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A contribution to the symmetry classification problem for second-order PDEs in $z(x, y)$

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I report on a contribution to the point *symmetry classification problem* for second-order partial differential equations (PDEs) in $z(x, y)$, i.e. to an overview over all possible symmetry groups admitted by this class of equations. The article also contains a concise introduction into *classical symmetry analysis.*

Sophus Lie (1842–1899) determined all continuous transformation groups of the 2D plane and gave normal forms for any ordinary differential equation that is invariant under one of those groups. I deal with the extension of Lie's program to second-order PDEs in $z(x, y)$. The starting point to this endeavour is a previously unknown paper by Amaldi from 1901, which claims to have completed Lie's classification of groups acting in (x, y, z)-space. I also present a Maple procedure ('LHSO1_PDE_Solver') for solving systems of linear, homogeneous first-order PDEs that performs better on this class than Maple's built-in PDE system solver.

Keywords: differential invariants; Lie groups; (partial) differential equations; symmetry analysis; symmetry group classification.

1. Introduction

In this paper, I report on a contribution (Hillgarter, 2002) to the point *symmetry classification problem* for second-order partial differential equations (PDEs) in $z(x, y)$, i.e. to an overview over all possible symmetry groups admitted by this class of equations. In addition, Section 2 is written as a concise introduction into *classical symmetry analysis* and hence addresses a broader readership, not necessarily only readers interested in the remaining part.

The study of symmetries has been initiated by Sophus Lie (1842–1899). Roughly speaking, a symmetry group is a group of transformations that takes solutions to solutions. The *symmetry classification problem* aims at obtaining a complete survey of all possible symmetries for a class of given differential equations (DEs), e.g. DEs of a given order in a predetermined number of dependent and independent variables. The starting point of this approach is a list of groups whose differential invariants determine the general form of a DE that may be invariant under the respective group.

Lie determined all continuous transformation groups of the 2D plane (Lie, 1888) and gave normal forms for any ordinary differential equation (ODE) that is invariant under one of those groups. Thereby, higher order equations are given implicitly, since they can be derived recursively from the corresponding equations of lower order. I deal with the extension of Lie's program to second-order PDEs in $z(x, y)$, following the strategy given below.

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(1) List all finite continuous transformation groups of the (x, y, z)-space.
(2) Find the differential invariants of the groups given in (1) where $z = z(x, y)$. For a given group, the DEs invariant under this group are functions of these invariants.

The contributions in Hillgarter (2002) to the symmetry classification problem for second-order PDEs in $z = z(x, y)$ are as follows.

(1) A list containing all point transformation groups acting on (x, y, z)-space, based on the work of Lie and Amaldi, with corrections and compact notation (see Section 3). This list can also serve as the starting point for the symmetry classification problem for *systems of ODEs* in $y = y(x)$, $z = z(x)$!
(2) A complete overview over the differential invariants up to order two, with the exception of Amaldi's groups of type [B] (see Sections 4 and 5).

The paper by Amaldi (1901, 1902), which claims to have completed Lie's classification of groups acting in (x, y, z)-space, has been discovered by Fritz Schwarz. This work has not been referenced in the modern symmetry group literature. Indeed, there are many statements that this classification problem is still unsolved, and some have claimed that the problem is insoluble in principle.

Differential Invariants are solutions of a system of linear, homogeneous first-order PDEs. I implemented a Maple procedure ('LHSO1_PDE_Solver') for solving such systems by means of the narrowing transformation, presented in Subsection 2.7 and Section 5.

In Section 6, I indicate some classical equations to be found among the computed invariants as well as some comments on applications.

Concluding remarks are given in Section 7. There I show how to compute higher order invariants from lower ones, and refer to some open problems. Several topics that would destroy the flow of presentation have been moved to the appendix: lower invariants, basic notions for Lie algebras (LAs) of vector fields (isomorphism and similarity) and systems of imprimitivity.

2. Basic concepts: symmetries of DEs

In this section, I introduce basic notions for symmetries of ODEs. These concepts extend to the case of PDEs, too. I introduce *transformation groups* and their *differential invariants*, which determine the *invariant equations* corresponding to the group. The differential invariants are solutions of a system of PDEs, called the *system of differential invariants*. Our presentation follows partly Bluman & Kumei (1996), Ibragimov (1999) and Schwarz (2005). Other recent textbooks on symmetry analysis are those of Baumann (2000), Cantwell (2002) and Hydon (2000). A comprehensive reference for group analysis of DEs in general is Ibragimov (1994–1996).

2.1 *Transformation groups of DEs*

Introducing new variables into a given (O)DE, as taught in standard university courses, is a widely used method in order to facilitate the solution process. Usually this is done in an ad hoc manner without guaranteed success. In particular, there is no criterion to decide whether a certain class of transformations will lead to an integrable equation or not. A critical examination of these methods was the starting point for Lie's symmetry analysis. We will now have a look at the behaviour of DEs under special kinds of transformations.

Let an ODE of order n be given as

$$\omega(x, y, y', \ldots, y^{(n)}) = 0. \tag{1}$$

The general solution of such an equation is a set of curves in the x-y-plane depending on n-parameters $C_1, \ldots, C_n$, given by

$$\Theta(x, y, C_1, \ldots, C_n) = 0. \tag{2}$$

Invertible analytic transformations between two planes with coordinates $\mathbf{x} = (x, y)$ and $\mathbf{u} = (u, v)$, respectively, that are of the form

$$\mathbf{u} = T(\mathbf{x}) = (\sigma(\mathbf{x}), \rho(\mathbf{x})) \tag{3}$$

are called *point transformations*. We will encounter them in the form of *one-parameter groups of point transformations*

$$\mathbf{u} = T(\mathbf{x},\varepsilon) = (\sigma(\mathbf{x},\varepsilon), \rho(\mathbf{x},\varepsilon)). \tag{4}$$

Here the real parameter ε ranges over an open interval including 0, such that for any fixed value of ε, (4) represents a point transformation. In addition, there exists a real group composition Φ such that

$$\mathbf{x}_1 = T(\mathbf{x}_0, \varepsilon_0), \quad \mathbf{x}_2 = T(\mathbf{x}_1, \varepsilon_1) \Longrightarrow \mathbf{x}_2 = T(\mathbf{x}_0, \Phi(\varepsilon_0, \varepsilon_1)).$$

Group transformations of this kind may be reparametrized such that we have $\Phi(\varepsilon, \bar{\varepsilon}) = \varepsilon + \bar{\varepsilon}$, and such that $\varepsilon = 0$ represents the identity element.

Equation (1) is said to be *invariant* under the change of variables $\mathbf{u} = T(\mathbf{x})$ if it retains its form under this transformation, i.e. if the functional dependence of the transformed equation on $\mathbf{u}$ is the same as in the original equation (1), i.e. $\omega(u, v, v', \ldots, v^{(n)}) = 0$, where $v' = \mathrm{d}v/\mathrm{d}u$ etc. Such a transformation T is called a *symmetry* of the DE. The same transformation acts on the curves (2). If it is a symmetry, the functional dependence of the transformed curves on $\mathbf{u}$ must be the same as in (2), i.e. $\Theta(\mathbf{u}, \bar{C}_1, \ldots, \bar{C}_n) = 0$. This is not necessarily true for the parameters $C_1, \ldots, C_n$ because they do not occur in the DE itself. This means, the entirety of curves described by the two equations is the same, however, to any fixed values for the constants two different curves may correspond. In other words, the solution curves are permuted among themselves by a symmetry transformation. It is fairly obvious that all symmetry transformations of a given DE form a group, the *symmetry group* of that equation.

EXAMPLE 1 The following table illustrates the notions introduced above. Provided are three simple one-parameter transformation groups in canonical parametrization and their invariant equations. Invariant DEs are provided at a later stage. The last column is explained in the following subsection.

Name	Transformation	Invariant Equation	Infinitesimal Generator
Translation	$u = x + \varepsilon$, $v = y$	$f(y) = 0$	$x = \partial_x$
Dilation	$u = \mathrm{e}^{\varepsilon} x$, $v = \mathrm{e}^{\varepsilon} y$	$f(x/y) = 0$	$X = x\partial_x + y\partial_y$
Rotation	$u = \cos(\varepsilon)x + \sin(\varepsilon)y$, $v = -\sin(\varepsilon)x + \cos(\varepsilon)y$	$f(x^2 + y^2) = 0$	$X = -y\partial_x + x\partial_y$

2.2 *Infinitesimal generators*

A Taylor expansion of (4) around $\epsilon = 0$ gives

$$\mathbf{u} = \mathbf{x} + \varepsilon \frac{\partial T}{\partial \varepsilon}(\mathbf{x}, \varepsilon)|_{\varepsilon=0} + \frac{\varepsilon^2}{2} \frac{\partial^2 T}{\partial^2 \varepsilon}(\mathbf{x}, \varepsilon)|_{\varepsilon=0} + \mathrm{O}(\varepsilon^3).$$

The components of $\xi(\mathbf{x}) = (\xi(\mathbf{x}), \eta(\mathbf{x})) := \frac{\partial T}{\partial \varepsilon}(\mathbf{x}, \varepsilon)|_{\varepsilon=0}$ are called *infinitesimals* of (4) and $\mathbf{u} = \mathbf{x} + \varepsilon\xi(\mathbf{x})$ is called the *infinitesimal transformation* of (4). From the *infinitesimal generator*

$$X = \xi(\mathbf{x})\partial_x + \eta(\mathbf{x})\partial_y, \tag{5}$$

the original group $\mathbf{u} = T(\mathbf{x}, \varepsilon)$ can be recovered as solution of the *initial value problem*

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\varepsilon} = \xi(\mathbf{u}), \quad (\mathbf{u} = \mathbf{x})|_{\varepsilon=0}.$$

Formally, this solution is also given by the *Lie series* $\mathbf{u} = \sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!} X^k \mathbf{x}$. The infinitesimal generator encodes the original transformation group in an economic way.

