpha(t-s)K(t,s)$. Various nonlinear and higher-order (neutral) counterparts of (7.1.1) will be considered in Sections 7.1.2 and 7.1.3.

The regularity analysis can be based on either of two second-kind VIEs that are equivalenti to the original initial-value problem (7.1.1). Its first reformulation has the form

$$y(t) = g_0(t) + \int_0^t K_\alpha^I(t,s)y(s)ds, \quad t \in I, \tag{7.1.3}$$

where

$$g_0(t) := y_0 + \int_0^t g(s)ds, \quad K_\alpha^I(t,s) := a(s) + \int_s^t H_\alpha(v,s)dv.$$

Alternatively, we may consider the equivalent VIE for $z(t) := y'(t)$, namely,

$$z(t) = f_0(t) + \int_0^t K_\alpha^{II}(t,s)z(s)ds, \quad t \in I, \tag{7.1.4}$$

with

$$f_0(t) := g(t) + \left(a(t) + \int_0^t H_\alpha(t,s)ds\right) y_0,$$

$$K_\alpha^{II}(t,s) := a(t) + \int_s^t H_\alpha(t,v)dv.$$

Note that if $a(t) \equiv 0$ and $K(t,s) \equiv 1$, we obtain

$$K_\alpha^I(t,s) = K_\alpha^{II}(t,s) = \begin{cases} \frac{1}{1-\alpha}(t-s)^{1-\alpha} & \text{if } 0 < \alpha < 1, \\ (t-s)[\log(t-s)-1] & \text{if } \alpha = 1. \end{cases}$$

Before stating the fundamental result on existence and representation of solutions we look at a representative example, as we did in Section 6.1.1. It is the VIDE

$$y'(t) = g(t) + \lambda \int_0^t (t-s)^{-\alpha} y(s)ds, \quad t \in I := [0,T] \quad (0 < \alpha < 1), \tag{7.1.5}$$

with initial condition $y(0) = y_0$. This initial-value problem is equivalent to the second-kind VIE

$$y(t) = g_0(t) + \lambda_0 \int_0^t (t-s)^{1-\alpha} y(s)ds, \quad t \in I,$$

with

$$\lambda_0 := \frac{\lambda}{1-\alpha} \quad \text{and} \quad g_0(t) := y_0 + \int_0^t g(s)ds.$$

Its (unique) solution is, according to Corollary 6.1.9 with $\nu = 1-\alpha$,

$$y(t) = g_0(t) + \int_0^t \left(\frac{d}{dt} E_{2-\alpha}(\lambda_0 \Gamma(2-\alpha)(t-s)^{2-\alpha})\right) g_0(s)ds.$$

A more explicit expression can be obtained by carrying out the differentiation and then applying integration by parts. For $g(t) \equiv 0$ this yields

$$y(t) = E_{2-\alpha}(\lambda_0 \Gamma(2-\alpha) t^{2-\alpha}) y_0, \quad t \in I.$$

The definition of the Mittag-Leffler function tells us that the solution y of (7.1.5) is in $C^1(I)$; however, its second derivative typically behaves like $|y''(t)| \leq C t^{-\alpha}$ near $t = 0^+$. As Theorem 7.1.4 will show, this reflects the general situation: solutions to (7.1.1) with smooth data $a,\ g$ and K and $0 < \alpha < 1$ exhibit the regularity behaviour just described; that is, the solution of (7.1.1) is in the Hölder space $C^{1,1-\alpha}(I)$.

We first give the representation of the solution of the initial-value problem (7.1.1), thus extending Theorem 3.1.1 to linear VIDEs with weakly singular kernels.

Theorem 7.1.1 *Assume that $a, g \in C(I)$ and $K \in C(D)$, and let $\alpha \in (0, 1]$. Then for any initial value y_0 the VIDE (7.1.1) possesses a unique solution $y \in C^1(I)$ satisfying $y(0) = y_0$. Moreover, there exists a unique function $r_\alpha = r_\alpha(t, s)$ satisfying $r_\alpha \in C^1(D)$, so that this solution has the representation*

$$y(t) = r_\alpha(t, 0) y_0 + \int_0^t r_\alpha(t, s) g(s) ds, \quad t \in I. \tag{7.1.6}$$

The resolvent kernel r_α can be defined as the solution of the resolvent equation

$$\frac{\partial r_\alpha(t, s)}{\partial s} = -r_\alpha(t, s) a(s) - \int_s^t r_\alpha(t, v) H_\alpha(v, s) dv, \quad (t, s) \in D, \tag{7.1.7}$$

with $r_\alpha(t, t) = 1$ for $t \in I$.

Proof We will use the second-kind VIE (7.1.3) to establish results on the properties of solutions of the weakly singular VIDE (7.1.1). Let $R_\alpha^I(t, s)$ denote the resolvent kernel of the kernel $K_\alpha^I(t, s)$ in the integral equation (7.1.3). Since $K_\alpha^I \in C(D)$ we can use the results of Section 2.1.1: R_α^I solves the resolvent equation (2.1.10),

$$R_\alpha^I(t, s) = K_\alpha^I(t, s) + \int_s^t R_\alpha^I(t, v) K_\alpha^I(v, s) dv, \quad (t, s) \in D, \tag{7.1.8}$$

and the (unique) solution $y \in C^1(I)$ of (7.1.3) is thus given by

$$y(t) = g_0(t) + \int_0^t R_\alpha^I(t, s) g_0(s) ds, \quad t \in I. \tag{7.1.9}$$

(note that we have $g_0 \in C(I)$ and $K_\alpha^I \in C(D)$, with integrable partial derivatives). Using the above definitions of g_0 and K_α^I we obtain

$$y(t) = \left(1 + \int_0^t R_\alpha^I(t,s)ds\right) y_0 + \int_0^t \left(1 + \int_s^t R_\alpha^I(t,v)dv\right) g(s)ds.$$

This shows that the desired function r_α in (7.1.5) is given by

$$r_\alpha(t,s) := 1 + \int_s^t R_\alpha^I(t,v)dv, \quad (t,s) \in D. \tag{7.1.10}$$

Its uniqueness, and the uniqueness of y, follow from that of the resolvent kernel R_α^I and from Theorem 2.1.2 in Section 2.1.1. Note that $r_\alpha \in C^1(D)$; in particular, we have $\partial r_\alpha(t,s)/\partial s = -R_\alpha^I(t,s) \in C(D)$, and $r_\alpha(t,t) = 1$ for all $t \in I$. Since (7.1.1) and (7.1.3) are equivalent, this completes the first part of the proof.

The above also reveals that the resolvent $r_\alpha(t,s)$ associated with the linear VIDE (7.1.1) satisfies

$$\begin{aligned}\frac{\partial r_\alpha(t,s)}{\partial s} &= -R_\alpha^I(t,s) = -K_\alpha^I(t,s) - \int_s^t R_\alpha^I(t,v)K_\alpha^I(v,s)dv \\ &= -a(s) - \int_s^t H_\alpha(v,s)dv - \int_s^t R_\alpha^I(t,v)\left(a(s) + \int_s^v H_\alpha(z,s)dz\right)dv \\ &= -\left(1 + \int_s^t R_\alpha^I(t,v)dv\right)a(s) \\ &\quad - \int_s^t \left(1 + \int_v^t R_\alpha^I(t,z)dz\right) H_\alpha(v,s)dv,\end{aligned}$$

and hence, by (7.1.9),

$$\frac{\partial r_\alpha(t,s)}{\partial s} = -r_\alpha(t,s)a(s) - \int_s^t r_\alpha(t,v)H_\alpha(v,s)dv, \quad (t,s) \in D. \tag{7.1.11}$$

The resolvent kernel $r_\alpha(t,s)$ can also be defined by the (unique) solution of an *adjoint resolvent equation*, in complete analogy to the result in Chapter 3 (Theorem 3.1.2). We summarise this for the sake of completeness in Theorem 7.1.2 and leave the details of its proof as an exercise.

Theorem 7.1.2 *Assume that $a \in C(I)$ and $K \in C(D)$, and let $\alpha \in (0,1]$. Then the resolvent kernel $r_\alpha = r_\alpha(t,s)$ of the linear weakly singular VIDE (7.1.1) is also the (unique) solution of the adjoint resolvent equation,*

$$\frac{\partial r_\alpha(t,s)}{\partial t} = r_\alpha(t,s)a(t) + \int_s^t H_\alpha(t,v)r_\alpha(v,s)dv, \quad (t,s) \in D, \tag{7.1.12}$$

with initial condition $r_\alpha(s,s) = 1$ for $s \in I$.

Corollary 7.1.3 *The resolvent equations associated with the special weakly singular VIDE*

$$y'(t) = g(t) + (\mathcal{V}_\alpha y)(t), \quad t \in I, \tag{7.1.13}$$

with $\mathcal{V}_\alpha$ as in (7.1.2), are

$$\frac{\partial r_\alpha(t,s)}{\partial s} = -\int_s^t r_\alpha(t,v)H_\alpha(v,s)dv, \quad (t,s) \in D,$$

and

$$\frac{\partial r_\alpha(t,s)}{\partial t} = \int_s^t H_\alpha(t,v)r_\alpha(v,s)dv, \quad (t,s) \in D,$$

with $r_\alpha(t,t) = 1$ ($t \in I$) and $r_\alpha(s,s) = 1$ ($s \in I$), respectively.

We now return to (7.1.1) and show that solutions corresponding to smooth data will in general not be smooth at $t = 0^+$: they lie in the Hölder space $C^{1,1-\alpha}(I)$.

Theorem 7.1.4 *Assume that $a, g \in C^m(I)$ and $K \in C^m(D)$ ($m \geq 1$), with $K(t,t) \neq 0$ on I, and let $\alpha \in (0,1)$. Then:*

(i) The regularity of the solution y of the linear VIDE (7.1.1) with weak kernel singularity $p_\alpha(t-s)$ is described by

$$y \in C^1(I) \cap C^{m+1}((0,T]),$$

with y'' being unbounded at $t = 0^+$:

$$|y''(t)| \leq Ct^{-\alpha} \quad \text{for} \quad t \in (0,T].$$

(ii) The solution y can be written in the form

$$y(t) = \sum_{(j,k)_\nu} \gamma_{j,k}(\nu)t^{j+k(1+\nu)} + Y_{m+1}(t;\nu), \quad t \in I, \tag{7.1.14}$$

where $\nu = 1 - \alpha$ and, slightly abusing the notation in Theorem 6.1.6,

$$(j,k)_\nu := \{(j,k) : \ j,k \in \mathbb{N}_0, \ j + k(1+\nu) < m+1\}.$$

Moreover, $Y_{m+1}(\cdot;\nu) \in C^{m+1}(I)$, and the coefficients $\gamma_{k,j}(\nu)$ are defined in analogy to the $\gamma_{j,k}(\alpha)$ in the proof of Theorem 6.1.6.

Proof We have seen that for $0 < \alpha < 1$, the weakly singular VIDE (7.1.1) is equivalent to the second-kind VIE (7.1.3) whose kernel is bounded and whose convolution part is essentially $(t-s)^{1-\alpha}$. The regularity of solutions to VIEs of this type has been analysed in Theorem 6.1.8: the present case corresponds to $\rho = 1$ in $\nu = \rho - \alpha$. Thus, the result of Theorem 7.1.4 follows immediately from that theorem.

Regularity results for linear VIDEs with logarithmic kernel singularity or with non-smooth but bounded kernels can be found in the papers by Brunner, Pedas and Vainikko (2001a, 2001b). Exercise 7.7.6 also deals with some of those cases.

7.1.2 Nonlinear VIDEs with weakly singular kernels

The nonlinear VIDE with weakly singular kernel,

$$y'(t) = f(t, y(t)) + \int_0^t h_\alpha(t, s, y(s))ds, \tag{7.1.15}$$

where $h_\alpha(t, s, y) := p_\alpha(t - s)k(t, s, y)$ and $0 < \alpha \le 1$, with smooth $f(t, y)$ and $k(t, s, y)$, is equivalent to a nonlinear VIE with *bounded* kernel,

$$y(t) = y(0) + \int_0^t \left(f(s, y(s)) + \int_s^t h_\alpha(v, s, y(s))dv \right) ds.$$

If (7.1.15) is of *Hammerstein* type, that is, if $k(t, s, y) = K(t, s)G(s, y)$, then the equivalent VIE is

$$y(t) = y(0) + \int_0^t (f(s, y(s)) + H_\alpha(t, s)G(s, y(s)))ds,$$

with

$$H_\alpha(t, s) := \int_s^t p_\alpha(v - s)K(v, s)dv, \quad (t, s) \in D.$$

Therefore, the existence and uniqueness of its solution follow from Theorem 2.1.10: if the nonlinearities $f(t, y)$ and $G(s, y)$ are (Lipschitz) continuous on $I \times \Omega$ for some $\Omega \subset \mathbb{R}$, then there is a unique local solution on some interval $[0, \delta_0)$.

The semilinear VIDE

$$y'(t) = a(t)y(t) + g(t) + \int_0^t H_\alpha(t, s)(y(s) + G(s, y(s)))ds \tag{7.1.16}$$

represents, as in Section 3.1.2, a first step towards more general nonlinear VIDEs: the *linear* Volterra integral operator $\mathcal{V}_\alpha$ has been perturbed by the *Hammerstein term*

$$(\mathcal{H}_\alpha y)(t) := \int_0^t H_\alpha(t, s)G(s, y(s))ds$$

corresponding to $H_\alpha(t,s) := p_\alpha(t-s)K(t,s)$, with $K(t,t) \neq 0$ on I. The following result can be found for example in Grossman and Miller (1970).

Theorem 7.1.5 *Assume that the initial-value problem for the semilinear VIDE (7.1.16) possesses a unique solution $y \in C^1(I)$, and let*

$$y_\ell(t) := r_\alpha(t,0)y_0 + \int_0^t r_\alpha(t,s)g(s)ds, \quad t \in I,$$

denote the solution of the linear VIDE

$$y'(t) = a(t)y(t) + g(t) + (\mathcal{V}_\alpha y)(t), \quad y(0) = y_0.$$

Then y and y_ℓ are related by

$$y(t) = y_\ell(t) - \int_0^t \left(r_\alpha(t,s)a(s) + \frac{\partial r_\alpha(t,s)}{\partial s} \right) G(s, y(s))ds, \quad t \in I. \tag{7.1.17}$$

Here, $r_\alpha(t,s)$ denotes the resolvent kernel associated with a and H_α describing the linear part of (7.1.16).

Proof Setting $Q_\alpha(t) := g(t) + (\mathcal{H}_\alpha y)(t)$, the semilinear VIDE (7.1.16) can be written as

$$y'(t) = a(t)y(t) + Q_\alpha(t) + (\mathcal{V}_\alpha y)(t), \quad t \in I.$$

According to Theorem 7.1.1 the solution of this perturbed linear VIDE is formally given by

$$y(t) = r_\alpha(t,0)y_0 + \int_0^t r_\alpha(t,s)Q_\alpha(s)ds, \quad t \in I.$$

The representation of y in Theorem 7.1.2 now follows readily by observing the resolvent equation (7.1.6) and by writing

$$\int_s^t r_\alpha(t,v)H_\alpha(v,s)dv = -r_\alpha(t,s)a(s) - \frac{\partial r_\alpha(t,s)}{\partial s}.$$

We observe that when $\alpha = 0$ the above result reduces to the one in Theorem 3.1.5.

7.1.3 Neutral and higher-order VIDEs

If the kernel h_α in the VIDE (7.1.15) also depends on y', that is, if the VIDE has the form

$$y'(t) = f(t, y(t)) + \int_0^t h_\alpha(t,s,y(s),y'(s))ds, \quad t \in I, \tag{7.1.18}$$

with

$$h_\alpha(t, s, y, z) := p_\alpha(t-s)k(t, s, y, z) \quad (0 < \alpha \le 1),$$

then this functional equation is the weakly singular counterpart of the (neutral) first-order VIDE (3.1.13). It can be viewed as a particular case of a weakly singular kth-order VIDE

$$y^{(k)}(t) = f(t, y(t), y'(t), \ldots, y^{(k-1)}(t)) + (\mathcal{V}_\alpha y)(t), \quad t \in I := [0, T],$$
$$y^{(\nu)}(0) = y_0^{(\nu)} \quad (\nu = 0, 1, \ldots, k-1), \tag{7.1.19}$$

with $k \ge 2$. Here, $\mathcal{V}_\alpha$ stands for

$$(\mathcal{V}_\alpha y)(t) := \int_0^t h_\alpha t, s, y(s), y'(s), \ldots, y^{(k)}(s))ds$$

and corresponds to the kernel

$$h_\alpha(t, s, y, \ldots, y^{(k)}) = p_\alpha(t-s)k(t, s, y, \ldots, y^{(k)}) \quad (0 < \alpha \le 1).$$

We will often use the linear counterpart of this VIDE as the basis for our subsequent convergence analysis: it is described by

$$f(t, y, y', \ldots, y^{(k-1)}) = \sum_{\nu=0}^{k-1} a_\nu(t) y^{(\nu)} + g(t), \tag{7.1.20}$$

$$h_\alpha(t, s, y, y', \ldots, y^{(k)}) = p_\alpha(t-s) \sum_{\nu=0}^{k} K_\nu(t, s) y^{(\nu)}. \tag{7.1.21}$$

The given functions $g,\ a_\nu$ and K_ν are assumed to be continuous on I and D, respectively. We note again that we allow the derivative of order k of y to occur as argument in the kernel of the VIDE. Compare also Exercise 7.7.7 for the more general form of (7.1.21),

$$h_\alpha(t, s, y, y', \ldots, y^{(k)}) := \sum_{\nu=0}^{k} p_{\alpha_\nu}(t-s) K_\nu(t, s) y^{(\nu)},$$

with $0 < \alpha_0 < \alpha_1 < \ldots < \alpha_k \le 1$.

We shall now briefly show that, for $\alpha \in (0, 1)$, the *linear VIDE* corresponding to (7.1.20) and (7.1.21) possesses a unique solution $y \in C^{k,1-\alpha}(I)$ satisfying a prescribed set of initial conditions. An analogous (generally only local) existence and uniqueness result can be obtained for the nonlinear VIDE (7.1.19), by a straightforward adaptation of the arguments presented below.

Let $\mathbf{w}(t) := (w_0(t), w_1(t), \ldots, w_k(t))^T := (y(t), y'(t), \ldots, y^{(k)}(t))^T$, and write

$$w_\nu(t) = y_0^{(\nu)} + \int_0^t w_{\nu+1}(s)ds, \quad t \in I \quad (\nu = 0, 1, \ldots, k-1).$$

Hence,

$$w_k(t) = \sum_{\nu=0}^{k-1} a_\nu(t) \left(y_0^{(\nu)} + \int_0^t w_{\nu+1}(s)ds \right) + g(t)$$
$$+ \int_0^t p_\alpha(t-s) \sum_{\nu=0}^{k} K_\nu(t,s) w_\nu(s)ds. \qquad (7.1.22)$$

Set

$$\gamma(t) := \left(y_0^{(0)}, y_0^{(1)}, \ldots, y_0^{(k-1)}, g(t) + \sum_{\nu=0}^{k-1} a_\nu(t) y_0^{(\nu)} \right)^T,$$

$H_{\alpha,\nu}(t,s) := p_\alpha(t-s)K_\nu(t,s)$, and define the matrix $\mathbf{H}_\alpha \in L(\mathbb{R}^{k+1})$ by

$$\mathbf{H}_\alpha(t,s) := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & & 1 \\ H_{\alpha,0}(t,s) & a_0(t) + H_{\alpha,1}(t,s) & \cdots & \cdots & a_{k-1}(t) + H_{\alpha,k}(t,s) \end{bmatrix}.$$

The given VIDE (7.1.19), with (7.1.20) and (7.1.21), is thus equivalent to a system of second-kind VIEs, namely

$$\mathbf{w}(t) = \gamma(t) + \int_0^t \mathbf{H}_\alpha(t,s)\mathbf{w}(s)ds, \quad t \in I. \qquad (7.1.23)$$

It follows from Section 6.1.4 that due to the continuity of γ and the integrability of $\mathbf{H}_\alpha$, this system possesses a unique solution $\mathbf{w} \in C(I)$ whose representation,

$$\mathbf{w}(t) = \gamma(t) + \int_0^t \mathbf{R}_\alpha(t,s)\gamma(s)ds, \quad t \in I, \qquad (7.1.24)$$

is based on the (matrix) resolvent kernel $\mathbf{R}_\alpha \in L(\mathbb{R}^{k+1})$ of $\mathbf{H}_\alpha$. If we write this matrix resolvent kernel as

$$\mathbf{R}_\alpha(t,s) := \begin{pmatrix} R_{0,0}(t,s;\alpha) & \ldots & R_{0,k}(t,s;\alpha) \\ \vdots & & \vdots \\ R_{k,0}(t,s;\alpha) & \ldots & R_{k,k}(t,s;\alpha) \end{pmatrix},$$

then the representation (7.1.24) permits the explicit derivation of the expressions for the $k+1$ components of the solution vector $\mathbf{w}(t)$, e.g. for $w_0(t) = y(t)$, in analogy to Section 3.1.2.

This equivalence between the initial-value problem for the kth-order VIDE (7.1.19)–(7.1.21) and the system of $k+1$ linear Volterra integral equations of the second kind (7.1.22) allows us, by appealing to Theorem 6.1.6, to obtain the following.

Theorem 7.1.6 *Let $\alpha \in (0,1)$ and assume that the a_ν $(\nu = 0, 1, \ldots, k-1)$ and g are in $C(I)$, and $K_\nu \in C(D)$, with $K_\nu(t,t) \neq 0$ on I $(\nu = 0, 1, \ldots, k)$. Then for any initial values $y_0^{(\nu)}$ $(\nu = 0, 1, \ldots, k-1)$ the kth-order VIDE (7.1.19) corresponding to (7.1.20), (7.1.21) possesses a unique solution $y \in C^{k,1-\alpha}(I)$ satifying the given initial conditions.*

We leave it to the reader (see Exercise 7.7.8) to write down the analogue of the solution representation (7.1.14) in Theorem 7.1.4.

The arguments leading from (7.1.18) to the system (7.1.23) remain valid for $\alpha = 1$ (when $p_\alpha(t-s) = \log(t-s)$). Hence, we see by extending Theorem 6.1.7 to *systems* of second-kind VIEs with logarithmic kernel singularity that, for any set of initial values, the the initial-value problem

$$y^{(k)}(t) = \sum_{\nu=0}^{k-1} a_\nu(t) y^{(\nu)}(t) + g(t) + \int_0^t \log(t-s) \sum_{\nu=0}^{k} K_\nu(t, s y^{(\nu)}(s) ds, \quad t \in I,$$

$$y^{(\nu)}(0) = y_0^{(\nu)} \quad (\nu = 0, 1, \ldots, k-1),$$

possesses a unique solution $y \in C^k(I)$ with

$$|y^{(k+1)}(t)| \leq C \cdot t |\log(t)|, \quad t \in (0, T],$$

provided the given functions are in $C(I)$.

7.1.4 Weakly singular VIDEs with delay arguments

Let the Volterra integral operator $\mathcal{V}_{\theta,\alpha}$ be defined as in Section 6.1.7, with the delay function θ subject to (D1)–(D3). From what we have seen in Section 6.1.7 it is clear that the presence of this delay operator in the VIDE

$$y'(t) = a(t)y(t) + b(t)y(\theta(t)) + g(t) + (\mathcal{V}_\alpha y)(t) + (\mathcal{V}_{\theta,\alpha} y)(t), \quad t \in I, \tag{7.1.25}$$

with initial condition $y(t) = \phi(t)$ on $[\theta(t_0), t_0]$, will affect the regularity of the solution y at the points $t = \xi_\mu^+$ $(\mu \geq 0)$.

Table 7.1. *Regularity and smoothing of solutions to weakly singular delay VIDEs*

Delay Volterra integro-differential equation (with arbitrarily smooth data)	Regularity on $I^{(\mu)} = (\xi_\mu, \xi_{\mu+1}]$ ($\mu = 0, 1, \ldots, M$)
• $y'(t) = f(t, y(t)) + (\mathcal{V}_{\theta,\alpha} y)(t)$	$C^{2\mu+1,1-\alpha}$ ('super-smoothing')
• $y'(t) = f(t, y(t), y(\theta(t))) + (\mathcal{V}_{\theta,\alpha} y)(t)$	$C^{\mu+1,1-\alpha}$
• $y'(t) = f(t, y(t), y(\theta(t))) + (\mathcal{W}_{\theta,\alpha} y)(t)$	$C^{\mu+1,1-\alpha}$
• $y'(t) = f(t, y(t), y(\theta(t)), y'(\theta(t)))$ $+ (\mathcal{V}_{\theta,\alpha} y)(t)$ $C^{1,1-\alpha}$	(no smoothing at $t = \xi_\mu$) $C^{1,1-\alpha}$
• $y'(t) = f(t, y(t), y(\theta(t)), y'(\theta(t)))$ $+ (\mathcal{W}_{\theta,\alpha} y)(t)$	(no smoothing at $t = \xi_\mu$)

This is also true for neutral VIDEs, for example for the class of equations described by

$$\frac{d}{dt}[a_0 y(t) - (\mathcal{V}_{\theta,\alpha} y)(t)] = f(t, y(t), y(\theta(t))), \tag{7.1.26}$$

where the coefficient a_0 is from $\{1, 0\}$.

The regularity results summarised in Table 7.1 are analogous to those in Table 6.1 for weakly singular VIEs and generalise those described in Table 4.1 (Section 4.1.4). Their proofs can be found in Ma (2004). In analogy to Table 6.1 the range of the values μ is such that when the exponent of the indicated Hölder space reaches $m + 1$, the reguarity on the remaining subintervals $I^{(\mu)}$ is C^{m+1}. We will not specify this in Table 7.1.

7.1.5 A generalisation of Gronwall's Lemma

The kth-order VIDE

$$y^{(k)}(t) = g(t) + \int_0^t (t-s)^{-\alpha} K(t,s) y(s) ds, \quad t \in I \quad (\alpha < 1,\ k \geq 1)) \tag{7.1.27}$$

with continuous g and K, is equivalent to the second-kind VIE

$$y(t) = g_0(t) + \int_0^t \int_0^{\tau_k} \cdots \int_0^{\tau_1} (\tau_1 - s)^{-\alpha} K(t,s) y(s) ds\, d\tau_1 \ldots d\tau_k, \quad t \in I,$$

where

$$g_0(t) := w_0(t) + \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} g(s) ds$$

and

$$w_0(t) := \sum_{\nu=0}^{k-1} \frac{y^{(\nu)}(0)}{\nu!} t^\nu.$$

Hence, the uniqueness of the solution of the initial-value problem for the VIDE (7.1.27) can be established by means of the following Gronwall-type result due to Dixon and McKee (1984).

Theorem 7.1.7 *Assume that $\gamma \in C(I)$ is non-negative and non-decreasing on $I := [0, T]$, and let $z \in C(I)$ be a non-negative function on I satisfying*

$$z(t) \le \gamma(t) + K_0 \int_0^t \int_0^{\tau_k} \cdots \int_0^{\tau_1} \frac{z(s)}{(\tau_1 - s)^\alpha} ds\, d\tau_1 \ldots d\tau_k, \quad t \in I, \quad (7.1.28)$$

with $\alpha < 1$ and $K_0 > 0$. If we define $\beta := k + 1 - \alpha$ then

$$z(t) \le E_\beta(K_0 \Gamma(1-\alpha) t^\beta)\gamma(t), \quad t \in I.$$

Proof The $(k+1)$-fold integral in (7.1.28) can be rewritten, using Dirichlet's formula, as

$$\int_0^t \int_0^{\tau_k} \cdots \int_0^{\tau_1} \frac{z(s)}{(\tau_1 - s)^\alpha} ds\, d\tau_1 \ldots d\tau_k$$
$$= \frac{\Gamma(1-\alpha)}{\Gamma(k+1-\alpha)} \int_0^t (t-s)^{k-\alpha} z(s) ds, \quad t \in I.$$

The result of Theorem 7.1.7 now follows from Theorem 6.1.17.

