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Reducible Ordinary Differential Equations

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Summary. The class of reducible differential equations under consideration here includes the class of symmetric systems, and examples show that the inclusion is proper. We first discuss reducibility, as well as the stronger concept of complete reducibility, from the viewpoint of Lie algebras of vector fields and their invariants, and find Lie algebra conditions for reducibility which generalize the conditions in the symmetric case. Completely reducible equations are shown to correspond to a special class of abelian Lie algebras. Then we consider the inverse problem of determining all vector fields which are reducible by some given map. We find conditions imposed on the vector fields by the map, and present an algorithmic access for a given polynomial or local analytic map to $\mathbb{R}$. Next, reducibility of polynomial systems is discussed, with applications to local reducibility near a stationary point. We find necessary conditions for reducibility, including restrictions for possible reduction maps to a one-dimensional equation.

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Introduction

The notion of "reduction" of an ordinary differential equation may be given many legitimate meanings. Here we consider a type of reduction which may be called "algebraic reduction": A vector field and a corresponding reduced vector field are related by some differentiable map, and this map reduces dimension in the sense that its rank is smaller than the dimension of its domain.

Many classical equations are reducible in this sense. Symmetric differential equations are well-known examples. For these, reduction can be carried out via the invariants of a local symmetry group or of a Lie algebra of infinitesimal symmetries. However, algebraic reducibility is a more general concept which covers many other interesting equations (e.g., from mathematical biology) having no apparent symmetries. See also the recent paper by Muriel and Romero [27] on reducible higher-order equations that do not admit Lie symmetries in the classical sense. We will discuss only reduction of finite dimensional systems, but we note that there are relevant and interesting infinite dimensional systems which admit a reduction to finite dimensional systems or to infinite dimensional systems with simpler structure; e.g. center manifold reduction, reduction of delay equations with exponentially distributed delays and of population models with distributed birth rates (the so-called chain trickery), reduction of hyperbolic systems to delay differential equations [7].

An important feature of reducible systems is that some interesting qualitative properties, as well as certain distinguished invariant sets, are accessible via reduction. For symmetric systems there is a large body of work on such topics; see the classical monographs by Olver [29], [30], Golubitsky et al. [16], [17], the book by Gaeta [14], the investigation of qualitative properties in homogeneous systems in [20], and several of the contributions to the Proceedings volumes [1], [5], [12], [15]. Special reduction maps for symmetric Hamiltonian systems were introduced by Marsden and Weinstein [25]. Ortega and Ratiu [31] recently extended these results. Reducibility, like symmetry, may force "nongeneric" behavior upon a differential equation; hence certain qualitative properties may preclude reducibility.

Two types of problems appear in the context of reducibility: First, we must decide whether a given system is reducible, and then find a reduction map if the answer is affirmative. The second, "inverse" problem is to determine all equations that are reducible by a prescribed map. In the present paper we will address both questions. One has to be aware that the straightening theorem (which states that any vector field near a nonstationary point is locally equivalent to a constant vector field) seems to trivialize such problems, but this theorem is of no practical help unless a vector field can be explicitly integrated. An appropriate scenario for our questions is concerned with the local setting near stationary points, or with the global setting, or with algorithmic problems.

The paper is organized as follows. Section 1 contains basic notions, some known results, and various examples. In Section 2, we establish the connection between reducibility and certain Lie algebras of vector fields; thus the relation of reducible systems to symmetric systems is clarified. Moreover, we discuss a correspondence between completely reducible systems and certain abelian Lie algebras. In Section 3 we show how certain invariant sets are enforced by reducing maps, and we adopt a classical strategy from the study of symmetric systems: Given some map, find the vector fields which are reducible by this map. For polynomial, resp. local analytic, maps to $\mathbb{R}$, we give definitive characterizations. In Section 4 we discuss the reduction of polynomial systems by polynomials, and derive criteria which make computations feasible. Applications include Volterra-Lotka systems as well as reduction of analytic systems near nondegenerate stationary points. Moreover we characterize the vector fields which admit reduction by some linear map, and the possible reduction maps. The emphasis is on computations.

1. Definitions and Examples

Let $U \subseteq \mathbb{R}^n$ be nonempty and open, and let $f\colon U \to \mathbb{R}^n$ be continuously differentiable. We consider the autonomous ordinary differential equation $\dot{x} = f(x)$. The solution of the initial value problem for $x(0) = x_0$ (the local flow of the vector field) will be denoted by $F(t, x_0)$. The notion of a solution-preserving map (morphism) to another differential equation is familiar and straightforward: If $\dot{y} = g(y)$ is defined on the open and nonempty subset V of $\mathbb{R}^m$ (with g continuously differentiable) and $G(t, y_0)$ is the associated local flow, then a map Φ from some open and nonempty $\tilde{U} \subseteq U$ to V is *solution-preserving* from $\dot{x} = f(x)$ to $\dot{y} = g(y)$ if the identity $\Phi(F(t, z)) = G(t, \Phi(z))$ holds. If Φ is differentiable (which we will always assume), then it is solution-preserving if and only if the identity

$$D\Phi(x)\, f(x) \;=\; g(\Phi(x)) \qquad (*)$$

holds. Following established terminology, we will then say that f and g are *related* by Φ. Since this is a local property, it can be directly transferred to vector fields on manifolds.

Definition 1.1. The differential equation $\dot{x} = f(x)$ is called *reducible* on $\tilde{U} \subseteq U$ if there is a nonconstant solution-preserving map Φ to some equation $\dot{y} = g(y)$ with the property that rank $D\Phi(x) < n$ for all $x \in \tilde{U}$. In that case, Φ will be referred to as a *reducing map* for f. The analogous notion of a reducible equation on a manifold is immediate.

It is sometimes appropriate to restrict attention to C^∞, or analytic, or polynomial, vector fields and maps.

Similar properties and notions have been considered previously. Arnold [3] introduced the closely related notion of lowerable (and of liftable) vector fields. (The only difference is that he requires the dimension m of the target space to be smaller than the dimension n of the source space, which implies our rank condition.) Schwarz [35] investigated the reduction of a vector field that admits a compact linear symmetry group to the corresponding orbit space. In particular, he proved that every C^∞ vector field on the orbit space can be lifted. Du Plessis and Wall [13] discussed maps Φ and employed lowerable and liftable vector fields to investigate the discriminant set of a map.

Example 1.2.

(a) Let $r < n$. An equation on $U \subseteq \mathbb{R}^r \times \mathbb{R}^{n-r}$ of the type

$$\dot{x}_1 = f_1(x_1), \qquad \dot{x}_2 = f_2(x_1, x_2) \quad \text{with } x_1 \in \mathbb{R}^r,\ x_2 \in \mathbb{R}^{n-r},$$

is obviously reducible by $\Phi(x_1, x_2) = x_1$. The implicit function theorem guarantees that this is a "generic" scenario for reduction: Locally, near any point where the derivative of the reducing map has maximal rank, a reducible equation can be transformed to this type. On the other hand, one sees that singular points of reducing maps deserve special attention.

(b) A first integral ψ of an equation $\dot{x} = f(x)$ (i.e., a nonconstant function which is constant along solution orbits) may be viewed as a solution-preserving map to the trivial equation $\dot{y} = 0$ in $\mathbb{R}$.

(c) The (single substrate, single user) chemostat system

$$\begin{aligned} \dot{x}_1 &= r \cdot (k - x_1) - a(x_1) \cdot x_2, \\ \dot{x}_2 &= (a(x_1) - r) \cdot x_2, \end{aligned}$$

with r and k positive constants and a some nonnegative function, is reducible by $\Phi(x) = x_1 + x_2$. This is one representative of many systems in mathematical biology which are reducible but possess no apparent symmetries. (Note that a linear coordinate transformation puts this system into the form given in (a). We will discuss the question of how such transformations and reducing maps can be determined if they exist.)

(d) Consider the matrix equation $\dot{X} = A \cdot X$ for $q \times q$ matrices. The Wronskian $\Phi(x) = \det X$ is a reducing map to $\dot{y} = \operatorname{tr} A \cdot y$.

(e) Consider the equation $\dot{X} = X^2$ for 2×2-matrices. The map defined by $\Phi(X) = (\operatorname{tr}(X), \det(X))$ is a reduction map to the differential equation

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1^2 - 2y_2 \\ y_1 y_2 \end{pmatrix},$$

as follows from the matrix identity $\operatorname{tr}(X^2) = \operatorname{tr}(X)^2 - 2\det(X)$ and the identity

$$D \det(X) Y = \operatorname{tr}(X^{\#} \cdot Y),$$

for the derivative of the determinant at X, applied to Y. (Here $X^{\#}$ denotes the adjoint of X.) There exist analogous reducing maps for every homogeneous quadratic matrix Riccati equation. These maps are built from the coefficients of the characteristic polynomial of the matrix, and are induced by symmetries, viz., conjugation by invertible matrices. Moreover it is known that for every homogeneous quadratic vector field with rational local flow there exists a solution-preserving (not necessarily reducing) map of similar type; thus the existence of a symmetry group is not indispensable. See [28] for more details.

Generally, one will not expect that a given equation is globally reducible, but actually proving nonreducibility may be difficult. In the following example, a short argument is available.

Example 1.3. The equation

$$\dot{x}_1 = x_1^2 - x_2^2, \qquad \dot{x}_2 = 2x_1x_2$$

may also be written as $\dot{z} = z^2$ for complex z. It is not reducible to an equation in $\mathbb{R}$: The solutions of this equation can be determined explicitly; every solution with an initial value (z_1, z_2), $z_2 \neq 0$, converges to 0 both for $t \to \infty$ and for $t \to -\infty$. Assume that there is a reducing map Φ, with $\Phi(0) = c$. By monotonicity of solutions of one-dimensional equations, every (z_1, z_2) with $z_2 \neq 0$ must be mapped to c; hence Φ is constant. This argument also works locally, whence local reduction to a curve (smooth or with isolated singular points) is also impossible. Since complex numbers can be embedded in 2×2-matrices, one also sees that the reduced system from Example 1.2(e) is not reducible to dimension one.

Remark 1.4. A reducing map does not always yield a simplification of a differential equation, nor is it necessarily intended to. The simplification may actually occur in the reverse direction, as in the following cases.

(i) The map $\mathbb{R}^{(m,n)} \times \mathbb{R}^{(n,p)} \to \mathbb{R}^{(m,p)}$, $(X_1, X_2) \mapsto X_1 \cdot X_2$, sends solutions of the matrix equation system $\dot{X}_1 = A \cdot X_1$, $\dot{X}_2 = X_2 \cdot B$ (with $A \in \mathbb{R}^{(m,m)}$ and $B \in \mathbb{R}^{(p,p)}$) to solutions of $\dot{Y} = A \cdot Y + Y \cdot B$.

(ii) The rational map $X \mapsto X_1 X_2^{-1}$ sends solutions of the linear (matrix) equation

$$\dot{X} = \begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

to solutions of the (matrix) Riccati equation $\dot{Y} = -YCY + AY - YD + B$ (see Reid [32]).

Next we consider the "inverse problem" with regard to reducibility: Determine all vector fields which are reducible by a prescribed map. In the following example, this problem can be solved in a straightforward manner.