2.3 *Multi-parameter groups*

The definitions for one-parameter transformation groups (4) extend to the case of r-parameter groups by simply replacing ε by $\varepsilon = (\varepsilon_1, \dots, \varepsilon_r)$ and the real group composition $\Phi : \mathbb{R} \to \mathbb{R}$ by $\mathbf{\Phi}: \mathbb{R}^r \to \mathbb{R}^r$. The main result sufficient to meet the needs of applied group analysis for practical constructions of multi-parameter groups is as follows: let L_r be an r-dimensional vector space spanned by r generators $\{X_1, \dots, X_r\}$. The composition $\mathbf{T}_\epsilon = T_{\varepsilon_r} \dots T_{\varepsilon_1}$ of r one-parameter groups of transformations T_{ϵ_i} generated individually by each of the basis operators X_i via the Lie equations

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\varepsilon_i} = \xi(\mathbf{u}), \quad (\mathbf{u} = \mathbf{x})|_{\varepsilon_i=0}, \quad 1 \leqslant i \leqslant r, \tag{6}$$

is an r-parameter group G_r if and only if L_r is a LA. The definition of LAs of vector fields and related basic notions are provided in the appendix.

EXAMPLE 2 We consider the 3D vector space spanned by $L_3 = \{\partial_x, \partial_y, y\partial_x\}$. Solution of the Lie equations (6) leads to $T_{\varepsilon_1}(x, y) = (x + \varepsilon_1, y)$, $T_{\varepsilon_2}(x, y) = (x, y + \varepsilon_2)$ and $T_{\varepsilon_3}(x, y) = (x + \varepsilon_3 y, y)$. Hence, the composition $\mathbf{T}_\epsilon = T_{\varepsilon_3} T_{\varepsilon_2} T_{\varepsilon_1}$, where $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$, has the form $\mathbf{T}_\epsilon(x, y) = (x + \varepsilon_3 y + \varepsilon_1 + \varepsilon_2, y + \varepsilon_2\varepsilon_3)$. Consecutive application of $\mathbf{T}_\epsilon$ and $\mathbf{T}_\delta$, where $\delta = (\delta_1, \delta_2, \delta_3)$, yields the transformation $(\mathbf{T}_\delta\mathbf{T}_\epsilon)(x, y) = (x + y(\epsilon_3 + \delta_3) + \epsilon_2(\epsilon_3 + \delta_3) + \delta_2\delta_3 + \epsilon_1 + \delta_1, y + \epsilon_2 + \delta_2)$. Now solving $\mathbf{T}_\delta\mathbf{T}_\epsilon = \mathbf{T}_\gamma$, where $\gamma = (\gamma_1, \gamma_2, \gamma_3)$, for $\gamma_1, \gamma_2, \gamma_3$ yields the group composition law $\mathbf{\Phi}(\varepsilon, \delta) = (\Phi_1(\varepsilon, \delta), \Phi_2(\varepsilon, \delta), \Phi_3(\varepsilon, \delta))$:

$$\Phi_1(\varepsilon, \delta) = \epsilon_1 + \delta_1 - \epsilon_3\delta_2, \quad \Phi_2(\varepsilon, \delta) = \epsilon_2 + \delta_2, \quad \Phi_3(\varepsilon, \delta) = \epsilon_3 + \delta_3.$$

2.4 *Prolongations*

Let a curve in the $\mathbf{x}$-plane described by $y = f(x)$ be transformed under a point transformation of the form (3) into $v = g(u)$. Now the question arises as to how the derivative $y' = \frac{\mathrm{d}f}{\mathrm{d}x}$ corresponds to $v' = \frac{\mathrm{d}g}{\mathrm{d}u}$ under this transformation. A simple calculation leads to the *first prolongation*

$$v' = \frac{\mathrm{d}v}{\mathrm{d}u} = \frac{\rho_x + \rho_y y'}{\sigma_x + \sigma_y y'} \equiv \chi(x, y, y').$$

Note that the knowledge of (x, y, y') and the equations of the point transformation (3) already determine v' uniquely, and the knowledge of the equation of the curve is not required. This may be expressed by saying that the line element (x, y, y') is transformed into the line element (u, v, v') under the action of a point transformation. Similarly, the transformation law for derivatives of second order is obtained as

$$v'' = \frac{\mathrm{d}v'}{\mathrm{d}u} = \frac{\chi_x + \chi_y y' + \chi_{y'} y''}{\sigma_x + \sigma_y y'}.$$

For later applications it would be useful to express the second derivative v'' explicitly in terms of σ and ρ. This more lengthy formula has not been provided here, so instead the prolongation formulas for one-parameter groups of point transformations are given below. The transformation properties of the derivatives are expressed in terms of the prolongations of the infinitesimal generator (5). These prolongations are defined via the operator of total differentiation w.r.t. x

$$D_x \equiv \partial_x + \sum_{k\geqslant 1} y^{(k)} \partial_{y^{(k-1)}},$$

as

$$X^{(n)} = X + \sum_{k=1}^{n} \zeta^{(k)}(y', \ldots, y^{(k)})\partial_{y^{(k)}}, \quad n \geqslant 1, \text{ where}$$
$$\zeta^{(1)} = D_x(\eta) - y' D_x(\xi),$$
$$\zeta^{(k)} = D_x(\zeta^{(k-1)}) - y^{(k)} D_x(\xi), \quad k \geqslant 2.$$

The two lowest ζ's are given explicitly:

$$\zeta^{(1)} = \eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2,$$
$$\zeta^{(2)} = \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 + (\eta_y - 2\xi_x)y'' - 3\xi_y y' y''.$$

These two innocent looking expressions should not divert from the fact that the number of terms in $\zeta^{(k)}$ grows roughly as 2^k. But $\zeta^{(k)}$ is at least linear and homogeneous in $\xi(\mathbf{x})$ and $\eta(\mathbf{x})$ and its derivatives up to order k. In addition, $\zeta^{(k)}$ does not depend explicitly on x and y but only on $y', y'', \ldots, y^{(k)}$. For $k > 1$, $y^{(k)}$ occurs linearly and y' occurs with power $k + 1$ in $\zeta^{(k)}$.

EXAMPLE 3 The infinitesimal generator of the canonical rotation given in Example 1 was $X = -y\partial_x + x\partial_y$. By $\zeta^{(1)} = D_x(\eta) - y' D_x(\xi) = D_x(x) - y' D_x(-y) = 1 + y'^2$ we obtain $X^{(1)} = X + \zeta^{(1)}\partial_{y'} = -y\partial_x + x\partial_y + (1 + y'^2)\partial_{y'}$. The first-order prolongations for the canonical translation and dilation are trivial, i.e. $\zeta^{(1)} = 0$.

2.5 *The system of differential invariants*

Consider an r-parameter plane group $G = \{X_1, \ldots, X_r\}$ represented by r infinitesimal generators $X_i = \xi_i(\mathbf{x})\partial_x + \eta_i(\mathbf{x})\partial_y$, $i = 1, \ldots, r$. Their m-th order prolongations

$$X_i^{(m)} = X_i + \sum_{k=1}^{m} \zeta_i^{(k)} \partial_{y^{(k)}}, \quad i = 1, \ldots, r$$

determine the *system of differential invariants*

$$X_i^{(m)}\Phi \equiv 0, \quad i = 1, \ldots, r. \tag{7}$$

Lie has discussed these systems in detail, for a recent presentation see Schwarz (2005). The group property guarantees that (7) is a complete system for Φ with $d := m + 2 - r$ fundamental solutions

$$\Phi_i(x, y, y', \ldots, y^{(m)}), \quad i = 1, \ldots, d,$$

called *differential invariants* of order m. They are linear in the highest derivative. Any ODE $f(x, y, y', \ldots, y^{(m)}) = 0$ invariant under G in the actual variables can be written as

$$f = F(\Phi_1, \ldots, \Phi_d),$$

where F is an arbitrary function depending on d arguments. System (7) may be brought into Jacobian normal form, an analog of the triangular form for matrices, before attempting to solve it. The dependencies of the fundamental solutions may then be chosen such that $\Phi_1 \equiv \Phi_1(x, y, y', \ldots, y^{(r-1)})$, $\Phi_2 \equiv \Phi_2(x, y, y', \ldots, y^{(r)})$, ..., $\Phi_d \equiv \Phi_d(x, y, y', \ldots, y^{(m)})$.

EXAMPLE 4 The following table provides the first-order differential invariants for the three canonical transformation groups considered before.

Name	$X^{(1)}$	First-order invariants $X^{(1)}\Phi = 0$
Translation	∂_x	y, y'
Dilation	$x\partial_x + y\partial_y$	$x/y, y'$
Rotation	$-y\partial_x + x\partial_y + (1 + y'^2)\partial_{y'}$	$x^2 + y^2$, $\arctan\left(\frac{x}{y}\right) + \arctan(y')$

EXAMPLE 5 We consider the three-parameter transformation group G_3 acting on the (x, y)-plane represented by $L_3 = \{\partial_x, \partial_y, y\partial_x\}$ from Example 2. Prolongation of its generators up to the third order yields the following system of differential invariants:

$$0 = \partial_x\Phi = \partial_y\Phi = (y\partial_x - y'^2\partial_{y'} - 3y'y''\partial_{y''} - (3y''^2 + 4y'y''')\partial_{y'''})\Phi.$$

Using some strategy for solving systems of linear PDEs (compare Subsection 2.7) we might arrive at the following two fundamental solutions:

$$\Phi_1 \equiv y''/y'^3, \quad \Phi_2 \equiv (y'y''' - 3y''^2)/y'^5.$$

The DEs of order not higher than three that have the respective Lie group G as symmetry group have the general form $F(\Phi_1, \Phi_2)$.

2.6 *Symmetries of PDEs*

All concepts and notions discussed in the preceding subsections extend to the case of PDEs, too. The only difference is that we deal with one dependent and n independent variables, i.e. $\mathbf{x} = (x_1, \ldots, x_n, u)$. Partial derivatives $\partial_{x_{i_1}} \ldots \partial_{x_{i_k}} u$ are represented by formal variables $u_{i_1 \ldots i_k}$, called *differential indeterminates*. They are symmetric in their indices. The differential variables of order k are denoted by $\mathbf{u}^{(k)}$.

