EXTERIOR – A MAPLE 10/11/12 library for computations in exterior calculus

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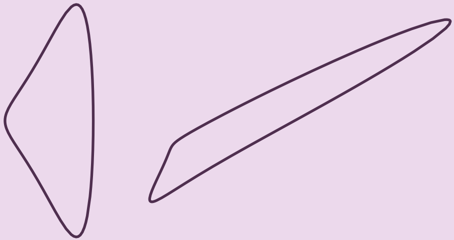

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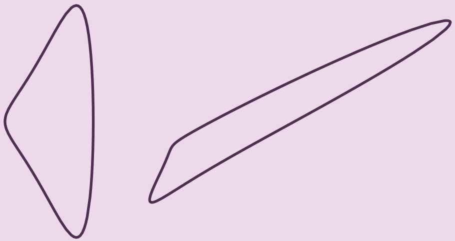

- What do we mean by the same?
- Is the second curve the image of the first curve but viewed from a different position?

Mathematical Formulation

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- For image recognition applications, G is the (Lie) group of projective transformations (or a subgroup of this group). The action on $\mathbf{R}^2$ is given by

$$(x,\ u) \mapsto \left(\frac{\alpha x + \beta u + \gamma}{\rho x + \sigma u + \tau},\ \frac{\lambda x + \mu u + \nu}{\rho x + \sigma u + \tau} \right)$$

with

$$\det \begin{bmatrix} \alpha & \beta & \gamma \\ \lambda & \mu & \nu \\ \rho & \sigma & \tau \end{bmatrix} = 1.$$

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- Obtain (eventually!) the differential invariants.
- **Two curves are equivalent if (and only if) their differential invariants agree.**

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- **Resolution**
    - Use a algebraic computing package (MAPLE in this case).
        - The computations are challenging for computer algebra packages due to term explosion (far more severe than in "standard" point symmetry computations) and branching.
    - One can reformulate the equivalence problem to yield invariants less sensitive to noise.

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- Input and output from EXTERIOR matches "hand computations" as much as possible.
- EXTERIOR is ideal for equivalence type computations (as well as Lie symmetries, Cauchy characteristics, ... computations).
- EXTERIOR is not restricted to polynomial dependencies.

An Example

Problem

Equivalence of second order ODEs

$$u_{xx} = F(x,\ u,\ u_x)$$

under fibre preserving transformations

$$(x,\ u) \mapsto (f(x),\ g(x,\ u)).$$

- This problem illustrates the features finding the invariants of the projective group.

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Equivalence of second order ODEs

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- This problem illustrates the features finding the invariants of the projective group.
- Hopefully(!) we can do this computation live in the allocated time frame.

- Coframe:

$$\omega = \begin{bmatrix} du - u_x\, dx \\ du_x - F\, dx \\ dx \end{bmatrix}$$

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- Group action:

$$G = \begin{bmatrix} a_1 & 0 & 0 \\ a_2 & a_3 & 0 \\ 0 & 0 & a_4 \end{bmatrix}$$

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- Lifted coframe:

$$\theta = G \cdot \omega = \begin{bmatrix} a_1\, du - a_1\, u_x\, dx \\ a_2\, du + a_3\, du_x - (a_2\, u_x + a_3\, F)\, dx \\ a_4\, dx \end{bmatrix}$$

- "Absorbed" form:

$$d\theta = \begin{bmatrix} \Phi_1 \wedge \theta_1 + T\,\theta_2 \wedge \theta_3 \\ \Phi_2 \wedge \theta_1 + \Phi_3 \wedge \theta_2 \\ \Phi_4 \wedge \theta_3 \end{bmatrix}$$

with

$$\Phi = \begin{bmatrix} \dfrac{da_1}{a_1} - \dfrac{a_2\,dx}{a_3} \\ \dfrac{da_2}{a_1} - \dfrac{a_2\,da_3 + (a_2^2 - a_2\,a_3\,F_{u_x} - a_3^2\,F_u)\,dx}{a_1\,a_3} \\ \dfrac{da_3}{a_3} + \left(\dfrac{a_2}{a_3} + F_{u_x}\right) dx \\ \dfrac{da_4}{a_4} \end{bmatrix}$$

and

$$T = -\frac{a_1}{a_3\,a_4}.$$

An Example

- Freedom in absorbed form:

$$\Phi \to \Phi + \begin{bmatrix} \chi_{1,1} & 0 & 0 \\ \chi_{2,1} & \chi_{2,2} & 0 \\ \chi_{2,2} & \chi_{3,2} & 0 \\ 0 & 0 & \chi_{4,3} \end{bmatrix} \theta.$$

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- Normalize non-constant torsion:

$$T = -1$$

that is,

$$a_3 = \frac{a_1}{a_4}.$$

An Example

- Recompute absorbed form:

$$d\theta = \begin{bmatrix} \Phi_1 \wedge \theta_1 - \theta_2 \wedge \theta_3 \\ \Phi_2 \wedge \theta_1 + (\Phi_1 - \Phi_3) \wedge \theta_2 \\ \Phi_3 \wedge \theta_3 \end{bmatrix}$$

with

$$\Phi = \begin{bmatrix} \dfrac{da_1}{a_1} - \dfrac{a_2\, a_4\, dx}{a_1} \\ \dfrac{1}{a_1^2\, a_4}\Big(a_1\, a_4\, da_2 - a_2\, a_4\, da_1 + a_1\, a_2\, da_4 \\ \qquad - (a_2^2\, a_4^2 + a_1\, a_2\, a_4\, F_{u_x} - a_1^2\, F_u)\, dx\Big) \\ \dfrac{da_4}{a_4} - \left(\dfrac{2a_2\, a_4}{a_1} + F_{u_x}\right) dx \end{bmatrix}.$$

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- Cartan test:
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- Freedom:

$$\Phi \rightarrow \Phi + \begin{bmatrix} \chi_{1,1} & 0 & 0 \\ \chi_{2,1} & \chi_{1,1} & 0 \\ 0 & 0 & 0 \end{bmatrix} \theta = \Phi + Z\,\theta.$$

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- Normalize non-constant torsion.
- This prolonged coframe is involutive.

- We have

$$d\theta^{(1)} = \begin{bmatrix} \Phi_1 \wedge \theta_1 - \theta_2 \wedge \theta_3 \\ \Phi_2 \wedge \theta_1 + (\Phi_1 - \Phi_3) \wedge \theta_2 \\ \Phi_3 \wedge \theta_3 \\ \pi_1 \wedge \theta_1 - \Phi_2 \wedge \theta_3 \\ \pi_2 \wedge \theta_1 + \pi_1 \wedge \theta_2 + \Phi_2 \wedge \Phi_3 \\ - 2\Phi_2 \wedge \theta_3 \end{bmatrix}$$

with

$$\pi_1 = J_1\,\theta_2 + J_2\,\theta_3$$
$$\pi_2 = J_3\,\theta_3$$

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- The coefficients of $\theta_a \wedge \theta_b$ on the equations for $d\theta_c$ are the structure invariants.
- However they are not necessarily independent or in "optimal" form.