Advances in Nonlinear Waves and Symbolic Computation

Edited By

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Direct Methods and Symbolic Software for Conservation Laws of Nonlinear Equations

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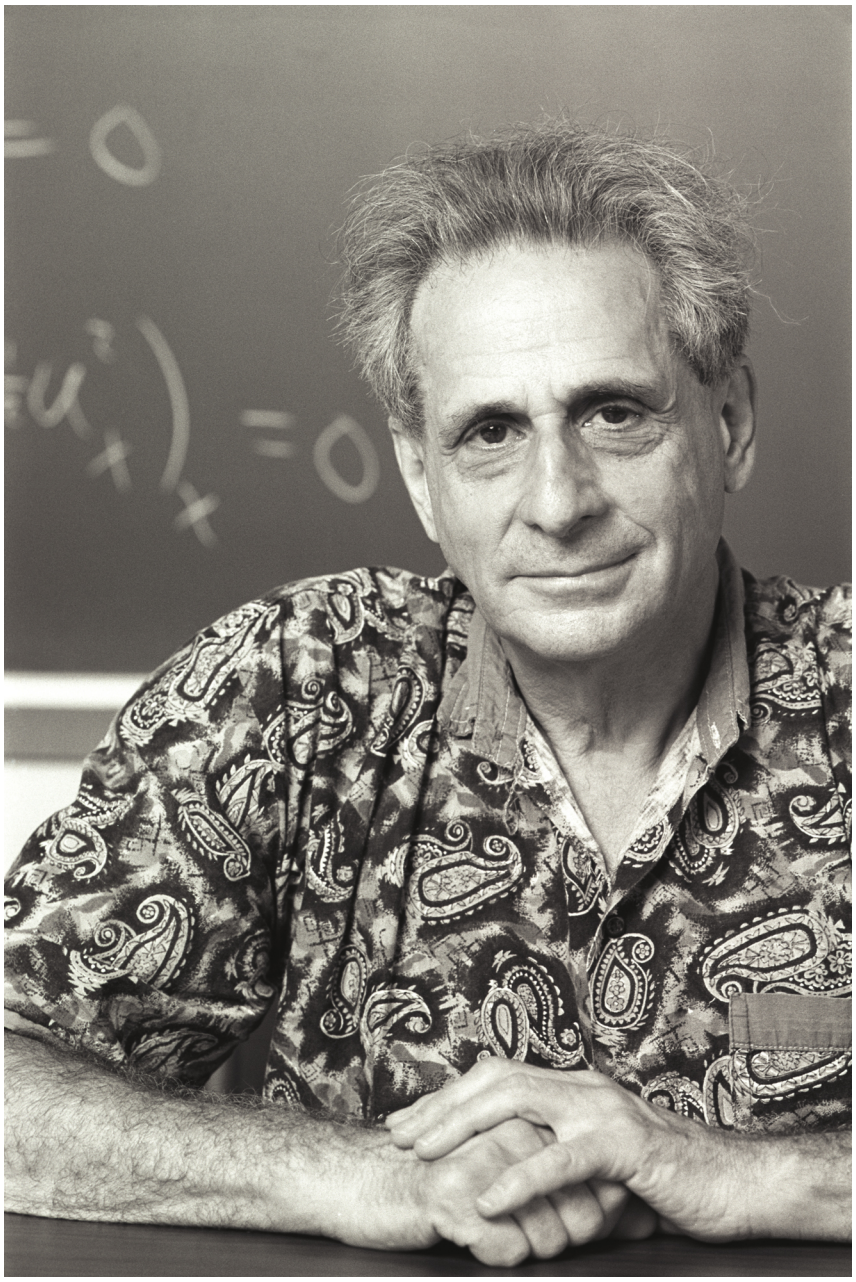

IN MEMORY OF MARTIN D. KRUSKAL (1925-2006)

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Abstract

We present direct methods, algorithms, and symbolic software for the computation of conservation laws of nonlinear partial differential equations (PDEs) and differential-difference equations (DDEs).

Our method for PDEs is based on calculus, linear algebra, and variational calculus. First, we compute the dilation symmetries of the given nonlinear system. Next, we build a candidate density as a linear combination with undetermined coefficients of terms that are scaling invariant. The variational derivative (Euler operator) is used to derive a linear system for the undetermined coefficients. This system is then analyzed and solved. Finally, we compute the flux with the homotopy operator.

The method is applied to nonlinear PDEs in $(1+1)$ dimensions with polynomial nonlinearities which include the Korteweg-de Vries (KdV), Boussinesq, and Drinfel'd-Sokolov-Wilson equations. An adaptation of the method is applied to PDEs with transcendental nonlinearities. Examples include the sine-Gordon, sinh-Gordon, and Liouville equations. For equations in laboratory coordinates, the coefficients of the candidate density are undetermined functions which must satisfy a mixed linear system of algebraic and ordinary differential equations.

For the computation of conservation laws of nonlinear DDEs we use a splitting of the identity operator. This method is more efficient that an approach based on the discrete Euler and homotopy operators. We apply the method of undetermined coefficients to the Kac-van Moerbeke, Toda, and Ablowitz-Ladik lattices. To overcome the shortcomings of the undetermined coefficient technique, we designed a new method that first calculates the leading order term and then the required terms of lower order. That method, which is no longer restricted to polynomial conservation laws, is applied to discretizations of the KdV and modified KdV equations, and a combination thereof. Additional examples include lattices due to Bogoyavlenskii, Belov-Chaltikian, and Blaszak-Marciniak.

The undetermined coefficient methods for PDEs and DDEs have been implemented in `Mathematica`. The code `TransPDEDensityFlux.m` computes densities and fluxes of systems of PDEs with or without transcendental nonlinearities. The code `DDEDensityFlux.m` does the same for polynomial nonlinear DDEs. Starting from the leading order terms, the new `Maple` library `discrete` computes densities and fluxes of nonlinear DDEs.

The software can be used to answer integrability questions and to gain insight in the physical and mathematical properties of nonlinear models. When applied to nonlinear systems with parameters, the software computes the conditions on the parameters for conservation laws to exist. The existence of a hierarchy of conservation laws is a predictor for complete integrability of the system and its solvability with the Inverse Scattering Transform.

1.1 Introduction

This chapter focuses on symbolic methods to compute polynomial conservation laws of partial differential equations (PDEs) in $(1+1)$ dimensions and differential-difference equations (DDEs), which are semi-discrete lattices. For the latter we treat systems where time is continuous and the spatial variable has been discretized.

Nonlinear PDEs that admit conservation laws arise in many disciplines of the applied sciences including physical chemistry, fluid mechanics, particle and quantum physics, plasma physics, elasticity, gas dynamics, electromagnetism, magneto-hydro-dynamics, nonlinear optics, and the bio-sciences. Conservation laws are fundamental laws of physics that maintain that a certain quantity will not change in time during physical processes. Familiar conservation laws include conservation of momentum, mass (matter), electric charge, or energy. The continuity equation of electromagnetic theory is an example of a conservation law which relates charge to current. In fluid dynamics, the continuity equation expresses conservation of mass, and in quantum mechanics the conservation of probability of the density and flux functions also yields a continuity equation.

There are many reasons to compute conserved densities and fluxes of PDEs explicitly. Invariants often lead to new discoveries as was the case in soliton theory. One may want to verify if conserved quantities of physical importance (e.g. momentum, energy, Hamiltonians, entropy, density, charge) are intact after constitutive relations have been added to close a system. For PDEs with arbitrary parameters one may wish to compute conditions on the parameters so that the model admits conserved quantities. Conserved densities also facilitate the study of qualitative properties of PDEs [86, 97], such as recursion operators, bi- or tri-Hamiltonian structures, and the like. They often guide the choice of solution methods or reveal the nature of special solutions. For example, an infinite sequence of conserved densities is a predictor of the existence of solitons [7] and complete integrability [2] which means that the PDE can be solved with the Inverse Scattering Transform (IST) method [2].

Conserved densities aid in the design of numerical solvers for PDEs [87, 88] and their stability analysis (see references in [23]). Indeed, semi-discretizations that conserve discrete conserved quantities lead to *stable* numerical schemes that are free of nonlinear instabilities and blowup. While solving DDEs, which arise in nonlinear networks and as semi-discretizations of PDEs, one should check that their conserved quantities indeed remain unchanged as time steps are taken.

Computer algebra systems (CAS) like `Mathematica`, `Maple`, and `REDUCE`, can greatly assist the computation of conservation laws of nonlinear PDEs and DDEs. Using CAS interactively, one can make a judicious guess (ansatz) and find a few simple densities and fluxes. Yet, that approach is fruitless for complicated systems with nontrivial conservation laws with increasing complexity. Furthermore, completely integrable equations PDEs [2, 7, 74, 89] and DDEs [10, 75] admit infinitely many independent conservation laws. Computing them is a challenging task. It involves tedious computations which are prone to

error if done with pen and paper. Kruskal and collaborators demonstrated the complexities of calculating conservation laws in their seminal papers [67, 78, 79] on the Korteweg-de Vries (KdV) equation from soliton theory [2,7,30]. We use this historical example to introduce the method of undetermined coefficients.

In the first part of this chapter we cover the symbolic computation of conservation laws of completely integrable PDEs in $(1+1)$ dimensions (with independent variables x and t). Our approach [8, 38, 51, 53] uses the concept of dilation (scaling) invariance and the method of undetermined coefficients. Our method proceeds as follows. First, build a candidate density as a linear combination (with undetermined coefficients) of "building blocks" that are homogeneous under the scaling symmetry of the PDE. If no such symmetry exists, construct one by introducing parameters with scaling. Next, use the Euler operator (variational derivative) to derive a linear algebraic system for the undetermined coefficients. After the system is analyzed and solved, use the homotopy operator to compute the flux. When applied to systems with parameters, our codes can determine the conditions on the parameters so that a sequence of conserved densities exists.

The method is applied to nonlinear PDEs in $(1+1)$ dimensions with polynomial terms which include the KdV, Boussinesq, and Drinfel'd-Sokolov-Wilson equations. An adaptation of the method is applied to PDEs with transcendental nonlinearities. Examples include the sine-Gordon, sinh-Gordon, and Liouville equations. For equations written in laboratory coordinates, the coefficients of the candidate density are undetermined functions which must satisfy a mixed linear system of algebraic and ordinary differential equations (ODEs).

Capitalizing on the analogy between PDEs and DDEs, the second part of this chapter deals with the symbolic computation of conservation laws of nonlinear DDEs [33, 39, 42, 51,54,57]. Again, we use scaling symmetries and the method of undetermined coefficients. One could use discrete versions of the Euler operator (to verify exactness) and the homotopy operator (to invert the forward difference). Although these operators might be valuable in theory, they are highly inefficient as tools for the symbolic computation of conservation laws of DDEs. We advocate the use of a "splitting and shifting" technique, which allows us to compute densities and fluxes simultaneously at minimal cost. The undetermined coefficient method for DDEs is illustrated with the Kac-van Moerbeke, Toda, and Ablowitz-Ladik lattices.

There is a fundamental difference between the continuous and discrete cases in the way densities are constructed. The total derivative has a weight whereas the shift operator does not. Consequently, a density of a PDE is bounded in order with respect to the space variable. Unfortunately, there is no *a priori* bound on the number of shifts in the density, unless a leading order analysis is carried out. To overcome this difficulty and other shortcomings of the undetermined coefficient method, we present a new method to compute conserved densities of DDEs. That method no longer uses dilation invariance and is no longer restricted to polynomial conservation laws. Instead of building a candidate density with undetermined coefficients, one first computes the leading order term in the density

and, secondly, generates the required terms of lower order. The method is fast and efficient since unnecessary terms are never computed. The new method is illustrated using a modified Volterra lattice as an example. The new method performs exceedingly well when applied to lattices due to Bogoyavlenskii, Belov-Chaltikian, and Blaszak-Marciniak. The new method is also applied to completely integrable discretizations of the KdV and modified KdV (mKdV) equations, and a combination thereof, known as the Gardner equation. Starting from a discretized eigenvalue problem, we first derive the Gardner lattice and then compute conservation laws.

There are several methods (see [51]) to compute conservation laws of nonlinear PDEs and DDEs. Some methods use a generating function [2,7], which requires the knowledge of key pieces of the IST. Another common approach uses the link between conservation laws and symmetries as stated in Noether's theorem [14, 15, 66, 81]. However, the computation of generalized (variational) symmetries, though algorithmic, is as daunting a task as the direct computation of conservation laws. Most of the more algorithmic methods [12, 13, 20, 25, 63, 101], require the solution of a determining system of ODEs or PDEs. Despite their power, only a few of these methods have been implemented in CAS. We devote a section to symbolic software for the computation of conservation laws. Additional reviews can be found in [38, 51, 101].

Over the past decade, in collaboration with students and researchers, we have designed and implemented direct algorithms for the computation of conservation laws of nonlinear PDEs and DDEs. We purposely avoid Noether's theorem, pre-knowledge of symmetries, and a Lagrangian formulation. Neither do we use differential forms or advanced differential-geometric tools. Instead, we concentrate on the undetermined coefficient method for PDEs and DDEs, which uses tools from calculus, linear algebra, and the variational calculus. Therefore, the method is easy to implement in `Mathematica` and easy to use by scientists and engineers. The code `TransPDEDensityFlux.m` computes densities and fluxes of systems of PDEs with or without transcendental nonlinearities. The code `DDEDensityFlux.m` does the same for polynomial nonlinear DDEs. Starting from the leading order terms, the new `Maple` library `discrete` computes densities and fluxes of nonlinear DDEs very efficiently. The software can thus be used to answer integrability questions and to gain insight in the physical and mathematical properties of nonlinear models.

Our software is in the public domain. The `Mathematica` packages and notebooks are available at [48] and Hickman's code in `Maple` is available at [56]. We are currently working on a comprehensive package to compute conservation laws of PDEs in multiple space dimensions [45, 51, 83].

Part I: Partial Differential Equations in $(1 + 1)$ Dimensions

In this first part we cover PDEs in $(1 + 1)$ dimensions, that is, PDEs in one space variable and time. Starting from a historical example, we introduce the concept of dilation invariance

and use the method of undetermined coefficients to compute conservation laws of evolution equations. Later on, we adapt the method of undetermined coefficients to cover PDEs with transcendental terms.

1.2 The Most Famous Example in Historical Perspective

The story of conservation laws for nonlinear PDEs begins with the discovery of an infinite number of conservation laws of the ubiquitous Korteweg-de Vries equation which models a variety of nonlinear wave phenomena, including shallow water waves [46] and ion-acoustic waves in plasmas [2, 7, 30]. The KdV equation can be recast in dimensionless variables as

$$u_t + \alpha u u_x + u_{3x} = 0, \tag{1.1}$$

where the subscripts denote partial derivatives, i.e. $u_t = \frac{\partial u}{\partial t}, u_x = \frac{\partial u}{\partial x}$, and $u_{3x} = \frac{\partial^3 u}{\partial x^3}$. The parameter α can be scaled to any real number. Commonly used values are $\alpha = \pm 1$ or $\alpha = \pm 6$.

Equation (1.1) is an example of a scalar $(1+1)-$dimensional evolution equation,

$$u_t = F(x, t, u, u_x, u_{2x}, \cdots, u_{nx}), \tag{1.2}$$

of order n in the independent space variable x and of first order in time t. Obviously, the dependent variable is $u(x, t)$. If parameters are present in (1.2), they will be denoted by lower-case Greek letters. A *conservation law* of (1.2) is of the form

$$\mathsf{D}_t\, \rho + \mathsf{D}_x\, J = 0, \tag{1.3}$$

which is satisfied for all solutions $u(x, t)$ of the PDE. In physics, ρ is called the *conserved density* (or charge); J is the associated *flux* (or current). In general, both are differential functions (functionals), i.e. functions of x, t, u, and partial derivatives of u with respect to x. In (1.3), D_x denotes the total derivative with respect to x, that is,

$$\mathsf{D}_x J = \frac{\partial J}{\partial x} + \sum_{k=0}^{N} \frac{\partial J}{\partial u_{kx}} u_{(k+1)x}, \tag{1.4}$$

where N is the order of J, and D_t is the total derivative with respect to t, defined by

$$\mathsf{D}_t\, \rho = \frac{\partial \rho}{\partial t} + \rho'[u_t] = \frac{\partial \rho}{\partial t} + \sum_{k=0}^{M} \frac{\partial \rho}{\partial u_{kx}} \mathsf{D}_x^k u_t, \tag{1.5}$$

where $\rho'[u_t]$ is the Fréchet derivative of ρ in the direction of u_t and M is the order of ρ.

The densities $\rho^{(1)} = u$ and $\rho^{(2)} = u^2$ of (1.1) were long known. In 1965, Whitham [98] had found a third density, $\rho^{(3)} = u^3 - \frac{3}{\alpha} u_x^2$, which, in the context of water waves,

corresponds to Boussinesq's moment of instability [76]. One can readily verify that

$$\mathsf{D}_t(u) + \mathsf{D}_x(\tfrac{1}{2}\alpha u^2 + u_{2x}) = 0, \tag{1.6}$$

$$\mathsf{D}_t(u^2) + \mathsf{D}_x(\tfrac{2}{3}\alpha u^3 - u_x^2 + 2uu_{2x}) = 0, \tag{1.7}$$

$$\mathsf{D}_t(u^3 - \tfrac{3}{\alpha}u_x^2) + \mathsf{D}_x(\tfrac{3}{4}\alpha u^4 - 6uu_x^2 + 3u^2u_{2x} + \tfrac{3}{\alpha}u_{2x}^2 - \tfrac{6}{\alpha}u_xu_{3x}) = 0. \tag{1.8}$$

Indeed, (1.6) is the KdV equation written as a conservation law; (1.7) is obtained after multiplying (1.1) by $2u$; (1.8) requires more work. Hence, the first three density-flux pairs of (1.1) are

$$\rho^{(1)} = u, \qquad J^{(1)} = \tfrac{1}{2}\alpha u^2 + u_{2x}, \tag{1.9}$$

$$\rho^{(2)} = u^2, \qquad J^{(2)} = \tfrac{2}{3}\alpha u^3 - u_x^2 + 2uu_{2x}, \tag{1.10}$$

$$\rho^{(3)} = u^3 - \tfrac{3}{\alpha}u_x^2, \quad J^{(3)} = \tfrac{3}{4}\alpha u^4 - 6uu_x^2 + 3u^2u_{2x} + \tfrac{3}{\alpha}u_{2x}^2 - \tfrac{6}{\alpha}u_xu_{3x}. \tag{1.11}$$

Integrals of motion readily follow from the densities. Indeed, assuming that J vanishes at infinity (for example due to sufficiently fast decay of u and its x derivatives), upon integration of (1.3) with respect to x one obtains that

$$P = \int_{-\infty}^{\infty} \rho \, dx \tag{1.12}$$

is *constant* in time. Such constants of motion also arise when u is periodic, in which case one integrates over the finite period. Depending on the physical setting, the first few constants of motion (i.e. integrals (1.12)) express conservation of mass, momentum, and energy.

Martin Kruskal and postdoctoral fellow Norman Zabusky discovered the fourth and fifth densities for the KdV equation [111]. However, they failed in finding a sixth conservation law due to an algebraic mistake in their computations. Kruskal asked Robert Miura, also postdoctoral fellow at the Princeton Plasma Physics Laboratory at New Jersey, to search for further conservation laws of the KdV equation. Miura [78] computed the seventh conservation law. After correcting the mistake mentioned before, he also found the sixth and eventually three additional conservation laws. Rumor [80] has it that in the summer of 1966 Miura went up into the Canadian Rockies and returned from the mountains with the first 10 conservation laws of the KdV equation engraved in his notebook. This biblical metaphor probably does not do justice to Miura's intense and tedious work with pen and paper.

With ten conservation laws in hand, it was conjectured that the KdV equation had an infinite sequence of conservation laws, later proven to be true [67, 79]. Aficionados of explicit formulas can find the first ten densities (and seven of the associated fluxes) in [79] and the eleventh density (with 45 terms) in [67], where a recursion formula is given to generate all further conserved densities. As an aside, in 1966 the first five conserved densities were computed on an IBM 7094 computer with FORMAC, an early CAS. The sixth density could no longer be computed because the available storage space was exceeded. In contrast, using

a method of undetermined coefficients, the first eleven densities were computed in 1969 on a AEC CDC-6600 computer in a record time of 2.2 seconds. Due to limitations in handling large integers, the computer could not correctly produce any further densities.

Undoubtedly, the discovery of conservation laws played a pivotal role in the comprehensive study of the properties and solutions of nonlinear completely integrable PDEs (like the KdV equation) and the development of the IST (see e.g. [80] for the history). Clifford Gardner, John Greene, Martin Kruskal, and Robert Miura received the 2006 Leroy P. Steele Prize [115], awarded by the American Mathematical Society, for their seminal contribution to research on the KdV equation. In turn, Martin Kruskal has received numerous honors and awards [114] for his fundamental contributions to the understanding of integrable systems and soliton theory. This chapter is dedicated to Martin Kruskal (1925-2006).

1.3 The Method of Undetermined Coefficients

We now sketch the method of undetermined coefficients to compute conservation laws [38, 96], which draws on ideas and observations in before mentioned work by Kruskal and collaborators.

1.3.1 Dilation Invariance of Nonlinear PDEs

Crucial to the computation of conservation laws is that (1.3) must hold on the PDE. This is achieved by substituting u_t (and u_{tx}, u_{txx}, etc.) from (1.1) in the evaluation of (1.6)-(1.8) and in all subsequent conservation laws of degree larger than 3. The elimination of all $t-$derivatives of u in favor of x derivatives has two important consequences: (i) any symmetry of the PDE, in particular, the dilation symmetry, will be adopted by the conservation law, (ii) once D_t is computed and evaluated on the PDE, t becomes a parameter in the computation of the flux.

We will first investigate the dilation (scaling) symmetry of evolution equations. The KdV equation is *dilation invariant* under the scaling symmetry

$$(t, x, u) \to (\lambda^{-3}t, \lambda^{-1}x, \lambda^2 u), \tag{1.13}$$

where λ is an arbitrary parameter. Indeed, after a change of variables with $\tilde{t} = \lambda^{-3}t, \tilde{x} = \lambda^{-1}x, \tilde{u} = \lambda^2 u$, and cancellation of a common factor λ^5, the KdV for $\tilde{u}(\tilde{x}, \tilde{t})$ arises. The dilation symmetry of (1.1) can be expressed as

$$u \sim \frac{\partial^2}{\partial x^2}, \qquad \frac{\partial}{\partial t} \sim \frac{\partial^3}{\partial x^3}, \tag{1.14}$$

which means that u corresponds to two $x-$derivatives and the time derivative corresponds to three $x-$derivatives. If we define the *weight*, W, of a variable (or operator) as the exponent of λ in (1.13), then $W(x) = -1$ or $W(\frac{\partial}{\partial x}) = 1; W(t) = -3$ or $W(\frac{\partial}{\partial t}) = 3$, and $W(u) = 2$.

All weights of dependent variables and the weights of $\partial/\partial x, \partial/\partial t$, are assumed to be non-negative and rational. The *rank* of a monomial is defined as the total weight of the monomial. Such monomials may involve the independent and dependent variables and the operators $\frac{\partial}{\partial x}, \mathsf{D}_x, \frac{\partial}{\partial t}$, and D_t. Ranks must be positive integers or positive rational numbers.

