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Noncommutative Burgers Equation

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Abstract

We present a noncommutative version of the Burgers equation which possesses the Lax representation and discuss the integrability in detail. We find a noncommutative version of the Cole-Hopf transformation and succeed in the linearization of it. The linearized equation is the (noncommutative) diffusion equation and exactly solved. We also discuss the properties of some exact solutions. The result shows that the noncommutative Burgers equation is completely integrable even though it contains infinite number of time derivatives. Furthermore, we derive the noncommutative Burgers equation from the noncommutative (anti-)self-dual Yang-Mills equation by reduction, which is an evidence for the noncommutative Ward conjecture. Finally, we present a noncommutative version of the Burgers hierarchy by both the Lax-pair generating technique and the Sato's approach.

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1 Introduction

The extension of ordinary integrable systems to noncommutative (NC) spaces has been studied intensively for the last several years [1]-[24]. In the recent developments of NC field theories, various new physical aspects of gauge theories were revealed [25] and several long-standing problems in real physics were solved.

NC field theories can be described as deformed theories from commutative ones. In terms of gauge theories, the deformation is essentially unique because it corresponds to the presence of the background magnetic fields [25]. Among them, NC (anti-)self-dual Yang-Mills equations are integrable and important [26]. The first breakthrough was the great work [27] of Atiyah-Drinfeld-Hitchin-Manin construction [28] of NC $U(1)$ instantons.

On the other hand, in the lower-dimensional theories, there are many typical integrable equations such as the Korteweg-de Vries (KdV) equation [29]. These equations contain no gauge field and the NC extension of them perhaps might have no physical picture. Furthermore, the NC extension of (1+1)-dimensional equations introduces infinite number of time derivatives and it becomes very hard to define the integrability.

Nevertheless, NC versions of them have been proposed in various contexts. They actually possess some integrable properties, such as the existence of infinite number of conserved quantities [1, 13]. Furthermore, a few of them can be derived from NC (anti-)self-dual Yang-Mills equations by suitable reductions [7]. This fact may give some physical meanings to the lower-dimensional NC field equations. Now it is time to investigate various aspects of them more in detail in order to confirm whether the NC field equations presented are really integrable or not.

For this purpose, the Burgers equation [30] would be the best example. On the commutative space-time, it can be derived from the Navier-Stokes equation and describes real phenomena, such as the turbulence and shock waves. In this point, the Burgers equation draws much attention amongst many integrable equations. Furthermore, it can be linearized by the Cole-Hopf transformation [31]. The linearized equation is the diffusion equation and can be solved by Fourier transformation for given boundary conditions. This shows that the Burgers equation is completely integrable. The Burgers equation actually sat in the central position at the early stage of integrable systems and have given much influence on the subsequent studies. For example, the Hirota's bilinear transformation [32], which is a simple generalization of the Cole-Hopf transformation, plays a crucial role in the construction of the exact multi-soliton solution of various soliton equations. Therefore if the NC Burgers equation is linearizable and integrable even on the NC space-time, it would be the first example of completely integrable NC equations and has much significance for further studies on the topics.

In this article, we present NC versions of the Burgers equation and the Burgers hierarchy which possess the Lax representations. We prove that the NC Burgers equation is linearizable by a NC version of the Cole-Hopf transformation. This shows that the NC Burgers equation is really integrable even though the NC Burgers equation contains infinite number of time derivatives in the nonlinear term. The linearized equation is the (NC) diffusion equation of first order with respect to time and the initial value problem is well-defined. The NC Lax representation is derived from the compatibility condition of NC versions of linear systems. Hence the integrability of the NC Burgers equation with the Lax representation could relate to some symmetry and the existence of the NC Burgers hierarchy might suggest a hidden infinite-dimensional symmetry which is considered as a deformed symmetry from commutative one. We also obtain the exact solutions which actually reflect the effects of the NC deformation. Finally we derive the NC Burgers equation from both NC (anti-)self-dual Yang-Mills equation and the framework of NC extension of Sato theory. This would be the first step to the confirmation of NC Ward conjecture and the completion of NC Sato theory. We also discuss general properties of integrability in NC field theories mainly in section 3 and 6.

Note added: After submitting the present article and our paper [20], we were aware of the paper [33] by L. Martina and O. K. Pashaev on arXiv e-print server, which contains some overlaps with ours.

2 Review of Noncommutative Field Theories

NC spaces are defined by the noncommutativity of the coordinates:

$$[x^i, x^j] = i\theta^{ij}, \tag{2.1}$$

where θ^{ij} are real constants and called the *NC parameters*. This relation looks like the canonical commutation relation in quantum mechanics and leads to "space-space uncertainty relation." Hence the singularity which exists on commutative spaces could resolve on NC spaces. This is one of the prominent features of NC field theories and yields various new physical objects.

NC field theories have the following two equivalent descriptions:

- the star-product formalism
- the operator formalism

These are connected one-to-one by the Weyl transformation, which is briefly summarized in Appendix A. In the present article, we mainly use the star-product formalism.

The star-product formalism

The star-product is defined for ordinary fields on commutative spaces. For Euclidean spaces, it is explicitly given by

$$\begin{aligned} f \star g(x) &:= \exp\left(\frac{i}{2}\theta^{ij}\partial_i^{(x')}\partial_j^{(x'')}\right) f(x')g(x'')\Big|_{x'=x''=x} \\ &= f(x)g(x) + \frac{i}{2}\theta^{ij}\partial_i f(x)\partial_j g(x) + \mathcal{O}(\theta^2), \end{aligned} \tag{2.2}$$

where $\partial_i^{(x')} := \partial/\partial x'^i$ and so on. This explicit representation is known as the *Moyal product* [34].

The star-product has associativity: $f \star (g \star h) = (f \star g) \star h$, and returns back to the ordinary product in the commutative limit: $\theta^{ij} \to 0$. The modification of the product makes the ordinary spatial coordinate "noncommutative," that is, $[x^i, x^j]_\star := x^i \star x^j - x^j \star x^i = i\theta^{ij}$.

NC field theories are given by the exchange of ordinary products in the commutative field theories for the star-products and realized as deformed theories from the commutative ones. In this context, they are often called the *NC-deformed theories*.

We note that the fields themselves take c-number values as usual and the differentiation and the integration for them are well-defined as usual. A nontrivial point is that NC field equations contain infinite number of derivatives in general. Hence the integrability of the equations are not so trivial as commutative cases.

