Ricci nilsoliton black holes

Sigbjørn Hervik *

Dalhousie University, Department of Mathematics and Statistics, Halifax, NS, Canada B3H 3J5

A R T I C L E   I N F O

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We follow a constructive approach and find higher-dimensional black holes with Ricci nilsoliton horizons. The spacetimes are solutions to the Einstein equation with a negative cosmological constant and generalise therefore, Anti-de Sitter black hole spacetimes. The approach combines a work by Lauret – which relates the so-called Ricci nilsolitons and Einstein solvmanifolds – and an earlier work by the author. The resulting black hole spacetimes are asymptotically Einstein solvmanifolds and thus, are examples of solutions which are not asymptotically Anti-de Sitter. We show that any nilpotent group in dimension $n \leq 6$ has a corresponding Ricci nilsoliton black hole solution in dimension $(n+2)$. Furthermore, we show that in dimensions $(n+2) > 8$, there exist an infinite number of locally distinct Ricci nilsoliton black hole metrics.

1. Introduction

The last decade has seen the interest in negatively curved spaces growing considerably. From a mathematical point of view, the negatively curved spaces have an extremely rich structure; for example, in three dimensions "most" manifolds are negatively curved [1–3]. From a physical point of view, negatively curved spaces have arisen both in superstring theories and in higher-dimensional theories of our universe (see e.g., [4–6]). The maximally symmetric Anti-de Sitter space (AdS) – which is a solution to the Einstein equations with a negative cosmological constant – is the space that has attracted the most attention. In this paper, however, we will draw attention to some other negatively curved solutions to the Einstein equations with a negative cosmological constant. In the mathematics literature they are known as Einstein solvmanifolds [7–9], and unlike the AdS spaces, are not maximally symmetric. We will study a class of Einstein solvmanifolds and show that they allow for a simple generalisation which can be interpreted as black hole solutions with a horizon geometry being that of a nilmanifold.

For a Lie algebra, $\mathfrak{g}$, we can construct the two descending series,

$$\mathfrak{g}_D^{(0)} = \mathfrak{g}, \qquad \mathfrak{g}_D^{(i+1)} = [\mathfrak{g}, \mathfrak{g}_D^{(i)}],$$
$$\mathfrak{g}_C^{(0)} = \mathfrak{g}, \qquad \mathfrak{g}_C^{(i+1)} = [\mathfrak{g}_C^{(i)}, \mathfrak{g}_C^{(i)}],$$

called the derived and the lower central series, respectively. If the derived series terminates, i.e., $\mathfrak{g}_D^{(k)} = 0$ for an integer k, we call the Lie algebra $\mathfrak{g}$ *nilpotent*. Similarly, if $\mathfrak{g}_C^{(k)} = 0$ for an integer k, we call the Lie algebra $\mathfrak{g}$ *solvable*. Clearly, any nilpotent Lie algebra is also solvable. Here, we will denote a generic nilpotent Lie algebra[1] by $\mathfrak{n}$. Any Lie algebra, $\mathfrak{g}$, gives rise to a unique connected and simply connected Lie group, G, such that the tangent space of G (as a manifold) at the unit element is $\mathfrak{g}$: $\mathfrak{g} = T_e G$ [10]. Any such Lie group can be equipped with a left-invariant metric which turns G into a Riemannian space having a metric which is invariant under the left action of G (see e.g., [11]). In the case of a nilpotent Lie algebra $\mathfrak{n}$, this gives rise to a nilpotent Lie group N. Such a nilpotent Lie group equipped with a left-invariant metric is commonly denoted as a *nilmanifold*. We will assume that this metric is Riemannian, unless stated otherwise.

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* Corresponding address: University of Stavanger, Faculty of Science and Technology, N-4036 Stavanger, Norway. Tel.: +47 51 83 21 72.

E-mail addresses: herviks@mathstat.dal.ca, sigbjorn.hervik@uis.no.

[1] The Abelian algebras are trivially nilpotent; however, we will assume that $\mathfrak{n}$ is non-Abelian.

A Lie group usually possesses *many* non-isometric left-invariant metrics. Therefore a natural question would be: Is there a particularly nice or distinguished left-invariant metric? Such a distinguished metric can, for example, be an *Einstein metric* [12]; that is, a metric $g_{\mu\nu}$ that obeys

$$R_{\mu\nu} = \lambda g_{\mu\nu}, \tag{1}$$

where $R_{\mu\nu}$ is the Ricci tensor. However, this is not appropriate for nilmanifolds since a well-known result states that nilmanifolds do not allow for a left-invariant metric which is Einstein [11]. On the other hand, Lauret [13] noted that some nilpotent groups allow for metrics which obey

$$R_{\mu\nu} = \lambda g_{\mu\nu} + D_{\mu\nu}, \tag{2}$$

where D^μ_ν as a linear map, $D : \mathfrak{n} \mapsto \mathfrak{n}$, is a *derivation of* $\mathfrak{n}$; i.e.

$$D([X, Y]) = [D(X), Y] + [X, D(Y)].$$

These metrics have a nice interpretation in terms of special solutions of the *Ricci flow* [14]. For a curve $\mathbf{g}(t)$ of Riemannian metrics on a manifold M, the Ricci flow is defined by the equation

$$\frac{\partial g_{\mu\nu}}{\partial t} = -2R_{\mu\nu}. \tag{3}$$

If a solution to the Ricci flow (3) moves by a diffeomorphism and is also scaled by a factor at the same time, we call the solution a *homothetic Ricci soliton* [15]. In other words, if ϕ_t is a one-parameter family of diffeomorphisms generated by some vector field and

$$\mathbf{g}(t) = c(t)\phi_t^* \mathbf{g}$$

is a solution of the Ricci flow, then $\mathbf{g}$ is a homothetic Ricci soliton.

Ricci nilsolitons are nilmanifolds with left-invariant metrics being homothetic Ricci solitons. In addition, a Ricci nilsoliton has a unique decomposition as given by Eq. (2); hence, Ricci nilsolitons are in some way a generalisation of Einstein metrics to nilpotent groups. These Ricci nilsolitons are also unique up to isometry and scaling and can therefore be taken to be distinguished left-invariant metrics on nilmanifolds [13].[2]

In this paper we will study black hole solutions where the horizon is locally a nilmanifold, while the total spacetime is a solution to the Einstein equations with a negative cosmological constant. Given an *n*-dimensional nilmanifold *N*, we will see that we can construct such black hole solutions (with compact horizons) of dimension $(n + 2)$ provided that:

1. The nilmanifold, *N*, allows for a nilsoliton metric.
2. The nilmanifold, *N*, allows for a compact quotient; i.e., there exists a lattice $\Gamma \subset N$ such that N/Γ is compact.

The nilsoliton metric will correspond to the (local) horizon geometry and consequently these solutions are Ricci nilsoliton black holes. In particular, we will see that for any nilmanifold of dimension ≤ 6, both requirements are fulfilled (thus, there exists a corresponding black hole solution of dimension ≤ 8). On the other hand, for nilmanifolds of dimension > 6, these requirements are not always fulfilled, however, we will show that there exists an infinite family of nilmanifolds for which they do. This implies that, for spacetime dimension > 8, there exists an infinite number of locally distinct Ricci nilsoliton black holes.