7.2 Collocation for linear weakly singular VIDEs

7.2.1 The exact collocation equations

The weakly singular counterpart of the VIDE (3.2.1) is given by

$$y'(t) = f(t, y(t)) + (\mathcal{V}_\alpha y)(t), \quad t \in I := [0, T], \quad y(0) = y_0 \quad (0 < \alpha \le 1). \quad (7.2.1)$$

The Volterra integral operator $\mathcal{V}_\alpha : C(I) \to C(I)$ is the one given in (7.1.2),

$$(\mathcal{V}_\alpha \phi)(t) := \int_0^t H_\alpha(t, s)\phi(s) ds := \int_0^t p_\alpha(t-s) K(t, s)\phi(s) ds, \quad t \in I.$$

Recall that the kernel singularity is either of algebraic type, $p_\alpha(t-s) := (t-s)^{-\alpha}$ if $0 < \alpha < 1$, or of logarithmic type, $p_\alpha(t-s) := \log(t-s)$ when $\alpha = 1$. We will assume that $K \in C(D)$ and $K(t, t) \ne 0$ $(t \in I)$. Later in this chapter we shall also turn to fully *nonlinear* versions of (7.2.1), in particular to the one associated with the Volterra–Hammerstein operator

$$(\mathcal{H}_\alpha \phi)(t) := \int_0^t p_\alpha(t-s)K(t,s)G(s,\phi(s))ds.$$

The collocation solution $u_h \in S_m^{(0)}(I_h)$ for (7.2.1) satisfies the collocation equation

$$u_h'(t) = f(t, u_h(t)) + (\mathcal{V}_\alpha u_h)(t), \quad t \in X_h, \quad u_h(0) = y_0, \tag{7.2.2}$$

with the familiar set X_h of collocation points,

$$X_h := \{t_{n,i} := t_n + c_i h_n : \; 0 \leq c_1 < \ldots < c_m \leq 1 \; (0 \leq n \leq N-1)\}.$$

If we admit sets $\{c_i\}$ where $c_1 = 0$ and $c_m = 1$ ($m \geq 2$), then – as in the case $\alpha = 0$ – the collocation solution lies in the smoother space $S_m^{(0)}(I_h) \cap C^1(I) =: S_m^{(1)}(I_h)$, provided the given functions f and k in (7.2.1) are continuous. However, since

$$\dim S_m^{(1)}(I_h) = N(m-1) + 2,$$

we need a second, 'artificial', initial condition, $u_h'(0) = y'(0) = f(0, y_0)$, in order to start the recursive process given by the computational form of (7.2.2).

The memory term $(\mathcal{V}_\alpha u_h)(t)$ corresponding to $t = t_{n,i}$ may be written as

$$(\mathcal{V}_\alpha u_h)(t_{n,i}) = F_n(t_{n,i}; \alpha) + h_n \int_0^{c_i} H_\alpha(t_{n,i}, t_n + sh_n) u_h(t_n + sh_n) ds,$$

with lag term $F_n(t;\alpha)$ defined as in (6.2.8),

$$F_n(t;\alpha) := \int_0^{t_n} H_\alpha(t,s) u_h(s) ds, \quad t = t_n + vh_n \in \bar{\sigma}_n. \tag{7.2.3}$$

We will use again the local (Lagrange) representation of $u_h \in S_m^{(0)}(I_h)$ on $\bar{\sigma}_n$, namely,

$$u_h(t_n + vh_n) = y_n + h_n \sum_{j=1}^m \beta_j(v) Y_{n,j}, \quad v \in [0,1],$$
$$\text{with} \;\; Y_{n,j} := u_h'(t_n + c_j h_n), \tag{7.2.4}$$

with $y_n := u_h(t_n)$ and $\beta_j(v) := \int_0^v L_j(s)ds$. The computational form of the collocation equation (7.2.2) on $\bar{\sigma}_n$ then becomes

$$\begin{aligned} Y_{n,i} = {} & f(t_{n,i}, y_n + h_n \sum_{j=1}^m a_{i,j} Y_{n,j}) \\ & + h_n^2 \sum_{j=1}^m \left(\int_0^{c_i} H_\alpha(t_{n,i}, t_n + sh_n) \beta_j(s) ds \right) Y_{n,j} \\ & + F_n(t_{n,i}; \alpha) + h_n \left(\int_0^{c_i} H_\alpha(t_{n,i}, t_n + sh_n) ds \right) y_n \;\; (i = 1, \ldots, m). \end{aligned} \tag{7.2.5}$$

Due to the continuity of u_h on I, the value y_n is given by

$$y_n = u_h(t_n) = y_{n-1} + h_{n-1} \sum_{j=1}^{m} b_j Y_{n-1,j} \quad (n = 1, \ldots, N),$$

with $b_j := \beta_j(1)$ and $y_0 = y(0)$.
In the remainder of this section we will assume that f in (7.2.1) is *linear*,

$$f(t, y) = a(t)y + g(t), \quad \text{with} \quad a, g \in C(I). \tag{7.2.6}$$

The collocation equation to (7.2.5) then assumes the form

$$\begin{aligned} Y_{n,i} - h_n a(t_{n,i}) \sum_{j=1}^{m} a_{i,j} Y_{n,j} - h_n^2 \sum_{j=1}^{m} \left(\int_0^{c_i} H_\alpha(t_{n,i}, t_n + s h_n) \beta_j(s)\, ds \right) Y_{n,j} \\ = g(t_{n,i}) + F_n(t_{n,i}; \alpha) + \left(a(t_{n,i}) + h_n \int_0^{c_i} H_\alpha(t_{n,i}, t_n + s h_n ds \right) y_n \end{aligned} \tag{7.2.7}$$

$(i = 1, \ldots, m)$, where the lag term $F_n(t_{n,i}; \alpha)$ may now be written as

$$\begin{aligned} F_n(t_{n,i}; \alpha) &= \sum_{\ell=0}^{n-1} h_\ell \int_0^1 H_\alpha(t_{n,i}, t_\ell + s h_\ell) \left(y_\ell + h_\ell \sum_{j=1}^{m} \beta_j(s) Y_{\ell,j} \right) ds \\ &= \sum_{\ell=0}^{n-1} h_\ell \left(\int_0^1 H_\alpha(t_{n,i}, t_\ell + s h_\ell) ds \right) y_\ell \\ &\quad + \sum_{\ell=0}^{n-1} h_\ell^2 \sum_{j=1}^{m} \left(\int_0^1 H_\alpha(t_{n,i}, t_\ell + s h_\ell) \beta_j(s) ds \right) Y_{\ell,j}. \end{aligned} \tag{7.2.8}$$

We will employ the vectors

$$\mathbf{Y}_n := (Y_{n,1}, \ldots, Y_{n,m})^T, \quad \mathbf{a}_n := (a(t_{n,1}), \ldots, a(t_{n,m}))^T,$$

$$\mathbf{g}_n := (g(t_{n,1}), \ldots, g(t_{n,m}))^T, \quad \mathbf{G}_n(\alpha) := (F_n(t_{n,1}; \alpha), \ldots, F_n(t_{n,m}; \alpha))^T,$$

and the matrices in $L(\mathbb{R}^m)$,

$$A := \begin{pmatrix} a_{i,j} \\ (i, j = 1, \ldots, m) \end{pmatrix}, \quad A_n := \operatorname{diag}(a(t_{n,i}))A \,,$$

$$C_n(\alpha) := \begin{pmatrix} \int_0^{c_i} H_\alpha(t_{n,i}, t_n + s h_n) \beta_j(s)\, ds \\ (i, j = 1, \ldots, m) \end{pmatrix},$$

$$C_n^{(\ell)}(\alpha) := \begin{pmatrix} \int_0^1 H_\alpha(t_{n,i}, t_\ell + s h_\ell) \beta_j(s)\, ds \\ (i, j = 1, \ldots, m) \end{pmatrix} \quad (\ell < n)$$

(see also (3.2.9)), with $a_{i,j} = \beta_j(c_i)$. Moreover, set

$$\boldsymbol{\kappa}_n(\alpha) := \mathbf{a}_n + h_n \left(\int_0^{c_i} H_\alpha(t_{n,i}, t_n + sh_n)\, ds \;\; (i = 1, \ldots, m) \right)^T \in \mathbb{R}^m$$

and, for $0 \leq \ell < n \leq N - 1$,

$$\boldsymbol{\kappa}_n^{(\ell)}(\alpha) := \left(\int_0^1 H_\alpha(t_{n,i}, t_\ell + sh_\ell) ds \;\; (i = 1, \ldots, m) \right)^T \in \mathbb{R}^m.$$

The system of linear algebraic equations (7.2.7) then becomes

$$[\mathcal{I}_m - h_n(A_n + h_n C_n(\alpha))]\mathbf{Y}_n = \mathbf{g}_n + \mathbf{G}_n(\alpha) + \boldsymbol{\kappa}_n(\alpha) y_n, \tag{7.2.9}$$

where $n = 0, 1, \ldots, N - 1$. Observe that the lag term $\mathbf{G}_n(\alpha)$ has the form

$$\mathbf{G}_n(\alpha) = \sum_{\ell=0}^{n-1} h_\ell^2 C_n^{(\ell)}(\alpha)\mathbf{Y}_\ell + \sum_{\ell=0}^{n-1} h_\ell \boldsymbol{\kappa}_\ell(\alpha) y_\ell.$$

When the solution $\mathbf{Y}_n$ of (7.2.9) has been found, the collocation solution on the interval $\bar{\sigma}_n$ is determined by

$$u_h(t_n + vh_n) = y_n + h_n \boldsymbol{\beta}^T(v)\mathbf{Y}_n, \quad v \in [0, 1], \tag{7.2.10}$$

where $\boldsymbol{\beta}(v) := (\beta_1(v), \ldots, \beta_m(v))^T \in \mathbb{R}^m$.

Theorem 7.2.1 *Assume that the functions a, g and K in the VIDE (7.2.1), with f given by (7.2.6), are continuous on their respective domains I and D. Then for any $\alpha \in (0, 1]$ there exists an $\bar{h} = \bar{h}(\alpha) > 0$ so that for any mesh I_h with mesh diameter $h \in (0, \bar{h})$, each of the linear algebraic systems (7.2.9) has a unique solution $\mathbf{Y}_n \in \mathbb{R}^m$. Hence the collocation equation (7.2.2) defines a unique collocation solution $u_h \in S_m^{(0)}(I_h)$ for the initial-value problem (7.2.1), (7.2.6), and its representation on the subinterval $\bar{\sigma}_n$ is given by (7.2.10).*

Proof It follows from the assumptions on a and K, and because the kernel H_α is integrable for all $\alpha \in (0, 1]$, that the matrices

$$\mathcal{C}_n(\alpha) := A_n + h_n C_n(\alpha) \quad (0 \leq n \leq N - 1)$$

in (7.2.9) have bounded elements for any mesh I_h. The argument in the proof of Theorem 2.2.1 can thus be used to deduce that the inverses $[\mathcal{I}_m - h_n\mathcal{C}_n(\alpha)]^{-1}$ exist and are uniformly bounded for $h_n \in (0, \bar{h})$, with sufficiently small $\bar{h} > 0$. This implies that each of the systems $[\mathcal{I}_m - h_n\mathcal{C}_n(\alpha]\mathbf{Y}_n = \mathbf{g}_n + \mathbf{G}_n(\alpha) + h_n\boldsymbol{\kappa}_n(\alpha)y_n$ is uniquely solvable for $\mathbf{Y}_n \in \mathbb{R}^m$ when $h = \max_{(n)} h_n < \bar{h}$. Hence, for each $n = 0, 1, \ldots, N - 1$ the local representation (7.2.10) is uniquely determined.

Example 7.2.1 $u_h \in S_1^{(0)}(I_h)$ $(m = 1)$, $0 < c_1 =: \theta \le 1$, $t_{n,1} = t_n + \theta h_n$: Here we have, as in Example 3.2.1, $\beta_1(v) = v$, $A = a_{1,1} = \theta$, and

$$u_h(t_n + vh_n) = (1 - v)y_n + vy_{n+1}, \quad v \in [0, 1], \quad y_n = u_h(t_n) \tag{7.2.11}$$

(since $u_h(t_n + vh_n) = y_n + vh_n Y_{n,1}$ yields, for $v = 1$, $h_n Y_{n,1} = y_{n+1} - y_n$). It thus follows that, in analogy to (3.2.6), y_{n+1} is given by the solution of the linear algebraic equation

$$\begin{aligned}&\left(1 - \theta h_n a(t_{n,1}) - h_n^2 \int_0^\theta H_\alpha(t_{n,1}, t_n + sh_n)s\,ds\right) y_{n+1}\\ &= h_n g(t_{n,1}) + h_n F_n(t_{n,1};\alpha)) + \Bigg(1 + (1-\theta)h_n a(t_{n,1})\\ &\quad + h_n^2 \int_0^\theta H_\alpha(t_{n,1}, t_n + sh_n)(1-s)ds\Bigg) y_n\end{aligned}$$

with

$$H_\alpha(t_{n,1}, t_n + sh_n) = p_\alpha((\theta - s)h_n)K(t_n + \theta h_n, t_n + sh_n),$$

and with lag term

$$F_n(t_{n,1};\alpha) = \sum_{\ell=0}^{n-1} h_\ell \int_0^1 H_\alpha(t_{n,1}, t_\ell + sh_\ell)[(1-s)y_\ell + sy_{\ell+1}]ds\ .$$

This method will be referred to as the *(exact) continuous θ-method* for the linear weakly singular VIDE (7.2.1),(7.2.6). Its *nonlinear* counterpart is given by (7.2.11) and by

$$\begin{aligned}y_{n+1} &= y_n + h_n f(t_{n,1}, (1-\theta)y_n + \theta y_{n+1}) + F_n(t_{n,1};\alpha)\\ &\quad + h_n^2 \int_0^\theta h_\alpha(t_{n,1}, t_n + sh_n, (1-s)y_n + sy_{n+1})ds,\end{aligned}$$

where now

$$F_n(t_{n,1};\alpha) := \sum_{\ell=0}^{n-1} h_\ell \int_0^1 h_\alpha(t_{n,1}, t_\ell + sh_\ell, (1-s)y_\ell + sy_{\ell+1})ds.$$

For $\theta = 1/2$ we obtain the *continuous implicit (product) midpoint method.*

Example 7.2.2 $u_h \in S_2^{(0)}(I_h)$ $(m = 2)$, $0 < c_1 < c_2 \le 1$:
Here, as in Example 3.2.2,

$$\beta_1(v) = \int_0^v L_1(s)ds = \frac{v(2c_2 - v)}{2(c_2 - c_1)},$$

$$\beta_2(v) = \int_0^v L_2)ds = \frac{v(v - 2c_1)}{2(c_2 - c_1)},$$

which permits the computation of the elements of the matrix A, $a_{i,j} = \beta_j(c_i)$ $(i, j = 1, 2)$ (compare also Example 1.1.2). The elements of the matrix $C_n(\alpha) \in L(\mathbb{R}^2)$ in (7.2.9) are

$$(C_n(\alpha))_{i,1} = \frac{1}{2(c_2 - c_1)} \int_0^{c_i} H_\alpha(t_{n,i}, t_n + sh_n)s(2c_2 - s)\, ds \quad (i = 1, 2),$$

and

$$(C_n(\alpha))_{i,2} = \frac{1}{2(c_2 - c_1)} \int_0^{c_i} H_\alpha(t_{n,i}, t_n + sh_n)s(s - 2c_1)\, ds \quad (i = 1, 2).$$

7.2.2 The fully discretised collocation equations

The (exact) collocation equation (7.2.5) for the VIDE (7.2.1) is amenable to numerical computations of u_h only if the integrals in the equation (and in the lag term (7.2.8)) can be found analytically. Since this will in general not be possible, they will have to be approximated by appropriate numerical (product) quadrature processes which, as in Section 6.2.1, will again be interpolatory m-point *product quadrature formulas* whose abscissas are based on the collocation parameters $\{c_i\}$. Hence, using the notation of (6.2.20) and (6.2.21), the fully discretised version of (7.2.5) is

$$\hat{Y}_{n,i} - h_n a(t_{n,i}) \sum_{j=1}^{m} a_{i,j} \hat{Y}_{n,j} - h_n^2 (\hat{Q}_n(\alpha)\hat{u}_h)(t_{n,i})$$
$$= g(t_{n,i}) + \hat{F}_n(t_{n,i}) + \left(a(t_{n,i}) + h_n \sum_{j=1}^{m} w_{n,j}(c_i; \alpha) K(t_{n,i}, t_n + c_i c_j) \right) \hat{y}_n \tag{7.2.12}$$

$(i = 1, \ldots, m)$, where the discretised lag term has the form

$$\hat{F}_n(t_{n,i}; \alpha) := \sum_{\ell=0}^{n-1} h_\ell (\hat{Q}_n^{(\ell)}(\alpha)\hat{u}_h)(t_{n,i}). \tag{7.2.13}$$

As mentioned above, we have employed the product quadrature approximations of Section 6.2.3,

$$(\hat{Q}_n(\alpha)\hat{u}_h)(t_{n,i}) := \sum_{k=1}^{m} w_{n,k}(c_i; \alpha) K(t_{n,i}, t_n + c_i c_k h_n)\hat{u}_h(t_n + c_i c_k h_n) \tag{7.2.14}$$

and, for $\ell < n$,

$$(\hat{Q}_n^{(\ell)}(\alpha)\hat{u}_h)(t_{n,i}) := \sum_{\ell=0}^{n-1} \sum_{k=1}^{m} w_{n,k}(c_i; \alpha) K(t_{n,i}, t_\ell + c_k h_\ell)\hat{u}_h(t_\ell + c_k h_\ell). \tag{7.2.15}$$

Here, the product quadrature weights are as in (6.2.22) and (6.2.23). Since $\hat{u}_h$ is locally given by

$$(\hat{u}_h(t_\ell + vh_\ell) = \hat{y}_\ell + h_\ell \sum_{j=1}^{m} \beta_j(v)\hat{Y}_{\ell,j}), \quad v \in [0, 1], \quad \text{with } \hat{Y}_{n,j} := \hat{u}_h(t_{n,j}),$$

we can now write

$$(\hat{Q}_n(\alpha) = \sum_{k=1}^{m} w_{n,k}(c_i;\alpha)K(t_{n,i}, t_n + c_i c_k h_n)\hat{y}_n$$
$$+ h_n \sum_{j=1}^{m} \left(\sum_{k=1}^{m} w_{n,k} K(t_{n,i}, t_n + c_i c_k h_n)\beta_j(c_i c_k) \right) \hat{Y}_{n,j},$$

$$(\hat{Q}_n^{(\ell)}(\alpha) = \sum_{\ell=0}^{n-1} \sum_{k=1}^{m} w_{n,k}(c_i;\alpha)K(t_{n,i}, t_\ell + c_k h_\ell)\hat{y}_\ell$$
$$+ \sum_{\ell=0}^{n-1} h_\ell \sum_{j=1}^{m} \left(\sum_{k=1}^{m} w_{n,k}(c_i;\alpha)K(t_{n,i}, t_\ell + c_k h_\ell)\beta_j(c_k) \right) \hat{Y}_{\ell,j} \quad (\ell < n).$$

The solution $\hat{\mathbf{Y}}_n := (\hat{Y}_{n,1}, \ldots, \hat{Y}_{n,m})^T \in \mathbb{R}^m$ of the linear algebraic system (7.2.12) determines the discretised collocation solution on the subinterval $\bar{\sigma}_n$:

$$\hat{u}_h(t_n + vh_n) = \hat{y}_n + h_n \sum_{j=1}^{m} \beta_j(v)\hat{Y}_{n,j}, \quad v \in [0, 1], \tag{7.2.16}$$

with

$$\hat{y}_n := \hat{u}_h(t_n) = \hat{y}_{n-1} + h_{n-1} \sum_{j=1}^{m} b_j \hat{Y}_{n-1,j}.$$

In order to state and prove the result on the existence and uniqueness of the discretised collocation solution on I, we write (7.2.12) in a more concise form that reflects the fully discretised analogue of (7.2.9), namely

$$[\mathcal{I}_m - h_n(A_n + h_n \hat{C}_n(\alpha))]\hat{\mathbf{Y}}_n = \mathbf{g}_n + \hat{\mathbf{G}}_n + \hat{\boldsymbol{\kappa}}(\alpha)\hat{y}_n \quad (n = 0, 1, \ldots, N-1), \tag{7.2.17}$$

with

$$\hat{C}_n(\alpha) := \begin{pmatrix} \sum_{k=1}^{m} w_{n,k}(c_i;\alpha)K(t_{n,i}, t_n + c_i c_k h_n)\beta_j(c_i c_k) \\ (i, j = 1, \ldots, m) \end{pmatrix},$$

and

$$\hat{\mathbf{G}}_n(\alpha) := (\hat{F}_n(t_{n,1};\alpha), \ldots, \hat{F}_n(t_{n,m};\alpha))^T.$$

The latter may be written as

$$\hat{\mathbf{G}}_n(\alpha) = \sum_{\ell=0}^{n-1} h_\ell \hat{\boldsymbol{\kappa}}_n^{(\ell)} \hat{y}_\ell + \sum_{\ell=0}^{m} h_\ell \hat{C}_n^{(\ell)} \hat{\mathbf{Y}}_\ell;$$

here,

$$\hat{C}_n^{(\ell)}(\alpha) := \begin{pmatrix} \sum_{k=1}^{m} w_{n,k}^{(\ell)}(c_i;\alpha) K(t_{n,i}, t_\ell + c_k h_\ell) \beta_j(c_k) \\ (i, j = 1, \ldots, m) \end{pmatrix} \quad (\ell < n),$$

with $\beta_j(c_k) = a_{k,j}$. The vectors $\hat{\kappa}_n(\alpha)$, $\hat{\boldsymbol{\kappa}}_n^{(\ell)}(\alpha) \in \mathbb{R}^m$ are defined by

$$\hat{\boldsymbol{\kappa}}_n(\alpha) := \mathbf{a}_n + h_n \left(\sum_{k=1}^{m} w_{n,k}(c_i;\alpha) K(t_{n,i}, t_n + c_i c_k h_n) \;\; (i = 1, \ldots, m) \right)^T \tag{7.2.18}$$

and, for $\ell < n$, by

$$\hat{\boldsymbol{\kappa}}_n^{(\ell)}(\alpha) := \left(\sum_{k=1}^{m} w_{n,k}^{(\ell)}(c_i;\alpha) K(t_{n,i}, t_\ell + c_k h_\ell) \;\; (i = 1, \ldots, m) \right)^T, \tag{7.2.19}$$

respectively.

Theorem 7.2.2 *Assume that the given functions a, g and K in the linear weakly singular VIDE (7.2.1), (7.2.6) satisfy the conditions of Theorem 7.2.1, and let $\alpha \in (0, 1]$. If the corresponding exact collocation equation (7.2.7) is discretised by means of the interpolatory m-point product quadrature formulas (7.2.14), (7.2.15), then there exists an $\hat{h} = \hat{h}(\alpha) > 0$ so that for any mesh I_h with mesh diameter $h \in (0, \hat{h})$, each of the linear systems (7.2.17) has a unique solution $\hat{\mathbf{Y}}_n \in \mathbb{R}^m$. Hence the discretised collocation equation (7.2.12) defines a unique discrete collocation solution $\hat{u}_h \in S_m^{(0)}(I_h)$ whose restriction to $\bar{\sigma}_n$ is given by (7.2.16).*

The **proof** is a straightforward adaptation of the one for Theorem 7.2.1 (or Theorem 6.2.2): for fixed $m \geq 1$ the weights of the above interpolatory m-point quadrature formulas are bounded for all $h > 0$, and hence, by the assumed continuity of a and K, the matrices $\hat{\mathcal{C}}_n(\alpha) := A_n + h_n \hat{C}_n(\alpha) \in L(\mathbb{R}^m)$ have bounded elements for any h_n. This implies that the inverses of the matrices characterising the systems (7.2.16), $\mathcal{I}_m - h_n \hat{\mathcal{C}}_n(\alpha)$ $(n = 0, 1, \ldots, N-1)$, exist and are uniformly bounded for $h_n \in (0, \hat{h})$ for some $\hat{h} > 0$ which depends on α and will in general be different from $\bar{h}$ defined in Theorem 7.2.1.

Example 7.2.3 m = 1 (*discretised θ-method*): It follows from Example 7.2.1 that this method is given by

$$\hat{u}_h(t_n + vh_n) = \hat{y}_n + (1 - v)\hat{y}_n + v\hat{y}_{n+1}, \quad v \in [0, 1],$$

and

$$\begin{aligned}
&\left(1 - \theta h_n a(t_{n,1}) - h_n^2\theta^2 w_{n,1}(\theta;\alpha)K(t_n + \theta^2 h_n)\right)\hat{y}_{n+1} \\
&\quad = h_n g(t_{n,1}) + h_n \hat{F}_n(t_{n,1};\alpha) + \Bigg(1 + (1-\theta)h_n a(t_{n,1}) \\
&\quad + h_n^2(1-\theta^2)w_{n,1}(\theta;\alpha)K(t_{n,1}, t_n + \theta^2 h_n)\Bigg)\hat{y}_n.
\end{aligned}$$

In the nonlinear case when $h_\alpha(t, s, y) := p_\alpha(t - s)k(t, s, y)$, the method is described by

$$\begin{aligned}
\hat{y}_{n+1} &= \hat{y}_n + h_n f(t_{n,1}, (1-\theta)\hat{y}_n + \theta\hat{y}_{n+1}) + h_n\hat{F}_n(t_{n,1};\alpha) \\
&\quad + h_n^2 w_{n,1}(\theta;\alpha)k(t_{n,1}, t_n + \theta^2 h_n, (1-\theta^2)\hat{y}_n + \theta^2\hat{y}_{n+1}),
\end{aligned}$$

with discretised lag term

$$\hat{F}_n(t_{n,1};\alpha) := \sum_{\ell=0}^{n-1} h_\ell w_{n,1}^{(\ell)}(\theta;\alpha)k(t_{n,1}, t_\ell + \theta h_\ell, (1-\theta)\hat{y}_\ell + \theta\hat{y}_{\ell+1}).$$

The product quadrature weights are

$$w_{n,1}(\theta;\alpha) = \int_0^\theta p_\alpha((v - s)h_n)ds$$

and

$$w_{n,1}^{(\ell)}(\theta;\alpha) = \int_0^1 p_\alpha((t_n + vh_n - t_\ell)/h_\ell - s)h_\ell)ds \quad (\ell < n)$$

(compare also Example 6.2.3). The method corresponding to $\theta = 1/2$ is the *discretised implicit (product) midpoint method* for (7.2.1).

7.2.3 Global convergence results

We have seen in Section 7.1 that, typically, VIDEs with weakly singular kernels but otherwise smooth data possess solutions that have an unbounded second derivative at the left endpoint of the interval of integration. Thus, in analogy to the results for weakly singular VIEs of the second kind, collocation solutions in $S_m^{(0)}(I_h)$ with *uniform* mesh I_h will not converge with optimal (global) order $p = m$. This result, and the one on how to recover optimal order, can be established in two different ways: we can either take the proof of the convergence result for VIDEs with *smooth solutions* (Theorem 3.2.3) as our starting point, with appropriate modifications – similar to the proof of Theorem 6.2.9 for weakly singular VIEs – of the remainder terms $R_{m+1,0}(v)$ and $R^{(1)}_{m+1,0}(v)$ in the local representations of e_h and e'_h on the initial interval σ_0 (recall (3.2.26) and (3.2.27)). Alternatively, we may use the integrated form of the VIDE, as discussed in Section 6.1.1, whose kernel is bounded but non-smooth, and then apply Theorem 6.2.12 to the resulting second-kind VIE.

