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Toward the classification of third-order integrable evolution equations

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Abstract. A non-standard way of representing an evolution equation in the form of a system is proposed. This representation allows us to investigate all the different classes of third-order integrable evolution equations simultaneously. Using this approach, a preliminary classification of these equations is made.

1. Introduction

The goal of this paper is to give a preliminary classification of the integrable equations of the form

$$w_t = H(x, w, w_x, w_{xx}, w_{xxx}) \tag{1}$$

based on its representation as a system

$$(v_i)_x = \Phi_i(x, v_1, \ldots, v_n, u) \qquad i = 1, \ldots, n \tag{2}$$

$$(v_i)_t = G_i(x, v_1, \ldots, v_n, u, u_x, \ldots) \tag{3}$$

$$u_t = F(x, v_1, \ldots, v_n, u, u_x, \ldots). \tag{4}$$

The complete classification of integrable equations (1) involves a huge number of computations and is, therefore, a very difficult task. Representation (2)–(4) allows these computations to be reduced by several times. We think that using this approach, the classification problem could be performed in its entirety.

Relations (2) are an under-determined system of ordinary differential equations with a number of equations equal to the number of unknowns minus one. A known example [1] is the Cartan–Hilbert equation $p_x = q_{xx}^2$, which can be written in the form

$$q = v_1 \qquad q_x = v_2 \qquad p = v_3 \qquad p_x = u$$

$$(v_1)_x = v_2 \qquad (v_2)_x = \sqrt{u} \qquad (v_3)_x = u.$$

Relations (3) and (4) can be considered [2] as an infinitesimal symmetry of (2).

The system (2)–(4) admits several interpretations. For example, we can consider the variables v_i as non-local (if $v_x = u$ then $v = \int u\,dx$). Or we can see (2) as a differential constraint on the system of evolution equations (3) and (4). A third point of view, if equation (4) does not depend on v_i, is to view (2) and (3) as a pseudopotential of (4) in the sense of Wahlquist–Estabrook [3].

With the help of local invertible transformations† *it is sometimes possible to eliminate* (2) and reduce the system (2)–(4) to only one evolution equation

$$w_t = H(x, w, w_x, \ldots). \tag{5}$$

The Cartan–Hilbert's equation gives us one of the simplest examples of constraints that cannot be eliminated by invertible transformations.

One of the main aims of this work is to demonstrate that it is often very useful to replace evolution equation (5) by its equivalent system (2)–(4).

Example 1. Let us consider the following integrable equation

$$w_t = -2w_x \left(\frac{w_{xxx}}{w_x} - \frac{3}{2}\frac{w_{xx}^2}{w_x^2} \right)^{-1/2}. \tag{6}$$

Putting $v_1 = w$, $v_2 = w_x$, $v_3 = w_{xx}$ and $u = \left(\frac{w_{xxx}}{w_x} - \frac{3}{2}\frac{w_{xx}^2}{w_x^2} \right)^{1/2}$ we obtain

$$(v_1)_x = v_2 \qquad (v_2)_x = v_3 \qquad (v_3)_x = v_2 u^2 + \frac{3}{2}\frac{(v_3)^2}{v_2} \tag{7}$$

$$\begin{cases} (v_1)_t = -2\dfrac{v_2}{u} \\ (v_2)_t = \dfrac{2v_2 u_x}{u^2} - \dfrac{2v_3}{u} \\ (v_3)_t = \dfrac{2v_2 u_{xx}}{u^2} - \dfrac{4v_2 u_x^2}{u^3} + \dfrac{4v_3 u_x}{u^2} 2v_2 u - 3\dfrac{v_3^2}{v_2 u} \end{cases} \tag{8}$$

$$(u)_t = \frac{u_{xxx}}{u^3} - 6\frac{u_x u_{xx}}{u^4} + 6\frac{u_x^3}{u^5} = \left(\frac{u_{xx}}{u^3} - \frac{3}{2}\frac{u_x^2}{u^4} \right)_x. \tag{9}$$

Why is the form (7)–(9) more convenient than equation (6)? Two reasons can be noted. The first is a practical one. The system becomes quasipolynomial‡ even though (6) is not quasipolynomial. In the search for such important properties such as higher symmetries, conservation laws, zero-curvature representation, pseudopotentials, etc, very heavy computations must be done. The use of computer algebra becomes necessary and if the functions involved are quasipolynomial, the computation is much more productive than in the case of algebraic functions.