The paper is organised as follows. First, we show how to construct Ricci nilsoliton metrics through a variational procedure. Then, using a method of Lauret, we construct Einstein solvmanifolds which constitute the "background" spacetime. A simple generalisation allows us to construct black hole solutions having Ricci nilsolitons as horizon geometries. Some aspects of these solutions are discussed, among them, the asymptotic geometry. The paper is constructive in nature and therefore, in the Appendix, a full list of the nilpotent Lie algebras of dimension ≤ 6 is given, along with their corresponding Ricci nilsoliton solution.

2. Finding Ricci nilsoliton metrics through a variational procedure

Let us consider a vector space $\mathfrak{n}$ with a fixed inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{n}$. Define a nilpotent Lie algebra μ on $\mathfrak{n}$ by the structure constants; i.e.,

$$[e_i, e_j] = \mu(e_i, e_j) = \mu_{ij}^k e_k, \quad \langle e_i, e_j \rangle = \delta_{ij}. \tag{4}$$

The set of nilpotent Lie algebras can be considered as an algebraic subset of $V = \wedge^2 \mathfrak{n}^* \otimes \mathfrak{n}$, the vectorspace of all skew-symmetric maps from $\mathfrak{n} \times \mathfrak{n}$ into $\mathfrak{n}$. Any nilpotent Lie algebra μ defines a corresponding simply connected nilpotent Lie group, N_μ, endowed with the left-invariant Riemannian metric determined by $\langle \cdot, \cdot \rangle$.

Moreover, define the action of $\mathbf{A} \in GL(n)$ on μ by

$$\mathbf{A} * \mu(X, Y) = \mathbf{A}\mu(\mathbf{A}^{-1}X, \mathbf{A}^{-1}Y), \quad X, Y \in \mathfrak{n}. \tag{5}$$

If $\tilde{\mu}$ and μ are two Lie algebras, then $\tilde{\mu}$ and μ are isomorphic as Lie algebras if and only if they are in the same $GL(n)$-orbit. Furthermore, the corresponding nilmanifolds $N_{\tilde{\mu}}$ and N_μ are isometric if and only if they are in the same $O(n)$-orbit.

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[2] Interestingly, the Ricci flow has shown to be of importance in resolving the Thurston geometrisation conjecture [1] and thereby the famous Poincaré conjecture (see e.g., [16]).

The Ricci operator of N_μ can be calculated to be

$$\langle \mathbf{R}_\mu e_i, e_j \rangle = \frac{1}{4} \sum_{kl} \left[\langle \mu(e_k, e_l), e_i \rangle \langle \mu(e_k, e_l), e_j \rangle - 2 \langle \mu(e_i, e_k), e_l \rangle \langle \mu(e_j, e_k), e_l \rangle \right]. \tag{6}$$

Also, consider the two functionals, $R(\mu)$ and $F(\mu)$ defined by

$$R(\mu) \equiv \mathrm{Tr}(\mathbf{R}_\mu),$$

$$F(\mu) \equiv \mathrm{Tr}(\mathbf{R}_\mu^2). \tag{7}$$

We note that these functionals are the Ricci scalar and the "square" of the Ricci tensor, $R_{ij}R^{ij}$, respectively, of the left-invariant metric. The inner product $\langle \cdot, \cdot \rangle$ defines an inner product on V, also denoted $\langle \cdot, \cdot \rangle$, by

$$\langle \mu, \tilde{\mu} \rangle = \sum_{ijk} \langle \mu(e_i, e_j), e_k \rangle \langle \tilde{\mu}(e_i, e_j), e_k \rangle = \sum_{ijk} (\mu_{ij}^k)(\tilde{\mu}_{ij}^k). \tag{8}$$

This inner product defines a natural normalisation of V:

$$\mathsf{S} = \{ \mu \in V \mid \langle \mu, \mu \rangle = 1 \}. \tag{9}$$

Note that this normalises $R(\mu)$ since $R(\mu) = -\langle \mu, \mu \rangle / 4$. Since a constant rescaling of the metric will rescale $R(\mu)$, there is no loss of generality to restrict to S.

It is desirable to find a distinguished metric on a nilmanifold. Since nilpotent groups do not allow for Einstein metrics, we can try the next best thing, namely minimize the functional

$$\mathrm{Tr} \left[\mathbf{R}_\mu - \frac{1}{n} \mathrm{Tr}(\mathbf{R}_\mu) \mathbf{1} \right]^2 = F(\mu) - \frac{1}{n} R(\mu)^2, \tag{10}$$

which measures how far N_μ is from being an Einstein space. Therefore, fixed points of F restricted to S are of particular significance. In fact, we have the following theorem by Lauret [17]:

Theorem. *For a nilpotent $\mu \in \mathsf{S}$ the following statements are equivalent:*

1. N_μ *is a Ricci nilsoliton.*
2. μ *is a critical point of $F : \mathsf{S} \mapsto \mathbb{R}$.*
3. μ *is a critical point of $F : GL(n) * \mu \cap \mathsf{S} \mapsto \mathbb{R}$.*
4. $\mathbf{R}_\mu \in \mathbb{R}\mathbf{1} \oplus \mathrm{Der}(\mu)$.

This theorem intimately connects the critical points of F and the Ricci nilsolitons. The Ricci nilsoliton metrics can therefore be considered to be particularly nice metrics on nilpotent groups.

For a Ricci nilsoliton there exists a symmetric derivation $\mathbf{D} \in \mathrm{Der}(\mu)$ such that

$$\mathbf{R}_\mu = c_\mu \mathbf{1} + \mathrm{Tr}(\mathbf{D})\mathbf{D}, \quad \mathrm{Tr}(\mathbf{R}_\mu \mathbf{D}) = 0. \tag{11}$$

Thus,

$$c_\mu = \frac{F(\mu)}{R(\mu)} = -\mathrm{Tr}(\mathbf{D}^2).$$

Since nilmanifolds are never Einstein, we necessarily have $\mathbf{D} \neq 0$ for a Ricci nilsoliton. Moreover, the scalar curvature never vanishes so that $R(\mu) < 0$.

A necessary condition for a Ricci nilsoliton metric to exist for a given nilpotent Lie algebra μ, is therefore that μ has a non-zero symmetric derivation $\mathbf{D}$. In particular, if μ is *characteristically nilpotent* (i.e., $\mathrm{Der}(\mu)$ is nilpotent) then there cannot exist such a $\mathbf{D}$ and no Ricci nilsoliton metric exists. Therefore, not all nilmanifolds allow for a Ricci nilsoliton metric. On the other hand, it has been proven that all nilmanifolds of dimension ≤ 6 allow for one [19].

We now have an algorithm for finding Ricci nilsolitons for a given nilpotent Lie algebra μ (if it exists):

1. Find the critical points of $F : GL(n) * \mu \cap \mathsf{S} \mapsto \mathbb{R}$. Any $\mu \in V$ such that $\mu / \langle \mu, \mu \rangle^{1/2}$ is a critical point will then correspond to a Ricci nilsoliton metric.
2. The Ricci tensor can be calculated from Eq. (6). Using Eq. (11) the derivation $\mathbf{D}$ can then be determined.