Example 1.5. Let $\Phi(x) = x/\|x\|$, with $\|x\| = \sqrt{\langle x, x \rangle}$ as the norm associated with the standard scalar product. This map sends $\mathbb{R}^n \setminus \{0\}$ to the unit sphere $\mathbf{S}^{n-1}$. We will determine all smooth vector fields on $\mathbb{R}^n \setminus \{0\}$ (or some open subcone) that are reducible by Φ.

Given a vector field f, one has

$$D\Phi(x) f(x) = \frac{1}{\|x\|} \left(\langle x, x \rangle f(x) - \langle x, f(x) \rangle x \right).$$

Now recall that for every nonzero z and every w there is a decomposition $w = w_0 + w_1$, with w_0 parallel to z, and w_1 orthogonal to z:

$$w_0 := \frac{\langle z, w \rangle}{\langle z, z \rangle} \cdot z, \qquad w_1 := w - w_0.$$

This yields a decomposition

$$f(x) = \alpha(x) \cdot x + \hat{f}(x),$$

with $< x, \hat{f}(x) >= 0$, and both α and $\hat{f}$ as smooth as f, and

$$D\Phi(x) f(x) = \frac{1}{\|x\|^3} \left(\langle x, x \rangle \hat{f}(x) - < x, \hat{f}(x) > x \right) = \frac{1}{\|x\|} \hat{f}(x).$$

Since $\Phi(x) = \Phi(\lambda x)$ for all $\lambda > 0$, reducibility implies

$$\frac{1}{\|x\|} \hat{f}(x) = \frac{1}{\|\lambda x\|} \hat{f}(\lambda x),$$

and

$$\hat{f}(\lambda x) = \lambda \hat{f}(x).$$

To summarize: If the differential equation $\dot{x} = f(x)$ is reducible by Φ, then $f(x) = \alpha(x) \cdot x + \hat{f}(x)$, with $\hat{f}$ positively homogeneous of degree one.

It is easy to verify that every vector field of this type is in fact reducible by Φ, with the reduced vector field $g = \hat{f}$.

We close this section with another basic observation.

Proposition 1.6. *In a suitable neighborhood $\widetilde{U}$ of any nonstationary point of $\dot{x} = f(x)$, the following holds: For any open $V \subseteq \mathbb{R}^m$ and $g\colon V \to \mathbb{R}^m$, there is a nonconstant map $\Phi\colon \widetilde{U} \to V$ that satisfies the identity*

$$D\Phi(x)f(x) = g(\Phi(x)).$$

In particular, $\dot{x} = f(x)$ is reducible on $\widetilde{U}$ to any differential equation on $\mathbb{R}^m$, with arbitrary $m < n$.

Proof. We may assume that $f = (1, 0, \dots, 0)^t$ by the straightening theorem; thus one has $D\Phi(x)f(x) = \partial\Phi(x)/\partial x_1$. Then the assertion follows readily by invoking standard results about initial value and parameter dependence of solutions of ordinary differential equations. □

Thus, the question of local reducibility arises only at stationary points. Example 1.3 shows that local reducibility near stationary points is not automatic. Generally, it will be seen below that constraints exist on reducing maps near a stationary point that are useful for computational purposes, even though they may preclude their existence.

2. Reducibility and Lie Algebras of Vector Fields

The invariants of a Lie algebra of infinitesimal symmetries for a given equation $\dot{x} = f(x)$ give rise to a canonical construction of reducing maps; see Hermann [21], Olver [29], and [41]. In this section we will exhibit the correspondence between arbitrary reducing maps and Lie algebras of vector fields. We will also discuss completely reducible equations, which correspond to certain abelian Lie algebras and to local diffeomorphisms. The results will be stated for the analytic case, but they also hold, with obvious modifications, for smooth vector fields.

Recall that the Lie derivative of a scalar-valued function ψ with respect to a vector field h is defined by $L_h(\psi)(x) = D\psi(x)h(x)$, and that the Lie bracket of two vector fields h and g is given by $[h, g](x) = Dg(x)h(x) - Dh(x)g(x)$.

Proposition 2.1. *Let $U \subseteq \mathbb{R}^n$ be open and connected, and let $A(U)$ be the algebra of real analytic functions from U to $\mathbb{R}$. Let $\mathcal{L}$ be a nonzero Lie algebra of analytic vector fields on U, and denote by $\mathcal{L}^*$ the $A(U)$-module generated by $\mathcal{L}$.*

Moreover, let $\tilde{U} \subseteq U$ be open and nonempty, and suppose that there are finitely many invariants $\phi_1, \dots, \phi_r$ of $\mathcal{L}$ on $\tilde{U}$ such that every invariant of $\mathcal{L}$ on $\tilde{U}$ can be expressed as $\sigma(\phi_1, \dots, \phi_r)$, with an analytic function σ of r variables.

Then $\Phi := (\phi_1, \dots, \phi_r)$ is a reducing map on $\tilde{U}$ for each vector field f that satisfies $[f, \mathcal{L}] \subseteq \mathcal{L}^*$.

Proof. Let ψ be an invariant of $\mathcal{L}$. For every $g \in \mathcal{L}$, there are analytic $\mu_1, \ldots, \mu_s$, and vector fields $h_1, \ldots, h_s \in \mathcal{L}$ such that

$$[g, f] = \mu_1 h_1 + \cdots + \mu_s h_s.$$

Using the identity $L_{[g,f]} = L_g L_f - L_f L_g$, and $L_g(\psi) = 0$, one finds

$$L_g L_f(\psi) = L_{[g,f]}(\psi) = \mu_1 L_{h_1}(\psi) + \cdots + \mu_s L_{h_s}(\psi) = 0.$$

Thus, $L_f(\psi)$ is again an invariant of $\mathcal{L}$, and hence there is an analytic function ρ of r variables such that

$$L_f(\psi) = \rho(\phi_1, \ldots, \phi_r).$$

An application to $\phi_1, \ldots, \phi_r$ shows

$$D\phi_j(x) f(x) = q_j(\phi_1(x), \ldots, \phi_r(x)), \qquad 1 \leq j \leq r,$$

with suitable analytic functions q_j; hence there is an analytic q such that $D\Phi(x) f(x) = q(\Phi(x))$ for all $x \in \tilde{U}$.

Since $\mathcal{L} \neq 0$, the rank of $D\Phi(x)$ is smaller than n at every point x. Thus, Φ is indeed reducing. □

Remark 2.2.

(a) The classical reduction theorem starts with a Lie algebra of symmetries, then $[\mathcal{L}, f] = 0$; see Hermann [21]. The proof is essentially the same. The "finite generation property" for the invariants of $\mathcal{L}$ in Proposition 2.1 holds automatically near every regular point of $\mathcal{L}$, due to Frobenius's theorem on involutive systems.

(b) The vector fields f satisfying $[f, \mathcal{L}] \subseteq \mathcal{L}^*$ form a Lie algebra $\mathcal{M}$. If $\mathcal{C}$ denotes the centralizer of $\mathcal{L}$, then one readily verifies

$$\mathcal{C} + \mathcal{L}^* \subseteq \mathcal{M}.$$

This fact illustrates the broader scope of reducible systems versus symmetric systems. For various Lie algebras, equality $\mathcal{M} = \mathcal{C} + \mathcal{L}^*$ holds; see the examples below.

Example 2.3. Let $\mathcal{L}$ be the Lie algebra of linear vector fields on $\mathbb{R}^p$ that are skew-symmetric with respect to the quadratic form $\phi(x) = \sum \epsilon_i x_i^2$, all $\epsilon_i \in \{1, -1\}$. It is well known that every analytic invariant of $\mathcal{L}$ can be written as an analytic function of $\phi(x)$. The condition $[f, \mathcal{L}] \subseteq \mathcal{L}^*$ is satisfied if and only if

$$f(x) = \rho(\phi(x)) \cdot x + \sum \nu_j(x) B_j x,$$

with analytic functions ρ and ν_j, and linear ϕ-skew-symmetric B_j. (One may view the element $\sum \nu_j(x) B_j$ of $\mathcal{L}^*$ as a ϕ-skew-symmetric matrix with x-dependent entries. See [39] for a proof of necessity.) Thus every vector field of this type is reduced by ϕ: $\mathbb{R}^p \to \mathbb{R}$, while only $f(x) = \rho(\phi(x)) \cdot x$ admits $\mathcal{L}$ as a symmetry algebra when $p \geq 2$.

Example 2.4. We sketch a "generic" approach to reducible vector fields for the case that $\mathcal{L}$ is the Lie algebra of a compact linear group acting on $\mathbb{R}^n$, with invariant scalar product $\langle\cdot,\cdot\rangle$. We refer to Golubitsky et al. [16], [17] for some of the background material used in the following.

Denote the generic orbit dimension of the action by s (with $0 < s < n$), and let $B_1, \dots, B_s \in \mathcal{L}$ be such that $B_1 y, \dots, B_s y$ are linearly independent in $\mathbb{R}^n$ for some y (and hence for all y in an open and dense subset of $\mathbb{R}^n$). The polynomial invariant algebra of $\mathcal{L}$ then contains $n-s$ algebraically independent elements $\psi_1, \dots, \psi_{n-s}$ (which form a maximal algebraically independent set). Define their gradients q_j via

$$\langle q_j(x), v\rangle = D\psi_j(x)v.$$

Note that the q_j are $\mathcal{L}$-symmetric, and that $q_1(z), \dots, q_{n-s}(z)$ are linearly independent in $\mathbb{R}^n$ for some z (hence all z on an open and dense set). The identity

$$\langle B_i x, q_j(x)\rangle = D\psi_j(x)B_i x = L_{B_i}(\psi_j)(x) = 0,$$

for all i and j, shows that there is an open and dense subset U of $\mathbb{R}^n$ such that

$$B_1 z, \dots, B_s z, q_1(z), \dots, q_{n-s}(z)$$

are linearly independent in $\mathbb{R}^n$ for all $z \in U$. One verifies that the polynomial

$$\theta(x) := \det(B_1 x, \dots, B_s x, q_1(x), \dots, q_{n-s}(x))$$

is an invariant of $\mathcal{L}$.

By Cramer's rule, any vector field f on U has a representation

$$f(x) = \sum \mu_i(x) B_i x + \sum \rho_j(x) q_j(x),$$

with $\mu_i = \tilde{\mu}_i/\theta$, $\rho_j = \tilde{\rho}_j/\theta$, and the $\tilde{\mu}_i$, $\tilde{\rho}_j$ as smooth as f. For instance,

$$\tilde{\mu}_1(x) = \det(f(x), B_2 x, \dots, B_s x, q_1(x), \dots, q_{n-s}(x)).$$

Such a vector field f is reducible by the invariants of $\mathcal{L}$ if and only if each Lie derivative $L_f(\psi_k)$ is $\mathcal{L}$-invariant, and the latter holds if and only if each ρ_j is $\mathcal{L}$-invariant. To see the necessity of the last condition, observe

$$L_f(\psi_k) = \sum \rho_j L_{q_j}(\psi_k) = \sum \rho_j \langle q_j, q_k\rangle,$$

and invertibility of the matrix with entries $\langle q_j, q_k\rangle$.

Thus, on U we have equality $\mathcal{C} + \mathcal{L}^* = \mathcal{M}$.

A partial converse to Proposition 2.1 follows next.