An expression (or equation) is *uniform in rank* if its monomial terms have equal rank. For example, (1.1) is uniform in rank since each of the three terms has rank 5.

Conversely, if one does not know the dilation symmetry of (1.1), then it can be readily computed by requiring that (1.1) is uniform in rank. Indeed, setting $W(\partial/\partial x) = 1$ and equating the ranks of the three terms in (1.1) gives

$$W(u) + W(\frac{\partial}{\partial t}) = 2W(u) + 1 = W(u) + 3, \tag{1.15}$$

which yields $W(u) = 2, W(\partial/\partial t) = 3$, and, in turn, confirms (1.13). So, requiring *uniformity in rank* of a PDE allows one to compute the weights of the variables (and thus the scaling symmetry) with linear algebra.

Dilation symmetries, which are special Lie-point symmetries, are common to many nonlinear PDEs. Needless to say, not every PDE is dilation invariant, but non-uniform PDEs can be made uniform by extending the set of dependent variables with auxiliary parameters with appropriate weights. Upon completion of the computations one can set these parameters to one. In what follows, we set $W(\partial/\partial x) = W(\mathsf{D}_x) = 1$ and $W(\partial/\partial t) = W(\mathsf{D}_t)$. Applied to (1.6), $\mathrm{rank}\,\rho^{(1)} = 2$, $\mathrm{rank}\,J^{(1)} = 4$. Hence, $\mathrm{rank}\,(\mathsf{D}_t\,\rho^{(1)}) = \mathrm{rank}\,(\mathsf{D}_x\,J^{(1)}) = 5$. Therefore, (1.6) is uniform of rank 5. In (1.7), $\mathrm{rank}\,\rho^{(2)} = 4$ and $\mathrm{rank}\,J^{(2)} = 6$, consequently, (1.7) is uniform of rank 7.

In (1.11), each term in $\rho^{(3)}$ has rank 6 and each term in $J^{(3)}$ has rank 8. Consequently, $\mathrm{rank}\,(\mathsf{D}_t\,\rho^{(3)}) = \mathrm{rank}\,(\mathsf{D}_x\,J^{(3)}) = 9$, which makes (1.8) is uniform of rank 9. All densities of (1.1) are uniform in rank and so are the associated fluxes and the conservation laws.

Equation (1.1) also has density-flux pairs that depend explicitly on t and x; for example,

$$\tilde{\rho} = tu^2 + \frac{2}{\alpha}xu, \quad \tilde{J} = t\left(\frac{2}{3}\alpha u^3 - u_x^2 + 2uu_{2x}\right) - x\left(u^2 - \frac{2}{\alpha}u_{2x}\right) + \frac{2}{\alpha}u_x. \tag{1.16}$$

Since $W(x) = -1$ and $W(t) = -3$, one has $\mathrm{rank}\,\tilde{\rho} = 1$, and $\mathrm{rank}\,\tilde{J} = 3$. The methods and algorithms discussed in subsequent sections have been adapted to compute densities and fluxes explicitly dependent on x and t. Instead of addressing this issue in this chapter, we refer the reader to [38, 53].

1.3.2 The Method of Undetermined Coefficients Applied to a Scalar Nonlinear PDE

We outline how densities and fluxes can be constructed for a scalar evolution equation (1.2). To keep matters transparent, we illustrate the steps for the KdV equation resulting in $\rho^{(3)}$ of rank $R = 6$ with associated flux $J^{(3)}$ of rank 8, both listed in (1.11). The tools needed for the computations will be presented in the next section.

• Select the rank R of ρ. Make a list, $\mathcal{R}$, of all monomials in u and its x-derivatives so that each monomial has rank R. This can be done as follows. Starting from the set $\mathcal{V}$ of dependent variables (including parameters with weight, when applicable), make a set $\mathcal{M}$ of all non-constant monomials of rank R or less (but without $x-$derivatives). Next, for each term in $\mathcal{M}$, introduce the right number of $x-$derivatives to adjust the rank of that term. Distribute the $x-$derivatives, strip off the numerical coefficients, and gather the resulting terms in a set $\mathcal{R}$. For the KdV equation and $R = 6$, $\mathcal{V} = \{u\}$ and $\mathcal{M} = \{u^3, u^2, u\}$. Since u^3, u^2, and u have ranks $6, 4$ and 2, respectively, one computes

$$\frac{\partial^0 u^3}{\partial x^0} = u^3, \quad \frac{\partial^2 u^2}{\partial x^2} = 2u_x^2 + 2uu_{2x}, \quad \frac{\partial^4 u}{\partial x^4} = u_{4x}. \tag{1.17}$$

Ignoring numerical coefficients in the right hand sides of the equations in (1.17), one gets $\mathcal{R} = \{u^3, u_x^2, uu_{2x}, u_{4x}\}$.

• Remove from $\mathcal{R}$ all monomials that are total $x-$derivatives. Also remove all "equivalent" monomials, i.e. the monomials that differ from another by a total $x-$derivative, keeping the monomial of lowest order. Call the resulting set $\mathcal{S}$. In our example, u_{4x} must be removed (because $u_{4x} = \mathsf{D}_x u_{3x}$) and uu_{2x} must be removed since uu_{2x} and u_x^2 are equivalent. Indeed, $uu_{2x} = \mathsf{D}_x(uu_x) - u_x^2$. Thus, $\mathcal{S} = \{u^3, u_x^2\}$.

• Linearly combine the monomials in $\mathcal{S}$ with constant undetermined coefficients c_i to obtain the candidate ρ. Continuing with the example,

$$\rho = c_1\, u^3 + c_2\, u_x^2, \tag{1.18}$$

which is of first order in x.

• Using (1.5), compute $\mathsf{D}_t\, \rho$. Applied to (1.18) where $M = 1$, one gets

$$\mathsf{D}_t\, \rho = (3c_1 u^2 \mathsf{I} + 2c_2 u_x \mathsf{D}_x)[u_t]. \tag{1.19}$$

As usual, $\mathsf{D}_x^0 = \mathsf{I}$ is the identity operator.

• Evaluate $-\mathsf{D}_t\, \rho$ on the PDE (1.2) by replacing u_t by F. The result is a differential function E in which t is a parameter. For the KdV equation (1.1), $F = -(\alpha u u_x + u_{3x})$. After reversing the sign, the evaluated form of (1.19) is

$$\begin{aligned} E &= (3c_1 u^2 \mathsf{I} + 2c_2 u_x \mathsf{D}_x)(\alpha u u_x + u_{3x}) \\ &= 3c_1 \alpha u^3 u_x + 2c_2 \alpha u_x^3 + 2c_2 \alpha u u_x u_{2x} + 3c_1 u^2 u_{3x} + 2c_2 u_x u_{4x}. \end{aligned} \tag{1.20}$$

• To obtain a conservation law, E must be a total derivative. Starting with highest orders, repeatedly integrate E by parts. Doing so, allows one to write E as the sum of a total $x-$derivative, $\mathsf{D}_x\, J$, and a non-integrable part (i.e. the obstructing terms). J is the (candi-

date) flux with $\text{rank}\, J = R + W(\mathsf{D}_t) - 1$. Integration by parts of (1.20) gives

$$\begin{aligned} E &= \mathsf{D}_x(\frac{3}{4}c_1\alpha u^4 + 3c_1u^2u_{2x} + c_2\alpha u u_x^2 + 2c_2u_xu_{3x}) - 6c_1uu_xu_{2x} \\ &\quad -2c_2u_{2x}u_{3x} + c_2\alpha u_x^3 \\ &= \mathsf{D}_x(\frac{3}{4}c_1\alpha u^4 - 3c_1uu_x^2 + c_2\alpha u u_x^2 + 3c_1u^2u_{2x} + 2c_2u_xu_{3x} - c_2u_{2x}^2) \\ &\quad +(3c_1 + c_2\alpha)u_x^3. \end{aligned} \tag{1.21}$$

The candidate flux therefore is

$$J = \frac{3}{4}c_1\alpha u^4 - 3c_1uu_x^2 + c_2\alpha u u_x^2 + 3c_1u^2u_{2x} + 2c_2u_xu_{3x} - c_2u_{2x}^2. \tag{1.22}$$

• Equate the coefficients of the obstructing terms to zero. Solve the linear system for the undetermined coefficients c_i. In the example $(3c_1 + c_2\alpha)u_x^3$ is the only obstructing term which vanishes for $c_2 = -\frac{3}{\alpha}c_1$, where c_1 is arbitrary.

• Substitute the coefficients c_i into ρ and J to obtain the final forms of the density and associated flux (with a common arbitrary factor which can be set to 1). Setting $c_1 = 1$ and substituting $c_2 = -\frac{3}{\alpha}$ into (1.18) and (1.22) yields $\rho^{(3)}$ and $J^{(3)}$ as given in (1.11).

Constructing "minimal" densities, i.e. densities which are free of equivalent terms and total derivatives terms, becomes challenging if the rank R is high. Furthermore, integration by parts is cumbersome and prone to mistakes if done by hand. Moreover, it would be advantageous if the integration by parts could be postponed until the undetermined coefficients c_i have been computed and substituted in E. Ideally, the computations of the density and the flux should be decoupled. There is a need for computational tools to address these issues, in particular, if one wants to compute conservation laws of systems of evolution equations.

1.4 Tools from the Calculus of Variations and Differential Geometry

A scalar differential function E of order M is called *exact* (integrable) if and only if there exists a scalar differential function J of order $M-1$ such that $E = \mathsf{D}_x J$. Obviously, $J = \mathsf{D}_x^{-1}E = \int E\, dx$ is then the primitive (or integral) of E. Two questions arise: (i) How can one test whether or not E is exact? (ii) If E is exact, how can one compute J without using standard integration by parts? To answer the first question we will use the variational derivative (Euler operator) from the calculus of variations. To perform integration by parts we will use the homotopy operator from differential geometry.

1.4.1 The Continuous Variational Derivative (Euler Operator)

The continuous *variational derivative*, also called the *Euler operator of order zero* , $\mathcal{L}^{(0)}_{u(x)}$, for variable $u(x)$ is defined [81] by

$$\begin{aligned}\mathcal{L}^{(0)}_{u(x)}E &= \sum_{k=0}^{M}(-\mathsf{D}_x)^k\frac{\partial E}{\partial u_{kx}}\\ &= \frac{\partial E}{\partial u}-\mathsf{D}_x\frac{\partial E}{\partial u_x}+\mathsf{D}_x^2\frac{\partial E}{\partial u_{2x}}-\mathsf{D}_x^3\frac{\partial E}{\partial u_{3x}}+\cdots+(-1)^M\mathsf{D}_x^M\frac{\partial E}{\partial u_{Mx}},\end{aligned}\tag{1.23}$$

where E is a differential function in $u(x)$ of order M.

A necessary and sufficient condition for a differential function E to be exact is that $\mathcal{L}^{(0)}_{u(x)}E\equiv 0$. A proof of this statement is given in e.g. [67]. If $\mathcal{L}^{(0)}_{u(x)}E\neq 0$, then E is not a total $x-$derivative due to obstructing terms.

Application 1. Returning to (1.20), we now use the variational derivative to determine c_1 and c_2 so that E of order $M=4$ will be exact. Using nothing but differentiations, we readily compute

$$\begin{aligned}\mathcal{L}^{(0)}_{u(x)}E &= \frac{\partial E}{\partial u}-\mathsf{D}_x\frac{\partial E}{\partial u_x}+\mathsf{D}_x^2\frac{\partial E}{\partial u_{2x}}-\mathsf{D}_x^3\frac{\partial E}{\partial u_{3x}}+\mathsf{D}_x^4\frac{\partial E}{\partial u_{4x}}\\ &= 9c_1\alpha u^2u_x+2c_2\alpha u_xu_{2x}+6c_1uu_{3x}-\mathsf{D}_x(3c_1\alpha u^3+6c_2\alpha u_x^2\\ &\quad +2c_2\alpha uu_{2x}+2c_2u_{4x})+\mathsf{D}_x^2(2c_2\alpha uu_x)-\mathsf{D}_x^3(3c_1u^2)+\mathsf{D}_x^4(2c_2u_x)\\ &= (9c_1\alpha u^2u_x+2c_2\alpha u_xu_{2x}+6c_1uu_{3x})-(9c_1\alpha u^2u_x+14c_2\alpha u_xu_{2x}\\ &\quad +2c_2\alpha uu_{3x}+2c_2u_{5x})+(6c_2\alpha u_xu_{2x}+2c_2\alpha uu_{3x})\\ &\quad -(18c_1u_xu_{2x}+6c_1uu_{3x})+(2c_2u_{5x})\\ &= -6(3c_1+c_2\alpha)u_xu_{2x}.\end{aligned}\tag{1.24}$$

Note that the terms in u^2u_x, uu_{3x}, and u_{5x} dropped out. Hence, requiring that $\mathcal{L}^{(0)}_{u(x)}E\equiv 0$ leads to $3c_1+c_2\alpha=0$. Substituting $c_1=1$, $c_2=-\frac{3}{\alpha}$, into (1.18) yields $\rho^{(3)}$ in (1.11).

Application 2. It is paramount that the candidate density is free of total $x-$derivatives and equivalent terms. If such terms were present, they could be moved into the flux J, and their coefficients c_i would be arbitrary. $\rho^{(1)}$ and $\rho^{(2)}$, are *equivalent* if and only if $\rho^{(1)}+k\rho^{(2)}=\mathsf{D}_x J$, for some J and non-zero scalar k. We write $\rho^{(1)}\equiv\rho^{(2)}$. Clearly ρ is equivalent to any non-zero multiple of itself and $\rho\equiv 0$ if and only if ρ is exact.

Instead of working with different densities, we investigate the equivalence of terms t_i in the same density. For example, returning to the set $\mathcal{R}=\{u^3,u_x^2,uu_{2x},u_{4x}\}$, terms $t_2=u_x^2$ and $t_3=uu_{2x}$ are equivalent because $t_3+t_2=uu_{2x}+u_x^2=\mathsf{D}_x(uu_x)$.

The variational derivative can be used to detect equivalent and exact terms. Indeed, note that $v_1=\mathcal{L}^{(0)}_{u(x)}(uu_{2x})=2u_{2x}$ and $v_2=\mathcal{L}^{(0)}_{u(x)}u_x^2=-2u_{2x}$ are linearly dependent. Also, for $t_4=u_{4x}=\mathsf{D}_xu_{3x}$ one gets $v_3=\mathcal{L}^{(0)}_{u(x)}u_{4x}=0$. To weed out the terms t_i in $\mathcal{R}$ that are

equivalent or total derivatives, it suffices to check the linear independence of their images v_i under the Euler operator.

One can optimize this procedure by starting from a set $\mathcal{R}$ where some of the equivalent and total derivatives terms have been removed *a priori*. Indeed, in view of (1.17), one can ignore the highest-order terms (typically the last terms) in each of the right hand sides. Therefore, $\mathcal{R} = \{u^3, u_x^2\}$ and, for this example, no further reduction would be necessary. Various algorithms are possible to construct minimal densities. Details are given in [38,51].

1.4.2 The Continuous Homotopy Operator

We now discuss the homotopy operator [12, 13, 52, 81] which will allow one to reduce the computation of $J = \mathsf{D}_x^{-1}E = \int E\,dx$ to a single integral with respect to an auxiliary variable denoted by λ (not to be confused with λ in Section 1.3.1). Hence, the homotopy operator circumvents integration by parts and reduces the inversion of D_x to a problem of single-variable calculus.

The *homotopy operator* [81, p. 372] for variable $u(x)$, acting on an exact expression E of order M, is given by

$$\mathcal{H}_{u(x)}E = \int_0^1 (I_u E)\,[\lambda u]\,\frac{d\lambda}{\lambda}, \tag{1.25}$$

where the integrand $I_u E$ is given by

$$I_u E = \sum_{k=1}^{M} \left(\sum_{i=0}^{k-1} u_{ix} (-\mathsf{D}_x)^{k-(i+1)} \right) \frac{\partial E}{\partial u_{kx}}. \tag{1.26}$$

In (1.25), $(I_u E)[\lambda u]$ means that in $I_u E$ one replaces $u \to \lambda u$, $u_x \to \lambda u_x$, etc. This is a special case of the homotopy, $\lambda(u^{(1)} - u^{(0)}) + u^{(0)}$, between two points, $u^{(0)} = (u^0, u_x^0, u_{2x}^0, \cdots, u_{Mx}^0)$ and $u^{(1)} = (u^1, u_x^1, u_{2x}^1, \cdots, u_{Mx}^1)$, in the jet space. For our purposes we set $u^{(0)} = (0, 0, \cdots, 0)$ and $u^{(1)} = (u, u_x, u_{2x}, \cdots, u_{Mx})$. Formula (1.26) is equivalent to the one in [52], which in turn is equivalent to the formula in terms of higher Euler operators [45, 51].

Given an exact differential function E of order M one has $J = \mathsf{D}_x^{-1}E = \int E\,dx = \mathcal{H}_{u(x)}E$. A proof of this statement can be found in [52].

Application. After substituting $c_1 = 1$ and $c_2 = -\frac{3}{\alpha}$ into (1.20) we obtain the exact expression

$$E = 3\alpha u^3 u_x - 6u_x^3 - 6uu_x u_{2x} + 3u^2 u_{3x} - \frac{6}{\alpha} u_x u_{4x}, \tag{1.27}$$

of order $M = 4$. First, using (1.26), we compute

$$\begin{aligned} I_u E &= \sum_{k=1}^{4} \left(\sum_{i=0}^{k-1} u_{ix} (-\mathsf{D}_x)^{k-(i+1)} \right) \frac{\partial E}{\partial u_{kx}} \\ &= (u\mathsf{I})(\frac{\partial E}{\partial u_x}) + (u_x\mathsf{I} - u\mathsf{D}_x)(\frac{\partial E}{\partial u_{2x}}) + (u_{2x}\mathsf{I} - u_x\mathsf{D}_x + u\mathsf{D}_x^2)(\frac{\partial E}{\partial u_{3x}}) \\ &\quad + (u_{3x}\mathsf{I} - u_{2x}\mathsf{D}_x + u_x\mathsf{D}_x^2 - u\mathsf{D}_x^3)(\frac{\partial E}{\partial u_{4x}}). \end{aligned} \tag{1.28}$$

After substitution of (1.27), one gets

$$\begin{aligned} I_u E &= (u\mathsf{I})(3\alpha u^3 + 18u_x^2 - 6uu_{2x} - \frac{6}{\alpha}u_{4x}) + (u_x\mathsf{I} - u\mathsf{D}_x)(-6uu_x) \\ &\quad + (u_{2x}\mathsf{I} - u_x\mathsf{D}_x + u\mathsf{D}_x^2)(3u^2) + (u_{3x}\mathsf{I} - u_{2x}\mathsf{D}_x + u_x\mathsf{D}_x^2 - u\mathsf{D}_x^3)(-\frac{6}{\alpha}u_x) \\ &= 3\alpha u^4 - 18uu_x^2 + 9u^2u_{2x} + \frac{6}{\alpha}u_{2x}^2 - \frac{12}{\alpha}u_x u_{3x}, \end{aligned} \tag{1.29}$$

which has the correct terms of $J^{(3)}$ but incorrect coefficients. Finally, using (1.25),

$$\begin{aligned} J = \mathcal{H}_{u(x)} E &= \int_0^1 (I_u E)[\lambda u] \, \frac{d\lambda}{\lambda} \\ &= \int_0^1 \left(3\alpha\lambda^3 u^4 - 18\lambda^2 uu_x^2 + 9\lambda^2 u^2 u_{2x} + \frac{6}{\alpha}\lambda u_{2x}^2 - \frac{12}{\alpha}\lambda u_x u_{3x} \right) d\lambda \\ &= \frac{3}{4}\alpha u^4 - 6uu_x^2 + 3u^2 u_{2x} + \frac{3}{\alpha}u_{2x}^2 - \frac{6}{\alpha}u_x u_{3x}, \end{aligned} \tag{1.30}$$

which matches $J^{(3)}$ in (1.11).

The crux of the homotopy operator method [12, 13, 26, 81] is that the integration by parts of a differential expression like (1.27), which involves an arbitrary function $u(x)$ and its $x-$derivatives, can be reduced to a standard integration of a polynomial in λ.

1.5 Conservation Laws of Nonlinear Systems of Polynomial PDEs

Thus far we have dealt with the computation of density-flux pairs of scalar evolution equations, with the KdV equation as the leading example. In this section we show how the method and tools can be generalized to cover systems of evolution equations. We will use the Drinfel'd-Sokolov-Wilson system and the Boussinesq equation to illustrate the steps.

1.5.1 Tools for Systems of Evolution Equations

For differential functions (like densities and fluxes) of two dependent variables (u, v) and their $x-$derivatives, the total derivatives are

$$\mathsf{D}_t\rho = \frac{\partial \rho}{\partial t} + \sum_{k=0}^{M_1} \frac{\partial \rho}{\partial u_{kx}} \mathsf{D}_x^k\, u_t + \sum_{k=0}^{M_2} \frac{\partial \rho}{\partial v_{kx}} \mathsf{D}_x^k\, v_t, \tag{1.31}$$

$$\mathsf{D}_x J = \frac{\partial J}{\partial x} + \sum_{k=0}^{N_1} \frac{\partial J}{\partial u_{kx}} u_{(k+1)x} + \sum_{k=0}^{N_2} \frac{\partial J}{\partial v_{kx}} v_{(k+1)x}, \tag{1.32}$$

where M_1 and M_2 are the (highest) orders of u and v in ρ, and N_1 and N_2 are the (highest) orders of u and v in J.

To accommodate two dependent variables, we need Euler operators $\mathcal{L}^{(0)}_{u(x)}$ and $\mathcal{L}^{(0)}_{v(x)}$ for each dependent variable separately. For brevity, we will use vector notation, that is, $\mathcal{L}^{(0)}_{\mathbf{u}(x)}E = (\mathcal{L}^{(0)}_{u(x)}E, \mathcal{L}^{(0)}_{v(x)}E)$. Likewise, the homotopy operator in (1.25) must be replaced by

$$\mathcal{H}_{\mathbf{u}(x)}E = \int_0^1 (I_u E + I_v E)[\lambda \mathbf{u}]\, \frac{d\lambda}{\lambda}, \tag{1.33}$$

where

$$I_u E = \sum_{k=1}^{M_1} \left(\sum_{i=0}^{k-1} u_{ix} (-\mathsf{D}_x)^{k-(i+1)} \right) \frac{\partial E}{\partial u_{kx}}, \tag{1.34}$$

and

$$I_v E = \sum_{k=1}^{M_2} \left(\sum_{i=0}^{k-1} v_{ix} (-\mathsf{D}_x)^{k-(i+1)} \right) \frac{\partial E}{\partial v_{kx}}, \tag{1.35}$$

where M_1, M_2 are the orders of E in u, v, respectively. In (1.33), $u \to \lambda u$, $u_x \to \lambda u_x, \cdots, v \to \lambda v$, $v_x \to \lambda v_x$, etc.

1.5.2 The Drinfel'd-Sokolov-Wilson System: Dilation Invariance and Conservation Laws

We consider a parameterized family of the Drinfel'd-Sokolov-Wilson (DSW) equations

$$u_t + 3vv_x = 0, \quad v_t + 2uv_x + \alpha u_x v + 2v_{3x} = 0, \tag{1.36}$$

where α is a nonzero parameter. The system with $\alpha = 1$ was first proposed by Drinfel'd and Sokolov [31, 32] and Wilson [99]. It can be obtained [59] as a reduction of the Kadomtsev-Petviashvili equation (i.e. a two-dimensional version of the KdV equation) and is a completely integrable system. In [109], Yao and Li computed conservation laws of (1.36), where they had introduced four arbitrary coefficients. Using scales on x, t, u and v, all but one coefficients in (1.36) can be scaled to any real number. Therefore, to cover the

entire family of DSW equations it suffices to leave one coefficient arbitrary, e.g. α in front of $u_x v$.