We begin by stating the basic global convergence result for the collocation solution $u_h \in S_m^{(0)}(I_h)$ to the linear VIDE

$$y'(t) = a(t)y(t) + g(t) + (\mathcal{V}_\alpha y)(t), \quad t \in I := [0, T], \tag{7.2.20}$$

where

$$(\mathcal{V}_\alpha y)(t) := \int_0^t p_\alpha(t-s)K(t,s)y(s)ds \quad (0 < \alpha < 1),$$

with $K(t,t) \neq 0,\ t \in I$. VIDEs with logarithmic kernel singularity ($\alpha = 1$) will be considered in Theorem 7.2.5. Recall that our graded meshes on $I := [0, T]$ are defined by

$$I_h := \{t_n := (n/N)^r T : 0 \le n \le N;\ r = r(\alpha) \ge 1\}.$$

Theorem 7.2.3 *Assume*

(a) The given functions in (7.2.20) satisfy $a, g \in C^m(I)$, $K \in C^m(D)$, with $K(t,t) \neq 0$ for $t \in I$.
(b) In the weakly singular part of $H_\alpha(t,s) := p_\alpha(t-s)K(t,s)$ we have $0 < \alpha < 1$.
(c) $u_h \in S_m^{(0)}(I_h)$ is the (unique) collocation solution to (7.2.20) defined by (7.2.9), (7.2.10), with $h \in (0, \bar{h})$ and collocation points X_h.
(d) The grading exponent $r = r(\alpha)$ has the form

$$r = \frac{\mu}{1-\alpha}, \quad \text{with} \quad \mu \ge 1-\alpha.$$

Then, setting $h := T/N$, the estimates

$$||y^{(\nu)} - u_h^{(\nu)}||_\infty \le C_\nu(r) \begin{cases} h^\mu & \text{if } 1-\alpha \le \mu < m, \\ h^m & \text{if } \mu \ge m \end{cases} \tag{7.2.21}$$

hold for $\nu = 0, 1$ and any set X_h of collocation points with $0 \le c_1 < \ldots < c_m \le 1$. The constants $C_\nu(r)$ depend on the collocation parameters $\{c_i\}$ and on the grading exponent $r = r(\alpha)$, but not on h.

Proof The collocation error $e_h := y - u_h$ satisfies the initial-value problem

$$e'_h(t) = a(t)e_h(t) + (\mathcal{V}e_h)(t) - \delta_h(t), \quad t \in I, \quad e_h(0) = 0. \tag{7.2.22}$$

The defect δ_h is defined by

$$\delta_h(t) := -u_h'(t) + a(t)u_h(t) + g(t) + (\mathcal{V}_\alpha u_h)(t), \quad t \in I,$$

and vanishes on X_h. Recall now the analogous error equation (3.2.25) for VIDEs, as well as the local representations (3.2.27) and (3.2.26) for e_h and e_h', respectively:

$$e_h(t_n + vh_n) = e_h(t_n) + h_n \sum_{j=1}^{m} \beta_j(v)\mathcal{E}_{n,j} + h_n^{m+1} R_{m+1,n}(v), \quad v \in [0, 1], \tag{7.2.23}$$

and

$$e_h'(t_n + vh_n) = \sum_{j=1}^{m} L_j(v)\mathcal{E}_{n,j} + h_n^m R_{m+1,n}^{(1)}(v), \quad v \in (0, 1], \tag{7.2.24}$$

with $\mathcal{E}_{n,j} := Z_{n,j} - Y_{n,j}$. Since, according to Theorem 7.1.4, the solution y has an unbounded second derivative at $t = 0^+$, these representations are only valid for $n \geq 1$. On the first subinterval $\bar{\sigma}_0 = [0, h_0]$ we resort to the representation (7.1.14) (Theorem 7.1.4) for the exact solution of (7.2.20) and the resulting analogue to the error representation (6.2.39), with m replaced by $m + 1$. The convergence analysis proceeds now along the lines we have mapped out in the proofs of Theorem 3.2.3 and Theorem 6.2.9.

In order to avoid these repetitive arguments (the reader may wish to consult Brunner (1985b, 1985c, 1986a) or Brunner and van der Houwen (1986, Chapter 6), we will describe a somewhat different approach to establishing the results of Theorem 7.2.3. It is based on the easily verified fact that we may without loss of generality consider the VIDE

$$y'(t) = g(t) + (\mathcal{V}_\alpha y)(t), \quad t \in I,$$

since the deleted term $a(t)y(t)$ of (7.2.20) has, according to Theorem 7.1.4, no smoothing effect on the solution. The error equation is then, for $0 < \alpha < 1$,

$$e_h'(t) = \delta_h(t) + \int_0^t (t - s)^{-\alpha} K(t, s) e_h(s) ds, \quad t \in I.$$

Using the initial condition $e_h(0) = 0$ we may rewrite it as

$$\begin{aligned} e_h'(t) &= \delta_h(t) + \int_0^t (t - s)^{-\alpha} K(t, s) \left(\int_0^s e_h'(v) dv \right) ds \\ &= \delta_h(t) + \int_0^t \left(\int_v^t (t - s)^{-\alpha} K(t, s) ds \right) e_h'(v) dv, \end{aligned}$$

or, setting

$$K_0(t, s; \alpha) := \int_s^t (t - v)^{-\alpha} K(t, v) dv,$$

as

$$e_h'(t) = \delta_h(t) + \int_0^t K_0(t,s;\alpha)e_h'(s)ds, \quad t \in I. \tag{7.2.25}$$

This equation is similar to (6.2.36) except that now the role of e_h is assumed by e_h'. Note that

$$\begin{aligned} K_0(t,s;\alpha) &= (t-s)^{1-\alpha}\int_0^1 (1-z)^{-\alpha}K(t,(t-s)z+s)dz \\ &=: (t-s)^{1-\alpha}H_0(t,s;\alpha), \quad (t,s) \in D, \end{aligned}$$

where $H_0(\cdot,\cdot;\alpha)$ inherits the assumed regularity of K in (7.2.20).

We know from Theorem 7.1.4 that $y' \in C^{m,1-\alpha}(I)$. Hence, on $\bar{\sigma}_0$, $e_h'(t)$ admits a local representation of the form (6.2.38) (cf. Theorem 6.1.6)), namely

$$e_h'(t_0 + vh_0) = \sum_{j=0}^{m-1}\beta_{j,0}(\alpha)v^j + h_0^{1-\alpha}\Phi_{m,0}(v;\alpha) + h_0^m R_{m,0}(v;\alpha), \quad v \in [0,1],$$

with appropriately adapted meaning of the coefficients $\beta_{j,0}(\alpha)$ and the remainder terms $R_{m,0}(v;\alpha)$ (recall (6.2.39)). These observations imply that we may now proceed exactly as in the proof of Theorem 6.2.9, to show that

$$|e_h'(t_0 + vh_0)| \le ||\beta_0(\alpha)||_1 + \gamma_0(\alpha)h_0^{1-\alpha} + \gamma_1(\alpha)h_0^m, \quad v \in [0,1],$$

where $||\beta_0(\alpha)||_1 \le Bh_0^{1-\alpha}$. If the grading exponent defining the graded mesh I_h is given by $r = \mu/(1-\alpha)$ then we obtain first, in analogy to the proof of Theorem 6.2.9, the estimate $||e_h'||_{0,\infty} = \mathcal{O}(h^\mu)$, and then, continuing as indicated,

$$||e_h'||_\infty \le C_1(r)\begin{cases} h^\mu & \text{if } 1-\alpha \le \mu \le m, \\ h^m & \text{if } \mu \ge m. \end{cases}$$

This holds for any set $\{c_i\}$ of collocation parameters.

Consider now the collocation error e_h itself: we have, for $n = 0, 1, \ldots, N-1$,

$$e_h(t_n + vh_n) = e_h(t_n) + h_n\int_0^v e_h'(t_n + sh_n)ds, \quad v \in [0,1],$$

with $e_h(t_0) = e_h(0) = 0$. Since

$$|e_h(t_1)| \le h_0||e_h'||_{0,\infty} \le C_1(r)h_0^{\mu+1} \quad (1-\alpha \le \mu \le m),$$

and, for $2 \le n \le N-1$,

$$e_h(t_n) = e_h(t_1) + \sum_{\ell=1}^{n-1}h_\ell\int_0^1 e_h'(t_\ell + sh_\ell)ds,$$

it follows that

$$|e_h(t_n)| \leq |e_h(t_1)| + \sum_{\ell=1}^{n-1} h_\ell ||e_h'||_\infty \leq C_1(r)h_0^{1-\alpha} + C_1(r)h^\mu T.$$

Hence, for $v \in [0, 1]$, we find

$$\begin{aligned} |e_h(t_n + vh_n)| &\leq |e_h(t_n)| + h_n \int_0^v |e_h'(t_n + sh_n)| ds \\ &\leq |e_h(t_1)| + \sum_{\ell=1}^{n-1} h_\ell ||e_h'||_\infty + h_n v ||e_h'||_\infty. \end{aligned}$$

The asserted $\mathcal{O}(h^\mu)$-convergence of $||e_h||_\infty$, with $1 - \alpha \leq \mu \leq m$, now follows immediately since $h_0^{1-\alpha} = (N^{-r}T)^{1-\alpha} = T^{1-\alpha}N^{-\mu}$ (where we have set $h := T/N$).

Remarks

1. It is possible to show that on *uniform* meshes one obtains in fact the slightly better order estimate $||e_h||_\infty = \mathcal{O}(h^{2-\alpha})$ (see Tang (1992, 1993a)).
2. Tang (1992) showed that the collocation solution $u_h \in S_m^{(0)}(I_h)$ corresponding to collocation parameters with $J_0 = 0$ and graded meshes with $r \geq (m + 1 - \alpha)/(2 - \alpha)$ satisfies

   $$||e_h||_\infty = \mathcal{O}(N^{-(m+1-\alpha)}) \quad \text{and} \quad \max_{t \in X_h} |e_h'(t)| = \mathcal{O}(N^{-(m+1-\alpha)}).$$

   Moreover, the choice $r > m/(2 - \alpha)$ implies $||e_h||_\infty = \mathcal{O}(N^{-m})$ for any set $\{c_i\}$ (Tang (1993a)).

The papers by Brunner, Pedas and Vainikko (2001a, 2001b) contain a complete global convergence and *superconvergence analysis* for linear VIDEs with weakly singular kernels (see also Kangro and Parts (2003) for related results). It complements the analysis in Tang (1992, 1993a) not only by admitting logarithmic kernel singularities but also by giving optimal L^p-estimates. We cite two typical results. Their proofs can be found in the above-mentioned papers.

Theorem 7.2.4 *Let* $0 < \alpha < 1,\ d \geq m$, *and assume:*

(a) $a,\ g \in C^d(I)$*;*
(b) $K \in C^d(D)$*, and* $K(t, t) \neq 0$ *on* I*;*
(c) u_h *is the collocation solution to (7.2.20) in* $S_m^{(0)}(I_h)$*, with graded mesh* I_h *governed by some grading exponent* $r \geq 1$*.*

The following estimates are true:

(i) If $d = m \geq 2$*, then*

$$||y - u_h||_\infty \leq C_0(r) \begin{cases} h^{r(2-\alpha)} & \text{if} \quad 1 \leq r < m/(2-\alpha), \\ h^m(1 + |\log(h)|) & \text{if} \quad r = m/(2-\alpha), \\ h^m & \text{if} \quad r > m/(2-\alpha), \end{cases}$$

where we have set $h := T/N$*. This holds for any choice of the set* $\{c_i\}$*.*

(ii) If, in addition to the above assumptions, the set $\{c_i\}$ *is such that*

$$J_0 := \int_0^1 \prod_{i=1}^{m} (s - c_i)ds = 0,$$

then we obtain

$$||y - u_h||_\infty \leq C_0(r)h^{m+1-\alpha},$$

provided we have $d \geq m + 1$ *and* $r \geq (m + 1 - \alpha)/(2 - \alpha)$*.*

Consider now the case where the weak singularity is of *logarithmic type*. The following result was also established in Brunner, Pedas and Vainikko (2001a, 2001b).

Theorem 7.2.5 *Let* $\alpha = 1$ *and assume that (a), (b) and (c) of Theorem 7.2.4 hold.*

(i) If $m = 1$*, then*

$$||y^{(\nu)} - u_h^{(\nu)}||_\infty \leq C_\nu(r) \begin{cases} h \cdot (1 + |\log(h)|) & \text{if} \quad r = 1, \\ h & \text{if} \quad r > 1, \end{cases}$$

is true for $\nu = 0, 1$ *and any* $c_1 \in [0, 1]$*.*

(ii) If $m \geq 2$*, then*

$$||y^{(\nu)} - u_h^{(\nu)}||_\infty \leq C_\nu(r) \begin{cases} h^r & \text{if} \quad 1 \leq r \leq m, \\ h^m & \text{if} \quad r > m, \end{cases}$$

holds for $\nu = 0, 1$ *and arbitrary* $0 \leq c_1 < \ldots < c_m \leq 1$*.*

(iii) The first of the preceding estimates for $m \geq 2$ *can be refined:*

$$||y - u_h||_\infty \leq C_0(r) \begin{cases} h^{2r} & \text{if} \quad 1 \leq r < m/2, \\ h^m(1 + |\log(h)|) & \text{if} \quad r = m/2, \\ h^m & \text{if} \quad r > m/2. \end{cases}$$

7.3 Hammerstein-type VIDEs with weakly singular kernels

We briefly consider the nonlinear VIDE

$$y'(t) = g(t) + \int_0^t p_\alpha(t-s)K(t,s)G(s,y(s))ds, \quad t \in I, \qquad (7.3.1)$$

with $y(0) = y_0$, $\alpha \in (0,1]$ and $K(t,t) \neq 0$ on I. Systems of this kind arise for example in the spatial semidiscretisation of certain partial VIDEs with weakly singular kernels, as studied by, e.g. Lubich, Sloan and Thomée (1996) and McLean and Thomée (1997). Compare also Chapter 7 in the monograph by Chen and Shih (1998).

As an alternative to using 'direct' collocation in $S_m^{(0)}(I_h)$, with appropriate mesh grading as discussed in Section 7.2.1, we rewrite this VIDE as a second-kind VIE,

$$y(t) = g_0(t) + \int_0^t K_0(t,s;\alpha)G(y(s))ds, \quad t \in I, \qquad (7.3.2)$$

where

$$g_0(t) := y_0 + \int_0^t g(s)ds \quad \text{and} \quad K_0(t,s;\alpha) := \int_s^t p_\alpha(v-s)K(v,s)dv.$$

If the solution of this VIE is approximated by $u_h \in S_{m-1}^{(-1)}(I_h)$, followed by an iteration step to generate u_h^{it}, then (by Theorem 6.2.12)

$$||y - u_h^{it}||_\infty \leq C(r)h^m \quad \text{if} \quad r \geq m/(2-\alpha).$$

Since the VIE (7.4.2) is of *Hammerstein* type, it can also be solved by *implicitly linear collocation*, especially if $K(t,s)$ is constant on D. Setting $z(t) := G(t, y(t))$ we obtain

$$z(t) = G\left(t, g_0(t) + \int_0^t K_0(t,s;\alpha)z(s)ds\right),$$

and hence

$$y(t) = g_0(t) + \int_0^t K_0(t,s;\alpha)z(s)ds, \quad t \in I.$$

As we have seen in Section 6.2.9, this approach will often avoid the need of having to resort to quadrature approximations in order to make the 'direct' collocation equations amenable to numerical computations. For details, including convergence estimates based on optimally graded meshes, we refer to Section 6.2.9 and Theorem 6.2.16: in the latter, the role of the optimal grading exponent is now assumed by

$$r = \mu/(2-\alpha), \quad \mu \geq m.$$

7.4 Higher-order weakly singular VIDEs

In Section 7.1.3 we introduced the first-order VIDE,

$$y'(t) = f(t, y(t)) + \int_0^t p_\alpha(t-s)k(t, s, y(s), y'(s))ds, \quad t \in I, \quad y(0) = y_0, \tag{7.4.1}$$

and its linear version,

$$y'(t) = a(t)y(t) + g(t) + (\mathcal{V}_{\alpha,1}y)(t) + (\mathcal{V}_{\alpha,2}y')(t), \tag{7.4.2}$$

with

$$(\mathcal{V}_{\alpha,1}\phi)(t) := \int_0^t p_\alpha(t-s)K_1(t,s)\phi(s)ds,$$

and

$$(\mathcal{V}_{\alpha,2}\phi)(t) := \int_0^t p_\alpha(t-s)K_2(t,s)\phi(s)ds,$$

as special cases of higher-order neutral VIDEs. In this section we shall derive the collocation equations and corresponding convergence results for the latter, and so obtain the analogues of Theorems 3.2.11–3.2.13 (Section 3.2.6) for (3.2.44). They then yield as special cases convergence order estimates for (7.4.1) and (7.4.2).

Let $k \geq 2$ be a given integer and consider the initial-value problem

$$\begin{aligned} &y^{(k)}(t) = f(t, y(t), y'(t), \ldots, y^{(k-1)}(t)) + (\mathcal{V}_\alpha y)(t), \quad t \in I := [0, T], \\ &y^{(\nu)}(0) = y_0^{(\nu)} \quad (\nu = 0, 1, \ldots, k-1), \end{aligned} \tag{7.4.3}$$

where, as in Section 7.1.3,

$$(\mathcal{V}_\alpha y)(t) := \int_0^t h_\alpha(t, s, y(s), y'(s), \ldots, y^{(k)}(s))ds,$$

with

$$h_\alpha(t, s, y, y', \ldots, y^{(k)}) := p_\alpha(t-s)k(t, s, y, y', \ldots, y^{(k)})$$

and $0 < \alpha \leq 1$.

We will first focus on its linear counterpart, described by

$$f(t, y, y', \dots, y^{(k-1)}) = \sum_{\nu=0}^{k-1} a_\nu(t) y^{(\nu)}, \tag{7.4.4}$$

$$k(t, s, y, y', \dots, y^{(k)}) = \sum_{\nu=0}^{k} H_{\alpha,\nu}(t, s) y^{(\nu)}, \tag{7.4.5}$$

where

$$H_{\alpha,\nu}(t, s) := p_\alpha(t - s) K_\nu(t, s) \quad (\nu = 0, 1, \dots, k).$$

The given functions a_ν and K_ν are assumed to be continuous on I and D, respectively.

The collocation solution for (7.4.3) will be sought in the 'natural' smooth piecewise polynomial space

$$S_{m+d}^{(d)}(I_h) := \{u_h \in C^d(I) : \; u_h|_{\bar{\sigma}_n} \in \pi_{m+d} \; (0 \le n \le N-1)\}$$

with $d = k - 1 \ge 1$ and, as the reader will recall, dim $S_{m+d}^{(d)}(I_h) = Nm + d + 1 = Nm + k$. This collocation solution u_h is thus defined by

$$u_h^{(k)}(t) = f(t, u_h(t), u_h'(t), \dots, u_h^{(k-1)}(t)) + (\mathcal{V}_\alpha u_h)(t), \quad t \in X_h, \tag{7.4.6}$$
$$u_h^{(\nu)}(0) = y_0^{(\nu)} \quad (\nu = 0, 1, \dots, k-1),$$

where $X_h := \{t_n + c_i h_n : \; 0 \le c_1 < \dots < c_m \le 1 \; (0 \le n \le N-1)\}$. Setting

$$y_n^{(\nu)} := u_h^{(\nu)}(t_n), \quad y_n := y_n^{(0)}, \quad Y_{n,j} := u_h^{(k)}(t_{n,j}),$$

and

$$u_h^{(k)}(t_n + v h_n) = \sum_{j=1}^{m} L_j(v) Y_{n,j}, \quad v \in (0, 1],$$

the local Lagrange representation of $u_h^{(\nu)}$ $(\nu = k-1, \dots, 0)$ on $\bar{\sigma}_n$ is given by

$$u_h^{(\nu)}(t_n + v h_n) = \sum_{\ell=0}^{k-\nu-1} \frac{y_n^{(\nu+\ell)}}{\ell!} (h_n v)^\ell + h_n^{k-\nu} \sum_{j=1}^{m} \beta_{\nu,j}(v) Y_{n,j}, \quad v \in [0, 1], \tag{7.4.7}$$

where we have defined, as in Section 3.2.5,

$$\beta_{\nu,j}(v) := \int_0^v \frac{(v - s)^{k-\nu-1}}{(k - \nu - 1)!} L_j(s) ds. \tag{7.4.8}$$

For $\nu = 0$, (7.4.7) yields

$$u_h(t_n + v h_n) = \sum_{\ell=0}^{k-1} \frac{y_n^{(\ell)}}{\ell!} (h_n v)^\ell + h_n^k \sum_{j=1}^{m} \beta_{0,j}(v) Y_{n,j}, \quad v \in [0, 1]. \tag{7.4.9}$$

Substitution of these local representations in (7.4.6), with $t = t_{n,i}$ (i.e. $v = c_i,\ i = 1, \ldots, m$) yields a system of algebraic equations for $\mathbf{Y}_n \in \mathbb{R}^m$, and its solution determines the values of the collocation solution and its k derivatives on σ_n, via (7.4.7).

We will illustrate this for $k = 2$, that is, for the linear weakly singular VIDE

$$y''(t) = \sum_{\nu=0}^{1} a_\nu(t) y^{(\nu)}(t) + g(t) + \int_0^t h_\alpha(t, s, y(s), y'(s), y''(s))ds. \quad (7.4.10)$$

The reader may wish to compare this with Illustration 3.2.1 ($\alpha = 0$).

Illustration 7.4.1 *The continuous m-stage Volterra–Runge–Kutta–Nyström (VRKN) method:*
Consider (7.4.3) with $k = 2$ and $\alpha \in (0, 1]$. It follows from

$$Y_{n,i} = f(t_{n,i}, u_h(t_{n,i}), u_h'(t_{n,i})) + (\mathcal{V}_\alpha u_h)(t_{n,i}), \quad i = 1, \ldots, m, \quad (7.4.11)$$

that the components of the vector $\mathbf{Y}_n := (Y_{n,1}, \ldots, Y_{n,m})^T$, with $Y_{n,j} := u_h''(t_{n,j})$, are given by the solution of the nonlinear algebraic system

$$\begin{aligned} Y_{n,i} = {} & f\Big(t_{n,i}, y_n + h_n v y_n^{(1)} + h_n^2 \sum_{j=1}^{m} \beta_{0,j}(c_i) Y_{n,j}, y_n^{(1)} + h_n \sum_{j=1}^{m} \beta_{1,j}(v) Y_{n,j}\Big) \\ & + F_n(t_{n,i}; \alpha) \qquad (7.4.12) \\ & + h_n \int_0^{c_i} h_\alpha\big(t_{n,i}, t_n + sh_n, u_h(t_n + sh_n), u_h'(t_n + sh_n), u_h''(t_n + sh_n)\big)\, ds \end{aligned}$$

($i = 1, \ldots, m$), with lag term approximation

$$F_n(t_{n,i}; \alpha) := \int_0^{t_n} p_\alpha(t_{n,i} - s) k(t_{n,i}, s, u_h(s), u_h'(s), u_h''(s))ds. \quad (7.4.13)$$

Once the solution $\mathbf{Y}_n := (Y_{n,1}, \ldots, Y_{n,m})^T$ has been computed, the values of u_h and u_h' on $\bar{\sigma}_n$ are determined by the interpolation formulas

$$u_h(t_n + vh_n) = y_n + h_n v y_n^{(1)} + h_n^2 \sum_{j=1}^{m} \beta_{0,j}(v) Y_{n,j}, \quad v \in [0, 1], \quad (7.4.14)$$

and

$$u_h'(t_n + vh_n) = y_n^{(1)} + h_n \sum_{j=1}^{m} \beta_{1,j}(v) Y_{n,j}, \quad v \in [0, 1], \quad (7.4.15)$$

where

$$\beta_{1,j}(v) := \int_0^v L_j(s)ds \quad \text{and} \quad \beta_{0,j}(v) := \int_0^v (v - s) L_j(s)ds.$$

For the *linear* version of this VIDE, corresponding to

$$f(t, y, y') = a_0(t)y + a_1(t)y' + g(t)$$

and

$$h_\alpha(t, s, y, y', y'') = \sum_{\nu=0}^{2} H_{\alpha,\nu}(t, s)y^{(\nu)}(s)ds, \quad t \in I \quad (0 < \alpha \le 1), \quad (7.4.16)$$

the linear algebraic system corresponding to (7.4.12) is seen to have the form

$$[\mathcal{I}_m - h_n(\mathcal{A}_n + \mathcal{C}_n(\alpha))]\mathbf{Y}_n = \mathbf{g}_n + \mathbf{G}_n(\alpha) + \boldsymbol{\kappa}_{n,0}(\alpha)y_n + \boldsymbol{\kappa}_{n,1}(\alpha)y_n^{(1)}, \quad (7.4.17)$$

where now

$$\mathcal{A}_n := A_{n,1} + h_n A_{n,0}, \quad \mathcal{C}_n(\alpha) := C_{n,2}(\alpha) + h_n C_{n,1}(\alpha) + h_n^2 C_{n,0}(\alpha).$$

The five matrices in $L(\mathbb{R}^m)$ defining $\mathcal{A}_n$ and $\mathcal{C}_n(\alpha)$ have the forms

$$A_{n,0} := \text{diag}(a_0(t_{n,i})) \begin{pmatrix} \beta_{0,j}(c_i) \\ (i, j = 1, \dots, m) \end{pmatrix},$$

$$A_{n,1} := \text{diag}(a_1(t_{n,i})) \begin{pmatrix} \beta_{1,j}(c_i) \\ (i, j = 1, \dots, m) \end{pmatrix},$$

$$C_{n,0}(\alpha) := \begin{pmatrix} \int_0^{c_i} H_{\alpha,0}(t_{n,i}, t_n + sh_n)\beta_{0,j}(s)ds \\ (i, j = 1, \dots, m) \end{pmatrix},$$

$$C_{n,1}(\alpha) := \begin{pmatrix} \int_0^{c_i} H_{\alpha,1}(t_{n,i}, t_n + sh_n)\beta_{1,j}(s)ds \\ (i, j = 1, \dots, m) \end{pmatrix},$$

$$C_{n,2}(\alpha) := \begin{pmatrix} \int_0^{c_i} H_{\alpha,2}(t_{n,i}, t_n + sh_n)L_j(s)ds \\ (i, j = 1, \dots, m) \end{pmatrix}.$$

The right-hand side terms $\mathbf{g}_n$ and $\mathbf{G}_n(\alpha)$ are as before, and the terms reflecting the C^1-regularity of the collocation solution u_h at $t = t_n$ are

$$\boldsymbol{\kappa}_{n,0}(\alpha) := \left(a_0(t_{n,i}) + h_n \int_0^{c_i} H_{\alpha,0}(t_{n,i}, t_n + sh_n)ds \quad (i = 1, \dots, m)\right)^T$$

$$\boldsymbol{\kappa}_{n,1}(\alpha) := \left(a_1(t_{n,i}) + h_n c_i a_0(t_{n,i}) + h_n \int_0^{c_i} H_{\alpha,1}(t_{n,i}, t_n + sh_n)dsi \right.$$
$$\left. + h_n^2 \int_0^{c_i} H_{\alpha,0}(t_{n,i}, t_n + sh_n)s\, ds \quad (i = 1, \dots, m)\right)^T.$$

Example 7.4.1 $m = 1$ (see also Example 3.2.1)
Setting $\theta := c_1 \in (0, 1]$, $t_{n,1} := t_n + \theta h_n$, and observing that $\beta_{1,1}(v) = v$, $\beta_{0,1}(v) = v^2/2$, the resulting continuous one-stage VRKN method is described by the collocation equation

$$\begin{aligned} Y_{n,1} &= f(t_{n,1}, u_h(t_{n,1}), u_h'(t_{n,1})) + F_n(t_{n,1}; \alpha) \\ &\quad + h_n \int_0^\theta h_\alpha(t_{n,1}, t_n + sh_n), u_h(t_n + sh_n), u_h'(t_n + sh_n), Y_{n,1})ds. \end{aligned}$$

Here,

$$Y_{n,1} := u_h''(t_n + vh_n) = \frac{1}{h_n}[y_{n+1}^{(1)} - y_n^{(1)}], \quad v \in (0, 1],$$

and this can be employed to express the local representations of u_h, u_h',

$$\begin{aligned} u_h(t_n + vh_n) &= y_n + h_n v y_n^{(1)} + \frac{h_n^2}{2} v^2 Y_{n,1}, \\ u_h'(t_n + vh_n) &= y_n^{(1)} + h_n v Y_{n,1}, \quad v \in [0, 1], \end{aligned}$$

in the form

$$\begin{aligned} u_h(t_n + vh_n) &= y_n + \frac{h_n v}{2}\left((2 - v)y_n^{(1)} + v y_{n+1}^{(1)}\right), \\ u_h'(t_n + vh_n) &= (1 - v)y_n^{(1)} + v y_{n+1}^{(1)}, \quad v \in [0, 1]. \end{aligned}$$

For the linear VIDE (7.4.10) the elements of the matrices characterising the left-hand side of the algebraic equation for $Y_{n,1}$ are found to be

$$\begin{aligned} A_{n,0} &= \frac{\theta^2}{2} a_0(t_n + \theta h_n), \qquad A_{n,1} = \theta a_1(t_n + \theta h_n), \\ C_{n,0}(\alpha) &= \frac{1}{2}\int_0^\theta H_{\alpha,0}(t_n + \theta h_n, t_n + sh_n)s^2 ds, \\ C_{n,1}(\alpha) &= \int_0^\theta H_{\alpha,1}(t_n + \theta h_n, t_n + sh_n)s\, ds, \\ C_{n,1}(\alpha) &= \int_0^\theta H_{\alpha,2}(t_n + \theta h_n, t_n + sh_n)ds. \end{aligned}$$

It is clear from the regularity results and the convergence analysis in earlier sections that the results of Theorems 7.2.4 and 7.2.5 can be extended in an obvious way to higher-order VIDEs with weakly singular kernels. As an example we cite a theorem (due to Tang and Yuan (1990)) for (7.3.10),(7.3.16) ($k = 2$). (The original result was proved only for kernels of the form $h_\alpha(t, s, y)$; however, it is readily extended to the general case where $h_\alpha(t, s, y, y', y'')$.) Note that for smooth data the exact solution of (7.4.10) lies in the space $C^{2,1-\alpha}(I)$ and has an unbounded third derivative at $t = 0^+$ that behaves like $|y'''(t)| \leq Ct^{-\alpha}$.