The operator formalism

In order to make some comments on the integrability of NC field equations later, let us introduce another formalism, the *operator formalism*, which is equivalent to the star-product formalism.

This time, we start with the noncommutativity of the spatial coordinates (2.1) and define NC gauge theory considering the coordinates as operators. From now on, we write the hats on the fields in order to emphasize that they are operators. For simplicity, we treat NC plane with the coordinates $\hat{x}^1, \hat{x}^2$ which satisfy $[\hat{x}^1, \hat{x}^2] = i\theta,\ \theta > 0$.

Defining new variables $\hat{a}, \hat{a}^\dagger$ as

$$\hat{a} := \frac{1}{\sqrt{2\theta}}\hat{z},\ \hat{a}^\dagger := \frac{1}{\sqrt{2\theta}}\hat{\bar{z}}, \tag{2.3}$$

where $\hat{z} = \hat{x}^1 + i\hat{x}^2,\ \hat{\bar{z}} = \hat{x}^1 - i\hat{x}^2$, we get the Heisenberg's commutation relation:

$$[\hat{a}, \hat{a}^\dagger] = 1. \tag{2.4}$$

Hence the spatial coordinates can be considered as the operators acting on Fock space $\mathcal{H}$ which is spanned by the occupation number basis $|n\rangle := \left\{(\hat{a}^\dagger)^n/\sqrt{n!}\right\}|0\rangle,\ \hat{a}|0\rangle = 0$:

$$\mathcal{H} = \oplus_{n=0}^{\infty}\mathbf{C}|n\rangle. \tag{2.5}$$

Fields on the space depend on the spatial coordinates and are also the operators acting on the Fock space $\mathcal{H}$. They are represented by the occupation number basis as

$$\hat{f} = \sum_{m,n=0}^{\infty} f_{mn}|m\rangle\langle n|. \tag{2.6}$$

If the fields have rotational symmetry on the plane, that is, the fields commute with the number operator $\hat{n} := \hat{a}^\dagger\hat{a} \sim (\hat{x}^1)^2 + (\hat{x}^2)^2$, they become diagonal:

$$\hat{f} = \sum_{n=0}^{\infty} f_n|n\rangle\langle n|. \tag{2.7}$$

The derivation is defined as follows:

$$\partial_i\hat{f} := [\hat{\partial}_i, \hat{f}] := [-i(\theta^{-1})_{ij}\hat{x}^j, \hat{f}], \tag{2.8}$$

which satisfies the Leibniz rule and the desired relation:

$$\partial_i\hat{x}^j = [-i(\theta^{-1})_{ik}\hat{x}^k, \hat{x}^j] = \delta_i^{\ j}. \tag{2.9}$$

The operator $\hat{\partial}_i$ is called the *derivative operator*. Hence we can define "differential equations" which are realized as recursion relations for the matrix element f_{mn}.

The integration can also be defined as the trace of the Fock space $\mathcal{H}$:

$$\int dx^1dx^2\ \hat{f}(\hat{x}^1,\hat{x}^2) \ := \ 2\pi\theta\mathrm{Tr}_{\mathcal{H}}\hat{f}. \tag{2.10}$$

Hence the solutions for "differential equations" are also well-defined.

3 Comments on Integrability of Noncommutative Field Equations

Before NC extension of the Burgers equation, let us discuss what is the definition of integrability of NC field equations.

Even on commutative spaces, it is hard to define what is integrability of field equations. (See e.g. [35].) There are various definitions for it according to situations. Typical definitions are as follows. Field equations are called integrable if they possess, for example, the linearizability, the Lax representation, the existence of infinite number of conserved quantities, the bi-Hamiltonian structure, the exact multi-soliton solutions, the Painlevé properties, the presence of algebraic geometry, and so on. (See e.g. [36, 37]) Among them, the linearizability is the best definition because the linearized equation can be solved by Fourier transformation for arbitrary given initial conditions.

On NC spaces, it becomes harder to define what is integrability of field equations. In this case, there are two situations which should be discussed separately:

- space-space noncommutativity
- space-time noncommutativity

In the former, the situation is just the same as the ordinary commutative case because NC field theories can be considered as just deformed theories. The fields are c-number valued functions (or infinite-side matrices in the operator formalism) and the derivation and the integration is well-defined as usual. The notions of time evolutions, Hamiltonian structures, action-angle variables and inverse scattering methods are also well-defined. Hence the above definitions for commutative field equations are also reasonable for those NC equations with space-space noncommutativities.

In the latter, however, the situation changes drastically. The obstruction arises in the notion of time evolution in nonlinear equations. For simplicity, let us consider the $(1+1)$-dimensional NC space-time whose coordinates and noncommutativity are (x,t) and $[t,x]=i\theta$, respectively. The noncommutativity introduces infinite number of time derivatives in nonlinear terms as

$$f\star g(t,x) = e^{\frac{i}{2}\theta(\partial_{t'}\partial_{x''}-\partial_{x'}\partial_{t''})} f(t',x')g(t'',x'')\Big|_{\substack{t'=t''=t\\ x'=x''=x,}} \tag{3.1}$$

where $\partial_t = \partial/\partial t$ and so on. Hence the notion of time evolution becomes vague and the infinite number of derivatives of time might lead to acausal structure into the theories. The initial value problem also seems to be hard to define. Therefore it becomes seriously hard to discuss the integrability. In this case, only one possible definition of integrability is the linearizability because linearized equations contain neither nonlinear term nor the star-product.

The Burgers equation is defined on $(1+1)$-dimensional space-time and the NC extension belongs to the latter case. Hence it is worth studying whether it is linearizable and how the initial value problem is solved.

4 Noncommutative Burgers Equation

In order to do it, we first present some NC version of the Burgers equation on $(1+1)$-dimensional noncommutative space-time with the noncommutativity: $[t,x]=i\theta$. In this section, we construct a NC version of the Burgers equation which possesses the Lax representation.

A given NC differential equation is said to have the Lax representation if there exists a suitable pair of operators (L, T) so that the following equation (the *Lax equation*)

$$[L, T + \partial_t]_\star = 0 \tag{4.1}$$

is equivalent to the given NC differential equation. Here the star-product does not affect the derivative operator, for example, $\partial_t \star \partial_x = \partial_t \partial_x$. The pair of operators (L, T) and the equation (4.1) are called the *Lax pair* and the *NC Lax equation*, respectively.