Proposition 2.5. *Let $\dot{x} = f(x)$ be reducible on the connected open subset $\tilde{U}$ of U, with reducing map $\Phi = (\phi_1, \dots, \phi_r)$. Then the set $\mathcal{L}^*$ of all analytic vector fields g on $\tilde{U}$ which satisfy $L_g(\phi_j) = 0$ for $j = 1, \dots, r$ is a Lie algebra as well as a module over $A(\tilde{U})$, and*

$$[f, \mathcal{L}^*] \subseteq \mathcal{L}^*.$$

Moreover, $\mathcal{L}^*$ is nonzero.

Proof. (i) To verify the asserted inclusion, note that there are analytic functions $q_1, \ldots, q_m$ such that $L_f(\phi_j) = q_j(\phi_1, \ldots, \phi_r)$, all j; hence,

$$L_g L_f(\phi_j) = 0, \quad \text{all } g \in \mathcal{L}^*, \quad 1 \le l \le m.$$

Therefore $L_{[g,f]}(\phi_j) = 0$ for all j, and $[g, f] \in \mathcal{L}^*$.

(ii) For the proof of $\mathcal{L}^* \neq 0$, we first show: If $\psi_1, \ldots, \psi_{n-1}$ are analytic on $\tilde{U}$, and their functional matrix has rank $n-1$ on an open-dense subset of $\tilde{U}$, then there is a nonzero vector field g which has first integrals $\psi_1, \ldots, \psi_{n-1}$. We invoke an auxiliary result from linear algebra: Let $\langle \cdot, \cdot \rangle$ denote the standard scalar product in $\mathbb{R}^n$. For each system $v_1, \ldots, v_{n-1}$ of vectors there is a unique vector $v_1 \times \cdots \times v_{n-1}$ with the property that

$$\langle v_1 \times \cdots \times v_{n-1}, y \rangle = \det(v_1, \ldots, v_{n-1}, y)$$

for all y. If the v_j are linearly independent, then $\langle v_1 \times \cdots \times v_{n-1}, y \rangle = 0$ if and only if y lies in the span of the v_j.

Now define

$$g(x) := \nabla \psi_1 \times \cdots \times \nabla \psi_{n-1},$$

where ∇ denotes the gradient. Then $g \neq 0$, and

$$D\psi_j(x) g(x) = \big\langle g(x), \nabla \psi_j(x) \big\rangle = 0,$$

for all j, whence $g \in \mathcal{L}^*$.

(iii) Let $\phi_1, \ldots, \phi_s$ be a maximal functionally independent subsystem of $\phi_1, \ldots, \phi_r$ (hence the Jacobians of $(\phi_1, \ldots, \phi_s)$ and of $(\phi_1, \ldots, \phi_r)$ have rank s) on some open and dense subset of $\tilde{U}$. Then (iii) shows the existence of a nonzero g such that $L_g(\phi_1) = \cdots = L_g(\phi_s) = 0$. Since each ϕ_j with $j > s$ is locally a function of $\phi_1, \ldots, \phi_s$, every $L_g(\phi_j) = 0$, and thus $g \in \mathcal{L}^*$. □

We now turn to complete reducibility, a considerably stronger concept. The corresponding notion for Hamiltonian systems, known by the name of "separability," has attracted much attention; see for instance Benenti [6] and several contributions to [1]. As we will see, completely reducible systems correspond to certain classes of abelian Lie algebras, and in turn, to local diffeomorphisms.

Definition 2.6. The ordinary differential equation $\dot{x} = f(x)$ is called *completely reducible* on the open set $\tilde{U} \subseteq U \subseteq \mathbb{R}^n$ if there is a solution-preserving map $\Psi\colon \tilde{U} \to \mathbb{R}^n$, locally invertible at every point of $\tilde{U}$, to a system $\dot{x} = q(x)$ which is a product of one-dimensional equations; thus

$$q(x) = \begin{pmatrix} q_1(x_1) \\ \vdots \\ q_n(x_n) \end{pmatrix} = \sum q_i(x_i) e_i,$$

with the standard basis $e_1, \ldots, e_n$.

Due to the straightening theorem, any vector field is locally completely reducible near a nonstationary point. We will see that there is a computational approach to characterizing and constructing completely reducible systems.

Proposition 2.7. *If the differential equation $\dot{x} = f(x)$ is completely reducible on $\tilde{U} \subseteq U$, then there are pairwise commuting vector fields $g_1, \dots, g_n$ which satisfy the following conditions:*

(i) *There is a $v \in \tilde{U}$ such that $g_1(v), \dots, g_n(v)$ are linearly independent in $\mathbb{R}^n$ ("regularity in v").*
(ii) *f is a linear combination of the g_i.*
In particular, the centralizer $\{g\colon [g, f] = 0\}$ of a completely reducible vector field is not trivial.

Proof. Assume that $f = q$ is completely reduced, and let $g_i(x) = q_i(x_i)e_i$ if $q_i \neq 0$, and $g_i(x) = e_i$ (the i^{th} standard basis vector) if $q_i = 0$. □

There is an alternative local characterization (and construction principle) for the "regular" abelian Lie algebras from Proposition 2.7:

Proposition 2.8. *Given a system of n pairwise commuting vector fields $g_1, \dots, g_n$, which is regular in $v \in U$, there is a local diffeomorphism Γ such that*

$$g_i(x) = D\Gamma(x)^{-1} e_i \quad (1 \leq i \leq n),$$

for all x in a neighborhood of v.

Proof. Define the matrix $Q(x) = (g_1(x), \dots, g_n(x))^{-1}$; thus $g_i(x) = Q(x)^{-1} e_i$ for $1 \leq i \leq n$. Since the g_i span an abelian Lie algebra, we have

$$\left[Q(x)^{-1}(a), Q(x)^{-1}(b)\right] = 0,$$

for all a and b. We evaluate the Lie bracket, using

$$D\left(Q(x)^{-1}\right) y = -Q(x)^{-1} \left(DQ(x)y\right) Q(x)^{-1},$$

and find

$$0 = -Q(x)^{-1} \left(DQ(x)\left(Q(x)^{-1}b\right) Q(x)^{-1}a - DQ(x)\left(Q(x)^{-1}a\right) Q(x)^{-1}b\right).$$

By invertibility of $Q(x)$ near v, we obtain the closedness condition

$$\left(DQ(x)u\right) w = \left(DQ(x)w\right) u,$$

for all u and w in $\mathbb{R}^n$. Therefore $Q(x)$ is the derivative of a local diffeomorphism Γ in some simply connected neighborhood of v. □

Remark 2.9. One may also reverse the course: Start with a local diffeomorphism Γ, and define the vector fields g_i by the formula in Proposition 2.8. Since this will produce an abelian Lie algebra, we obtain a construction principle for completely reducible systems.

It should be noted that Proposition 2.8 alone is in general not sufficient to ensure complete reducibility. One obstacle is the fact that there exist even linear maps that are not completely reducible.

Example 2.10.

(a) For $n\times n$ matrices, consider the rational map $\Gamma(X)=X^{-1}$. The identity $D\Gamma(X)^{-1}A=-XAX$ shows that the vector fields $q_A(X)=XAX$ form an abelian Lie algebra. More generally, inversion in unital Jordan algebras provides such examples of birational straightening maps which yield polynomial vector fields. See [18] about homogeneous rational maps of this type.

(b) A particular example in dimension two is as follows: Let p_1 and p_2 be functions of one variable. Then

$$\begin{aligned} g_1(x) &= \begin{pmatrix} p_1(e^{x_1}\cos x_2)\cdot e^{-x_1}\cos x_2 \\ -p_1(e^{x_1}\cos x_2)\cdot e^{-x_1}\sin x_2 \end{pmatrix}, \\ g_2(x) &= \begin{pmatrix} p_2(e^{x_1}\sin x_2)\cdot e^{-x_1}\sin x_2 \\ p_2(e^{x_1}\sin x_2)\cdot e^{-x_1}\cos x_2 \end{pmatrix} \end{aligned}$$

span a two-dimensional abelian Lie algebra. The map

$$\Psi(x) = \begin{pmatrix} e^{x_1}\cos x_2 \\ e^{x_1}\sin x_2 \end{pmatrix}$$

is solution-preserving from $\dot{x} = g_i(x)$ to $\dot{x} = p_i(x_i)e_i$, and thus separating. Clearly, any map from $\mathbb{R}^2$ to $\mathbb{R}^2$ which is a local diffeomorphism at every point will give rise to a separable system.

(c) Consider the map $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$, with

$$\Phi(x) = \begin{pmatrix} x_1^2 + x_2^2 \\ 2x_1x_2 \end{pmatrix}, \quad \text{and} \quad D\Phi(x)^{-1} = \frac{1}{2(x_1^2 - x_2^2)} \begin{pmatrix} x_1 & -x_2 \\ -x_2 & x_1 \end{pmatrix}.$$

Note that Φ is not a local diffeomorphism at every point of $\mathbb{R}^2$. Now let γ be an odd analytic function on $\mathbb{R}$. Then

$$f(x) := D\Phi(x)^{-1} \begin{pmatrix} \gamma(x_1^2 + x_2^2) \\ \gamma(2x_1x_2) \end{pmatrix}$$

is actually analytic on the whole plane: The term

$$x_1\gamma(x_1^2 + x_2^2) - x_2\gamma(2x_1x_2)$$

vanishes whenever $x_1 = x_2$ or $x_1 = -x_2$, and is therefore (e.g., by the Hilbert-Rückert Nullstellensatz; see Ruiz [34]) the product of $(x_1 - x_2)\cdot(x_1 + x_2)$ and some analytic function. Hence the first entry of f is analytic, and a similar observation holds for the second entry. By construction, Φ is solution-preserving from $\dot{x} = f(x)$ to

$$\dot{x} = \begin{pmatrix} \gamma(x_1) \\ \gamma(x_2) \end{pmatrix},$$

and thus f is completely reducible.

3. Prescribed Reducing Maps

A standard strategy in the discussion of symmetric systems is to consider the following "inverse" problem: Start with a Lie algebra $\mathcal{L}$ of vector fields and determine those equations which admit $\mathcal{L}$ as a Lie algebra of infinitesimal symmetries. This procedure will yield a class of differential equations that are reducible by the invariants of $\mathcal{L}$. Lie, in his classical work [24], thus illuminated the underlying common feature for a multitude of differential equations which are integrable in an elementary manner. In more recent times, there has been a large volume of work on symmetric systems where the (linear; compact or reductive) symmetry group is given at the start, symmetric systems are constructed, and their qualitative properties are being investigated. It seems natural to extend this approach to reducibility. Hence we will start with a map Φ and consider the vector fields reducible by Φ.

We will discuss two items, viz. qualitative properties of reducible equations enforced by the reducing map Φ, and construction of Φ-reducible vector fields.

The first item is related to properties of the (singular) foliation defined by Φ. One generally finds certain invariant sets from this foliation, similar to the stratification in the linear compact symmetry case.

The first statement of the following proposition is quoted from [41], while the second follows directly from the behavior of invariant sets under solution-preserving maps.

Proposition 3.1. *Let Φ: $\tilde{U} \to \mathbb{R}^m$ be solution-preserving from $\dot{x} = f(x)$ to $\dot{y} = g(y)$.*

(a) *For every integer $k \geq 0$, the set*

$$Z_k := \{x \in \tilde{U}\colon \operatorname{rank} D\Phi(x) = k\}$$

is invariant for $\dot{x} = f(x)$.