Theorem 7.4.1 *Assume*:

(a) The given functions a_ν, g *and* K_ν *in (7.4.10) and (7.4.16) possess continuous derivatives of order* m *on their respective domains* I *and* D.
(b) $u_h \in S^{(1)}_{m+1}(I_h)$ *is the collocation solution defined by (7.4.12)–(7.4.15), and the underlying mesh* I_h *is graded, with* $r = \mu/(1-\alpha) \geq 1$.

Then for any $\alpha \in (0, 1)$ *the estimates*

$$||y^{(\nu)} - u_h^{(\nu)}||_\infty \leq C_\nu(r) \begin{cases} h^\mu & \text{if} \;\; 1-\alpha \leq \mu \leq m, \\ h^m & \text{if} \;\; \mu \geq m \end{cases}$$

hold for $\nu = 0, 1, 2$ *and all* $\{c_i\}$ *with* $0 \leq c_1 < \ldots < c_m \leq 1$.

Refined estimates, analogous to those given in Theorems 7.2.4 and 7.2.5, can also be derived. We leave this as a research exercise (Exercise 7.7.13).

7.5 Non-polynomial spline collocation methods

The solution representation (7.1.14) in Section 7.1.5 suggests that, on uniform I_h, it may be more natural to seek the collocation solution to the weakly singular VIDE (7.2.20) in a special *non-polynomial spline space* based on the expansion (7.1.14) of the exact solution y. In analogy to Section 6.4.1 we choose a collocation space $Z^{(0)}_m(I_h)$ with the property that on $\bar{\sigma}_0 = [t_0, t_1]$ ($t_0 = 0$) any element z_h from this space reduces to

$$z_h(t_0 + vh) = \sum_{(j,k)_{2-\alpha}} b_{j,k} v^{j+k(2-\alpha)}, \quad v \in [0, 1], \tag{7.5.1}$$

where $(j, k)_{2-\alpha} := \{(j, k) : \; j + k(2-\alpha) < m+1, \; j, k \in \mathbb{N}_0\}$ and $b_{j,k} = b_{j,k}(h)$. This local representation thus exactly matches the terms in the first expression on the right-hand side of (7.1.14)). The error analysis for the corresponding collocation solution z_h can then be carried out along familiar lines (using a standard Gronwall argument, as in the proof of Theorem 7.2.3), and it reveals that $\|y^{(\nu)} - z_h^{(\nu)}\|_\infty = \mathcal{O}(h^m)$ for $\nu = 0, 1$. The details can be found in Brunner (1983, pp. 1116–1119).

Variants of this non-polynomial collocation method were studied by Hu (1996a, 1998b): he employed collocation solutions based on so-called piecewise β-polynomials (employing integer powers of t^β, with suitable β, as basis functions); the mesh I_h is a specially chosen *geometric mesh*.

7.6 Weakly singular Volterra functional integro-differential equations

7.6.1 Weakly singular VIDEs with non-vanishing delays

The result in Lemma 6.5.1 shows that the attainable order of (super-) convergence of collocation solutions to VIDEs with non-vanishing delays and weakly singular kernels will depend on whether the lag function θ is linear or nonlinear. Hence, in analogy to Theorem 6.5.2, if θ in the equations

$$y'(t) = a(t)y(t) + b(t)y(\theta(t)) + g(t) + (\mathcal{V}_\alpha y)(t) + (\mathcal{V}_{\theta,\alpha} y)(t), \quad t \in I, \tag{7.6.1}$$

or

$$y'(t) = a(t)y(t) + b(t)y(\theta(t)) + g(t) + (\mathcal{W}_{\theta,\alpha} y)(t), \quad t \in I, \tag{7.6.2}$$

is linear, the optimal orders derived in Theorem 7.2.3 are also attained by the collocation solutions $u_h \in S_m^{(0)}(I_h)$ for (7.6.1) and (7.6.2), provided the mesh I_h is θ-invariant and the first submesh $I_h^{(0)}$ is optimally graded. For nonlinear θ this is no longer valid. We summarise these fact in

Theorem 7.6.1 *Assume*

(a) The given functions a, b, K_1, K_2, K in (7.6.1) and (7.6.2) are $d \geq m$-times continuously differentiable on their respective domains, $\phi \in C^{d+1}[\theta(t_0, t_0]$, and the lag function satisfies (D1)–(D3).

(b) $u_h \in S_m^{(0)}(I_h)$ is the collocation solution to (7.6.1) or (7.6.2), with θ-invariant mesh I_h.

(c) The first submesh $I_h^{(0)}$ is optimally graded:

$$t_n^{(0)} := t_0 + \left(\frac{n}{N}\right)^{r_0} (\xi_0 - t_0) \quad (n = 0, 1, \ldots, N), \quad \textit{with} \quad r_0 = \frac{m}{1-\alpha}.$$

(I) *If θ is* linear, *the results of Theorems 7.2.3 and 7.2.4 remain valid on each subinterval $I^{(\mu)} := [\xi_\mu, \xi_{\mu+1}]$:*

$$\|y^{(\nu)} - u_h^{(\nu)}\|_{\mu,\infty} := \sup_{t \in I^{(\mu)}} |y^{(\nu)}(t) - u_h^{(\nu)}(t)| \leq C_\nu N^{-m} \quad (0 \leq \mu \leq M; \nu = 0, 1).$$

If the collocation parameters are such that $J_0 = 0$ holds, and if $d \geq m+1$, then we obtain

$$\|y - u_h\|_{\mu,\infty} \leq C(\alpha) N^{-(m+1-\alpha)} \quad \textit{whenever} \quad r \geq \frac{m+1-\alpha}{2-\alpha}.$$

(II) *If θ is* nonlinear, *the results of Theorems 7.2.3 and 7.2.4 are in general valid only on $I^{(0)}$. On the subsequent subintervals $I^{(\mu)}$ ($\mu \geq 1$) the attainable orders* *of* $\|y^{(\nu)} - u_h^\nu\|_{\mu,\infty}$ $(\nu = 0, 1)$ *will be less than m and lie between* $1 - \alpha$ *and m, except when* $m = 1$ *and we have smoothing in the exact solution.*

Remarks

1. For *linear* lag functions the (super-) convergence results of Brunner, Pedas and Vainikko (2001a, 2001b), in particular those involving logarithmic kernel singularities, remain true for second-kind delay VIDEs with weakly singular, or bounded but non-smooth, kernels. The same is true for the convergence estimates corresponding to a more refined choice of the grading exponent (recall Remark 2 following the proof of Theorem 7.2.3).
2. If θ is *nonlinear*, we can – as for weakly singular VIEs with weakly singular kernels – again achieve global convergence order $p = m$, by resorting to submeshes that are *individually graded*:
   - If the solution y has $C^{\mu+1,1-\alpha}$-regularity at $t = \xi_\mu^+$ (cf. Table 7.1) then – according to Theorem 7.2.3 – the optimal grading exponent for $I_h^{(\mu)}$ is

$$r_\mu = \begin{cases} \dfrac{m}{\mu + 1 - \alpha} & \text{for } \mu = 0, 1, \ldots, \min\{m, M\}, \\ \\ 1 & \text{for } \mu = m + 1, \ldots, M. \end{cases}$$

   - If y has only $C^{1,1-\alpha}$-regularity at each ξ_μ^+ (no smoothing), then we choose $r_\mu = m/(1 - \alpha)$ for all $\mu = 0, 1, \ldots, M$.

   We recall from Section 6.5.1 that the corresponding global mesh I_h is now no longer θ-invariant.

7.7 Exercises and research problems

Exercise 7.7.1 Use the reformulation (7.1.4) to prove Theorems 7.1.1 and 7.1.4.

Exercise 7.7.2 Derive the adjoint resolvent equation (7.1.11) and prove the C^1-regularity of r_α.

Exercise 7.7.3 Describe the Hölder space containing the resolvent $r_\alpha = r_\alpha(t, s)$ defined in (7.1.7) or (7.1.12), under the assumptions of Theorem 7.1.1. Is the special resolvent r_α of Corollary 7.1.3 in the same Hölder space?

Exercise 7.7.4 Find the resolvent kernel $r_\alpha(t, s)$ for the special VIDE (7.1.12) when $H_\alpha(t, s) = p_\alpha(t - s)$ $(0 < \alpha \leq 1)$.

Exercise 7.7.5 Extend the regularity result of Theorem 7.1.4 to linear VIDEs (7.1.1) with Volterra integral operator

$$(\mathcal{V}_\nu y)(t) := \int_0^t (t-s)^\nu K(t,s)y(s)ds,$$

where $\nu := \rho - \alpha,\ \rho \in \mathbb{N},\ 0 < \alpha < 1$, and

$$g(t) = g_1(t) + t^\beta g_2(t) \quad (\beta > 0,\ \beta \notin \mathbb{N},$$

with smooth functions g_i and $g_2(0) \neq 0$.

Exercise 7.7.6 Analyse the regularity of the solutions of the linear VIDE

$$y'(t) = g(t) + \int_0^t (t-s)^k \log(t-s)K(t,s)y(s)ds, \quad t \in I := [0,T],$$

where $k \in \mathbb{N}_0$ and $K(t,t) \neq 0$ on I.

Exercise 7.7.7 What can be said about the regularity of the solution to the VIDE in Exercise 7.7.6 if $k = 0$ and g is replaced by the more general (non-smooth) function g of Exercise 7.7.5?

Exercise 7.7.8 Derive the analogue of the VIE (7.1.24) when the kernel of the integral operator in (7.1.19) has the more general form

$$h_\alpha(t,s,y,y',\ldots,y^{(k)}) := \sum_{\nu=0}^{k} p_{\alpha_\nu}(t-s)K_\nu(t,s),$$

with $0 < \alpha_0 < \alpha_1 < \ldots < \alpha_k \leq 1$.

Exercise 7.7.9 Derive the solution representation for (7.1.19)–(7.1.21); i.e., prove the corresponding analogue of Theorem 7.1.4.

Exercise 7.7.10 Consider the semilinear VIDE

$$y'(t) = \lambda y(t) + \int_0^t p_\alpha(t-s)G(y(s))ds, t \geq 0,$$

where $\lambda \leq 0$ and $G(y) = u^p$ $(p > 1)$. Discuss the existence and possible *blow-up* of solutions corresponding to initial conditions of the form $y(0) = y_0 > 0$. (The above equation is a non-local analogue of the ODE studied in Section 2.1.5 (Theorem 2.1.11).)

Exercise 7.7.11 In Example 7.2.3, does the use of the right rectangle product rule lead to the same order of convergence as the the product midpoint rule?

Exercise 7.7.12 Derive the fully discretised version of the Rung–Kutta–Nyström method for the (linear) weakly singular second-order VIDE (7.4.10), first for general m, then for $m = 1$ (cf. Example 7.4.1) and for $m = 2$.

Exercise 7.7.13 Extend the results of Theorem 7.2.4 ($0 < \alpha < 1$) and Theorem 7.2.5 ($\alpha = 1$) to collocation solutions $u_h \in S^{(1)}_{m+1}(I_h)$ for the second-order weakly singular VIDE (7.4.10).

Exercise 7.7.14 Prove the analogues of Theorems 7.2.3, 7.2.4 and 7.2.5 for the approximation y_h generated by using implicitly linear collocation for the integrated form (7.4.2) of the Hammerstein type VIDE (7.4.1).

Exercise 7.7.15 In (7.4.1) choose $K(t,s) = \lambda < 0,\ G(s,y) = s\exp(-y)$. For a prescribed 'test solution' $y(t)$ (of your choice), with corresponding $g(t)$, carry out a numerical comparison when the VIDE is solved, on appropriately graded meshes I_h,

(i) by direct collocation in $S^{(0)}_m(I_h)$;
(ii) by direct collocation, followed by u_{it}, for the integrated form (7.4.2);
(iii) by implicitly linear collocation for (7.4.2).

Discuss the relative merits of these methods.

Exercise 7.7.16 (Research problem)
High-order convergence on uniform meshes for solutions of weakly singular VIDEs is only possible if the collocation solution lies in some feasible non-polynomial spline space. The solution representation given in Theorem 7.1.4 gives a hint on how to choose this space: on the first subinterval $\bar{\sigma}_0 = [0, h]$ it will have to be spanned by the functions

$$\phi^{(0)}_{j,k}(t) := t^{j+k(2-\alpha)} \quad (j + k(2-\alpha) < m+1,\ \ j,k \in \mathbb{N}_0).$$

Describe the collocation equation for such collocation solutions and show that they exhibit $\mathcal{O}(h^m)$-convergence on uniform meshes, for any choice of the collocation parameters. Is (global and local) superconvergence possible for judicious choices of these parameters?

Exercise 7.7.17

(a) Prove the regularity results summarised in Table 7.1.
(b) Use these results to establish results on the regularity of the neutral VIDEs

$$\frac{d}{dt}[y(t) - (\mathcal{V}_{\theta,\alpha}y)(t)] = f(t, y(t), y(\theta(t))),$$

$$\frac{d}{dt}[y(t) - (\mathcal{W}_{\theta,\alpha}y)(t)] = f(t, y(t), y(\theta(t))),$$

and

$$\frac{d}{dt}[(\mathcal{W}_{\theta,\alpha}y)(t)] = g(t).$$

(See also Burns, Herdman and Stech (1983), Kappel and Zhang (1986), and Clément, Desch and Homan (2003) for a (different) semigroup framework for the last of these three functional equations.)

(c) (Research problem) Establish convergence results, similar to those in Theorem 7.6.1, for collocation solutions to the FVIDEs in (b).

(The paper by Ito and Turi (1991) employs the semigroup framework of Burns, Herdman and Stech (1983) to derive and analyse a corresponding numerical method for the last VIDE in (b). It will be interesting to compare this with an analogous one exploiting the ideas in Clément, Desch and Homan (2003).)

Exercise 7.7.18 (Research problem)
Consider the state-dependent DDE

$$y'(t) = y(y(t)) + g(t), \quad t \in [0, 1], \quad y(0) = 0,$$

with

$$g(t) = (3 + \alpha)t^{2+\alpha} - t^{(3+\alpha)^2} \quad (0 < \alpha < 1)$$

(Tavernini (1978, p. 1049)). Show that its (unique) solution is given by $y(t) = t^{3+\alpha}$. Discuss the application of collocation in $S_m^{(0)}(I_h)$ $(m \geq 1)$: for which m, and how, does one need to grade the mesh I_h in order to obtain optimal order of convergence of u_h on I?

7.8 Notes

7.1: Review of basic Volterra theory (IV)

The regularity properties of solutions to VIDEs with weakly singular kernels are analysed in Lubich (1983a), Brunner (1983, 1985b, 1985c) and – especially – in Brunner, Pedas and Vainikko (2001a, 2001b). See also Kiryakova (1994) and Meehan and O'Regan (1999) for related results.

There is an extensive literature on the regularity of solutions to partial VIDEs of parabolic type. We mention DaPrato, Iannelli and Sinestrari (1985), Lunardi and Sinestrari (1986), Sanz-Serna (1988), Choi and MacCamy (1989), Grasselli and Lorenzi (1991), Sforza (1991), Prüss (1993), Chen and Shih (1998, Chapter 7), Clément and Londen (2000), and Gripenberg, Clément and Londen (2000). See also the two papers by Fujita (1990).

The mathematics underlying Volterra functional integro-differential equations with non-vanishing delays has received much attention since the early 1980s. In particular, the semigroup framework for $(d/dt)[(\mathcal{W}_{\theta,\alpha} y)(t)] = f(t)$ $(\theta(t) = t - \tau,\ 0 < \alpha < 1)$ is discussed in, e.g. Burns, Herdman and Stech

(1983), Kappel and Zhang (1986), and (in a wider context) in Staffans (1985b, Section 10). See also the more recent paper by Clément, Desch and Homan (2003) and the monograph by Ito and Kappel (2002).

Applications of weakly singular VIDEs
A good source of information (including numerous additional references) on applications of weakly singular VIDE is the monograph by Prüss (1993). As in Chapter 6 we will list a representative sample of application areas, together with typical papers.

- *Viscoelasticity / materials with memory:* Hrusa, Nohel and Renardy (1988), Renardy, Hrusa and Nohel (1988), Choi and MacCamy (1989), Brewer and Powers (1990).
- *Biosciences:* Dixon (1987), Jumarhon (1994), Jones, Jumarhon, McKee and Scott (1996), Jumarhon and Pidcock (1996), Jumarhon, Lamb, McKee and Tang (1996), Clements and Smith (1996).
- *Diffusion of discrete particles in turbulent fluids:* McKee and Stokes (1983) (see also for references on the Basset equation), Brunner and Tang (1989).
- *Vapour-bubble growth in superheated liquid:* Prosperetti (1982) (the paper contains numerous references on the underlying physical model).
- *Capillarity theory:* A. Corduneanu and Morosanu (1996).
- *Aero-elastic systems:* Burns, Cliff and Herdman (1983, 1987), Burns, Herdman and Stech (1983), Burns, Herdman and Turi (1987), Herdman and Turi (1991a, b), Cerezo (1996).

7.2: Collocation for linear weakly singular VIDEs
A comprehensive analysis of global and local superconvergence in collocation solutions on graded meshes for linear weakly singular VIDEs (with $0 < \alpha \leq 1$) is given in Brunner, Pedas and Vainikko (2001a, 2001b). Earlier results (for $0 < \alpha < 1$) were given by Brunner (1985b, 1985c, 1986a), Brunner and Tang (1989) (for the Basset equation), Tang (1992, 1993a). The paper by Kangro and Parts (refines some of the results by Brunner, Pedas and Vainikko.

The discontinuous Galerkin method for such VIDEs is studied in Brunner and Schötzau (2002) (*hp*-method) and in the dissertation by Ma (2004).

Waveform and timepoint relaxation methods for solving large systems of nonlinear systems of weakly singular VIDEs (and their discrete versions) are described and analysed in Parsons (1999); also VIEs: Brunner, Crisci, Russo and Vecchio (2003).

7.3: Hammerstein-type VIDEs with weakly singular kernels
Ladopoulos (1997) discusses collocation methods for general nonlinear VIDEs with weakly singular kernels.

7.4: Higher-order weakly singular VIDEs
The convergence and numerical performance of collocation methods for such problems were studied by Papatheodorou and Jesanis (1980) (general mth-order VIDEs), Prosperetti (1982), and by Tang and Yuan (1990) ($k = 2$: extension of results of Brunner (1986a)).

7.5: Non-polynomial spline collocation methods
Collocation solutions in special non-polynomial spline spaces (and uniform meshes) were first analysed by Brunner (1983). Hu (1996a, 1998b) combines so-called β-polynomials with geometric meshes to obtain superconvergent non-polynomial spline collocation solutions.

We note that Keller (1982) used special non-polynomial collocation spaces for (stiff) ODEs; these spaces also reflect certain known properties of the solution of the given problem.

7.6: Weakly singular Volterra functional integro-differential equations
The survey papers by Brunner (1999a, 1999c) describe the many open problems in the numerical analysis of VIEs and VIDEs with weakly singular kernels and non-vanishing delays.

Parabolic VIDEs with weakly singular kernels
There are numerous papers on time-stepping in spatially semidiscretised versions of such PVIDEs; see, e.g. Sanz-Serna (1988), López-Marcos (1990), and Tang (1993b) (Burgers' equation with weakly singular Volterra memory term, $0 < \alpha < 1$), Chen, Thomée and Wahlbin (1992), Xu (1993, 1998), Y. Lin (1994), Chen and Shih (1998), and Larsson, Thomée and Wahlbin (1998) (also for additional references). The hp-discontinuous Galerkin method described in Brunner and Schötzau (2002) extends the approach by Schötzau and Schwab (2000, 2001) to parabolic VIDEs with weakly singular memory term; see also the Ph.D. thesis of Ma (2004). Related papers of interest are by Kim and Choi (1998) (spectral collocation) and by Cuesta and Palencia (2003) (fractional trapezoidal method for abstract VIDEs).

8

Outlook: integral-algebraic equations and beyond

Summary: As we mentioned in the Preface the voyage through the previous seven chapters has now brought us in many ways to the 'frontier' of what is known about the analysis of collocation methods. Thus, in this chapter we will make this more precise, first by reviewing recent and current work on collocation methods for DAEs and Volterra-type integral-algebraic equations (IAEs) of index 1. This will be followed by an exploration of various directions for future research on IAEs in particular, and collocation methods in general, in more abstract settings that may contain the key to the solution of many of the open problems encountered earlier.

8.1 Basic theory of DAEs and IAEs

The purpose of this section, especially Section 8.1.1, is to present some of the modern tools that will be required in the analysis of collocation methods for integral-algebraic equations of Volterra type. Thus, we present a fairly detailed introduction to the basic theory of (index-1) DAEs: this will allow us better to appreciate the complexity behind the analysis of collocation methods for IAEs and, especially, IDAEs of higher index. As we have just said, much of the quantitative and qualitative analysis of collocation solutions to such problems remains to be carried out.

The reader not familar with the theory and numerical analysis of DAEs will find good introductions to these subjects in the monographs by Griepentrog and März (1986), Hairer, Lubich and Roche (1989), Brenan, Campbell and Petzold (1996), and the surveys by März (1992, 1994), Rabier and Rheinboldt (2002), and Schulz (2003).

8.1.1 DAEs: a brief introduction

A system of implicit ODEs,

$$\Phi(x'(t), x(t), t) = 0, \quad t \in I := [0, T], \tag{8.1.1}$$

where $\Phi : \mathbb{R}^d \times \mathbb{R}^d \times I \to \mathbb{R}^d$ $(d \geq 2)$ is said to be a system of *differential-algebraic equations* (or simply: a *DAE*) if the Jacobian $\partial\Phi/\partial x'$ is singular for all values of its arguments. If $\partial\Phi/\partial x'$ is regular on $\mathbb{R}^d \times \mathbb{R}^d \times I$, then (8.1.1) is a regular (implicit) ODE.

If the DAE (8.1.1) has the form

$$\begin{aligned} y'(t) &= F(t, y(t), z(t)), \\ 0 &= G(t, y(t), z(t)), \quad t \in I, \end{aligned} \tag{8.1.2}$$

with (continuous) functions $F : I \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}^{d_1}$ and $G : I \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}^{d_2}$, it is called is a *semi-explicit* DAE. The component $z(t)$ in the solution $x(t) = (y(t),\ z(t))^T$ (with $y \in \mathbb{R}^{d_1}$ and $z \in \mathbb{R}^{d_2}$) is referred to as its *algebraic component*. In this chapter we will restrict the discussion to semi-explicit DAEs (and analogous integral-algebraic and integro-differential-algebraic equations; see Sections 8.1.2 and 8.1.3).

The system (8.1.2) is complemented by a given set of *initial values*, $x(0) = (y(0),\ z(0))^T = (y_0,\ z_0)^T$: it will be assumed that these values are *consistent*, that is, they satisfy

$$G(0, y_0, z_0) = 0 \tag{8.1.3}$$

(see also Griepentrog and März (1989)). The general *semilinear* version of the DAE (8.1.1) is given by

$$A(t)x'(t) + b(x(t), t) = 0, \quad t \in I, \tag{8.1.4}$$

where $A(\cdot) \in L(\mathbb{R}^d)$ is continuous and singular (but has constant rank at least one) for all $t \in I$, and $b : \mathbb{R}^d \times I \to \mathbb{R}^d$ is (Lipschitz) continuous. The *semi-explicit* form of the *linear DAE*

$$A(t)x'(t) + B(t)x(t) = q(t), \quad t \in I. \tag{8.1.5}$$

with continuous $B(\cdot) : I \to \mathbb{R}^d$, corresponds to

$$A(t) = \text{diag}\,(\mathcal{I}_{d_1}, O_{d_2})$$

and is thus given by the more structured system

$$y'(t) + B_{11}(t)y(t) + B_{12}(t)z(t) = q_1(t), \tag{8.1.6}$$

$$B_{21}(t)y(t) + B_{22}(t)z(t) = q_2(t), \quad t \in I. \tag{8.1.7}$$

The matrix functions $B_{kk}(\cdot) \in L(\mathbb{R}^{d_k})$ $(k = 1, 2)$ and $B_{12}(\cdot) \in L(\mathbb{R}^{d_2}, \mathbb{R}^{d_1})$, $B_{21}(\cdot) \in L(\mathbb{R}^{d_1}, \mathbb{R}^{d_2})$ are assumed to be continuous on I, as are $q_1 : I \to \mathbb{R}^{d_1}$ and $q_2 : I \to \mathbb{R}^{d_2}$.

It was shown by Rheinboldt (1984) (see also Hairer and Wanner (1996, pp. 457–458) and Rabier and Rheinboldt (1994, 2002)) that DAEs may be viewed as differential equations on manifolds, with the manifolds described by the given algebraic constraints in the DAE. This geometric interpretation adds considerable insight into the behaviour of solutions to DAEs and into the properties a feasible numerical method must have. (See also the related remark in Section 8.3 and Exercise 8.6.12 on the geometry of IDAEs!) While the geometry is relatively simple for so-called index-1 DAEs like

$$\begin{aligned} y'(t) &= F(y(t), z(t)), \\ 0 &= G(y(t), z(t)), \quad t \in I \end{aligned} \tag{8.1.8}$$

(where G is smooth and has non-vanishing Jacobian $\partial G/\partial z$), it becomes much more complex if the DAE has the ('index-2') form

$$\begin{aligned} y'(t) &= F(y(t), z(t)), \\ 0 &= G(y(t)), \quad t \in I. \end{aligned} \tag{8.1.9}$$

The notion of *index* is crucial for the classification of DAEs, as the above examples indicate. There exist several different (but often closely related) definitions of the index of a DAE. Somewhat loosely speaking, we say that the semi-linear DAE (8.1.4) has

- *differentiation index* 1 if, and only if, a single differentiation of the algebraic constraints yields a system of (implicit) regular ODEs;
- *perturbation index* 1 if, and only if, perturbations in the right-hand side of (8.1.4) lead to perturbations in the solution that can be estimated in terms of the original perturbations, with the estimate not depending on derivatives of the input.
- *tractability index 1* if, and only if, the algebraic constraints are locally solvable for the algebraic components of the solution x.