† The invertible transformations and symmetries for system (2) have been investigated in detail in [2] (see also [1]).

‡ By quasipolynomial we mean an algebraic sum of terms of the form $cx_1^{m_1} \cdots x_s^{m_s}$, where $c, m_1, \ldots, m_s$ are constants.

The second reason is a theoretical one. It turns out that the transition to system (2)–(4) allows a unification of different classes of evolution equation. This becomes very useful in the classification of integrable equations. It is known [4] that integrable evolution equations of the form (1) must have one of the following dependences on the third derivative w_{xxx}:

$$w_t = f_1 w_{xxx} + f_2 \tag{10}$$

$$w_t = (f_1 w_{xxx} + f_2)^{-2} + f_3 \tag{11}$$

$$w_t = (2f_1 w_{xxx} + f_2)(f_1 w_{xxx}^2 + f_2 w_{xxx} + f_3)^{-1/2} + f_4 \tag{12}$$

where $f_i = f_i(x, w, w_x, w_{xx})$. Equation (6) in example 1 belongs to class (12).

In order to construct what we will call the associated system (2)–(4) for the general equation (1), we will always choose

$$u = \left(\frac{\partial H}{\partial w_{xxx}}\right)^{-1/3} \tag{13}$$

and

$$v_1 = w, \ldots, v_n = w_{\underbrace{x\ldots x}_{n-1}}$$

where n is the order of the function u in the highest derivative of w. It turns out that after such a choice of u, the associated systems for (10)–(12) are very similar in spite of the very different forms of the original equations. Expression (13) often appears in work concerning integrable equations of type (1). But the fact that, if we use (13) the right-hand side of the associated system always becomes quasipolynomial, is still a mystery to us. A rigorous description of the construction of the associated system can be found in the appendix.

If $n < 3$ then equation (1) is quasilinear, i.e. it has the form (10). For quasilinear equations of the form

$$w_t = \alpha(x, w) w_{xxx} + \beta(x, w, w_x, w_{xx}) \tag{14}$$

we have $n = 0$, so the v_i are absent. In the cases $n = 1$ and $n = 2$, the equations reduce to the form

$$w_t = \alpha(x, w, w_x) w_{xxx} + \beta(x, w, w_x, w_{xx}) \tag{15}$$

$$w_t = \alpha(x, w, w_x, w_{xx}) w_{xxx} + \beta(x, w, w_x, w_{xx}) \tag{16}$$

respectively. For equations of the form

$$w_t = \alpha(x) w_{xxx} + \beta(x, w, w_x, w_{xx}) \tag{17}$$

u depends only on x and it is not a valid dynamical variable. Nevertheless, we can construct an associated system for them in the following way. Two equations can be considered equivalent if there exists a point transformation transforming one into the other. Performing, for instance, a transformation such as $\bar{x} = w$, $\bar{w} = x$, equation (17) can be brought into an equation of type (15). Finally, for non-quasilinear equations, $n = 3$ (cf example 1).

It is clear that the source equation (1) turns into the first equation of system (3). Our crucial idea is to concentrate on equation (4) instead of the source equation. In all integrable cases we will see that, for any n, equation (4) has the unique form

$$u_t = \left(\frac{u_{xx}}{u^3} + f(v_1, \dots, v_n, u, u_x)\right)_x . \tag{18}$$

Moreover, in terms of the vector fields

$$\Phi = \frac{\partial}{\partial x} + \sum_{i=1}^{n} \Phi_i \frac{\partial}{\partial v_i} \qquad G = \sum_{i=1}^{n} G_i \frac{\partial}{\partial v_i} \tag{19}$$

where Φ_i and G_i are the functions in (2) and (3), all formulae concerning equation (1) for all cases (10)–(12) can be written in a unified way.

2. Integrability conditions

In recent years the problem of classifying integrable partial-differential equations has received considerable attention. There are several ways of defining integrability: using generalized symmetries, conservation laws, Painlevé test and so on. These approaches have been discussed in [5], where references can be found.