To compute the dilation symmetry of (1.36), we assign weights, $W(u)$ and $W(v)$, to both dependent variables and express that each equation separately must be uniform in rank (i.e. the ranks of the equations in (1.36) may differ from each other).

For the DSW equations (1.36), one has

$$\begin{aligned} W(u) + W(\partial/\partial t) &= 2W(v) + 1, \\ W(v) + W(\partial/\partial t) &= W(u) + W(v) + 1 = W(v) + 3, \end{aligned} \tag{1.37}$$

which yields $W(u) = W(v) = 2$, $W(\partial/\partial t) = 3$. The DSW system (1.36) is thus invariant under the scaling symmetry

$$(x, t, u, v) \to (\lambda^{-1}x, \lambda^{-3}t, \lambda^2 u, \lambda^2 v), \tag{1.38}$$

where λ is an arbitrary scaling parameter.

The first three density-flux pairs for the DSW equations (1.36) are

$$\rho^{(1)} = u, \qquad J^{(1)} = \frac{3}{2}v^2, \tag{1.39}$$

$$\rho^{(2)} = v, \qquad J^{(2)} = 2(uv + v_{2x}), \quad \text{if } \alpha = 2, \tag{1.40}$$

$$\rho^{(3)} = (\alpha - 1)u^2 + \frac{3}{2}v^2, \qquad J^{(3)} = 3(\alpha u v^2 - v_x^2 + 2v v_{2x}), \tag{1.41}$$

Both $\rho^{(1)}$ and $\rho^{(2)}$ have rank 2; their fluxes have rank 4. The pair $(\rho^{(1)}, J^{(1)})$ exists for any α, whereas $(\rho^{(2)}, J^{(2)})$ only exists if $\alpha = 2$. Density $\rho^{(3)}$ of rank 4 and flux $J^{(3)}$ of rank 6 are valid for any α. At rank $R = 6$,

$$\rho^{(4)} = (\alpha + 1)(\alpha - 2)u^3 - \frac{9}{2}(\alpha + 1)uv^2 - \frac{3}{2}(\alpha - 2)u_x^2 - \frac{27}{2}v_x^2. \tag{1.42}$$

The corresponding flux (not shown) has 7 terms. At rank $R = 8$,

$$\rho^{(5)} = u^4 - \frac{9}{2}u^2v^2 - \frac{27}{8}v^4 - \frac{9}{2}uu_x^2 + \frac{3}{4}u_{2x}^2 + \frac{45}{2}vu_x v_x + 27uv_x^2 - \frac{81}{4}v_{2x}^2, \tag{1.43}$$

provided $\alpha = 1$. The corresponding flux (not shown) has 15 terms. There exists a density-flux pair for all even ranks $R \leq 10$ provided $\alpha = 1$, for which (1.36) is completely integrable.

1.5.3 Computation of a Conservation Law of the Drinfel'd-Sokolov Wilson System

To illustrate how the presence of a parameter, like α, affects the computation of densities, we compute $\rho^{(1)}$ and $\rho^{(2)}$ of rank $R = 2$ given in (1.39) and (1.40).

Step 1: **Construct the form of the density**

The set of dependent variables is $\mathcal{V} = \{u, v\}$. Both elements are of rank 2 so, no $x-$derivatives are needed. Thus, $\mathcal{M} = \mathcal{R} = \mathcal{S} = \{u, v\}$. Linearly combining the elements in $\mathcal{S}$ gives $\rho = c_1 u + c_2 v$.

Step 2: **Compute the undetermined coefficients** c_i

Evaluating $E = -\mathsf{D}_t \rho = -(c_1 u_t + c_2 v_t)$ on (1.36), yields

$$E = 3c_1 v v_x + c_2(2uv_x + \alpha u_x v + 2v_{3x}), \tag{1.44}$$

which will be exact if $\mathcal{L}^{(0)}_{\mathbf{u}(x)} E = (\mathcal{L}^{(0)}_{u(x)} E, \mathcal{L}^{(0)}_{v(x)} E) \equiv (0, 0)$. Since E is of order $M_1 = 1$ in u and order $M_2 = 3$ in v,

$$\mathcal{L}^{(0)}_{u(x)} E = \frac{\partial E}{\partial u} - \mathsf{D}_x \frac{\partial E}{\partial u_x} = 2c_2 v_x - \mathsf{D}_x(c_2 \alpha v) = (2 - \alpha) c_2 v_x, \tag{1.45}$$

and

$$\begin{aligned} \mathcal{L}^{(0)}_{v(x)} E &= \frac{\partial E}{\partial v} - \mathsf{D}_x \frac{\partial E}{\partial v_x} + \mathsf{D}_x^2 \frac{\partial E}{\partial v_{2x}} - \mathsf{D}_x^3 \frac{\partial E}{\partial v_{3x}} \\ &= 3c_1 v_x + c_2 \alpha u_x - \mathsf{D}_x(3c_1 v + 2c_2 u) - \mathsf{D}_x^3(2c_2) \\ &= (\alpha - 2) c_2 u_x. \end{aligned} \tag{1.46}$$

Both (1.45) and (1.46) will vanish identically if and only if $(\alpha - 2)c_2 = 0$. This equation (with unknowns c_1 and c_2) is parameterized by $\alpha \neq 0$. The solution algorithm [38] considers all branches of the solution and possible compatibility conditions. Setting $c_1 = 1$, leads to either (i) $c_2 = 0$ if $\alpha \neq 2$, or (ii) c_2 arbitrary if $\alpha = 2$. Setting $c_2 = 1$ leads to the compatibility condition, $\alpha = 2$, and c_1 arbitrary. Substituting the solutions into $\rho = c_1 u + c_2 v$ gives $\rho = u$ which is valid for any α; and $\rho = u + c_2 v$ or $\rho = c_1 u + v$ provided $\alpha = 2$. In other words, $\rho^{(1)} = u$ is the only density of rank 2 for arbitrary values of α. For $\alpha = 2$ there exist two independent densities, $\rho^{(1)} = u$ and $\rho^{(2)} = v$.

Step 3: **Compute the associated flux** J

As an example, we compute the flux in (1.40) associated with $\rho^{(2)} = v$ and $\alpha = 2$. In this case, $c_1 = 0, c_2 = 1$, for which (1.44) simplifies into

$$E = 2(uv_x + u_x v + v_{3x}), \tag{1.47}$$

which is of order $M_1 = 1$ in u and order $M_2 = 3$ in v. Using (1.34) and (1.35), we obtain

$$I_u E = (u\mathsf{I}) \frac{\partial E}{\partial u_x} = (u\mathsf{I})(2v) = 2uv, \tag{1.48}$$

and

$$\begin{aligned} I_v E &= (v\mathsf{I})\left(\frac{\partial E}{\partial v_x}\right) + (v_x \mathsf{I} - v\mathsf{D}_x)\left(\frac{\partial E}{\partial v_{2x}}\right) + (v_{2x}\mathsf{I} - v_x \mathsf{D}_x + v\mathsf{D}_x^2)\left(\frac{\partial E}{\partial v_{3x}}\right) \\ &= (v\mathsf{I})(2u) + (v_{2x}\mathsf{I} - v_x \mathsf{D}_x + v\mathsf{D}_x^2)(2) \\ &= 2uv + 2v_{2x}. \end{aligned} \tag{1.49}$$

Hence, using (1.33),

$$J = \mathcal{H}_{u(x)}E = \int_0^1 (I_u E + I_v E)[\lambda \mathbf{u}]\,\frac{d\lambda}{\lambda} = \int_0^1 (4\lambda uv + 2v_{2x})\,d\lambda = 2(uv + v_{2x}), \quad (1.50)$$

which is $J^{(2)}$ in (1.40). The integration of (1.47) could easily be done by hand. The homotopy operator method pays off if the expression to be integrated has a large number of terms.

1.5.4 The Boussinesq Equation: Dilation Invariance and Conservation Laws

The wave equation,

$$u_{2t} - u_{2x} + 3u_x^2 + 3uu_{2x} + \alpha u_{4x} = 0, \quad (1.51)$$

for $u(x,t)$ with real parameter α, was proposed by Boussinesq to describe surface waves in shallow water [2]. For what follows, we rewrite (1.51) as a system of evolution equations,

$$u_t + v_x = 0, \quad v_t + u_x - 3uu_x - \alpha u_{3x} = 0, \quad (1.52)$$

where $v(x,t)$ is an auxiliary dependent variable.

The Boussinesq system (1.52) is not uniform in rank because the terms u_x and αu_{3x} lead to an inconsistent system of weight equations. To circumvent the problem we introduce an auxiliary parameter β with (unknown) weight, and replace (1.52) by

$$u_t + v_x = 0, \quad v_t + \beta u_x - 3uu_x - \alpha u_{3x} = 0. \quad (1.53)$$

Requiring uniformity in rank, we obtain (after some algebra)

$$W(u) = 2, \quad W(v) = 3, \quad W(\beta) = 2, \quad W(\frac{\partial}{\partial t}) = 2. \quad (1.54)$$

Therefore, (1.53) is invariant under the scaling symmetry

$$(x, t, u, v, \beta) \to (\lambda^{-1}x, \lambda^{-2}t, \lambda^2 u, \lambda^3 v, \lambda^2 \beta). \quad (1.55)$$

As the above example shows, a PDE that is not dilation invariant can be made so by extending the set of dependent variables with one or more auxiliary parameters with weights. Upon completion of the computations one can set each of these parameters equal to 1.

The Boussinesq equation (1.51) has infinitely many conservation laws and is completely integrable [2, 7]. The first four density-flux pairs [8] for (1.53) are

$$\rho^{(1)} = u, \quad J^{(1)} = v, \quad (1.56)$$

$$\rho^{(2)} = v, \quad J^{(2)} = \beta u - \tfrac{3}{2}u^2 - \alpha u_{2x}, \quad (1.57)$$

$$\rho^{(3)} = uv, \quad J^{(3)} = \tfrac{1}{2}\beta u^2 - u^3 + \tfrac{1}{2}v^2 + \tfrac{1}{2}\alpha u_x^2 - \alpha u u_{2x}, \quad (1.58)$$

$$\rho^{(4)} = \beta u^2 - u^3 + v^2 + \alpha u_x^2,$$

$$J^{(4)} = 2\beta uv - 3u^2 v + 2\alpha u_x v_x - 2\alpha u_{2x} v. \quad (1.59)$$

These densities are of ranks $2, 3, 5$ and 6, respectively. The corresponding fluxes are of one rank higher. After setting $\beta = 1$ we obtain the conserved quantities of (1.52) even though initially this system was not uniform in rank.

1.5.5 Computation of a Conservation Law for the Boussinesq System

We show the computation of $\rho^{(4)}$ and $J^{(4)}$ of ranks 6 and 7, respectively. The presence of the auxiliary parameter β with weight complicates matters. At a fixed rank R, conserved densities corresponding to lower ranks might appear in the result. These lower-rank densities are easy to recognize for they are multiplied with arbitrary coefficients c_i. Consequently, when parameters with weight are introduced, the densities corresponding to distinct ranks are no longer linearly independent. As the example below will show, densities must be split into independent pieces.

Step 1: **Construct the form of the density**

Augment the set of dependent variables with the parameter β (with non-zero weight). Hence, $\mathcal{V} = \{u, v, \beta\}$. Construct $\mathcal{M} = \{\beta^2 u, \beta u^2, \beta u, \beta v, u^3, u^2, u, v^2, v, uv\}$, which contains all non-constant monomials of (chosen) rank 6 or less (without derivatives). Next, for each term in $\mathcal{M}$, introduce the right number of x-derivatives so that each term has rank 6. For example,

$$\frac{\partial^2 \beta u}{\partial x^2} = \beta u_{2x}, \quad \frac{\partial^2 u^2}{\partial x^2} = 2u_x^2 + 2uu_{2x}, \quad \frac{\partial^4 u}{\partial x^4} = u_{4x}, \quad \frac{\partial (uv)}{\partial x} = vu_x + uv_x, \quad etc.. \tag{1.60}$$

Gather the terms in the right hand sides of the equations in (1.60) to get

$$\mathcal{R} = \{\beta^2 u, \beta u^2, u^3, v^2, vu_x, u_x^2, \beta v_x, uv_x, \beta u_{2x}, uu_{2x}, v_{3x}, u_{4x}\}. \tag{1.61}$$

Using (1.23) and a similar formula for v, for every term t_i in $\mathcal{R}$ we compute $\mathbf{v}_i = \mathcal{L}^{(0)}_{\mathbf{u}(x)} t_i = (\mathcal{L}^{(0)}_{u(x)} t_i, \mathcal{L}^{(0)}_{v(x)} t_i)$. If $\mathbf{v}_i = (0, 0)$ then t_i is discarded and so is $\mathbf{v}_i$. If $\mathbf{v}_i \neq (0, 0)$ we verify whether or not $\mathbf{v}_i$ is linearly independent of the non-zero vectors $\mathbf{v}_j,\ j = 1, 2, \cdots, i-1$. If independent, the term t_i is kept, otherwise, t_i is discarded and so is $\mathbf{v}_i$.

Upon application of $\mathcal{L}^{(0)}_{\mathbf{u}(x)}$, the first six terms in $\mathcal{R}$ lead to linearly independent vectors $\mathbf{v}_1$ through $\mathbf{v}_6$. Therefore, t_1 through t_6 are kept (and so are the corresponding vectors). For $t_7 = \beta v_x$ we compute $\mathbf{v}_7 = \mathcal{L}^{(0)}_{\mathbf{u}(x)}(\beta v_x) = (0, 0)$. So, t_7 is discarded and so is $\mathbf{v}_7$. For $t_8 = uv_x$ we get $\mathbf{v}_8 = \mathcal{L}^{(0)}_{\mathbf{u}(x)}(uv_x) = (v_x, -u_x) = -\mathbf{v}_5$. So, t_8 is discarded and so is $\mathbf{v}_8$.

Proceeding in a similar fashion, t_9, t_{10}, t_{11} and t_{12} are discarded. Thus, $\mathcal{R}$ is replaced by

$$\mathcal{S} = \{\beta^2 u, \beta u^2, u^3, v^2, vu_x, u_x^2\}, \tag{1.62}$$

which is free of divergences and divergence-equivalent terms. Ignoring the highest-order terms (typically the last terms) in each of the right hand sides of the equations in (1.60)

optimizes the procedure. Indeed, $\mathcal{R}$ would have had six instead of twelve terms. Coincidentally, in this example no further eliminations would be needed to obtain $\mathcal{S}$. Next, linearly combine the terms in $\mathcal{S}$ to get

$$\rho = c_1\beta^2 u + c_2\beta u^2 + c_3 u^3 + c_4 v^2 + c_5 v u_x + c_6 u_x^2. \tag{1.63}$$

Step 2: **Compute the undetermined coefficients** c_i

Compute $\mathsf{D}_t\rho$. Here, ρ is of order $M_1 = 1$ in u and order $M_2 = 0$ in v. Hence, application of (1.31) gives

$$\begin{aligned} \mathsf{D}_t\rho &= \frac{\partial\rho}{\partial u}\mathsf{I}u_t + \frac{\partial\rho}{\partial u_x}\mathsf{D}_x u_t + \frac{\partial\rho}{\partial v}\mathsf{I}v_t \\ &= (c_1\beta^2 + 2c_2\beta u + 3c_3u^2)u_t + (c_5 v + 2c_6 u_x)u_{tx} + (2c_4 v + c_5 u_x)v_t. \end{aligned} \tag{1.64}$$

Use (1.53) to eliminate u_t, u_{tx}, and v_t. Then, $E = -\mathsf{D}_t\,\rho$ evaluates to

$$\begin{aligned} E = {} & (c_1\beta^2 + 2c_2\beta u + 3c_3u^2)v_x + (c_5 v + 2c_6 u_x)v_{2x} \\ & + (2c_4 v + c_5 u_x)(\beta u_x - 3u u_x - \alpha u_{3x}), \end{aligned} \tag{1.65}$$

which must be exact. Thus, require that $\mathcal{L}^{(0)}_{\mathbf{u}(x)}E = (\mathcal{L}^{(0)}_{u(x)}E, \mathcal{L}^{(0)}_{v(x)}E) \equiv (0,0)$. Group like terms. Set their coefficients equal to zero to obtain the parameterized system

$$\beta(c_2 - c_4) = 0,\ \ c_3 + c_4 = 0,\ \ c_5 = 0,\ \ \alpha c_5 = 0,\ \ \beta c_5 = 0,\ \ \alpha c_4 - c_6 = 0, \tag{1.66}$$

where $\alpha \neq 0$ and $\beta \neq 0$. Investigate the eliminant of the system. Set $c_1 = 1$ and obtain the solution

$$c_1 = 1, \quad c_2 = c_4, \quad c_3 = -c_4, \quad c_5 = 0, \quad c_6 = \alpha c_4, \tag{1.67}$$

which holds without condition on α and β. Substitute (1.67) into (1.63) to get

$$\rho = \beta^2 u + c_4(\beta u^2 - u^3 + v^2 + \alpha u_x^2). \tag{1.68}$$

The density must be split into independent pieces. Indeed, since c_4 is arbitrary, set $c_4 = 0$ or $c_4 = 1$, thus splitting (1.68) into two independent densities

$$\rho = \beta^2 u \equiv u, \quad \rho = \beta u^2 - u^3 + v^2 + \alpha u_x^2, \tag{1.69}$$

which are $\rho^{(1)}$ and $\rho^{(4)}$ in (1.56)-(1.59).

Step 3: **Compute the flux** J

Compute the flux corresponding to ρ in (1.69). Substitute (1.67) into (1.65). Take the terms in c_4 and set $c_4 = 1$. Thus,

$$E = 2\beta u v_x + 2\beta v u_x - 3u^2 v_x - 6uvu_x + 2\alpha u_x v_{2x} - 2\alpha v u_{3x}, \tag{1.70}$$

which is of order $M_1 = 3$ in u and order $M_2 = 2$ in v. Using (1.34) and (1.35), one readily obtains

$$I_u E = 2\beta uv - 6u^2 v + 2\alpha u_x v_x - 2\alpha u_{2x} v, \tag{1.71}$$

and

$$I_v E = 2\beta uv - 3u^2 v + 2\alpha u_x v_x - 2\alpha u_{2x} v. \tag{1.72}$$

Hence, using (1.33),

$$\begin{aligned} J &= \mathcal{H}_{u(x)} E = \int_0^1 (I_u E + I_v E)[\lambda \mathbf{u}] \, \frac{d\lambda}{\lambda} \\ &= \int_0^1 \left(4\beta\lambda uv - 9\lambda^2 u^2 v + 4\alpha\lambda u_x v_x - 4\alpha\lambda u_{2x} v\right) d\lambda \\ &= 2\beta uv - 3u^2 v + 2\alpha u_x v_x - 2\alpha u_{2x} v, \end{aligned} \tag{1.73}$$

which is $J^{(4)}$ in (1.59). One can set $\beta = 1$ at the end of the computations.

1.6 Conservation Laws of Systems of PDEs with Transcendental Nonlinearities

We now turn to the symbolic computation of conservation laws of certain classes of PDEs with transcendental nonlinearities. We only consider PDEs where the transcendental functions act on one dependent variable u (and not on $x-$derivatives of u). In contrast to the examples in the previous sections, the candidate density will no longer have *constant* undetermined coefficients but *functional* coefficients which depend on the variable u. Furthermore, we consider only PDEs which have one type of nonlinearity. For example, sine, or cosine, or exponential terms are fine but not a mixture of these functions.

1.6.1 The sine-Gordon Equation: Dilation Invariance and Conservation Laws

The sine-Gordon (sG) equation appears in the literature [17, 69] in two different ways:

• In light-cone coordinates the sG equation, $u_{xt} = \sin u$, has a mixed derivative term, which complicates matters. We return to this type of equation in Section 1.7.1.

• The sG equation in laboratory coordinates , $u_{2t} - u_{2x} = \sin u$, can be recast as

$$u_t + v = 0, \quad v_t + u_{2x} + \sin u = 0, \tag{1.74}$$

where $v(x, t)$ is an auxiliary variable. System (1.74) is amenable to our approach, subject to modifications to accommodate the transcendental nonlinearity.

The sG equation describes the propagation of crystal dislocations, superconductivity in a Josephson junction, and ultra-short optical pulse propagation in a resonant medium [69]. In mathematics, the sG equation is long known in the differential geometry of surfaces of constant negative Gaussian curvature [30, 80].

The sine-Gordon equation (1.74) is not uniform in rank unless we replace it by

$$u_t + v = 0, \quad v_t + u_{2x} + \alpha \sin u = 0, \tag{1.75}$$

where α is a real parameter with weight. Indeed, substituting the Maclaurin series, $\sin u = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \cdots$, and requiring uniformity in rank yields

$$\begin{aligned} W(u) + W(\partial/\partial t) &= W(v), \\ W(v) + W(\partial/\partial t) &= W(u) + 2 = W(\alpha) + W(u) \\ &= W(\alpha) + 3W(u) = W(\alpha) + 5W(u) = \cdots \end{aligned} \tag{1.76}$$

This forces us to set $W(u) = 0$ and $W(\alpha) = 2$. Consequently, (1.75) is scaling invariant under the symmetry

$$(x, t, u, v, \alpha) \to (\lambda^{-1}x, \lambda^{-1}t, \lambda^0 u, \lambda^1 v, \lambda^2 \alpha), \tag{1.77}$$

corresponding to $W(\partial/\partial x) = W(\partial/\partial t) = 1, W(u) = 0, W(v) = 1, W(\alpha) = 2$. The first and second equations in (1.75) are uniform of ranks 1 and 2, respectively.

The first few (of infinitely many) density-flux pairs [8, 29] for the sG equation (1.75) are

$$\rho^{(1)} = 2\alpha \cos u + v^2 + u_x^2, \qquad J^{(1)} = 2vu_x, \tag{1.78}$$

$$\rho^{(2)} = 2vu_x, \qquad J^{(2)} = -2\alpha \cos u + v^2 + u_x^2, \tag{1.79}$$

$$\rho^{(3)} = 6\alpha v u_x \cos u + v^3 u_x + v u_x^3 - 8 v_x u_{2x}, \tag{1.80}$$

$$\begin{aligned} \rho^{(4)} = {} & 2\alpha^2 \cos^2 u - 2\alpha^2 \sin^2 u + 4\alpha v^2 \cos u + v^4 + 20\alpha u_x^2 \cos u \\ & + 6v^2 u_x^2 + u_x^4 - 16 v_x^2 - 16 u_{2x}^2, \end{aligned} \tag{1.81}$$

$J^{(3)}$ and $J^{(4)}$ are not shown due to length. Again, all densities and fluxes are uniform in rank (before α is set equal to 1).