For details and, especially, the extension of the above definitions to DAEs with index 2 and higher, and to fully nonlinear DAEs, we refer the reader to

Griepentrog and März (1986), März (1987, 1989, 1992, 2002a), Gear (1990), Hairer, Lubich and Roche (1989), Brenan, Campbell and Petzold (1996), and the recent survey by Schulz (2003). Lamour (2001) presents a general algorithm, based on the notion of the tractability index, for computing this index.

In the following we will follow the route chosen by R. März and her collaborators and employ the *tractability index*, not least because it requires minimal regularity assumptions. Also, it can easily be seen that if (8.1.4) possesses tractability index 1, then its perturbation index is also 1; the same is true for the differentiation index, provided b has sufficient regularity. This close relationship no longer remains true for DAEs of index 2 or higher (compare also Hairer, Lubich and Roche (1989, pp. 12–13), and Schulz (2003)).

Definition 8.1.1

(i) The *matrix pencil* $p(\lambda) := \det(A + \lambda B)$ $(A, B \in L(\mathbb{R}^d),\ \lambda \in \mathbf{C})$ associated with the linear DAE with *constant coefficients*,

$$Ax'(t) + Bx(t) = q(t), \quad t \in I \quad (\det(A) = 0,\ \operatorname{rank}(A) \geq 1), \qquad (8.1.10)$$

is called a *regular matrix pencil* if $p(\lambda) \not\equiv 0$ (that is, if there is a λ for which $A + \lambda B$ is a regular matrix). We will denote the matrix pencil associated with the matrices A and B by $\{A, B\}$.

(ii) The DAE (8.1.10) is said to be *tractable* if its matrix pencil $\{A, B\}$ is *regular*.

We leave it to the reader to discuss the phenomena that can occur if the matrix pencil associated with (8.1.10) is singular.

In order to make the meaning of Definition 8.1.1 more transparent, assume that $\{A, B\}$ is a regular matrix pencil. It then follows from a result by Weierstrass (see, e.g. Griepentrog and März (1986, pp. 14–23) or Brenan, Campbell and Petzold (1996, pp. 18–22) for details) that there exist regular matrices $E, F \in L(\mathbb{R}^d)$ so that

$$\tilde{A} := EAF = \begin{bmatrix} \mathcal{I} & 0 \\ 0 & \mathcal{J} \end{bmatrix}, \qquad \tilde{B} := EBF = \begin{bmatrix} W & 0 \\ 0 & \mathcal{I} \end{bmatrix},$$

where $\mathcal{I}$ denotes the appropriate identity matrix, $W \in L(\mathbb{R}^k)$, and

$$\mathcal{J} := \begin{bmatrix} \mathcal{J}_1 & \dots & 0 \\ & \ddots & \\ 0 & \dots & \mathcal{J}_\nu \end{bmatrix} \in L(\mathbb{R}^{d-k}):$$

here, the $\mathcal{J}_i$ denote (nilpotent) Jordan blocks in $L(\mathbb{R}^{m_i})$. The pencil $\{\tilde{A}, \tilde{B}\}$ is called the *Kronecker normal form* of the regular matrix pencil $\{A, B\}$, and the corresponding transformed DAE is

$$\tilde{A}\tilde{x}'(t) + \tilde{B}\tilde{x}(t) = \tilde{q}(t), \quad t \in I,$$

where $\tilde{x}(t) := F^{-1}x(t),\ \tilde{q}(t) := Eq(t)$.

Definition 8.1.2 The *index of nilpotency* of $\{A, B\}$ is

$$\mu_p := \max\{m_i :\ i = 1, \ldots, \nu\}.$$

It is not difficult to see that the perturbation index of the linear DAE (8.1.10) equals its index of nilpotency.

The given DAE (8.1.10) can therefore be decoupled into a system of (regular) differential equations of the form

$$\tilde{y}'(t) + W\tilde{y}(t) = \tilde{q}_1(t), \tag{8.1.11}$$

$$\mathcal{J}\tilde{z}'(t) + \tilde{z}(t) = \tilde{q}_2(t), \quad t \in I, \tag{8.1.12}$$

with obvious meaning of $\tilde{y}(t)$ and $\tilde{z}(t)$. This DAE is referred to as the *Kronecker normal form* of the DAE (8.1.10). It is described by the *regular ODE* (8.1.11) for $\tilde{y}$ and the 'backward system' (8.1.12) for the *algebraic components* $\tilde{z}$. Note that if we have $m_i = 1$ for $i = 1, \ldots, \nu$, then the components of $\tilde{z}$ are given in terms of those of $\tilde{q}_2$, and no derivatives of $\tilde{z}$ are needed. In other words, the index of nilpotency equals one, and this is also the index of tractability of the original DAE (8.1.10).

While the Kronecker normal form also exists for DAEs (8.1.5) with *variable coefficients* $A(t)$, $B(t)$ (see, e.g. Gear and Petzold (1984)), there is a more elegant (and practically very feasible – see Lamour (2001, 2003)) way to describe the index of tractability. It was introduced by Griepentrog and März (1986) and is based on the null space of $A(t)$ and certain matrix chains associated with it. Details can be found in, e.g. Griepentrog and März (1986), März (1992, 2002a, 2002b), and Schulz (2003). Here we will describe the basic ideas for the index-1 case. See also Lamour (2001, 2003) on the computational determination of the tractability index of a DAE.

Assume that $A(\cdot) \in L(\mathbb{R}^d)$ is singular for all $t \in I := [0, T]$ but has *constant rank* $r \geq 1$. We introduce the subspaces

$$N(t) := \ker A(t) := \{w \in \mathbb{R}^d :\ A(t)w = 0\}$$

(the *null space* of $A(t)$) and, for $B(\cdot) \in L(\mathbb{R}^d)$ in (8.1.5),

$$S(t) := \{w \in \mathbb{R}^d :\ B(t)w \in \operatorname{im} A(t)\},$$

where $\operatorname{im} A(\cdot) := \{A(\cdot)w :\ w \in \mathbb{R}^d\}$ denotes the image of $A(\cdot) \in L(\mathbb{R}^d)$. This space obviously contains every solution of the homogeneous DAE (8.1.5).

Moreover, we assume that $N(t)$ ($t \in I$) is spanned by $d - r$ continuously differentiable basis functions. Then there exists a matrix function $Q(\cdot) \in L(\mathbb{R}^d)$, with $Q \in C^1(I)$, that projects $\mathbb{R}^d$ pointwise onto $N(t)$:

$$(Q(t))^2 = Q(t), \quad \text{im } Q(t) = N(t), \quad t \in I. \tag{8.1.13}$$

For such a projector $Q(t)$ define $P(t) := \mathcal{I}_d - Q(t)$, $t \in I$. It follows that $A(t) = A(t)P(t)$, since we have, by definition of $Q(t)$,

$$0 = A(t)Q(t) = A(t)[\mathcal{I}_d - P(t)] = A(t) - A(t)P(t), \quad t \in I.$$

Hence, the DAE (8.1.5) can be rewritten as

$$A(t)[(P(t)x(t))' - P'(t)x(t)] + B(t)x(t) = q(t),$$

or as

$$A(t)[P(t)x(t)]' + [B(t) - A(t)P'(t)]x(t) = q(t), \quad t \in I. \tag{8.1.14}$$

This reformulation, incidentally, also yields information about the feasible function space in which the solution x is to be sought: instead of requiring that $x \in C^1(I)$, we define a solution to be an element of the space

$$C_N^1(I) := \{x \in C(I) : \; P(\cdot)x \in C^1(I)\}.$$

Lemma 8.1.1 *Let $N(t)$ be the null space of $A(t)$ in (8.1.5), and assume that the projector $Q(t)$ satifies (8.1.13). Then the DAE (8.1.14) decomposes into the system*

$$[P(t)x]' - P'(t)P(t)x + P(t)A_1^{-1}(t)B_0(t)P(t)x = P(t)A_1^{-1}(t)q(t), \tag{8.1.15}$$
$$Q(t)x + Q(t)A_1^{-1}(t)B_0(t)P(t)x = Q(t)A_1^{-1}(t)q(t). \tag{8.1.16}$$

Here, we have introduced the matrix functions

$$B_0(t) := B(t) - A(t)P'(t), \qquad A_1(t) := A(t) + B_0(t)Q(t).$$

Proof Using the definition of $Q(t)$ and $P(t)$ (and omitting the argument t) we first write (8.1.14) as

$$A(Px)' + (B - AP')(Px + Qx) = q.$$

Since $AP = A$, $Q^2 = 0$ and $QP = 0$ we obtain

$$\{A + (B - AP')Q\}\{P(Px)' + Qx\} + (B - AP')Px = q. \tag{8.1.17}$$

This motivates the definition of the matrix functions B_0 and A_1. If A_1 is *nonsingular* for all $t \in I$, we multiply (8.1.17) by PA_1^{-1} and QA_1^{-1}, respectively, and this yields the system

$$(Px)' - P'Px + PA_1^{-1}B_0Px = PA_1^{-1}q, \tag{8.1.18}$$

$$Qx + QA_1^{-1}B_0Px = QA_1^{-1}q. \tag{8.1.19}$$

We see that the second component (8.1.19) of this system is derivative free and determines the null space component Qx of the solution once Px is known. The non-null space component $w := Px$ (note that P is non-singular) is given by the solution of the so-called *inherent regular ODE*,

$$w' - P'w + PA_1^{-1}B_0w = PA_1^{-1}q,$$

associated with the DAE (8.1.5).

The proof of the following criterion for $A_1(t)$ to be non-singular on I is left as an exercise.

Lemma 8.1.2 *The matrix* $A_1(t) = B(t) + B_0(t)Q(t)$ *is non-singular for all* $t \in I$ *if, and only if, the direct sum of the spaces* $N_0(t) := N(t)$ *and*

$$S_0(t) := s(t)\{w \in \mathbb{R}^d : B_0(t)w \in \text{im } A(t)\} = \{w \in \mathbb{R}^d : B(t)w \in \text{im } A(t)\}$$

spans $\mathbb{R}^d$*:*

$$S_0(t) \oplus N_0(t) = \mathbb{R}^d \quad \textit{for all} \quad t \in I. \tag{8.1.20}$$

We are now ready to characterise linear DAEs with variable coefficients that are *index-1 tractable*.

Definition 8.1.3 Consider the linear DAE (8.1.5) whose coefficients are continuous matrix functions in $L(\mathbb{R}^d)$, and assume that on I, $\det A(t) = 0$, $A(t)$ has constant rank, and its null space $N(t)$ is smooth. Then (8.1.5) is said to be *index-1 tractable* if

$$\det A_1(t) \neq 0 \quad \text{for all} \quad t \in I.$$

Here, $A_1(t)$ is defined in Lemma 8.1.1.

It can be shown that the tractability index does not depend on the choice of the projector Q.

We will use these insights into the 'inner' structure of linear DAEs to obtain a better understanding of the *semi-explicit* DAE given by (8.1.6) and (8.1.7). We start with

Theorem 8.1.3 *The semi-explicit DAE (8.1.6),(8.1.7) is index-1 tractable if* $B_{22}(t)$ *is non-singular on* I*.*

Proof Choose

$$Q := \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{I}_{d_2} \end{bmatrix}, \quad A_1 := A + BQ = \begin{bmatrix} \mathcal{I}_{d_1} & B_{12} \\ 0 & B_{22} \end{bmatrix}.$$

We now formalise the above discussion by introducing the important concept of a *(numerically) properly stated DAE.* Our starting point is the linear DAE (8.1.5) which we now write in the form

$$\bar{A}(t)(D(t)x(t))' + \bar{B}(t)x(t) = g(t), \quad t \in I, \tag{8.1.21}$$

where $\bar{A}(\cdot) \in L(\mathbb{R}^{d_0}, \mathbb{R}^d)$, $D(\cdot) \in L(\mathbb{R}^d, \mathbb{R}^{d_0})$ and $\bar{B}(\cdot) \in L(\mathbb{R}^d)$ are continuous on I. Here, we usually (but not always) have $d_0 = d$. The term $\bar{A}(t)(D(t)x(t))$ is called the *leading term* of the DAE (8.1.21).

Definition 8.1.4 The leading term of the DAE (8.1.21) is said to be *properly stated* if the matrices $\bar{A}(t)$ and $D(t)$ have the property that

$$\text{im } D(t) \oplus \ker \bar{A}(t) = \mathbb{R}^d \quad \text{for all } \ t \in I,$$

with the subspaces im $D(t)$ and ker $\bar{A}(t)$ spanned by C^1 bases.

The matrices $\bar{A}(t)$ and $D(t)$ in (8.1.21) are then called *well matched.*

The following statements are readily verified.

Lemma 8.1.4 *Assume that the matrices $\bar{A}(t)$ and $D(t)$ are well matched. Then:*

(i) rank $\bar{A}(t) =$ rank $D(t) =: r$ *is constant on I;*
(ii) im $\bar{A}(t)D(t) =$ im $A(t)$, ker $\bar{A}(t)D(t) =$ ker $D(t)$*;*
(iii) ker $\bar{A}(t) \bigcap$ im $D(t) = \emptyset$.

If the matrix $A(\cdot) \in L(\mathbb{R}^d)$ in the DAE (8.1.5) has constant rank and admits a factorisation $A(t) = \bar{A}(t)D(t)$ $(t \in I)$ so that $\bar{A}(\cdot)$ and $D(\cdot)$ are continuously differentiable matrix functions on I that are well matched, then the left-hand side of the DAE can be written in the form

$$\bar{A}(t)D(t)x(t) + B(t)x(t) = \bar{A}(t)(D(t)x(t))' + [B(t) - \bar{A}(t)D'(t)]x(t), \tag{8.1.22}$$

in analogy to (8.1.14).

8.1.2 IAEs with smooth kernels

We now turn to ‘mixed’ systems of Volterra integral equations,

$$A(t)x(t) = q(t) + \int_0^t k(t, s, x(s))ds, \quad t \in I, \tag{8.1.23}$$

with continuous $A(\cdot) \in L(\mathbb{R}^d)$ as in (8.1.4) and (8.1.5) (that is, $\det A(t) = 0$ and $\operatorname{rank} A(t) \geq 1$ on I). The *semi-explicit* linear version of this system is

$$y(t) = q_1(t) + (\mathcal{V}_{11}y)(t) + (\mathcal{V}_{12}z)(t), \tag{8.1.24}$$

$$0 = q_2(t) + (\mathcal{V}_{21}y)(t) + (\mathcal{V}_{22}z)(t), \quad t \in I, \tag{8.1.25}$$

where the Volterra integral operators $\mathcal{V}_{kl}$ are given by

$$(\mathcal{V}_{kl}\phi)(t) := \int_0^t K_{kl}(t,s)\phi(s)ds \quad (k, l = 1, 2). \tag{8.1.26}$$

The matrix kernels $K_{kl}(\cdot,\cdot)$ $(k, l = 1, 2)$: $K_{kk}(\cdot,\cdot) \in L(\mathbb{R}^{d_k})$, $K_{12}(\cdot,\cdot) \in L(\mathbb{R}^{d_2}, \mathbb{R}^{d_1})$, and $K_{21}(\cdot,\cdot) \in L(\mathbb{R}^{d_1}, \mathbb{R}^{d_2})$ are assumed to be continuous (or possibly unbounded but integrable). We will always assume that $q_2(0) = 0$.

As the nonlinear analogue we will choose the one based on Hammerstein operators,

$$y(t) = q_1(t) + \int_0^t K_1(t,s)G_1(s, y(s), z(s))ds, \tag{8.1.27}$$

$$0 = q_2(t) + \int_0^t K_2(t,s)G_2(s, y(s), z(s))ds, \quad t \in I, \tag{8.1.28}$$

with continuous $K_k(\cdot,\cdot) \in L(\mathbb{R}^{d_k})$ $(k = 1, 2)$; the functions $G_1 : I \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}^{d_1}$ and $G_2 : I \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}^{d_2}$ are assumed to be smooth. We set again $x(t) := (y(t),\ z(t))^T$.

In accordance with the terminology introduced by Gear (1990) we will refer to these systems of Volterra integral equations as *integral-algebraic equations* of Volterra type (or IAEs in short since we will not discuss Fredholm-type integral equations). Here, the word 'algebraic' assumes a wider meaning, in that it refers to the 'non-differential' constraints forming part of the system.

Definition 8.1.5 The semi-explicit IAE (8.1.24/25) is said to be *index-1 tractable* if the first-kind VIE corresponding to the Volterra operator $\mathcal{V}_{22}$,

$$(\mathcal{V}_{22}w)(t) = g(t), \quad t \in I, \tag{8.1.29}$$

is uniquely solvable in $C(I)$ whenever $g \in C^1(I)$ and $g(0) = 0$.

In the following we will focus again on *index-1 IEAs*, in analogy to DAEs in Section 8.1.1. Thus, the key question concerns the unique solvability for $z(t)$ of the (linear or nonlinear) first-kind Volterra integral equations (8.1.25) and (8.1.28) replacing the algebraic equations in (8.1.7) and (8.1.2).

Note first that when we have

$$\partial K_{2l}(t,s)/\partial t = 0 \quad \text{on} \quad D$$

if $\mathcal{V}_{21}$ and $\mathcal{V}_{22}$ are linear, or

$$\partial K_k(t,s)/\partial t = 0 \quad \text{on } D$$

in the nonlinear (Hammerstein) case, then differentiation of these first-kind integral equations leads to (linear or nonlinear) *algebraic* constraints,

$$0 = g'(t) + K_{21}(t)y(t) + K_{22}(t)z(t)$$

and

$$0 = g'(t) + K_2(t)G_2(t, y(t), z(t)),$$

respectively.

In order to deal with the general case of integral constraints, we resort to Theorems 2.1.8 and 6.1.16 (with $\alpha = 0$). It follows from the former that the *linear* VIE (8.1.25) is (formally) uniquely solvable for $z \in C(I)$ if we assume $q_2 \in C^1(I)$, with $q_2(0) = 0$, and if given matrix functions $K_{21}(\cdot,\cdot) \in L(\mathbb{R}^{d_1}, L(\mathbb{R}^{d_2}),\ K_{22}(\cdot,\cdot) \in L(\mathbb{R}^{d_2})$ describing the Volterra operators $\mathcal{V}_{21},\ \mathcal{V}_{22}$ in (8.1.25) are all continuous on their domains D and in addition satisfy

$$K_{21} \in C^1(D); \quad K_{22} \in C^1(D), \quad \text{with} \quad |\det K_{22}(t,t)| \geq k_0 > 0, \quad t \in I. \tag{8.1.30}$$

We now turn to the *nonlinear* IAE (8.1.28): in this case, Theorem 6.1.16) ($\alpha = 0$) reveals the additional conditions we have to impose on G_2. To see this in the present context, consider the differentiated form of (8.1.28),

$$0 = q_2'(t) + K_2(t,t)G_2(t, y(t), z(t)) + \int_0^t \frac{\partial K_2(t,s)}{\partial t} G_2(s, y(s), z(s))ds. \tag{8.1.31}$$

Thus, this implicit VIE is uniquely solvable for $z \in C(I)$ if $K_2(\cdot,\cdot) \in L(\mathbb{R}^{d_2})$ is continuously differentiable, with $|\det K_2(t,t)| \geq k_0 > 0$ $(t \in I)$, and if G_2 satisfies, for $y \in \mathbb{R}^{d_1},\ z,\ \tilde{z} \in \mathbb{R}^{d_2}$ $(z \neq \tilde{z})$,

$$\langle G_2(t,y,z) - G_2(t,y,\tilde{z}), z - \tilde{z}\rangle > 0, \quad t \in I, \tag{8.1.32}$$

$$\lim_{||z||\to\infty} \frac{G_2(t,y,z)z}{||z||} = \infty, \quad t \in I. \tag{8.1.33}$$

Here, $\langle\cdot,\cdot\rangle$ denotes the standard inner product in $\mathbb{R}^{d_2}$, with induced norm $||\cdot||$.

We will now have a closer look at the *linear* IAE (8.1.25) and describe its solvability and the regularity of its solution.

Theorem 8.1.5 *Let* $\nu \geq 0$ *and assume that*

(a) $K_{1l} \in C^\nu(D)$ *for* $l = 1, 2$;
(b) $K_{2l} \in C^{\nu+1}(D)$ $(l = 1, 2)$, *and* K_{22} *satisfies the condition (8.1.30);*
(c) $q_1 \in C^\nu(I)$ *and* $q_2 \in C^{\nu+1}(I)$, *with* $q_2(0) = 0$.

Then the IAE (8.1.24),(8.1.25) possesses a unique solution $x = (y,\ z)^T$ on I, with $y,\ z \in C^\nu(I)$.

The **proof** of the regularity of x can also be deduced from the following representation theorem (which we state and prove, for ease of notation, for the case $\mathcal{V}_{21} = 0$).

Theorem 8.1.6 *Assume that the hypotheses given in Theorem 8.1.5 hold with $\nu \geq 0$. Then for $K_{21}(t,s) = 0$ on D the (unique) solution of the linear IAE (8.1.24),(8.1.25) is given by the representation*

$$y(t) = q_1(t) + \int_0^t R_{11}(t,s)q_1(s)ds \qquad (8.1.34)$$

$$+ \kappa_{12}(t)q_2(t) + \int_0^t Q_{12}(t,s)q_2(s)ds,$$

$$z(t) = \kappa_{21}(t)q_2(t) + \kappa_{22}(t)q_2'(t) + \int_0^t Q_{22}(t,s)q_2(s)ds, \quad t \in I, \qquad (8.1.35)$$

where $R_{11}(t,s)$ denotes the matrix resolvent kernel of $K_{11}(t,s)$ in (8.1.26). It and the matrix functions $Q_{12},\ Q_{22}$ lie in $C^\nu(D)$. (The proof will give an indication of the connection between these functions and the given kernels K_{kl}.)

Proof If we denote by $R_{11}(t,s)$ and $R_{22}(t,s)$ the resolvent kernels associated respectively with the kernel $K_{11}(t,s)$ in (8.1.24) and the kernel

$$H_{22}(t,s) := -(K_{22}(t,t))^{-1}\frac{\partial K_{22}(t,s)}{\partial t}$$

arising when the differentiated form of (8.1.25),

$$0 = q_2'(t) + K_{22}(t,t)z(t) + \int_0^t \frac{\partial K_{22}(t,s)}{\partial t}z(s)ds, \qquad (8.1.36)$$

is rewritten as a standard second-kind VIE. Recall from Chapter 2 that these resolvent kernels can be defined by suitably adapted versions of the resolvent equations (2.1.9) or (2.1.10). The solutions of the second-kind VIEs (8.1.31) (with $K_{21} = 0$) and (8.1.24) are then given by

$$z(t) = g_2(t) + \int_0^t R_{22}(t,s)g_2(s)ds, \quad t \in I,$$

and

$$y(t) = q_1(t) + \int_0^t R_{11}(t,s)q_1(s)ds$$

$$+ \int_0^t \left(K_{12}(t,s) + \int_s^t R_{11}(t,v)K_{12}(v,s)dv\right)z(s)ds, \quad t \in I, \qquad (8.1.37)$$

respectively, where $g_2(t) := -(K_{22}(t,t))^{-1}q_2'(t)$ and $R_{22}(t,s)$ denotes the resolvent kernel for $H_{22}(t,s)$. It follows from the definition of g_2 and integration by parts that

$$\begin{aligned}\int_0^t R_{22}(t,s)g_2(s)ds &= -\int_0^t \frac{R_{22}(t,s)}{K_{22}(s,s)}q_2'(s)ds =: \int_0^t \tilde{Q}_{22}(t,s)q_2'(s)ds \\ &= -\tilde{Q}_{22}(t,t)q_2(t) + \int_0^t \frac{\partial \tilde{Q}_{22}(t,s)}{\partial s}q_2(s)ds \\ &=: \kappa_{21}(t)q_2(t) + \int_0^t Q_{22}(t,s)q_2(s)ds\end{aligned}$$

(note that, by assumption, $q_2(0) = 0$). The representation (8.1.35) for $z(t)$ now readily follows, by setting $\kappa_{22}(t) := -(K_{22}(t,t))^{-1}$.

The representation (8.1.34) for the solution component $y(t)$ results by replacing $z(s)$ by the right-hand side of (8.1.37), and by applying Dirichlet's formula and an integration by parts step. This representation also yields the desired information on the regularity of the solution y of the IAE (8.1.24/25).

8.1.3 IDAEs with smooth kernels

In this section we will consider the semilinear Volterra 'integro-differential-algebraic' equation (IDAE),

$$A(t)x'(t) + b(x(t),t) = \int_0^t k(t,s,x(s))ds, \quad t \in I, \tag{8.1.38}$$

(cf. (8.1.4) and (8.1.23)), as well as its linear version described by

$$b(x,t) = B(t)x - q(t) \quad \text{and} \quad k(t,s,x) = K(t,s)x.$$

We write the corresponding *linear semi-explicit* IDAE as

$$y'(t) + B_{11}(t)y(t) + B_{12}(t)z(t) = q_1(t) + (\mathcal{V}_{11}y)(t) + (\mathcal{V}_{12}z)(t), \tag{8.1.39}$$

$$0 = q_2(t) + (\mathcal{V}_{21}y)(t) + (\mathcal{V}_{22}z)(t), \tag{8.1.40}$$

with B_{11}, B_{12} and the integral operators $\mathcal{V}_{kl}$ as in (8.1.6/7) and (8.1.24/25), and subject to the hypotheses stated earlier. We assume again that $q_2(0) = 0$. An interesting special case – the link between a DAE and a 'full' IDAE, so to speak – is obtained when $\mathcal{V}_{11} = \mathcal{V}_{12} = 0$: we then have the coupling of an *ODE* with a first-kind VIE. If we are looking for a solution $x(t) = (y(t),\ z(t))^T$ to the IDAE system (8.1.39),(8.1.40) satisfying an initial condition $x_0 = (y_0,\ z_0)^T$, we observe that these initial values are *consistent* if

$$K_{21}(0,0)y_0 + K_{22}(0,0)z_0 = -q_2'(0).$$

Definition 8.1.6 The semi-explicit IDAE (8.1.39/40) is said to be *index*-1 *tractable* if the first-kind VIE corresponding to the Volterra operator $\mathcal{V}_{22}$ in (8.1.40),

$$(\mathcal{V}_{22}w)(t) = g(t), \quad t \in I, \tag{8.1.41}$$

is uniquely solvable in $C(I)$ whenever $g \in C^1(I)$ and $g(0) = 0$.

We will focus again on *index-1 IDEAs*, in analogy to DAEs in Section 8.1.1.

It follows from the resolvent theory of Section 3.1.1 and the analysis of the previous section on the representation of solutions of IAEs that the solution $x(t) := (y(t),\ z(t))^T$ of the IDAE (8.1.39),(8.1.40) can be expressed in a way completely analogous to that of Theorem 8.1.6. We state this in Theorem 8.1.7 but leave the details of the proof to the reader.