The NC Lax equation (4.1) is derived from the compatible condition of the following NC version of the linear system

$$L \star \psi = \lambda \psi, \tag{4.2}$$

$$\frac{\partial \psi}{\partial t} + T \star \psi = 0, \tag{4.3}$$

where the eigenvalue λ is a constant. On commutative spaces, Eq. (4.3) is an evolution equation and the existence of the Lax representation (4.1) suggests the compatibility of the linear systems. On NC spaces, however, the RHS of the equations (4.3) could contain infinite number of derivatives of some variables and the geometrical meaning might be vague. Therefore at this stage, the integrability of the Lax equation is not trivial. In the next section, we will see a NC Burgers equation is actually linearizable, which suggests the NC deformation would have good properties. Furthermore, we would like to comment that the NC (anti-)self-dual Yang-Mills equation is integrable and derived from the compatibility of linear systems with spectral parameters. (e.g. [40])

Now let us construct the NC Burgers equation with the Lax representation by the *Lax-pair generating technique* [38]. The technique is a method to find a corresponding T-operator for a given L-operator and based on the following ansatz for the T-operator

$$T = \partial_i^n L^m + T'. \tag{4.4}$$

Then the problem reduces to that for the T'-operator and becomes enough easy to solve in many cases including NC cases [16, 20].

Let us apply this technique to the NC extension of the Burgers equation. The L-operator of the Burgers equation is given by

$$L_{\text{Burgers}} = \partial_x + u(t, x). \tag{4.5}$$

The ansatz for the T-operator is now taken as

$$T_{\text{Burgers}} = \partial_x L_{\text{Burgers}} + T'_{\text{Burgers}}, \tag{4.6}$$

which is the case for $n = 1$ in the general ansatz (4.4). The ansatz for $n = 2, 3, \dots$ gives rise to the Burgers hierarchy, which is discussed in section 4.

Then the NC Lax equation becomes

$$[\partial_x + u, T'_{\text{Burgers}}]_\star = u_x \partial_x + u_t + u_x \star u, \tag{4.7}$$

where $u_x := \partial u/\partial x$ and so on. Here the term $u_x \partial_x$ in the RHS of Eq. (4.7) is troublesome because the Lax equation should be a differential equation without bare derivatives ∂_i. Hence we have to delete this term to find an appropriate T'-operator so that the bare derivative term in the LHS of Eq. (4.7) should be canceled out.In order to do this, we can take the T'-operator as the following form:

$$T'_{\text{Burgers}} = A\partial_x + B, \tag{4.8}$$

where A and B are polynomials of u, u_x, u_t, etc. Then the Lax equation becomes $f\partial_x + g = 0$ and the condition $f = 0$ determines some part of A, B and finally a differential equation $g = 0$ is expected to be the Burgers equation which possesses the Lax representation.

The condition $f = 0$ is

$$A_x + [u, A]_\star = u_x. \tag{4.9}$$

The solution is $A = u$.[3] Then the differential equation $g = 0$ becomes

$$B_x + [u, B]_\star = u_t + u \star u_x + u_x \star u. \tag{4.10}$$

Taking the dimensions of the variables into account, that is, $[x] = -1, [u] = 1, [t] = -2$, hence, $[B] = 2$, we can take the unknown B as[4]

$$B = au_x + bu^2, \tag{4.11}$$

where a and b are constants.[5]

Finally we get the NC version of the Burgers equation with parameters:

$$u_t - au_{xx} + (1 + a - b)u_x \star u + (1 - a - b)u \star u_x = 0, \tag{4.12}$$

whose Lax pair is

$$\begin{cases} L_{\text{Burgers}} &= \partial_x + u, \\ T_{\text{Burgers}} &= \partial_x^2 + 2u\partial_x + (a + 1)u_x + bu^2. \end{cases} \tag{4.13}$$

[3]Exactly speaking, the general solution is $A = u + \alpha$ where α is a constant. However this constant can be absorbed by the scale transformation $u \to u + \alpha/(2 - 2\beta)$. In this article, we omit such constants.

[4]Here we don't consider fractional terms such as $u_{xx} \star u^{-1}$ and so on. This constraint corresponds to $B_2 := (L^2)_{\star\geq 0}$ in the framework of Sato theory. (cf. section 9.)

[5]We note that $u \star u \equiv u^2$.

In the commutative limit, it reduces to

$$u_t - au_{xx} + 2(1-b)uu_x = 0. \tag{4.14}$$

We note that the nonlinear term uu_x in the Burgers equation should be extended not as symmetric forms but as Eq. (4.12) so that the NC Burgers equation should possess the Lax representation.

This parameter family is a general form with Lax representation. Of course, some parameters can be absorbed by a scale transformation.

However, on NC spaces, it is not clear whether the Lax representation has good properties or not in the integrable sense. In the next section, let us seek for the condition on the constants that the Burgers equations should be linearizable.

5 Noncommutative Cole-Hopf Transformation

In commutative case, it is well known that the Burgers equation is linearized by the Cole-Hopf transformation

$$u = \frac{1}{c}\partial_x \log\psi = \frac{1}{c}\frac{\psi_x}{\psi}. \tag{5.1}$$

Taking the transformation for the Burgers equation (4.12), we get[6]

$$\psi_t = a\psi_{xx} - \left(a - \frac{b-1}{c}\right)\frac{\psi_x^2}{\psi}. \tag{5.2}$$

Hence we can see that only when $ac = b-1$, the Burgers equation reduces to the linear equation $\psi_t = a\psi_{xx}$.[7] The linearizable Burgers equation becomes

$$u_t - au_{xx} - 2acuu_x = 0. \tag{5.3}$$

We note that the scale transformations $t \to (1/a)t$ and $u \to (1/c)u$ absorb the constants a and c in Eqs. (5.1) and (5.3), respectively.

This transformation (5.1) still works well in NC case. Then we have to treat the inverse of ψ carefully. There are typically two possibilities to define the NC version of the

[6]Here we can set the constant $C(t)$ zero in Eq. (5.2) after the linearization without loss of generality because it can be absorbed by the scale transformation $\psi \to \psi \exp\left\{\pm\int^t C(t')dt'\right\}$.

[7]Without loss of generality, we can take $a > 0$. Then the linear equation is just the diffusion equation or the heat equation where a shows the coefficient of viscosity.