1.6.2 Computation of a Conservation Law for the sine-Gordon System

We show how to compute densities $\rho^{(1)}$ and $\rho^{(2)}$, both of rank 2, and their associated fluxes $J^{(1)}$ and $J^{(2)}$. The candidate density will no longer have *constant* undetermined coefficients c_i but *functional* coefficients $h_i(u)$ which depend on the variable with weight zero [8]. To avoid having to solve PDEs, we tacitly assume that there is only *one* dependent variable with weight zero. As before, the algorithm proceeds in three steps:

Step 1: **Construct the form of the density**

Augment the set of dependent variables with α (with non-zero weight) and replace u by u_x (since $W(u) = 0$). Hence, $\mathcal{V} = \{\alpha, u_x, v\}$. Compute $\mathcal{R} = \{\alpha, v^2, v^2, u_{2x}, v u_x, u_x^2\}$

and remove divergences and equivalent terms to get $\mathcal{S} = \{\alpha, v^2, u_x^2, vu_x\}$. The candidate density

$$\rho = \alpha h_1(u) + h_2(u)v^2 + h_3(u)u_x^2 + h_4(u)vu_x, \tag{1.82}$$

with undetermined functional coefficients $h_i(u)$.

Step 2: **Compute the functions** $h_i(u)$

Compute

$$\begin{aligned} \mathsf{D}_t\rho &= \frac{\partial\rho}{\partial u}\mathsf{I}u_t + \frac{\partial\rho}{\partial u_x}\mathsf{D}_x u_t + \frac{\partial\rho}{\partial v}\mathsf{I}v_t \\ &= (\alpha h_1' + v^2 h_2' + u_x^2 h_3' + vu_x h_4')u_t + (2u_x h_3 + vh_4)u_{tx} \\ &\quad +(2vh_2 + u_x h_4)v_t, \end{aligned} \tag{1.83}$$

where h_i' denotes $\frac{dh_i}{du}$. After replacing u_t and v_t from (1.75), $E = -\mathsf{D}_t\rho$ becomes

$$\begin{aligned} E &= (\alpha h_1' + v^2 h_2' + u_x^2 h_3' + vu_x h_4')v + (2u_x h_3 + vh_4)v_x \\ &\quad +(2vh_2 + u_x h_4)(\alpha\sin u + u_{2x}). \end{aligned} \tag{1.84}$$

E must be exact. Therefore, require that $\mathcal{L}^{(0)}_{u(x)}E \equiv 0$ and $\mathcal{L}^{(0)}_{v(x)}E \equiv 0$. Set the coefficients of like terms equal to zero to get a mixed linear system of algebraic equations and ODEs:

$$\begin{aligned} &h_2(u) - h_3(u) = 0, \quad h_2'(u) = 0, \quad h_3'(u) = 0, \quad h_4'(u) = 0, \quad h_2''(u) = 0, \\ &h_4''(u) = 0, \quad 2h_2'(u) - h_3'(u) = 0, \quad 2h_2''(u) - h_3''(u) = 0, \\ &h_1'(u) + 2h_2(u)\sin u = 0, \quad h_1''(u) + 2h_2'(u)\sin u + 2h_2(u)\cos u = 0. \end{aligned} \tag{1.85}$$

Solve the system [8] and substitute the solution

$$h_1(u) = 2c_1\cos u + c_3, \quad h_2(u) = h_3(u) = c_1, \quad h_4(u) = c_2, \tag{1.86}$$

(with arbitrary constants c_i) into (1.82) to obtain

$$\rho = c_1(2\alpha\cos u + v^2 + u_x^2) + c_2 vu_x + c_3\alpha. \tag{1.87}$$

Step 3: **Compute the flux** J

Compute the flux corresponding to ρ in (1.87). Substitute (1.86) into (1.84), to get

$$E = c_1(2u_x v_x + 2vu_{2x}) + c_2(\alpha u_x\sin u + vv_x + u_x u_{2x}). \tag{1.88}$$

Since $E = \mathsf{D}_x J$, one must integrate to obtain J. Using (1.26) and (1.35) one gets $I_u E = 2c_1 vu_x + c_2(\alpha u\sin u + u_x^2)$ and $I_v E = 2c_1 vu_x + c_2 v^2$. Using (1.33),

$$\begin{aligned} J &= \mathcal{H}_{\mathbf{u}(x)}E = \int_0^1 (I_u E + I_v E)[\lambda\mathbf{u}]\,\frac{d\lambda}{\lambda} \\ &= \int_0^1 \left(4c_1\lambda vu_x + c_2(\alpha u\sin(\lambda u) + \lambda v^2 + \lambda u_x^2)\right)\,d\lambda \\ &= c_1(2vu_x) + c_2\left(-\alpha\cos u + \tfrac{1}{2}v^2 + \tfrac{1}{2}u_x^2\right). \end{aligned} \tag{1.89}$$

Finally, split density (1.87) and flux (1.89) into independent pieces (for c_1 and c_2) to get

$$\rho^{(1)} = 2\alpha\cos u + v^2 + u_x^2, \qquad J^{(1)} = 2vu_x, \tag{1.90}$$

$$\rho^{(2)} = vu_x, \qquad J^{(2)} = -\alpha\cos u + \tfrac{1}{2}v^2 + \tfrac{1}{2}u_x^2. \tag{1.91}$$

For E in (1.88), J in (1.89) can easily be computed by hand [8]. However, the computation of fluxes corresponding to densities of ranks ≥ 2 is cumbersome and requires integration with the homotopy operator.

1.7 Conservation Laws of Scalar Equations with Transcendental and Mixed Derivative Terms

Our method to compute densities and fluxes of scalar equations with transcendental terms and a mixed derivative term (i.e. u_{xt}) is an adaptation of the technique shown in Section 1.5. We only consider single PDEs with one type of transcendental nonlinearity. Since we are no longer dealing with evolution equations, densities and fluxes could dependent on u_t, u_{2t}, etc. We do not cover such cases; instead, we refer the reader to [101].

1.7.1 The sine-Gordon Equation in Light-Cone Coordinates

In light-cone coordinates (or characteristic coordinates) the sG equation,

$$u_{xt} = \sin u, \tag{1.92}$$

has a mixed derivative as well as a transcendental term. A change of variables, $\Phi = u_x, \Psi = -1 + \cos u$, allows one to replace (1.92) by

$$\Phi_{xt} - \Phi - \Phi\Psi = 0, \quad 2\Psi + \Psi^2 + \Phi_t^2 = 0, \tag{1.93}$$

without transcendental terms. Unfortunately, neither (1.92) nor (1.93) can be written as a system of evolution equations. As shown in Section 1.6.1, to deal with the transcendental nonlinearity, which imposes $W(u) = 0$, one has to replace (1.92) by

$$u_{xt} = \alpha\sin u, \tag{1.94}$$

where α is an auxiliary parameter with weight. Indeed, (1.94) is dilation invariant under the scaling symmetry

$$(x, t, u, \alpha) \to (\lambda^{-1}x, \lambda^{-1}t, \lambda^0 u, \lambda^2\alpha), \tag{1.95}$$

corresponding to $W(\partial/\partial x) = W(\partial/\partial t) = 1, W(u) = 0$, and $W(\alpha) = 2$. The density-flux pairs [8, 29] of ranks $2, 4, 6$, and 8 (which are independent of u_t, u_{2t}, etc.), are

$$\rho^{(1)} = u_x^2, \qquad J^{(1)} = 2\alpha\cos u, \tag{1.96}$$

$$\rho^{(2)} = u_x^4 - 4u_{2x}^2 \qquad J^{(2)} = 4\alpha u_x^2\cos u, \tag{1.97}$$

$$\rho^{(3)} = u_x^6 - 20u_x^2u_{2x}^2 + 8u_{3x}^2,$$

$$J^{(3)} = 2\alpha(3u_x^4\cos u + 8u_x^2u_{2x}\sin u - 4u_{2x}^2\cos u), \tag{1.98}$$

$$\rho^{(4)} = 5u_x^8 - 280u_x^4u_{2x}^2 - 112u_{2x}^4 + 224u_x^2u_{3x}^2 - 64u_{4x}^2, \tag{1.99}$$

$$\begin{aligned} J^{(4)} = {} & 8\alpha\left(5u_x^6\cos u + 40u_x^4u_{2x}\sin u + 20u_x^2u_{2x}^2\cos u + 16u_{2x}^3\sin u\right. \\ & \left. -16u_x^3u_{3x}\cos u - 48u_xu_{2x}u_{3x}\sin u + 8u_{3x}^2\cos u\right). \end{aligned} \tag{1.100}$$

There are infinitely many density-flux pairs (all of even rank). Since $u_{xt} = (u_x)_t$, one can view (1.94) as an evolution equation in a new variable, $U = u_x$, and construct densities as linear combinations with constant coefficients of monomials in U and its $x-$derivatives. As before, each monomial has a (pre-selected) rank. To accommodate the transcendental term(s) one might be incorrectly tempted to linearly combine such monomials with functional coefficients $h_i(u)$ instead of constant coefficients c_i. For example, however, for rank $R = 2$, one should take $\rho = c_1u_x^2$ instead of $\rho = h_1(u)u_x^2$, because the latter would lead to $\mathsf{D}_t\rho = h_1'u_tu_x^2 + 2h_1u_xu_{2x}$ and u_t cannot be replaced from (1.94).

1.7.2 Examples of Equations with Transcendental Nonlinearities

In this section we consider additional PDEs of the form $u_{xt} = f(u)$, where $f(u)$ has transcendental terms. Using the Painlevé integrability test, researchers [12] have concluded that the only PDEs of that type that are completely integrable are equivalent to one of the standard forms of the nonlinear Klein-Gordon equation [2, 12]. These standard forms (in light-cone coordinates) include the sine-Gordon equation, $u_{xt} = \sin u$, discussed in Section 1.6.1, the sinh-Gordon equation $u_{xt} = \sinh u$, the Liouville equation $u_{xt} = \mathrm{e}^u$, and the double Liouville equations, $u_{xt} = \mathrm{e}^u \pm \mathrm{e}^{-2u}$. The latter is also referred to in the literature as the Tzetzeica and Mikhailov equations. For each of these equations one can compute conservation laws with the method discussed in Section 1.7.1. Alternatively, if these equations were transformed into laboratory coordinates, one would apply the method of Section 1.6.2. The multiple sine-Gordon equations, e.g. $u_{xt} = \sin u + \sin 2u$, have only a finite number of conservation laws and are not completely integrable, as supported by other evidence [2].

The sinh-Gordon equation, $u_{xt} = \sinh u$, arises as a special case of the Toda lattice discussed in Section 1.11.1. It also describes the dynamics of strings in constant curvature space-times [70]. In thermodynamics, the sinh-Gordon equation can be used to calculate partition and correlation functions, and thus support Langevin simulations [64]. In Table 1.1, we show a few density-flux pairs for the sinh-Gordon equation in light-cone coordinates, $u_{xt} = \alpha\sinh u$. As with the sG equation (1.92), $W(\partial/\partial x) = W(\partial/\partial t) = 1$,

$W(u) = 0$, and $W(\alpha) = 2$. The ranks in the first column of Table 1.1 correspond to the ranks of the densities, which are polynomial in $U = u_x$ and its $x-$derivatives. The sinh-Gordon equation has infinitely many conservation laws and is known to be completely integrable [2].

Table 1.1: Conservation Laws of the sinh-Gordon equation, $u_{xt} = \alpha \sinh u$

Rank	Density (ρ)	Flux (J)
2	u_x^2	$-2\alpha \cosh u$
4	$u_x^4 + 4u_{2x}^2$	$-4\alpha u_x^2 \cosh u$
6	$u_x^6 + 20u_x^2u_{2x}^2 + 8u_{3x}^2$	$-2\alpha[(3u_x^4 + 4u_{2x}^2)\cosh u + 8u_x^2u_{2x}\sinh u]$
8	$5u_x^8 + 280u_x^4u_{2x}^2 + 64u_{4x}^2$ $+224u_x^2u_{3x}^2 - 112u_{2x}^4$	$-8\alpha\left[(5u_x^6 - 20u_x^2u_{2x}^2 + 16u_x^3u_{3x} + 8u_{3x}^2)\cosh u\right.$ $\left.+(40u_x^4u_{2x} - 16u_{2x}^3 + 48u_xu_{2x}u_{3x})\sinh u\right]$.

The Liouville equation, $u_{xt} = \mathrm{e}^u$, plays an important role in modern field theory [68], e.g. in the theory of strings, where the quantum Liouville field appears as a conformal anomaly [65]. The first few (of infinitely many) density-flux pairs for the Liouville equation in light-cone coordinates, $u_{xt} = \alpha \mathrm{e}^u$, are given in Table 1.2. As before, $W(\partial/\partial x) = W(\partial/\partial t) = 1$, $W(u) = 0$, and $W(\alpha) = 2$. The ranks in the table refer to the ranks of the densities. Dodd and Bullough [29] have shown that the Liouville equation has no densities of ranks $3, 5$, and 7. As shown in Table 1.2, there are two densities of rank 6, and three densities of rank 8. Our results agree with those in [29], where one can also find the unique density of rank 9 and the four independent densities of rank 10.

The double Liouville equations,

$$u_{xt} = \mathrm{e}^u \pm \mathrm{e}^{-2u}, \tag{1.101}$$

arise in the field of "laser-induced vibrational predesorption of molecules physisorbed on insulating substrates." More precisely, (1.101) is used to investigate the dynamics of energy flow of excited admolecules on insulating substrates [82]. Double Liouville equations are also relevant in studies of global properties of scalar-vacuum configurations in general relativity and similarly systems in alternative theories of gravity [22].

In Table 1.3, we show some density-flux pairs of $u_{xt} = \alpha(\mathrm{e}^u - \mathrm{e}^{-2u})$. There are no density-flux pairs for ranks 4 and 10. We computed a density-flux pair for rank 12 (not shown due to length). The results for (1.101) with the plus sign are similar.

Part II: Nonlinear Differential-Difference Equations

In the second part of this chapter we discuss two distinct methods to construct conservation laws of nonlinear DDEs. The first method follows closely the technique for PDEs discussed in Part I. It is quite effective for certain classes of DDEs, including the Kac-van Moerbeke

Table 1.2: Conservation Laws of the Liouville equation, $u_{xt} = \alpha \mathrm{e}^u$

R	Density (ρ)	Flux (J)
2	u_x^2	$-2\alpha \mathrm{e}^u$
4	$u_x^4 + 4u_{2x}^2$	$-4\alpha u_x^2 \mathrm{e}^u$
6	$c_1(u_x^6 - 20u_{2x}^3 - 12u_{3x}^2)$	$-\alpha\left[6c_1(u_x^4 - 4u_x^2u_{2x} - 2u_{2x}^2)\right.$
	$+c_2(u_x^2u_{2x}^2 + u_{2x}^3 + u_{3x}^2)$	$\left.+c_2u_{2x}(2u_x^2 + u_{2x})\right] \mathrm{e}^u$
8	$c_1(u_x^8 - 56u_x^2u_{2x}^3 - 168u_x^2u_{3x}^2$	$-\alpha\left[8c_1(u_x^6 - 6u_x^4u_{2x} + 3u_x^2u_{2x}^2 - 20u_{2x}^3\right.$
	$-672u_{2x}u_{3x}^2 - 144u_{4x}^2)$	$\left.-36u_x^3u_{3x} - 108u_xu_{2x}u_{3x} - 18u_{3x}^2)\right.$
	$+c_2(u_x^4u_{2x}^2 + u_x^2u_{2x}^3 + 5u_x^2u_{3x}^2$	$+c_2(2u_x^4u_{2x} - u_x^2u_{2x}^2 + 4u_{2x}^3 + 8u_x^3u_{3x}$
	$+18u_{2x}u_{3x}^2 + 4u_{4x}^2) + c_3(u_{2x}^4$	$+24u_xu_{2x}u_{3x} + 4u_{3x}^2)$
	$+3u_x^2u_{3x}^2 + 15u_{2x}u_{3x}^2 + 3u_{4x}^2)$	$\left.+c_3(4u_{2x}^3 + 6u_x^3u_{3x} + 18u_xu_{2x}u_{3x} + 3u_{3x}^2)\right] \mathrm{e}^u$

and Toda lattices, but far less effective for more complicated lattices, such as the Bogoyavlenskii and the Gardner lattices. The latter examples are treated with a new method based on a leading order analysis proposed by Hickman [55].

1.8 Nonlinear DDEs and Conservation Laws

We consider autonomous nonlinear systems of DDEs of the form

$$\dot{\mathbf{u}}_n = \mathbf{F}(\mathbf{u}_{n-l}, ..., \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, ..., \mathbf{u}_{n+m}), \tag{1.102}$$

where $\mathbf{u}_n$ and $\mathbf{F}$ are vector-valued functions with N components. We only consider DDEs with one discrete variable, denoted by integer n. In many applications, n comes from a discretization of a space variable. The dot stands for differentiation with respect to the continuous variable which frequently is time, t. We assume that $\mathbf{F}$ is polynomial with constant coefficients, although this restriction can be waived for the method presented in Section 1.12. No restrictions are imposed on the degree of nonlinearity of $\mathbf{F}$. If parameters are present in (1.102), they will be denoted by lower-case Greek letters.

$\mathbf{F}$ depends on $\mathbf{u}_n$ and a finite number of forward and backward shifts of $\mathbf{u}_n$. We identify l with the furthest negative shift of any variable in the system, and m with the furthest positive shift of any variable in the system. No restrictions are imposed on the integers l and m, which measure the degree of non-locality in (1.102).

By analogy with D_x, we define the shift operator D by $\mathsf{D}\mathbf{u}_n = \mathbf{u}_{n+1}$. The operator D is often called the *up-shift operator* or forward- or right-shift operator. Its inverse, D^{-1}, is the

Table 1.3: Conservation Laws of the double Liouville equation, $u_{xt} = \alpha(\mathrm{e}^{u} - \mathrm{e}^{-2u})$

Rank	Density (ρ)	Flux (J)
2	u_x^2	$-\alpha(2\mathrm{e}^{u} + \mathrm{e}^{-2u})$
4	——	——
6	$u_x^6 + 15u_x^2u_{2x}^2 - 5u_{2x}^3 + 3u_{3x}^2$	$-3\alpha\left[(2u_x^4 + 2u_x^2u_{2x} + u_{2x}^2)\mathrm{e}^{u} + (u_x^4 - 8u_x^2u_{2x} + 2u_{2x}^2)\mathrm{e}^{-2u}\right]$
8	$u_x^8 + 42u_x^4u_{2x}^2 - 14u_x^2u_{2x}^3 - 7u_{2x}^4 + 21u_x^2u_{3x}^2 - 21u_{2x}u_{3x}^2 + 3u_{4x}^2$	$-\alpha\left[(8u_x^6 + 36u_x^4u_{2x} - 18u_x^2u_{2x}^2 - 20u_{2x}^3 + 6u_x^3u_{3x} + 18u_xu_{2x}u_{3x} + 3u_{3x}^2)\mathrm{e}^{u} + (4u_x^6 - 72u_x^4u_{2x} - 18u_x^2u_{2x}^2 - 4u_{2x}^3 + 48u_x^3u_{3x} - 72u_xu_{2x}u_{3x} + 6u_{3x}^2)\mathrm{e}^{-2\,u}\right]$
10	——	——

down-shift operator or backward- or left-shift operator, $\mathsf{D}^{-1}\mathbf{u}_n = \mathbf{u}_{n-1}$. The action of the shift operators is extended to functions by acting on their arguments. For example,

$$\begin{aligned} & \mathsf{D}\mathbf{F}(\mathbf{u}_{n-l}, ..., \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, ..., \mathbf{u}_{n+m}) \\ = \; & \mathbf{F}(\mathsf{D}\mathbf{u}_{n-l}, ..., \mathsf{D}\mathbf{u}_{n-1}, \mathsf{D}\mathbf{u}_n, \mathsf{D}\mathbf{u}_{n+1}, ..., \mathsf{D}\mathbf{u}_{n+m}) \\ = \; & \mathbf{F}(\mathbf{u}_{n-l+1}, ..., \mathbf{u}_n, \mathbf{u}_{n+1}, \mathbf{u}_{n+2}, ..., \mathbf{u}_{n+m+1}). \end{aligned} \tag{1.103}$$

Following [57], we generate (1.102) from

$$\dot{\mathbf{u}}_0 = \mathbf{F}(\mathbf{u}_{-l}, \mathbf{u}_{-l+1}, ..., \mathbf{u}_{-1}, \mathbf{u}_0, \mathbf{u}_1, ..., \mathbf{u}_{m-1}, \mathbf{u}_m), \tag{1.104}$$

where $\dot{\mathbf{u}}_n = \mathsf{D}^n\dot{\mathbf{u}}_0 = \mathsf{D}^n\mathbf{F}$. To further simplify the notation, we denote the zero-shifted dependent variable, $\mathbf{u}_0$, by $\mathbf{u}$. Shifts of $\mathbf{u}$ are generated by repeated application of D. For instance, $\mathbf{u}_k = \mathsf{D}^k\mathbf{u}$. The identity operator is denoted by I, where $\mathsf{D}^0\mathbf{u} = \mathsf{I}\mathbf{u} = \mathbf{u}$.

A *conservation law* of (1.104),

$$\mathsf{D}_t\,\rho + \Delta\,J = 0, \tag{1.105}$$

links a *conserved density* ρ to an *associated flux* J, where both are scalar functions depending on $\mathbf{u}$ and its shifts. In (1.105), which holds on solutions of (1.104), D_t is the total derivative with respect to time and $\Delta = \mathsf{D} - \mathsf{I}$ is the forward difference operator.

For readability (in particular, in the examples), the components of $\mathbf{u}$ will be denoted by u, v, w, etc. In what follows we consider only autonomous functions, i.e. $\mathbf{F}$, ρ, and J do not explicitly depend on t.

The time derivatives are defined in a similar way as in the continuous case, see (1.5) and (1.31). We show the discrete analog of (1.31). For a density $\rho(u_p, u_{p+1}, \cdots, u_q, v_r, v_{r+1}, \cdots, v_s)$, involving two dependent variables (u, v) and their shifts, the time derivative is computed as

$$\begin{aligned} \mathsf{D}_t\rho &= \sum_{k=p}^{q} \frac{\partial \rho}{\partial u_k}\dot{u}_k + \sum_{k=r}^{s} \frac{\partial \rho}{\partial v_k}\dot{v}_k \\ &= \left(\sum_{k=p}^{q} \frac{\partial \rho}{\partial u_k}\mathsf{D}^k\right)\dot{u} + \left(\sum_{k=r}^{s} \frac{\partial \rho}{\partial v_k}\mathsf{D}^k\right)\dot{v}, \end{aligned} \tag{1.106}$$

since D and $\mathrm{d}/\mathrm{d}t$ commute. Obviously, the difference operator extends to functions. For example, $\Delta J = \mathsf{D}\,J - J$ for a flux, J.

A density is *trivial* if there exists a function ψ so that $\rho = \Delta\psi$. Similar to the continuous case, we say that two densities, $\rho^{(1)}$ and $\rho^{(2)}$, are *equivalent* if and only if $\rho^{(1)} + k\rho^{(2)} = \Delta\psi$, for some ψ and some non-zero scalar k.