In this paper we use the approach of a canonical series of conservation laws [4, 6–8]. For any integrable equation there exists a sequence of so-called canonical local conservation laws

$$(\rho_i)_t = (\sigma_i)_x \qquad i = 1, 2, \dots . \tag{20}$$

The canonical densities ρ_i can be expressed explicitly in terms of the right-hand side of the equation and the fluxes σ_j and $j = 1, \dots, i-1$. The mechanism for obtaining such formulae is described in detail in [9]. For a system of type (2)–(4), this mechanism was generalized in [2]. The first five canonical densities, calculated in this way for the system associated with (1), are given by

$$\rho_1 = u \tag{21}$$

$$\rho_2 = u^3 \frac{\partial F}{\partial u_{xx}} \tag{22}$$

$$\rho_3 = \left(2u^{-2}u_x + u^2 \frac{\partial F}{\partial u_{xx}}\right)_x + u^{-3}u_x^2 + \tfrac{1}{3}u^5 \left(\frac{\partial F}{\partial u_{xx}}\right)^2 + uu_x \frac{\partial F}{\partial u_{xx}} - u^2 \frac{\partial F}{\partial u_x} + u\sigma_1 \tag{23}$$

$$\begin{aligned}\rho_4 = {} & -\tfrac{1}{3}u_{xx}\frac{\partial F}{\partial u_{xx}} - u_x \frac{\partial F}{\partial u_x} + u\frac{\partial F}{\partial u} + u^{-1}u_x^2 \frac{\partial F}{\partial u_{xx}} - \tfrac{1}{3}u^4 \frac{\partial F}{\partial u_x}\frac{\partial F}{\partial u_{xx}} + \tfrac{1}{3}u^3 u_x \left(\frac{\partial F}{\partial u_{xx}}\right)^2 \\ & + \tfrac{2}{27}u^7 \left(\frac{\partial F}{\partial u_{xx}}\right)^3 + \tfrac{1}{3}u\sigma_2\end{aligned} \tag{24}$$

$$\rho_5 = u\sigma_3 - \rho_3\sigma_1 - 3\frac{\partial \Phi}{\partial u}(F). \tag{25}$$

Here, and in what follows, vector notation will be used. Thus, the associated system (2)–(4) has the form

$$v_x = \Phi(v, u) \tag{26}$$

$$v_t = G(v, u, u_x, u_{xx}) \tag{27}$$

$$u_t = F(v, u, u_x, u_{xx}, u_{xxx}) \tag{28}$$

where $v = (x, v_1, \ldots, v_n)$ and

$$\Phi = (1, \Phi_1, \ldots, \Phi_n) \qquad G = (0, G_1, \ldots, G_n). \tag{29}$$

Sometimes we will identify these vectors with their vector field counterparts (19), hoping that this will not lead to confusion.

In this paper the fact that expressions (21)–(23) must be densities of local conservation laws is exploited; this implies strong restrictions on the form of the functions Φ, F and G in the right-hand side of the associated system (26)–(28). Step by step, we will determine their dependence on u_{xxx}, u_{xx}, etc.

In the first step of this sharpening process, we take into account the condition that $\rho_1 = u$ must be a conserved density according to (21). This means that

$$F = (\sigma_1(v, u, u_x, u_{xx}))_x.$$

On the other hand, as is proven in the appendix (cf (73)), with the chosen form of u given by (13), F must be of the form $F = u^{-3}u_{xxx}$ + lower terms. Thus

$$F = (u^{-3}u_{xx} + f(v, u, u_x))_x. \tag{30}$$

This formula describes the dependence of F in u_{xxx} and u_{xx}. Using the compatibility conditions

$$(G)_x = (\Phi)_t \tag{31}$$

that must hold between (26) and (27), we are led to

$$\frac{\partial G}{\partial u_{xx}} u_{xxx} + \cdots = \frac{\partial \Phi}{\partial u}\frac{u_{xxx}}{u^3} + \cdots \Rightarrow \frac{\partial G}{\partial u_{xx}} = \frac{1}{u^3}\frac{\partial \Phi}{\partial u}.$$

Therefore

$$G = u^{-3}u_{xx}\frac{\partial \Phi}{\partial u} + g(v, u, u_x). \tag{32}$$

The general procedure to further determine the integrable equations is comprised by steps analogous to the one outlined above. In each step, one of the integrability conditions (20) firstly allows the form of F to be refined. Secondly, compatibility condition (31) is used to refine the form of G and Φ.

In the first step of our procedure, we used the first canonical density $\rho_1 = u$. In the second step, we are going to consider the second canonical density $\rho_2 = u^3(\partial F/\partial u_{xx})$. But first, it is useful to derive some formulae concerning the form of a general conservation law (26)–(28).

