ON SOME KINDS OF FUZZY CONNECTED SPACES

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Abstract. In this paper we introduce new results in fuzzy connected spaces. Among the results obtained we can mention the good extension of local connectedness. Also we prove that in a T_1-fuzzy compact space the notions c-zero dimensional, strong c-zero dimensional and totally c_i-disconnected are equivalent.

Keywords: fuzzy connected space, fuzzy strong connected, fuzzy super connected, c-zero dimensional, strong c-zero dimensional, totally c_i-disconnected

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1. Introduction

In 1965 Zadeh [11] in his classical paper generalized characteristic functions to fuzzy sets. Chang [4] in 1968 introduced the topological structure fuzzy sets in a given set. Pu and Liu [9] defined connectedness by using the concept of the fuzzy closed set. Lowen [8] also defined an extension of connectedness in a restricted family of fuzzy topologies. Fatteh and Bassam [7] studied further the notions of fuzzy super connected and fuzzy strongly connected spaces. However, they defined connectedness only for a crisp set in a fuzzy topological space. Ajmal and Kohli [1] extended the notions of connectedness to an arbitrary fuzzy set, and defined the notions of c-zero dimensional, total disconnected and strong zero connected. Several authors have tried to extend to the fuzzy set theory the main notions of general topology, see [3], [5], [6], [10], [13].

In this paper we introduce more results on these spaces. Among the results obtained, we introduce the good extension of a locally connected space, namely a locally fuzzy connected space. Also, we prove that in a T_1-fuzzy compact space the notions c-zero dimensional, strong c-zero dimensional and totally c_i-disconnected are equivalent.

2. Basic definitions

The following definitions have been used to obtain the results and properties presented in this paper.

Definition 2.1 ([7]). A fuzzy topological space X is said to be fuzzy connected if it has no proper fuzzy clopen set. (A fuzzy set λ in X is proper if $\lambda \neq 0$ and $\lambda \neq 1$, clopen means closed-open.)

Definition 2.2 ([7]). A fuzzy topological space X is said to be fuzzy superconnected if X does not have non-zero fuzzy open sets λ and μ such that $\lambda + \mu \leqslant 1$.

Definition 2.3 ([7]). A fuzzy topological space X is said to be fuzzy strong connected if it has no non-zero fuzzy closed sets f and k such that $f + k \leqslant 1$.

Definition 2.4 ([1]). A fuzzy topological space X is said to be c-zero dimensional if for every crisp fuzzy point x_1 in X and every fuzzy open set μ containing x_1 there exists a crisp clopen fuzzy set δ in X such that $x_1 \leqslant \delta \leqslant \mu$.

Definition 2.5 ([1]). A fuzzy topological space X is said to be totally c_i-disconnected ($i = 1, 2, 3, 4$) if the support of every non zero c_i-connected fuzzy set ([1, Definition 2.7]) in X is a singleton.

Definition 2.6 ([1]). A fuzzy topological space X is said to be strong c-zero dimensional if it is not c-connected between any pair of its disjoint fuzzy closed sets.

Definition 2.7 ([2]). A fuzzy topological space X is said to be fuzzy locally connected at a fuzzy point x_α in X if for every fuzzy open set μ in X containing x_α, there exists a connected fuzzy open set δ in X such that $x_1 \leqslant \delta \leqslant \mu$.

Definition 2.8 ([2]). Let (X, T) be a fuzzy topological space, suppose $A \subset X$ and let $T_A = \{\mu|_A \colon \mu \in T\}$. Then (A, T_A) is called a fuzzy subspace of (X, T). In short we shall denote (A, T_A) by A. A fuzzy subspace A is said to be a fuzzy open subspace if its characteristic function χ_A is fuzzy open in (X, T).

Let μ be a fuzzy set in X and let $A \subset X$. Then we denote $\mu|_A$ by μ^a. In particular, if μ is a fuzzy point x_α in X, then we denote the fuzzy set x_{α_A} by x^a_α.

Definition 2.9 ([1]). A fuzzy topological space (X, T) is said to be c-connected between fuzzy sets μ and δ if there exists no crisp clopen fuzzy set η such that $\mu \leqslant \eta \leqslant 1_X - \delta$.

Definition 2.10 ([1]). A fuzzy topological space (X, T) is sait to be Lindelöff if every covering of X by fuzzy open sets in X has a countable subcover.

Definition 2.11 ([7]). Let (X, T) be a fuzzy topological space, let $A \subset X$. Then A is said to be a fuzzy connected subset of X if A is a fuzzy connected space as a fuzzy subspace of X.