Cole-Hopf transformation:[8]

$$\text{(i)} \qquad u = \frac{1}{c}\psi_x \star \psi^{-1} \tag{5.4}$$

$$\text{(ii)} \qquad u = \frac{1}{c}\psi^{-1} \star \psi_x \tag{5.5}$$

In the case (i), we can see that when $a + b = 1, c = -1$,[9] the NC Burgers equation reduces to the equation: $(\partial_x - \psi_x \star \psi^{-1}) \star (\psi_t - a\psi_{xx}) = 0$. Hence the solutions of the NC diffusion equation[10]

$$\psi_t = a\psi_{xx}, \tag{5.6}$$

give rise to the exact solutions of the NC Burgers equation via the NC Cole-Hopf transformation (5.4). The naive solution of the NC diffusion equation (5.6) is

$$\psi(t,x) = 1 + \sum_{i=1}^{N} h_i e^{ak_i^2 t} \star e^{\pm k_i x} = 1 + \sum_{i=1}^{N} h_i e^{\frac{i}{2}ak_i^3\theta} e^{ak_i^2 t \pm k_i x}, \tag{5.7}$$

where h_i, k_i are complex constants. The final form in (5.7) shows that the naive solution on commutative space is deformed by $e^{\frac{i}{2}ak_i^3\theta}$ due to the noncommutativity. This reduces to the N-shock wave solution in fluid dynamics. Hence we call it the *NC N-shock wave solution*. The explicit representation in terms of u is hard to obtain because the derivation of τ^{-1} is non-trivial. However we can discuss the asymptotic behaviors at $t \to \pm\infty$. The effect of the NC deformation is easily seen. In fact , exact solutions for $N = 1, 2$ are obtained by L. Martina and O. Pashaev [33] and nontrivial effects of the NC-deformation are actually reported.

If we want to know more general solutions, it would be appropriate to take the Fourier transformation under some boundary conditions. The calculation is the same as the commutative case. The initial value problem is also well-defined, that is, the initial condition $u(t = 0, x) = -\psi_x \star \psi^{-1}\Big|_{t=0}$ is an appropriate one.

Let us comment on multi-soliton solutions with no scattering process. Defining $z := x + vt, \bar{z} := x - vt$, we easily see

$$f(z) \star g(z) = f(z)g(z) \tag{5.8}$$

[8]There would be other candidates, such as $u = \psi^{-\alpha} \star \psi_x \star \psi^{-\beta}$ where $\alpha + \beta = 1$. However they do not seem to lead to linear equations because $\partial_i\psi^{-\alpha} = -\psi^{-\alpha} \star \partial_i\psi^{\alpha} \star \psi^{-\alpha}$ makes it complicated.

[9]Here we omit the possibility: $a = 0, b = 1$ because the NC Burgers equation (4.12) becomes trivial in this case.

[10]Here we can also put $a > 0$ as in commutative case.

because the star-product (3.1) is rewritten in terms of $(z,\bar{z})$ as

$$f(z,\bar{z})\star g(z,\bar{z})=e^{iv\theta(\partial_{\bar{z}'}\partial_{z''}-\partial_{z'}\partial_{\bar{z}''})}f(z',\bar{z}')g(z'',\bar{z}'')\Big|_{\substack{z'=z''=z\\ \bar{z}'=\bar{z}''=\bar{z}.}} \tag{5.9}$$

This situation is realized when all k_i are the same: $k_1=k_2=\cdots=k_N=k(=:v/a)$ including one-soliton solutions. The NC one shock-wave solution [33] is essentially the same as the commutative one because of the above observation. In fact, their one shock-wave solution is reduced to our solution (5.7) by putting $k_1=0$ in [33]. The condition $k_1=0$ is taken without loss of generality and then the effect of NC-deformation disappears.

We note that the solution $\psi^{\rm sol}$ of the diffusion equation (5.6) do not yield all solutions of the NC Burgers equation because the NC Cole-Hopf transformation (5.4) is a one-way map. However the transformation $\psi^{\rm sol}\to g_\psi\star\psi^{\rm sol}$ gives rise to the solution of the directly transformed equation $(\partial_x-\psi_x\star\psi^{-1})\star(\psi_t-a\psi_{xx})=0$ from the NC Burgers equation, where g_ψ is the so-called *NC transition operator* which satisfies $\partial_x g_\psi=(\psi_x\star\psi^{-1})\star g_\psi$. The existence of g_ψ would be guaranteed [25] and in principle we can construct all solutions of the NC Burgers equation via the inverse of the NC Cole-Hopf transformation.

In the case (ii), the same discussion leads us to the similar conclusion that when $a-b=-1, c=1$,[11] the solutions of the NC diffusion equation (5.6) yields the exact solutions of the NC Burgers equation via the NC Cole-Hopf transformation (5.5).

The region where the NC Burgers equation (4.12) can be linearized is shown in Fig. 1.

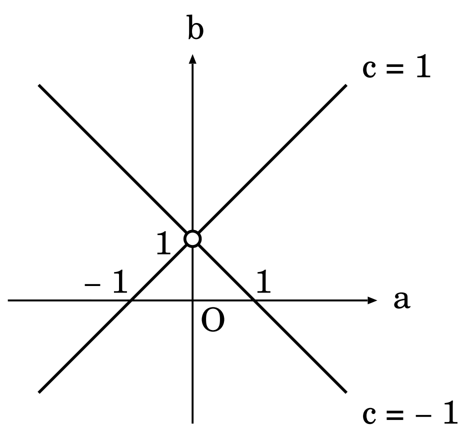

Figure 1: The region where the NC Burgers equation can be linearized

It is interesting that the condition on a,b is equivalent to that each part of two coefficients of $u_x\star u$ and $u\star u_x$ in the NC Burgers equation (4.12) vanishes. The result is summarized in Table 1.

[11]Here we also omit the possibility: $a=0, b=1$ for the same reason as in the case (i).

Table 1: The Linearizable NC Burgers Equation

	NC Cole-Hopf transformation	NC Burgers Equation
(i)	$u = -\psi_x \star \psi^{-1}$	$u_t - au_{xx} + 2au_x \star u = 0$
(ii)	$u = \psi^{-1} \star \psi_x$	$u_t - au_{xx} - 2au \star u_x = 0$

This is formally consistent with the condition that the matrix Burgers equation should be integrable [41], which would be reasonable because the variable u in the NC deformed Burgers equation can be rewritten as the infinite-size matrix by the Weyl transformation. Of course, the notions of time evolution are different.

In the commutative limit, the linearizable NC Burgers equation reduces to commutative one (5.3) with $c = \pm 1$.

6 Conserved Quantities of the Noncommutative Burgers Equation

Here we would like to comment on conserved quantities of NC Burgers equation. The discussion is basically the same as commutative case because both the differentiation and the integration are the same as commutative ones in the Moyal representation.