An important observation is that the eigenvalues of $\mathbf{D}$, up to a scalar multiplication, can be arranged into a tuple:

$$(k; d) = (k_1 < k_2 < \cdots < k_r; d_1, d_2, \ldots, d_r), \tag{12}$$

where the k_l are integers without common divisors and d_l are their corresponding multiplicities. This tuple is called the eigenvalue type. Usually the Ricci nilsolitons are given in terms of its eigenvalue type as above due to the relation to the classification of Einstein solvmanifolds (see, e.g, [9]).

If μ is an n-dimensional nilpotent Lie algebra for which $\mu / \langle \mu, \mu \rangle^{1/2}$ corresponds to a fixed point of $F(\mu)$ as explained above, then

$$\mathbf{D}_\mu = \frac{\langle \mu, \mu \rangle^{\frac{1}{2}}}{2} \left[n \left(k_1^2 d_1 + \cdots + k_r^2 d_r \right) - (k_1 d_1 + \cdots + k_r d_r)^2 \right]^{-\frac{1}{2}} \tilde{\mathbf{D}}, \tag{13}$$

where $\tilde{\mathbf{D}}$ is the derivation of μ with eigenvalues k_i of multiplicities d_i.

Let $\tilde{\mu}$ be the extension of μ by adding an Abelian factor: $\tilde{\mathfrak{n}} = \mathfrak{n} \oplus \mathbb{R}^m$; i.e., $\tilde{\mu}|_{\mathfrak{n} \times \mathfrak{n}} = \mu$ and $[\mathbb{R}^m, \mathfrak{n}] = 0$. Then $F(\tilde{\mu}) = F(\mu)$ and the critical point has the eigenvalue type

$$\left(\alpha k_1 < \cdots < \frac{k_1^2 d_1 + \cdots + k_r^2 d_r}{d} < \cdots < \alpha k_r; d_1 < \cdots < m < \cdots < d_r \right), \tag{14}$$

where $d = \mathrm{mcd}(k_1 d_1 + \cdots + k_r d_r, k_1^2 d_1 + \cdots + k_r^2 d_r)$ and $\alpha = \frac{k_1 d_1 + \cdots + k_r d_r}{d}$. In the case that $\frac{k_1^2 d_1 + \cdots + k_r^2 d_r}{d} = \alpha k_i$ for some i, then the multiplicity is $m + d_i$. This result will be useful for us since adding a time direction will add an additional 1-dimensional Abelian factor to the nilpotent group.

3. From Ricci nilsolitons to black holes

The reason for stressing the properties of the Ricci nilsolitons is because of the importance it has in constructing Einstein solvmanifolds. This relation between Ricci nilsolitons and Einstein solvmanifolds seems to have been noticed by Lauret [13,17,18,20]. Moreover, when going from the homogeneous solvmanifold to the inhomogeneous black hole solutions, the isometry group of the Ricci nilmanifolds will survive the construction of the black hole spacetime. The Ricci flow will therefore have a particular role in black hole spacetimes.[3]

3.1. Einstein solvmanifolds and Ricci nilsolitons

Let us first state a theorem due to Lauret [13]:

Theorem (*Lauret*). *A homogeneous nilmanifold* $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ *is a Ricci nilsoliton if and only if* $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ *admits a metric solvable extension* $(\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}, \mathbf{g})$ *with* $\mathfrak{a}$ *Abelian whose corresponding solvmanifold* $(S, \mathbf{g})$ *is Einstein.*

This solvmanifold can be constructed as follows: Consider the following metric solvable extension of the nilpotent algebra $\mathfrak{n}$ (with brackets μ):

$$\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}, \qquad [\mathfrak{s}, \mathfrak{s}]_{\mathfrak{s}} = \mathfrak{n}, \qquad [\cdot, \cdot]_{\mathfrak{s}}|_{\mathfrak{n} \times \mathfrak{n}} = [\cdot, \cdot]_{\mathfrak{n}}, \tag{15}$$

and

$$\langle \cdot, \cdot \rangle_{\mathfrak{s}}|_{\mathfrak{n} \times \mathfrak{n}} = \langle \cdot, \cdot \rangle_{\mathfrak{n}}, \qquad \langle \mathfrak{a}, \mathfrak{n} \rangle_{\mathfrak{s}} = 0. \tag{16}$$

Moreover, let $\mathfrak{a} = \mathbb{R}$, $\mathbf{D} \in \mathrm{Der}(\mu)$ and let $\mathsf{E} \in \mathfrak{a}$ such that $\langle \mathsf{E}, \mathsf{E} \rangle_{\mathfrak{s}} = 1$. Then we can define the solvable Lie algebra by

$$[\mathsf{E}, \mathsf{e}_i] = \mathbf{D}\mathsf{e}_i, \qquad [\mathsf{e}_i, \mathsf{e}_j] = \mu(\mathsf{e}_i, \mathsf{e}_j). \tag{17}$$

Let $(S, \mathbf{g})$ be the corresponding solvmanifold equipped with the left-invariant metric. The Ricci tensor of S can found to be [22]

$$\mathbf{g}(\mathbf{R}_{\mathfrak{s}} \mathsf{E}, \mathsf{E}) = -\mathrm{Tr}(\mathbf{D}^2), \tag{18}$$

$$\mathbf{g}(\mathbf{R}_{\mathfrak{s}} \mathsf{E}, \mathsf{e}_i) = 0, \tag{19}$$

$$\mathbf{g}(\mathbf{R}_{\mathfrak{s}} \mathsf{e}_i, \mathsf{e}_j) = \mathbf{g}([-\mathrm{Tr}(\mathbf{D})\mathbf{D} + \mathbf{R}_{\mu}]\mathsf{e}_i, \mathsf{e}_j). \tag{20}$$

Hence, $(S, \mathbf{g})$ is Einstein if μ is a Ricci nilsoliton with $\mathbf{D} \in \mathrm{Der}(\mu)$ given in Eq. (11):

$$\mathbf{R}_{\mathfrak{s}} = c_{\mu} \mathbf{1}, \quad c_{\mu} = -\mathrm{Tr}(\mathbf{D}^2). \tag{21}$$

These Einstein solvmanifolds will correspond to the asymptotic metric for the black holes. In particular, this means that the black holes are not asymptotically AdS, but rather asymptotically a solvmanifold.

The Einstein solvmanifolds have been the centre of a long outstanding question regarding the classification of negatively curved homogeneous Einstein manifolds [12,9]. All known examples of negatively curved homogeneous Einstein manifolds are isometric to a Einstein solvmanifold, however, it is not proven that all necessarily are. Only in low dimensions some progress has been made (see [23] for dimension 5).[4] In spite of the lack of a classification result, numerous examples of Einstein solvmanifolds exist. The most commonly known are the real, the complex and the quaternionic hyperbolic spaces $\mathbb{H}^n$, $\mathbb{H}^n_{\mathbb{C}}$, $\mathbb{H}^n_{\mathbb{H}}$, and the Cayley hyperbolic plane $\mathbb{H}^2_{\mathbf{Cay}}$. These hyperbolic spaces can be further generalised to the so-called Damek–Ricci spaces [26] which are solvable extensions of generalised Heisenberg spaces.