Definition 2.12. A fuzzy topological space (X, T) is said to be locally fuzzy super connected (locally fuzzy strong connected) at a fuzzy point x_a in X if for every fuzzy open set μ in X containing x_a there exists a fuzzy super connected (fuzzy strong connected) open set η in X such that $x_a \leqslant \eta \leqslant \mu$.

Definition 2.13. A fuzzy quasi-component of a fuzzy point x_1 in a fuzzy topological space (X, T) is the smallest fuzzy clopen subset of X containing x_1. We denote it by Q.

Definition 2.14. A fuzzy path-component of a fuzzy point x_1 in a fuzzy topological space (X, T) is the maximal fuzzy path connected in (X, T) containing x_1. We denote it by C.

Definition 2.15 ([9], [1]). A fuzzy topological space X is said to be a T_1-fuzzy topological space if every fuzzy point in X is fuzzy closed.

3. Fuzzy connectedness and its stronger forms

In this section we study some stronger forms of connectedness such as fuzzy super connected, totally c_i-disconnected and fuzzy strongly connected introduced by Fatteh and Bassam. We get some additional results and properties for these spaces. Among the results we prove that the local fuzzy connectedness is a good extension of local connectedness.

Theorem 3.1. *A topological space (X, τ) is locally connected if and only if $(X, \omega(\tau))$ is locally connected (where $\omega(\tau)$ is the set of all lower semi-continuous functions from (X, τ) to the unit interval $I = [0, 1]$).*

Proof. Let μ be a fuzzy open set in $\omega(\tau)$ containing a fuzzy point x_α. Since μ is a lower semicontinuous function, by local connectedness of (X, τ) there exists an open connected set U in X containing x and contained in the support of μ (i.e. $x \in U \subset \operatorname{Supp} \mu$). Now χ_U is the characteristic function of U and it is lower semicontinuous, hence $\chi_U \wedge \mu$ is a fuzzy open set in $\omega(\tau)$. We claim $\delta = \chi_U \wedge \mu$ is a fuzzy connected set containing x_α. If not then by [8, Theorem 3.1], there exist a non zero lower semicontinuous functions μ_1, μ_2 in $\omega(\tau)$ such that

$$\mu_1|_\delta + \mu_2|_\delta = 1.$$

Now $\operatorname{Supp}\delta = U$ and $\operatorname{Supp}\mu_1$, $\operatorname{Supp}\mu_2$ are open sets in τ such that

$$U \subset \operatorname{Supp}\mu_1 \cup \operatorname{Supp}\mu_2,$$

hence,

$$U \cap \operatorname{Supp}\mu_1 \neq \emptyset$$

and

$$U \cap \operatorname{Supp}\mu_2 \neq \emptyset$$

and consequently

$$(U \cap \operatorname{Supp}\mu_1) \cup (U \cap \operatorname{Supp}\mu_2) = U \cap (\operatorname{Supp}\mu_1 \cup \operatorname{Supp}\mu_2) = U$$

is not connected. Conversely, let U be an open set in τ containing x, $x_\alpha \in \chi_U$, (χ_U is the characteristic function of U), χ_U is a fuzzy open set in $\omega(\tau)$. By fuzzy connectedness of $(X, \omega(\tau))$ there exists a fuzzy open connected set μ in $\omega(\tau)$ such that

$$x_\alpha \leqslant \mu \leqslant \chi_U.$$

We claim that $\operatorname{Supp}\mu$ is connected ($x \in \operatorname{Supp}\mu \subset U$). If not there exist two non empty open sets $G_1, G_2 \in \tau$ such that

$$\operatorname{Supp}\mu = G_1 \cup G_2 \quad \text{and} \quad G_1 \cap G_2 = \emptyset.$$

It is clear that

$$\chi_{G_1} + \chi_{G_2} = 1_\mu,$$

which is a contradiction, because μ is fuzzy connected. □

Theorem 3.2. *If G is a subset of a fuzzy topological space (X, T) such that μ_G (μ_G is the characteristic function of a subset G of X) is fuzzy open in X, then if X is a super connected space then G is a fuzzy super connected space.*

Proof. Suppose that G is not a fuzzy super connected space. Then by [7, Theorem 6.1] there exist fuzzy open sets λ_1, λ_2 in X such that

$$\lambda_1|_G \neq 0, \quad \lambda_2|_G \neq 0$$

and

$$\lambda_1|_G + \lambda_2|_G \leqslant 1,$$

and therefore

$$\lambda_1 \wedge \mu_G + \lambda_2 \wedge \mu_G \leqslant 1.$$

Then X is not a fuzzy super connected space, a contradiction. □

Theorem 3.3. *If A and B are fuzzy strong connected subsets of a fuzzy topological space (X, T) and $\overline{\mu_B}|_A \neq 0$ or $\overline{\mu_A}|_B \neq 0$, then $A \vee B$ is a fuzzy strong connected subset of X where μ_A, μ_B are the characteristic functions of A and B respectively.*