The fact that the integrability conditions have the form of a conservation law leads us to study this notion more deeply. A local conservation law for the associated system (26)–(28) is an expression of the form

$$(\rho(v, u, u_x, \ldots))_t = (\sigma(v, u, u_x, \ldots))_x \tag{33}$$

which must hold over all the solutions of (26)–(28). Or, equivalently, (33) must be an identity in the variables $v, u, u_x, u_{xx}, \ldots$, when we eliminate all derivatives with respect to t using (27)–(28). The functions ρ and σ are called the density and flux of the conservation law, respectively. From a conservation law $(\rho_1)_t = (\sigma_1)_x$, we obtain another conservation law $(\rho_2)_t = (\sigma_2)_x$ putting $\rho_2 = \rho_1 + \phi_x$ and $\sigma_2 = \sigma_1 + \phi_t$, where ϕ is an arbitrary function $\phi(v, u, u_x, \ldots)$. We will say that these conservation laws are equivalent and write

$$\rho_1 \sim \rho_2. \tag{34}$$

It is easily verified that any conserved density is equivalent to one of the form $\rho = R(v, u)$ or one of the form

$$\rho = R(v, u, u_x, \ldots, u_{(m)}) \qquad \frac{\partial^2 R}{\partial u_{(m)}^2} \neq 0 \tag{35}$$

where, for brevity, we denote by $u_{(m)}$ the derivative $u_{\underbrace{x\ldots x}_{m\,\text{times}}}$. The number m is called the order of the conserved density.

The usual way of eliminating the function σ from a conservation law is to use the variational derivative [10]. The variational derivative of a total x-derivative is zero and the conservation law can be written only in terms of ρ. However, here we prefer a more straightforward approach to restrict the form of a conserved density of (26)–(28).

Let R be a function as in (35). Then $(R)_t$ must be a total x-derivative. After a short calculation, we obtain that $(R)_t$ depends linearly on $u_{(m+3)}$, but we can subtract some total derivative of a function of order $m+2$ to get

$$(R)_t \sim \frac{1}{2}\frac{u_{(m+1)}^3}{u^3}\frac{\partial^3 R}{\partial u_{(m)}^3} + r_2 u_{(m+1)}^2 + r_1 u_{(m+1)} + r_0 \tag{36}$$

where $r_i = r_i(v, u, u_x, \ldots, u_{(m)})$. The right-hand side of (36) must be linear in $u_{(m+1)}$, so we have $\partial^3 R/\partial u_{(m)}^3 = 0$, i.e. $R = Au_{(m)}^2 + Bu_{(m)} + C$. Calculating again

$$(R)_t \sim \left(3(2m-1)u^{-4}Au_x + 3u^{-3}A_x - 2A\frac{\partial f}{\partial u_x}\right)u_{(m+1)}^2 + r_1 u_{(m+1)} + r_0. \tag{37}$$

This formula implies a relationship between $A(v, u, u_x, \ldots, u_{(m-1)})$ and the right-hand side of (30)

$$\frac{\partial f}{\partial u_x} = \tfrac{3}{2}(2m-1)u^{-4}u_x + \tfrac{3}{2}u^{-3}A^{-1}A_x. \tag{38}$$

Now we can proceed with the second step in the procedure, calculating the integrability conditions (20). The second canonical density is

$$\rho_2 = u^3 \frac{\partial F}{\partial u_{xx}} = (-3\ln u)_x + u^3 \frac{\partial f}{\partial u_x} \tag{39}$$

so ρ_2 is of order $m = 1$ whenever $(\partial^3 f/\partial u_x^3) \neq 0$. In this case, relation (38) with $m = 1$ shows that $u^3(\partial f/\partial u_x)$ is a total x-derivative so we must have $(\partial^3 f/\partial u_x^3) = 0$, arriving at a contradiction. Hence, f is quadratic in u_x. We will use the notation

$$f = \frac{\partial p}{\partial u} u^{-3} u_x^2 + (2\Phi(p) + E) u^{-3} u_x + q \tag{40}$$

with $p = p(v, u)$, $q = q(v, u)$ and $E = E(v, u)$. This notation is convenient since the second canonical density (39) is, up to a total x-derivative, equal to E.