Let us suppose the conservation law

$$\frac{\partial J(t,x)}{\partial t} = \frac{\partial K(t,x)}{\partial x}. \tag{6.1}$$

then the conserved quantity is given by an integral

$$Q(t) = \int_{-\infty}^{\infty} J(t,x)dx. \tag{6.2}$$

The proof is straightforward:

$$\frac{dQ}{dt} = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} J(t,x)dx = \int_{-\infty}^{\infty} \frac{\partial J(t,x)}{\partial t}dx = \int_{-\infty}^{\infty} \frac{\partial K(t,x)}{\partial x}dx = 0, \tag{6.3}$$

unless the integrand $K(t,x)$ vanishes or is periodic at spatial infinity. The convergence of the integral is also expected because the star-product naively reduces to the ordinary product at spatial infinity due to: $\partial_x \sim \mathcal{O}(x^{-1})$.

On commutative spaces, the existence of infinite number of conserved quantities would lead to infinite-dimensional hidden symmetry from Noether's theorem. In Liouville sense, the existence is necessary condition for complete integrability unlike dynamical systems with finite-dimensional degree of freedom.

On NC spaces, this is also true and the existence of infinite number of conserved quantities would be meaningful. Many NC field equations with infinite number of conserved quantities have been found by the bi-complex method [1, 2, 13]. The bi-complex method also seems to be applicable to the NC Burgers equation. However this time, it is not so trivial whether Noether's theorem is valid or not.

7 Reduction from the Noncommutative (Anti-)Self-Dual Yang-Mills Equation

On commutative spaces, there is a famous conjecture, *Ward conjecture* [39]. The statement is that almost all lower-dimensional integrable equations can be derived from (anti-)self-dual Yang-Mills equation by reductions. This conjecture is almost confirmed now [40].

It is very interesting to study whether this conjecture still holds on NC spaces or not. In this section, we show that the NC Burgers equation could be derived from a NC (anti-)self-dual Yang-Mills equation by reduction, which is one example of NC Ward conjecture [20].

Let us consider the following NC (anti-)self-dual Yang-Mills equation with $G = U(1)$ (Eq. (3.1.2) in [40]):

$$\begin{aligned} &\partial_w A_z - \partial_z A_w + [A_w, A_z]_\star = 0, \quad \partial_{\tilde{w}} A_{\tilde{z}} - \partial_{\tilde{z}} A_{\tilde{w}} + [A_{\tilde{w}}, A_{\tilde{z}}]_\star = 0, \\ &\partial_z A_{\tilde{z}} - \partial_{\tilde{z}} A_z + \partial_{\tilde{w}} A_w - \partial_w A_{\tilde{w}} + [A_{\tilde{z}}, A_z]_\star + [A_w, A_{\tilde{w}}]_\star = 0. \end{aligned} \tag{7.1}$$

where $(z, \tilde{z}, w, \tilde{w})$ and $A_{z,\tilde{z},w,\tilde{w}}$ denote the coordinates of the original $(2+2)$-dimensional space and the gauge fields, respectively. We note that the commutator part should remain though the gauge group is $U(1)$ because the elements of the gauge group could be operators and the gauge group could be considered to be non-abelian: $U(\infty)$. This commutator part actually plays an important role as usual in NC theories.

Now let us take the simple dimensional reduction $\partial_{\tilde{z}} = \partial_{\tilde{w}} = 0$ and put the following constraints:

$$A_{\tilde{z}} = A_{\tilde{w}} = 0, \quad A_z = u, \quad A_w = a u_z + (1-b)u^2. \tag{7.2}$$

Then the NC (anti-)self-dual Yang-Mills equation (7.1) reduces to

$$u_w - a u_{zz} + (1+a-b) u_z \star u + (1-a-b) u \star u_z = 0. \tag{7.3}$$

This is just the NC Burgers equation (4.12) with $w \equiv t,\ z \equiv x$. We note that without the commutator part $[A_w, A_z]_\star$, the nonlinear term should be symmetric: $(u_z \star u + u \star u_z)$, which cannot lead to the Lax representation as is commented below Eq. (4.12). This shows that the special feature in NC gauge theories plays a crucial role. Therefore the

NC Burgers equation is expected to have some non-trivial property special to NC spaces such as the existence of $U(1)$ instantons.

Essentially the same argument is presented in [33] from a different viewpoint.

8 Noncommutative Hierarchy Equations

Now let us look for the Lax representations of the NC Burgers equation with the higher-dimensional time evolution by the Lax-pair generating technique:

$$[L_{\text{Burgers}}, T_{n\text{th-h}} + \partial_{t_n}]_\star = 0, \tag{8.1}$$

where the dimensions are given by $[t_n] = -n, [T_{n\text{th-h}}] = n$ and the noncommutativity could be introduced as $[t_n, x] = i\theta_n$. The Lax representations (8.1) is derived from the compatible conditions of the NC linear systems:

$$L_{\text{Burgers}} \star \psi = \lambda\psi, \tag{8.2}$$

$$\frac{\partial \psi}{\partial t_n} + T_{n\text{th-h}} \star \psi = 0. \tag{8.3}$$

This time, Eq. (8.3) is not an evolution equation. However as the previous discussion, some geometrical meaning would be expected. Then, the existence of infinite number of hierarchy equations would suggest infinite-dimensional hidden symmetry which is expected to be deformed symmetry from commutative one.

Now let us take the other ansatz for the operator $L_{\text{Burgers}} = \partial_x + u$ as

$$T_{(n+1)\text{th-h}} = \partial_x^n L_{\text{Burgers}} + T'_{(n+1)\text{th-h}}. \tag{8.4}$$

Then the unknown part is reduced to $T'_{(n+1)\text{th-h}}$ which is determined so that Eq. (8.1) is a differential equation. The results are as follows.

- For $n = 1$, the NC Lax equation gives the (second-order) NC Burgers equation (4.12).

- For $n = 2$, the NC Lax equation gives the third-order NC Burgers equation.