(b) *Moreover, every $\Phi(Z_k)$ is invariant for $\dot{y} = g(y)$, and every $\Phi^{-1}(\Phi(Z_k))$ is invariant for $\dot{x} = f(x)$.*

A related nonlocal result about invariant sets follows next. Here, a formulation for manifolds seems appropriate.

Proposition 3.2. *Let $\dot{x} = f(x)$ be given on the manifold M, and $\dot{y} = g(y)$ on the manifold N, and suppose that f and g are related by Φ: $M \to N$. For $z \in N$, let*

$V_z := \{x \in M\colon \Phi(x) = z\}$.

(a) *Let* $z \in N$*, and assume that there is* $r > 0$ *such that* $F(t, x_0)$ *exists for all* $x_0 \in V_z$ *and all* t *with* $|t| < r$*. (This assumption is satisfied, in particular, if* Φ *is a proper map.) Then for all* $|t| < r$*, the level sets* V_z *and* $V_{G(t,z)}$ *are diffeomorphic.*

(b) *Let* $x_0 \in M$*, and* $z := \Phi(x_0)$*. Then for every relatively compact open neighborhood* W *of* x_0 *in* V_z*, there is an* $r > 0$ *such that* $F(t, x)$ *exists for all* $x \in W$ *and all* $|t| < r$*, and for any submanifold* Y *of* W*,* $F(t, Y)$ *is diffeomorphic to* Y *for all* $|t| < r$*.*

(c) *Assume that* M *is compact. Then for all* $z \in N$ *and all* $t \in \mathbb{R}$ *the level sets* V_z *and* $V_{G(t,z)}$ *are diffeomorphic.*

Proof. As for part (a), consider the map

$$V_z \to V_{G(t,z)}, \qquad w \mapsto F(t, w),$$

with inverse sending v to $F(-t, v)$. The remaining assertions follow by standard arguments. □

Example 3.3.

(a) If the derivative $D\Phi(x)$ has maximal rank on some open subset of M, then locally the corresponding level sets are diffeomorphic by the implicit function theorem. This illustrates the overlap of Propositions 3.1 and 3.2.

(b) Consider $\Phi(x) = x_1^3 + x_1^2 - x_2^2$, with derivative $D\Phi(x) = (3x_1^2 + 2x_1, -2x_2)$. The invariant set Z_0, according to Proposition 3.1, consists of the points $(0, 0)$ and $(-2/3, 0)$, and the invariant set $\Phi^{-1}(\Phi(Z_0))$ is the union of the sets defined by $\Phi(x) = 0$, resp. $\Phi(x) = \Phi(-2/3, 0) = 4/27$. The connected components are two curves and the point $(-2/3, 0)$. Using the approach via Proposition 3.2, one notes that the "loop" V_0 is not (not even locally) diffeomorphic to any V_c with c near 0, which forces its invariance. By the same token one obtains invariance of the isolated point contained in $V_{(-2/3,0)}$, but Proposition 3.1 yields a sharper result here.

(c) Consider the three-dimensional sphere $M = \{x \in \mathbb{R}^4\colon x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$, and the projection

$$\Phi\colon (x_1, x_2, x_3, x_4) \mapsto (x_1, x_2) \in \mathbb{R}^2 =: N.$$

The inverse image of (y_1, y_2) is a 1-sphere if $y_1^2 + y_2^2 < 1$ and a point if $y_1^2 + y_2^2 = 1$. By Proposition 3.2 the set $\{x \in M\colon x_3 = x_4 = 0\}$ is invariant for any differential equation on M which is reducible by Φ.

It is natural to ask about the behavior of limit sets in the presence of a reducing map. Generally, if $A = \omega(x_0)$ is an omega limit set for $\dot{x} = f(x)$, then $\Phi(A) \subseteq \omega(\Phi(x_0))$. If the positive semitrajectory of x_0 is contained in a compact subset of U, or if Φ is a proper map, then equality holds, as is easily verified. (Equality does not always hold; for instance, the limit set $\omega(x_0)$ may be empty while $\omega(\Phi(x_0))$ is not.) In general, $\omega(\Phi(x_0))$ has a simpler structure than $\omega(x_0)$; e.g., a periodic orbit may be mapped to a point. In mathematical biology one frequently deals with a dynamical system $\dot{x} = f(x)$ in $\mathbb{R}^n_+$ which describes the coevolution of n types, and is reducible by a linear functional $\phi(x)$

such that all solutions of the reduced equation converge to a point. (For instance, one may have total population size $\phi(x) = \sum_i x_i$ or one species $\phi(x) = x_n$.) Thus one obtains a subsystem in (a copy of) $\mathbb{R}_+^{n-1}$, which may be seen as a reduction in a different sense. This subsystem can be studied in its own right. It is not true in general that the limit sets of the large system are equal to those of the subsystem. For a systematic study of this question, within the framework of asymptotically autonomous systems, see [26], [22], and work cited therein.

In the following, we will discuss the construction of local analytic and of polynomial reducible vector fields. First, we record some structural properties. Let $\tilde{U} \subseteq \mathbb{R}^n$ be nonempty, open, and connected, and Φ an analytic map to some open and connected $\tilde{V} \subseteq \mathbb{R}^m$. Since Φ is a designated reducing map, we also require that rank $D\Phi(x) < n$ for all x.

Now consider pairs of analytic vector fields f on $\tilde{U}$ and g on $\tilde{V}$ which are related by Φ. Denote by $\mathcal{M}(\Phi)$ the set of all such vector fields f, and by $\mathcal{M}_{\text{red}}(\Phi)$ the set of all such reduced vector fields g.

Proposition 3.4. *The sets $\mathcal{M}(\Phi)$ and $\mathcal{M}_{\text{red}}(\Phi)$ are Lie algebras, and $\mathcal{M}_{\text{red}}(\Phi)$ is a module over $A(\tilde{V})$. Moreover, $\mathcal{M}(\Phi) \neq 0$.*

Proof. The Lie algebra property is well known. Furthermore, if f and g are Φ-related, and $\mu \in A(\tilde{V})$, then $(\mu \circ \Phi) f$ and μg are Φ-related. The last assertion was shown in Proposition 2.5. □

By pulling back via Φ, the Lie algebra $\mathcal{M}(\Phi)$ is also a module over $A(\tilde{V})$, but in general there is no sensible way to make $\mathcal{M}(\Phi)$ a module over $A(\tilde{U})$.

Example 3.5. Let $\Phi(x) = (x_1, \ldots, x_r)$ with $r < n$, $\tilde{U}$ and $\tilde{V}$ appropriate. Then $\mathcal{M}_{\text{red}}(\Phi)$ is the set of all analytic vector fields. The elements of $\mathcal{M}(\Phi)$ were characterized in Example 1.2.

While this example is almost trivial, it describes the local situation near regular points of any reducing map, and it also includes—up to linear coordinate transformations—all linear reducing maps.

Generally, it seems to be a subtle problem to determine $\mathcal{M}(\Phi)$ and $\mathcal{M}_{\text{red}}(\Phi)$ for a given map Φ; or even to find nontrivial elements in $\mathcal{M}_{\text{red}}(\Phi)$. In view of Proposition 3.1, one certainly has to consider the critical points of Φ. Du Plessis and Wall [13] determine these modules for a complex local (analytic or algebraic) map Φ which satisfies a certain finiteness condition. Consideration of the critical set (where $D\Phi(x)$ has nonmaximal rank) and its image, the discriminant set, is crucial in their argument. In the following, we will restrict attention to the algebraically distinguished settings of polynomial, respectively of local analytic, vector fields, and maps. With regard to reducibility to $\mathbb{R}$, we can give definitive characterizations.

Theorem 3.6. *Let Φ be a nonconstant polynomial map from $\mathbb{R}^n$ to $\mathbb{R}$, and denote the polynomial Φ-reducible vector fields, resp. their images, by $\mathcal{N}(\Phi)$, resp. $\mathcal{N}_{\text{red}}(\Phi)$. Then* *there is a nonzero polynomial g^* over $\mathbb{R}$ that generates the module $\mathcal{N}_{\text{red}}(\Phi)$ over the polynomial algebra $\mathbb{R}[x]$.*

Proof. The existence of g^* follows from the fact that $\mathbb{R}[x]$ is a principal ideal domain. It remains to show that $g^* \neq 0$.

It is harmless to complexify. Thus, consider in $\mathbb{C}^n$ the zero set Y of the ideal $\langle \partial\Phi/\partial x_1, \dots, \partial\Phi/\partial x_n \rangle$ generated by the partial derivatives. Then Y has only finitely many components $Y_1, \dots, Y_s$, and the map Φ is constant on each component. If Y_i is a point, this is obvious. Otherwise it is sufficient to show constancy on an open and dense subset Z_i of Y_i, and there exists such a subset which is a complex manifold; see Shafarevich [36]. For any pair of points of Z_i, take a piecewise smooth curve γ connecting them. Then

$$\frac{d}{ds}(\Phi(\gamma(s))) = \sum \frac{\partial \Phi}{\partial x_j}(\gamma(s))\gamma'(s) = 0,$$

since all $\partial\Phi/\partial x_j$ vanish on γ, and constancy follows. Now choose $y_k \in Y_k$ for $k = 1, \dots, s$, and define

$$h(x) := \prod (x - \Phi(y_k)) \in \mathbb{C}[x].$$

Then $p(x) := h(\Phi(x)) \in \mathbb{C}[x_1, \dots, x_n]$ satisfies the following: Whenever

$$\partial\Phi/\partial x_1(z) = \cdots = \partial\Phi/\partial x_n(z) = 0,$$

then z is contained in some Y_k, whence $\Phi(z) = \Phi(y_k)$ for some k, and $p(z) = 0$. Hilbert's Nullstellensatz (see Shafarevich [36]) shows

$$p^d \in \langle \partial\Phi/\partial x_1, \dots, \partial\Phi/\partial x_n \rangle,$$

for some $d \geq 1$, and therefore h^d is a nonzero element of $\mathcal{N}_{\text{red}}(\Phi)$. □

Example 3.7. Consider $\Phi(x) = x_1^3 + x_1^2 - x_2^2$, with $\partial\Phi/\partial x_1 = x_1(3x_1 + 2)$, $\partial\Phi/\partial x_2 = -2x_2$.

The singular points are $(0, 0)$, with $\Phi(0, 0) = 0$, and $(-2/3, 0)$, with $\Phi(-2/3, 0) = 4/27$. Since the ideal $\langle \partial\Phi/\partial x_1,\ \partial\Phi/\partial x_2 \rangle$ is clearly a radical ideal, the polynomial $h(x) = x(x - 4/27)$ is an element of $\mathcal{N}_{\text{red}}(\Phi)$. Here we have $g^* = h$.