3.2. Black holes

We can now proceed with the construction of Ricci nilsoliton black holes. More specifically, the constructed black hole spacetime, in spite of being inhomogeneous, will possess an isometry group arising from the Ricci nilsolitons. These black hole solutions were discussed in an earlier paper in a more general context [27]. For the time being it is advantageous to

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[3] Interestingly, the role of the Ricci flow and black holes has been studied in a different context in [21].

[4] Moveover, recently a paper by Lauret appeared [24] proving that all Einstein solvmanifolds are necessary standard (see also [25]).

keep the manifold Riemannian and assume that you can foliate the space using Ricci nilsoliton hypersurfaces. We introduce the extrinsic curvature **k** which is a bilinear and symmetric tensor residing on the hypersurfaces. We define the extrinsic curvature operator $\mathbf{K} : \mathfrak{n} \mapsto \mathfrak{n}$ by

$$\langle \mathbf{K}e_i, e_j \rangle = \mathbf{k}(e_i, e_j).$$

Let us also introduce the Gaussian coordinate y such that $\partial/\partial y$ is a unit normal vector to the nilmanifolds. Assume further that $\mathfrak{n}$ contains an Abelian factor spanned by e_1, say, so that $[e_1, \mathfrak{n}] = 0$. This Abelian factor will eventually correspond to the time direction. For the Ricci nilsoliton this implies

$$\mathbf{R}_\mu e_1 = 0. \tag{22}$$

Moreover, let **D** be the constant derivation given earlier. We can decompose **D** and **K** as

$$\mathbf{D} = \begin{bmatrix} D_{11} & 0 \\ 0 & \widetilde{\mathbf{D}} \end{bmatrix}, \qquad \mathbf{K} = \begin{bmatrix} K_{11} & 0 \\ 0 & \widetilde{\mathbf{K}} \end{bmatrix}. \tag{23}$$

Here, $D_{11} = \mathrm{Tr}(\mathbf{D}^2)/\mathrm{Tr}(\mathbf{D})$, which follows from Eq. (22). By assuming $\widetilde{\mathbf{K}} = \Lambda(y)\widetilde{\mathbf{D}}$ where $\Lambda(y)$ is some function of y, implies that **K** is also a derivation pf $\mathfrak{n}$. Since the derivations are the generators of the automorphism group, this choice of **K** implies that the geometry of the hypersurfaces is preserved as you go along the Gaussian coordinate y. So $\dot{\mathbf{R}}_\mu = 0$, where dot denotes the (Lie) derivative with respect to y, and hence, $\mathbf{R}_\mu = -\mathrm{Tr}(\mathbf{D}^2)\mathbf{1} + \mathrm{Tr}(\mathbf{D})\mathbf{D}$ where the derivation **D** can be considered to be a constant.

The Gauss' equations now reduce to

$$\dot{\mathbf{K}} + \mathrm{Tr}(\mathbf{K})\mathbf{K} - \mathbf{R}_\mu + \lambda\mathbf{1} = 0, \tag{24}$$

$$\mathrm{Tr}(\mathbf{K}^2) - [\mathrm{Tr}(\mathbf{K})]^2 + \mathrm{Tr}(\mathbf{R}_\mu) - (n-1)\lambda = 0. \tag{25}$$

We note first that the solution given by

$$\mathbf{K} = \mathbf{D} \tag{26}$$

is the Einstein solvmanifold given above, with $\lambda = -\mathrm{Tr}(\mathbf{D}^2)$.

Another set of solutions can be found by

$$\mathbf{K} = \coth[D(y - y_0)]\mathbf{D} + \frac{\sigma}{\sinh[D(y - y_0)]}, \tag{27}$$

where we have set $D = \mathrm{Tr}(\mathbf{D})$ and

$$\sigma = \begin{bmatrix} (D - D_{11}) & 0 \\ 0 & -\widetilde{\mathbf{D}} \end{bmatrix}. \tag{28}$$

We note that σ is trace-free and orthogonal to **D**; i.e.,

$$\mathrm{Tr}(\sigma) = 0, \qquad \mathrm{Tr}(\sigma\mathbf{D}) = 0. \tag{29}$$

The $\mathbf{R}_\mu$ and λ are constants and given as above.

The solutions given above are the Euclidean versions of Ricci nilsoliton black holes given in terms of the nilsoliton foliation of the solutions. It is useful to write down the metric for this solution in the standard form. This can be accomplished by introducing the coordinate w by

$$y - y_0 = \frac{2}{D}\mathrm{artanh}\sqrt{1 - M\exp(-Dw)}. \tag{30}$$

By diagonalising $\mathbf{D} = \mathrm{diag}(q_1, q_2, \ldots, q_n)$, we can write

$$ds^2 = \frac{dw^2}{1 - Me^{-Dw}} + (1 - Me^{-Dw})e^{2q_1w}(dx^1)^2 + \sum_{i=2}^{n} e^{2q_iw}\left(\omega^i\right)^2, \tag{31}$$

where $\{dx^1, \omega^i\}$ is an appropriate set of left-invariant vectors on $\mathfrak{n}$. These obey $d\omega^k = -(1/2)C_{ij}^k\omega^i \wedge \omega^j$ where C_{ij}^k are constants and are the structure constants of $\mathfrak{n}$.

A Lorentzian solution can now be found by Wick-rotating the coordinate x^1; i.e., by setting $t = ix^1$, we get

$$ds^2 = -(1 - Me^{-Dw})e^{2q_1w}dt^2 + \frac{dw^2}{1 - Me^{-Dw}} + \sum_{i=2}^{n} e^{2q_iw}\left(\omega^i\right)^2. \tag{32}$$

A more standard form can be obtained by defining a new variable r by $w = (1/q_1)\ln(q_1r)$ for which we get

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + h_{AB}(r)\omega^A\omega^B, \quad f(r) = q_1^2r^2 - M(q_1r)^{-\frac{D-2q_1}{q_1}}. \tag{33}$$

We therefore see that these nilsoliton black holes are generalisation of the standard toroidal AdS black holes. This is also clear from the fact that the toroidal black holes have flat horizon geometry; so in a sense, the toroidal AdS black hole is the trivial case where the nilpotent group is Abelian.

We usually assume that black holes have compact horizons. In order for the horizon to be compact, one must require that the nilmanifolds allow for a compact quotient; i.e., there exists a lattice $\Gamma \subset N_\mu$ such that N_μ / Γ is a compact manifold. For nilmanifolds the existence of such a lattice can be determined using the Lie algebra μ [28]:

Theorem. *A nilmanifold can be compactified if and only if there exists a frame such that $[e_i, e_j] = C_{ij}^k e_k$ where C_{ij}^k are all rational constants.*

By inspection of the nilpotent Lie algebras of dimension ≤ 6 we get an immediate consequence of this theorem:

Corollary. *All nilmanifolds of dimension ≤ 6 allow for a compact quotient.*

There are only a finite number of nilpotent Lie algebras of dimension ≤ 6 (50 including the Abelian ones), all of which allow for a Ricci nilsoliton metric [19] (a copy of Will's list is given in the Appendix).