  The Lax pair is given by

  $$L_{\text{Burgers}} = \partial_x + u(t, x), \quad T_{3\text{rd-h}} = \partial_x^2 L_{\text{Burgers}} + T'_{3\text{rd-h}}, \tag{8.5}$$

  where

  $$\partial_x^2 L_{\text{Burgers}} = \partial_x^3 + u\partial_x^2 + 2u_x\partial_x + u_{xx}. \tag{8.6}$$

Substituting this ansatz into the NC Lax equation, we can take more explicit form on $T'_{\text{3rd-h}}$ as

$$T'_{\text{3rd-h}} = A\partial_x^2 + B\partial_x + C, \tag{8.7}$$

where A, B and C are polynomials of u, u_x, u_t, etc. In the similar way to the $n = 1$ case, the unknown polynomials satisfy the following differential equations

$$\begin{aligned} &A_x + [u, A]_\star - 2u_x = 0, \\ &B_x + [u, B]_\star - 2A \star u_x - u_{xx} - 2u_x \star u = 0, \\ &C_x + [u, C]_\star - A \star u_{xx} - B \star u_x - u_t - 2u_x^2 - u_{xx} \star u = 0, \end{aligned} \tag{8.8}$$

and the solutions are:

$$\begin{aligned} &A = 2u, \quad B = u_x + 3u^2, \\ &C = au_{xx} + bu_x \star u + cu \star u_x + du^3, \end{aligned} \tag{8.9}$$

where the coefficients a, b, c and d are constants.

Then the last equation of (8.8) yields the third-order NC Burgers equation with parameters:

$$\begin{aligned} &u_t - au_{xxx} + (1 + a - b)u_{xx} \star u + (2 - a - c)u \star u_{xx} + (3 - b - c)u_x^2 \\ &+(b - d)u_x \star u^2 + (c - b - d)u \star u_x \star u + (3 - c - d)u^2 \star u_x = 0, \end{aligned} \tag{8.10}$$

whose Lax pair is

$$\begin{cases} L_{\text{Burgers}} &= \partial_x + u, \\ T_{\text{3rd-h}} &= \partial_x^3 + 3u\partial_x^2 + 3(u_x + u^2)\partial_x + (a + 1)u_{xx} + bu_x \star u + cu \star u_x + du^3. \end{cases} \tag{8.11}$$

The parameter family of this equation formally coincides with four integrable equations given in Theorem 3.6 in [41], that is, two type of the third-order NC Burgers equations and two type of NC modified KdV equations up to scale transformations:

– the third-order NC Burgers equation: $(a = -1, b = c = d = 0)$

$$u_t + u_{xxx} + 3u \star u_{xx} + 3u_x^2 + 3u^2 \star u_x = 0. \tag{8.12}$$

– the third-order (conjugated) NC Burgers equation: $(a = -1, b = c = 3, d = 0)$

$$u_t + u_{xxx} - 3u_{xx} \star u - 3u_x^2 + 3u_x \star u^2 = 0. \tag{8.13}$$

– NC modified KdV equation via NC Miura map from NC KdV equation [2]: $(a = -1, b = 0, c = d = 3)$

$$u_t + u_{xxx} - 3u_x \star u^2 - 3u^2 \star u_x = 0. \tag{8.14}$$

Our result gives the Lax representation of the Miura-mapped NC KdV equation.

– NC modified KdV equation: $(a = -1, c = 0, b = d = 3)$

$$u_t + u_{xxx} + 3[u, u_{xx}]_\star - 6u \star u_x \star u = 0. \tag{8.15}$$

This has another Lax representation:

$$\begin{cases} L = \partial_x^2 + 2u\partial_x, \\ T = 4\partial_x^3 + 12u\partial_x^2 + 6(u^2 + u_x)\partial_x. \end{cases} \tag{8.16}$$

Let us comment on the linearizability. This time, the linearizable condition by the NC version of Cole-Hopf transformation leads to the restricted situation $a = 0, b = 1, c = 2, d = 1$ where the third-order NC Burgers equation (8.10) becomes trivial. The result shows that the linearizable condition is too strict for the third-order NC Burgers equation (8.10) due to the nonlinear effect.

- For $n = 3$, the NC Lax equation gives the fourth-order NC Burgers equation.

The Lax pair is given by

$$L_{\text{Burgers}} = \partial_x + u(t, x), \quad T_{\text{4th-h}} = \partial_x^3 L_{\text{Burgers}} + T'_{\text{4th-h}}. \tag{8.17}$$

Substituting this ansatz into the Lax equation (4.1), we can take more explicit form on $T'_{\text{4th-h}}$ as

$$T'_{\text{4th-h}} = A\partial_x^3 + B\partial_x^2 + C\partial_x + D, \tag{8.18}$$

where A, B, C and D are polynomials of u, u_x, u_t, etc and are determined in the similar way to the cases for $n = 1, 2$ as differential equations

$$\begin{aligned} A &= 3u, \quad B = 3u_x + 6u^2, \\ C &= u_{xx} + 4u_x \star u + 8u \star u_x + 4u^3, \\ D &= au_{xxx} + bu_{xx} \star u + cu \star u_{xx} + du_x^2 \\ &\quad + eu_x \star u^2 + fu \star u_x \star u + gu^2 \star u_x + hu^4, \end{aligned} \tag{8.19}$$

where the coefficients $a, b, \ldots, h$ are constants. Then we can get the fourth-order NC Burgers equation with parameters:

$$
\begin{aligned}
&u_t - a u_{xxxx} + (1 + a - b) u_{xxx} \star u + (3 - a - c) u \star u_{xxx} \\
&+ (4 - b - d) u_{xx} \star u_x + (6 - c - d) u_x \star u_{xx} \\
&+ (b - e) u_{xx} \star u^2 + (c - b - f) u \star u_x \star u + (6 - c - g) u^2 \star u_{xx} \\
&+ (d - e - f) u_x^2 \star u + (4 - e - g) u_x \star u \star u_x + (8 - d - f - g) u \star u_x^2 \\
&+ (e - h) u_x \star u^3 + (f - e - h) u \star u_x \star u^2 \\
&+ (g - f - h) u^2 \star u_x \star u + (4 - g - h) u^3 \star u_x = 0.
\end{aligned}
\tag{8.20}
$$

In this way, we can generate the higher-order NC Burgers equations. The ansatz for the $(n + 1)$-th order is more explicitly given by

$$
T_{(n+1)\text{-th}} = \partial_x^n L + T'_{(n+1)\text{-th}} = \partial_x^{n+1} + \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} (\partial_x^k u) \partial_x^{n-k} + \sum_{l=0}^{n} A_l \partial_x^{n-l}, \tag{8.21}
$$

where A_l are homogeneous polynomials of u, u_x, u_{xx} and so on, whose degrees are $[A_l] = l + 1$. These unknown polynomials are determined one by one as $A_0 = nu$ and so on.