The k-th Prolongation of an infinitesimal generator

$$X = \sum_{i=1}^{n} \xi_i(\mathbf{x})\partial_{x_i} + \eta(\mathbf{x})\partial_u$$

is given by

$$X^{(k)} = X + \sum_{i=1}^{n} \zeta_i(\mathbf{u}^{(1)})\partial_{u_i} + \cdots + \sum_{i_1 i_2 \ldots i_k} \zeta_{i_1 i_2 \ldots i_k}(\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(k)})\partial_{u_{i_1 i_2 \ldots i_k}},$$

where recursive formulas for the extended infinitesimals $\zeta_{i_1 i_2 \ldots i_k}$ are given in terms of the operators of total differentiation w.r.t. the variables x_i

$$D_i = \frac{\partial}{\partial x_i} + \sum_{k \geqslant 1} \sum_{i_1 i_2 \ldots i_k} u_{i i_1 \ldots i_k} \frac{\partial}{\partial u_{i_1 \ldots i_k}}, \tag{8}$$

by

$$\zeta_i = D_i \eta - \sum_{j=1}^{n} (D_i \xi_j) u_j, \quad 1 \leqslant i \leqslant n, \tag{9}$$

$$\zeta_{i_1 i_2 \ldots i_k} = D_{i_k} \zeta_{i_1 i_2 \ldots i_{k-1}} - \sum_{j=1}^{n} (D_{i_k} \xi_j) u_{i_1 i_2 \ldots i_{k-1} j}, \tag{10}$$

where $k \geqslant 2$. Specializing to the case of one dependent variable z and two independent variables x, y, the second-order prolongation of

$$X = \xi_1(x, y, z)\partial_x + \xi_2(x, y, z)\partial_y + \eta(x, y, z)\partial_z$$

is given by

$$X^{(2)} = X + \zeta_1 \partial_x + \zeta_2 \partial_y + \zeta_{11} \partial_{xx} + \zeta_{12} \partial_{xy} + \zeta_{22} \partial_{yy},$$

where the extended infinitesimals $\zeta_1(\mathbf{z}^{(1)})$, $\zeta_2(\mathbf{z}^{(1)})$, $\zeta_{11}(\mathbf{z}^{(1)}, \mathbf{z}^{(2)})$, $\zeta_{12}(\mathbf{z}^{(1)}, \mathbf{z}^{(2)})$, $\zeta_{22}(\mathbf{z}^{(1)}, \mathbf{z}^{(2)})$, with $\mathbf{z}^{(1)} = (z_x, z_y)$, $\mathbf{z}^{(2)} = (z_{xx}, z_{xy}, z_{yy})$, can be computed by (9) and (10). Explicit formulas can be found, e.g. in Bluman & Kumei (1996), Hillgarter (2002), Ibragimov (1999) and Olver (1993).

EXAMPLE 6 Consider the six-parameter group denoted in Hillgarter (2002) by

$$\mathbf{ip}_{22} = \{\partial_x, \partial_y, x\partial_y, x\partial_x - y\partial_y, y\partial_x, x\partial_x + y\partial_y + \partial_z\}.$$

It is taken from Lie's listing of imprimitive groups acting on (x, y, z)-space. Prolongations up to order two leads to the following extended generators:

$$\begin{aligned}
&\partial_x,\ \partial_y,\ x\partial_y - z_y\partial_{z_x} - 2z_{xy}\partial_{z_{xx}} - z_{yy}\partial_{z_{xy}},\\
&x\partial_x - y\partial_y - z_x\partial_{z_x} + z_y\partial_{z_y} - 2z_{xx}\partial_{z_{xx}} + 2z_{yy}\partial_{z_{yy}},\\
&y\partial_x - z_x\partial_{z_y} - z_{xx}\partial_{z_{xy}} - 2z_{xy}\partial_{z_{yy}},\\
&x\partial_x + y\partial_y + \partial_z - z_x\partial_{z_x} - z_y\partial_{z_y} - 2z_{xx}\partial_{z_{xx}} - 2z_{xy}\partial_{z_{xy}} - 2z_{yy}\partial_{z_{yy}}.
\end{aligned}$$

2.7 *Solving systems of differential invariants*

The infinitesimal generators $X_k = \sum_{i=1}^{r} \xi_i(\mathbf{x})\partial_{x_i}$, for $k = 1, \dots, r$, corresponding to a point transformation group form an LA and hence satisfy $[X_i, X_j] = \sum_{k=1}^{r} c_{i,j,k} X_k$, for some constants $c_{i,j,k}$, compare Appendix B.1. The same is true for their prolongations, i.e. $[X_i^{(m)}, X_j^{(m)}] = \sum_{k=1}^{r} c_{i,j,k} X_k^{(m)}$. Hence, the corresponding system of differential invariants $X_k^{(m)}\Phi \equiv 0$, for $k = 1, \dots, r$ is complete, i.e. it satisfies the *integrability conditions*

$$[X_i^{(m)}, X_j^{(m)}] = \sum_{k=1}^{r} h_{i,j,k}(\mathbf{x}) X_k^{(m)}, \quad i, j = 1, \dots, r,$$

with constant coefficients $h_{i,j,k}(\mathbf{x}) = c_{i,j,k}$. We introduce *narrowing transformation* for solving such systems of linear homogeneous first-order PDEs

$$X_k\Phi \equiv 0, \quad X_k = \sum_{i=1}^{n} \xi_{k,i}(\mathbf{x})\partial_{x_i}, \quad k = 1, \dots, r, \tag{11}$$

where $\mathbf{x} = (x_1, \dots, x_n)$, and the $\xi_{k,i}$ are continuous. The idea is to reduce this problem to solving single linear homogeneous PDEs, compare Kamke (1965). More precisely, suppose that we know $n - 1$ functionally independent solutions[1] $\Psi_1(\mathbf{x}), \dots, \Psi_{n-1}(\mathbf{x})$ for the r-th equation. Then we have for an arbitrary continuously differentiable function $\zeta(y_1, \dots, y_{n-1})$ that $\zeta(\Psi_1(\mathbf{x}), \dots, \Psi_{n-1}(\mathbf{x}))$ forms an integral of the m-th equation, too. One may now try to narrow down the domain of functions ζ in such a way that $\zeta(\Psi_1(\mathbf{x}), \dots, \Psi_{n-1}(\mathbf{x}))$ also satisfies the remaining DEs of the system (11). For this purpose, $z(\mathbf{x}) = \zeta(y_1, \dots, y_{n-1})$ is plugged via the assumption

$$\mathbf{y} = (y_1, \dots, y_n) = (\Psi_1(\mathbf{x}), \dots, \Psi_{n-1}(\mathbf{x}), x_n) =: T(\mathbf{x}),$$

into (11), thereby trying to get a system of DEs for $\zeta(y_1, \dots, y_{n-1})$. Working out the details shows that the integrals of (11) are exactly those functions $z(\mathbf{x}) = \zeta(\Psi_1, \dots, \Psi_{n-1})$ where ζ runs through the solutions of the system

$$\left(\sum_{i=1}^{n-1} f_{k,i}(\mathbf{y})\partial_{y_i}\right)\zeta = 0, \quad k = 1, \dots, r-1, \tag{12}$$

where $f_{k,i}(\mathbf{y}) := X_k(\Psi_i)_{|\mathbf{x}=T^{-1}(\mathbf{y})}$. System (12) satisfies the integrability conditions and contains $r - 1$ equations in $n - 1$ variables (the identical component $x_n = y_n$ only serves as a dummy). By a recursive application of this reduction, we will obtain a fundamental system for (11).

I implemented a Maple procedure[2] (‘LHSO1_PDE_Solver’) for solving systems of linear homogeneous first-order PDEs by means of the narrowing transformation. The need arose since Maple (version

[1] We assume that $\Psi_1, \dots, \Psi_{n-1}$ are twice continuously differentiable functions with

$$\frac{\partial(\Psi^1, \dots, \Psi^{n-1})}{\partial(x_1, \dots, x_{n-1})} \neq 0.$$

.

[2] Actually, there is a wealth of software for the computation of the Lie symmetries of a given (system) of equation(s) (Hereman, 1996; Butcher *et al.*, 2003) but the reverse problem of computing invariant equations from given symmetries has received much less attendance.

five) did not handle systems of PDEs when I started computing invariants. In between, since version seven, Maple can deal with systems of PDEs via pdsolve but performs weaker on the indicated class of equations (it is generally slower and cannot handle several examples that LHSO1_PDE_Solver does). The same is true for the newest routines of pdesolve (written by E. Cheb-Terrab) in version nine. The computation of differential invariants of order two corresponding to the space group $\mathbf{ip}_{22}$ introduced is demonstrated in Example 6. The procedure PDEInvariants sets up the system of differential invariants and solves it by means of LHSO1_PDE_Solver.

```
> read ‘C:/Maple Files/PDEInvariants.mpl‘;
> read ‘C:/Maple Files/LieGroups.mpl‘;

> Gr := IP22;
```

$$Gr := [Dx, Dy, x\,Dy, x\,Dx - y\,Dy, y\,Dx, x\,Dx + y\,Dy + Dz]$$

```
> Sol := PDEInvariants(Gr,[],2);
```

$$6, 8, 6$$

$$[[0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0]]$$

$$Sol := [(z_x^2 z_{yy} - 2z_x z_y z_{xy} + z_y^2 z_{xx})e^{(4z)}, (-z_{xx} z_{yy} + z_{xy}^2)e^{(4z)}]$$

In this example, the arguments of PDEInvariants are first a list of generators representing the group, second an empty list indicating that no special options are chosen and finally an integer indicating the maximal order of any differential invariant to be computed. The final output assigned to *Sol* is a list containing a basis of differential invariants up to order two for $\mathbf{ip}_{22}$. The intermediate output 6, 8, 6 denotes the number of equations, the number of variables and the number of unconnected equations in the system of differential invariants, respectively. Finally, each list of six zeros in [[0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0]] represents the result of plugging an element of *Sol* into the six equations of the system of differential invariants. This test assures that everything evaluates to zero.