P r o o f. Suppose $Y = A \vee B$ is not a fuzzy strong connected subset of X. Then there exist fuzzy closed sets δ and λ such that $\delta|_Y \neq 0$ and $\lambda|_Y \neq 0$ and $\delta|_Y + \lambda|_Y \leqslant 1$. Since A is a fuzzy strong connected subset of X, then either $\delta|_A = 0$ or $\lambda|_A = 0$. Without loss of generality assume that $\delta|_A = 0$. In this case, since B is also fuzzy strong connected, we have

$$\delta|_A = 0, \quad \lambda|_A \neq 0, \quad \delta|_B \neq 0, \quad \lambda|_B = 0$$

and therefore

$$\lambda|_A + \overline{\mu_B}|_A \leqslant 1. \tag{1}$$

If $\overline{\mu_B}|_A \neq 0$, then $\lambda|_A \neq 0$ with (1) imply that A is not a fuzzy connected subset of X. In the same way if $\overline{\mu_A}|_B \neq 0$ then $\delta|_B \neq 0$ and $\lambda|_B + \overline{\mu_A}|_B \leqslant 1$ imply that B is not a fuzzy strong connected subset of X, a contradiction. □

Theorem 3.4. *If A and B are subsets of a fuzzy topological space (X, T) and $\mu_A \leqslant \mu_B \leqslant \overline{\mu_A}$, then if A is a fuzzy strong connected subset of X then B is also fuzzy strong connected.*

P r o o f. Let B be not fuzzy strong connected. Then there exist two non zero fuzzy closed sets $f|_B$ and $k|_B$ such that

$$f|_B + k|_B \leqslant 1. \tag{2}$$

If $f|_A = 0$ then $f + \mu_A \leqslant 1$ and this implies

$$f + \mu_A \leqslant f + \mu_B \leqslant f + \overline{\mu_A}. \tag{3}$$

Hence $f + \mu_B \leqslant 1$, thus $f|_B = 0$, a contradiction, and therefore $f|_A \neq 0$. Similarly we can show that $k|_A \neq 0$. By (2) and with the relation $\mu|_A \leqslant \mu_B$ we conclude that

$$f|_A + k|_A \leqslant 1,$$

so A is not fuzzy strong connected, which is again a contradiction. □

Theorem 3.5. *A fuzzy topological space (X, T) is locally fuzzy connected if and only if every fuzzy open subspace of X is locally connected.*

Proof. Let A be a fuzzy open subspace of X and let η be a fuzzy open set in X. To prove A is fuzzy connected, let x_α^a be a fuzzy point in A and let $\eta|_A$ be a fuzzy open set in A containing x_α^a. We must prove that there exists a connected fuzzy open set $\mu|_A$ in A such that

$$x_\alpha^a \leqslant \mu|_A \leqslant \eta|_A.$$

Clearly, the fuzzy point x_α in X lies in η. Since X is locally fuzzy connected, there exists an open fuzzy connected μ such that

$$x_\alpha \leqslant \mu \leqslant \eta \quad \text{and} \quad \mu \leqslant \eta \wedge \chi_A.$$

It is easy to prove that

$$x_\alpha^a \leqslant \mu|_A \leqslant \eta|_A.$$

If $\mu|_A$ is not fuzzy connected, then there exists a proper fuzzy clopen $\lambda|_A$ in $\mu|_A$ (λ is proper fuzzy clopen in μ). This is a contradiction with the fact that μ is fuzzy connected and hence A is fuzzy connected. □

In the same way we can prove an analogue of Theorem 3.5 provided (X, T) is a strong connected or a super connected space.

Theorem 3.6. *Let X be a locally super connected and Y a fuzzy topological space, let F be fuzzy continuous from X onto Y. Then Y is locally super connected.*

Proof. Let y_λ be a fuzzy point of Y. To prove Y is locally fuzzy super connected means to show that for every open set μ in Y containing y_λ ($y_\lambda \leqslant \mu$) there exists a super connected fuzzy open set η such that $y_\lambda \leqslant \eta \leqslant \mu$. Let $F\colon X \longrightarrow Y$ be fuzzy continuous. Then there exists a fuzzy point x_δ of X such that $F(x_\delta) = y_\lambda$, $F^{-1}(\mu)$ is fuzzy open set in X. Consequently,

$$F^{-1}(\mu)(x_\delta) = \mu(F(x_\delta)) = \mu(y_\lambda), \quad F(x_\delta) \leqslant \mu$$

and thus $x_\delta \leqslant F^{-1}(\mu)$. Since X is locally fuzzy super connected there exists a fuzzy super connected open set η such that

$$x_\delta \leqslant \eta \leqslant F^{-1}(\mu),$$

then

$$F(x_\delta) \leqslant F(\eta) \leqslant \mu$$

and $F(\eta)$ is super connected fuzzy [7, Theorem 6.5]. □

In the same way we can prove an analogue of Theorem 3.6 for locally fuzzy strong connected spaces.