It is paramount that the density is free of equivalent terms for if such terms were present, they could be moved into the flux J. Instead of working with different densities, we will use the equivalence of monomial terms t_i in the same density (of a fixed rank). Compositions of D or D^{-1} define an *equivalence relation* $(\equiv)$ on monomial terms. Simply stated, all shifted terms are *equivalent*, e.g. $u_{-1}v_1 \equiv uv_2 \equiv u_2v_4 \equiv u_{-3}v_{-1}$ since

$$\begin{aligned} u_{-1}v_1 = uv_2 - \Delta\left(u_{-1}v_1\right) &= u_2v_4 - \Delta\left(u_1v_3 + uv_2 + u_{-1}v_1\right) \\ &= u_{-3}v_{-1} + \Delta\left(u_{-2}v + u_{-3}v_{-1}\right). \end{aligned} \tag{1.107}$$

This equivalence relation holds for any function of the dependent variables, but for the construction of conserved densities we will apply it only to monomials.

In the algorithm used in Sections 1.9.2, 1.11.2, and 1.11.4, we will use the following *equivalence criterion*: two monomial terms, t_1 and t_2, are equivalent, $t_1 \equiv t_2$, if and only if $t_1 = \mathsf{D}^r\,t_2$ for some integer r. Obviously, if $t_1 \equiv t_2$, then $t_1 = t_2 + \Delta J$ for some polynomial J, which depends on $\mathbf{u}$ and its shifts. For example, $u_{-2}u \equiv u_{-1}u_1$ because $u_{-2}u = \mathsf{D}^{-1}u_{-1}u_1$. Hence, $u_{-2}u = u_{-1}u_1 + [-u_{-1}u_1 + u_{-2}u] = u_{-1}u_1 + \Delta J$, with $J = -u_{-2}u$.

For efficiency we need a criterion to choose a unique representative from each equivalence class. There are a number of ways to do this. We define the *canonical* representative as that member that has (i) no negative shifts and (ii) a non-trivial dependence on the *local* (that is, zero-shifted) variable. For example, uu_2 is the canonical representative of the class $\{\cdots, u_{-2}u, u_{-1}u_1, uu_2, u_1u_3, \cdots\}$. In the case of e.g. two variables (u and v), u_2v is the canonical representative of the class $\{\cdots, u_{-1}v_{-3}, uv_{-2}, u_1v_{-1}, u_2v, u_3v_1, \cdots\}$.

Alternatively, one could choose a variable ordering and then choose the member that depends on the zero-shifted variable of lowest lexicographical order. The code in [48] uses

lexicographical ordering of the variables, i.e. $u \prec v \prec w$, etc. Thus, uv_{-2} (instead of $u_2 v$) is chosen as the canonical representative of $\{\cdots, u_{-1}v_{-3}, uv_{-2}, u_1 v_{-1}, u_2 v, u_3 v_1, \cdots\}$.

It is easy to show [55] that if ρ is a density then $\mathsf{D}^k \rho$ is also a density. Hence, using an appropriate "up-shift" all negative shifts in a density can been removed. Thus, without loss of generality, we may assume that a density that depends on q shifts has *canonical* form $\rho(\mathbf{u}, \mathbf{u}_1, \cdots, \mathbf{u}_q)$.

1.9 The Method of Undetermined Coefficients for DDEs

In this section we show how polynomial conservation laws can be computed for a scalar DDE,

$$\dot{u} = F(u_{-l}, u_{-l+1}, \cdots, u, \cdots, u_{m-1}, u_m). \tag{1.108}$$

The Kac-van Moerbeke example is used to illustrate the steps.

1.9.1 A Classic Example: The Kac-van Moerbeke Lattice

The Kac-van Moerbeke (KvM) lattice [60, 62], also known as the Volterra lattice,

$$\dot{u}_n = u_n(u_{n+1} - u_{n-1}), \tag{1.109}$$

arises in the study of Langmuir oscillations in plasmas, population dynamics, etc. Eq. (1.109) appears in the literature in various forms, including $\dot{R}_n = \frac{1}{2}(\exp(-R_{n-1}) - \exp(-R_{n+1}))$, and $\dot{w}_n = w_n(w_{n+1}^2 - w_{n-1}^2)$, which relate to (1.109) by simple transformations [92]. We continue with (1.109) and, adhering to the simplified notation, write it as

$$\dot{u} = u(u_1 - u_{-1}), \tag{1.110}$$

or, with the conventions adopted above, $\dot{u} = u(\mathsf{D}u - \mathsf{D}^{-1}u)$.

Lattice (1.110) is invariant under the scaling symmetry $(t, u) \to (\lambda^{-1}t, \lambda u)$. Hence, u corresponds to one derivative with respect to t, i.e. $u \sim \frac{\mathrm{d}}{\mathrm{dt}}$. In analogy to the continuous case, we define the *weight* W of a variable as the exponent of λ that multiplies the variable [39, 40]. We assume that shifts of a variable have the same weights, that is, $W(u_{-1}) = W(u) = W(u_1)$. Weights of dependent variables are nonnegative and rational. The *rank* of a monomial equals the total weight of the monomial. An expression (or equation) is *uniform in rank* if all its monomial terms have equal rank.

Applied to (1.110), $W(\mathrm{d}/\mathrm{d}t) = W(\mathsf{D}_t) = 1$ and $W(u) = 1$. Conversely, the scaling symmetry can be computed with linear algebra as follows. Setting $W(\mathrm{d}/\mathrm{d}t) = 1$ and requiring that (1.110) is uniform in rank yields $W(u) + 1 = 2W(u)$. Thus, $W(u) = 1$, which agrees with the scaling symmetry.

Many integrable nonlinear DDEs are scaling (dilation) invariant. If not, they can be made so by extending the set of dependent variables with parameters with weights. Examples of such cases are given in Sections 1.11.3 and 1.13.2.

The KvM lattice has infinitely many polynomial density-flux pairs. We give the conserved densities of rank $R \leq 4$ with associated fluxes ($J^{(4)}$ is omitted due to length):

$$\begin{aligned}
\rho^{(1)} &= u, & J^{(1)} &= -uu_{-1}, && (1.111)\\
\rho^{(2)} &= \tfrac{1}{2}u^2 + uu_1, & J^{(2)} &= -(u_{-1}u^2 - u_{-1}uu_1), && (1.112)\\
\rho^{(3)} &= \tfrac{1}{3}u^3 + uu_1(u + u_1 + u_2), & & && \\
J^{(3)} &= -(u_{-1}u^3 + 2u_{-1}u^2u_1 + u_{-1}uu_1^2 + u_{-1}uu_1u_2), && && (1.113)\\
\rho^{(4)} &= \tfrac{1}{4}u^4 + u^3u_1 + \tfrac{3}{2}u^2u_1^2 + uu_1^2(u_1 + u_2) && && \\
&\quad + uu_1u_2(u + u_1 + u_2 + u_3). && && (1.114)
\end{aligned}$$

In addition to infinitely many polynomial conserved densities, (1.110) has a non-polynomial density, $\rho^{(0)} = \ln u$ with flux $J^{(0)} = -(u + u_{-1})$. We discuss the computation of non-polynomial densities in Section 1.12.

1.9.2 The Method of Undetermined Coefficients Applied to a Scalar Nonlinear DDE

We outline how densities and fluxes can be constructed for a scalar DDE (1.108). Using (1.110) as an example, we compute $\rho^{(3)}$ of rank $R = 3$ and associated flux $J^{(3)}$ of rank $R = 4$, both listed in (1.113).

• Select the rank R of ρ. Start from the set $\mathcal{V}$ of dependent variables (including parameters with weight, when applicable), and form a set $\mathcal{M}$ of all non-constant monomials of rank R or less (without shifts). For each monomial in $\mathcal{M}$ introduce the right number of $t-$derivatives to adjust the rank of that term. Using the DDE, evaluate the $t-$derivatives, strip off the numerical coefficients, and gather the resulting terms in a set $\mathcal{R}$. For the KvM lattice (1.110), $\mathcal{V} = \{u\}$ and $\mathcal{M} = \{u^3, u^2, u\}$. Since u^3, u^2, and u have ranks $3, 2$, and 1, respectively, one computes

$$\frac{\mathrm{d}^0u^3}{\mathrm{d}t^0} = u^3, \quad \frac{\mathrm{d}u^2}{\mathrm{d}t} = 2u\dot{u} = 2u^2(u_1 - u_{-1}) = 2u^2u_1 - 2u_{-1}u^2, \tag{1.115}$$

and

$$\begin{aligned}
\frac{\mathrm{d}^2u}{\mathrm{d}t^2} &= \frac{\mathrm{d}\dot{u}}{\mathrm{d}t} = \frac{\mathrm{d}\left(u(u_1 - u_{-1})\right)}{\mathrm{d}t} = \dot{u}(u_1 - u_{-1}) + u(\dot{u}_1 - \dot{u}_{-1})\\
&= u(u_1 - u_{-1})^2 + u(u_1(u_2 - u) - u_{-1}(u - u_{-2}))\\
&= uu_1^2 - 2u_{-1}uu_1 + u_{-1}^2u + uu_1u_2 - u^2u_1 - u_{-1}u^2 + u_{-2}u_{-1}u, \qquad (1.116)
\end{aligned}$$

where (1.110) (and its shifts) has been used to remove the time derivatives. Build $\mathcal{R}$ using the terms from the right hand sides of the equations in (1.115) and ignoring numerical coefficients,

$$\mathcal{R} = \{u^3, u^2u_1, u_{-1}u^2, uu_1^2, u_{-1}uu_1, u_{-1}^2u, uu_1u_2, u_{-2}u_{-1}u\}. \tag{1.117}$$

• Identify the elements in $\mathcal{R}$ that belong to the same equivalence classes, replace them by their canonical representatives, and remove all duplicates. Call the resulting set $\mathcal{S}$, which has the building blocks of a candidate density. Continuing with (1.117), $u_{-2}u_{-1}u \equiv u_{-1}uu_1 \equiv uu_1u_2$. Likewise, $u_{-1}u^2 \equiv uu_1^2$ and $u_{-1}^2u \equiv u^2u_1$. Thus, $\mathcal{S} = \{u^3, u^2u_1, uu_1^2, uu_1u_2\}$.

• Form an arbitrary linear combination of the elements in $\mathcal{S}$. This is the candidate ρ. Continuing with the example,

$$\rho = c_1\, u^3 + c_2\, u^2u_1 + c_3\, uu_1^2 + c_4\, uu_1u_2. \tag{1.118}$$

• Compute

$$\mathsf{D}_t\rho = \sum_{k=0}^{q} \frac{\partial \rho}{\partial u_k}\dot{u}_k = \left(\sum_{k=0}^{q} \frac{\partial \rho}{\partial u_k}\mathsf{D}^k\right)\dot{u}, \tag{1.119}$$

where q is the highest shift in ρ. Using (1.118) where $q = 2$,

$$\begin{aligned} \mathsf{D}_t\rho &= \left(\frac{\partial \rho}{\partial u}\mathsf{I} + \frac{\partial \rho}{\partial u_1}\mathsf{D} + \frac{\partial \rho}{\partial u_2}\mathsf{D}^2\right)\dot{u} \\ &= [(3c_1u^2 + 2c_2uu_1 + c_3u_1^2 + c_4u_1u_2)\mathsf{I} + \\ &\quad (c_2u^2 + 3c_3uu_1 + c_4uu_2)\mathsf{D} + c_4uu_1\mathsf{D}^2]\dot{u}. \end{aligned} \tag{1.120}$$

• Evaluate $\mathsf{D}_t\rho$ on the DDE (1.108) by replacing $\dot{u}$ by F. Call the result E. In (1.110), $F = u(u_1 - u_{-1})$. The evaluated form of (1.120) is

$$\begin{aligned} E &= (3c_1u^2 + 2c_2uu_1 + c_3u_1^2 + c_4u_1u_2)u(u_1 - u_{-1}) + (c_4uu_1)u_2(u_3 - u_1) \\ &\quad + (c_2u^2 + 2c_3uu_1 + c_4uu_2)u_1(u_2 - u) \\ &= (3c_1 - c_2)u^3u_1 - 3c_1u_{-1}u^3 + 2(c_2 - c_3)u^2u_1^2 - 2c_2u_{-1}u^2u_1 \\ &\quad + c_3uu_1^3 - c_3u_{-1}uu_1^2 - c_4u_{-1}uu_1u_2 + (c_2 - c_4)u^2u_1u_2 \\ &\quad + 2c_3uu_1^2u_2 + c_4uu_1u_2^2 + c_4uu_1u_2u_3. \end{aligned} \tag{1.121}$$

• Set $J = 0$. Transform E into its canonical form. In doing so modify J so that $E + \Delta\, J$ remains unchanged. For example in (1.121), replace $-3c_1u_{-1}u^3$ in E by $-3c_1uu_1^3$ and add $-3c_1u_1u^3$ to J since $uu_1^3 - [uu_1^3 - u_{-1}u^3]$. Do the same for all the other terms which are not in canonical form. After grouping like terms, (1.121) becomes

$$\begin{aligned} E &= (3c_1 - c_2)u^3u_1 + (c_3 - 3c_1)uu_1^3 + 2(c_2 - c_3)u^2u_1^2 \\ &\quad + 2(c_3 - c_2)uu_1^2u_2 + (c_4 - c_3)uu_1u_2^2 + (c_2 - c_4)u^2u_1u_2, \end{aligned} \tag{1.122}$$

with

$$J = -(3c_1u_{-1}u^3 + 2c_2u_{-1}u^2u_1 + c_3u_{-1}uu_1^2 + c_4u_{-1}uu_1u_2). \tag{1.123}$$

• E is now the obstruction to ρ being a density. Set $E = 0$ and solve for the undetermined coefficients c_i. Thus,

$$3c_1 - c_2 = 0, \;\; 3c_1 - c_3 = 0, \;\; c_2 - c_3 = 0, \;\; c_3 - c_4 = 0, \;\; c_2 - c_4 = 0, \tag{1.124}$$

which yields $c_2 = c_3 = c_4 = 3c_1$, where c_1 is arbitrary.

• Substitute the solution for the c_i into the candidates for ρ and J to obtain the final density and associated flux (up to a common arbitrary factor which can be set to 1 or any other nonzero value). For the example, setting $c_1 = \frac{1}{3}$ and substituting $c_2 = c_3 = c_4 = 1$ into (1.118) and (1.123) yields $\rho^{(3)}$ and $J^{(3)}$ as given in (1.113).

1.10 Discrete Euler and Homotopy Operators

For simplicity, we will consider the case of only one discrete (dependent) variable u. First, we remove negative shifts from E. Thus, $\tilde{E} = \mathsf{D}^l E$ where l corresponds to the lowest shift in E.

The discrete variational derivative (discrete Euler operator)

An expression E is *exact* if and only if it is a total difference. The following exactness test is well-known [10,57]: A necessary and sufficient condition for a function E, with positive shifts, to be exact is that $\mathcal{L}_u^{(0)} E \equiv 0$, where $\mathcal{L}_u^{(0)}$ is the *discrete variational derivative* (discrete Euler operator of order zero) [10] defined by

$$\begin{aligned} \mathcal{L}_u^{(0)} E &= \frac{\partial}{\partial u}\left(\sum_{k=0}^{m} \mathsf{D}^{-k}\right) E = \frac{\partial}{\partial u}\left(\mathsf{I} + \mathsf{D}^{-1} + \mathsf{D}^{-2} + \mathsf{D}^{-3} + \cdots + \mathsf{D}^{-m}\right) E, \\ &= \mathsf{I}\frac{\partial E}{\partial u} + \mathsf{D}^{-1}\frac{\partial E}{\partial u_1} + \mathsf{D}^{-2}\frac{\partial E}{\partial u_2} + \mathsf{D}^{-3}\frac{\partial E}{\partial u_3} + \cdots + \mathsf{D}^{-m}\frac{\partial E}{\partial u_m}, \end{aligned} \tag{1.125}$$

where m is highest shift occurring in E.

Application. We return to (1.121) where $l = 1$. Therefore,

$$\begin{aligned} \tilde{E} = \mathsf{D}E \;=\; & (3c_1 - c_2)u_1^3 u_2 - 3c_1 u u_1^3 + 2(c_2 - c_3)u_1^2 u_2^2 - 2c_2 u u_1^2 u_2 \\ & + c_3 u_1 u_2^3 - c_3 u u_1 u_2^2 - c_4 u u_1 u_2 u_3 + (c_2 - c_4)u_1^2 u_2 u_3 \\ & + 2c_3 u_1 u_2^2 u_3 + c_4 u_1 u_2 u_3^2 + c_4 u_1 u_2 u_3 u_4, \end{aligned} \tag{1.126}$$

which is free of negative shifts. Applying (1.125) to (1.126), where $m = 4$, gives

$$\begin{aligned}
\mathcal{L}_u^{(0)}\tilde{E} &= \frac{\partial}{\partial u}\left(\mathsf{I} + \mathsf{D}^{-1} + \mathsf{D}^{-2} + \mathsf{D}^{-3} + \mathsf{D}^{-4}\right)\tilde{E} \\
&= 3(3c_1 - c_2)u^2u_1 + 3(c_3 - 3c_1)u_{-1}u^2 + 2(c_2 - c_4)uu_1u_2 \\
&\quad +4(c_2 - c_3)uu_1^2 + 4(c_3 - c_2)u_{-1}uu_1 + 2(c_3 - c_2)u_1^2u_2 \\
&\quad +(c_3 - 3c_1)u_1^3 + (c_4 - c_3)u_{-1}u_1^2 + (c_4 - c_3)u_1u_2^2 \\
&\quad +(3c_1 - c_2)u_{-1}^3 + (c_2 - c_4)u_{-1}^2u_1 + 4(c_2 - c_3)u_{-1}^2u \\
&\quad +2(c_3 - c_2)u_{-2}u_{-1}^2 + 2(c_4 - c_3)u_{-2}u_{-1}u + (c_2 - c_4)u_{-2}^2u_{-1},
\end{aligned} \tag{1.127}$$

which, as expected, vanishes identically when $c_1 = \frac{1}{3}, c_2 = c_3 = c_4 = 1$.

Due to the large amount of terms generated by the Euler operator, this method for finding the undetermined coefficients is much less efficient than the "splitting and shifting" approach illustrated on the same example in Section 1.9.2.

The discrete homotopy operator

As in the continuous case, the discrete homotopy operator reduces the inversion of the difference operator, $\Delta = \mathsf{D} - \mathsf{I}$, to a problem of single-variable calculus. Indeed, assuming that E is exact and free of negative shifts, the flux $J = \Delta^{-1}E$ can be computed without "summation by parts." Instead, one computes a single integral with respect to an auxiliary variable denoted by λ (not to be confused with λ in Section 1.9.1).

Consider an exact expression E (of one variable u), free of negative shifts, and with highest shift m. The *discrete homotopy operator* is defined [61, 71, 72] by

$$\mathcal{H}_u E = \int_0^1 (I_u E)\,[\lambda u]\,\frac{d\lambda}{\lambda}, \tag{1.128}$$

with

$$I_u E = \sum_{k=1}^{m}\left(\sum_{i=0}^{m-k} u_i\frac{\partial}{\partial u_i}\right)\mathsf{D}^{-k}E = \sum_{k=1}^{m}\left(\sum_{i=1}^{k}\mathsf{D}^{-i}\right)u_k\frac{\partial E}{\partial u_k}, \tag{1.129}$$

where $(I_u E)[\lambda u]$ means that in $I_u E$ one replaces $u \to \lambda u$, $u_1 \to \lambda u_1$, $u_2 \to \lambda u_2$, etc. The formulas in (1.129) are equivalent to the one in [52], which in turn is equivalent to the formula in terms of discrete higher Euler operators [51, 54]. Given an exact function E without negative shifts one has $J = \Delta^{-1}E = \mathcal{H}_u E$. A proof can be found in [61, 72].

Application. Upon substitution of $c_1 = \frac{1}{3}, c_2 = c_3 = c_4 = 1$ into (1.126), we obtain

$$\begin{aligned}
\tilde{E} &= \mathsf{D}E \\
&= -uu_1^3 - 2uu_1^2u_2 + u_1u_2^3 - uu_1u_2^2 - uu_1u_2u_3 \\
&= +2u_1u_2^2u_3 + u_1u_2u_3^2 + u_1u_2u_3u_4,
\end{aligned} \tag{1.130}$$

where the highest shift is $m = 4$. Using (1.129),

$$\begin{aligned} I_u\tilde{E} &= \sum_{k=1}^{4}\left(\sum_{i=1}^{k}\mathsf{D}^{-i}\right)u_k\frac{\partial\tilde{E}}{\partial u_k} = (\mathsf{D}^{-1})u_1\frac{\partial\tilde{E}}{\partial u_1} + (\mathsf{D}^{-1}+\mathsf{D}^{-2})u_2\frac{\partial\tilde{E}}{\partial u_2} \\ &\quad +(\mathsf{D}^{-1}+\mathsf{D}^{-2}+\mathsf{D}^{-3})u_3\frac{\partial\tilde{E}}{\partial u_3} + (\mathsf{D}^{-1}+\mathsf{D}^{-2}+\mathsf{D}^{-3}+\mathsf{D}^{-4})u_4\frac{\partial\tilde{E}}{\partial u_4} \\ &= \mathsf{D}^{-1}u_1(-3uu_1^2 - 4uu_1u_2 + u_2^3 - uu_2^2 - uu_2u_3 + 2u_2^2u_3 + u_2u_3^2 + u_2u_3u_4) \\ &\quad +(\mathsf{D}^{-1}+\mathsf{D}^{-2})u_2(-2uu_1^2 + 3u_1u_2^2 - 2uu_1u_2 - uu_1u_3 \\ &\quad +4u_1u_2u_3 + u_1u_3^2 + u_1u_3u_4) \\ &\quad +(\mathsf{D}^{-1}+\mathsf{D}^{-2}+\mathsf{D}^{-3})u_3(-uu_1u_2 + 2u_1u_2^2 + 2u_1u_2u_3 + u_1u_2u_4) \\ &\quad +(\mathsf{D}^{-1}+\mathsf{D}^{-2}+\mathsf{D}^{-3}+\mathsf{D}^{-4})u_4(u_1u_2u_3) \\ &= 4(uu_1^3 + 2uu_1^2u_2 + uu_1u_2^2 + uu_1u_2u_3), \end{aligned} \tag{1.131}$$

which has the correct terms of $J^{(3)}$ but incorrect coefficients. Finally, using (1.128)

$$\begin{aligned} \tilde{J} &= \mathcal{H}_u(-\tilde{E}) = -\int_0^1 (I_u\tilde{E})[\lambda u]\,\frac{d\lambda}{\lambda} \\ &= -4\int_0^1 \left(uu_1^3 + 2uu_1^2u_2 + uu_1u_2^2 + uu_1u_2u_3\right)\lambda^3\,d\lambda \\ &= -(uu_1^3 + 2uu_1^2u_2 + uu_1u_2^2 + uu_1u_2u_3). \end{aligned} \tag{1.132}$$

Recall that $\tilde{E} = \mathsf{D}E$. Hence,

$$J = \mathsf{D}^{-1}\tilde{J} = -(u_{-1}u^3 + 2u_{-1}u^2u_1 + u_{-1}uu_1^2 + u_{-1}uu_1u_2), \tag{1.133}$$

which corresponds to $J^{(3)}$ in (1.11).

The homotopy method is computationally inefficient. Even for a simple example, like (1.131), the integrand has a large number of terms, most of which eventually cancel. To compute the flux we recommend the "splitting and shifting" approach which was illustrated (on the same example) in Section 1.9.2.