3. The groups of the (x, y, z)-space

Lie used the term *type* of a point group for the full equivalence class of the respective group w.r.t. point transformations. Therefore, the type is completely determined by providing a canonical representative for the class. The group classification of ODEs $f(x, y, y', \ldots, y^{(m)}) = 0$ in Lie (1888) is based on the enumeration of all possible LAs (infinitesimal groups in Lie's terminology) in the (x, y)-plane. These algebras are maximally simplified by a proper choice of bases and by means of a suitable change of variables. Associated with these two types of simplifying transformations are two distinctly different notions: isomorphic and similar LAs, compare Appendix B.2 for more details.

In this section, the notation $\mathbf{x} = (x, y, z)$ is used. We are concerned with the point groups of the $\mathbf{x}$-space. They are divided into primitive groups and three categories of imprimitive groups (compare Appendix C for this notion). The first two categories of imprimitive groups are given by Lie (1970a,b) and the third (and by far the largest one) was given by Amaldi (1901, 1902). These three categories are not claimed to be disjoint, but to represent a complete overview. The listing given in Hillgarter (2002) includes corrections and features like the group size; it is the basis for the classification problem for PDEs in $z(x, y)$.

The following subsections deal with conventions on notation, Lie's primitive groups, Lie's two categories of imprimitive groups and Amaldi's category of imprimitive groups.

3.1 *Notation*

In this subsection, I present some notation introduced in Hillgarter (2002) to simplify the presentation of space groups.

For an integer variable υ and an integer constant or variable n we write $\upsilon \to n$, for $\upsilon = 1, \ldots, n$. We write $\upsilon \to^* n$, for $\upsilon = 0, \ldots, n$. For a term t, we denote by $t_{\upsilon=l}^{u}$ the list $t[\upsilon = l], t[\upsilon = l + 1], \ldots, t[\upsilon = u]$. Similarly, by $t_{\upsilon=l,\ldots,u}^{\bar\upsilon=\bar l,\ldots,\bar u}$ we denote $t[\upsilon = l, \bar\upsilon = \bar l], \ldots, t[\upsilon = l, \bar\upsilon = \bar u], \ldots, t[\upsilon = u, \bar\upsilon = \bar l], \ldots, t[\upsilon = u, \bar\upsilon = \bar u]$. Also several multi-parameter ranges are allowed, e.g. $t_{i\to m,\ j\to n}^{k\to l}$. Hence, we have, e.g. $\{(x^i\partial_y + ix^{i-1}\partial_z)_{i=0}^{s}\} = \{\partial_y, x\partial_y + \partial_z, \ldots, x^s\partial_y + sx^{s-1}\partial_z\}$ and $\{(x^i y^j)_{i\to m}^{j\to n}\} = \{xy, \ldots, xy^n, x^2y, \ldots, x^2y^n, \ldots, x^m y, \ldots, x^m y^n\}$.

For a truth value formula b, we denote by $[b]$ the truth value of b w.r.t. its actual arguments, e.g. $[i \geqslant t] = 1$ if $i \geqslant t$, $[i \geqslant t] = 0$ otherwise. For a generator $g = \xi_1\partial_x + \xi_2\partial_y + \xi_3\partial_z$ and $\upsilon \in \{x, y, z\}$ we denote by g_υ the generator received by applying the operator ∂_υ to the coefficients of g, e.g. $g_x = 2x\partial_x + sy\partial_y + (s-2)z\partial_z$ for $g := x^2\partial_x + sxy\partial_y + [(s-2)zx + sy]\partial_z$.

3.2 *The* (x, y, z)*-space groups*

Lie gave a partial classification of the point groups of the $\mathbf{x}$-space. Among them are all primitive space groups. Any primitive r-parameter group G_r of the $\mathbf{x}$-space is also transitive (Lie, 1970b, p. 221), and hence contains an $(r-3)$-parameter subgroup G_{r-3} that leaves an arbitrary point $\mathbf{x}_0 = (x_0, y_0, z_0)$ invariant. We can w.l.o.g.[3] assume $\mathbf{x}_0 = (0, 0, 0)$, and hence the generators of this linear and homogeneous group G_{r-3} are of the form

$$X_i = (\lambda_i \cdot \mathbf{x})\partial_x + (\mu_i \cdot \mathbf{x})\partial_y + (\eta_i \cdot \mathbf{x})\partial_z, \quad i = 1, \ldots, r-3,$$

where λ_i, μ_i, η_i are (vector) constants and '$\cdot$' denotes scalar multiplication. If we regard the *projective* tangent space of the $\mathbf{x}$-space, denoted by $\mathbf{dx} = (\mathrm{d}x{:}\mathrm{d}y{:}\mathrm{d}z)$, as being transformed under G_{r-3}, there are three possible cases[4] (Lie, 1970b, p. 94).

(I) G_{r-3} acts on this projective space as an eight-parameter group.

(II) G_{r-3} acts on this projective space as the most general three-parameter group leaving a certain non-degenerate conic of degree 2 invariant.

(III) G_{r-3} leaves a linear complex through $\mathbf{x}_0$ invariant.

The table on the next page gives an overview of the *structure and geometry* of the eight arising groups.

Here $IC_7 := \mathrm{d}z^2 + y^2\mathrm{d}x^2 + x^2\mathrm{d}y^2 + (4z - 2xy)\mathrm{d}x\mathrm{d}y - 2x\mathrm{d}y\mathrm{d}z - 2y\mathrm{d}z\mathrm{d}x$, and S. T. denotes 'similarity transformations'. The listing given below is taken from Chapter 7 in Lie (1970b, 'Bestimmung aller primitiven Gruppen des dreifach ausgedehnten Raumes'). The presentation follows (Hillgarter, 2002). Let $G := \{\partial_x, \partial_y, \partial_z\}$.

$\mathbf{p}_1$: $G \cup \{(u\partial_\upsilon)_{\{u,\upsilon\}=\{x,y,z\}}, (u^2\partial_u + u\upsilon\partial_\upsilon + uw\partial_w)_{\{u,\upsilon,w\}=\{x,y,z\}}\}$, size: 15.

$\mathbf{p}_2$: $G \cup \{(u\partial_\upsilon)_{\{u,\upsilon\}=\{x,y,z\}}\}$, size: 12.

[3]The two arising groups are isomorphic.

[4]Actually there are four cases, but the fourth one does not lead to a primitive group.

Not.	Type	Name	Invariant complex
$\mathbf{p}_1$	I	General Projective Group	
$\mathbf{p}_2$	I	General Linear Group	
$\mathbf{p}_3$	I	Special Linear Group	
$\mathbf{p}_4$	III	Projective Group	$y\mathrm{d}x - x\mathrm{d}y - \mathrm{d}z = 0$
$\mathbf{p}_5$	II	Euclidean Group	$\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = 0$
$\mathbf{p}_6$	II	Euclidean Group w. S. T.	$\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = 0$
$\mathbf{p}_7$	II	Non-Euclidean Group	$IC_7 = 0, z - xz = 0$
$\mathbf{p}_8$	II	Conformal Group	$\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = 0$

$\mathbf{p}_3$: $G \cup \{(x\partial_x - v\partial_v)_{v=y,z}, (v\partial_u, w\partial_u)_{\{u,v,w\}=\{x,y,z\}}\}$, size: 11.

$\mathbf{p}_4$: $\{\partial_z, x\partial_y, y\partial_x, x\partial_x - y\partial_y, x\partial_x + y\partial_y + 2z\partial_z, z(x\partial_x + y\partial_y + z\partial_z), g, g_z, \bar{g}, \bar{g}_z\}$,
$g := z\partial_x - y(x\partial_x + y\partial_y + z\partial_z)$, $\bar{g} := z\partial_y + x(x\partial_x + y\partial_y + z\partial_z)$, size: 10.

$\mathbf{p}_5$: $G \cup \{(u\partial_v - v\partial_u)_{(u,v)=(x,y),(x,z),(y,z)}\}$, size: 6.

$\mathbf{p}_6$: $\{x\partial_x + y\partial_y + z\partial_z\} \cup \mathbf{p}_5$, size: 7.

$\mathbf{p}_7$: $\{(\partial_u + v\partial_z)_{\{u,v\}=\{x,y\}}, (v\partial_v + z\partial_z)_{v=x,y}, (u^2\partial_u + (xy - z)\partial_v + uz\partial_z)_{\{u,v\}=\{x,y\}}\}$,
size: 6.

$\mathbf{p}_8$: $G \cup \{(u\partial_v - v\partial_u)_{(u,v)=(x,y),(x,z),(y,z)}, g, (2ug - S(\mathbf{x})\partial_v)_{(u,v)=(x,y),(x,z),(y,z)}\}$,
$g := x\partial_x + y\partial_y + z\partial_z$, $S(\mathbf{x}) := x^2 + y^2 + z^2$, size: 10.

3.3 *The imprimitive* (x, y, z)*-groups given by Lie*

In his partial classification of the point groups of the **x**-space, Lie divided the imprimitive groups into three categories, according to their imprimitivity systems. He computed the groups of the first two categories explicitly. The first category is characterized by an imprimitivity system of the form $\varphi(\mathbf{x}) =$ *const*. The second category has two imprimitivity systems of the form $\varphi(\mathbf{x}) =$ *const*., $\psi(\mathbf{x}) =$ *const*. that cannot be written as a system of surfaces of the form $\Omega(\varphi(\mathbf{x}), \psi(\mathbf{x})) =$ *const*. Both categories encompass 33 groups, most of them having a parameter-dependent size.

We do not repeat here the listing of these groups given in Hillgarter (2002); they were extracted from Chapter 8 in Lie (1970b, ‘Bestimmung gewisser imprimitiver Gruppen des dreifach ausgedehnten Raumes’).

3.4 *The imprimitive* (x, y, z)*-groups given by Amaldi*

The imprimitive point groups of the **x**-space belonging to the third category have two systems of imprimitivity of the form $\varphi(\mathbf{x}) =$ *const*., $\psi(\mathbf{x}) =$ *const*. that can be written as a system of surfaces of the form $\Omega(\varphi(\mathbf{x}), \psi(\mathbf{x})) =$ *const*. The Italian mathematician Ugo Amaldi (1901, 1902) explicitly computed the representatives by a method proposed by Lie (1970b). The main result is that there are four different types of groups of the form presented in the following table.