Among the 7-dimensional Lie algebras, there exists a curve of non-isometric Lie algebras. These allow for a nilsoliton metric of type (see [18])

$$(1 < 2 < 3 < 4 < 5 < 6 < 7; 1, \ldots, 1).$$

The curve of nilsoliton Lie algebras can be given by

$$\mu_{12}^3 = \mu_{34}^7 = (1 - t)^{\frac{1}{2}}, \qquad \mu_{14}^5 = \mu_{23}^7 = t^{\frac{1}{2}}$$
$$\mu_{13}^4 = \mu_{15}^6 = \mu_{16}^7 = \mu_{23}^5 = \mu_{24}^6 = 1. \tag{34}$$

This algebra can be shown to be isomorphic to the algebra denoted by $1, 2, 3, 4, 5, 7_l : t$ in Seeley's list of 7-dimensional nilpotent Lie algebras [29]. Hence, the above algebra is isomorphic to the Lie algebra given by

$$\tilde{\mu}_{34}^7 = (1 - t), \qquad \tilde{\mu}_{23}^7 = t$$
$$\tilde{\mu}_{12}^3 = \tilde{\mu}_{14}^5 = \tilde{\mu}_{13}^4 = \tilde{\mu}_{15}^6 = \tilde{\mu}_{16}^7 = \tilde{\mu}_{23}^5 = \tilde{\mu}_{24}^6 = 1. \tag{35}$$

Thus, by virtue of the above theorem, if t is rational then the corresponding nilsoliton metric allows for a compact quotient. This implies that there exists an infinite number of model nilmanifolds which allows for a compact quotient.

Therefore, if we classify the black hole solutions in terms of the model geometries, we have that for every nilpotent group of dimension ≤ 6, there exists a corresponding black hole solution in dimension ≤ 8. Moreover, for any dimension > 8, there is an infinite number of locally distinct black hole solutions with a nilsoliton metric as a horizon.

Note that there may be many different lattices Γ for a given nilmanifold (for example, there is an infinite number of possible non-homeomorphic quotients of $N_{3,1}$). Also, we have not addressed the issue of moduli space of non-isometric quotients.

3.3. Making the Euclidean solution regular

To make the Euclidean solution regular we must ensure that the solution behaves regularly at the horizon. We can use the Gaussian coordinate y and approximate the solution close to $y = y_0$. This yields

$$ds^2 \approx dy^2 + (y - y_0)^2 \left(\frac{D}{2} M^{\frac{q_1}{D}} dx^1 \right)^2 + \sum_{i=2}^{n} M^{\frac{2q_i}{D}} (\omega^i)^2. \tag{36}$$

Hence, if we identify x^1 under the map

$$x^1 \longmapsto x^1 + \frac{4\pi}{DM^{\frac{q_1}{D}}},$$

the solution closes off regularly at $y = y_0$ and the Euclidean solution is everywhere regular. If we write this identification as $x^1 \mapsto x^1 + \beta$ then β is usually interpreted as the inverse temperature of the black hole; i.e., $\beta = 1/T$. This implies that $T \propto M^{q_1/D}$ and so the temperature increases as the mass increases.

3.4. Generalisations

Let us recapitulate what assumptions were made in order for Eq. (27) to be a solution:

1. $\mathbf{K}$ is a derivation of $\mathfrak{n}$.
2. $\mathrm{Tr}(\sigma) = \mathrm{Tr}(\sigma \mathbf{D}) = 0$.
3. $\mathrm{Tr}(\sigma^2) = [\mathrm{Tr}(\mathbf{D})]^2 - \mathrm{Tr}(\mathbf{D}^2)$.

We can therefore generalise the above solution as long as these criteria are satisfied.

So, for example, consider an m-dimensional Abelian factor of $\mathfrak{n}$ such that $[\mathbb{R}^m, \mathfrak{n}] = 0$. Furthermore, assume that the Abelian factor is spanned by $e_1, \ldots, e_m$. Then $\langle \mathbf{R}_\mu e_i, e_i \rangle = 0$ for $i = 1, \ldots, m$ and the derivation $\mathbf{D}$ can be decomposed as

$$\mathbf{D} = \begin{bmatrix} D_{11}\mathbf{1}_{m \times m} & 0 \\ 0 & \tilde{\mathbf{D}} \end{bmatrix}. \tag{37}$$

Then Eq. (27) is a solution as long as we choose σ the following way:

$$\sigma = \begin{bmatrix} \dfrac{1}{m}(D - mD_{11})\mathbf{1}_{m \times m} + \mathbf{A} & 0 \\ 0 & -\tilde{\mathbf{D}} \end{bmatrix}, \tag{38}$$

where the $m \times m$ matrix $\mathbf{A}$ obeys

$$\mathrm{Tr}(\mathbf{A}) = 0, \qquad \mathrm{Tr}(\mathbf{A}^2) = \frac{m-1}{m}D^2. \tag{39}$$

Since the solutions are only defined locally, it is not clear what the interpretation of these solutions are or whether they can be made regular by an appropriate identification.

4. Properties of Ricci nilsoliton black holes

Let us consider the Lorentzian Ricci nilsoliton black hole metric:

$$ds^2 = -(1 - Me^{-Dw})e^{2q_1w}dt^2 + \frac{dw^2}{1 - Me^{-Dw}} + \sum_{i=2}^{n} e^{2q_iw}\left(\omega^i\right)^2. \tag{40}$$

First we note that $w = \infty$ corresponds to an infinite value of the Gaussian coordinate y; thus spatial infinity is infinitely far away. Moreover, the horizon is located at $y = y_0$ which corresponds to $w = (1/D)\ln M$.

4.1. Geodesics

For outbound null-geodesics travelling in the w-direction we get

$$\frac{dt}{dw} = \frac{e^{-q_1w}}{1 - Me^{-Dw}}. \tag{41}$$

So by integration

$$t - t_0 = \int_{w_0}^{w} \frac{e^{-q_1w}dw}{1 - Me^{-Dw}} \leq \int_{w_0}^{w} e^{-q_1w}dw \leq \frac{e^{-q_1w_0}}{q_1}, \tag{42}$$

and hence, $t - t_0$ is bounded. This implies that light rays reach spatial infinity within finite coordinate time. In this way light signals can leak through spatial infinity. This is analogous to the Anti-de Sitter spacetime.

Consider now timelike geodesics and let p_t be the canonical momentum of t:

$$p_t = \frac{\partial L}{\partial \dot{t}} = \text{constant}. \tag{43}$$

Then, by using the identity $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -1$, we get

$$\dot{w}^2 = p_t^2 e^{-2q_1w} - (1 - Me^{-Dw})\left(1 + h_{ab}\dot{x}^a\dot{x}^b\right) \leq p_t^2 e^{-2q_1w} - (1 - Me^{-Dw}). \tag{44}$$

Hence, for any timelike geodesic, there exists a hypersurface given by $w = w_{\max}$ for which the geodesic can never pass; i.e., $w(\tau) \leq w_{\max}$. The outward going geodesics ultimately stop and start to go inwards. Any timelike geodesic will ultimately cross the horizon at some time in the future.