We also determine $\mathcal{N}(\Phi)$ for this map. To do so, first solve

$$x_1(3x_1 - 2)f_1^* - 2x_2 f_2^* = g^*(\Phi(x)) = (x_1^3 + x_1^2 - x_2^2)(x_1^3 + x_1^2 - x_2^2 - 4/27).$$

Over $\mathbb{R}(x_2)[x_1]$, by linear algebra, every solution can be written as

$$f_2^* = -(x_1^3 + x_1^2 - x_2^2)(x_1^3 + x_1^2 - x_2^2 - 4/27)/(2x_2) + x_1(3x_1 + 2)\mu, \qquad f_1^* = 2x_2\mu,$$

with μ a rational function in x_2. It is easy to verify that

$$\mu = (x_1^2 + x_1)(x_1^2/3 + x_1/9 - 2/27)/(2x_2)$$

yields the polynomial vector field

$$f^* = \begin{pmatrix} (x_1^2 + x_1)(x_1^2/3 + x_1/9 - 2/27) \\ -x_2^3/2 + x_2(x_1^3 + x_1^2 - 2/27) \end{pmatrix},$$

which satisfies $D\Phi(x) f^*(x) = g^*(\Phi(x))$. Moreover, since the polynomials $x_1(3x_1+2)$ and $2x_2$ are relatively prime, one gets $D\Phi(x)q(x) = 0$ if and only if $q = \rho\cdot(2x_2, x_1(3x_1+2))$. Therefore,

$$\mathcal{N}(\Phi) = \left\{ \sigma(\Phi(x)) f^*(x) + \rho(x) \cdot \begin{pmatrix} 2x_2 \\ x_1(3x_1+2) \end{pmatrix};\ \sigma \in \mathbb{R}[t]\,,\ \rho \in \mathbb{R}[x_1, x_2] \right\}.$$

Generally, the equation

$$\sum f_i^* \partial\Phi/\partial x_i = g^*(\Phi(x))$$

is accessible in an algorithmic manner: For instance, determine a Groebner basis S of the ideal generated by the $\partial\Phi/\partial x_i$; then $g^*(\Phi(x))$ can be expressed via the elements of S by the division algorithm. Since determining a Groebner basis also yields the elements of S as linear combinations of the $\partial\Phi/\partial x_i$, one obtains a solution of the equation.

In the local analytic, and also in the formal case, the arguments are similar. Here $\mathcal{M}(\Phi)$ and $\mathcal{M}_{\text{red}}(\Phi)$ will denote the corresponding local, or formal, objects.

Theorem 3.8. *Let Φ be (real-) analytic in a neighborhood of 0, with image contained in $\mathbb{R}$ and $\Phi(0) = 0$. Then there is a unique integer $e \geq 0$ such that $\mathcal{M}_{\text{red}}(\Phi)$ is generated by x^e. The same statement holds in the setting of formal power series.*

Proof. Since every nonzero ideal of the power series algebra in one variable x is generated by a power of x, one only needs to show that $\mathcal{M}_{\text{red}}(\Phi) \neq 0$. This follows directly from the property

$$\Phi \in \sqrt{\langle \partial\Phi/\partial x_1, \ldots, \partial\Phi/\partial x_n \rangle}$$

of power series; see Ruiz [34], ch. IV, Proof of Cor. 2.4. Hence there exists f^* such that $D\Phi(x) f^*(x) = \Phi(x)^e$. □

Example 3.9.

(a) Whenever the derivative of Φ at 0 is nonzero, then the exponent e equals 0.
(b) Whenever $D\Phi(0) = 0$ and the Hessian $D^2\Phi(0)$ is regular, then $e = 1$, since the $\partial\Phi/\partial x_i$ generate the maximal ideal of the power series ring.
(c) There are various cases for which $e = 1$ is automatic: For instance, let $\Phi(x_1, \ldots, x_n) = \gamma_1(x_1) + \cdots + \gamma_n(x_n)$, and note that γ_i is contained in the ideal generated by γ_i' in the algebra of power series. One also has $e = 1$ for quasi-homogeneous Φ, since then there exists some nonzero p such that $D\Phi(x)p(x) = \Phi(x)$. In particular, all the two-dimensional modality 0 singularities in Arnold's list (see [4], Part II, Thm. 1) lead to $e = 1$, as they fall into one of the above classes.

(d) The case $e > 1$ does occur, however. Let

$$\begin{aligned}\Phi(x) &= x_1x_2(x_1+x_2)(x_1^2+x_2^3) = x_1^3x_2(x_1+x_2) + x_1x_2^4(x_1+x_2)\\ &=: \Phi_5(x) + \Phi_6(x),\end{aligned}$$

and assume that $D\Phi(x)f(x) = \Phi(x)$ for some f. Then the Taylor expansion of $f = B + f_2 + \cdots$ necessarily starts with a nonzero linear term B that satisfies the identity $D\Phi_5(x)Bx = \Phi_5(x)$, which forces $B = \frac{1}{5}\mathrm{id}$. (Here the observation is useful that the sets defined by $x_1 = 0$, $x_2 = 0$, resp. $x_1 + x_2 = 0$ are invariant for B.) Comparing degree six terms then shows

$$D\Phi_5(x)f_2(x) = -\Phi_6(x),$$

with some homogeneous quadratic f_2, and a simple verification shows that this equation for f_2 has no solution.

The results we give in Theorems 3.6 and 3.8 are restricted to image dimension one, but are not necessarily local, and no finiteness condition is required. Thus they cannot be obtained as a consequence of [13].

Remark 3.10. One may furthermore consider *prescribed completely reducing maps*. But in contrast to the situation discussed above, prescription of such a map may force the vector fields to be trivial. As an example, consider

$$\Phi(x) = \begin{pmatrix} x_1^2 + x_2^2 \\ x_1^2 + 2x_2^2 \end{pmatrix}, \quad \text{with} \quad D\Phi(x) = \begin{pmatrix} 2x_1 & 2x_2 \\ 2x_1 & 4x_2 \end{pmatrix},$$

and suppose that Φ is completely reducing from $\dot{x} = f(x)$ to $\dot{y} = g(y)$. The derivative $D\Phi(x)$ is not invertible precisely for

$$x \in V := \mathbb{R}\cdot\begin{pmatrix}1\\0\end{pmatrix} \cup \mathbb{R}\cdot\begin{pmatrix}0\\1\end{pmatrix},$$

and thus by Proposition 3.1 V is invariant for f, and

$$\Phi(V) = \mathbb{R}\cdot\begin{pmatrix}1\\1\end{pmatrix} \cup \mathbb{R}\cdot\begin{pmatrix}1\\2\end{pmatrix}$$

is invariant for g. This implies $g_1(s) = g_2(s)$ for all s (invariance of $\mathbb{R}\cdot(1,1)$), and $2g_1(s) = g_2(2s)$ for all s (invariance of $\mathbb{R}\cdot(1,2)$), and thus $g(x) = \alpha x$ for some α, as well as $f(x) = (\alpha/2)x$. We see that the only vector fields completely reducible by Φ are already completely reduced. (There also exist maps for which only $f = 0$ is completely reducible.) We note that the vector fields in Example 2.10(c) were constructed from the invariance requirements in Proposition 3.1, and that the conditions imposed on them are both necessary and sufficient.

4. Reducibility Criteria

In this section the main focus is on criteria for, and obstructions to, reducibility of a given vector field. The goal is to actually compute a reducing map or to prove that no such map exists. We will deal with homogeneous polynomial vector fields and reducing maps in some detail, and apply the results to polynomial and to local analytic vector fields. In addition, we characterize general vector fields that are reducible by a linear map.

By $\mathbb{K}$ we denote the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. We will always consider vector fields f and g related by a map Φ, so that $(*)$ is satisfied, and will discuss this relation when additional conditions are imposed. We start with an elementary observation.

Lemma 4.1. *Let f be a homogeneous polynomial vector field of degree r on $\mathbb{K}^n$, g a homogeneous polynomial vector field of degree s on $\mathbb{K}^m$, and Φ: $\mathbb{K}^n \to \mathbb{K}^m$ a polynomial map which is homogeneous of degree q. If $g \circ \Phi \neq 0$, then $r - 1 = (s - 1)q$.*

Proposition 4.2. *Let $f(x) = Bx$ be linear, and q a positive integer.*

(a) *Every L_B-invariant subspace of S_q, the vector space of homogeneous polynomials of degree q, gives rise to a homogeneous polynomial map Φ: $\mathbb{K}^n \to \mathbb{K}^m$ of degree q which satisfies the identity $D\Phi(x)\,Bx = C\Phi(x)$ for some linear C. Conversely, Φ and C define an L_B-invariant subspace of S_q.*

(b) *Let Φ and C be as in part (a). If the eigenvalues of B are $\lambda_1, \ldots, \lambda_n$, then every eigenvalue of C has the form*

$$d_1\lambda_1 + \cdots + d_n\lambda_n,$$

with nonnegative integers d_i such that $\sum d_i = q$. If B is semisimple, then so is L_B, and up to an invertible linear transformation in the target space one may assume that the complex matrix representation of C is diagonal.

Proof. (a) We may assume that $\mathbb{K} = \mathbb{C}$. If there is a homogeneous Φ and a linear C satisfying the identity, then one has (with the components ϕ_i of Φ):

$$L_B(\phi_i) = \sum \gamma_{ij}\phi_j \quad (1 \leq i \leq m),$$

with suitable $\gamma_{ij} \in \mathbb{K}$; hence the subspace $V := \mathbb{K}\phi_1 + \cdots + \mathbb{K}\phi_m$ of S_q is L_B-invariant. On the other hand, one may choose a set of generators for an L_B-invariant subspace of S_q to construct a map Φ.

(b) Let $B = B_s + B_n$ be the decomposition into semisimple and nilpotent part. Then it is known (see, for example, [38]) that $L_B = L_{B_s} + L_{B_n}$ is the decomposition into semisimple and nilpotent part. Hence, there is a basis $\psi_1, \ldots, \psi_m$ of V such that

$$L_{B_s}(\psi_i) = \alpha_i\psi_i, \qquad L_{B_n}(\psi_i) \in \sum_{l<i} \mathbb{K}\psi_l,$$

for suitable scalars α_i. Moreover, each $\alpha_i = \sum d_{ij}\lambda_j$, with nonnegative integers d_{ij} such that $\sum_j d_{ij} = q$, according to [38]. □

Since S_1 is just the dual space of $\mathbb{K}^n$, we have:

Corollary 4.3. *Let $\dot{x} = Bx$ be linear on $\mathbb{K}^n$. Then there is a correspondence between the collection of linear reducing maps T from $\dot{x} = Bx$ to some other linear equation $\dot{y} = Cy$ on some $\mathbb{K}^m$ and the collection of subspaces of $\mathbb{K}^n$ that are invariant with respect to the transpose of B.*

Remark 4.4. In the generic case when B has simple eigenvalues, there are exactly 2^n (and thus finitely many) B^T- invariant subspaces of $\mathbb{C}^n$, viz., all sums of eigenspaces. In other words, there are essentially only $2^n - 2$ different linear reducing maps. If the eigenvalues of B are sufficiently generic, for instance linearly independent over the rationals, a similar finiteness property holds for homogeneous nonlinear reducing maps between linear differential equations of any degree; see Proposition 4.2 and its proof. In this situation there are only finitely many candidates for the invariant subspaces.