9 Sato's Approach to the Noncommutative Hierarchy

In this final section, we present NC versions of the Burgers equation and the Burgers hierarchy in the framework of the Sato theory [42] by using the pseudo-differential operator. We look for the Lax representation of the NC Burgers hierarchy.

Let us introduce the following Lax operator as a pseudo-differential operator:

$$
L = \partial_x + u_1 + u_2 \partial_x^{-1} + u_3 \partial_x^{-2} + u_4 \partial_x^{-3} + \cdots, \tag{9.1}
$$

where the infinite number of fields u_m $(m = 1, 2, \ \ldots)$ depend on infinite number of variables $(t_1, t_2, t_3, \ \ldots)$. The action of the operator ∂_x^n on a multiplicity function $f(x)$ is given by

$$
\partial_x^n \cdot f := \sum_{i \geq 0} \begin{pmatrix} n \\ i \end{pmatrix} (\partial_x^i f) \partial^{n-i}, \tag{9.2}
$$

where the binomial coefficient is defined as

$$
\begin{pmatrix} n \\ i \end{pmatrix} := \frac{n(n-1)\cdots(n-i+1)}{i(i-1)\cdots 1}. \tag{9.3}
$$

We note that the definition can be extended to the negative powers of ∂_x.The examples are,

$$\begin{aligned}
\partial_x^{-1} \cdot f &= f\partial_x^{-1} - f_x\partial_x^{-2} + f_{xx}\partial_x^{-3} - \cdots, \\
\partial_x^{-2} \cdot f &= f\partial_x^{-2} - 2f_x\partial_x^{-3} + 3f_{xx}\partial_x^{-4} - \cdots, \\
\partial_x^{-3} \cdot f &= f\partial_x^{-3} - 3f_x\partial_x^{-4} + 6f_{xx}\partial_x^{-5} - \cdots,
\end{aligned}$$

where ∂_x^{-1} in the RHS acts as an integration operator $\int^x dx$. For more on foundation of the pseudo-differential operators and the Sato theory, see e.g. [43, 44, 45].

The noncommutativity for them can be introduced arbitrarily. Thus we do not fix the noncommutativity here. At the end of the present section, we comment on this point.

The Lax representation for the NC Burgers hierarchy in Sato's framework is given by

$$[\partial_{t_m} - B_m, L]_\star = 0, \tag{9.4}$$

where B_m is given here by

$$B_m := (\underbrace{L \star \cdots \star L}_{m \text{ times}})_{\geq 1} =: (L^m)_{\star \geq 1}. \tag{9.5}$$

The suffix "≥ 1" represents the positive power part of L^m. A few concrete examples are as follows:

$$\begin{aligned}
B_1 &= \partial_x, \\
B_2 &= \partial_x^2 + 2u_1\partial_x, \\
B_3 &= \partial_x^3 + 3u_1\partial_x^2 + 3(u_2 + u_1^2 + (u_1)_x)\partial_x.
\end{aligned} \tag{9.6}$$

The replacement of the products of fields in the commutator with the star products means the NC extension. In this approach, the geometrical meaning of the Lax representation is also vague. However we can expect that the Lax equations actually contain integrable equations as in the previous sections.

Now let us discuss the existence of the NC Burgers hierarchy. The NC Burgers hierarchy could be derived by putting the following constraint for the Lax equations (9.4)

$$L = B_1 \ (=: L_{\text{Burgers}}), \tag{9.7}$$

which implies

$$u_k = 0, \quad k = 2, 3, 4, \ \ldots. \tag{9.8}$$

This means that the Lax equations (9.4) can be represented in terms of one kind of field $u_1 \equiv u$, which guarantees the existence of the hierarchy. Now we can see that the Lax equation (9.4) just gives a differential equation.

The hierarchy equations are as follows:

- For $m=1$, the Lax equation (9.4) reduces to $u_{t_1} = u_x$, which means $t_1 = x$.
- For $m=2$, the Lax equation (9.4) becomes the second order NC Burgers equation

$$\frac{\partial u}{\partial t_2} = [B_2, L_{\text{Burgers}}]_\star = [\partial_x^2 + 2u\partial_x, \partial_x + u]_\star = u_{xx} + 2u \star u_x. \tag{9.9}$$

  This is just one of the linearizable NC Burgers equation with $t_2 \equiv t$. (See Table 1.)

- For $m=3$, the Lax equation (9.4) is the third order NC Burgers equation

$$\begin{aligned}\frac{\partial u}{\partial t_3} &= [B_3, L_{\text{Burgers}}]_\star \\ &= [\partial_x^3 + 3\partial_x^2 + 3(u^2 + u_x)\partial_x, \partial_x + u]_\star \\ &= u_{xxx} + 3u \star u_{xx} + 3u_x^2 + 3u^2 \star u_x.\end{aligned} \tag{9.10}$$

  This just coincides with the third order NC Burgers equation (8.12).

- For $m=4$, the Lax equation (9.4) is the fourth order NC Burgers equation

$$\begin{aligned}\frac{\partial u}{\partial t_4} &= [\partial_x^4 + 4u\partial_x^3 + 6(u^2 + u_x)\partial_x^2 + 4(u^3 + u_x \star u + 2u \star u_x)\partial_x, \partial_x + u]_\star \\ &= u_{xxxx} + 4u \star u_{xxx} + 4u_{xx} \star u_x + 6u_x \star u_{xx} \\ &\quad +6u^2 \star u_{xx} + 4u_x u u_x + 8u \star u_x^2 + 4u^3 \star u_x.\end{aligned} \tag{9.11}$$

In this way, we can obtain infinite number of NC hierarchy equations which possess the Lax representations with no parameter. The results for $m = 2, 3$ suggest that this approach gives rise to integrable equations directly. The relationship between this approach and the Lax-pair generating technique is $B_m = (T_{\text{mth-h}})_{\geq 1}$. Hence the Lax-pair generating technique would yield wider class of integrable equations than Sato's approach such as the modified KdV equations (8.14) and (8.15).