4.2. The mass of the black hole

Analogous to the Ashtekar–Magnon–Das conformal mass, we can define the 'mass' of the black hole as follows (see, e.g., [30]):

$$\tilde{M} = -\lim_{w \to \infty} \oint m^\rho m^\sigma \left(R_{\mu\rho\sigma\nu} - \bar{R}_{\mu\rho\sigma\nu}\right) n^\mu \xi^\nu d\mathbf{S}. \tag{45}$$

Here, the d$\mathbf{S}$ is the volume element of the surfaces defined by the Ricci nilsolitons; m^μ and n^μ are orthonormal vectors orthogonal to S:

$$g^{\mu\nu}n_\mu n_\nu = -1, \qquad g^{\mu\nu}m_\mu m_\nu = 1, \qquad g^{\mu\nu}m_\mu n_\nu = 0;$$

ξ^μ is the timelike Killing vector field ∂_t; $R_{\mu\rho\sigma\nu}$ is the Riemann tensor of the black hole metric; and $\bar{R}_{\mu\rho\sigma\nu}$ is the Riemann tensor of the 'background' Einstein solvmanifold (this will be justified later).

For the Ricci nilsoliton black holes, we get

$$\widetilde{M} = \frac{1}{2}M(D - 2q_1)(D - q_1)\mathrm{Vol}(N_\mu/\Gamma), \tag{46}$$

where $\mathrm{Vol}(N_\mu/\Gamma)$ is the volume of the compact hypersurfaces at $w = 0$. This shows that the parameter M, up to a constant, can indeed be interpreted as a 'mass' relative to the background solvmanifold.

4.3. The asymptotic geometry

Consider the (Euclidean) spacetime close to spatial infinity, $y = \infty$. The extrinsic curvature can be approximated by

$$\mathbf{K} = \mathbf{D} + 2\boldsymbol{\sigma}\mathrm{e}^{-D(y-y_0)} + \mathcal{O}(\mathrm{e}^{-2D(y-y_0)}). \tag{47}$$

Hence, asymptotically, the spacetime approaches the corresponding solvmanifold as claimed. The solvmanifold spacetime can therefore be considered as the background spacetime in which there is a black hole. Sufficiently far away from the black hole, the spacetime can be approximated as a solvmanifold. These solutions are therefore black hole solutions with a negative cosmological constant, which are not (locally) asymptotically Anti-de Sitter.

In the AdS case, the isometry group acts on the conformal boundary as conformal transformations. This is directly related to the fact that for real hyperbolic space, which is the Euclidean version of AdS space, we have the relation [3]

$$\mathrm{Isom}(\mathbb{H}^n) = \mathrm{Conf}(\partial\mathbb{H}^n), \tag{48}$$

where $\partial\mathbb{H}^n$ is the conformal boundary of $\mathbb{H}^n$. The conformal boundary of $\mathbb{H}^n$ can be identified as the one-point compactification of flat space, $\mathbb{E}^{n-1}$; hence, we can write $\mathrm{Isom}(\mathbb{H}^n) = \mathrm{Conf}(\mathbb{E}^{n-1})$. This relation lies as a foundation of many works on AdS space.

For the nilsoliton black holes, we believe there is an analogous (but not identical) relation for the asymptotic geometry. Firstly, it is easy to see that the isometries of the horospheres are preserved as $y \to \infty$. Secondly, the derivation, $\mathbf{D}$, generates a one-parameter group of automorphisms acting on the asymptotic nilsolitonic geometry. This one-parameter group of automorphisms can be viewed as a *dilaton*, ϕ_t, acting on the nilsolitons. This dilaton and the isometries of the nilsolitons act transitively on the 'background' solvmanifold. In addition to these symmetries, the background solvmanifold may possess some additional isometries. In the case of $\mathbb{H}^n$, these additional symmetries act as 'inversions' on the conformal boundary $\mathbb{E}^{n-1}$ [3].

4.3.1. Complex hyperbolic space

For a general solvmanifold, it is not known whether there exists a similar identity as (48). However, let us consider the complex hyperbolic space, $\mathbb{H}^n_{\mathbb{C}}$, where a related relation is known to exist. Let us also, for simplicity, restrict to 2 complex dimensions even though the following can easily be generalised to $\mathbb{H}^n_{\mathbb{C}}$ for any n.

For $\mathbb{H}^2_{\mathbb{C}} = SU(1, 2)/U(2)$ the group $SU(1, 2)$ acts as isometries with $U(2)$ as an isotropy group. For $\mathbb{H}^2_{\mathbb{C}}$ the boundary can be considered to be the one-point compactification of the Heisenberg group Heis_3. Let us introduce the coordinates (x, y, z) on Heis_3 such that the one-forms

$$\boldsymbol{\omega} = \mathrm{d}x + 2(y\mathrm{d}z - z\mathrm{d}y), \quad \mathrm{d}y, \quad \mathrm{d}z, \tag{49}$$

are the left-invariant one-forms on Heis_3.[5] By considering the dual vector to $\boldsymbol{\omega}$, $\mathbf{e}_x$, we see that $\mathbf{e}_x \in \mathfrak{g}_D^{(1)}$ where $\mathfrak{g}_D^{(i)}$ is the derived series of the Heisenberg algebra. Moreover, $\mathfrak{g}_D^{(2)} = 0$ so $\mathfrak{g}_D^{(1)}$ is Abelian.

Consider therefore a general nilpotent Lie algebra, $\mathfrak{n}$, and let $\mathfrak{g}_D^{(i)}$ be its derived series. Moreover, let k be the largest number such that $\mathfrak{g}_D^{(k)} \neq 0$ and $\mathfrak{g}_D^{(k+1)} = 0$. This implies that $\mathfrak{g}_D^{(k)}$ is an Abelian ideal in $\mathfrak{n}$ and is in the centre of $\mathfrak{n}$. By considering a left-invariant $\mathbf{e} \in \mathfrak{g}_D^{(k)}$, and using Eq. (6), we note that,

$$\langle \mathbf{R}_\mu \mathbf{e}, \mathbf{e} \rangle = \frac{1}{4}\sum_{kl}\langle \mu(\mathbf{e}_k, \mathbf{e}_l), \mathbf{e} \rangle^2 \geq 0, \tag{51}$$

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[5] The one-form $\boldsymbol{\omega}$ defines a *contact structure* on Heis_3, i.e., $\boldsymbol{\omega} \wedge (\mathrm{d}\boldsymbol{\omega})^m$, $m \in \mathbb{N}$ is the volume form. A transformation $f : M \mapsto M$ is a *contact transformation* if

$$f^*\boldsymbol{\omega} = \lambda\boldsymbol{\omega}, \tag{50}$$

for a scalar function λ. The group action of $SU(2, 1)$ will act on Heis_3 as contact transformations. However, the contact structure does not generalise to all the nilpotent black hole spacetimes.

where $= 0$ if and only if $\mathfrak{n}$ is Abelian. This implies, using Eq. (11), that the biggest eigenvalues for **D** will correspond to the the ideal $\mathfrak{g}_D^{(k)}$. Hence, the ideal $\mathfrak{g}_D^{(k)}$ will dominate the asymptotic geometry of the solvmanifold. This ideal therefore plays an important role in the asymptotic geometry; more specifically, assume that q_n is the largest eigenvalue of **D**, then the conformal transformation

$$\mathrm{d}s^2 \mapsto \mathrm{e}^{-2q_n w}\mathrm{d}s^2, \tag{52}$$

renders the limit $\mathbf{g}_\partial \equiv \lim_{w \to \infty} \mathrm{e}^{-2q_n w}\mathrm{d}s^2$ well defined. We can therefore study the symmetry group of the solvmanifold acting on the symmetric two-tensor $\mathbf{g}_\partial$ which resides on the boundary.