In the third step of the procedure, the third integrability condition (23), in conjunction with the compatibility conditions, allow the right-hand sides of equations (27) and (28) to be stated precisely as follows

$$u_t = \left(u^{-3}u_{xx} - \tfrac{3}{2}u^{-4}u_x^2 - \tfrac{3}{4}P^{-1}\frac{\partial P}{\partial u}u^{-3}u_x^2 - \tfrac{3}{2}P^{-1}\Phi(P)u^{-3}u_x + Eu^{-3}u_x + q(v, u) \right)_x \tag{41}$$

$$\begin{aligned} v_t = {} & \left(u^{-3}u_{xx} - \tfrac{3}{2}u^{-4}u_x^2 - \tfrac{3}{4}P^{-1}\frac{\partial P}{\partial u}u^{-3}u_x^2 - \tfrac{3}{2}P^{-1}\Phi(P)u^{-3}u_x + E\, u^{-3}u_x \right) \frac{\partial \Phi}{\partial u} \\ & - \tfrac{1}{2}u^{-3}u_x^2 \frac{\partial^2 \Phi}{\partial u^2} - u^{-3}u_x \left[\Phi, \frac{\partial \Phi}{\partial u} \right] + r(v, u) \end{aligned} \tag{42}$$

where

$$P = \alpha(v)u^2 + \beta(v)u + \gamma \qquad \gamma = \text{constant} \tag{43}$$

is a quadratic polynomial, defined up to a multiplicative constant.

Let us make some remarks about the function $E(v, u)$. If there exists some conserved density of order $m \geqslant 1$ then we see from (38)–(40) that, without losing generality, we can take $E = 0$. From the integrability condition of ρ_2, we obtain

$$\frac{\partial^3 E}{\partial u^3} + \tfrac{3}{2}P^{-1}\frac{\partial P}{\partial u}\frac{\partial^2 E}{\partial u^2} = 0. \tag{44}$$

From the integrability condition of ρ_3 and (37), we conclude that E and the polynomial P are connected by

$$\gamma E = 0. \tag{45}$$

The dependence of Φ on u can also be found from the compatibility condition (31). It is determined by the relations

$$\frac{\partial^3 \Phi}{\partial u^3} + \tfrac{3}{2}P^{-1}\frac{\partial P}{\partial u}\frac{\partial^2 \Phi}{\partial u^2} = 0 \tag{46}$$

$$\left[\Phi, \frac{\partial^2 \Phi}{\partial u^2} \right] + \left(\tfrac{1}{2}P^{-1}\frac{\partial P}{\partial u} - u^{-1} \right) \left[\Phi, \frac{\partial \Phi}{\partial u} \right] + \left(P^{-1}\Phi(P) - \tfrac{2}{3}E \right) \frac{\partial^2 \Phi}{\partial u^2} = 0. \tag{47}$$

The preliminary classification is performed by studying the different solutions of equations (46) and (47) which lead to different classes of integrable equations.

3. Preliminary classification

To start the classification, the easiest way is to solve (46). In the generic case, the solution is $\Phi = (P^{1/2})\Phi_2 + u\Phi_1 + \Phi_0$. In the degenerate case $P = (au+b)^2$ we obtain two more types of solution corresponding to $a = 0$ and $a \neq 0$. Therefore, we obtain three types of solutions of (46) related to the different dependences on u of Φ:

$$\Phi = u^2\Phi_2(v) + u\Phi_1(v) + \Phi_0(v) \tag{48}$$

$$\Phi = (a(v)u+b)^{-1}\Phi_2(v) + u\Phi_1(v) + \Phi_0(v) \tag{49}$$

$$\Phi = (\alpha(v)u^2 + \beta(v)u + \gamma)^{1/2}\Phi_2(v) + u\Phi_1(v) + \Phi_0(v). \tag{50}$$

Let us first study the case (48). Here, $P = \text{constant} = \gamma \neq 0$. Then, according to (45), $E = 0$. Substituting Φ in (47) and equating the coefficients of the different powers of u to zero, we obtain the conditions

$$[\Phi_1, \Phi_2] = 0 \qquad [\Phi_0, \Phi_1] = 0. \tag{51}$$

In order to investigate these conditions, we must take into account the structure of the vector field Φ in (19). Since $v_1 = w$ and $v_2 = w_x, \ldots$, this vector field has the form

$$\Phi = \frac{\partial}{\partial x} + \sum_{i=1}^{n-1} v_{i+1}\frac{\partial}{\partial v_i} + R(v,u)\frac{\partial}{\partial v_n}. \tag{52}$$