Let us now turn to the situation of Lemma 4.1 with homogeneous vector fields that are not necessarily linear. Given the vector field f, one may view the identity $(*)$ as a nonlinear system of determining equations for the coefficients of Φ and g with respect to the given coordinates. However, such a system is, even for small m and n, high-dimensional and quite formidable. The following observations help make computations feasible. In case $\mathbb{K} = \mathbb{C}$, one knows that a homogeneous vector field f on $\mathbb{C}^n$ of degree r admits $(r^n - 1)/(r - 1)$ or infinitely many one-dimensional invariant subspaces (with appropriate counting of multiplicities), and generically there are exactly $(r^n - 1)/(r - 1)$ of these; see Röhrl [33]. One-dimensional invariant subspaces yield particular solutions of the differential equation: If $f(c) = \alpha c$, then one obtains solutions $x(t) = \gamma(t) \cdot c$ with $\dot{\gamma}(t) = \alpha\gamma(t)^m$. Moreover, these subspaces yield a partial linearization of $(*)$. Note that such a subspace is spanned by some $c \neq 0$ such that $f(c) \in \mathbb{C} \cdot c$.

Proposition 4.5. *Let the homogeneous polynomial vector fields f and g be of respective degrees r and s, and let Φ be a nonzero homogeneous polynomial map of degree q such that the identity $(*)$ holds.*

(a) *If $c \neq 0$ satisfies $f(c) = \alpha c$ for some $\alpha \in \mathbb{K}$, then $\tilde{c} := \Phi(c)$ satisfies $g(\tilde{c}) = q\alpha\tilde{c}$.*
(b) *There is a smallest integer $k \geq 1$ such that $D^k\Phi(c) \neq 0$. The homogeneous map Φ^* of degree k defined by*

$$\Phi^*(x) = D^k\Phi(c)(x, \ldots, x)$$

is solution-preserving from the linear equation $\dot{x} = Df(c)\,x$ to the linear equation

$$\dot{y} = (Dg(\tilde{c}) - (q - k)\alpha\,\mathrm{id})\;y.$$

If $\alpha \neq 0$ and $\tilde{c} \neq 0$, then $k = 1$.

Proof. Note that $D^\ell\Phi$ is homogeneous of degree $q - \ell$ for $\ell \geq 0$. Part (a) is immediate from the defining identity, and Euler's equation. The existence of k follows from $D^q\Phi(y)(x, \ldots, x) = q! \cdot \Phi(x)$ (all y and x) and $\Phi \neq 0$. If $\alpha \neq 0$ and $\tilde{c} \neq 0$, then $D\Phi(c)\,c \neq 0$; hence $k = 1$.

Differentiate the identity $(*)$ ℓ times to obtain

$$D^{\ell+1}\Phi(x)\,(f(x), y_1, \dots, y_\ell) + \sum_{i=1}^{\ell} D^\ell\Phi(x)\,(y_1, \dots, Df(x)y_i, \dots, y_\ell) + \cdots$$
$$= Dg(\Phi(x))D^\ell\Phi(x)\,(y_1, \dots, y_\ell) + \cdots,$$

where all summands that are not written down explicitly contain a lower-order term $D^j\Phi(x)$ for some $j < \ell$. Now substitute $x = c$, let $\ell = k$, and use Euler's equation

$$D^{k+1}\Phi(c)\,(c, y_1, \dots, y_k) = (q-k)D^k\Phi(c)\,(y_1, \dots, y_k)$$

to obtain the assertion. □

The following consequence for first integrals has been noted previously; see [39] and Tsygvintsev [37]. The proof is immediate with Proposition 4.2.

Corollary 4.6. *Let Φ: $\mathbb{K}^n \to \mathbb{K}$ be a nonzero homogeneous polynomial first integral, with degree q, of the homogeneous polynomial vector field f, and $c \neq 0$ such that $f(c) = \alpha\, c$. Then there is an integer $k \geq 1$ such that*

$$D\Phi^*(x)Df(c)\,x = -(q-k)\alpha\Phi^*(x).$$

If $\beta_1, \dots, \beta_n$ are the eigenvalues of $Df(c)$, then there exist nonnegative integers $d_1, \dots, d_n$ such that $\sum d_i = k$ and

$$\sum d_i\beta_i + (q-k)\alpha = 0.$$

Example 4.7.

(a) We first consider complex planar homogeneous polynomial vector fields with regard to reducibility to $\mathbb{C}$. In view of Lemma 4.1, the critical question is concerned with the existence of a polynomial first integral, since there are degree bounds if no such integral exists. We use some results from Gröbner and Knapp [19], and from [39] and [40]. Thus let f: $\mathbb{C}^2 \to \mathbb{C}^2$ be homogeneous of degree $m \geq 2$. Then $\det(x,\ f(x))$ is a form of degree $m+1$ in two variables; hence,

$$\det(x,\ f(x)) = \gamma_1(x)\dots\gamma_{m+1}(x),$$

with suitable linear forms γ_j that are unique up to a nonzero scalar. There exist $(m-1)$-forms $\lambda_1, \dots, \lambda_{m+1}$ such that

$$\gamma_j(f(x)) = \lambda_j(x)\cdot\gamma_j(x), \quad 1 \leq j \leq m+1.$$

If $\gamma_j(c) = 0$, then $f(c)$ is a scalar multiple of c, and in case $f(c) \neq 0$, one can achieve $f(c) = c \neq 0$ by scaling. We call such elements idempotents of f. We will restrict attention to the generic case when the γ_j are mutually relatively prime and there are idempotents $c_1, \dots, c_{m+1}$ such that $\gamma_j(c_j) = 0$ for all j. Note that $\lambda_j(c_k) = 1$

whenever $j \neq k$. Since the space of $(m-1)$-forms on $\mathbb{C}^2$ is m-dimensional, there is a nontrivial relation

$$\mu_1\lambda_1 + \cdots + \mu_{m+1}\lambda_{m+1} = 0,$$

with constants μ_j, giving rise to a first integral $\gamma_1^{\mu_1} \ldots \gamma_{m+1}^{\mu_{m+1}}$. Obviously there exists a polynomial first integral if and only if all μ_j can be chosen as nonnegative integers. We will show how the μ_j can be determined directly from the c_j, without computing the λ_j. Substitute c_k in the relation for the μ_j to obtain

$$\mu_1 + \cdots + \mu_{k-1} + \mu_k\lambda_k(c_k) + \mu_{k+1} + \cdots + \mu_{m+1} = 0,$$

which shows that

$$\begin{pmatrix} \lambda_1(c_1) & 1 & \cdots & \cdots & 1 \\ 1 & \lambda_2(c_2) & 1 & \cdots & 1 \\ \vdots & 1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 1 & 1 & \cdots & 1 & \lambda_{m+1}(c_{m+1}) \end{pmatrix} \cdot \begin{pmatrix} \mu_1 \\ \vdots \\ \vdots \\ \mu_{m+1} \end{pmatrix} = 0,$$

and since this system has a nontrivial solution, the determinant of the matrix equals zero. Introducing $\alpha_j := \lambda_j(c_j) - 1$ and noting that all $\alpha_j \neq 0$ in the generic case (see below), the determinant vanishes if and only if

$$\frac{1}{\alpha_1} + \cdots \frac{1}{\alpha_{m+1}} + 1 = 0,$$

as can be verified by a simple induction, and then

$$\begin{pmatrix} \mu_1 \\ \vdots \\ \mu_{m+1} \end{pmatrix} \in \mathbb{C} \cdot \begin{pmatrix} 1/\alpha_1 \\ \vdots \\ 1/\alpha_{m+1} \end{pmatrix}.$$

Moreover, the differentiated identity

$$\gamma_j\left(Df(x)y\right) = D\lambda_j(x)y \cdot \gamma_j(x) + \lambda_j(x)\gamma_j(y)$$

yields

$$\gamma_j\left(Df(c_j)y\right) = \lambda_j(c_j)\gamma_j(y),$$

and therefore $\lambda_j(c_j)$ is the second eigenvalue of $Df(c_j)$ (in addition to the eigenvalue m with eigenvector c_j). In our generic scenario $\lambda_j(c_j) \neq 1$, since otherwise $\det(x,\ f(x))$ would have a multiple factor. To summarize, it is sufficient to know the idempotents and the linearizations of f at the idempotents to decide whether a polynomial first integral exists. Together with Lemma 4.1, the question of reducibility can thus be handled effectively.

(b) In particular we discuss homogeneous quadratic vector fields, in the generic scenario. We may assume that the standard basis elements $c_1,\ c_2$ are idempotents; hence

$$f(x) = \begin{pmatrix} x_1(x_1 - r_1x_2) \\ x_2(-r_2x_1 + x_2) \end{pmatrix},$$

with suitable $r_1,\ r_2$. The second eigenvalue of $Df(c_1)$ is equal to $-r_2$; thus $\lambda_1(c_1) = -r_2$ and $\alpha_1 = -(1+r_2)$. Likewise, $\alpha_2 = -(1+r_1)$. The third idempotent is

$$c_3 = (1 - r_1r_2)^{-1} \begin{pmatrix} 1 + r_1 \\ 1 + r_2 \end{pmatrix}, \quad \text{with } \alpha_3 = \frac{r_1 + r_2}{1 - r_1r_2} - 1.$$

(Note that $r_1r_2 \neq 1$ in the generic case.) By (a) and some elementary computations, there exists a polynomial first integral if and only if r_1 and r_2 are rational and positive with $r_1r_2 > 1$; see also Tsygvintsev [37]. If there is no polynomial first integral, by Lemma 4.1 a homogeneous reducing map Φ to dimension one must be linear, and moreover we may assume $g(y) = y^2$ for the reduced vector field. An idempotent is mapped either to 0 or to 1, and $\Phi(c_j) = 0$ can hold for at most one idempotent, since $\Phi \neq 0$.

Assume that $\Phi(c_1) = \Phi(c_2) = 1$; hence $\Phi(x) = x_1 + x_2$. Using Proposition 4.5(b) for c_3, we find the necessary condition $2x_1 - (r_1+r_2)x_2 = 2(x_1+x_2)$; thus $r_1+r_2 = -2$. One checks directly that this condition makes Φ a reducing map.

Essentially there is only one other case, viz. $\Phi(c_1) = 1$ and $\Phi(c_2) = 0$; thus $\Phi(x) = x_1$. A similar analysis yields the necessary and sufficient condition $r_1 = 0$.

To summarize: For a homogeneous quadratic vector field on $\mathbb{C}^2$, there is generally no reducing map to $\mathbb{C}$. A similar statement is true for homogeneous polynomial vector fields of arbitrary degree > 1 on $\mathbb{C}^n$, but the proof is more involved.