Another possible generalisation is the following observation for complex hyperbolic spaces (see, e.g., [31,32]). Define the following 'gauge' on Heis$_3$:

$$\|g\| = \left[x^2 + (y^2 + z^2)^2\right]^{\frac{1}{4}}, \quad \text{where } g = (x, y, z) \in \text{Heis}_3. \tag{53}$$

We define the left-invariant distance between g and g' by

$$d_H(g, g') = \|g^{-1}g'\|. \tag{54}$$

We note that this metric is *not Riemannian*. On the other hand, we do note that $SU(2, 1)$ acts conformally with respect to this metric; in fact,

$$\text{Isom}(\mathbb{H}_\mathbb{C}^2) = \text{Conf}_{d_H}(\text{Heis}_3). \tag{55}$$

By introducing Eq. (53) we can thus manifestly generalise Eq. (48) to complex hyperbolic spaces. The key observation is that the gauge preserves the symmetries of Heis$_3$ and that the dilaton acts homogeneously on the gauge.

We can speculate whether either of these paths can be followed to generalise Eq. (48) to the black hole spacetimes considered here. More work is clearly needed here.

5. Discussion

Here we have discussed how we can construct Ricci nilsoliton black holes from nilpotent groups. The corresponding black hole spacetimes are solutions to the Einstein equations with a negative cosmological constant. We have given conditions for when such solutions exist and, in particular, we have shown that any nilpotent group of dimension ≤ 6 has a corresponding Ricci nilsoliton black hole in dimension ≤ 8. In dimensions higher than 8, there are, in each dimension, an infinite number of locally distinct Ricci nilsoliton black hole spacetimes.

The lowest dimension where there exists a non-trivial nilpotent group is 3. In this regard, Cadeau and Woolgar [33] seem to be the first to construct the corresponding black hole in dimension 5. However, apart from this solution, the nilpotent black holes seem to have gone unnoticed in the literature (they are also pointed out by the author in [27] but in a more general context).

The negatively curved spaces seem to have an incredible rich structure, some of which are displayed in this work. This rich structure makes the negatively curved spaces difficult to study in general which is probably the main reason for the lack of understanding of such spaces. However, in the same token, the wealth of different phenomena these spaces possess is also what makes them so interesting.[6] It is clear that we only have unveiled the tip of the iceberg and that many more treasures remain to be discovered.

Appendix. Ricci nilsolitons of low dimension

The following list contains all nilpotent Lie algebras with their critical points. The eigenvalue type is also included. These tables are taken from [18,19].

The notation used is best illustrated with an example. The tuple

$$(0, 0, \sqrt{3}[12], \sqrt{3}[13], \sqrt{2}[14] + \sqrt{2}[23])$$

represents the Lie algebra

$$[e_1, e_2] = \sqrt{3}e_3, \qquad [e_1, e_3] = \sqrt{3}e_4,$$
$$[e_1, e_4] = \sqrt{2}e_5, \qquad [e_2, e_3] = \sqrt{2}e_5.$$

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[6] Another application of solvmanifolds can be seen in [34].

A.1. Dimension 3

	Critical point	Eigenvalue type	Comments
$N_{3,1}$	$(0, 0, [12])$	$(1 < 2; 2, 1)$	Heis_3

A.2. Dimension 4

	Critical point	Eigenvalue type	Comments
$N_{4,1}$	$(0, 0, [12], [13])$	$(1 < 2 < 3 < 4; 1, 1, 1, 1)$	
$N_{4,2}$	$(0, 0, [12], 0)$	$(2 < 3 < 4; 2, 1, 1)$	$\text{Heis}_3 \oplus \mathbb{R}$

A.3. Dimension 5

	Critical point	Eigenvalue type	Comments
$N_{5,1}$	$(0, 0, 3[12], 4[13], 3[14])$	$(2 < 9 < 11 < 13 < 15; 1, \ldots, 1)$	
$N_{5,2}$	$(0, 0, \sqrt{3}[12], \sqrt{3}[13], \sqrt{2}[14] + \sqrt{2}[23])$	$(1 < 2 < 3 < 4 < 5; 1, \ldots, 1)$	
$N_{5,3}$	$(0, 0, 0, [12], \sqrt{2}[14] + \sqrt{2}[23])$	$(3 < 4 < 6 < 7 < 10; 1, \ldots, 1)$	
$N_{5,4}$	$(0, 0, 0, 0, [12]+[34])$	$(1 < 2; 4, 1)$	Heis_5
$N_{5,5}$	$(0, 0, 4[12], 3[13], 3[23])$	$(1 < 2 < 3; 2, 1, 2)$	
$N_{5,6}$	$(0, 0, 0, [12], [13])$	$(2 < 3 < 5; 1, 2, 2)$	
$N_{5,7}$	$(0, 0, 0, 0, [12])$	$(2 < 3 < 4; 2, 2, 1)$	$\text{Heis}_3 \oplus \mathbb{R}^2$
$N_{5,8}$	$(0, 0, 0, [12], [14])$	$(1 < 2 < 3 < 4; 1, 1, 2, 1)$	$N_{4,1} \oplus \mathbb{R}$

A.4. Dimension 6

A.4.1. 5 and 4 step:

	Critical point	Eigenvalue type
$N_{6,1}$	$(0, 0, \sqrt{13}[12], 4[13], \sqrt{12}[14] + 2[23], \sqrt{12}[34] + \sqrt{13}[52])$	$(1 < 2 < 3 < 4 < 5 < 7; 1, \ldots, 1)$
$N_{6,2}$	$(0, 0, [12], \sqrt{\frac{4}{3}}[13], [14], [34] + [52])$	$(1 < 3 < 4 < 5 < 6 < 9; 1, \ldots, 1)$
$N_{6,3}$	$(0, 0, 2[12], \sqrt{6}[13], \sqrt{6}[14], 2[15])$	$(1 < 9 < 10 < 11 < 12 < 13; 1, \ldots, 1)$
$N_{6,4}$	$(0, 0, \sqrt{22}[12], 6[13], \sqrt{22}[14] + \sqrt{30}[23], 5[24] + \sqrt{30}[15])$	$(1 < 2 < 3 < 4 < 5 < 6; 1, \ldots, 1)$
$N_{6,5}$	$(0, 0, \sqrt{7}[12], \sqrt{\frac{15}{2}}[13], 3[14], \sqrt{\frac{15}{2}}[23] + 2[15])$	$(1 < 3 < 4 < 5 < 6 < 7; 1, \ldots, 1)$
$N_{6,6}$	$(0, 0, [12], [13], [23], [14])$	$(1 < 2 < 3 < 4 < 5; 1, 1, 1, 1, 2)$
$N_{6,7}$	$(0, 0, 2[12], \sqrt{5}[13], \sqrt{5}[23], 2[14] - 2[25])$	$(1 < 2 < 3 < 4; 2, 1, 2, 1)$
$N_{6,8}$	$(0, 0, 2[12], \sqrt{5}[13], \sqrt{5}[23], 2[14] + 2[25])$	$(1 < 2 < 3 < 4; 2, 1, 2, 1)$
$N_{6,9}$	$(0, 0, 0, \sqrt{\frac{5}{4}}[12], [14] - [23], \sqrt{\frac{5}{4}}[15] + [34])$	$(6 < 11 < 12 < 17 < 23 < 29; 1, \ldots, 1)$
$N_{6,10}$	$(0, 0, 0, [12], \sqrt{\frac{5}{3}}[14], [15] + [23])$	$(4 < 9 < 12 < 13 < 17 < 21; 1, \ldots, 1)$
$N_{6,11}$	$(0, 0, -\sqrt{\frac{35}{136}}[12], \sqrt{\frac{21}{34}}[12], \sqrt{\frac{25}{68}}[14] - \sqrt{\frac{15}{17}}[13], \sqrt{\frac{3}{4}}[15] + \sqrt{\frac{7}{8}}[24])$	$(1 < 2 < 3 < 4 < 5; 1, 1, 2, 1, 1)$
$N_{6,12}$	$(0, 0, 0, \sqrt{3}[12], \sqrt{3}[14], \sqrt{2}[15] + \sqrt{2}[24])$	$(3 < 6 < 9 < 11 < 12; 1, \ldots, 1)$
$N_{6,13}$	$(0, 0, 0, \sqrt{3}[12], 2[14], \sqrt{3}[15])$	$(2 < 9 < 11 < 12 < 13 < 15; 1, \ldots, 1)$