Theorem 8.1.7 *Assume that the matrix functions $B_{11}(\cdot,\cdot)$, $B_{12}(\cdot,\cdot)$ are in $C^\nu(I)$, with $\nu \geq 0$, and that $q_1,\ q_2$ and the kernels $K_{k,l}$ $(k, l = 1, 2)$ are subject to the assumptions (a)–(c) of Theorem 8.1.5. Then for any set of consistent initial values $\{y_0,\ z_0\}$ the IDEA (8.1.39),(8.1.40) possesses a unique solution $x = (y,\ z)^T$, with $y \in C^{\nu+1}(I)$ and $z \in C^\nu(I)$, satisfying $x(0) = (y_0,\ z_0)^T$.*

This solution can be represented in the form

$$y(t) = r_{11}(t,0)y_0 + \int_0^t r_{11}(t,s)q_1(s)ds \tag{8.1.42}$$

$$\qquad + \kappa_{12}(t)q_2(t) + \int_0^t q_{12}(t,s)q_2(s)ds,$$

$$z(t) = \kappa_{21}(t)q_2(t) + \kappa_{22}(t)q_2'(t) + \int_0^t q_{22}(t,s)q_2(s)ds, \quad t \in I. \tag{8.1.43}$$

Here, $r_{11} \in C^{\nu+1}(D)$ denotes the matrix resolvent kernel corresponding to the homogeneous data $B_{11}(\cdot)$ and $K_{11}(\cdot,\cdot)$ in (8.1.39), and the remaining functions κ_{12}, q_{12} and κ_{21}, κ_{22}, q_{22} inherit, respectively, the regularity of the data in (8.1.39) and (8.1.40).

Proof We proceed along the lines of the proof of Theorem 8.1.6 for IAEs. The argument regarding the solvability of the first-kind equation (8.1.40) and the representation of its solution z by (8.1.35) remains the same. If we view the VIDE (8.1.40) as an equation for y, then we may use the classical resolvent representation (see (8.1.46) below) to express y in terms of z: we know that formally this representation is the same for both equations. Substitution of the previously derived expression for z, and the adaptation of the techniques used in the proof of Theorem 8.1.6 then readily lead to the result of the theorem.

Remark The form of the solution representation in Theorem 8.1.7 remains of course valid for IDAEs of the special form

$$y'(t) + B_{11}(t)y(t) + B_{12}(t)z(t) = q_1(t) \tag{8.1.44}$$
$$(\mathcal{V}_{21}y)(t) + (\mathcal{V}_{22}z)(t) = q_2(t), \tag{8.1.45}$$

since the resolvent representation (3.1.4),

$$y(t) = r(t,0)y_0 + \int_0^t r(t,s)g(s)ds, \quad t \in I, \tag{8.1.46}$$

with $r(t,s)$ defined by

$$\frac{\partial r(t,s)}{\partial s} = -r(t,s)a(s) - \int_s^t r(t,v)K(v,s)dv, \quad (t,s) \in D,$$

of the solution to

$$y'(t) = a(t)y(t) + g(t) + \int_0^t K(t,s)y(s)ds, \quad y(0) = y_0,$$

is formally identical with the one for the ODE corresponding to $K(t,s) \equiv 0$.

Question Can we extend the notion of *inherent regular ODE* and *properly stated leading term*, described in Section 8.1.1 (recall Lemma 8.1.1, its proof, and Definition 8.1.4), to the linear version of the IDAE (8.1.38),

$$A(t)x'(t) + B(t)x(t) = q(t) + (\mathcal{V}x)(t), \quad t \in I, \tag{8.1.47}$$

where we have set

$$(\mathcal{V}x)(t) := \int_0^t K(t,s)x(s)ds,$$

with continuous matrix function $K(\cdot,\cdot) \in L(\mathbb{R}^d)$? In other words, what is the *inherent regular VIDE* of (8.1.38) or (8.1.47)?

Proceeding formally, starting from Lemma 8.1.1 (and omitting the argument t in $P(t)$, $A_1(t)$, $B_0(t)$), we may rewrite (8.1.47) as

$$(Px)' - P'Px + PA_1^{-1}B_0Px = PA_1^{-1}[q(t) + (\mathcal{V}x)(t)], \tag{8.1.48}$$
$$Qx + QA_1^{-1}B_0Px = QA_1^{-1}[q(t) + (\mathcal{V}x)(t)]. \tag{8.1.49}$$

This observation clearly reveals the limitations of the previous framework: the analogous decoupling analysis for IDAEs requires an *infinite-dimensional* (Hilbert or Banach space) setting, as has been pointed out in März (2002) and Lamour, März and Tischendorf (2001). Compare also the remarks in Section 9.3.

8.1.4 IAEs and IDAEs with weakly singular kernels

Except for the regularity results in Theorems 8.1.5 and 8.1.7, the analysis of Sections 8.1.2 and 8.1.3 carries over to IAEs and IDAEs in which the Volterra integral operators possess weakly singular kernels. In the linear case the definition (8.1.26) is then replaced by

$$(\mathcal{V}_{kl}^{\alpha}\phi)(t) := \int_0^t p_\alpha(t-s)K_{kl}(t,s)\phi(s)ds, \quad t \in I \quad (k,l = 1,2),$$

with $0 < \alpha \leq 1$. As in Chapter 6 the weakly singular factor p_α is given by $p_\alpha(t) := t^{-\alpha}$ when $0 < \alpha < 1$, and by $p_1(t) := \log(t)$. The continuous matrix kernels $K_{kl}(\cdot,\cdot)$ $(k,l = 1,2)$ satisfy $K_{kk}(\cdot,\cdot) \in L(\mathbb{R}^{d_k})$, $K_{12}(\cdot,\cdot) \in L(\mathbb{R}^{d_2},\mathbb{R}^{d_1})$, and $K_{21}(\cdot,\cdot) \in L(\mathbb{R}^{d_1},\mathbb{R}^{d_2})$, with $K_{kl}(t,t) \neq 0$ on I.

Hence, the weakly singular analogue of the *semi-explicit* linear IAE (8.1.24/25) reads

$$y(t) = q_1(t) + (\mathcal{V}_{11}^{\alpha}y)(t) + (\mathcal{V}_{12}^{\alpha}z)(t), \tag{8.1.50}$$

$$0 = q_2(t) + (\mathcal{V}_{21}^{\alpha}y)(t) + (\mathcal{V}_{22}^{\alpha}z)(t), \quad t \in I. \tag{8.1.51}$$

We will always assume that $q_2(0) = 0$, and we set again $x(t) := (y(t),\ z(t))^T$.

In the nonlinear case corresponding to (8.1.27/28) the Volterra operators have the Hammerstein forms

$$\int_0^t p_\alpha(t-s)K_k(t,s)G_k(s,y(s),z(s))ds \quad (k = 1,2),$$

with $0 < \alpha \leq 1$, continuous kernels K_k not vanishing along $t = s$, and smooth G_k as in (2.1.27) and (2.1.28). The corresponding weakly singular IAE is then

$$y(t) = q_1(t) + \int_0^t p_\alpha(t-s)K_1(t,s)G_1(s,y(s),z(s))ds, \tag{8.1.52}$$

$$0 = q_2(t) + \int_0^t p_\alpha(t-s)K_2(t,s)G_2(s,y(s),z(s))ds, \quad t \in I. \tag{8.1.53}$$

As in Section 8.1.2 we will again focus on *index-1 IEAs*.

Definition 8.1.7 The semi-explicit IAE system (8.1.50/51) is said to be *index*-1 *tractable* if the first-kind VIE corresponding to the Volterra operator $\mathcal{V}_{22}^{\alpha}$,

$$(\mathcal{V}_{22}^{\alpha}w)(t) = g(t), \quad t \in I, \tag{8.1.54}$$

is uniquely solvable in $C(I)$ whenever $g \in C^1(I)$ and $g(0) = 0$.

Index-1 tractability for the nonlinear system (8.1.52/53) is defined analogously, based on the unique solvability of (8.1.53) in $C(I)$ with respect to z.

Conditions under which the first-kind VIEs (8.1.51) and (8.1.53) possess unique continuous solutions can be found in, or deduced from, Theorems 6.1.13

and 6.1.16. In the following we will restrict our analysis to values $\alpha \in (0, 1)$; its (rather straightforward) extension to logarithmic kernel singularities is left to the reader.

The degree of regularity of y and z follows essentially from Theorems 6.1.11 and 6.1.14. To see this, assume for ease of exposition that $\mathcal{V}_{21}^{\alpha} = 0$ in (8.1.51) (compare Exercise 8.6.6 for the general case). If $q_2 \in C^{\nu+1}(I)$, with $q_2(0) = 0$, and $K_{22} \in C^{\nu+1}(D)$ satisfies $|K_{22}(t, t)| \geq k_0 > 0$ on I, then it follows from Theorem 6.1.14 that the solution z of the first-kind VIE (8.1.51) lies in the Hölder space $C^{\alpha}(I)$ and possesses continuous derivatives up to order ν on $(0, T]$.

Consider now (8.1.50): since

$$\int_0^t (t-s)^{-\alpha} s^{\alpha} ds = t^{1-\alpha+\alpha} \int_0^1 (1-v)^{-\alpha} v^{\alpha} dv = B(1-\alpha, 1+\alpha)t,$$

the contribution of the term $(\mathcal{V}_{12}^{\alpha} z)(t)$ is smooth: it is in $C^{\nu}(I)$ (but see also the Remark below). Hence, Theorem 6.1.11 tells us that the (unique) solution of

$$y(t) = f(t) + (\mathcal{V}_{11}^{\alpha} y)(t), \quad t \in I,$$

with $f(t) := q_1(t) + (\mathcal{V}_{12}^{\alpha} z)(t)$, lies in $C^{1-\alpha}(I)$ but has continuous derivatives up to order ν on $(0, T]$.

We summarise these observations in

Theorem 8.1.8 *Assume that* $0 < \alpha < 1$, *and let the given functions* q_k *and* K_{kl} $(k, l = 1, 2)$ *in the index-1 IAE system (8.1.50/51) satisfy the hypotheses stated in Theorem 8.1.5. If* $K_{21} \equiv 0$, *then the regularity of its solution* $x = (y,\ z)^T$ *is described by*

$$y \in C^{1-\alpha}(I), \quad \textit{with} \quad y \in C^{\nu}(0, T],$$

and

$$z \in C^{\alpha}(I), \quad \textit{with} \quad z \in C^{\nu}(0, T].$$

Remark Suppose that the Volterra integral operators describing the IAE system (8.1.50/51) are replaced respectively by $\mathcal{V}_{1l}^{\alpha_1}$ $(0 < \alpha_1 < 1)$ (in (8.1.50)) and $\mathcal{V}_{2l}^{\alpha_2}$ $(0 < \alpha_2 < 1)$ (in (8.1.51)). Let again $K_{21} \equiv 0$. Since $z \in C^{\alpha_2}(I)$, it follows from the analysis preceding Theorem 8.1.8 that

$$\int_0^t (t-s)^{-\alpha_1} s^{\alpha_2} ds = B(1-\alpha_1, 1+\alpha_2) t^{1-\alpha_1+\alpha_2}.$$

Hence, according to Theorem 6.1.11, we have $y \in C^{\alpha_2}(I)$, regardless of the value of α_1.

A similar regularity result is true for the semi-explicit IDAE system

$$y'(t) + B_{11}(t)y(t) + B_{12}(t)z(t) = q_1(t) + (\mathcal{V}^{\alpha}_{11}y)(t) + (\mathcal{V}^{\alpha}_{12}z)(t), \quad (8.1.55)$$

$$0 = q_2(t) + (\mathcal{V}^{\alpha}_{21}y)(t) + (\mathcal{V}^{\alpha}_{22}z)(t), \quad (8.1.56)$$

with B_{11}, B_{12} and the integral operators $\mathcal{V}^{\alpha}_{kl}$ as in (8.1.6) and (8.1.50/51), respectively, and subject to the hypotheses stated earlier. We assume again that $q_2(0) = 0$ and that the tractability index of (8.1.55/56) equals one.

Theorem 8.1.9 *Let $0 < \alpha < 1$ and assume that the given functions in (8.1.55/56) satisfy the assumptions stated in Theorem 8.1.7. If $\mathcal{V}^{\alpha}_{21} = 0$, then the regularity of the solution $x = (y,\ z)^T$ to the IDAE system (8.1.55/56) is given by*

$$y \in C^{1,1-\alpha}(I), \quad \text{with} \quad y \in C^{\nu+1}(0, T],$$

and

$$z \in C^{\alpha}(I), \quad \text{with} \quad z \in C^{\nu}(0, T].$$

An interesting special case corresponds to $\mathcal{V}^{\alpha}_{11} = \mathcal{V}^{\alpha}_{12} = 0$: we then have the coupling of an *ODE* with a weakly singular first-kind VIE.

Corollary 8.1.10 *Let $0 < \alpha < 1$. If $K_{1l} \equiv 0\ (l = 1, 2)$ and $K_{21} \equiv 0$ in (8.1.55/56), then $z \in C^{\alpha}(I)$, with $z \in C^{\nu}(0, T]$, and $y \in C^{1,\alpha}(I)$, with $y \in C^{\nu+1}(0, T]$.*

The proofs of the above regularity results are a direct consequence of Theorems 6.1.14 and Theorem 7.1.4. Details are left to the reader (who should also study Exercise 8.9.2(ii) to obtain analogous results for more general IDAE systems).

8.2 Collocation for DAEs: a brief review

8.2.1 The collocation equations for index-1 problems

Let $x(t) := (y(t),\ z(t))^T$ denote the solution of the semi-explicit index-1 DAE

$$y'(t) = F(t, y(t), z(t)), \quad (8.2.1)$$

$$0 = G(t, y(t), z(t)), \quad t \in I := [0, T], \quad (8.2.2)$$

with consistent initial values $\{y_0, z_0\}$. Assume that it is approximated by the collocation solution $x_h(t) := (u_h(t),\ v_h(t)^{\ T})$, where $u_h,\ v_h \in S^{(0)}_m(I_h)$ are determined by the collocation equations

$$u'_h(t) = F(t, u_h(t), v_h(t)), \quad t \in X_h, \quad (8.2.3)$$

$$0 = G(t, u_h(t), v_h(t)), \quad t \in X_h, \quad (8.2.4)$$

and the initial conditions $u_h(0) = y_0,\ v_h(0) = z_0$.

Let $X_h := \{t_{n,i} := t_n + c_i h_n : \ 0 < c_1 < \ldots < c_m \leq 1 \ (0 \leq n \leq N-1)\}$ be the set of the collocation points. In analogy to Chapter 1, the computational form of the collocation equations (8.2.3),(8.2.4) will be based on the local Lagrange representations of u_h and v_h:

$$u_h(t_n + sh_n) = y_n + h_n \sum_{j=1}^{m} \beta_j(s) Y_{n,j}, \tag{8.2.5}$$

$$v_h(t_n + sh_n) = z_n + h_n \sum_{j=1}^{m} \beta_j(s) Z_{n,j}, \quad s \in [0,1], \tag{8.2.6}$$

where $y_n := u_h(t_n), \ z_n := v_h(t_n), \ Y_{n,j} := u_h'(t_{n,i}), \ Z_{n,j} := v_h(t_{n,i})$. Thus, for $t = t_{n,i}$ the collocation equations become the stage equations for the resulting *continuous m-stage implicit Runge–Kutta method* for the given semi-explicit DAE, namely

$$Y_{n,i} = F(t_{n,i}, y_n + h_n \sum_{j=1}^{m} a_{i,j} Y_{n,j}, z_n + h_n \sum_{j=1}^{m} a_{i,j} Z_{n,j}), \tag{8.2.7}$$

$$0 = G(t_{n,i}, y_n + h_n \sum_{j=1}^{m} a_{i,j} Y_{n,j}, z_n + h_n \sum_{j=1}^{m} a_{i,j} Z_{n,j}) \tag{8.2.8}$$

$(i = 1, \ldots, m)$, with $a_{i,j} := \beta_j(c_i)$.

Since classical Runge–Kutta methods and collocation methods for DAEs are well understood (detailed treatments, also for DAES with index 2 and higher, can be found in the books by Griepentrog and März (1986), Brenan, Campbell and Petzold (1996), Hairer, Lubich and Roche (1989), Strehmel and Weiner (1992) (index-1 DAEs), and Hairer and Wanner (1996); see also Ascher and Petzold (1991) and Jay (1993)), we will only state a typical (super-) convergence result. Additional convergence results will be given in Section 8.4 for IDAEs (see in particular Theorem 8.4.2) of which our DAEs are particular cases.

Theorem 8.2.1 *Let $x_h = (u_h, \ v_h)^T$ be the collocation solution to the semi-explicit DAE (8.2.1),(8.2.2), with u_h and v_h in $S_m^{(0)}(I_h)$. Define, as in Theorem 2.4.2,*

$$\rho_m := (-1)^m \prod_{i=1}^{m} \frac{1-c_i}{c_i}.$$

If the given functions are sufficiently regular, and such that (8.2.1),(8.2.2) is an index-1 DAE, the following is true:

(i) *The collocation solution* $x_h = (u_h,\ v_h)^T$ *converges to* $x = (y,\ z)^T$ *if, and only if,* $|\rho_m| \le 1$.

(ii) *At the mesh points* $I_h' := I_h \setminus \{0\}$ *we have*

$$\max_{t \in I_h'} |y(t) - u_h(t)| \le C_1 h^{2m-1},$$
$$\max_{t \in X_h'} |z(t) - v_h(t)| \le C_2 h^m,$$

provided the collocation parameters are the Radau II points (for which $\rho_m = 0$*).*

(iii) *If collocation is at the Gauss points (where* $\rho_m = \pm 1$*), the local superconvergence property is lost: the optimal orders of convergence are*

$$\max_{t \in X_h'} |y(t) - u_h(t)| \le C_1 h^m,$$
$$\max_{t \in X_h'} |z(t) - v_h(t)| \le C_2 \begin{cases} h^m & \text{if } -1 \le \rho_m < 1, \\ h^{m-1} & \text{if } \rho_m = 1. \end{cases}.$$

Remarks

1. The the eigenvalue ρ_m (recall the proof of Theorem 2.4.2) is closely related to the stability function $R(z)$ associated with the (collocation-based) implicit Runge–Kutta method using as abscissas the collocation parameters $\{c_i\}$ with $c_1 > 0$ (see, e.g. Dekker and Verwer (1984) or Hairer and Wanner (1996)): it coincides with $R(\infty)$. The reader may also wish to consult Kauthen and Brunner (1997) for details on this connection.
2. As Ascher (1989) has shown, the collocation approximation v_h to the algebraic component z of x could also have been sought in $S_{m-1}^{(-1)}(I_h)$.

8.2.2 Collocation for semi-explicit index-2 DAEs

We will briefly illustrate, mainly for the sake of comparison, the collocation equations for the semi-explicit index-2 DAE

$$y'(t) = F(t, y(t), z(t)),$$
$$0 = G(t, y(t)), \quad t \in I,$$

where F and G are smooth and G_y is non-singular. The collocation approximations u_h and v_h in $S_m^{(0)}(I_h)$ to y and z are defined by their local representations

(8.2.5) and (8.2.6) and the collocation equations

$$Y_{n,i} = f(t_{n,i}, y_n + h_n \sum_{j=1}^{m} a_{i,j} Y_{n,j}, z_n + h_n \sum_{j=1}^{m} a_{i,j} Z_{n,j}),$$

$$0 = G(t_{n,i}, y_n + h_n \sum_{j=1}^{m} a_{i,j} Y_{n,j}) \quad (i = 1, \dots, m).$$

Detailed analyses of the convergence properties of Runge–Kutta methods for index-2 DAEs can be found in, e.g. Griepentrog and März (1986), Petzold (1986) and, especially, in März and Rodríguez-Santiesteban (2002). Convergence results for piecewise polynomial collocation methods (and corresponding implicit Runge–Kutta methods) applied to index-2 DAEs are discussed in Hairer and Wanner (1996, pp. 498–501); a summary of local superconvergence results is given on p. 504. Ascher and Petzold (1991) introduced projected collocation methods and studied their properties; see also Lubich (1991) and März (1996) for an in-depth analysis and wider prespective of projected collocation methods.

8.2.3 Numerically properly formulated DAEs

The above (superconvergence order) results for index-1 DAEs do not necessarily imply that the collocation solution possesses the correct dynamics as $t \to \infty$ (and $h > 0$ fixed). This fact is closely connected with the notion of a properly formulated DAE described in Section 8.1.1. We will illustrate this by analysing the collocation in the space $S_1^{(0)}(I_h)$ and with $c_1 = 1$ for a simple index-1 DAE: while the resulting continuous implicit Euler method shows the familiar unconditional asymptotic behaviour when the DAE has constant coefficients, it is no longer asymptotically stable for any stepsize $h > 0$ in the case of variable coefficients. This deficiency can be rectified if the DAE is rewritten so that its leading terms are well matched, as described at the end of Section 8.1.1 (Definition 8.1.4).

We use the following linear DAE (see also Gear and Petzold (1984), März (1996), März and Rodríguez-Santiesteban (2002)) to show that even in an index–1 DAE a usually 'foolproof' collocation method (like the continuous implicit Euler method) may not reflect the asymptotic behaviour of the exact solution unconditionally for any stepsize $h > 0$.

Example 8.2.1

$$A(t)x'(t) + B(t)x(t) = q(t), \quad t \in I, \tag{8.2.9}$$

with

$$A(t) = \begin{bmatrix} \delta - 1 & \delta t \\ 0 & 0 \end{bmatrix}, \qquad B(t) = \sigma \begin{bmatrix} \delta - 1 & \delta t \\ \delta - 1 & \delta t - 1 \end{bmatrix}, \tag{8.2.10}$$

where $\delta, \sigma \in \mathbb{R}$, with $\delta \neq 1,\ \sigma \neq 0$. It is easy to see that under these assumptions the DAE has (tractability) *index one*. Note also that for $\delta = 0$ the DAE has constant coefficients. For $q(t) \equiv 0$ the solution $x(t) = (y(t),\ z(t))^T$ is given by

$$z(t) = \exp((\delta - \sigma)t)z_0, \quad y(t) = \frac{1 - \delta t}{\delta - 1} z(t).$$

(Note that the initial values $x(0) = x_0 = (y_0,\ z_0)^T$ are consistent if $(\delta - 1)y_0 - z_0 = 0$, or $y_0 = z_0/(\delta - 1)$.) It thus follows that

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} y(t) = 0$$

whenever $\delta < \sigma$.

Assume now that we solve (8.2.10) by collocation in $S_1^{(0)}(I_h)$, with uniform mesh I_h and $0 < c_1 \leq 1$. We use the local representations

$$u_h(t_n + sh) = y_n + shY_{n,1}, \quad v_h(t_n + sh) = z_n + shZ_{n,1} \quad (v \in [0, 1]),$$

where $y_n := u_h(t_n),\ z_n := v_h(t_n),\ Y_{n,1} := u_h'(t_n + c_1 h),\ Z_{n,1} := v_h'(t_n + c_1 h)$. The collocation equation defining the collocation solution $x_h = (u_h,\ v_h)^T$,

$$\begin{bmatrix} \delta - 1 & \delta t_{n,1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_h'(t_{n,1}) \\ v_h'(t_{n,1}) \end{bmatrix} + \sigma \begin{bmatrix} \delta - 1 & \delta t_{n,1} \\ \delta - 1 & \delta t_{n,1} - 1 \end{bmatrix} \begin{bmatrix} u_h(t_{n,1}) \\ v_h(t_{n,1}) \end{bmatrix} = 0,$$

can be rewritten as

$$\begin{bmatrix} \delta - 1 & \delta t_{n,1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_{n+1} - y_n \\ z_{n+1} - z_n \end{bmatrix} + \sigma h \begin{bmatrix} \delta - 1 & \delta t_{n,1} \\ \delta - 1 & \delta t_{n,1} - 1 \end{bmatrix}$$
$$\times \begin{bmatrix} (1 - c_1)y_n + c_1 y_{n+1} \\ (1 - c_1 z_n + c_1 z_{n+1} \end{bmatrix} = 0$$

(since $Y_{n,1} = (y_{n+1} - y_n)/h$, etc.). For $c_1 = 1$ (which yields the continuous *implicit Euler method*) we find the recursion

$$z_{n+1} = \frac{1 + \delta h}{1 + \sigma h} z_n, \quad y_{n+1} = \frac{1 - \delta t_{n+1}}{\delta - 1} z_{n+1}.$$

It reveals the following:

- For $\delta = 0$ (DAE with *constant coefficients*) and $\sigma < 0$ we see that

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} y_n = 0$$

  for any stepsize $h > 0$. This reflects the 'typical' unconditional asymptotic stability of the implicit Euler method.

- For $\delta \neq 0$ (DAE with *variable coefficients*) the situation is very different: the approximation z_n to the algebraic component $z(t)$ at $t = t_n$ does not remain bounded for every $h > 0$. More precisely, x_h is asymptotically stable only under the *stepsize restriction*

$$|1 + \delta h| < |1 + \sigma h|.$$

This shows that, in contrast to ODEs ('DAEs are not ODEs' – Petzold (1984)) DAE test equations with *constant coefficients* do not properly model the general qualitative behaviour of solutions to *variable coefficient* DAEs, even if they are linear: variable coefficients in DAEs have a much stronger effect on the numerical solution than in ODEs.

This phenomenon is even more pronounced in DAEs of higher index. Illuminating examples (and additional references) can be found in März (1992) and Schulz (2003).

However, choosing $\bar{A} := \text{diag}\,(1,\ 0),\ \ D = A$, we obtain $\bar{B} := B - \bar{A}D$ (cf. (8.1.22)), and the use of the continuous implicit Euler method to the decoupled DAE

$$\bar{A}(D(t)x(t))' + \bar{B}(t)x(t) = 0, \quad t \geq 0, \tag{8.2.11}$$

with $\delta \neq 0,\ \delta < \sigma$, yields an approximation that is asymptotically stable for any stepsize $h > 0$. Its components are given by

$$z_{n+1} = \frac{1}{1 - (\delta - \sigma)h} z_n,$$
$$y_{n+1} = \frac{1 - \delta t_{n+1}}{\delta - 1} z_{n+1}.$$

We leave the proof of this statement to the reader.

8.3 Collocation for IAEs with smooth kernels

8.3.1 The collocation equations for Volterra IAEs

Consider the IAE system given by

$$y(t) = q_1(t) + (\mathcal{V}_{11}y)(t) + (\mathcal{V}_{12}z)(t), \tag{8.3.1}$$
$$0 = q_2(t) + (\mathcal{V}_{21}y)(t) + (\mathcal{V}_{22}z)(t), \quad t \in I := [0, T], \tag{8.3.2}$$

and based on the (linear) Volterra integral operators $\mathcal{V}_{kl} :\ C)I) \to C(I)$,

$$(\mathcal{V}_{kl}\phi)(t) := \int_0^t K_{kl}(t, s)\phi(s)ds, \quad K_{kl} \in C(D) \tag{8.3.3}$$

(see (8.1.24/25)). We approximate its solution $x := (y,\ z)^T$ by $x_h := (u_h,\ v_h)^T$, with u_h and v_h in $S_{m-1}^{(-1)}(I_h)$ (and $u_h :\ I \to \mathbb{R}^{d_1},\ v_h :\ I \to \mathbb{R}^{d_2}$). This collocation solution is defined by the collocation equations

$$u_h(t) = q_1(t) + (\mathcal{V}_{11}u_h)(t) + (\mathcal{V}_{12}v_h)(t), \tag{8.3.4}$$

$$0 = q_2(t) + (\mathcal{V}_{21}u_h)(t) + (\mathcal{V}_{22}v_h)(t), \quad t \in X_h, \tag{8.3.5}$$

with the set of collocation points, X_h, as before. When x_h has been found we can use it to define the iterate of u_h,

$$u_h^{it}(t) := q_1(t) + (\mathcal{V}_{11}u_h)(t) + (\mathcal{V}_{12}v_h)(t), \quad t \in I. \tag{8.3.6}$$

In order to describe the key ideas without having to resort to complex notation involving Kronecker products of matrices and vectors (for example in the linear system (8.3.12) below), we will assume that $d_1 = d_2 = 1$.