(c) Quadratic Volterra-Lotka systems are a classical topic in mathematical biology; see Hofbauer and Sigmund [23]. The homogeneous systems in this class are of interest because of their relation to replicator equations. In spite of their apparently simple form, these equations are not particularly well understood; hence, determining the reducible ones is a step towards a better understanding. We will discuss homogeneous systems. Let us first introduce some notation. For a real $n \times n$-matrix $A = (\alpha_{ij})$, the homogeneous Volterra-Lotka system associated with A is given by

$$\dot{x} = f(x) := \big(x_j \cdot (Ax)_j\big)_{1 \leq j \leq n},$$

where $(\cdot)_j$ denotes entry $\#j$. The two-dimensional systems from (b) are Volterra-Lotka systems. A special property of Volterra-Lotka systems is that every coordinate subspace

$$V_{j_1,\dots,j_k} := \big\{x\colon\ x_{j_1} = \cdots x_{j_k} = 0\big\}$$

is invariant. Determining the idempotents of Volterra-Lotka systems amounts to linear algebra: A vector $c = \big(c_j\big)_{1 \leq j \leq n}$ with all entries $c_j \neq 0$ is an idempotent if and only if $Ac = (1, \dots, 1)^T$, and by analogous reasoning one finds all the idempotents in the cordinate subspaces. Let us assume $\alpha_{11} \neq 0$ and discuss the idempotent

$$c_1 = \frac{1}{\alpha_{11}} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad Df(c_1) = \begin{pmatrix} 2 & \beta_{12} & \cdots & \cdots & \beta_{1n} \\ 0 & \beta_{21} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \beta_{n1} \end{pmatrix},$$

with $\beta_{ij} = \alpha_{ij}/\alpha_{11}$. In the generic case that the eigenvalues $2, \beta_{21}, \ldots, \beta_{n1}$ are pairwise distinct, there is an essentially unique eigenbasis

$$\psi_1 := x_1 + \frac{\beta_{12}}{2-\beta_{21}} x_2 + \cdots \frac{\beta_{1n}}{2-\beta_{n1}} x_n,\ \psi_2 := x_2, \ldots, \psi_n := x_n,$$

for the action of $T := Df(c_1)$ on S_1, and for $\phi = \psi_1^{d_1} \ldots \psi_n^{d_n}$ (with integers $d_1, \ldots, d_n \geq 0$), one has

$$D\phi(x)Tx = (2d_1 + \beta_{21}d_2 + \cdots + \beta_{n1}d_n)\,\phi(x).$$

Thus (4.5) and (4.6) are readily applicable, and similarly this holds for the other idempotents. Thus one obtains restrictions that make computations feasible.

Let us discuss a particular example. Given a rational number $r > 1$, let

$$A = \begin{pmatrix} 1 & -r & -r \\ -r & 1 & -r \\ -r & -r & 1 \end{pmatrix}.$$

According to (b), the restriction to every two-dimensional coordinate subspace admits a polynomial first integral. Consider the idempotent

$$c = -\frac{1}{2r-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

with eigenvalues 2 and $\frac{r-2}{2r-1}$ (double) for $Df(c)$. Now assume that $r > 2$. Then

$$2d_1 + \frac{r-2}{2r-1}(d_2 + d_3) > 0,$$

for every nontrivial triple of nonnegative integers. By (4.6) there exists no polynomial first integral, and reducibility to the one-dimensional equation $\dot{y} = y^2$ by a linear form is the only remaining possibility. Considering c_1, one easily shows with (4.5) that no such map exists for $r > 2$. On the other hand, for $r = 2$ the system admits the first integral

$$x_1x_2(x_1 - x_2) + x_2x_3(x_2 - x_3) + x_3x_1(x_3 - x_1),$$

which can be obtained by first considering the restrictions to the two-dimensional coordinate subspaces. This particular example shows that (4.5) and (4.6) are useful for finding reduction maps as well as precluding their existence in a systematic manner.

Deciding about reducibility of a general nonlinear vector field by a linear map is a related problem. Here a precise characterization is possible.

Proposition 4.8. *Let f be a vector field on $U \subseteq \mathbb{K}^n$. A surjective linear map $T\colon \mathbb{K}^n \to \mathbb{K}^m$ (with $m < n$) is a reduction map for $\dot{x} = f(x)$ only if*

$$Df(x)(\mathrm{Ker}(T)) \subseteq \mathrm{Ker}(T),$$

for all $x \in U$. If U is star-shaped, or if f is analytic and U is connected, then the condition is also sufficient for reducibility of f on U. Hence there is a correspondence between all surjective linear reducing maps for f and joint invariant subspaces of all the $Df(x)$, with $x \in U$.

Proof. The reducibility condition here is $Tf(x) = g(Tx)$ for some g. In particular, for any $z \in \mathrm{Ker}(T)$ and all x in U, this implies $Tf(x+z) = Tf(x)$. Conversely, if this identity holds for all x and all $z \in \mathrm{Ker}(T)$, then g is well-defined by $g(Tx) = Tf(x)$. Now the necessity of the asserted condition, as well as local sufficiency, follows by differentiation. The remaining statements are proved by standard arguments. □

Example 4.9. Consider a homogeneous quadratic Volterra-Lotka system, with the notation as in Example 4.7(c), and assume (to simplify notation) that all $\alpha_{jj} = 1$. Then the first standard basis vector c_1 is an idempotent, and a basis of eigenvectors for $Df(c_1)$ is given by

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} \alpha_{12} \\ \alpha_{21} - 2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, v_n = \begin{pmatrix} \alpha_{1n} \\ 0 \\ \vdots \\ 0 \\ \alpha_{n1} - 2 \end{pmatrix},$$

provided that no $\alpha_{j1} = 2$. If furthermore all the α_{j1} are pairwise distinct, then there are only $2^n - 2$ possible (kernels of) linear reduction maps. Let us consider dimension $n = 3$ in detail. Here

$$Df(x) = \begin{pmatrix} 2x_1 + \alpha_{12}x_2 + \alpha_{13}x_3 & \alpha_{12}x_1 & \alpha_{13}x_1 \\ \alpha_{21}x_2 & \alpha_{21}x_1 + 2x_2 + \alpha_{23}x_3 & \alpha_{23}x_2 \\ \alpha_{31}x_3 & \alpha_{32}x_3 & \alpha_{31}x_1 + \alpha_{32}x_2 + 2x_3 \end{pmatrix},$$

and we will consider the case that the kernel is one-dimensional (for the other case, see Example 4.7).

The condition $Df(x)v_1 \in \mathbb{R} \cdot v_1$ is satisfied for all x if and only if $\alpha_{21} = \alpha_{31} = 0$, and $x \mapsto (x_2,\ x_3)^T$ defines a corresponding reduction map.

The condition $Df(x)v_2 \in \mathbb{R} \cdot v_2$, all x, needs more work. With

$$Df(x)v_2 = \begin{pmatrix} \alpha_{12}\alpha_{21}x_1 + \alpha_{12}^2 x_2 + \alpha_{12}\alpha_{13}x_3 \\ (\alpha_{21} - 2)\alpha_{21}x_1 + (2(\alpha_{21} - 2) + \alpha_{21}\alpha_{12})\, x_2 + (\alpha_{21} - 2)\alpha_{23}x_3 \\ (\alpha_{31}\alpha_{12} + \alpha_{32}(\alpha_{21} - 2))\, x_3 \end{pmatrix},$$

one obtains the following set of necessary and sufficient conditions:

$$\begin{aligned} 2\alpha_{21}(\alpha_{12}+\alpha_{21}-2) &= 0, \\ (2-\alpha_{21})\alpha_{12}(\alpha_{13}-\alpha_{23}) &= 0, \\ \alpha_{31}\alpha_{12}+\alpha_{32}(\alpha_{21}-2) &= 0. \end{aligned}$$

By assumption $\alpha_{21} \neq 2$; hence, $\alpha_{12} = 0$ or $\alpha_{13} = \alpha_{23}$ from the second condition. A simple further analysis yields three subcases of necessary and sufficient conditions:

(i) $\alpha_{12} = \alpha_{21} = \alpha_{32} = 0$.
(ii) $\alpha_{13} = \alpha_{23}$ and $\alpha_{21} = 2\alpha_{32} - \alpha_{31}\alpha_{12} = 0$.
(iii) $\alpha_{13} = \alpha_{23}$ and $\alpha_{12} + \alpha_{21} - 2 = \alpha_{31} - \alpha_{32} = 0$.

The existence of a linear reducing map with kernel $\mathbb{R} \cdot v_3$ can be discussed in a similar manner. The point of this example is to show that Proposition 4.8 makes a systematic approach possible.

Next, we consider the extension of Lemma 4.1 to nonhomogeneous polynomial and local analytic vector fields. Some additional conditions are necessary here. For the proof, compare terms of highest, resp. lowest, order.

Lemma 4.10.

(a) *Let $f = f^{(0)} + \cdots + f^{(r)}$ be a polynomial vector field in $\mathbb{K}^n$, with $f^{(i)}$ denoting the homogeneous part of degree i and $f^{(r)} \neq 0$, likewise $g = g^{(0)} + \cdots + g^{(s)}$ a polynomial vector field in $\mathbb{K}^m$, with $g^{(s)} \neq 0$, and $\Phi = \Phi^{(0)} + \cdots + \Phi^{(q)}$ a polynomial map from $\mathbb{K}^n$ to $\mathbb{K}^m$, with $q > 0$ and $D\Phi^{(q)}(z)$ of rank m for some z.*
Then $r - 1 \geq (s-1)q$. If $f^{(r)}$ admits no polynomial first integral, then equality holds.

(b) *Let $f = f^{(r)} + \cdots$ be a formal or analytic vector field near 0 in $\mathbb{K}^n$, with $f^{(i)}$ denoting the homogeneous part of degree i and $f^{(r)} \neq 0$; likewise $g = g^{(s)} + \cdots$ a formal or analytic vector field near 0 in $\mathbb{K}^m$, with $g^{(s)} \neq 0$, and $\Phi = \Phi^{(q)} + \cdots$ from $\mathbb{K}^n$ to $\mathbb{K}^m$, with $q > 0$ and $D\Phi^{(q)}(z)$ of rank m for some z.*
Then $r - 1 \leq (s-1)q$. If $f^{(r)}$ admits no polynomial first integral, then equality holds.

We emphasize that, due to Proposition 4.5 and Corollary 4.6, there are efficient means to verify the hypotheses of this Lemma. The condition on the derivative of $\Phi^{(q)}$ guarantees that $g^{(s)}(\Phi^{(q)}(x))$ is not identically zero. It is automatically satisfied as soon as $\Phi^{(q)} \neq 0$ in the case $m = 1$, and only then. There are examples (for instance certain symmetric systems) for reductions where such a condition is not satisfied, and no degree comparison is possible.

With regard to the general investigation of local reducibility, Poincaré-Dulac normal forms (see, e.g., Bruno [8], and [38]) provide a valuable tool for stationary points that are not too degenerate. Recall that a local analytic or formal vector field $f = B + \sum_{j \geq 2} f^{(j)}$, with linear B and each $f^{(j)}$ homogeneous of degree j, is in (Poincaré-Dulac) normal form if $\left[B_s,\ f^{(j)}\right] = 0$ for all j, where B_s denotes the semisimple part of B. It is known that any local analytic vector field can be transformed by a formal power series to some formal vector field in normal form, but a convergent transformation does not necessarily exist. The bracket condition $[B_s,\ f] = 0$ is actually a symmetry condition, and thus one can construct a reducing map for a system in normal form from invariants of B_s; see [38] for details. We will call such a reducing map a symmetry reduction of the equation in normal form.

Theorem 4.11. *Let the local analytic or formal vector field*

$$f(x) = Bx + \sum_{j\geq 2} f^{(j)}(x)$$

be in normal form. Let $\Phi = \Phi^{(q)} + \cdots$, $q > 0$, *be a reducing map to* $\dot{y} = g(y)$ *on* $\mathbb{K}$, *with* $g(y) = \alpha y + \cdots$.