Let us comment how to introduce the noncommutativity of infinite-dimensional parameter spaces: $[t_i, t_j] = i\theta^{ij}$. A natural one is $\theta^{12} \neq 0, \theta^{34} \neq 0, \ldots, \theta^{2n-1,2n} \neq 0, \ldots$ and otherwise $= 0$. However the noncommutativity $\theta^{2n-1,2n} \neq 0$ introduces infinite number of derivatives with respect to t_{2n-1}, t_{2n}. If we respect the notion of time evolutions, it is reasonable to take the special possibility: $\theta^{12} \neq 0$ and otherwise $= 0$. Then problematic directions are t_1 and t_2-directions only. However, we know that the NC Burgers equation is actually integrable and the integrability is guaranteed in t_1, t_2-direction. In the other directions, time evolution is well-defined and the hierarchy equations can be considered as infinite-side matrix evolution equations because of $\hat{u} = \sum_{m,n=0}^{\infty} u_{mn}(t_3, t_4, \ldots)|m\rangle\langle n|$. Then the situation belongs to infinite-dimensional version of matrix Sato theory and various discussions would be possible. This time, the results in [41] are applicable to the present discussion and the third order NC Burgers equation is proved to be integrable.

10 Conclusion and Discussion

In this article, we gave NC versions of the Burgers equation and the Burgers hierarchy which possess the Lax representations. We proved that the NC Burgers equation is linearizable by the NC versions of the Cole-Hopf transformations (5.4) and (5.5). The linearized equation is the (NC) diffusion equation of first order with respect to time and the initial value problem is well-defined. We obtained the exact solutions of the linearized equation and discussed the effects of the NC deformation. Furthermore, we rederived the NC Burgers equation from both NC (anti-)self-dual Yang-Mills equation, which would be an evidence of NC Ward conjecture. Finally we showed the NC Burgers hierarchy in the frameworks of the Lax-pair generating technique and the Sato theory, which would lead to the completion of NC Sato theory.

The results show that the NC Burgers equation is completely integrable even though the NC Burgers equation contains infinite number of time derivatives in the nonlinear term. The linearized equation is the (NC) diffusion equation of first order with respect to time and the initial value problem is well-defined. This is a surprising discovery and a good news for the study of NC extension of integrable equations. There would be many further directions such as NC extension of the Hirota's bilinear method which is a simple generalization of the Cole-Hopf transformation. The existence of NC hierarchies for other integrable equations have been already proved in [16, 46]. Hence if we succeed in the bilinearization, the NC Sato theory must be constructed. Then we can discuss the structure of the solution spaces and the symmetry underlying the integrability. The discretization of the NC Burgers equation is also considerably interesting in recent developments of integrable systems.[12] The further study is worth investigating.

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[12]The semi-discretization of the Burgers equation is discussed in detail, for example, by T. Tsuchida [47].

A Equivalence between the star-product formalism and the operator formalism

The two formalisms of NC field theories, the star-product formalism and the operator formalism are equivalent and connected by the Weyl transformation. The Weyl transformation transforms the field $f(x^1, x^2)$ in the star-product formalism into the infinite-size matrix $\hat{f}(\hat{x}^1, \hat{x}^2)$ as

$$\hat{f}(\hat{x}^1, \hat{x}^2) \;:=\; \frac{1}{(2\pi)^2} \int dk_1 dk_2 \; \tilde{f}(k_1, k_2) e^{-i(k_1\hat{x}^1 + k_2\hat{x}^2)}, \tag{A.1}$$

where

$$\tilde{f}(k_1, k_2) \;:=\; \int dx^1 dx^2 \; f(x^1, x^2) e^{i(k_1 x^1 + k_2 x^2)}. \tag{A.2}$$

This map is the composite of twice Fourier transformations replacing the commutative coordinates x^1, x^2 in the exponential with the NC coordinates $\hat{x}^1, \hat{x}^2$ in the inverse transformation:

$$\begin{array}{ccc} & & f(x^1, x^2) \\ & \swarrow & \quad | \\ \tilde{f}(k_1, k_2) & & \text{Weyl transformation} \\ & \searrow & \quad \downarrow \\ & & \hat{f}(\hat{x}^1, \hat{x}^2). \end{array}$$

The Weyl transformation preserves the product:

$$\widehat{f \star g} = \hat{f} \cdot \hat{g}. \tag{A.3}$$

The inverse transformation of the Weyl transformation is given directly by

$$f(x^1, x^2) = \int dk_2 \; e^{-ik_2 x^2} \Big\langle x^1 + \frac{k_2}{2} \Big| \hat{f}(\hat{x}^1, \hat{x}^2) \Big| x^1 - \frac{k_2}{2} \Big\rangle. \tag{A.4}$$

The transformation also maps the derivation and the integration one-to-one . Hence the field equation and the solution are also transformed one-to-one. The correspondences are the following (See e.g. [48]):

The star-product formalism	← Weyl transformation →	The operator formalism
ordinary functions $f(x^1, x^2)$	[field]	infinite-size matrices $\hat{f}(\hat{x}^1, \hat{x}^2) = \sum_{m,n=0}^{\infty} f_{mn} \vert m\rangle\langle n \vert$
star-products $\left(f \star (g \star h) = (f \star g) \star h\right)$	[product] (associativity)	multiplications of matrices $\left(\hat{f}(\hat{g}\hat{h}) = (\hat{f}\hat{g})\hat{h} \text{ (trivial)}\right)$
$[x^i, x^j]_\star = i\theta^{ij}$	[noncommutativity]	$[\hat{x}^i, \hat{x}^j] = i\theta^{ij}$
$\partial_i f$ $\left(\text{especially, } \partial_i x^j = \delta_i{}^j\right)$	[derivation]	$\partial_i \hat{f} := [\underbrace{-i(\theta^{-1})_{ij}\hat{x}^j}_{=:\,\hat{\partial}_i}, \hat{f}]$ $\left(\text{especially, } \partial_i \hat{x}^j = \delta_i{}^j\right)$
$\int dx^1 dx^2 \; f(x^1, x^2)$	[integration]	$2\pi\theta \mathrm{Tr}_{\mathcal{H}} \hat{f}(\hat{x}^1, \hat{x}^2)$
$\sqrt{\frac{n!}{m!}}\left(2r^2/\theta\right)^{\frac{m-n}{2}} e^{i(m-n)\varphi} \times 2(-1)^n L_n^{m-n}(2r^2/\theta) e^{-\frac{r^2}{\theta}}$	[matrix element]	$\vert n\rangle\langle m \vert$

where (r, φ) is the usual polar coordinate $(r = \{(x^1)^2 + (x^2)^2\}^{\frac{1}{2}})$ and $L_n^\alpha(x)$ is the Laguerre polynomial:

$$L_n^\alpha(x) := \frac{x^{-\alpha} e^x}{n!} \left(\frac{d}{dx}\right)^n (e^{-x} x^{n+\alpha}). \tag{A.5}$$

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