Suppose then that the local representations of u_h and v_h are again

$$u_h(t_n + vh_n) = \sum_{j=1}^{m} L_j(v)U_{n,j}, \quad v \in (0, 1], \quad \text{with} \quad U_{n,j} := u_h(t_{n,j}), \tag{8.3.7}$$

$$v_h(t_n + vh_n) = \sum_{j=1}^{m} L_j(v)V_{n,j}, \quad v \in (0, 1], \quad \text{with} \quad V_{n,j} := v_h(t_{n,j}). \tag{8.3.8}$$

The computational forms of the collocation equations (8.3.4), (8.3.5) at $t = t_{n,i}$ then become

$$U_{n,i} = h_n \sum_{j=1}^{m} \left(\int_0^{c_i} K_{11}(t_{n,i}, t_n + sh_n)L_j(s)ds \right) U_{n,j} + h_n \sum_{j=1}^{m} \left(\int_0^{c_i} K_{12}(t_{n,i}, t_n + sh_n)L_j(s)ds \right) V_{n,j} + q_1(t_{n,i}) + F_n^{(1)}(t_{n,i}), \tag{8.3.9}$$

$$0 = h_n \sum_{j=1}^{m} \left(\int_0^{c_i} K_{21}(t_{n,i}, t_n + sh_n)L_j(s)ds \right) U_{n,j} + h_n \sum_{j=1}^{m} \left(\int_0^{c_i} K_{22}(t_{n,i}, t_n + sh_n)L_j(s)ds \right) V_{n,j} + q_2(t_{n,i}) + F_n^{(2)}(t_{n,i}), \tag{8.3.10}$$

with lag term approximations given by

$$F_n^{(k)}(t_{n,i}) = \int_0^{t_n} \big(K_{k1}(t_{n,i}, s)u_h(s) + K_{k2}(t_{n,i}, s)v_h(s)\big)\, ds \quad (k = 1, 2). \tag{8.3.11}$$

In order to formulate the resulting systems of linear algebraic equations for the vector $\mathbf{U}_n := (U_{n,1}, \ldots, U_{n,m})^T$ and $\mathbf{V}_n := (V_{n,1}, \ldots, V_{n,m})^T$ we adapt the notation introduced in Section 2.2.2 and define the matrices

$$B_n^{(k,l)} := \begin{pmatrix} \int_0^{c_i} K_{kl}(t_{n,i}, t_n + sh_n) L_j(s) ds \\ (i, j = 1, \ldots, m) \end{pmatrix} \quad (k, l = 1, 2)$$

in $L(\mathbb{R}^m)$. Moreover, we set

$$\mathbf{f}_n := (q_1(t_{n,1}), \ldots, q_1(t_{n,m}))^T, \quad \mathbf{g}_n := (q_2(t_{n,1}), \ldots, q_2(t_{n,m}))^T,$$

and

$$\mathbf{G}_n^{(k)} := (F_n^{(k)}(t_{n,1}), \ldots, F_n^{(k)}(t_{n,m}))^T \quad (k = 1, 2).$$

The algebraic system defining $\mathbf{U}_n$ and $\mathbf{V}_n$ can then be written as

$$\begin{bmatrix} \mathcal{I}_m - h_n B_n^{(1,1)} & -h_n B_n^{(1,2)} \\ B_n^{(2,1)} & B_n^{(2,2)} \end{bmatrix} \begin{bmatrix} \mathbf{U}_n \\ \mathbf{V}_n \end{bmatrix} = \begin{bmatrix} \mathbf{f}_n + \mathbf{G}_n^{(1)} \\ -h_n^{-1}[\mathbf{g}_n + \mathbf{G}_n^{(2)}] \end{bmatrix}. \tag{8.3.12}$$

It is clear that due to our assumptions on the kernels K_{kl} the left-hand side (block-) matrix is non-singular for all sufficiently small h_n: there exists a $\bar{h} > 0$ so that the linear algebraic system (8.3.12) has a unique solution $\mathbf{U}_n,\ \mathbf{V}_n$ for $n = 0, 1, \ldots, N-1$ whenever I_h is a mesh with $h \in (0, \bar{h})$. This is a consequence of the structure of the matrix block $\mathcal{I}_m - h_n B_n^{(1,1)}$ (cf. Theorem 2.2.1) and the form of $B_n^{(2,2)}$ (condition (8.2.20) on $K_{22}(t, s)$ and Theorem 2.4.1).

8.3.2 Convergence results

Once we know the collocation solution components u_h and v_h we can compute the *iterated collocation solution* u_h^{it} at $t = t_n + vh_n$ ($v \in [0, 1]$) via (8.3.6). However, as we shall see below (Theorem 8.3.2) this will now not have any advantage (except for generating a *continuous* approximation on I). Since the given system of IAEs contains a *first-kind* Volterra integral equation, the convergence properties of the collocation solution w_h will be governed by the (necessary and sufficient) conditions on the $\{c_i\}$ given in Theorem 2.4.2. The 'coupling' of VIEs of the second and first kind will also mean that collocation at the Gauss points (for which we have $c_m < 1$) will no longer lead to global superconvergence (of order $m + 1$) on I, or to local superconvergence (of order $2m$) on I_h in the iterated collocation solution u_h^{it}.

We start with the following global convergence result (due to Kauthen (2001)).

Theorem 8.3.1 *Let the assumptions of Theorem 8.1.5 hold with $\nu \geq m$, and suppose that I_h is a mesh with $h \in (0, \bar{h})$. Then the following statements are true for the collocation solution x_h, with $u_h,\ v_h \in S_{m-1}^{(-1)}(I_h)$:*

(i) For every choice of the $\{c_i\}$ with $0 < c_1 < \ldots < c_m = 1$ we have

$$||y - u_h||_\infty \leq C_1 h^m, \qquad ||z - v_h||_\infty \leq C_2 h^m.$$

(ii) If $0 < c_1 < \ldots < c_m < 1$, the attainable order of convergence is given by

$$||y - u_h||_\infty \leq C_1 h^m,$$
$$||z - v_h||_\infty \leq C_2 \begin{cases} h^m & \text{if } \rho_m \in [-1, 1), \\ h^{m-1} & \text{if } \rho = 1. \end{cases}$$

Here,

$$\rho_m := (-1)^m \prod_{i=1}^m \frac{1 - c_i}{c_i}.$$

Proof The proof of this result combines elements of the proofs for Theorems 2.2.3 and 2.4.2. For $c_m = 1$ the assertion that both u_h and v_h converge with the same order $p = m$ is easily verified. In the case where $c_m < 1$ we have to resort to the differencing procedure (used in the proof of Theorem 2.4.2) for the collocation equation corresponding to the first-kind VIE (8.2.3). The reader is referred to Kauthen (2001, pp. 1509–1511) for the details.

We now turn to the question on the attainable orders of (global and local) superconvergence for u_h and v_h. As in Section 2.2, let

$$e_h(t_n + vh_n) := \sum_{j=1}^m L_j(v)\mathcal{E}_{n,j} + h_n^m R_{m,n}^{(1)}(v), \quad v \in (0, 1],$$

and

$$\epsilon_h(t_n + vh_n) := \sum_{j=1}^m L_j(v)E_{n,j} + R_{m,n}^{(2)}(v) \quad v \in (0, 1],$$

with $\mathcal{E}_{n,j} := e_h(t_{n,j})$ and $E_{n,i} := \epsilon_h(t_{n,j})$, be the local representations of the collocation errors $e_h := y - u_h$ and $\epsilon_h := z - v_h$ on σ_n. These errors solve the IAE system

$$e_h(t) = \delta_h(t) + (\mathcal{V}_{11}e_h)(t) + (\mathcal{V}_{12}\epsilon_h)(t), \tag{8.3.13}$$
$$0 = d_h(t) + (\mathcal{V}_{21}e_h)(t) + (\mathcal{V}_{22}\epsilon_h)(t), \quad t \in I, \tag{8.3.14}$$

where the defects δ_h and d_h vanish at all points of X_h. Since the above system has the same structure as (8.1.39) and (8.1.40), with $e_h,\ \epsilon_h,\ \delta_h$ and d_h replacing

$y,\ z,\ f$ and g, Theorem 8.1.6 tells us that the solution of the system of IAEs for the errors has the representation

$$e_h(t) = \delta_h(t) + \int_0^t R_{11}(t,s)\delta_h(s)ds \tag{8.3.15}$$
$$+\kappa_{12}(t)d_h(t) + \int_0^t Q_{12}(t,s)d_h(s)ds,$$
$$\epsilon_h(t) = \kappa_{21}(t)d_h(t) + \kappa_{22}(t)d_h'(t) + \int_0^t Q_{22}(t,s)d_h(s)ds, \quad t \in I. \tag{8.3.16}$$

Moreover, we have $e_h^{it} := y - u_h^{it} = e_h - \delta_h$. Setting $t = t_n$ $(n = 1, \ldots, N)$ we are led to two observations: firstly, the expression for $e_h(t_n)$ reduces to a sum of integrals (containing the defects δ_h and d_h in their integrands) if, and only if, $t_n \in X_h$, that is, if $c_m = 1$. In this case, the well-known quadrature argument we have employed in the proofs of superconvergence results for second-kind VIEs can again be applied: they show that if the $\{c_i\}$ are the Radau II points then e_h exhibits $\mathcal{O}(h^{2m-1})$-convergence on $I_h \setminus \{0\}$. We also see that local superconvergence in e_h^{it} of order $2m$ is no longer possible when the Gauss points are used, due to the presence of the term $\kappa_{12}(t)d_h(t)$ in (8.3.15).

A similar reason prevents the occurrence of local superconvergence in $\epsilon_h(t_n)$. Here, (8.2.3) contains, in addition to $\kappa_{21}(t_n)$ (which vanishes when $c_m = 1$) the term $\kappa_{22}(t_n)d_h'(t_n)$: it would only be zero in the case of *Hermite*-type collocation at $c_m = 1$. This is of course not surprising after what we have learned in Section 2.4.2.

We summarise the results obtained from these observations in

Theorem 8.3.2 *Assume:*

(a) $f \in C^\nu(I),\ K_{1l} \in C^\nu(D)$ $(l = 1, 2)$.
(b) $g \in C^{\nu+1}(I),\ g(0) = 0$, *and* $K_{2l} \in C^{\nu+1}(D)$, *with* $|K_{22}(t,t)| \geq k_0 > 0$ *for* $t \in I$.
(c) $u_h \in S_{m-1}^{(-0)}(I_h)$ *and* $v_h \in S_{m-1}^{(-1)}(I_h)$ *are the collocation solutions to the IAE (8.3.1),(8.3.2) determined by (8.3.4) and (8.3.5).*

If the collocation points X_h *correspond to the Radau II* $\{c_i\}$ *points (where* $c_m = 1$*) and if* $\nu \geq 2m-1$, *then*

$$\max_{t \in I_h \setminus \{0\}} |y(t) - u_h(t)| \leq C_1 h^{2m-1}, \tag{8.3.17}$$
$$\max_{t \in I_h \setminus \{0\}} |z(t) - v_h(t)| \leq C_2 h^m. \tag{8.3.18}$$

If collocation is based on the Gauss points, then the (local) order of u_h^{it} on I_h cannot in general exceed

$$\max_{t \in X_h \setminus \{0\}} |y(t) - u_h^{it}(t)| \le C_1 h^m,$$

even if $\nu \ge 2m$. The order of v_h depends on the value of ρ_m: if $\rho_m \in [-1, 0)$ then we obtain $\mathcal{O}(h^m)$-convergence, while for $\rho_m = 1$ the order reduces to $p = m - 1$.

8.4 Collocation for IDAEs with smooth kernels

8.4.1 The collocation equations for Volterra IDAEs

Recall the IDAE system given by (8.1.39) and (8.1.40):

$$y'(t) + B_{11}(t)y(t) + B_{12}(t)z(t) = q_1(t) + (\mathcal{V}_{11}y)(t) + (\mathcal{V}_{12}z)(t), \quad (8.4.1)$$

$$0 = q_2(t) + (\mathcal{V}_{21}y)(t) + (\mathcal{V}_{22}z)(t). \quad (8.4.2)$$

We will also consider the corresponding nonlinear (Volterra–Hammerstein) version,

$$y'(t) = F(y(t), z(t)) + \int_0^t K_1(t,s)G_1(s, y(s), z(s))ds, \quad (8.4.3)$$

$$0 = q_2(t) + \int_0^t K_2(t,s)G_2(s, y(s), z(s))ds, \quad t \in I. \quad (8.4.4)$$

It is natural to approximate their solutions $x(t) = (y(t),\ z(t))^T$ by the collocation solution $x_h(t) = (u_h(t),\ v_h(t))^T$ with $u_h \in S_m^{(0)}(I_h)$ and $v_h \in S_{m-1}^{(-1)}(I_h)$. Since, as we know from Section 2.2.1, the dimensions of these two linear spaces differ only by one, we can use the set

$$X_h := \{t_{n,i} := t_n + c_i h_n :\ 0 < c_1 < \ldots < c_m \le 1 \ (0 \le n \le N-1)\}$$

as collocation points in the collocation equations for both VIEs in (8.4.1/2) and (8.4.3/4). For the linear IDAE system (8.4.1/2) these equations read

$$u_h'(t) + B_{11}(t)u_h(t) + B_{12}(t)v_h(t) = q_1(t) + (\mathcal{V}_{11}u_h)(t) + (\mathcal{V}_{12}v_h)(t), \quad (8.4.5)$$

$$0 = q_2(t) + (\mathcal{V}_{21}u_h)(t) + (\mathcal{V}_{22}v_h)(t), \quad (8.4.6)$$

where $t \in X_h$.

As in Section 8.3.1 we will again assume that $d_1 = d_2 = 1$ in (8.4.1/2); this permits a more transparent exposition of the main ideas underlying the convergence analysis. The reader is again invited to derive the more general analysis for arbitrary d_1 and d_2, employing the familiar Kronecker product notation used in the description of Runge–Kutta methods for systems of linear ODEs (see, for example, Dekker and Verwer (1984)) for the linear system (8.4.11) below.

Let the collocation solutions have the local representations

$$u_h(t_n + vh_n) = y_n + h_n \sum_{j=1}^{m} \beta_j(v) Y_{n,j}, \quad v \in [0, 1], \tag{8.4.7}$$

$$v_h(t_n + vh_n) = \sum_{j=1}^{m} L_j(v) V_{n,j}, \quad v \in (0, 1], \tag{8.4.8}$$

with $Y_{n,j} := u_h'(t_{n,j})$ and $V_{n,j} := v_h(t_{n,j})$. Hence, the computational forms of the collocation equations (8.4.5), (8.4.6) on $\bar{\sigma}_n$ become respectively,

$$\begin{aligned} Y_{n,i} &+ B_{11}(t_{n,i})[y_n + h_n \sum_{j=1}^{m} a_{i,j} Y_{n,j}] + B_{12}(t_{n,i}) V_{n,i} \\ &+ h_n^2 \sum_{j=1}^{m} \left(\int_0^{c_i} K_{11}(t_{n,i}, t_n + sh_n) \beta_j(s) ds \right) Y_{n,j} \\ &+ h_n \sum_{j=1}^{m} \left(\int_0^{c_i} K_{12}(t_{n,i}, t_n + sh_n) L_j(s) ds \right) V_{n,j} \\ &+ q_1(t_{n,i}) + h_n \left(\int_0^{c_i} K_{11}(t_{n,i}, t_n + sh_n) ds \right) y_n \\ &+ F_n^{(1)}(t_{n,i}) \quad (i = 1, \dots, m), \end{aligned} \tag{8.4.9}$$

and

$$\begin{aligned} 0 = h_n^2 &\sum_{j=1}^{m} \left(\int_0^{c_i} K_{21}(t_{n,i}, t_n + sh_n) \beta_j(s) ds \right) Y_{n,j} \\ &+ h_n \sum_{j=1}^{m} \left(\int_0^{c_i} K_{22}(t_{n,i}, t_n + sh_n) L_j(s) ds \right) V_{n,j} \\ &+ q_2(t_{n,i}) + h_n \left(\int_0^{c_i} K_{21}(t_{n,i}, t_n + sh_n) ds \right) y_n \\ &+ F_n^{(2)}(t_{n,i}) \quad (i = 1, \dots, m), \end{aligned} \tag{8.4.10}$$

In analogy to the previous section we define

$$F_n^{(k)}(t_{n,i}) := \int_0^{t_n} \big(K_{1k}(t_{n,i}, s) u_h(s) + K_{k2}(t_{n,i}, t_n + sh_n) v_h(s) \big)\, ds \quad (k = 1, 2),$$

and, for future use, we set $\mathbf{G}_n^{(k)} := (F_n^{(k)}(t_{n,1}), \ldots, F_n^{(k)}(t_{n,m}))^T$ $(k = 1, 2)$, and

$$\kappa_{ni}^{(k)} := \int_0^{c_i} K_{k1}(t_{n,i}, t_n + sh_n)ds \quad (i = 1, \ldots, m;\ k = 1, 2).$$

The linear algebraic system whose solution determines the collocation solutions u_h and v_h on σ_n can be written concisely as

$$\begin{bmatrix} \mathcal{I}_m - h_n^2 C_n^{(1,1)} & -h_n B_n^{(1,2)} \\ h_n C_n^{(2,1)} & B_n^{(2,2)} \end{bmatrix} \begin{bmatrix} \mathbf{Y}_n \\ \mathbf{V}_n \end{bmatrix} = \begin{bmatrix} \mathbf{f}_n + \kappa_n^{(1)} + \mathbf{G}_n^{(1)} \\ -h_n^{-1}[\mathbf{g}_n + \mathbf{G}_n^{(2)}] - \boldsymbol{\kappa}_n^{(2)} \end{bmatrix}. \tag{8.4.11}$$

Observe the similarity with the analogous system (8.3.12) for IAEs. The meaning of the vectors on the right-hand side of (8.4.11) is clear from the above description of the computational forms of the two collocation equations. The matrices $B_n^{(k,2)} \in L(\mathbb{R}^m)$ $(k = 1, 2)$ are as in (8.3.12), while $C_n^{(k,1)} \in L(\mathbb{R}^m)$ is given by

$$C_n^{(k,1)} := \begin{pmatrix} \displaystyle\int_0^{c_i} K_{k1}(t_{n,i}, t_n + sh_n)\beta_j(s)ds \\ (i, j = 1, \ldots, m) \end{pmatrix} \quad (k = 1, 2).$$

Thanks to the assumed continuity of the kernels K_{kl} on D there exists again an $\bar{h} > 0$ so that the (block-) matrix describing the left-hand side of (8.4.11) is invertible for all $h \in (0, \bar{h})$. Thus, for meshes whose diameters satisfy this condition, the algebraic systems (8.4.11) and the local representations (8.4.7/8) define a unique collocation solution x_h to the IDAE (8.4.1/2).

8.4.2 Convergence results

Theorems 8.2.1 and 8.3.2 give an indication of what the analogous results on the optimal global and local orders of (super-) convergence for the collocation solutions defined by (8.4.5) and (8.4.6) will look like for IDAEs, since the orders will again be governed by the presence of a first-kind VIE. We first state, without proof, the result on the attainable order of global convergence on I. A closely related result for implicit Runge–Kutta methods was derived by Kauthen (1993).

Theorem 8.4.1 *Assume that*

(a) $B_{11},\ B_{12},\ q_1 \in C^m(I)$*, and* $q_2 \in C^{m+1}(I)$*, with* $q_2(0) = 0$*;*
(b) $K_{1l} \in C^m(D)$ $(l = 1, 2)$*;*
(c) $K_{2l} \in C^{m+1}(D)$*, with* $|K_{22}(t, t)| \geq k_0 > 0$ *on* I*;*

(d) $x_h = (u_h,\ v_h)^T$, *with* $u_h \in S_m^{(0)}(I_h)$ *and* $v_h \in S_{m-1}^{(-1)}(I_h)$, *is the collocation solution defined by (8.4.7/8) and (8.4.11).*

If the collocation points X_h *correspond to any* $\{c_i\}$ *with* $0 < c_1 < \ldots < c_m = 1$, *then* x_h *induces the estimate*

$$||y - u_h||_\infty \le C_1 h^m, \quad ||z - v_h||_\infty \le C_2 h^m.$$

For $c_m < 1$ *the second of the above estimates does not necessarily remain true: we now have*

$$||y - u_h||_\infty \le C_1 h^m,$$
$$||z - v_h||_\infty \le C_2 \begin{cases} h^m & \text{if } -1 \le \rho_m < 1, \\ h^{m-1} & \text{if } \rho_m = 1. \end{cases}$$

As in Theorem 8.2.1, ρ_m $(=\ R(\infty))$ *is given by*

$$\rho_m := (-1)^m \prod_{i=1}^{m} \frac{1 - c_i}{c_i}.$$

An order reduction in the optimal order of local superconvergence is, after Theorem 8.2.1 (index-1 DAEs), Theorem 8.3.2 (index-1 IAEs) and the above result, no longer surprising: collocation at the Gauss points does not lead to $p^* = 2m$ for u_h on I_h, and the optimal order is given by $2m - 1$.

Theorem 8.4.2 *Let the assumptions of Theorem 8.4.1 hold, but with m in the regularity hypotheses replaced by* $\nu \ge 2m - 1$. *If the collocation points* X_h *are those corresponding to the Radau II points* $\{c_i\}$, *we obtain the estimates*

$$\max_{t \in I_h \setminus \{0\}} |y(t) - u_h(t)| \le C_1 h^{2m-1}, \tag{8.4.12}$$

$$\max_{t \in I_h \setminus \{0\}} |z(t) - v_h(t)| \le C_2 h^m. \tag{8.4.13}$$

If collocation is at the Gauss points, then the local order of u_h *coincides with the global one: we only attain*

$$\max_{t \in I_h \setminus \{0\}} |y(t) - u_h(t)| \le C_1 h^m.$$

The second estimate (8.4.13) remains valid if m is odd; for even values of m it becomes

$$\max_{t \in I_h \setminus \{0\}} |z(t) - v_h(t)| \le C_2 h^{m-1}.$$

Proof Setting again $e_h := y - u_h$ and $\epsilon_h := z - v_h$, and denoting the defects induced by collocation by δ_h and d_h, respectively, we see that the errors solve

$$e_h'(t) = B_{11}(t)e_h(t) + B_{12}(t)\epsilon_h(t) + \delta_h(t) + (\mathcal{V}_{11}e_h)(t) + (\mathcal{V}_{12}\epsilon_h)(t), \quad t \in I,$$
$$0 = d_h(t) + (\mathcal{V}_{21}e_h)(t) + (\mathcal{V}_{22}\epsilon_h)(t), \quad t \in I,$$

with $e_h(0) = 0$. Thus, the key to the above order results is contained in Theorem 8.1.7 and the representation of the collocation errors e_h and ϵ_h. We readily obtain – along the lines of the proof of Theorem 8.3.2 – the error representations

$$e_h(t) = r_{11}(t,0)e_h(0) + \int_0^t r_{11}(t,s)\delta_h(s)ds \qquad (8.4.14)$$

$$+\kappa_{12}(t)d_h(t) + \int_0^t q_{12}(t,s)d_h(s)ds,$$

and

$$\epsilon_h(t) = \kappa_{21}(t)d_h(t) + \kappa_{22}(t)d_h'(t) + \int_0^t q_{22}(t,s)d_h(s)ds, \quad t \in I. \quad (8.4.15)$$

Setting $t = t_n$ $(1 \le n \le N)$ and employing once more the familiar quadrature arguments, the assertions in Theorem 8.4.2 follow: we observe that $c_m = 1$ implies that $\delta_h(t_n) = d_h(t_n) = 0$.

If the $\{c_i\}$ are the Gauss points, then $d_h(t_n) \neq 0$ (in fact, this term is $\mathcal{O}(h^m)$). Thus, the local order on I_h cannot exceed the global order, $p = m$, of u_h.

Corollary 8.4.3 *The optimal order results of Theorem 8.4.2 hold for the semi-explicit index-1 system given by*

$$y'(t) = F(t, y(t), z(t)), \qquad (8.4.16)$$

$$0 = g(t) + \int_0^t K(t,s)G(t, y(s), z(s))ds. \qquad (8.4.17)$$

8.5 IAEs and IDAEs with weakly singular kernels

8.5.1 Collocation for weakly singular IAEs

Assume that the solution $x = (y,\ z)^T$ of the linear semi-explicit IAE system (8.1.50/51) is approximated by the collocation solution $x_h = (u_h,\ v_h)^T$, as already described in Section 8.3.1 for $\alpha = 0$. The collocation equations defining $u_h, v_h \in S_{m-1}^{(-1)}(I_h)$ are now

$$u_h(t) = q_1(t) + (\mathcal{V}_{11}^\alpha u_h)(t) + (\mathcal{V}_{12}^\alpha v_h)(t), \quad t \in X_h, \qquad (8.5.1)$$

$$0 = q_2(t) + (\mathcal{V}_{21}^\alpha u_h)(t) + (\mathcal{V}_{22}^\alpha v_h)(t), \quad t \in X_h. \qquad (8.5.2)$$

The iterated collocation solution u_h^{it} is then found from

$$u_h^{it}(t) := q_1(t) + (\mathcal{V}_{11}^\alpha u_h)(t) + (\mathcal{V}_{12}^\alpha v_h)(t), \quad t \in I. \qquad (8.5.3)$$

The computational forms of these three equations are readily derived from (6.2.13), (6.3.4), and (6.2.16), using again the familiar local Lagrange representations of u_h and v_h. The linear algebraic system defining the stage vectors $\mathbf{U}_n$ and $\mathbf{V}_n$ for u_h and v_h has of course the same structure as (8.3.12): for $d_1 = d_2 = 1$ it is given by

$$\begin{bmatrix} \mathcal{I}_m - h_n^{1-\alpha} B_n^{(1,1)}(\alpha) & -h_n^{1-\alpha} B_n^{(1,2)}(\alpha) \\ B_n^{(2,1)}(\alpha) & B_n^{(2,2)}(\alpha) \end{bmatrix} \begin{bmatrix} \mathbf{U}_n \\ \mathbf{V}_n \end{bmatrix} = \begin{bmatrix} \mathbf{f}_n + \mathbf{G}_n^{(1)}(\alpha) \\ -h_n^{\alpha-1}[\mathbf{g}_n + \mathbf{G}_n^{(2)}(\alpha)] \end{bmatrix}. \tag{8.5.4}$$

Here, we have set $\mathbf{U}_n := (U_{n,1}, \ldots, U_{n,m})^T$, $\mathbf{V}_n := (V_{n,1}, \ldots, V_{n,m})^T$, and

$$\mathbf{f}_n := (q_1(t_{n,1}), \ldots, q_1(t_{n,m}))^T, \quad \mathbf{g}_n := (q_2(t_{n,1}), \ldots, q_2(t_{n,m}))^T,$$

and

$$\mathbf{G}_n^{(k)}(\alpha) := (F_n^{(k)}(t_{n,1}; \alpha), \ldots, F_n^{(k)}(t_{n,m}; \alpha))^T \quad (k = 1, 2).$$

The lag term approximations are given by

$$F_n^{(k)}(t_{n,i}; \alpha) = \int_0^{t_n} p_\alpha(t_{n,i} - s)[K_{k1}(t_{n,i}, s)u_h(s) + K_{k2}(t_{n,i}, s)v_h(s)]ds \\ (k = 1, 2), \tag{8.5.5}$$

with the matrices in $L(\mathbb{R}^m)$,

$$B_n^{(k,l)}(\alpha) := \left(\begin{matrix} \int_0^{c_i} (c_i - s)^{-\alpha} K_{kl}(t_{n,i}, t_n + sh_n) L_j(s) ds \\ (i, j = 1, \ldots, m) \end{matrix} \right) \quad (k, l = 1, 2).$$

Due to our assumptions on the kernels K_{kl}, the existence of a unique solution for these linear algebraic systems follows along standard arguments, and is true for all meshes I_h with $h \in (0, \bar{h})$, for some $\bar{h} = \bar{h}(\alpha) > 0$.