4. Other types of connectedness

In this section we prove that in a T_1-fuzzy compact space the notions c-zero dimensional, strong c-zero dimensional and totally c_i-disconnected are equivalent.

Definition 4.1 ([4]). Let A be a subspace of a fuzzy topological space (X,T). A is fuzzy compact if for each family of fuzzy open sets $\{u_s\}_{s\in S}$ of (X,T) such that $A \leqslant \bigvee_{s\in S} u_s$ there exists a finite set $\{s_1, s_2, \ldots, s_k\}$ such that $A \leqslant \bigvee_{i=1}^{k} u_{s_i}$.

Theorem 4.2. *In a fuzzy topological space (X,T) a fuzzy path-component C is smaller than the fuzzy quasi-component Q for every point x_1.*

Proof. Let x_1 be a fuzzy point in (X,T), suppose $C \nleqslant Q$, let μ be any fuzzy clopen subset of X containing x_1, let us consider $C \wedge \mu$ and $C - \mu$. It is clear that $C \wedge \mu \neq 0$,

$$(C-\mu)(x) = \begin{cases} C(x) & \text{if } C(x) > \mu(x), \\ 0 & \text{otherwise.} \end{cases}$$

If $C - \mu = C$ then

$$(C \wedge \mu)(x) + (C - \mu)(x) = 1_c,$$

a contradiction, hence $C - \mu = 0$, then $C \leqslant \mu$ since μ is arbitrary, thus $C \leqslant Q$. □

Lemma 4.3. *Let μ be a fuzzy open subset of a topological space (X,T). If a family $\{F_s\}_{s\in S}$ of closed subsets of X contains at least one fuzzy compact set, in particular if X is fuzzy compact and if $\bigwedge_{s\in S} F_s < \mu$, then there exists a finite set $\{s_1, s_2, \ldots, s_k\}$ such that $\bigwedge_{i=1}^{k} F_{s_i} < \mu$.*

Proof. Let μ be a fuzzy open set, then $1-\mu = \mu^c$ is fuzzy closed and

$$\left(\bigwedge_{s\in S} F_s < \mu\right)^c = \bigvee_{s\in S} F_s^c > \mu^c = 1 - \mu$$

is fuzzy compact (every fuzzy closed subset of fuzzy compact set is fuzzy compact). Thus we have

$$1 - \mu < \bigvee_{s\in S} F_s^c.$$

Therefore

$$1-\mu<\bigvee_{i=1}^{k}F_{s_i}^{c}$$

and then

$$\bigwedge_{i=1}^{k}F_{s_i}<\mu.$$

□

Theorem 4.4. *Let (X,T) be a T_1-fuzzy compact space. Then the following conditions are equivalent:*

(1) *(X,T) is* c*-zero dimensional.*

(2) *(X,T) is strong* c*-zero dimensional.*

(3) *(X,T) is totally* c_i*-disconnected,* $i=1,2,3,4$.

Proof. (1) ⇒ (2): it is clear that every fuzzy compact set is a fuzzy Lindelöff topological space, hence by [1, Theorem 4.5], (X,T) is strongly c-zero dimensional.

(2) ⇒ (1): Since (X,T) is T_1-space, then every fuzzy point is fuzzy closed, and by [1, Theorem 4.4], (X,T) is c-zero dimensional.

(1) ⇒ (3): Let (X,T) be a c-zero dimensional, then by [1, Theorem 4.3], (X,T) is totaly c_i-disconnected, $i=1,2,3,4$.

(3) ⇒ (1): we have to prove that if (X,T) is fuzzy compact totally c_i-disconnected, then it is c-zero dimensional. Let x_1 be a crisp fuzzy point in X and let μ be a fuzzy open set containing x_1. We will prove that there exists a crisp clopen fuzzy set δ in X such that

$$x_1\leqslant\delta\leqslant\mu.$$

Let

$$\mu^*=\bigwedge\{\mu\colon\ \mu\ \text{is fuzzy clopen and}\ x_1\leqslant\mu\}$$

be a fuzzy quasi-component of x_1, let U be a neighborhood of x_1, $\mu^*<U$. Then by Lemma 4.3 and the fuzzy compactness of (X,T),

$$\mu^*=\mu_1\wedge\mu_2\wedge\ldots\wedge\mu_k<U$$

and we get

$$x_1\leqslant\mu^*<U.$$

□

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