Here $j = 1, 2, \ldots, h > 0$ and $\xi_i(x)\partial_x + \eta_i(x, y)\partial_y$ for $i = 1, 2, \ldots, l$ are the generators of the *corresponding imprimitive* (x, y)*-plane group*. In case [A], the $\zeta_i(\mathbf{x})$ are already determined by

Type	Generators
[A]	$\xi_i(x)\partial_x + \eta_i(x, y)\partial_y + \zeta_i(\mathbf{x})\partial_z$
[B]	$\xi_i(x)\partial_x + \eta_i(x, y)\partial_y + \{\zeta_{i,1}(x, y)z + \zeta_{i,2}(x, y)\}\partial_z,\ \varphi_j(x, y)\partial_z$
[C]	$\xi_i(x)\partial_x + \eta_i(x, y)\partial_y + \zeta_i(x, y)z\partial_z,\ z\partial_z,\ \varphi_j(x, y)\partial_z$
[D]	$\xi_i(x)\partial_x + \eta_i(x, y)\partial_y,\ \partial_z,\ z\partial_z,\ z^2\partial_z$

requiring that all commutators $[X_i, X_j]$ belong to the linear hull $L(X_1, \ldots, X_l)$. Amaldi also derived some theorems concerning the functions $\varphi_j(x, y)$ that occur in types [B] and [C], compare pages 279–282 in Amaldi (1901, 1902). This category of groups is actually much larger than the first two categories.

Amaldi's listing of the imprimitive groups of the (x, y)-plane is not organized according to systems of imprimitivity so that any parameter is given free variability and the number of groups is reduced to the minimum. The listing encompasses 21 groups and basically looks like this:

$\mathbf{ip}_1$: $\{\partial_y, x\partial_y, \ldots, x^s\partial_y, \partial_x, x\partial_x + cy\partial_y\}, \quad s \geqslant 1.$

$\mathbf{ip}_2$: $\{\partial_y, x\partial_y, \ldots, x^s\partial_y, \partial_x, x^2\partial_x + sxy\partial_y, 2x\partial_x + sy\partial_y\}, \quad s \geqslant 1.$

$\mathbf{ip}_3$: $\{\partial_y, x\partial_y, \ldots, x^s\partial_y, \partial_x, x\partial_x + (sy + x^s)\partial_y\}, \quad s \geqslant 1.$

$\vdots$

$\mathbf{ip}_{21}$: $\{(x^j e^{a_i x}\partial_y)_{i\to^* l,\ j\to^* s_i}^{a_0=0,\ a_1=1}, y\partial_y, \partial_x\}, \quad l \geqslant 0,\ l + \sum s_i \geqslant 0.$

The order in which the groups are listed has been chosen to shorten the calculations of **x**-space groups. Where it was possible, any group G_l is followed by the minimal group G_{l+t} in which it is contained. This has the advantage demonstrated in the following example. Group names are as in Hillgarter (2002).

EXAMPLE 7 We consider the calculation of $\mathbf{ip}_{5,A}$, a space group where Amaldi's calculation was slightly erroneous. First of all we note that the plane groups

$$\mathbf{ip}_4 = \{(x^s\partial_y)_{i=0}^s, y\partial_y, \partial_x, x\partial_x\}, \quad \mathbf{ip}_5 = \mathbf{ip}_4 \cup \{g\}$$

differ only by one additional generator $g := x^2\partial_x + sxy\partial_y$. We suppose to already know the space group of type [A] corresponding to $\mathbf{ip}_4$:

$$\mathbf{ip}_{4,\mathrm{A}}[s, t] = \left\{\left(x^i\partial_y + [i \geqslant t]\binom{i}{t}x^{i-t}\partial_z\right)_{i=0}^s, \partial_x, y\partial_y + z\partial_z, x\partial_x - tz\partial_z\right\}.$$

Let $\bar{g} = g + \zeta(\mathbf{x})\partial_z$. It is now possible to make an ansatz for $\mathbf{ip}_{5,\mathrm{A}}$ of the form $\mathbf{ip}_{5,\mathrm{A}}[s, t] = \mathbf{ip}_{4,\mathrm{A}}[s, t] \cup \{\bar{g}\}$. By considering $[\partial_y, \bar{g}] = sx\partial_x + \zeta_y(\mathbf{x})\partial_z$, we conclude that the cases $t > 1$ are not possible (this point was overlooked by Amaldi) and that $\zeta_y = s$, i.e. $\zeta(\mathbf{x}) = sy + \bar{\zeta}(x, z)$. Now by $[\partial_x, \bar{g}] = 2x\partial_x + sy\partial_y + \bar{\zeta}_x\partial_z$ we conclude that $\bar{\zeta}_x = (s-2)z$, i.e. $\bar{\zeta}(x, z) = (s-2)xz + \bar{\bar{\zeta}}(z)$. Finally, by $[x\partial_y + \partial_z, \bar{g}] = (s-1)x^2\partial_y + 2(s-1)x\partial_z + \bar{\bar{\zeta}}_z\partial_z$ we infer that $\bar{\bar{\zeta}}_z = 0$, i.e. $\bar{\bar{\zeta}}(z) = C$, w.l.o.g. $\bar{\bar{\zeta}}(z) = 0$. Hence, we have $\bar{g} = g + (sy + (s-2)xz)\partial_z$ and $\mathbf{ip}_{5,\mathrm{A}}[s] = \mathbf{ip}_{4,\mathrm{A}}[s, 1] \cup \{\bar{g}\} = \{\partial_x, (x^i\partial_y + ix^{i-1}\partial_z)_{i=0}^s, x\partial_x - z\partial_z, y\partial_y + z\partial_z, \bar{g}\}$.

4. Differential invariants up to order two

A special chapter in Hillgarter (2002) is devoted to a listing of the differential invariants up to order two for all space groups with less than eight generators (except for Amaldi's groups of type [B]), for groups that have more than seven generators we refer to the next section. Groups with less than eight generators usually have a *differential invariant basis*. This is so because the number of variables $V = \{x, y, z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}\}$ involved in (7) is eight, so any $n := 8 - r_*$ functionally independent solutions $\Psi_1(V), \dots, \Psi_n(V)$ of (7) form a basis of its solution space. Hence, any other solution of (7) has the form $F = \Phi(\Psi_1(V), \dots, \Psi_n(V))$, justifying the name differential invariant basis for $\Psi_1, \dots, \Psi_n$.

The list is organized according to the *derived series* (compare Appendix B.1) of the LAs. Since derived series are invariant under point transformations, this organization simplifies the identification of a given group. The groups themselves are represented by the generators of the corresponding LAs as usual. Groups with the same derived series are ordered according to the (Lie/Amaldi) class, their number and their parameters. The scheme of presentation is

$$\langle\text{group identifier}\rangle = \langle\text{generatorlist}\rangle{:}\langle\text{invariant basis}\rangle.$$

EXAMPLE 8 We take a look at one particular entry of the invariant list:

Derived Series (7, 4, 0)

⋮

$\mathbf{ip}_{11,\mathrm{C}}[\mathbf{L} = [[0], [0], [0], [0]]] = \{\partial_x, \partial_y, z\partial_z, (\mathrm{e}^{a_i x + b_i y}\partial_z)_{i=1}^{4}\}$: $d_{1,2,3,4,6}/d_{1,2,3,4,5}$.

Example 8 gives the differential invariant basis of a special instance of the group $\mathbf{ip}_{11,\mathrm{C}} = \{\partial_x, \partial_y, z\partial_z, (x^k y^l \mathrm{e}^{a_m x + b_m y}\partial_z)_{m\to p,\, k\to^* k_m}^{l\to^* l_{m,k}}\}$, whose parameters satisfy $0 \leqslant l_{m,k_m} \leqslant l_{m,k_m-1} \leqslant \cdots \leqslant l_{m,0}$. The parameter-dependent size of this group is $3 + p + \sum_{m=1}^{p} k_m + \sum_{m\to p}^{k\to^* k_m} l_{m,k}$. The special instance in Example 8 occurs for the parameter choice $p = 4$, $k_1 = k_2 = k_3 = k_4 = 0$, $l_{1,0} = l_{2,0} = l_{3,0} = l_{4,0} = 0$. This parameter choice is encoded by the matrix $\mathbf{L} = [[l_{1,0}], [l_{2,0}], [l_{3,0}], [l_{4,0}]] = [[0], [0], [0], [0]]$, whose number of lines and their length corresponds to p and $k_i + 1$, respectively.

The single invariant indicated in Example 8 is defined via $d_J := \det([D_{i,j}]_{j=J}^{i\to n})$, where $J := j_1, \dots, j_n$ and $D := [[z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}], [1, a_i, b_i, a_i^2, a_i b_i, b_i^2]_{i=1}^{5}]$. In particular, we have

$$\frac{d_{1,2,3,4,6}}{d_{1,2,3,4,5}} = \det\begin{pmatrix} z & z_x & z_y & z_{xx} & z_{yy} \\ 1 & a_1 & b_1 & a_1^2 & b_1^2 \\ 1 & a_2 & b_2 & a_2^2 & b_2^2 \\ 1 & a_3 & b_3 & a_3^2 & b_3^2 \\ 1 & a_5 & b_5 & a_5^2 & b_5^2 \end{pmatrix} \Bigg/ \det\begin{pmatrix} z & z_x & z_y & z_{xx} & z_{yy} \\ 1 & a_1 & b_1 & a_1^2 & b_1^2 \\ 1 & a_2 & b_2 & a_2^2 & b_2^2 \\ 1 & a_3 & b_3 & a_3^2 & b_3^2 \\ 1 & a_4 & b_4 & a_4^2 & b_4^2 \end{pmatrix}.$$

This expression alone would fill a page if written down explicitly as a quotient of differential polynomials. Hence, expressions like d_J (that appear several times in the list) were defined for each (Lie/Amaldi) class of groups in order to give a compact presentation of the invariant list. The expression lists for Amaldi's groups of type [A] and [C] became quite involved, encompassing roughly 70 and 90 definitions, respectively. I used some conventions to keep their presentation concise, too. For example, if expression H_i contains a subexpression, that appears as H_j at some other place in the expression list for the groups of type [A], we refer to H_j in order to shorten the presentation . The recursion depth hereby usually is one, in a few cases it is two.