The convergence analysis for the collocation solution x_h remains open. This is due, as we recall from Section 6.3.3, to the fact that for $m > 1$ we do not yet know necessary and sufficient conditions on the collocation parameters $\{c_i\}$ ensuring the uniform convergence of v_h to z. However, assuming that we have set $\{c_i\}$ for which convergence holds, then we can say more about the attainable order of convergence of u_h and v_h, since we have already derived, in Theorem 8.1.8, results on the regularity of solution components y and z.

Suppose that I_h is a graded mesh of the form

$$I_h := \{t_n := (n/N)^r T : \; n = 0, 1, \ldots, N \; (r = r(\alpha) \geq 1)\},$$

and set

$$r^I := \frac{m}{1-\alpha}, \quad r^{II} := \frac{m}{\alpha} \quad (0 < \alpha < 1).$$

We remember from Chapter 6 that r^I and r^{II} are respectively the optimal grading exponents for weakly singular VIEs of the second and first kind (cf. Theorems 6.2.9 and 6.3.2). For $\alpha = 1/2$ we have $r^I = r^{II}$. If $\alpha \neq 1/2$, 'over-grading' still yields optimal convergence orders. Hence, the following result is now obvious.

Theorem 8.5.1 *Assume that the given functions describing the IAE system (8.1.50/51) with $0 < \alpha < 1$ are subject to the hypotheses in Theorem 8.1.5, where $\nu \geq m$ and $K_{21} \equiv 0$. Set $r^* := \max\{r^I,\ r^{II}\}$, and let the collocation parameters $\{c_i\}$ be such that the collocation solution $v_h \in S_{m-1}^{(-1)}(I_h)$ to weakly singular first-kind VIEs is uniformly convergent on I. Then the collocation solution $x_h := (u_h,\ v_h)^T$ determined by the collocation equations (8.5.1/2) satisfies*

$$||y - u_h||_\infty \leq C_1 h^m, \quad ||z - v_h||_\infty \leq C_2 h^m,$$

for any graded mesh I_h with grading exponent $r = r^*$ and sufficiently large N. Here we have set again $h := T/N$.

Remark The attainable order in the iterated collocation solution u_h^{it} given by (8.5.3) is not yet understood, since we do not know if collocation parameters satisfying the orthogonality condition $J_0 = 0$ (recall assumption (c) in Theorem 6.2.13) lead to convergent collocation solutions for weakly singular VIEs of the first kind.

8.5.2 Collocation for weakly singular IDAEs

The results in the previous section readily suggest that an optimal convergence result similar to that in Theorem 8.5.1 will hold for the semilinear IDAE system

$$y'(t) + B_{11}(t)y(t) + B_{12}(t)z(t) = q_1(t) + (\mathcal{V}_{11}^\alpha y)(t) + (\mathcal{V}_{12}^\alpha z)(t), \quad (8.5.6)$$

$$0 = q_2(t) + (\mathcal{V}_{21}^\alpha y)(t) + (\mathcal{V}_{22}^\alpha z)(t), \quad (8.5.7)$$

again with the proviso that the collocation parameters are feasible in the sense of that theorem. Thus, assume that the solution of (8.5.4/5) is approximated in $S_m^{(0)}(I_h)$ and $S_{m-1}^{(-1)}(I_h)$, respectively. According to Section 8.4.1 the resulting collocation equations are

$$u_h'(t) + B_{11}(t)u_h(t) + B_{12}(t)v_h(t) = q_1(t) + (\mathcal{V}_{11}^\alpha u_h)(t) + (\mathcal{V}_{12}^\alpha v_h)(t), \quad (8.5.8)$$

$$0 = q_2(t) + (\mathcal{V}_{21}^\alpha u_h)(t) + (\mathcal{V}_{22}^\alpha v_h)(t), \quad (8.5.9)$$

where $t \in X)h$. The collocation solutions will have the local representations

$$u_h(t_n + vh_n) = y_n + h_n \sum_{j=1}^{m} \beta_j(v) Y_{n,j}, \quad v \in [0, 1], \quad (8.5.10)$$

$$v_h(t_n + vh_n) = \sum_{j=1}^{m} L_j(v) V_{n,j}, \quad v \in (0, 1], \quad (8.5.11)$$

with $Y_{n,j} := u_h'(t_{n,j})$ and $V_{n,j} := v_h(t_{n,j})$. As we have done before, we will again describe the key ideas by assuming that $d_1 = d_2 = 1$ in (8.5.6/7). It therefore follows, as in Section 8.4.1, that the linear algebraic system whose solution determines the collocation solutions u_h and v_h on σ_n can be written concisely as

$$\begin{bmatrix} \mathcal{I}_m - h_n^2 C_n^{(1,1)}(\alpha) & -h_n B_n^{(1,2)}(\alpha) \\ h_n C_n^{(2,1)}(\alpha) & B_n^{(2,2)}(\alpha) \end{bmatrix} \begin{bmatrix} \mathbf{Y}_n \\ \mathbf{V}_n \end{bmatrix} = \begin{bmatrix} \mathbf{f}_n + \kappa_n^{(1)}(\alpha) + \mathbf{G}_n^{(1)}(\alpha) \\ -h_n^{-1}[\mathbf{g}_n + \mathbf{G}_n^{(2)}(\alpha)] - \boldsymbol{\kappa}_n^{(2)}(\alpha) \end{bmatrix}. \tag{8.5.12}$$

The matrices $B_n^{(k,2)}(\alpha) \in L(\mathbb{R}^m)$ $(k = 1, 2)$ are as in Section 8.5.1, while $C_n^{(k,1)}(\alpha) \in L(\mathbb{R}^m)$ is given by

$$C_n^{(k,1)}(\alpha) := \begin{pmatrix} \int_0^{c_i} (c_i - s)^{-\alpha} K_{k1}(t_{n,i}, t_n + sh_n)\beta_j(s)ds \\ (i, j = 1, \dots, m) \end{pmatrix} \quad (k = 1, 2).$$

Moreover, we have defined, in analogy to Section 8.4.1,

$$F_n^{(k)}(t_{n,i}; \alpha) := \int_0^{t_n} (t_{n,i} - s)^{-\alpha} [K_{1k}(t_{n,i}, s)u_h(s) + K_{k2}(t_{n,i}, t_n + sh_n)v_h(s)]ds$$
$$(k = 1, 2),$$

and set $\mathbf{G}_n^{(k)}(\alpha) := (F_n^{(k)}(t_{n,1}; \alpha), \dots, F_n^{(k)}(t_{n,m}; \alpha))^T$ $(k = 1, 2)$, and

$$\kappa_{ni}^{(k)}(\alpha) := \int_0^{c_i} (c_i - s)^{-\alpha} K_{k1}(t_{n,i}, t_n + sh_n)ds \quad (i = 1, \dots, m;\ k = 1, 2).$$

Theorem 8.5.2 *Let $0 < \alpha < 1$ and assume that the given functions describing the IDAE system (8.5.4/5) are subject to the hypotheses in Theorem 8.1.7, with $\nu \geq m$ and $K_{21} \equiv 0$. Set $r^* := m/\alpha$, and let the collocation parameters $\{c_i\}$ be such that the collocation solution $v_h \in S_{m-1}^{(-1)}(I_h)$ to weakly singular first-kind VIEs is uniformly convergent on I. Then the collocation solution $x_h := (u_h,\ v_h)^T$ determined by the collocation equations (8.5.1/2) satisfies*

$$||y - u_h||_\infty \leq C_1 h^m, \quad ||z - v_h||_\infty \leq C_2 h^m,$$

for any graded mesh I_h with grading exponent $r = r^*$ and sufficiently large N. Here we have set again $h := T/N$.

Proof Under the assumption that the chosen collocation parameters $\{c_i\}$ yield uniform convergence of collocation solutions in $S_{m-1}^{(-1)}(I_h)$ to weakly singular VIEs of the first kind, we can resort to Section 7.2.3 (see Remark 2 following the proof of Theorem 7.2.3) and to Theorem 6.3.2, to deduce that – in complete analogy to Theorem 8.5.1 – the optimal order of global convergence for u_h and

v_h is attained if the grading exponent r^* is given by

$$r^* := \max\{(m+1-\alpha)/(2-\alpha),\ m/\alpha\} = m/\alpha \quad (0 < \alpha < 1).$$

We conclude this section by briefly looking at a simple special case of the index-1 IDAE system (8.5.4), namely

$$y'(t) + B_{11}(t)y(t) + B_{12}(t)z(t) = q_1(t), \tag{8.5.13}$$

$$(\mathcal{V}_{22}^{\alpha} z)(t) + q_2(t) = 0, \quad t \in I. \tag{8.5.14}$$

Here, the kernel K_{22} in $\mathcal{V}_{22}^{\alpha}$ is assumed to satisfy the condition (c) in Theorem 8.4.1: $K_{22} \in C^{m+1}(D)$, with $|K_{22}(t,t)| \geq k_0 > 0$ on I. While the first part of this IDAE system now does not contain a Volterra operator with weakly singular kernel, the non-smooth contribution has its origin solely in the second equation: under the standard assumptions on q_2 and K_{22} we know that its solution lies in $C^{\alpha}(I)$ and is smooth on $(0, T]$. Hence, according to Theorem 7.1.1 and Exercise 7.7.5, the general solution of the VIDE (8.5.6) lies in the Hölder space $C^{1,\alpha}(I)$. This insight allows us to establish the attainable orders of u_h and v_h on suitably graded meshes.

Theorem 8.5.3 *Let the setting described in Theorem 8.5.2 hold. If the mesh I_h is graded, with grading exponent $r = r^{II} := m/\alpha$, then the collocation solution $x_h = (u_h,\ v_h)^T$ to (8.5.6/7), with $u_h \in S_m^{(0)}(I_h)$ and $v_h \in S_{m-1}^{(-1)}(I_h)$, exhibits the global orders given by*

$$||y - u_h||_\infty \leq C_1 h^m, \quad ||z - v_h||_\infty \leq C_2 h^m,$$

with constants $C_k = C_k(r)$ depending on α but not on N.

8.6 Exercises and research problems

Exercise 8.6.1 Prove Lemma 8.1.2. Show that $Q_s(t) := Q(t)A_1^{-1}(t)B_0(t)$ projects $\mathbb{R}^d$ onto $N_0(t)$ along $S_0(t)$.

Exercise 8.6.2 Prove Lemma 8.1.4.

Exercise 8.6.3 State and prove Theorem 8.1.6 for the case where $\mathcal{V}_{21} \neq 0$.

Exercise 8.6.4 Prove Theorem 8.1.7 on the representation of the solution $y,\ z$ of the IDAE (8.1.39/40).

Exercise 8.6.5

(a) Recall Example 8.2.1: analyse the asymptotic behaviour of the continuous θ-method (resulting from collocation in $S_1^{(0)}(I_h)$ with $\theta := c_1 \in (0, 1]$) for the DAE (8.2.10). Then apply the θ-method to the numerically properly formulated DAE (8.2.11) and discuss its asymptotic stability.
(b) Use the DAE (8.2.10) as the basis for the construction of an index-1 'test' IDAE (with convolution kernels) whose solution $x = (y,\ z)^T$ has the same asymptotic stability property as the one for the original DAE. Suppose that this IDAE is solved numerically in the same collocation space as the DAE, and analyse the asymptotic stability of the collocation solution $w_h = (u_h,\ v_h)^T$. Then derive the corresponding IDAE with properly stated leading term (recall (8.1.48/49)) and analyse the asymptotic behaviour of the θ-method for the reformulated IDAE.

Exercise 8.6.6 Consider the linear IDAE given by

$$2(y'(t) - z'(t)) + y(t) + 2(y(t) - z(t)) = q(t),$$
$$y(t) + 2z(t) + \int_0^t z(s)ds = q(t), \quad t \geq 0$$

(see Doležal (1960, p. 20)). Determine consistent initial values, and determine the solution of this IDAE. What is its tractability index?

Exercise 8.6.7

(a) Consider the generalisation of the weakly singular IAE system (5.1.50/51) where $\alpha \in (0, 1)$ has been replaced respectively by $\alpha_1 \in (0, 1)$ (in (8.1.50)) and $\alpha_2 \in (0, 1)$ (in (8.1.51)). Provide an analysis of the regularity of the corresponding solution.
(b) Solve the analogue of (a) for the IDAE system (8.1.55/56).

Exercise 8.6.8 Analyse the regularity of the solutions to the weakly singular index-2 IAE and IDAE systems

$$y^{(\nu)}(t) = q_1(t) + (\mathcal{V}_{11}^{\alpha} y)(t) + (\mathcal{V}_{12}^{\alpha} z)(t),$$
$$0 = q_2(t) + (\mathcal{V}_{21}^{\alpha} y)(t) + (\mathcal{V}_{22}^{\alpha} z)(t)$$

($\nu = 0, 1$), where $0 < \alpha < 1$ and $K_{21} \in C^{m+1}(D)$, with $|K_{21}(t, t)| \geq k_0 > 0$ on I. Consider in particular the systems with $\mathcal{V}_{22}^{\alpha} = 0$.

Exercise 8.6.9 Determine the collocation solution $x_h = (u_h,\ v_h)^T$ in Example 8.2.1 for arbitrary $c_1 \in (0, 1]$. For which c_1 is x_h asymptotically stable when (i) $\delta = 0$; (ii) $\delta < \sigma$?

Exercise 8.6.10 Consider the IAE (8.1.24/25) and suppose that the collocation solution $x_h = (u_h,\ v_h)^T$, with u_h and v_h in $S_{m-1}^{(-1)}(I_h)$, is based on the parameters $\{c_i\}$ given by the m *positive* Lobatto points from $0 = c_0 < c_1 < \ldots < c_m = 1$ (cf. Theorem 2.4.6). Discuss the resulting orders of convergence for u_h and v_h. In particular, can the results of Theorem 2.4.6 be extended to IAEs: if m is odd, then

$$\max_{(n)} |z(t_{n+1/2}) - v_h(t_{n+1/2})| \le C_2 h^{m+1} \ ?$$

Exercise 8.6.11 Prove Theorem 8.4.1

Exercise 8.6.12 Describe and analyse projected collocation methods for the linear version of the IDAE (8.4.38). Extend the results in Ascher and Petzold (1991), and the insight obtained by Lubich (1991) and März (1996) to these methods for IDAEs.

Exercise 8.6.13 Collocation for the IAE and IDAE systems of Exercise 8.6.8: assume that $\mathcal{V}_{21}^{\alpha} = 0$ and $|K_{22}(t,t)| \ge k_0 > 0$ on I. Analyse the attainable orders of global convergence in the collocation approximations to y and z, assuming that $0 < \alpha_1 < \alpha_2 < 1$. Do these orders change if $\alpha_2 < \alpha_1$?

Exercise 8.6.14 (Research problem)
'IDAEs on manifolds': give a geometrical interpretation (e.g. along the lines of Rheinboldt (1984) and Rabier and Rheinboldt (1994)) of the exact solution and its collocation approximation for the IDAE system (8.5.6/7). See also Hairer and Wanner (1996), pp. 457–458.

Exercise 8.6.15 (Research problem)
Consider the *semi-explicit index-2* analogue of the linear system (8.1.24/25):

$$y(t) = q_1(t) + (\mathcal{V}_{11} y(t) + (\mathcal{V}_{12} z(t),\ 0 = q_2(t) + (\mathcal{V}_{21}, y)(t), \quad t \in I,$$

where the Volterra integral operators V_{kl} are again given by (8.1.26),

$$(V_{kl}\phi(t) := \int_0^t K_{kl}(t,s)\phi(s)ds.$$

The matrix kernels $K_{kl}(.,.)$ are continuous and K_{21} is such that $K_{21} \in C^1(D)$, with $|\det K_{21}(t,t)| \ge k_0 > 0$ on I.

Analyse the convergence of collocation solutions for this index-2 IAE, along the lines of Section 8.3.2.

8.7 Notes

8.1: Basic theory of DAEs and IAEs
The most comprehensive and up-to-date survey of the theory and numerical analysis of DAEs is Rabier and Rheinboldt (2002). The books by Griepentrog and März (1986), Brenan, Campbell and Petzold (1996) (first published

in 1989), Hairer, Lubich and Roche (1989), and Hairer and Wanner (1996) (Chapters VI and VII) all contain good introductions to, and descriptions of, the respective state of the art in numerical DAEs and their applications. See, in addition, the survey papers by März (1985, 1990, 1992, 1994, 1998), the book by Boyarintsev and Chystyakov (1998) (who also consider the IAE forms of DAE systems), and the report by Schulz (2003). Chapter 6 of Strehmel and Weiner (1992) treats numerical methods for index-1 DAEs.

The geometry of DAEs is studied in Rheinboldt (1984) and Rabier and Rheinboldt (1994, 2002).

Applications of DAEs, IAEs and IDAEs:
The survey by Rabier and Rheinboldt (2002, pp. 197–218) contains a wide-ranging description of DAEs arising in applications, from network problems and constrained rigid-body systems to control problems. Good sources on applications are also Ascher (1989), Brenan, Campbell and Petzold (1996), Winkler (2003), and Tischendorf (2001). Consult also the many references in März (2001).

'Mixed' systems of IAEs consisting of second- and first-kind VIEs arise in many mathematical modelling processes; we mention memory kernel identification problems in heat conduction and viscoelasticity (v. Wolfersdorf (1994), Janno and v. Wolfersdorf (1997a, 1997b) and Kiss (1999)), evolution of a chemical reaction within a small cell (Jumarhon, Lamb, McKee and Tang (1996) and references), and Kirchhoff's laws (Doležal (1960); this appears to be the first source (except for a similar paper, in Czech, of 1959 by the same author) of a Volterra IDAE). (The author is grateful to Roswitha März for pointing out this paper to him.)

8.2: Collocation for DAEs
Hairer, Lubich and Roche (1989) present numerous superconvergence results for Runge–Kutta solutions to DAEs; see also Petzold (1986), März (1989), Lopez (1990), Hanke, Izquierdo Macana and März (1998), März and Rodríguez-Santiesteban (2002), and Rabier and Rheinboldt (2002, pp. 415–424). Two-step Runge–Kutta methods for index-1 DAEs are presented in Y. Chen (1995).

Collocation methods for higher-index DAEs are studied in Hairer, Lubich and Roche (1989), Jay (1993) and Hairer and Wanner (1996, Ch. VII). Ascher and Petzold (1991) introduced *projected collocation methods* for DAEs, to avoid 'drift-off' of the approximate solution; these methods are studied further in Lubich (1991), Hairer and Wanner (1996, pp. 512–515) and, especially, in März (1996). See also Rabier and Rheinboldt (2002, pp. 426–428).

Boundary-value problems for DAEs have also received considerable attention, not least owing to their importance in applications. We refer the reader to

Stöver (2001) and to Section 81 of Rabier and Rheinboldt (2002, pp. 507–513) (as well as the lists of references in these two articles).

Readers interested in the numerical analysis of partial DAEs should consult the paper by Lucht, Strehmel and Eichler-Liebenow (1999) and its references.

März (2001) and Lamour, März and Tischendorf (2001) have shown that partial DAEs and IDAEs can be re-formulated as *abstract DAEs* in an infinite-dimensional (Hilbert space) setting. This insight, combined with the work of März and her collaborators (see, e.g. the papers by März (1992, 2002a, 2002b), Higueras and März (2000), Higueras, März and Tischendorf (2001a, 2001b), and Balla and März (2002)) appears to provide a powerful tool for the qualitative and quantitative analysis of numerical solutions to partial DAEs and IDAEs.

DAEs with delay arguments

Ascher and Petzold (1995) and Hauber (1997) studied the convergence properties of Runge–Kutta and piecewise polynomial collocation solutions for DAEs with constant and (more general) non-vanishing delays. It would be interesting to investigate these problems within the general framework of abstract DAEs, as described in März (2001). The alternative approach by Bellen and Maset (1999) and Maset (1999, 2002) of recasting a DDE as a Cauchy problem for an abstract ODE may also be worth investigating.

Except for our (super-) convergence results for index-1 IAEs and IDAEs with delay arguments, the general study of the quantitative (and qualitative) properties of collocation solutions to such problems with higher index, especially the extension of the theory of März and her collaborators, has not yet been done.

8.3: Collocation for IAEs with smooth kernels

The results on local superconvergence of piecewise polynomial collocation solutions to index-1 Volterra IAEs are due to Kauthen (2001). Compare also Kauthen (1997b) for a related analysis. The analogous analysis for IAEs of tractability index 2 and higher is waiting to be carried out.

8.4: Collocation for IDAEs with smooth kernels

To the best of my knowledge, this is the first treatment of the question of global and local superconvergence of piecewise polynomial collocation solutions for index-1 IDAEs. The corresponding analysis for IDAEs with index-2 or higher is open.

Kauthen (1993) provided the first study of the convergence properties of implicit Runge–Kutta methods of Pouzet-type for IDAEs. These methods can be viewed as fully discretised collocation methods.

8.5: IDAEs with weakly singular kernels

As we have mentioned, the numerical analysis of IAEs and IDAEs with weakly singular kernels is largely incomplete, because it hinges on the open problem regarding the sets of collocation parameters $\{c_i\}$ for which the collocation solution in $S_{m-1}^{(-1)}(I_h)$ or $S_m^{(0)}(I_h)$ for first-kind VIEs with integrable kernel singularities is uniformly convergent.

The paper by Favini, Lorenzi and Tanabe (2002) deals with the analysis of IDAEs of the form

$$[Mu'(t)]' + Lu(t) = \int_0^t k(t-s)L_1 u(s)ds + f(t),$$

where L, L_1, M are closed linear operators in a Banach space, with L^{-1} bounded and M not necessarily invertible. The kernel $k(t-s)$ is either weakly singular or non-smooth (with some unbounded derivative when $t = s$). The numerical analysis of problems of this type appears to be open, too.

Singularly perturbed Volterra equations

Due to limitations of space we can only point to some recent advances in the theory and the numerical analysis of singularly perturbed Volterra equations; the comprehensive survey paper by Kauthen (1997a) gives a good idea about the 'state of the art' and has an extensive list of references also on applications. In addition, compare Kauthen (1995) and Bijura (2002a, 2002b, 2003).

The analysis regarding the attainable order of convergence in collocation solutions for singularly perturbed VEs possessing weakly singular kernels,

$$\varepsilon y(t) = g(t) + (V_\alpha y)(t), \tag{8.7.1}$$

and

$$\varepsilon y^{(r)}(t) = f(t, y(t)) + (V_\alpha y)(t), \quad (r = 1,\ 2), \tag{8.7.2}$$

with $0 < \varepsilon \ll 1,\ 0 < \alpha \leq 1$, and with $\mathcal{V}_\alpha$ as in Chapters 6 and 7, is essentially open. This is due to the fact that (i) we do not yet know under what conditions on the collocation parameters $\{c_i\}$ the collocation solutions for the limiting first-kind VIEs corresponding to $\varepsilon = 0$ in (8.10.1) and (8.10.2) are convergent; and (ii) for $\alpha \neq 1/2$ the optimal grading exponent have different values (recall Theorems 6.2.9, 6.3.2, and 7.2.4). Thus, a complete understanding of the dependence of the optimal grading exponent on ε, as $\varepsilon \to 0^+$, will be crucial for the analysis of the attainable order convergence.

9
Epilogue

Our voyage through the preceding eight chapters has shown that we have certainly not yet reached the end of the story on collocation methods for Volterra functional integral and integro-differential equations. Many important questions remain unanswered. It is my belief that we have to find new mathematical approaches and tools (likely from very unexpected areas) if we are to make substantial progress towards finding complete solutions to these open problems.

It is the purpose of this brief final chapter to point to some possible, and seemingly very promising, new approaches for the numerical analysis of collocation solutions to Volterra functional equations.

9.1 Semigroups and abstract resolvent theory

The long-time integration of Volterra integral and integro-differential equations by collocation methods, in particular the asymptotic behaviour of collocation solutions, is not yet understood. As a number of papers and books have shown (see, e.g. Ito and Kappel (1989, 1991, 2002), Ito and Turi (1991), Brunner, Kauthen and Ostermann (1995), Bellen and Maset (1999), Maset (1999, 2003), and Bellen and Zennaro (2003, pp. 56–60)) the appropriate reformulation of the given equation as an abstract Cauchy problem and the exploitation of the underlying semigroup or abstract resolvent framework (integrability and asymptotic behaviour of resolvents) will often lead to deep insight into the qualitative properties of approximate solutions.

The following books and papers will be helpful in acquiring the basic tools necessary to investigate qualitative properties of collocation solutions to VIEs and VIDEs, and their more general (delay and weakly singular) versions.

- *Semigroup theory and sectorial operators:*
  These aspects of modern functional analysis are well covered in the monographs by Aubin (1979), Henry (1981), Zeidler (1990), Prüss (1993), Lunardi (1995), and Ito and Kappel (2002). See also Delfour (1980), Burns, Herdman and Stech (1983), Staffans (1984, 1985a, 1985b), Kappel and Zhang (1986), Burns, Herdman and Turi (1990), and Clément, Desch and Homan (2003), and the references in these papers.
- *Abstract resolvent theory:*
  Abstract VIEs in Banach spaces and properties of resolvents are analysed in Friedman and Shinbrot (1967), Miller (1975), Chen and Grimmer (1980), Grimmer (1982), Grimmer and Pritchard (1983), Gripenberg and Prüss (1985), Gripenberg (1987), Gripenberg, Londen and Staffans (1990), Prüss (1993), and Engel and Nagel (2000, Ch. VI.6/7).
  Analogous results for abstract VIDEs can be found for example in Chen and Grimmer (1982), Desch and Schappacher (1985), Desch and Grimmer (1989); see also Gripenberg, Londen and Staffans (1990) and Prüss (1993).

9.2 C^*-algebra techniques and invertibility of approximating operator sequences

Suppose that the operator equation $\mathcal{A}y = g$, where $\mathcal{A}$ is a bounded linear operator acting between two infinite-dimensional Banach spaces, is approximated by a sequence of approximating equations $\mathcal{A}_h u_h = g_h$ (where, for example, $h = T/N$, with $N \to \infty$). Under what conditions on the approximating operator $\mathcal{A}_h$ does u_h converge (in an appropriate norm) to the solution y of the given operator equation?

We have encountered a typical problem of this kind in Sections 6.3 and 6.5.2, where $\mathcal{A}$ represents one of the Volterra integral operators $\mathcal{V}_\alpha$ or $\mathcal{W}_{\theta,\alpha}$, and u_h is a piecewise polynomial (or non-polynomial spline) collocation solution. As we have seen, we do not know necessary or sufficient conditions for the collocation parameters $\{c_i\}$ under which u_h converges uniformly to y on I, as $h \to 0$.

In recent years, Silbermann and his collaborators have convincingly shown that C^*-algebra techniques provide very powerful tools for answering such invertibility and convergence questions in the case of, e.g. spline projection methods for periodic pseudo-differential equations and other types of singular integral equations. This novel approach (based on a very surprising connection between the rather different worlds of C^*-algebra and numerical analysis) may well yield the key tools for successfully dealing with our open problems in collocation methods for first-kind VIEs with weakly singular kernels.

Excellent introductions to the application of C^*-algebra in the numerical analysis of operator equations are given in the monograph by Hagen, Roch and Silbermann (2001) and in the survey papers by Roch and Silbermann (1996), Silbermann, Hagen and Roch (1998), and Böttcher (2000). The related books by Prössdorf and Silbermann (1991) and Hagen, Roch and Silbermann (1995) provide much of the necessary functional analysis framework.

9.3 Abstract DAEs

As we briefly mentioned in Chapter 8, partial (parabolic) DAEs and (partial) IDAEs of Volterra (or Fredholm) type can be reformulated as abstract DAEs in an infinite-dimensional Hilbert space setting. This not only permits the extension of the notions of (tractability) index and properly stated leading terms to a much wider class of differential-algebraic problems but also appears to furnish the tools for the analysis of collocation methods for ordinary or partial IDAEs of higher index.

The mathematical framework underlying the numerical analysis of abstract DAEs is currently being developed by März and her collaborators at Humboldt University in Berlin; the reader can find an introduction to these ideas in the paper by März (2001, pp. 330–334) and its sequel by Lamour, März and Tischendorf (2001).

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