A.4.2. 3 and 2 step:

	Critical point	Eigenvalue type
$N_{6,14}$	$(0, 0, 0, \sqrt{3}[12], \sqrt{2}[13], \sqrt{2}[14] + \sqrt{3}[35])$	$(2 < 3 < 4 < 5 < 6 < 8; 1, \ldots, 1)$
$N_{6,15}$	$(0, 0, 0, [12], [23], [14] + [35])$	$(1 < 2 < 3; 3, 2, 1)$
$N_{6,16}$	$(0, 0, 0, [12], [23], [14] - [35])$	$(1 < 2 < 3; 3, 2, 1)$
$N_{6,17}$	$(0, 0, 0, 2[12], \sqrt{3}[14], \sqrt{3}[24])$	$(5 < 10 < 12 < 15; 2, 1, 1, 2)$
$N_{6,18}$	$(0, 0, 0, \sqrt{2}[12], \sqrt{\frac{1}{2}}[13]+\sqrt{\frac{3}{2}}[42], \sqrt{\frac{3}{2}}[14]+\sqrt{\frac{1}{2}}[23])$	$(1 < 2 < 3; 2, 2, 2)$
$N_{6,19}$	$(0, 0, 0, 2[12], \sqrt{3}[14], [13] + \sqrt{3}[42])$	$(5 < 6 < 11 < 12 < 16 < 17; 1, \ldots, 1)$
$N_{6,20}$	$(0, 0, -[12], \sqrt{3}[12], 2[14], [24] - \sqrt{3}[23])$	$(1 < 2 < 3; 2, 2, 2)$
$N_{6,21}$	$(0, 0, 0, \sqrt{2}[12], [13], \sqrt{2}[14] + [23])$	$(3 < 5 < 6 < 8 < 9; 1, \ldots, 1)$
$N_{6,22}$	$(0, 0, 0, \sqrt{\frac{3}{4}}[12], \sqrt{\frac{3}{4}}[13], [24])$	$(5 < 6 < 9 < 11 < 15 < 16; 1, \ldots, 1)$
$N_{6,23}$	$(0, 0, 0, \sqrt{2}[12], [13], \sqrt{2}[14])$	$(2 < 5 < 6 < 7 < 8 < 9; 1, \ldots, 1)$
$N_{6,24}$	$(0, 0, 0, [12], [13], [23])$	$(1 < 2; 3, 3)$
$N_{6,25}$	$(0, 0, 0, 0, 2[12], \sqrt{3}[15] + \sqrt{3}[34])$	$(5 < 8 < 9 < 13 < 18; 1, 1, 2, 1, 1)$
$N_{6,26}$	$(0, 0, 0, 0, [12], [15])$	$(1 < 2 < 3 < 4; 1, 1, 3, 1)$
$N_{6,27}$	$(0, 0, 0, 0, \sqrt{2}[12], [14] + \sqrt{2}[25])$	$(3 < 4 < 6 < 7 < 10; 1, 1, 1, 2, 1)$
$N_{6,28}$	$(0, 0, 0, 0.[13] + [42], [14] + [23])$	$(1 < 2; 4, 2)$
$N_{6,29}$	$(0, 0, 0, 0, [12], [14] + [23])$	$(13 < 4 < 6 < 7; 2, 2, 1, 1)$
$N_{6,30}$	$(0, 0, 0, 0, [12], [34])$	$(1 < 2; 4, 2)$
$N_{6,31}$	$(0, 0, 0, 0, [12], [13])$	$(2 < 3 < 4 < 5; 1, 2, 1, 2)$
$N_{6,32}$	$(0, 0, 0, 0, 0, [12] + [34])$	$(3 < 4 < 6; 4, 1, 1)$
$N_{6,33}$	$(0, 0, 0, 0, 0, [12])$	$(2 < 3 < 4; 2, 3, 1)$

We note that the following Lie algebras contain an Abelian factor:

$$N_{6,12} = N_{5,2} \oplus \mathbb{R}, \qquad N_{6,13} = N_{5,1} \oplus \mathbb{R},$$
$$N_{6,17} = N_{5,5} \oplus \mathbb{R}, \qquad N_{6,26} = N_{4,1} \oplus \mathbb{R}^2,$$
$$N_{6,27} = N_{5,3} \oplus \mathbb{R}, \qquad N_{6,31} = N_{5,6} \oplus \mathbb{R},$$
$$N_{6,32} = N_{5,4} \oplus \mathbb{R}, \qquad N_{6,33} = N_{3,1} \oplus \mathbb{R}^3. \tag{56}$$

Also worth noting are the following isomorphisms:

$$N_{6,28} = \text{Heis}_{3,\mathbb{C}}, \qquad N_{6,30} = \text{Heis}_3 \oplus \text{Heis}_3. \tag{57}$$

As an example, consider $N_{6,25}$. The components of the Ricci tensor can be found from Eq. (6):

$$\mathbf{R}_\mu = \text{diag}\left(-\frac{7}{2}, -2, -\frac{3}{2}, -\frac{3}{2}, \frac{1}{2}, 3\right), \tag{58}$$

which gives

$$R(\mu) = -5, \qquad F(\mu) = 30. \tag{59}$$

Since, $\mathbf{R}_\mu = c_\mu \mathbf{1} + \text{Tr}(\mathbf{D})\mathbf{D}$, and $c_\mu = F(\mu)/R(\mu)$, we have

$$\mathbf{R}_\mu = -6 \cdot \mathbf{1} + \frac{1}{2}\text{diag}(5, 8, 9, 9, 13, 18). \tag{60}$$

The eigenvalue type is therefore given in the above table (which corrects a typo in [19]).

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