Hence, vector fields Φ_i are of the form

$$\Phi_0 = \frac{\partial}{\partial x} + \sum_{i=1}^{n-1} v_{i+1}\frac{\partial}{\partial v_i} + R_0(v)\frac{\partial}{\partial v_n} \tag{53}$$

$$\Phi_1 = R_1(v)\frac{\partial}{\partial v_n} \tag{54}$$

$$\Phi_2 = R_2(v)\frac{\partial}{\partial v_n}. \tag{55}$$

If $[\Phi_0, \Phi_1] = 0$ then the explicit form of the vectors requires either $n < 2$ or $\Phi_1 = 0$. If $n < 2$ we have a quasilinear equation of type (14) or (15). The case $\Phi_1 = 0$ corresponds to the generic case

$$\Phi = u^2\Phi_2(v) + \Phi_0(v) \qquad P = 1 \qquad E = 0 \tag{56}$$

to which example 1 belongs.

Analogously, studying the remaining types of solutions (49) and (50), we obtain some cases with $n < 2$ and two additional generic cases

$$\Phi = u\Phi_1(v) + \Phi_0(v) \qquad P = u^2\theta(v) \qquad E = E(v) \tag{57}$$

$$\Phi = P^{1/2}\Phi_2(v) + \Phi_0(v) \qquad P = u^2\theta(v) - 1 \qquad E = 0. \tag{58}$$

We must note that in these three generic cases, any value for $n \leqslant 3$ is possible.

Now we show some examples with equations of the quasilinear types (14)–(16) and fully nonlinear equations corresponding to the three generic cases. In all of them, $q = 0$ and $r = 0$.

Examples of the generic case (56). The next equations are examples belonging to case (56) and, for all of them, the t-evolution of u, given by (41), is $u_t = \left(u^{-3}u_{xx} - \frac{3}{2}u^{-4}u_x^2\right)_x$.

$$w_t = \left(w^{-3}w_{xx} - \tfrac{3}{2}w^{-4}w_x^2\right)_x$$

$$w_t = w_x^{-3/2}w_{xxx} - \tfrac{3}{2}w_x^{-5/2}w_{xx}^2$$

$$w_t = w_{xx}^{-3/2}w_{xxx}$$

$$w_t = -2w_{xxx}^{-1/2}.$$

Examples of the generic case (57). In these examples $P = u^2$, $E = 0$ and (41) is $u_t = \left(u^{-3}u_{xx} - \frac{3}{2}u^{-4}u_x^2\right)_x$.

$$w_t = \left(w^{-3}w_{xx} - 3w^{-4}w_x^2\right)_x$$

$$w_t = w_x^{-3}w_{xxx} - 3w_x^{-4}w_{xx}^2 \tag{59}$$

$$w_t = w_{xx}^{-3}w_{xxx} \tag{60}$$

$$w_t = -\tfrac{1}{2}w_{xxx}^{-2}.$$

Equations (59) and (60) can be transformed using a contact transformation to the linear equation $w_t = w_{xxx}$.

Examples of the generic case (58).

$$w_t = \left(w^{-3}w_{xx} - \tfrac{3}{2}w^{-4}(w^2-1)^{-1}(2w^2-1)w_x^2\right)_x$$

$$w_t = (w_x^2+1)^{-3/2}w_{xxx} - 3(w_x^2+1)^{-5/2}w_{xx}^2$$

$$w_t = (w_{xx}^2+1)^{-3/2}w_{xxx} - 3(w_{xx}^2+1)^{-5/2}w_{xx}^2$$

$$w_t = (w_{xxx}^2+1)^{-3/2}w_{xxx}.$$

Here $P = u^2 - 1$ and (41) is $u_t = \left(u^{-3}u_{xx}\frac{3}{2}u^{-4}(u^2-1)^{-1}(2u^2-1)u_x^2\right)_x$.

Besides the three generic cases, there are several special quasilinear cases with $n < 2$. It is easy to see that any equation of the form (14), (15) and (17) can be reduced by a contact transformation to the form (16) and then, without loss of generality, it can be thought that any integrable equation of the form (1) belongs to one of the generic cases (56)–(58).