The generalization of the exactness test to an expression E with multiple dependent variables $(u, v, \cdots)$ is straightforward. For example, an expression E of discrete variables u, v and their forward shifts will be exact if and only if $\mathcal{L}_{\mathbf{u}}^{(0)}E = (\mathcal{L}_u^{(0)}E, \mathcal{L}_v^{(0)}E) \equiv (0, 0)$, where $\mathcal{L}_v^{(0)}$ is defined analogously to (1.125). Similar to the continuous case, the homotopy operator formulas (1.128) and (1.129) straightforwardly generalize to multiple dependent variables. The reader is referred to [51, 52, 54] for details.

1.11 Conservation Laws of Nonlinear Systems of DDEs

We use the method discussed in Section 1.9.2 to compute conservation laws for the Toda and Ablowitz-Ladik lattices. Using the latter lattice, we illustrate a "divide and conquer"

strategy, based on multiple scales, which allows on to circumvent difficulties in the computation of densities of DDEs that are not dilation invariant.

1.11.1 The Toda Lattice

One of the earliest and most famous examples of completely integrable DDEs is the Toda lattice [93, 94],

$$\ddot{y}_n = \exp\left(y_{n-1} - y_n\right) - \exp\left(y_n - y_{n+1}\right), \tag{1.134}$$

where y_n is the displacement from equilibrium of the nth particle with unit mass under an exponential decaying interaction force between nearest neighbors. With the change of variables, $u_n = \dot{y}_n, v_n = \exp\left(y_n - y_{n+1}\right)$, lattice (1.134) can be written in algebraic form

$$\dot{u}_n = v_{n-1} - v_n, \quad \dot{v}_n = v_n(u_n - u_{n+1}). \tag{1.135}$$

Adhering to the simplified notation, we continue with

$$\dot{u} = v_{-1} - v, \quad \dot{v} = v(u - u_1). \tag{1.136}$$

As before, we set $W(\mathrm{d}/\mathrm{d}t) = 1$, assign unknown weights, $W(u), W(v)$, to the dependent variables, and require that each equation in (1.136) is uniform in rank. This yields

$$W(u) + 1 = W(v), \qquad W(v) + 1 = W(u) + W(v). \tag{1.137}$$

The solution $W(u) = 1, W(v) = 2$ reveals that (1.136) is invariant under the scaling symmetry

$$(t, u, v) \to (\lambda^{-1}t, \lambda u, \lambda^2 v), \tag{1.138}$$

where λ is an arbitrary parameter.

The Toda lattice has infinitely many conservation laws [44]. The first two density-flux pairs are easy to compute by hand. Here we give the densities of rank $R \leq 4$ with associated fluxes, $J^{(4)}$ being omitted due to length:

$$\rho^{(1)} = u, \qquad J^{(1)} = v_{-1}, \tag{1.139}$$

$$\rho^{(2)} = \tfrac{1}{2}u^2 + v, \qquad J^{(2)} = uv_{-1}, \tag{1.140}$$

$$\rho^{(3)} = \tfrac{1}{3}u^3 + u(v_{-1} + v), \qquad J^{(3)} = u_{-1}uv_{-1} + v_{-1}^2, \tag{1.141}$$

$$\rho^{(4)} = \tfrac{1}{4}u^4 + u^2(v_{-1} + v) + uu_1v + \tfrac{1}{2}v^2 + vv_1. \tag{1.142}$$

1.11.2 Computation of a Conservation Law of the Toda Lattice

As an example, we compute density $\rho^{(3)}$ (of rank $R = 3$) and associated flux $J^{(3)}$ (of rank 4) in (1.141).

Step 1: Construct the form of the density

Start from $\mathcal{V} = \{u, v\}$, i.e. the set of dependent variables with weight. List all monomials in u and v of rank $R = 3$ or less: $\mathcal{M} = \{u^3, u^2, uv, u, v\}$. Next, for each monomial in $\mathcal{M}$, introduce the correct number of t-derivatives so that each term has rank 3. Using (1.136), compute

$$\begin{aligned} &\frac{\mathrm{d}^0 u^3}{\mathrm{d}t^0} = u^3, && \frac{\mathrm{d}^0 uv}{\mathrm{d}t^0} = uv, \\ &\frac{\mathrm{d}u^2}{\mathrm{d}t} = 2u\dot{u} = 2uv_{-1} - 2uv, && \frac{\mathrm{d}v}{\mathrm{d}t} = \dot{v} = uv - u_1 v, \\ &\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} = \frac{\mathrm{d}\dot{u}}{\mathrm{d}t} = \frac{\mathrm{d}(v_{-1} - v)}{\mathrm{d}t} = u_{-1}v_{-1} - uv_{-1} - uv + u_1 v. \end{aligned} \tag{1.143}$$

Gather the terms in the right hand sides in (1.143) to get $\mathcal{R} = \{u^3, uv_{-1}, uv, u_{-1}v_{-1}, u_1 v\}$.

Identify members belonging to the same equivalence classes and replace them by their canonical representatives. For example, $uv_{-1} \equiv u_1 v$. Adhering to lexicographical ordering, we will use uv_{-1} instead of $u_1 v$. Doing so, replace $\mathcal{R}$ by $\mathcal{S} = \{u^3, uv_{-1}, uv\}$, which has the building blocks of the density. Linearly combine the monomials in $\mathcal{S}$ with undetermined coefficients c_i to get the candidate density of rank 3 :

$$\rho = c_1\, u^3 + c_2\, uv_{-1} + c_3\, uv. \tag{1.144}$$

Step 2: Compute the undetermined coefficients c_i

Compute $\mathsf{D}_t \rho$ and use (1.136) to eliminate $\dot{u}$ and $\dot{v}$ and their shifts. Thus,

$$\begin{aligned} E \;=\; & (3c_1 - c_2)u^2 v_{-1} + (c_3 - 3c_1)u^2 v + (c_3 - c_2)v_{-1} v \\ & + c_2 u_{-1} u v_{-1} + c_2 v_{-1}^2 - c_3 u u_1 v - c_3 v^2. \end{aligned} \tag{1.145}$$

Step 3: Find the associated flux J

Transform (1.145) into canonical form to obtain

$$E = (3c_1 - c_2)u_1^2 v + (c_3 - 3c_1)u^2 v + (c_3 - c_2) v v_1 + c_2 u u_1 v + c_2 v^2 - c_3 u u_1 v - c_3 v^2 \tag{1.146}$$

with

$$J = (3c_1 - c_2)u^2 v_{-1} + (c_3 - c_2)v_{-1} v + c_2 u_{-1} u v_{-1} + c_2 v_{-1}^2. \tag{1.147}$$

Set $E = 0$ to get the linear system

$$3c_1 - c_2 = 0, \quad c_3 - 3c_1 = 0, \quad c_2 - c_3 = 0. \tag{1.148}$$

Set $c_1 = \frac{1}{3}$ and substitute the solution $c_1 = \frac{1}{3}, c_2 = c_3 = 1$, into (1.144) and (1.147) to obtain $\rho^{(3)}$ and $J^{(3)}$ in (1.141).

1.11.3 The Ablowitz-Ladik Lattice

In [4–6], Ablowitz and Ladik derived and studied the following completely integrable discretization of the nonlinear Schrödinger equation:

$$i\,\dot{u}_n = u_{n+1} - 2u_n + u_{n-1} \pm u_n^* u_n (u_{n+1} + u_{n-1}), \tag{1.149}$$

where u_n^* is the complex conjugate of u_n. We continue with (1.149) with the plus sign; the case with the negative sign is analogous. Instead of splitting u_n into its real and imaginary parts, we treat u_n and $v_n = u_n^*$ as independent variables and augment (1.149) with its complex conjugate equation,

$$\begin{aligned} \dot{u}_n &= (u_{n+1} - 2u_n + u_{n-1}) + u_n v_n (u_{n+1} + u_{n-1}), \\ \dot{v}_n &= -(v_{n+1} - 2v_n + v_{n-1}) - u_n v_n (v_{n+1} + v_{n-1}), \end{aligned} \tag{1.150}$$

where i has been absorbed into a scale on t. Since $v_n = u_n^*$ we have $W(v_n) = W(u_n)$. Neither equation in (1.150) is dilation invariant. To circumvent this problem we introduce an auxiliary parameter α with weight, and replace (1.150) by

$$\begin{aligned} \dot{u} &= \alpha(u_1 - 2u + u_{-1}) + uv(u_1 + u_{-1}), \\ \dot{v} &= -\alpha(v_1 - 2v + v_{-1}) - uv(v_1 + v_{-1}), \end{aligned} \tag{1.151}$$

presented in the simplified notation. Both equations in (1.151) are uniform in rank provided

$$\begin{aligned} W(u) + 1 &= W(\alpha) + W(u) = 2W(u) + W(v), \\ W(v) + 1 &= W(\alpha) + W(v) = 2W(v) + W(u), \end{aligned} \tag{1.152}$$

which holds when $W(u) + W(v) = W(\alpha) = 1$. Since $W(u) = W(v)$ we have $W(u) = W(v) = \frac{1}{2}$, and $W(\alpha) = 1$, which expresses that (1.151) is invariant under the scaling symmetry

$$(t, u, v, \alpha) \to (\lambda^{-1} t, \lambda^{\frac{1}{2}} u, \lambda^{\frac{1}{2}} v, \lambda\alpha). \tag{1.153}$$

We give the conserved densities of (1.151) of ranks 2 through 4. Only the flux of rank 3 associated to $\rho^{(1)}$ is shown. The others are omitted due to length.

$$\rho^{(1)} = \alpha(c_1 u v_{-1} + c_2 u v_1), \tag{1.154}$$

$$\begin{aligned} J^{(1)} = &-\alpha\,[c_1(\alpha u v_{-2} - \alpha u_{-1} v_{-1} + u_{-1} u v_{-2} v_{-1}) \\ &+ c_2(\alpha u v - \alpha u_{-1} v_1 - u_{-1} u v v_1)]\,, \end{aligned} \tag{1.155}$$

$$\begin{aligned} \rho^{(2)} = &\ \alpha\left[c_1\left(\tfrac{1}{2}u^2 v_{-1}^2 + u u_1 v_{-1} v + \alpha u v_{-2}\right)\right. \\ &\left. + c_2\left(\tfrac{1}{2}u^2 v_1^2 + u u_1 v_1 v_2 + \alpha u v_2\right)\right], \end{aligned} \tag{1.156}$$

$$\begin{aligned} \rho^{(3)} = &\ \alpha\left(c_1\left[\tfrac{1}{3}u^3 v_{-1}^3 + u u_1 v_{-1} v(u v_{-1} + u_1 v + u_2 v_1) + \alpha u v_{-1}(u v_{-2} + u_1 v_{-1})\right.\right. \\ &\left. + \alpha u v(u_1 v_{-2} + u_2 v_{-1}) + \alpha^2 u v_{-3}\right] + c_2\left[+u u_1 v_1 v_2 (u v_1 + u_1 v_2 + u_2 v_3)\right] \\ &\left. + \tfrac{1}{3}u^3 v_1^3 + \alpha u v_2 (u v_1 + u_1 v_2) + \alpha u v_3 (u_1 v_1 + u_2 v_2) + \alpha^2 u v_3\right]\Big), \end{aligned} \tag{1.157}$$

where c_1 and c_2 are arbitrary constants. Our results confirm those in [6]. Since our method is restricted to polynomial densities we cannot compute the density with a logarithmic term,

$$\rho_n = \left(u_n^*(u_{n-1} + u_{n+1}) - 2\ln(1 + u_n u_n^*)\right), \tag{1.158}$$

which corresponds [3, 6] to the Hamiltonian of (1.149), viz. $H = -i\sum_n \rho_n$.

1.11.4 Computation of a Conservation Law of the Ablowitz-Ladik Lattice

To make (1.150) scaling invariant we had to introduce an auxiliary parameter α. This complicates matters in two ways as we will show in this section. First, to compute rather simple conserved densities, like $\rho = uv_{-1}$ and $\rho = uv_1$, we will have to select rank $R = 2$ for which the candidate density has twenty terms. However, eighteen of these terms eventually drop out. Second, conserved densities corresponding to lower ranks might appear in the result. These lower-rank densities are easy to recognize for they are multiplied with arbitrary coefficients c_i. Consequently, when parameters with weight are introduced, the densities corresponding to distinct ranks are no longer linearly independent.

We compute $\rho^{(1)}$ of rank $R = 2$ in (1.154) with associated flux $J^{(1)}$ in (1.155). Note that $\rho^{(1)}$ and $J^{(1)}$ cannot be computed with the steps below when $R = 1$.

Step 1: **Construct the form of the density**

Augment the set of dependent variables with the parameter α (with non-zero weight). Hence, $\mathcal{V} = \{u, v, \alpha\}$. Construct

$$\begin{aligned}\mathcal{M} \;=\; \{&u, v, u^2, uv, v^2, \alpha u, u^3, \alpha v, u^2 v, uv^2, v^3, \alpha u^2,\\ &u^4, \alpha uv, u^3 v, \alpha v^2, u^2 v^2, uv^3, v^4\},\end{aligned} \tag{1.159}$$

which contains all non-constant monomials of (chosen) rank 2 or less (without shifts). As with the previous examples, for each element in $\mathcal{M}$ add the right number of $t-$derivatives. Use (1.151) to evaluate the $t-$derivatives, gather the terms in the right hand sides, introduce the canonical representatives (based on lexicographical ordering), and remove duplicates to get

$$\begin{aligned}\mathcal{S} \;=\; \{&\alpha u^2, u^4, \alpha u u_1, \alpha u v_{-1}, \alpha uv, u^3 v, u^2 u_1 v, u^2 v_{-1} v, \alpha v^2, u^2 v^2, u u_1 v^2,\\ &uv_{-1}v^2, uv^3, v^4, \alpha u v_1, u u_1^2 v_1, \alpha v v_1, u^2 v v_1, uv^2 v_1, u u_1 v_1^2\}.\end{aligned} \tag{1.160}$$

Linearly combine the monomials in $\mathcal{S}$ with undetermined coefficients c_i to get the candidate density of rank 2:

$$\begin{aligned}\rho \;=\; &c_1\,\alpha u^2 + c_2\,u^4 + c_3\,\alpha u u_1 + c_4\,\alpha u v_{-1} + c_5\,\alpha uv + c_6\,u^3 v + c_7\,u^2 u_1 v\\ &+c_8\,u^2 v_{-1} v + c_9\,\alpha v^2 + c_{10}\,u^2 v^2 + c_{11}\,u u_1 v^2 + c_{12}\,uv_{-1}v^2 + c_{13}\,uv^3\\ &+c_{14}\,v^4 + c_{15}\,\alpha u v_1 + c_{16}\,u u_1^2 v_1 + c_{17}\,\alpha v v_1 + c_{18}\,u^2 v v_1\\ &+c_{19}\,uv^2 v_1 + c_{20}\,u u_1 v_1^2.\end{aligned} \tag{1.161}$$

Step 2: Compute the undetermined coefficients c_i

The computations proceed as in the examples in Sections 1.9.2 and 1.11.2. Thus, compute $\mathsf{D}_t\rho$ and use (1.151) to eliminate $\dot{u}$ and $\dot{v}$ and their shifts. Next, bring the expression E into canonical form to obtain the linear system for the undetermined coefficients c_i.

Without showing the lengthy computations, one finds that all constants $c_i = 0$, except c_4 and c_{15} which are arbitrary. Substitute the coefficients into (1.161) to get $\rho^{(1)}$ in (1.154).

Step 3: **Find the associated flux** J

The associated flux comes for free when E is transformed into canonical form. Alternatively, one could apply the homotopy approach for multiple dependent variables [51, 54] to compute the flux. In either case, one gets $J^{(1)}$ in (1.155).

1.11.5 A "Divide and Conquer" Strategy

It should be clear from the example in Section 1.11.4 that our method is not efficient if the densities are not of the form (1.111)-(1.114) for the KvM lattice and (1.139)-(1.142) for the Toda lattice. Indeed, the densities for the AL lattice in (1.154)-(1.157) are quite different in structure. Therefore, in [33], Eklund presented alternate strategies to deal more efficiently with DDEs, in particular with those involving weighted parameters such as (1.151).

A first alternative is to work with *multiple scales* by setting either $W(\mathsf{D}_t) = 0$ or $W(\mathsf{D}_t) = 1$, the latter choice is what we have used thus far. A second possibility is to leave $W(\mathsf{D}_t)$ unspecified and, if needed, introduce extra parameters with weight into the DDE.

Let us explore these ideas for the AL lattice (1.151), which we therefore replace by

$$\begin{aligned} \dot{u} &= \alpha\beta(u_1 - 2u + u_{-1}) + \beta uv(u_1 + u_{-1}), \\ \dot{v} &= -\alpha\beta(v_1 - 2v + v_{-1}) - \beta uv(v_1 + v_{-1}), \end{aligned} \tag{1.162}$$

where β is a second auxiliary parameter with weight. Requiring uniformity in rank leads to

$$\begin{aligned} W(u) + W(\mathsf{D}_t) &= W(\alpha) + W(\beta) + W(u) = W(\beta) + 2W(u) + W(v), \\ W(v) + W(\mathsf{D}_t) &= W(\alpha) + W(\beta) + W(v) = W(\beta) + 2W(v) + W(u), \end{aligned} \tag{1.163}$$

which are satisfied when $W(\alpha) = W(u) + W(v)$ and $W(\mathsf{D}_t) = W(u) + W(v) + W(\beta)$. Therefore, we can set $W(\mathsf{D}_t) = a, W(u) = b$, and $W(\beta) = c$ with a, b, c rational numbers so that $W(v) = a - b - c$ and $W(\alpha) = a - c$ are strictly positive. Thus (1.162) is dilation invariant under a three-parameter family of scaling symmetries,

$$(t, u, v, \alpha, \beta) \to (\lambda^{-a}t, \lambda^b u, \lambda^{a-b-c}v, \lambda^{a-c}\alpha, \lambda^c\beta). \tag{1.164}$$

Scaling symmetry (1.153) corresponds to the case where $a = 1, b = \frac{1}{2}$, and $c = 0$, more precisely, $\beta = 1$. The fact that (1.162) is invariant under multiple scales is advantageous.

Indeed, one can use the invariance under one scale to construct a candidate density and, subsequently, use additional scale(s) to split ρ into smaller densities. This "divide and conquer" strategy drastically reduces the complexity of the computations. The use of multiple scales has proven to be successful in the computation of conservation laws for PDEs with more than one space variable [51].

Table 1.4: Candidate densities for the Ablowitz-Ladik lattice (1.162).

i	Rank	Candidate ρ_i	Final ρ_i	Final $\mathbf{J}_i$
1	$a+2b-c$	$c_1\alpha u^2 + c_3\alpha u u_1 + c_6 u^3 v$ $+c_7 u^2 u_1 v + c_{16} u u_1^2 v_1$	0	0
2	$4b$	$c_2 u^4$	0	0
3	$2(a-c)$	$c_4\alpha u v_{-1} + c_5\alpha u v + c_8 u^2 v_{-1} v$ $+c_{10} u^2 v^2 + c_{11} u u_1 v^2 + c_{15}\alpha u v_1$ $+c_{18} u^2 v v_1 + c_{20} u u_1 v_1^2$	$c_4\alpha u v_{-1}$ $+c_{15}\alpha u v_1$	$J^{(1)}$ in (1.155)
4	$3a-2b-3c$	$c_9\alpha v^2 + c_{12} u v_{-1} v^2 + c_{13} u v^3$ $+c_{17}\alpha v v_1 + c_{19} u v^2 v_1$	0	0
5	$4(a-b-c)$	$c_{14} v^4$	0	0

Candidate density (1.161) is uniform of rank 2 under (1.153) but can be split into smaller pieces, ρ_i, using (1.164), even without specifying values for a, b, and c. Indeed, as shown in Table 1.4, ρ in (1.161) can be split into ρ_1 through ρ_5 of distinct ranks under (1.164). Steps 2 and 3 of the algorithm are then carried out for each of these ρ_i, $(i = 1, \cdots, 5)$ separately. As shown in the table, all but one density lead to a trivial result. The longest density, ρ_3, with 8 terms, leads to $\rho^{(1)}$ and $J^{(1)}$ in (1.154) and (1.155), respectively.

1.12 A New Method to Compute Conservation Laws of Nonlinear DDEs

In the continuous case, the total derivative D_x has weight one. Consequently, any density is bounded with respect to the order in x. For example, as shown in Section 1.3.2 for the KdV case, the candidate density ρ of rank 6 in (1.18) is of first order (after removing the second and fourth order terms.) Obviously, a density of rank 6 could never have leading order terms of fifth or higher order in x because the rank of such terms would exceed 6.

In the discrete case, however, the shift operator D has no weight. Thus, any shift $\mathsf{D}^k u = u_k$ of a dependent variable u has the same weight as that dependent variable. Simply put, $W(u_k) = W(u)$ for any integer k. Consequently, using the total derivative D_t as a tool to construct a (candidate) density has a major shortcoming: the density may lack terms

involving sufficiently high shifts of the variables. As shown in Section 1.9.2 for the KvM lattice, the candidate density ρ of rank 3 in (1.118) has leading order term uu_1u_2, i.e. the term with the highest shift (2 in this example). It is *a priori* not excluded that ρ might have terms involving higher shifts. For example, uu_1u_3 has rank 3 and so do infinitely many other cubic terms. Note that for this example we constructed ρ starting from powers of u, viz. u^3, u^2, u; and, by repeated differentiation, worked our way "down" towards the leading order term uu_1u_2. In this section we outline the key features of a new method which goes in the opposite direction: (i) first compute the leading order term and subsequently (ii) compute the terms involving lower shifts. In step (ii) only the necessary terms are computed, nothing more, nothing less. This method is fast and powerful for it circumvents the use of the dilation invariance and the method of undetermined coefficients. More importantly, the new method is not restricted to densities and fluxes of polynomial form.

1.12.1 Leading Order Analysis

Consider a density, ρ, that depends on q shifts. Since $\mathsf{D}^q\,\rho$ is also a density, we may, without loss of generality, assume that ρ has canonical form $\rho(\mathbf{u},\ \mathbf{u}_1,\ \ldots,\ \mathbf{u}_q)$ with $\frac{\partial^2\rho}{\partial\mathbf{u}\,\partial\mathbf{u}_q} \neq 0$. In [55], Hickman derived necessary conditions on this leading term (which, in the system case, is a matrix). First all terms in the candidate density ρ that contribute directly to the flux are removed. Rather than applying the Euler operator on the remaining terms in ρ, the necessary condition [57],

$$\frac{\partial^2 g}{\partial\mathbf{u}\,\partial\mathbf{u}_q} = 0, \tag{1.165}$$

for g to be a total difference is applied to obtain a system of equations for the terms that depend on the maximal shift, $\mathbf{u}_q$, in ρ. This system is rewritten as a matrix equation. Solutions to this system will give us the form of the leading term in ρ.

We apply a splitting of the identity operator,

$$\mathsf{I} = (\mathsf{D} - \mathsf{I} + \mathsf{I})\,\mathsf{D}^{-1} = \Delta\,\mathsf{D}^{-1} + \mathsf{D}^{-1}, \tag{1.166}$$

to the part, ρ^*, of the candidate ρ that is independent of the variables with the lowest order shift. The first term $\Delta\,\mathrm{D}^{-1}\,\rho^*$ contributes to the flux while the second term $\mathrm{D}^{-1}\,\rho^*$ has a strictly lower shift than ρ^*. Applying this split repeatedly we get

$$\mathsf{I} = (\mathsf{D}^k - \mathsf{I} + \mathsf{I})\,\mathsf{D}^{-k} = \Delta\,\left(\mathsf{D}^{k-1} + \mathsf{D}^{k-2} + \cdots + \mathsf{D} + \mathsf{I}\right)\mathsf{D}^{-k} + \mathsf{D}^{-k}, \tag{1.167}$$

where, again, the first term contributes to the flux and the remainder has strictly lower shift.