EXAMPLE 9 We take a look at another particular entry of the invariant list:

Derived Series (4, 2, 0)

⋮

$\mathbf{ip}^2_{1,\mathrm{A}}[s = 1, t = 1] = \{\partial_y, x\partial_y + \partial_z, \partial_x, x\partial_x + y\partial_y + c\partial_z\}$: H_{11}, H_{12}, H_{13}, H_{14}.
The last expression in the invariant basis is defined in dependence on other expressions as $H_{14} := H_{18}/z_y^2 + cH_{33}(H_{39})_{a=2}$, where $H_{18} := z_{xx} + 2zz_{xy} + z^2 z_{yy}$, $H_{33} := \log(z_y)$, $H_{39} := (cz_{yy}H_{33} + aH_{35})/z_y^2$, $H_{35} := z_{xy} + zz_{yy}$.

Another convention used is that free variables in an invariant I match the parameters of the group G, in case that I appears in the invariant basis of G. A simple case is demonstrated in the next example.

EXAMPLE 10 Consider an entry in the invariant list of the following form:

$\mathbf{ip}_{6,\mathrm{C}}[\mathbf{m} = [1]] = \{\partial_x, y\partial_y, x\partial_x, x^2\partial_x + xy\partial_y + cxz\partial_z, z\partial_z, y^{c-1}\partial_z, xy^{c-1}\partial_z\}$: I_{29}.

The convention implies that the invariant basis of $\mathbf{ip}_{6,\mathrm{C}}[\mathbf{m} = [1]]$ consists of the single invariant

$$\frac{y^2 z_{yy} - c^2 z + 3cz - 2z}{yz_y - cz + z}.$$

Example 10 deals with a special instance of the group $\mathbf{ip}_{6,\mathrm{C}} = \{\partial_x, y\partial_y, x\partial_x, x^2\partial_x + xy\partial_y + cxz\partial_z, z\partial_z, (x^m y^{c-m_j}\partial_z)^{m\to^* m_j}_{j\to^* l}\}$, whose parameters satisfy $m_{j+1} > m_j \geqslant 0$. The parameter-dependent size of this group is $6 + l + \sum_{j=0}^{l} m_j$. The special instance in Example 10 occurs for the parameter choice $l = 0$, $m_0 = 1$. This parameter choice is encoded by the vector $\mathbf{m} = [m_0] = [1]$, whose length corresponds to $l + 1$ in general. The expression I_{29} is defined as

$$I_{29} := \frac{y^2 z_{yy} - c^2 z + (2m_0 + 1)cz - m_0(m_0 + 1)z}{yz_y - cz + m_0 z},$$

its free variable is m_0. According to the convention, this free variable should match the parameters of $\mathbf{ip}_{6,\mathrm{C}}[\mathbf{m} = [m_0]] = \mathbf{ip}_{6,\mathrm{C}}[\mathbf{m} = [1]]$. This implies that the invariant under consideration is given by $I_{29}[m_0 = 1]$, which is just the expression indicated at the end of Example 10.

5. Lower invariants

Groups with more than seven parameters are treated in Chapter 5 of Hillgarter (2002). Included are all types of (x, y, z)-space groups, except Amaldi's groups of type [B]. The lower invariants[5] were usually indicated (see Appendix A); in those exceptional cases where the rank r_* is less than eight, the invariant basis is indicated. The listing is organized lexicographically according to the derived series of the corresponding groups. To this end, the structure of any derived series is represented as a function of the unknown group size, which is always denoted by n.

EXAMPLE 11 The group $\mathbf{p}_3$ has the derived series (11). Its lower invariants are found in a listing with heading 'Derived Series (n)'.

[5] In modern nomenclature, they can be regarded as 2D versions of Lie-determinants, compare Olver (1995). They are only relatively invariant under the transformation group and hence do not represent elements of an invariant basis.

For example, the lower invariant listing for Lie's primitive (x, y, z)-space groups looks as follows. The groups $\mathbf{p}_i$ are defined in Section 3 and $F_1 := 1 + z_x^2 + z_y^2$, $F_2 := z_{xy}^2 - z_{xx}z_{yy}$ and $F_3 := z_x^2 z_{yy} + z_y^2 z_{xx} - 2z_x z_y z_{xy}$:

Derived Series (n)

$\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$: F_2.

$\mathbf{p}_4$: $F_2,\ F_3 + F$,

$$F := z_{xx}(x^2 + 2xz_y) + 2z_{xy}(xy - xz_x + yz_y) + z_{yy}(y^2 - 2yz_x).$$

$\mathbf{p}_8$: $F_1,\ F_3^2 + 2(F_2 + z_{xy}^2)F_1 + F$,

$$F := z_{xx}^2(1 + 2z_y^2) + z_{yy}^2(1 + 2z_x^2) - 4z_x z_y z_{xy}(z_{xx} + z_{yy}).$$

EXAMPLE 12 The computation of the lower invariants corresponding to $\mathbf{p}_3$ proceeds as indicated in Appendix A. The determinant of the system of differential invariants specified there turns out to be $d = -8z_x z_y(z_x z_{yy} - z_y z_{xy})F_2$. Among its four factors, only $F_2 = z_{xy}^2 - z_{xx}z_{yy}$ passes the test (A.2) in Appendix A.

The MAPLE procedure PDEInvariants mentioned in Subsection 2.7 also handles the computation of lower invariants; to this end the second argument has to be chosen as the list [d].

```
> read 'C:/Maple Files/PDEInvariants.mpl';
> read 'C:/Maple Files/LieGroups.mpl';
```

```
> Gr := P3;
```

$$Gr := [Dx, Dy, Dz, yDx, zDx, xDy, zDy, xDz, yDz, xDx - yDy, xDx - zDz]$$

```
> Sol := PDEInvariants(Gr,[d],2);
```

$$11, 8, 8$$

$$Sol := [z_{xy}^2 - z_{xx}z_{yy}]$$

The dependence of the derived series on the group parameters sometimes became quite complicated. *Group parameter-related numbers* for several Amaldi Type C groups were defined; they serve to simplify the presentation of derived series in the headings of the lower invariant listings. In the following example, $[\cdot]$ denotes the truth value function $[x] = 1$, if x is true, $[x] = 0$ otherwise. If x is not a truth value formula, the symbols '[', ']' simply denote square brackets.

EXAMPLE 13 For $\mathbf{ip}_{1,C}(s, \mathbf{m})$ with parameters $s \geqslant 0$, $\mathbf{m} = [m_0, \ldots, m_l]$, where $l, m_j \geqslant 0$ for $j = 0, \ldots, l$ and $m_{j+1} \geqslant m_j + s$ for $j = 0, \ldots, l-1$, we define, $d := \sum_{i=0}^{l-1}[m_{i+1} - m_i > s]$, $e := [s \neq 0] \cdot \max\left(0, \sum_{i=0}^{l-1} m_i + (s-1)(l - [m_0 = 0]) + \sum_{i=0}^{l-2}[m_{i+1} - m_i = s]\right)$. One of the lower invariant listings that contains instances of $\mathbf{ip}_{1,\mathrm{C}}$ looks as follows:

Derived Series $(n, n-2, n-5-d, e, 0)$
$\mathbf{ip}_{1,\mathrm{C}}[s = 0, \mathbf{m} = [0, m], m > 1]$: z_{xy}, z_{yy}.
⋮

This reads as follows: under this heading, only groups with derived series $(n, n-2, n-5-d, e, 0)$ for some $0 \leqslant e \leqslant d \leqslant n-5$ are listed; for each such group their parameter values determine the concrete values for d, e as defined above.

6. Classical equations, applications

In this section, we discuss some *classical equations* that appear in the invariant list of Hillgarter (2002) (*Burgers* equation and the *Korteweg de Vries* equation). We also present a specific example (the *pipe current energy equation*) chosen from the vast amount of possible applications of symmetry analysis to second-order PDEs. Generally, applications of Lie groups to DEs include reduction of order for ODEs, mapping solutions to other solutions, reduction of the number of independent variables of PDEs, construction of invariant solutions (also for boundary value problems), construction of conservation laws, detection of linearizing transformations of DEs and many other applications (Bluman & Kumei, 1996; Ibragimov, 1994–1996, 1999; Hydon, 2000; Olver, 1993; Steeb, 1996; Stephani, 1990). The last 30 years have seen a huge resurgence in interest of the application of Lie groups to DEs. Indeed, Daniel Zwillinger (1998) in his *Handbook of Differential Equations* remarks: ‘Lie groups analysis is the most useful and general of all the techniques presented in this book’.

6.1 *Classical equations*

The *Burgers* equation $z_x + zz_y + z_{yy} = 0$ appears among the invariants of $\mathbf{ip}^1_{2,\mathrm{A}}[s = 1]$, a group with derived series (5):

$\mathbf{ip}^1_{2,\mathrm{A}}[s = 1] = \{\partial_y, x\partial_y + \partial_z, \partial_x, 2x\partial_x + y\partial_y - z\partial_z, x^2\partial_x + xy\partial_y + (-xz + y)\partial_z\}$:
$H_{21}, H_{22}, H_{34}/z_{yy}$.

Adding the constant 1 to the third invariant basis element gives the Burgers equation divided by z_{yy}: $H_{34}/z_{yy} + 1 = (z_x + zz_y)/z_{yy} + 1 = 0$. The *Korteweg de Vries* equation $z_x = zz_y + z_{yyy}$ was considered the first time in the context of soliton solutions; its symmetry group is similar to

$\mathbf{ip}^1_{1,\mathrm{A}}[s = 1, t = 1] = \{\partial_y, x\partial_y + \partial_z, \partial_x, x\partial_x + cy\partial_y + (c-1)z\partial_z\}$

for $c = \frac{1}{3}$, a group with derived series (4, 3, 1, 0).