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Appendix. Non-standard dynamical variables

In this appendix we will use the notation

$$w_i = \frac{\partial^i w}{\partial x^i}$$

to stress the fact that in the algebraic study of an evolution equation

$$w_t = H\left(x, w, \frac{\partial w}{\partial x}, \dots, \frac{\partial^k w}{\partial x^k}\right) \tag{61}$$

these partial derivatives are regarded as independent variables (cf (33)). We will call the set $x, w, w_1, \dots, w_i, \dots$ the standard set of independent variables for (61). In terms of these variables, the evolution equation (61) is a pair of compatible infinite dynamical systems of equations

$$(x)_x = 1 \qquad (w_i)_x = w_{i+1} \tag{62}$$

$$(x)_t = 0 \qquad (w_i)_t = D^i(H(x, w, w_1, \dots, w_k)) \qquad i = 0, 1, \dots \tag{63}$$

where

$$D = \frac{\partial}{\partial x} + \sum_{i=0}^{\infty} w_{i+1}\frac{\partial}{\partial w_i}$$

is the total derivative with respect to x.

Formally, the transition from evolution equation (61) to the associated system is a change of variables in the systems (62)–(63), namely to the new set of independent variables

$$x, w, w_1, \dots, w_{n-1}, u, u_1, \dots \tag{64}$$

where

$$u = Q(x, w, w_1, \dots, w_n) \qquad \frac{\partial Q}{\partial w_n} \neq 0. \tag{65}$$

It is clear that $u_i = D^i(Q)$. This change from standard to non-standard variables is invertible. In fact, from equation (65), w_n can be expressed in the variables $x, w, w_1, \dots, w_{n-1}, u$:

$$w_n = R(x, w, w_1, \dots, w_{n-1}, u). \tag{66}$$

In order to express w_{n+1} in non-standard variables $x, w, w_1, \dots, w_{n-1}, u, u_1$, we have to apply the total derivative operator D to (66):

$$w_{n+1} = \frac{\partial R}{\partial x} + w_1\frac{\partial R}{\partial w} + \dots + w_{n-1}\frac{\partial R}{\partial w_{n-2}} + w_n\frac{\partial R}{\partial w_{n-1}} + u_1\frac{\partial R}{\partial u} \tag{67}$$

replacing w_n by R. We can express the variables w_i with $i > n+1$ analogously

$$w_{n+l} = \bar{D}^l(R) \tag{68}$$

where

$$\bar{D} = \frac{\partial}{\partial x} + \sum_{i=0}^{n-2} w_{i+1}\frac{\partial}{\partial w_i} + R\frac{\partial}{\partial w_{n-1}} + \sum_{i=0}^{\infty} u_{i+1}\frac{\partial}{\partial u_i} \tag{69}$$

is the total derivative with respect to x in the non-standard variables. The map between standard and non-standard variables is now completely described and routine calculations allow all objects such as vector fields, symmetries, conservation laws, etc to be rewritten. It is, in fact, an algorithmic procedure which can be implemented in computer algebra systems. We have implemented this on the MATHEMATICA package, which was used to perform the calculations involved in this paper.

It is easy to see that the infinite system (62)–(63) is thus rewritten in non-standard variables as

$$(x)_t = 0 \tag{70}$$

$$(w_i)_t = \bar{D}^i(G(x, w, w_1, \ldots, w_{n-1}, u, \ldots, u_{k-n})) \qquad i = 0, \ldots, n-1 \tag{71}$$

$$(u_j)_t = \bar{D}^j(F(x, w, w_1, \ldots, w_{n-1}, u, \ldots, u_k)) \qquad j = 0, \ldots. \tag{72}$$

To find the explicit form of G we replace $w_n, w_{n+1}, \ldots, w_k$ in the right-hand side of (61) using (68). To obtain F we can differentiate equation (65) with respect to t

$$\begin{aligned} u_t &= \frac{\partial Q}{\partial w_n}(w_n)_t + \text{lower terms} = \frac{\partial Q}{\partial w_n} D^n H + \text{lower terms} \\ &= \frac{\partial Q}{\partial w_n}\frac{\partial H}{\partial w_k} w_{k+n} + \text{lower terms} \end{aligned}$$

and differentiating (65) i times with respect to x

$$u_i = \frac{\partial Q}{\partial w_n} w_{i+n} + \text{ lower terms.}$$

Therefore, $w_{i+n} = (\partial Q/\partial w_n)^{-1} u_i + \cdots$ and

$$F = \frac{\partial H}{\partial w_k} u_k + \text{lower terms.} \tag{73}$$

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