- (i) *If* $\alpha \neq 0$, *then there exists a reduction to* $g(y) = \alpha y$; *in this case all* $\Phi^{(k)}$ *are in the* α*-eigenspace of the Lie derivative operator* L_{B_s}.
- (ii) *If* $\alpha = 0$, *then all* $\Phi^{(k)}$ *are annihilated by* L_{B_s}. *Thus* Φ *factors through any symmetry reduction of the system.*

Proof. In the case $\alpha \neq 0$, one may normalize $g(y)$ to αy by an invertible transformation near 0 in $\mathbb{K}$ (convergent if the series of g converges). Thus we may assume $g(y) = \alpha y$. We will show $L_{B_s}(\Phi^{(j)}) = \alpha\Phi^{(j)}$ by induction on j. For the case $j = q$, this follows from comparing lowest order terms. Comparing terms of degree $j > q$, one gets

$$L_B(\Phi^{(j)}) + L_{f^{(2)}}(\Phi^{(j-1)}) + \cdots L_{f^{(j+1-q)}}(\Phi^{(q)}) = \alpha\Phi^{(j)}.$$

Using the induction hypothesis together with the commuting property of L_{B_s} and the $L_{f^{(k)}}$, one sees that all the $L_{f^{(k)}}(\Phi^{(j-k+1)})$ are in the α-eigenspace of L_{B_s}. Therefore, the same holds for $L_B(\Phi^{(j)}) - \alpha\Phi^{(j)}$, and the semisimplicity of L_{B_s} shows that $\Phi^{(j)}$ itself is in this eigenspace.

In case $\alpha = 0$ we have $g = g^{(2)} + \cdots$; thus comparing terms of degree q yields $L_B(\Phi^{(q)}) = 0$ and hence $L_{B_s}(\Phi^{(q)}) = 0$. For degree $j > q$, we obtain

$$L_B(\Phi^{(j)}) + L_{f^{(2)}}(\Phi^{(j-1)}) + \cdots + L_{f^{(j+1-q)}}(\Phi^{(q)}) = \sigma_j(\Phi^{(q)}, \ldots, \Phi^{(j-1)}),$$

with certain polynomials σ_j. The σ_j do not depend on $\Phi^{(j)}$ since g contains no linear term. One has

$$L_{B_s}\left(\sigma_j(\Phi^{(q)}, \ldots, \Phi^{(j-1)})\right) = 0,$$

from standard differentiation rules and the induction hypothesis. Since L_{B_s} and the $L_{f^{(k)}}$ commute,

$$L_{B_s}L_{f^{(k)}}(\Phi^{(j+1-k)}) = 0 \quad \text{for} \quad k \geq 2,$$

whence

$$L_{B_s}L_B(\Phi^{(j)}) = 0 \quad \text{and} \quad L_{B_s}(\Phi^{(j)}) = 0.$$ □

We will refer to these possible reductions as type (i), resp. type (ii). For type (i) reductions, one also has

$$L_{B_n}(\Phi^{(j)}) + L_{f^{(2)}}(\Phi^{(j-1)}) + \cdots L_{f^{(j+1-q)}}(\Phi^{(q)}) = 0,$$

for all j, with the nilpotent part B_n of B. This is a very restrictive condition whenever the polynomial invariant algebra of B_s is nontrivial, and one should expect a type (i) reduction only in very special cases. If the invariant algebra cannot be generated by one element, then intuitively one would not expect reductions of type (ii), since the equation produced in the symmetry reduction by invariants has a degenerate stationary point (with nilpotent linearization).

Example 4.12. $f(x) = Bx + \cdots$ in $\mathbb{R}^4$, with

$$B = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Equations in normal form of this type are well understood: see e.g. Anosov et al. [2]. The eigenvalues of B are $i, -i, i\sqrt{2}, -i\sqrt{2}$, and the invariant algebra of B is generated by $\psi_1(x) = x_1^2 + x_2^2$ and $\psi_2(x) = x_3^2 + 2x_4^2$. The map $\Psi = (\psi_1, \psi_2)$ reduces this equation to an equation of type

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = h(y) = \begin{pmatrix} y_1(\gamma_{11}y_1 + \gamma_{12}y_2) + \cdots \\ y_2(\gamma_{21}y_1 + \gamma_{22}y_2) + \cdots \end{pmatrix},$$

and for suitable (generic) choice of f the quadratic part of h is not reducible to $\mathbb{R}$; see Example 4.7. Thus, there can be no type (ii) reduction to dimension one for $\dot{x} = f(x)$. But since the only real eigenvalue of L_B is zero, there is no type (i) reduction to $\mathbb{R}$ by default.

Another consequence of (4.11) is the existence of local obstructions to complete reducibility if a normal form contains nonlinear terms. If the eigenvalues of the linear part force the formal normal form to be linear, and the necessary condition for complete reducibility from (2.7) is satisfied, then it is known (see [9], [11]) that there exists a convergent transformation to normal form, and the latter is completely reduced. But if nontrivial terms are admissible in the normal form, a generic vector field is not even formally completely reducible. A simple example in dimension two is given by a non-diagonalizable linear map B with a nonzero double eigenvalue α. Here Theorem 4.11 implies that there is essentially just one reducing map to $\mathbb{K}$, which corresponds to the α-eigenspace of L_B on S_1. The next result is another application of Theorem 4.11.

Proposition 4.13. *Let p and q be relatively prime positive integers, and let $B = \mathrm{diag}(q,\ -p)$. The vector field in normal form*

$$f(x) = Bx + \sum_{i\geq 1} \psi(x)^i\,(\alpha_i x + \beta_i Bx)\,,$$

with $\psi(x) = x_1^p x_2^q$, *and* $\alpha_r \neq 0$ *for some* $r \geq 1$ *but* $\alpha_i = 0$ *for all* $i < r$, *admits a type (i) reduction if and only if there are nonnegative integers* u, v *such that* $qu - pv \neq 0$ *and*

$$\alpha_i(u+v) + \beta_i(qu - pv) = 0 \quad \text{for} \quad i = 1, \ldots, r.$$

In particular, there is no type (i) reduction if some $\beta_i \neq 0$ *for* $1 \leq i \leq r-1$, *or if* β_r/α_r *is not a rational number. Moreover, in this case* f *is not completely reducible.*

Proof. The polynomial invariant algebra of $B = B_s$ is generated by ψ. If there is a type (i) reduction map, then it necessarily has the form $\phi(x) = x_1^u x_2^v \cdot \rho(\psi(x))$, with ρ a nonzero power series in one variable, and u, v nonnegative integers such that $qu - pv \neq 0$, with $L_B(\phi) = (qu - pv)\phi$. This follows from Theorem 4.11 and the elementary result that $L_B(\sigma) = (qu - pv)\sigma$ if and only if σ has the above form. But then

$$\begin{aligned} D\phi(x)f(x) \;=\; & (qu - pv + \vartheta(\psi))\, x_1^u x_2^v \rho(\psi) \\ & + x_1^u x_2^v \rho'(\psi) \cdot L_f(\psi), \end{aligned}$$

with

$$\vartheta(\psi) := \sum_{i \geq 1} (\alpha_i(u+v) + \beta_i(qu - pv))\, \psi^i \quad \text{and} \quad L_f(\psi) = \sum_{i \geq 1} (p+q)\alpha_i \psi^{i+1}.$$

Therefore $D\phi(x)f(x)$ is equal to $(qu - pv)\phi(x)$ (see Proposition 4.8, and note that necessarily $\alpha = qu - pv$) only if

$$\vartheta(\psi)\rho(\psi) + L_f(\psi)\rho'(\psi) = 0,$$

and one finds that the order of $\vartheta(\psi)$ cannot be smaller than the order of $L_f(\psi)$. Evaluating this condition yields the necessity of the stated conditions. Moreover, they are also sufficient, since then ρ can be obtained as a solution of a linear analytic differential equation.

If there is no type (i) reduction, then the system cannot be completely reducible, since every symmetry reduction map is an analytic function of ψ, and thus there exists no completely reducing map with a derivative that is invertible at some point near 0. □

Finally, we present an example where global nonreducibility can be deduced from local properties.

Example 4.14. The equation (see [42], Example 2.6)

$$\dot{x} = f(x) = \begin{pmatrix} x_1 + 3x_2 + x_1^2 \\ 3x_1 + x_2 + \frac{9}{2}x_1^2 + \frac{3}{2}x_1 x_2 \end{pmatrix}$$

on $\mathbb{R}^2$ is not reducible to $\mathbb{R}$ by an analytic map.

Assume that there is a reducing map Φ, and $\Phi(0) = 0$ with no loss of generality. The equation has, by construction, the invariant set $\gamma(x) = 0$, with $\gamma(x) = x_1^2 + x_1^3 - x_2^2$. In particular this set contains a homoclinic loop Δ connecting the stationary point 0 to itself, and just as in (1.3) one sees that Δ must be mapped to 0. Thus, near every point

$z \neq 0$ of the loop there is a neighborhood $U = U(z)$, a positive integer d_z, and an analytic function σ_z that does not vanish identically on the zero set of γ in U, such that

$$\Phi = \gamma^{d_z} \cdot \sigma_z.$$

Both d_z and σ_z are uniquely determined by these properties. Now let V be an open neighborhood of 0. Then $\widetilde{\Delta} := \Delta \setminus V$ is compact, and one sees from the above and the identity theorem for analytic functions, that there is a neighborhood $\widetilde{U}$ of $\widetilde{\Delta}$ such that

$$\Phi(x) = \gamma(x)^d \cdot \sigma(x) \text{ on } \widetilde{U},$$

where d and the analytic function σ are uniquely determined by the property that σ has at most isolated zeros on $\widetilde{\Delta}$.

On the other hand, the eigenvalues of $Df(0)$ are equal to 4 and -2, and near 0 one has a decomposition $\gamma(x) = \gamma_1(x)\gamma_2(x)$, with $\gamma_j(x) = x_2 \pm x_1\sqrt{1+x_1}$. Computation of the normal form (up to degree 4, in this case) and Proposition 4.13, with $p = 1$ and $q = 2$, show that there is no type (i) reduction. Therefore, near 0 one must have $\Phi(x) = (\gamma_1(x)\gamma_2(x)^2)^e\tilde{\sigma}(x)$, for some positive integer e. This yields $e = d = 2e$, and we have a contradiction.

5. Conclusion

Section 4 indicates in which way our results are applicable: The constraints on possible reducing maps in the local scenario (or in a scenario with a priori restrictions on the admissible functions) are sufficiently restrictive to be useful for computations. Our focus on (homogeneous) polynomials is justified by the observation that these occur as initial terms in Taylor expansions. The question of reducibility of the first few terms in such an expansion (up to higher-order remainders) becomes manageable with the criteria presented above. As mentioned earlier, the criteria will frequently be too strong to admit any reducing map; but in parameter-dependent systems one may thus determine critical parameter values. A possible area of application is related to the search for nonlinear symmetries of a given system. To see that the class of systems admitting such symmetries is much larger than the class admitting linear symmetry groups, see for instance Christopher et al. [10] on two-dimensional Darboux-integrable systems. There is no systematic approach to detecting symmetries in this scenario (see Olver [29], Section 2.5), and the problem of determining a reducing map seems somewhat simpler than the problem of determining symmetries. The examples which we discussed most extensively are connected to Volterra-Lotka equations. As mentioned above, these equations occur in various areas of mathematical biology. Moreover, Volterra-Lotka systems also occur as lowest-order terms in the reduced system for nonresonant coupled oscillators. Very little is known about general properties of Volterra-Lotka systems. In such scenarios, it seems reasonable to have at least a few concepts and tools available for a case-by-case investigation. Reducibility is such a concept and provides such tools.

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