This decomposition is repeatedly applied to terms that do not involve the lowest order shifted variables. Any terms that remain will involve the lowest order shifted variable. These terms yield the constraints on the undetermined coefficients or unknown functions in

the density ρ. As shown in [55], the result of this split is that ρ is a density if and only if

$$\sigma = \mathsf{D}^l\left(\mathbf{F}\,\frac{\partial}{\partial \mathbf{u}}\right)\sum_{j=0}^{l}\mathsf{D}^j\rho + \sum_{j=l+1}^{q}\mathsf{D}^j\left(\mathbf{F}\,\frac{\partial}{\partial \mathbf{u}}\right)\rho \tag{1.168}$$

is a total difference, where the operator

$$\mathbf{F}\,\frac{\partial}{\partial \mathbf{u}} = \sum_{\alpha} F^{\alpha}\frac{\partial}{\partial u^{\alpha}} \tag{1.169}$$

involves a summation over the components in the system of DDEs. As before, in the examples we will use u, v, w, etc. to denote the dependent variables u^{α}.

Now, σ depends on the shifted variables $\mathbf{u},\ \ldots,\ \mathbf{u}_{q+L}$ where $L = \max(l,\ m)$. For $q > L$, (1.165) gives

$$\frac{\partial^2\sigma}{\partial \mathbf{u}\,\partial \mathbf{u}_{q+L}} = \mathsf{D}^L\left(\frac{\partial^2\rho}{\partial \mathbf{u}\,\partial \mathbf{u}_q}\,\frac{\partial \mathbf{F}}{\partial \mathbf{u}_{-L}}\right) + \mathsf{D}^q\left(\frac{\partial \mathbf{F}}{\partial \mathbf{u}_L}\right)^{\mathrm{T}}\frac{\partial^2\rho}{\partial \mathbf{u}\,\partial \mathbf{u}_q} = 0, \tag{1.170}$$

where T stands for transpose. The system case is treated in detail in [55]. For brevity, we continue with the scalar case. Let

$$\lambda = \frac{\partial F}{\partial u_{-L}}, \quad \mu = \frac{\partial F}{\partial u_L}, \tag{1.171}$$

then the leading term will satisfy

$$\mathcal{S}\,\frac{\partial^2\rho}{\partial u\,\partial u_q} = 0, \tag{1.172}$$

with $\mathcal{S} = \mathsf{D}^L\,\lambda\,\mathsf{D}^L + \mathsf{D}^q\,\mu$. We immediately see that if $l \neq m$ then either λ or μ is zero and (1.172) has no non-trivial solutions. Let $q = pL + r$ with p and r integers, $0 \leq r < L$, and

$$c = \left(\prod_{k=1}^{p-1}\mathsf{D}^{kL}\lambda\right)\mathsf{D}^{pL}\zeta. \tag{1.173}$$

Then

$$\mathcal{S}\,c = \prod_{k=1}^{p-1}\mathsf{D}^{kL}\lambda\,\mathsf{D}^{pL}\left(\lambda\,\mathsf{D}^L\zeta + \zeta\,\mathsf{D}^r\mu\right). \tag{1.174}$$

Thus $c \neq 0$ lies in the kernel of $\mathcal{S}$ if and only if

$$\lambda\,\mathsf{D}^L\zeta + \zeta\,\mathsf{D}^r\mu = 0 \tag{1.175}$$

has a non-zero solution ζ. Suppose (1.175) has two non-zero solutions, say, ζ_1 and ζ_2. Then,

$$\frac{\zeta_2}{\zeta_1} = \mathsf{D}^L\left(\frac{\zeta_2}{\zeta_1}\right). \tag{1.176}$$

So, since $L \neq 0$, $\zeta_2 = a\zeta_1$ for some constant a. Therefore, the kernel of $\mathcal{S}$ is, *at most*, one dimensional. This implies that a scalar DDE can have, at most, one conserved density that depends on q shifts for $q > L$. The leading term, $\tilde{\rho}$, will satisfy

$$\frac{\partial^2 \rho}{\partial u\, \partial u_q} = c, \tag{1.177}$$

which, upon integration, yields

$$\tilde{\rho} = \iint c\, du\, du_q. \tag{1.178}$$

The density (if it exists) may now be computed by a "split and shift" strategy on this leading term. Starting with $\rho = \tilde{\rho}$. the objective is to successively compute the terms (of lower shift!) that must be added to ρ until $\mathsf{D}_t\, \rho \equiv 0$. First, $\mathsf{D}_t\, \rho$ is computed and evaluated on the DDE. Next, all terms are shifted so that the resulting expression depends on u (and not on lower shifts of u). Then the leading terms, ξ, in $\mathsf{D}_t\, \rho$ are isolated. Last, we solve

$$\mathsf{D}_t\, \rho^{(1)} = \xi + \text{terms of lower shift}. \tag{1.179}$$

If (1.179) has no solution then a density with q shifts does not exist. On the other hand, if (1.179) has a solution, the "correction" term $\rho^{(1)}$ is then subtracted from ρ and we recompute $\mathsf{D}_t\, \rho$. By construction, the highest shift that occurs in the result will now be lower than before and we repeat the entire procedure to obtain a new correction term $\rho^{(2)}$. After a finite number of steps, we will either find an that the correction term does not exist (and so the density does not exist) or we will obtain $\mathsf{D}_t\, \rho \equiv 0$ and ρ will be a density.

This algorithm has been coded [56] in `Maple`. Further details and worked examples will be presented in [58]. For now, we illustrate the algorithm with a simple example.

1.12.2 A"Modified" Volterra Lattice

Consider the DDE,

$$\dot{u} = u^2\, (u_2 - u_{-2}), \tag{1.180}$$

which is related to the well-known modified Volterra Lattice [11]. Here $L = 2$ and, using (1.171),

$$\lambda = \frac{\partial F}{\partial u_{-2}} = -\, u^2, \quad \mu = \frac{\partial F}{\partial u_2} = u^2. \tag{1.181}$$

Thus, the condition (1.175) for a non-trivial density becomes $\zeta\, u_r^2 = u^2\, \mathsf{D}^2 \zeta$ for $r = 0,\ 1$. For $r = 0$, we have $\zeta = \mathsf{D}^2 \zeta$ and so we may choose $\zeta = 1$. For $r = 1$, we have $\zeta\, u_1^2 = u^2\, \mathsf{D}^2 \zeta$, which has no non-zero solutions. Thus, densities only exist for $r = 0$. Since $q = pL = 2p$, with p integer, we conclude that q must be even. In these cases the leading term will satisfy

$$\frac{\partial^2 \rho}{\partial u\, \partial u_q} = c = \left(\prod_{k=1}^{p-1} u_{2k} \right) = u\, u_2\, \cdots\, u_{q-2}. \tag{1.182}$$

Therefore, after scaling to remove constants, the leading term is $\tilde{\rho} = u^2\, u_2 \cdots u_{q-2}\, u_q$.

For example, let us compute the density that depends on $q = 4$ shifts. We set $\rho = \tilde{\rho} = u^2\, u_2\, u_4$. Using (1.180),

$$\begin{aligned} \mathsf{D}_t\, \rho &= u^2 u_2^2 u_4\, (u_2 - u_{-2}) + 2u\, u_2^3\, u_4\, (u_4 - u) + u\, u_2^2\, u_4^2\, (u_6 - u_2) \\ &\equiv u\, u_2^3\, u_4^2 - u^2\, u_2^3\, u_4. \end{aligned} \tag{1.183}$$

The highest shift is u_4 and so the leading terms are

$$\xi = u\, u_2^3\, u_4^2 - u^2\, {u_2}^3\, u_4. \tag{1.184}$$

Note that the terms in u_6 *must* cancel by the construction of $\tilde{\rho}$. Next, we note that u_4 as a leading term must arise as either $\dot{u}_2 = u_2^2\, (u_4 - u)$ or

$$u_2\, \dot{u} = u_2\, u^2\, (u_2 - u_{-2}) \equiv u^2\, u_2^2 - u\, u_2^2\, u_4. \tag{1.185}$$

The quadratic term must arise from (1.185) (as we have already determined all terms that involve u_4, so, we cannot have a quadratic term given by $u_4 \dot{u}_2$). Now, we solve (1.179) to get

$$\rho^{(1)} = -\frac{1}{2}\, u^2\, u_2^2 + \cdots . \tag{1.186}$$

Indeed,

$$\begin{aligned} \mathsf{D}_t \left(-\tfrac{1}{2}\, u^2\, u_2^2\right) &= -u\, u_2^2\, \dot{u} - u^2\, u_2\, \dot{u}_2 \\ &\equiv u\, u_2^3\, u_4^2 - u^2\, u_2^3\, u_4 + \text{lower shift terms} \\ &= \xi + \text{lower shift terms}, \end{aligned} \tag{1.187}$$

with ξ in (1.184). Thus, we update $\rho = u^2 u_2 u_4$ by subtracting $\rho^{(1)}$. This yields,

$$\rho = u^2\, u_2\, u_4 + \frac{1}{2}\, u^2\, u_2^2. \tag{1.188}$$

We readily verify that $\mathsf{D}_t\, \rho \equiv 0$ and, thus, ρ in (1.188) is the unique density of (1.180) that depends on 4 shifts.

1.13 The Gardner Lattice

In this section we consider the DDE described in [91],

$$\begin{aligned} \dot{u} &= \left(1 + \alpha h^2 u + \beta h^2 u^2\right) \left\{ \frac{1}{h^3} \left(\frac{1}{2} u_{-2} - u_{-1} + u_1 - \frac{1}{2} u_2 \right) \right. \\ &\quad + \frac{\alpha}{2h} \left[u_{-1} u_{-2} + u_{-1}^2 + u(u_{-1} - u_1) - u_1^2 - u_1 u_2 \right] \\ &\quad \left. + \frac{\beta}{2h} \left[u_{-1}^2 (u_{-2} + u) - u_1^2 (u + u_2) \right] \right\}, \end{aligned} \tag{1.189}$$

which is a completely integrable discretization of the Gardner equation,

$$u_t + 6\alpha u u_x + 6\beta u^2 u_x + u_{3x} = 0. \tag{1.190}$$

Therefore, we call (1.189) the Gardner lattice. Note that (1.190) is a combination of the KdV equation ($\beta = 0$) and the mKdV equation ($\alpha = 0$). Consequently, (1.189) includes the completely integrable discretizations of the KdV and mKdV equations as special cases.

1.13.1 Derivation of the Gardner Lattice

Based on work by Taha [91], we outline the derivation of the Gardner lattice from a discretized version [4] of the eigenvalue problem of Zakharov and Shabat. Consider the discrete system [1, 7],

$$V_{1,n+1} = zV_{1,n} + Q_n(t)V_{2,n}, \qquad V_{2,n+1} = \frac{1}{z}V_{2,n} + R_n(t)V_{1,n}, \tag{1.191}$$

$$\dot{V}_{1,n} = A_n(z)V_{1,n} + B_n(z)V_{2,n}, \quad \dot{V}_{2,n} = C_n(z)V_{1,n} + D_n(z)V_{2,n}, \tag{1.192}$$

where, in general, the coefficients A_n through D_n depend on the potentials, R_n and Q_n. The eigenvalue z is assumed to be time-independent.

Step 1: Expressing the compatibility conditions, $\dot{V}_{i,n+1} = \mathsf{D}\dot{V}_{i,n}$, for $i = 1, 2$, and equating the coefficients of $V_{1,n}$ and $V_{2,n}$, we get [7]

$$\begin{aligned} C_nQ_n - B_{n+1}R_n &= z(A_{n+1} - A_n), \\ B_nR_n - C_{n+1}Q_n &= \frac{1}{z}(D_{n+1} - D_n), \end{aligned} \tag{1.193}$$

$$\begin{aligned} \dot{Q}_n + D_nQ_n - A_{n+1}Q_n &= \frac{1}{z}B_{n+1} - zB_n, \\ \dot{R}_n + A_nR_n - D_{n+1}R_n &= \frac{1}{z}C_{n+1} - zC_n. \end{aligned} \tag{1.194}$$

Step 2: Substituting the expansions [91],

$$A_n = \sum_{j=-2}^{2} z^{2j} A_n^{(2j)}, \qquad D_n = \sum_{j=-2}^{2} z^{2j} D_n^{(2j)}, \tag{1.195}$$

$$B_n = \sum_{j=-1}^{2} z^{2j-1} B_n^{(2j-1)}, \quad C_n = \sum_{j=-1}^{2} z^{2j-1} C_n^{(2j-1)}, \tag{1.196}$$

into (1.193) and (1.194) and setting the coefficients of like powers of z equal to zero, we obtain a system of twenty equations for the eighteen unknowns functions $A_n^{(2j)}, D_n^{(2j)}$ with

$j = -2, -1, 0, 1, 2$; and $B_n^{(2j-1)}$, $C_n^{(2j-1)}$ with $j = -1, 0, 1, 2$. The simplest equations arise from the coefficients of the terms in $z^{\pm 5}$ and $z^{\pm 4}$:

$$A_{n+1}^{(4)} - A_n^{(4)} = 0, \quad D_{n+1}^{(-4)} - D_n^{(-4)} = 0, \tag{1.197}$$

$$Q_n(D_n^{(4)} - A_{n+1}^{(4)}) + B_n^{(3)} = 0, \quad R_n(A_n^{(-4)} - D_{n+1}^{(-4)}) + C_n^{(-3)} = 0, \tag{1.198}$$

$$Q_n(D_n^{(-4)} - A_{n+1}^{(-4)}) - B_{n+1}^{(-3)} = 0, \quad R_n(A_n^{(4)} - D_{n+1}^{(4)}) - C_{n+1}^{(3)} = 0. \tag{1.199}$$

Step 3: We outline the solution process which follows the strategy in [7, Section 2.2a]. From (1.197), we conclude that $A_n^{(4)}$ and $D_n^{(-4)}$ are independent of n. Hence, $A_n^{(4)} = \tilde{A}^{(4)}$ and $D_n^{(-4)} = \tilde{D}^{(-4)}$ are constants. The tilde will remind us that we are dealing with *constants*. Solving (1.198) and (1.199), we get

$$B_n^{(3)} = (\tilde{A}^{(4)} - D_n^{(4)})Q_n, \qquad C_n^{(-3)} = (\tilde{D}^{(-4)} - A_n^{(-4)})R_n, \tag{1.200}$$

$$B_n^{(-3)} = (\tilde{D}^{(-4)} - A_n^{(-4)})Q_{n-1}, \quad C_n^{(3)} = (\tilde{A}^{(4)} - D_n^{(4)})R_{n-1}. \tag{1.201}$$

Substituting these results into two of the equations coming from $z^{\pm 3}$, we find that $A_n^{(-4)} = \tilde{A}^{(-4)}$ and $D_n^{(4)} = \tilde{D}^{(4)}$ are constants. From equations corresponding to $z^{\pm 3}, z^{\pm 1}$, we obtain

$$\Delta A_n^{(2)} = A_{n+1}^{(2)} - A_n^{(2)} = \tilde{\alpha}(Q_n R_{n-1} - Q_{n+1} R_n), \tag{1.202}$$

$$\Delta D_n^{(-2)} = D_{n+1}^{(-2)} - D_n^{(-2)} = \tilde{\beta}(R_n Q_{n-1} - R_{n+1} Q_n), \tag{1.203}$$

$$\Delta A_n^{(-2)} = A_{n+1}^{(-2)} - A_n^{(-2)} = \tilde{\beta}(Q_n R_{n+1} - Q_{n-1} R_n), \tag{1.204}$$

$$\Delta D_n^{(2)} = D_{n+1}^{(2)} - D_n^{(2)} = \tilde{\alpha}(R_n Q_{n+1} - R_{n-1} Q_n), \tag{1.205}$$

where $\tilde{\alpha} = \tilde{A}^{(4)} - \tilde{D}^{(4)}$ and $\tilde{\beta} = \tilde{D}^{(-4)} - \tilde{A}^{(-4)}$. Equations (1.202)-(1.205) are satisfied for

$$A_n^{(2)} = \tilde{A}^{(2)} - \tilde{\alpha} Q_n R_{n-1}, \qquad D_n^{(-2)} = \tilde{D}^{(-2)} - \tilde{\beta} R_n Q_{n-1}, \tag{1.206}$$

$$A_n^{(-2)} = \tilde{A}^{(-2)} + \tilde{\beta} Q_{n-1} R_n, \quad D_n^{(2)} = \tilde{D}^{(2)} + \tilde{\alpha} R_{n-1} Q_n, \tag{1.207}$$

where $\tilde{A}^{(2)}, \tilde{D}^{(-2)}$ and $\tilde{A}^{(-2)}, \tilde{D}^{(2)}$ are constants. Next, we solve equations (from the terms in $z^{\pm 2}$) for $B_n^{(\pm 1)}$ and $C_n^{(\pm 1)}$, yielding

$$B_n^{(1)} = \tilde{\delta} Q_n + \tilde{\alpha} Q_{n+1} - \tilde{\alpha} Q_n (Q_{n+1} R_n + Q_n R_{n-1}), \tag{1.208}$$

$$C_n^{(-1)} = \tilde{\gamma} R_n + \tilde{\beta} R_{n+1} - \tilde{\beta} R_n (R_{n+1} Q_n + R_n Q_{n-1}), \tag{1.209}$$

$$B_n^{(-1)} = \tilde{\gamma} Q_{n-1} + \tilde{\beta} Q_{n-2} - \tilde{\beta} Q_{n-1} (R_{n-1} Q_{n-2} + R_n Q_{n-1}), \tag{1.210}$$

$$C_n^{(1)} = \tilde{\delta} R_{n-1} + \tilde{\alpha} R_{n-2} - \tilde{\alpha} R_{n-1} (Q_{n-1} R_{n-2} + Q_n R_{n-1}), \tag{1.211}$$

where $\tilde{\delta} = \tilde{A}^{(2)} - \tilde{D}^{(2)}$ and $\tilde{\gamma} = \tilde{D}^{(-2)} - \tilde{A}^{(-2)}$. Next, we solve equations coming from $z^{\pm 1}$, which involve $\Delta A_n^{(0)}$ and $\Delta D_n^{(0)}$. These equations, which are similar in structure to

(1.202)-(1.205), yield

$$\begin{aligned} A_n^{(0)} &= \tilde{\alpha}\,(-Q_{n+1}R_{n-1} - Q_nR_{n-2} + Q_nQ_{n+1}R_{n-1}R_n \\ &\quad + Q_{n-1}Q_nR_{n-2}R_{n-1} + Q_n^2R_{n-1}^2) + \tilde{A}^{(0)} - \tilde{\delta}R_{n-1}Q_n, \qquad (1.212) \\ D_n^{(0)} &= \tilde{\beta}\,(-R_{n+1}Q_{n-1} - R_nQ_{n-2} + R_nR_{n+1}Q_{n-1}Q_n \\ &\quad + R_{n-1}R_nQ_{n-2}Q_{n-1} + R_n^2Q_{n-1}^2) + \tilde{D}^{(0)} - \tilde{\gamma}Q_{n-1}R_n, \qquad (1.213) \end{aligned}$$

where $\tilde{A}^{(0)}$ and $\tilde{D}^{(0)}$ are constants. All the difference equations for the unknown coefficients are now satisfied. We are left with the two DDEs (coming from the terms in z^0),

$$\begin{aligned} &\dot{Q}_n - \tilde{\kappa}Q_n + (1 - Q_nR_n)\Big[-\tilde{\alpha}Q_{n+2} - \tilde{\delta}Q_{n+1} + \tilde{\gamma}Q_{n-1} + \tilde{\beta}Q_{n-2} \\ &\quad + \tilde{\alpha}Q_{n+1}(Q_{n+2}R_{n+1} + Q_{n+1}R_n) - \tilde{\beta}Q_{n-1}(Q_{n-2}R_{n-1} + Q_{n-1}R_n) \\ &\quad + \tilde{\alpha}Q_nQ_{n+1}R_{n-1} - \tilde{\beta}Q_nQ_{n-1}R_{n+1}\Big] = 0, \qquad (1.214) \\ &\dot{R}_n + \tilde{\kappa}R_n + (1 - R_nQ_n)\Big[-\tilde{\beta}R_{n+2} - \tilde{\gamma}R_{n+1} + \tilde{\delta}R_{n-1} + \tilde{\alpha}R_{n-2} \\ &\quad + \tilde{\beta}R_{n+1}(R_{n+2}Q_{n+1} + R_{n+1}Q_n) - \tilde{\alpha}R_{n-1}(R_{n-2}Q_{n-1} + R_{n-1}Q_n) \\ &\quad + \tilde{\beta}R_nR_{n+1}Q_{n-1} - \tilde{\alpha}R_nR_{n-1}Q_{n+1}\Big] = 0, \qquad (1.215) \end{aligned}$$

where $\tilde{\kappa} = \tilde{A}^{(0)} - \tilde{D}^{(0)}$.

Step 4: The terms $-\tilde{\kappa}Q_n$ in (1.214) and $\tilde{\kappa}R_n$ in (1.215) could be removed with a suitable transformation. Accomplishing the same, we set $\tilde{\kappa} = 0$. Next, we substitute

$$Q_n = u_n, \quad R_n = -h^2(\alpha + \beta u_n), \qquad (1.216)$$

into (1.214) and (1.215). Next, we set $\tilde{\beta} = \tilde{\alpha}$ and $\tilde{\gamma} = \tilde{\delta}$ to remove all constant terms. Doing so, (1.214) and (1.215) collapse into a single DDE,

$$\begin{aligned} &\dot{u}_n + (1 + \alpha h^2u_n + \beta h^2u_n^2)\Big\{\tilde{\alpha}(u_{n-2} - u_{n+2}) + \tilde{\delta}(u_{n-1} - u_{n+1}) \\ &\quad + \alpha\tilde{\alpha}h^2(u_{n-2}u_{n-1} + u_{n-1}^2 + u_n(u_{n-1} - u_{n+1}) - u_{n+1}^2 - u_{n+1}u_{n+2}) \\ &\quad + \beta\tilde{\alpha}h^2(u_{n-1}^2(u_{n-2} + u_n) - u_{n+1}^2(u_n + u_{n+2}))\Big\} = 0, \qquad (1.217) \end{aligned}$$

Step 5: To fix the scale on t, as well as the constants $\tilde{\alpha}$, $\tilde{\beta}$, and $\tilde{\delta}$, we consider the limit of (1.217) for $h \to 0$. Using, $\lim_{h\to 0} u_n(t) = u(x,t)$, $\lim_{h\to 0} \dot{u}_n(t) = u_t(x,t)$, and substituting

$$\begin{aligned} \lim_{h\to 0} u_{n+m}(t) &= u(x,t) + mh\,u_x(x,t) + \frac{1}{2}(mh)^2u_{2x}(x,t) \\ &\quad + \frac{1}{6}(mh)^3u_{3x}(x,t) + \cdots, \qquad (1.218) \end{aligned}$$

into (1.217), we obtain

$$u_t - 2h(\tilde{\delta} + 2\tilde{\alpha})u_x - 2h^3(\tilde{\delta} + 8\tilde{\alpha})\left(\alpha u u_x + \beta u^2 u_x + \tfrac{1}{6}u_{3x}\right) + \mathcal{O}(h^5) = 0. \quad (1.219)$$

The term in h disappears when $\tilde{\delta} = -2\tilde{\alpha}$. Substituting $\tilde{\alpha} = -\frac{1}{2}, \tilde{\delta} = 1$, into (1.217) and (1.219), we get

$$\begin{aligned} \dot{u} \;=\; & \left(1 + \alpha h^2 u + \beta h^2 u^2\right)\left\{\frac{1}{2}u_{-2} - u_{-1} + u_1 - \frac{1}{2}u_2\right. \\ & + \frac{1}{2}\alpha h^2\left[u_{-1}u_{-2} + u_{-1}^2 + u(u_{-1} - u_1) - u_1^2 - u_1 u_2\right] \\ & \left. + \frac{1}{2}\beta h^2\left[u_{-1}^2(u_{-2} + u) - u_1^2(u + u_2)\right]\right\}, \end{aligned} \quad (1.220)$$

and

$$u_t + h^3(6\alpha u u_x + 6\beta u^2 u_x + u_{3x}) = 0, \quad (1.221)$$

where the $\mathcal{O}(h^5)$ terms were ignored. Using a scale, $t \to h^3 t$, we get (1.189) and (1.190).