6.2 *Other applications*

I recently came across the *pipe flow energy equation*

$$(A - Br^2)\frac{\partial T}{\partial z} + \frac{\partial^2 T}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = (C - Dr^2), \tag{13}$$

where $T = T(r, z), r > 0$ and $A, B, C, D > 0$ are constant parameters. In order to study the dependence of solutions on the parameters, an analysable symbolic solution would be helpful. The symmetries of

(13) are $\{X_1, X_2, X_3\} = \{F(r,z)\partial_T,\ T\partial_T,\ \partial_z\}$, where $F(r,z)$ satisfies (13) with the right-hand side multiplied by an arbitrary constant. The commutator table of this group is

$$\begin{bmatrix} 0 & X_1 & -F_z(r,z)\partial_T \\ -X_1 & 0 & 0 \\ F_z(r,z)\partial_T & 0 & 0 \end{bmatrix}.$$

If we request the LA to be 3D via $F_z(r,z) = cF(r,z)$, we obtain[6] $F(r,z) = f(r)\mathrm{e}^{cz}$. Plugging this *similarity solution* into (13) gives

$$\mathrm{e}^{cz}[rf''(r) + f'(r) + c(Ar - Br^3 + cr)f(r)] - Cr + Dr^3 = 0,$$

i.e. we receive a second-order ODE for $f(r)$. Its general solution can be found and expressed in terms of *WhittakerM* and *WhittakerW* functions.

Some further directions of possible applications could be analysis of Darboux integrability of PDEs in two variables (Anderson & Kamran, 1997; Sokolov & Zhiber, 1995) and the classification of first-order differential operators (Draisma, 2003) with applications in quantum mechanics.

7. Related questions, outlook

In this last section, I show how to compute higher order invariants and finally mention the steps necessary to complete the symmetry classification for the considered class of PDEs.

7.1 *Higher invariants*

In this subsection, I explain why it suffices in principle to list the differential invariants of lower order. Lie (1888) gave for each transformation group of the (x, y)-plane the two lowest invariants Φ_1 and Φ_2. The higher order ones may then be obtained recursively by differentiation according to $\Phi_j = \Phi'_{j-1}/\Phi'_1$, for $j \geqslant 3$. The generalization to the higher dimensional case is given (Tresse, 1884; Ovsiannikov, 1982) in modern language as follows: let G_r be any r-parameter group of point transformations with infinitesimal generators

$$X_l = \sum_{i=1}^{n} \xi_{l,i}(\mathbf{x},\mathbf{u})\frac{\partial}{\partial x_i} + \sum_{j=1}^{m} \eta_{l,j}(\mathbf{x},\mathbf{u})\frac{\partial}{\partial u_j}, \quad l = 1,\ldots,r,$$

where $\mathbf{x} = (x_1,\ldots,x_n)$, $\mathbf{u} = (u_1,\ldots,u_m)$. Then there exist n independent *invariant derivations* (operators that transform invariants into higher order ones) $\mathcal{D}_j = \sum_{i=1}^{n} \lambda_i(\mathbf{x},\mathbf{u},\mathbf{u}^{(1)},\mathbf{u}^{(2)},\ldots)D_i$, where D_i is the operator of total differentiation w.r.t. x_i (8), and the λ_i are differential functions determined by

$$X_l^{(k)}(\lambda_i) = \sum_{j=1}^{n} \lambda_j D_j(\xi_{l,i}), \quad i = 1,\ldots,n, \ \ l = 1,\ldots,r, \tag{14}$$

[6] In Hillgarter (2002), this 3D symmetry group appears among the groups with derived series (3, 1, 0) as $\mathbf{ip}_{9,\mathrm{C}}[\mathbf{L} = [[1]]] = \{\partial_y, z\partial_z, \Psi_{1,1}(x)\mathrm{e}^{c_1 y}\partial_z\}$.

for a sufficiently high[7] k. That way, differential invariant bases of higher order can be produced from lower order ones.

EXAMPLE 14 The group $\mathbf{ip}^2_{15,\mathrm{A}}$, a group with derived series (4, 3), whose differential invariants up to order two are given (Hillgarter, 2002) by

$\mathbf{ip}^2_{15,\mathrm{A}} = \{\partial_x, \partial_y, y\partial_y, y^2\partial_y\}$: z, z_x, z_{xx}, z_y/z_{xy} is considered.

These invariants were computed by solving the system of differential invariants (7). To compute $n = 2$ independent invariant derivations of the form $\lambda_1 D_x + \lambda_2 D_y$, we set up the system (14) for[8] $\lambda_1 = \lambda_1(x, y, z, z_x, z_y)$, $\lambda_2 = \lambda_2(x, y, z, z_x, z_y)$:

$$\begin{gathered}
(\lambda_1)_x = (\lambda_2)_x = (\lambda_1)_y = (\lambda_2)_y = 0,\\
y(\lambda_1)_y - z_y(\lambda_1)_{z_y} = 0,\\
y(\lambda_2)_y - z_y(\lambda_2)_{z_y} = \lambda_2,\\
y^2(\lambda_1)_y - 2yz_y(\lambda_1)_{z_y} = 0,\\
y^2(\lambda_2)_y - 2yz_y(\lambda_2)_{z_y} = 2y\lambda_2.
\end{gathered}$$

The four equations in the first line just express that λ_1, λ_2 do not depend on x and y. The general solution for the remaining equations is given by $\lambda_1(z, z_x, z_y) = f(z, z_x)$, $\lambda_2(z, z_x, z_y) = z_y^{-1}g(z, z_x)$ for arbitrary functions f, g. Hence, we know that the two independent invariant derivations are of the form $\mathcal{D} = f(z, z_x)\partial_x + z_y^{-1}g(z, z_x)\partial_y$. Choosing (f, g) to be (1, 0) and (0, 1), respectively, we get the two independent invariant derivations

$$\mathcal{D}_1 = \partial_x, \qquad \mathcal{D}_2 = z_y^{-1}\partial_y.$$

Now the infinite-order differential invariant basis of $\mathbf{ip}^2_{15,\mathrm{A}}$ is represented by

$\mathbf{ip}^2_{15,\mathrm{A}} = \{\partial_x, \partial_y, y\partial_y, y^2\partial_y\}$: z, $z_y^{-4}(2z_y z_{yyy} - 3z_{yy}^2)$; $\mathcal{D}_1, \mathcal{D}_2$.

The implicit meaning is that the (iterated) applications of $\mathcal{D}_1, \mathcal{D}_2$ to the two given invariants successively produce all differential invariant basis elements of any order. To this end, let B_i be the (differential) invariant basis of order i for $\mathbf{ip}^2_{15,\mathrm{A}}$. Then, we have $B_0 = \{z\}$, $B_1 = \{z_x\}$, $B_2 = \{z_{xx}, z_y/z_{xy}\}$ and $B_3 = \{z_{xxx}, z_y^{-1}z_{xxy}, z_y^{-3}(z_y z_{xyy} - z_{xy}z_{yy}), z_y^{-4}(2z_y z_{yyy} - 3z_{yy}^2)\}$. The effect of the iterated applications of the invariant derivations $\mathcal{D}_1, \mathcal{D}_2$ on the invariant z is as follows:

$$\begin{gathered}
\mathcal{D}_1 B_0 = \{z_x\} = B_1,\\
\{\mathcal{D}_1, \mathcal{D}_2\} \circ B_1 = \{z_{xx}, z_y/z_{xy}\} = B_2,\\
\{\mathcal{D}_1, \mathcal{D}_2\} \circ B_2 = \{z_y^{-1}z_{xxy}, z_{xxx}, z_y^{-3}(z_y z_{xyy} - z_{xy}z_{yy}), z_y^{-2}(z_{xy}^2 - z_{xxy}z_y)\}.
\end{gathered}$$

The first three invariants in the last set are also in B_3, whereas the fourth one $z_y^{-2}(z_{xy}^2 - z_{xxy}z_y) = (z_y^{-1}z_{xy})^2 - z_y^{-1}z_{xxy}$ is functionally dependent on previous invariants. We see that successive application

[7] The number of variables N_v in $X_l^{(k)}$ (or equivalently, the number of variables on which the $\lambda_i(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(k)})$ explicitly depend) is $N_v = n + m\binom{n+k}{n}$. The rank R of the system (14) satisfies $R \leqslant l \cdot n$, since it is bound by the number of equations. The condition for the existence of n independent solutions λ_i is now $n \cdot N_v - R \geqslant n - 1$. Choosing $R = l \cdot n$, this leads to the simple criterion $N_v - l \geqslant 1$. For $m = 1$, $n = 2$ this amounts to $k = \lceil(\sqrt{8l - 7} - 3)/2\rceil$.

[8] The prolongation order k chosen according to my criterion in the previous footnote is $k = \lceil(\sqrt{8 \cdot 4 - 7} - 3)/2\rceil = 1$.

of $\mathcal{D}_1, \mathcal{D}_2$ to z delivers all differential invariant basis elements up to order three, with the exception of $z_y^{-4}(2z_y z_{yyy} - 3z_{yy}^2)$. Including this invariant in the infinite-order basis completes the latter, since problems of this kind do not occur anymore.

7.2 *Open problems*

The differential invariants for Amaldi's groups of type [B] have not been determined within Hillgarter (2002). In addition, since any group list contains just representatives of equivalence classes of similar groups, the group types have to be determined. This means one has to find criterions that allow to identify the symmetry group for a given second-order DE in $z(x, y)$. Due to the huge number of transformation groups of the (x, y, z)-space, this normal form problem is not considered in Hillgarter (2002).[9] This step has been carried out so far only for ODEs up to order three (see Schwarz, 2003).

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[9] This step would be accomplished by computing the Janet base of the determining system of any differential invariant, applying a general point transformation to it, which in general destroys the Janet base property, and reestablishing the Janet base property by applying the algorithm Janet base again. Thereby, a classification of Janet bases for determining systems of DEs for this class would be achieved.

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Appendix

A. *Lower invariants*

The notion of lower invariants is introduced. This is a type of invariant that may even exist when the system of differential invariants has as many unconnected equations as variables, thus allowing only trivial (constant) solutions.