Remarks. Step 2 is well suited for a CAS. Solving the system, as outlined in Step 3, is a challenging task, in particular, if attempted with pen and paper. The system consists of two DDEs and eighteen difference equations. None of the CAS has a build-in solver for such mixed systems. A fully automated solution is therefore impossible. We used a feedback mechanism which mimics what one would do by hand: Solve the simplest difference equations; enter that partial solution; let CAS simplify the entire system; repeat the process until all difference equations are satisfied and the two DDEs are simplified as far as possible. Continuing with (1.214) and (1.215), steps 4 and 5 were straightforward to implement. The five steps have been implemented in `Mathematica` [50]. Starting from (1.191), the code generates the Gardner lattice (1.220). The code could be modified to assist in the derivation of other completely integrable DDEs, such as discrete versions of the nonlinear Schrödinger and sine-Gordon equations.

1.13.2 Dilation Invariance of the Gardner Lattice

Since (1.189) is not uniform in rank we must introduce auxiliary parameters with weight. This can be done in several ways. One of the possibilities is to replace (1.189) by

$$\begin{aligned} \dot{u} \;=\; & \left(\gamma + \alpha u + \beta u^2\right)\left\{\gamma\left(\frac{1}{2}u_{-2} - u_{-1} + u_1 - \frac{1}{2}u_2\right)\right. \\ & + \frac{\alpha}{2}\left[u_{-1}u_{-2} + u_{-1}^2 + u(u_{-1} - u_1) - u_1^2 - u_1 u_2\right] \\ & \left. + \frac{\beta}{2}\left[u_{-1}^2(u_{-2} + u) - u_1^2(u + u_2)\right]\right\}, \end{aligned} \quad (1.222)$$

where we have set $h = 1$ (by scaling) and introduced a parameter γ.

Expressing uniformity of rank, setting $W(\mathsf{D}_t) = 1$, and solving the linear system for the weights, one finds that $W(u) = W(\alpha) = \frac{1}{4}$, $W(\beta) = 0$, and $W(\gamma) = \frac{1}{2}$. So, we do not need a scale on β and (1.222) is invariant under the scaling symmetry

$$(t, u, \alpha, \gamma) \to (\lambda^{-1}t, \lambda^{\frac{1}{4}}u, \lambda^{\frac{1}{4}}\alpha, \lambda^{\frac{1}{2}}\gamma). \tag{1.223}$$

For $\beta = 0$, (1.222) reduces to

$$\begin{aligned} \dot{u} \;=\; & (\gamma + \alpha u)\left\{\gamma\left(\frac{1}{2}u_{-2} - u_{-1} + u_1 - \frac{1}{2}u_2\right)\right. \\ & \left. + \frac{\alpha}{2}\left[u_{-1}u_{-2} + u_{-1}^2 + u(u_{-1} - u_1) - u_1^2 - u_1u_2\right]\right\}, \end{aligned} \tag{1.224}$$

which is a completely integrable discretization of the KdV equation, $u_t + 6\alpha u u_x + u_{3x} = 0$. Computing the weights, one can set $W(\alpha) = 0$, which leads to $W(u) = W(\gamma) = \frac{1}{2}$. So, (1.224) is invariant under the scaling symmetry $(t, u, \gamma) \to (\lambda^{-1}t, \lambda^{\frac{1}{2}}u, \lambda^{\frac{1}{2}}\gamma)$. For $\alpha = 0$, (1.222) reduces to

$$\begin{aligned} \dot{u} \;=\; & \left(\gamma + \beta u^2\right)\left\{\gamma\left(\frac{1}{2}u_{-2} - u_{-1} + u_1 - \frac{1}{2}u_2\right)\right. \\ & \left. + \frac{\beta}{2}\left[u_{-1}^2(u_{-2} + u) - u_1^2(u + u_2)\right]\right\}, \end{aligned} \tag{1.225}$$

which is a completely integrable discretization of the mKdV equation, $u_t + 6\beta u^2 u_x + u_{3x} = 0$. In this case, one can set $W(\beta) = 0$. Then $W(u) = \frac{1}{4}$ and $W(\gamma) = \frac{1}{2}$. Thus, (1.225) is invariant under the scaling symmetry $(t, u, \gamma) \to (\lambda^{-1}t, \lambda^{\frac{1}{4}}u, \lambda^{\frac{1}{2}}\gamma)$.

1.13.3 Conservation Laws of the Gardner Lattice

One can either apply the method of Section 1.12 directly to (1.189) or, alternatively, apply to (1.222) the technique based on dilation invariance outlined in Sections 1.9.2, 1.11.2, and 1.11.4. In particular, one can use the "divide and conquer" strategy of Section 1.11.5 to split candidate densities into smaller pieces. Computational details can be found in [33] The results below were computed [58] with the method in Section 1.12. For $q = 0$ shifts there are two (non-polynomial) density-flux pairs:

$$\rho_1^{(0)} \;=\; \ln\left(1 + \alpha h^2\, u + \beta h^2\, u^2\right), \tag{1.226}$$

$$\begin{aligned} J_1^{(0)} \;=\; & -\frac{1}{2}\left\{\frac{\alpha}{h}\left(u_{-2} - u_{-1} - u + u_1\right) + \frac{\beta}{h}\left(2u_{-2}u - 4u_{-1}u + 2u_{-1}u_1\right)\right. \\ & + \alpha^2 h\left(u_{-1}(u_{-2} + u_{-1} + u) + u(u_{-1} + u + u_1)\right) + \alpha\beta\, h\left(u_{-1}^2 u_{-2}\right. \\ & + 2u_{-2}u_{-1}u + 3u_{-1}^2 u + 3u_{-1}u^2 + 2u_{-1}uu_1 + u^2u_1\big) \\ & \left. + 2\beta^2\, h\, u_{-1}u\big(u_{-2}u_{-1} + u_{-1}u + uu_1\big)\right\}, \end{aligned} \tag{1.227}$$

and

$$\rho_2^{(0)} = \operatorname{arctanh}\left(\frac{h(\alpha+2\beta u)}{\sqrt{\alpha^2\,h^2-4\beta}}\right), \tag{1.228}$$

$$\begin{aligned} J_2^{(0)} = \frac{1}{4}\sqrt{\alpha^2h^2-4\beta}\Big\{ & \tfrac{1}{h^2}\,(2u_{-1}-u_{-2}-2u_1+u_2) \\ & +\alpha\left(u_{-2}u_{-1}+u_{-1}^2+2u_{-1}u+u^2+uu_1\right) \\ & +\beta\left(u_{-1}^2(u_{-2}+u)+u^2(u_{-1}+u_1)\right)\Big\}. \end{aligned} \tag{1.229}$$

The next two (of infinitely many polynomial) densities are

$$\rho^{(1)} = uu_1+\frac{\alpha}{\beta}\,u, \tag{1.230}$$

$$\begin{aligned} \rho^{(2)} = {} & uu_2\left(1+\alpha h^2u_1+\beta h^2u_1^2\right)+\alpha h^2uu_1(u+u_1) \\ & +\frac{1}{2}\beta h^2u^2u_1^2+\frac{\alpha}{\beta}(1-\frac{\alpha^2}{\beta}h^2)u+\frac{\alpha^2}{2\beta}h^2u^2, \end{aligned} \tag{1.231}$$

where the associated fluxes have been omitted due to length.

Special Cases.

We consider two important special cases. The first few densities for (1.189) with $\beta=0$ are:

$$\rho_1^{(0)} = \ln(1+\alpha h^2u),\ \ \rho_2^{(0)}=u, \tag{1.232}$$

$$\rho^{(1)} = \tfrac{1}{2}\,u^2+uu_1, \tag{1.233}$$

$$\rho^{(2)} = uu_2(1+\alpha h^2u_1)+\alpha h^2u(u_1^2+uu_1+\tfrac{1}{3}\,u^2).$$

The first few densities for (1.189) with $\alpha=0$ are

$$\rho_1^{(0)} = \ln(1+\beta h^2u^2), \qquad \rho_2^{(0)}=\arctan(\sqrt{\beta}hu), \tag{1.234}$$

$$\rho^{(1)} = uu_1, \qquad \rho^{(2)}=uu_2(1+\beta h^2u_1^2)+\tfrac{1}{2}\,\beta h^2u^2u_1^2. \tag{1.235}$$

1.14 Additional Examples of Nonlinear DDEs

1.14.1 The Bogoyavlenskii Lattice

The Bogoyavlenskii lattice [21] and [90, Eq. (17.1.2)],

$$\dot{u}=u\left(\prod_{j=1}^{p}u_j-\prod_{j=1}^{p}u_{-j}\right), \tag{1.236}$$

is a generalization of the KvM lattice (1.110). For $p=2$, lattice (1.236) becomes

$$\dot{u}=u(u_1u_2-u_{-1}u_{-2}), \tag{1.237}$$

which is invariant under the following scaling symmetry

$$(t, u) \to (\lambda^{-1}t, \lambda^{\frac{1}{2}}u). \tag{1.238}$$

Lattice (1.237) has the following density-flux pairs (of infinitely many):

$$\begin{aligned}
\rho^{(0)} &= \ln u, \qquad J^{(0)} = -(u_{-1}u_{-2} + u_{-1}u + uu_1), && (1.239)\\
\rho^{(1)} &= u, \qquad J^{(1)} = -uu_{-1}(u_{-2} + u_1), && (1.240)\\
\rho^{(2)} &= uu_1, \qquad J^{(2)} = -u_{-1}uu_1(u_{-2} + u + u_2), && (1.241)\\
\rho^{(3)} &= uu_1(\tfrac{1}{2}uu_1 + u_1u_2 + u_2u_3), && (1.242)\\
J^{(3)} &= -u_{-1}uu_1(u_{-2}uu_1 + u^2u_1 + u_{-2}u_1u_2 + 2uu_1u_2 + u_1u_2^2 \\
&\quad +u_{-2}u_2u_3 + uu_2u_3 + u_2^2u_3 + u_2u_3u_4). && (1.243)
\end{aligned}$$

For (1.237), we also computed the densities $\rho^{(4)}$ through $\rho^{(9)}$. Every time the rank increases by one, the number of terms in the density increases by a factor three. For example, $\rho^{(9)}$ has 2187 terms and the highest shift is 15.

1.14.2 The Belov-Chaltikian Lattice

The Belov-Chaltikian lattice [18, Eq. (12)],

$$\dot{u} = u(u_1 - u_{-1}) + v_{-1} - v, \quad \dot{v} = v(u_2 - u_{-1}), \tag{1.244}$$

in invariant under the scaling symmetry

$$(t, u, v) \to (\lambda^{-1}t, \lambda u, \lambda^2 v). \tag{1.245}$$

The first few density-flux pairs (of infinitely many) are

$$\begin{aligned}
\rho^{(1)} &= u, \qquad J^{(1)} = -u_{-1}u + v_{-1}, && (1.246)\\
\rho^{(2)} &= \tfrac{1}{2}u^2 + uu_1 - v, \\
J^{(2)} &= -u_{-1}u^2 - u_{-1}uu_1 + uv_{-1} + u_1v_{-1} + u_{-1}v, && (1.247)\\
\rho^{(3)} &= u(\tfrac{1}{3}u^2 + uu_1 + u_1^2 + u_1u_2 - v_{-2} - v_{-1} - v - v_1), && (1.248)
\end{aligned}$$

where $J^{(3)}$ has been omitted due to length. Our results match these in [84].

1.14.3 The Blaszak-Marciniak Lattices

In [19], Blaszak and Marciniak used the R matrix approach to derive families of integrable lattices involving three and four fields. Below we consider two cases involving three fields. Examples based on four fields could be dealt with in a similar fashion [113].

The Blaszak-Marciniak three field lattice I [84, Eq. (2)],

$$\dot{u} = w_1 - w_{-1}, \quad \dot{v} = u_{-1}w_{-1} - uw, \quad \dot{w} = w(v - v_1), \tag{1.249}$$

is invariant under the scaling symmetry

$$(t, u, v, w) \to (\lambda^{-1}t, \lambda^{\frac{1}{2}}u, \lambda v, \lambda^{\frac{3}{2}}w). \tag{1.250}$$

We computed the following density-flux pairs of (1.249), which is a completely integrable lattice:

$$\begin{aligned}
\rho^{(0)} &= \ln w, & J^{(0)} &= v, && (1.251)\\
\rho^{(1)} &= u, & J^{(1)} &= -w_{-1} - w, && (1.252)\\
\rho^{(2)} &= v, & J^{(2)} &= u_{-1}w_{-1}, && (1.253)\\
\rho^{(3)} &= \tfrac{1}{2}v^2 + uw, & J^{(3)} &= u_{-1}vw_{-1} - w_{-1}w, && (1.254)\\
\rho^{(4)} &= \tfrac{1}{3}v^3 + uvw + uv_1w - ww_1, && && (1.255)\\
J^{(4)} &= w_{-1}(u_{-1}v^2 + u_{-1}uw - vw - v_1w). && && (1.256)
\end{aligned}$$

Our results confirm those in [84] and [112]. The Blaszak-Marciniak three field lattice II [112, Eq. (1.4)],

$$\dot{u} = v_1 - v + u(w_{-1} - w), \quad \dot{v} = v(w_{-2} - w), \quad \dot{w} = u_1 - u, \tag{1.257}$$

is invariant under the scaling symmetry

$$(t, u, v, w) \to (\lambda^{-1}t, \lambda^2 u, \lambda^3 v, \lambda w). \tag{1.258}$$

The first few density-flux pairs for (1.257), which is completely integrable, are

$$\begin{aligned}
\rho^{(1)} &= w, & J^{(1)} &= -u, && (1.259)\\
\rho^{(2)} &= \tfrac{1}{2}w^2 - u, & J^{(2)} &= v - uw_{-1}, && (1.260)\\
\rho^{(3)} &= \tfrac{1}{3}w^3 + v - uw_{-1} - uw, & J^{(3)} &= u_{-1}u + vw_{-2} + vw_{-1} - uw_{-1}^2, && (1.261)\\
\rho^{(4)} &= \tfrac{1}{4}w^4 + \tfrac{1}{2}u^2 + uu_1 + vw_{-2} + vw_{-1} - uw_{-1}^2 && && \\
&\quad + vw - uw_{-1}w - uw^2, && && (1.262)\\
J^{(4)} &= -u_{-2}v - uv - u_{-1}v_1 + vw_{-2}^2 + 2u_{-1}uw_{-1} && && \\
&\quad + vw_{-2}w_{-1} + vw_{-1}^2 - uw_{-1}^3 + u_{-1}uw. && && (1.263)
\end{aligned}$$

Our results agree with those in [84] and [112].

1.15 Software to Compute Conservation Laws for PDEs and DDEs

We first discuss our packages for conservation laws of PDEs and DDEs, followed by a brief summary of symbolic codes developed by other researchers.

1.15.1 Our Mathematica and Maple Software

The package `TransPDEDensityFlux.m` [9] , automates the computation of conservation laws of nonlinear PDEs in $(1+1)$ dimensions. In addition to polynomial PDEs, the software can handle PDEs with transcendental nonlinearities. The results in Sections 1.3 and 1.5 were computed with `TransPDEDensityFlux.m` and cross-checked with the newest version of `condens.m`, introduced in [38]. We used `TransPDEDensityFlux.m` to compute the density-flux pairs for the examples in Sections 1.6 and 1.7. Details about the algorithm and a discussion of implementation issues can be found in [8].

The code `DDEDensityFlux.m` [34] was used to compute the conservation laws in Sections 1.9 and 1.11. The results were cross-checked with the latest version of `diffdens.m` , featured in [39]. Using multiple scales, the efficiency of `DDEDensityFlux.m` was drastically improved. Nonetheless, the algorithms [33] within `DDEDensityFlux.m` are impractical for finding densities and fluxes of high rank. Therefore, we used the new `Maple` library `discrete` [56] to compute the results in Sections 1.13 and 1.14.

Some of the features of earlier versions of `condens.m` and `diffdens.m` were combined into the `InvariantsSymmetries.m` [39,41], which allows one to compute generalized symmetries as well as conserved densities (but no fluxes). `InvariantsSymmetries.m` is available from MathSource, the `Mathematica` program bank of Wolfram Research, Inc.

Our `Mathematica` packages and notebooks are available at [48] and Hickman's `Maple` code is available at [56]. Our `Mathematica` codes for the continuous and discrete Euler and homotopy operators in one dimension are available at [49]. We are currently designing a comprehensive package to compute conservation laws of PDEs in multiple space dimensions [45, 51, 83].

Our codes have been used in a variety of research projects. For example, `condens.m` [38] was used by Sakovich and Tsuchida [85, 95] to compute conservation laws of nonlinear Schrödinger equations. In [84], Sahadevan and Khousalya use the algorithms of `diffdens.m` [42] and `InvariantsSymmetries.m` [39, 41] to compute conserved densities of the Belov-Chaltikian and Blaszak-Marciniak lattices. Ergenç and Karasözen [35] used our software in the design of Poisson integrators for Volterra lattices.

1.15.2 Software Packages of Other Researchers

Our `Mathematica` code for conservation laws of PDEs has been "translated" [108–110] into a `Maple` package, called `CONSLAW` , which only handles PDEs in $(1+1)$ dimensions. Based on the concept of dilation invariance and the method of undetermined coefficients, similar software was developed by Deconinck and Nivala [27] and Yang *et al.* [107]. Our algorithms [38, 42] for DDEs have been adapted to fully-discretized equations [36, 37].

There are several algorithms (see e.g. [101]) to symbolically compute conservation laws of nonlinear PDEs but few have been fully implemented in CAS. Wolf's package `ConLaw`

[101, 102, 106] computes first integrals of ODEs and conservation laws of PDEs. `ConLaw` uses the `REDUCE` package `CRACK` [103–105], which contains tools to solve overdetermined systems of PDEs. Wolf's application packages heavily rely on the capabilities of `CRACK`, which took years to develop and perfect. Unfortunately, no such package is available in `Mathematica`.

A common approach is to use the link between conservation laws and symmetries as stated in Noether's theorem [15, 66, 81]. Among the newest software based on that approach is the `Maple` code `GeM` [25] by Cheviakov [23, 24], which allows one to compute conservation laws of systems of ODEs and PDEs based on the knowledge of generalized symmetries. However, the computation of such symmetries [47] is as difficult a task as the direct computation of conservation laws for it requires solving systems of overdetermined PDEs with, e.g., the `Rif` package [47, 100]. Some methods circumvent the existence of a variational principle (required by Noether's theorem) [12, 20, 101, 106] but they still rely on software to solve ODEs or PDEs.

The package `DE_APPLS` [16, 77] also offers commands for constructing conservation laws from (variational) symmetries by Noether's theorem, but the computation is not fully automated. Likewise, the package `Noether` [43] in `Maple` allows one to compute conservation laws from infinitesimal symmetry generators corresponding to (simple) Lagrangians.

Based on the formal symmetry approach, Sokolov and Shabat [89], Mikhailov *et al.* [74, 75], and Adler *et al.* [10] classified completely integrable PDEs and DDEs in $(1 + 1)$ dimensions. Unfortunately, the software used (see [38]) in the classification is obsolete.

For completeness, we also mention the packages `Jets` by Marvan [73] and `Vessiot` by Anderson [16, 77]. Both are general purpose suites of `Maple` packages for computations on jet spaces. The commands within `Jets` and `Vessiot` use differential forms and advanced concepts from differential geometry. By avoiding differential forms, our codes were readily adaptable to nonlinear DDEs (not covered in `Jets` and `Vessiot`).

Finally, Deconinck and Nivala [26] developed `Maple` software for the continuous and discrete homotopy operators. Their code is available at [28].

1.16 Summary

We presented methods to symbolically compute conservation laws of nonlinear polynomial and transcendental systems of PDEs in $(1 + 1)$ dimensions and polynomial DDEs in one discrete variable.

The first part of this chapter dealt with nonlinear PDEs for which we showed the computation of densities and fluxes in detail. Using the dilation invariance of the given PDE, candidate polynomial densities are constructed as linear combinations with undetermined coefficients of scaling invariant building blocks. For polynomial PDEs, the undetermined coefficients are constants which must satisfy a linear system of algebraic equations. That system will be parameterized by constants appearing in the PDE, if any. For transcendental

PDEs, the undetermined coefficients are functions which much satisfy a linear system which is a mixture of algebraic equations and ODEs.

The continuous homotopy operator is a powerful, algorithmic tool to compute fluxes explicitly. Indeed, the homotopy operator handles integration by parts which allowed us to invert the total derivative operator. The methods for polynomial PDEs are illustrated with classical examples such as the KdV and Boussinesq equations and the Drinfel'd-Sokolov-Wilson system. The computation of conservation laws of system with transcendental nonlinearities is applied to sine-Gordon, sinh-Gordon, and Liouville equations.

In the second part we dealt with the symbolic computation of conservation laws of nonlinear DDEs. Again, we used the scaling symmetries of the DDE and the method of undetermined coefficients to find densities and fluxes. In analogy with the continuous case, to compute the flux one could use the discrete homotopy operator, which handles summation by parts and inverts the forward difference operator. However, in comparison with the "splitting and shifting" technique, the discrete Euler and homotopy operators are inefficient tools for the symbolic computation of conservation laws of DDEs. The undetermined coefficient method is illustrated with classical examples such as the Kac-van Moerbeke, Toda and Ablowitz-Ladik lattices.

There is a fundamental difference between the continuous and discrete cases in the way densities (of selected rank) are constructed. The total derivative has a weight but the shift operator does not. Consequently, a density of a PDE is bounded in order (with respect to x). Unfortunately, there is no *a priori* bound on the number of shifts in the density, unless a leading order analysis is carried out. To overcome this difficulty and other shortcomings of the undetermined coefficient method, we presented a new method to compute conserved densities of DDEs. That method no longer uses dilation invariance and is no longer restricted to polynomial conservation laws. Instead of building a candidate density with undetermined coefficients, one first computes the leading order term in the density and, second, generates the correction terms of lower order. The method is fast and efficient since no unnecessary terms are computed. The new method was illustrated using a modified Volterra lattice as an example, and applied to lattices due to Bogoyavlenskii, Belov-Chaltikian, Blaszak-Marciniak, and Gardner. A derivation of the latter lattice was also given.

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