Consider an r-parameter ($r \geqslant 8$) point transformation group G_r acting on (x, y, z)-space. We write the second-order prolongations of its generators X_i as $X_i^{(2)} = \sum_{\upsilon \in V} \xi_{i,\upsilon} \partial_\upsilon$, for $i = 1, \ldots, r$, where $V = \{x, y, z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}\}$. Note that $|V| = 8$. We assume that the (generic) rank $r_* := \operatorname{rank}([\xi_{i,\upsilon}]_{i=1,\ldots,r}^{\upsilon \in V})$ of the coefficient matrix of the system of differential invariants is eight, w.l.o.g. we assume that the first eight equations

$$X_i^{(2)}(F) \equiv 0, \quad i = 1, \ldots, 8 \tag{A.1}$$

are unconnected. A non-trivial solution of (A.1) can only exist if the determinant $d := \det([\xi_{i,\upsilon}]_{i=1,\ldots,8}^{\upsilon \in V})$ of its square coefficient matrix vanishes. We call any irreducible factor f of d that satisfies

$$X_i^{(2)}(f) \equiv_f 0, \quad i = 1, \ldots, 8, \tag{A.2}$$

a *lower invariant* of G_r. The computation proceeds by factoring the determinant d into irreducible factors and applying the test (A.2) for each factor. Lower invariants $f_1, \ldots, f_k$ are differential invariants of G_r, but they do not form an invariant basis, i.e. $\Phi(f_1, \ldots, f_k)$ in general is not an invariant of G_r.

B. *Basic notions for LAs*

I introduce several basic notions for LAs of vector fields, the only type of LAs considered in this work. The notions introduced are dimension, commutator table, structure constants, derived series, isomorphism and similarity. Details and the purely algebraic theory of LAs as introduced by Killing may be found in the book by Jacobson (1962).

B.1 *LAs of vector fields.* A *LA of vector fields* is a vector space L of operators $X = \sum_i \xi_i(x)\partial_{x_i}$ endowed with the *commutator* $[\cdot,\cdot]$ such that $[X, Y] := XY - YX \in L$ for $X, Y \in L$. The commutator is bilinear, skew-symmetric and satisfies the Jacobi identity $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]] = 0$ for $X, Y, Z \in L$. We say that the finite LA L has *dimension* r, written $\dim(L) = r$, if L is the linear span of r linearly independent operators $X_1, \dots, X_r$ with constant coefficients, written $L = \{X_1, \dots, X_r\}$. We call $X_1, \dots, X_r$ a *basis* of L. The matrix $[[X_i, X_j]]_{i=1,\dots,r}^{j=1,\dots,r}$ is called the *commutator table* of L w.r.t. $X_1, \dots, X_r$. The constants $c_{i,j,k}$ in the relations $[X_i, X_j] = \sum_{k=1}^{r} c_{i,j,k} X_k$ for $i, j = 1, \dots, r$ are called *structure constants*. The commutator table is skew-symmetric and has only zeros in the diagonal.

EXAMPLE B1 Let $L = \{X_1, X_2, X_3\}$, where $X_1 = \partial_x$, $X_2 = \partial_y$ and $X_3 = y\partial_x$. The commutator table of L is

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & X_1 \\ 0 & -X_1 & 0 \end{bmatrix}. \tag{B.1}$$

The *derived algebra* $L^{(1)}$ of a LA L is the LA generated by all commutators of L, i.e. $L^{(1)} := [L,L] = \{[X,Y] \mid X, Y \in L\}$. Derived algebras of higher order are defined recursively by $L^{(i+1)} = (L^{(i)})^{(1)}$, for $i \geqslant 1$. The *derived series* of a finite LA is the sequence of dimensions of its derived algebras. The derived series in the form of the finite sequence $(\dim(L), \dim(L^{(1)}), \dots, \dim(L^{(t)}))$ is presented, where t is the smallest number $0 \leqslant t \leqslant \dim(L) + 1$ such that $L^{(t)} = L^{(t+i)}$, for all $i \geqslant 1$.

EXAMPLE B2 Consider again $L = \{\partial_x, \partial_y, y\partial_x\}$ with $\dim(L) = 3$. From the commutator table (B.1) we see that $L^{(1)} = [L,L] = \{\partial_x\}$, and hence $\dim(L^{(1)}) = 1$. Since $L^{(2)} = [L^{(1)},L^{(1)}] = \{0\}$, we have $\dim(L^{(2)}) = 0$. Hence, the derived series is $(\dim(L), \dim(L^{(1)}), \dim(L^{(2)})) = (3, 1, 0)$.

B.2 *Isomorphic and similar LAs.* Let L and K be two LAs, and let $\dim(L) = \dim(K)$. A linear one-to-one map f of L onto K is called an *isomorphism* if it preserves the commutation relation, i.e. if $f([X,Y]) = [f(X), f(Y)]$ for $X, Y \in L$. If the LAs L and K can be related by an isomorphism, they are termed *isomorphic* LAs. Two finite-dimensional LAs are isomorphic if and only if one can choose bases for the algebras such that the algebras have, in these bases, equal structure constants, i.e. the same table of commutators.

The LAs of vector fields L and $\overline{L}$ are *similar* if one is obtained from the other by a change of variables. It means that the operators $X = \sum \xi_i(\mathbf{x})\partial_{x_i}$ and $\overline{X} = \sum \bar{\xi}_i(\bar{\mathbf{x}})\partial_{\bar{x}_i}$ of L and $\overline{L}$ are related by $\bar{x}_i = \bar{x}_i(\mathbf{x})$, $\bar{\xi}_i = X(\bar{x}_i)|_{\mathbf{x}=\bar{\mathbf{x}}^{-1}(\bar{\mathbf{x}})}$, for $i = 1, \dots, n$, where $\bar{\mathbf{x}}^{-1}(\mathbf{x})$ denotes the inverse of the change of variables $\bar{\mathbf{x}}(\mathbf{x})$.

EXAMPLE B3 Let $X = \partial_y$ be the operator of translation in y in the (x, y)-plane. The transformed operator $\overline{X}$ under the change of variables $\bar{x} = x+y$, $\bar{y} = x-y$ is computed. We get $\bar{\xi}_x = \partial_y(\bar{x})|_{x=\bar{x}^{-1}} = 1$, $\bar{\xi}_y = \partial_y(\bar{y})|_{x=\bar{x}^{-1}} = -1$, i.e. $\overline{X} = \partial_{\bar{x}} - \partial_{\bar{y}}$.

In order that two LAs with the same dimension and the same number of variables are similar, it is necessary that they are isomorphic. The converse is not true. It is precisely similarity that is of use in group analysis as a criterion of reducibility of one DE to another by a suitable change of variables. Nonetheless, establishing isomorphism is important as a first step for the determination of similarity.

C. *Systems of imprimitivity*

In this section, I introduce systems of imprimitivity. Lie used their number and type to obtain a classification of groups allowed by various manifolds Lie (1970a,b). The presentation of plane and space groups in Hillgarter (2002) is also organized that way. Transitivity and primitivity are notions from substitution theory; Lie extended them to transformation groups. Details may be found in Lie (1970a) and Schwarz (2005).

Let D be a domain in $\mathbb{R}^n$. An r-parameter transformation group $T_a\colon D \to D$ with parameter space $P \subseteq \mathbb{R}^r$ is called *transitive* iff for all $\bar{x}, \bar{y} \in D$ there exists $\bar{a} \in P$ such that $T_{\bar{a}}(\bar{x}) = \bar{y}$. Otherwise, we call T *intransitive*.

EXAMPLE C1 The translation group $T_{(a_1,a_2)}(x_1, x_2) = (x_1 + a_1, x_2 + a_2)$ of the plane is obviously transitive. For every choice of two points $(\bar{x}_1, \bar{x}_2)$, $(\bar{y}_1, \bar{y}_2)$ we have $T_{(\bar{y}_1-\bar{x}_1,\bar{y}_2-\bar{x}_2)}(\bar{x}_1, \bar{x}_2) = (\bar{y}_1, \bar{y}_2)$. The transformation group $T_{(a_1,a_2)}(x_1, x_2) = (x_1, x_2 + a_1 x_1 + a_2)$ is obviously intransitive. For every choice of two points $(\bar{x}_1, \bar{x}_2)$, $(\bar{y}_1, \bar{y}_2)$ with $\bar{x}_1 \neq \bar{y}_1$, there is no choice of parameters $(\bar{a}_1, \bar{a}_2)$ such that $T_{(\bar{a}_1,\bar{a}_2)}$ maps the first point into the latter.

A group of the n-dimensional space with r infinitesimal generators $X_k = \sum_{i=1}^{n} \xi_{k,i}(x)\partial_{x_i}$ for $k = 1, \ldots, r$ is transitive iff $\operatorname{rank}([\xi_{k,i}]_{k=1,\ldots,n}^{i=1,\ldots,r}) = n$, where rank denotes the maximal number of unconnected lines of the matrix.

EXAMPLE C2 Consider $\mathbf{p}_6 = \{\partial_x, \partial_y, \partial_z, x\partial_y - y\partial_x, x\partial_z - z\partial_x, y\partial_z - z\partial_y\}$. By

$$\operatorname{rank}\left(\begin{bmatrix} 1 & 0 & 0 & y & 0 & -z \\ 0 & 1 & 0 & -x & z & 0 \\ 0 & 0 & 1 & 0 & -y & x \end{bmatrix}^\top\right) = 3,$$

we conclude that $\mathbf{p}_6$ is transitive. Now consider $\mathbf{ip}_1 = \{\partial_x, \partial_y, x\partial_x, y\partial_x, x\partial_y, y\partial_y, x^2\partial_x + xy\partial_y, y^2\partial_y + xy\partial_x\}$. By

$$\operatorname{rank}\left(\begin{bmatrix} 1 & 0 & x & y & 0 & 0 & x^2 & xy \\ 0 & 1 & 0 & 0 & x & y & xy & y^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^\top\right) = 2,$$

we conclude that $\mathbf{ip}_1$ is intransitive.

Finally, a transitive group is called *primitive* iff its action leaves no *foliation* invariant, otherwise it is called *imprimitive*.