Introduction to General Topology

An Introductory Course
for the Fourth Semester
im Wahlpflichtbereich

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Summer 2003

Chapter 1
Topological spaces

Topological spaces generalize metric spaces. One uses metric spaces in analysis to work with continuous functions on what appears to be the "right" level of generality. But even in this context one notices, that many important concepts such as the continuity of functions between metric spaces itself can be expressed in the language of *open sets* alone. This observation has caused mathematicians, first Felix Hausdorff next Paul Alexandroff and Heinz Hopf to use the idea of *open sets* as the basis for a general theory of continuity in an axiomatic approach. In fact Hausdorff's definition was based on the concept of systems of *neighborhoods* for each point.

We shall begin by defining topological spaces and continuous functions in both ways and by showing that they are equivalent.

The *objects* of our study are the "spaces"; the *transformations* between them are the "continuous functions". One should always treat them in a parallel approach. This is what has become known as "category theoretical" procedure, but we shall not be very formal in this regard.

1. Topological spaces and continuous functions

If X is a set we let $\mathfrak{P}(X)$ denote the set $\{A : A \subseteq X\}$ of all subsets of X. This set is called the *power set* of X. The name derives from a natural bijection

$$(1)\qquad A \mapsto \chi_A : \mathfrak{P}(X) \to \{0,1\}^X, \quad \chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in X \setminus A. \end{cases}$$

The function χ_A is called the *characteristic function* of the subset A of X. The two element set $\{0,1\}$ is often abbreviated by $\mathbf{2}$ and thus $\mathbf{2}^X = \{0,1\}^X$. The power set of a set is never empty, because $\emptyset \in \mathfrak{P}(X)$ and $X \in \mathfrak{P}(X)$ for any set X.

The set theoretical operations of arbitrary unions and intersections are well defined on $\mathfrak{P}(X)$. If $\mathcal{A} = \{A_j : j \in J\}$, $A_j \subseteq X$ is a family of subsets of X, then

$$\bigcup \mathcal{A} = \bigcup_{j\in J} A_j \stackrel{\text{def}}{=} \{x \in X : (\exists j \in J)\, x \in A_j\}, \tag{2}$$

$$\bigcap \mathcal{A} = \bigcap_{j\in J} A_j \stackrel{\text{def}}{=} \{x \in X : (\forall j \in J)\, x \in A_j\}. \tag{3}$$

Exercise E1.1. (i) Verify that the function $A \mapsto \chi_A$ defined in (1) above is a bijection by exhibiting its inverse function $\mathbf{2}^X \to \mathfrak{P}(X)$.

(ii) Let $\mathcal{A}$ denote the empty set of subsets of a set X. Compute $\bigcup \mathcal{A}$ and $\bigcap \mathcal{A}$, using (2) and (3).

[Hint. Regarding (i), in very explicit terms, we have for instance $\bigcap \mathcal{A} = \{x \in X : (\forall A)\,(A \in \mathcal{A}) \Rightarrow (x \in A)\}$. So what?]

(iii) Verify the following distributive law for a subset A and a family $\{A_j : j \in J\}$ of subsets A_j of a set X:

(4)
$$A \cap \bigcup_{j \in J} A_j = \bigcup_{j \in J} (A \cap A_j).$$
□

In order to understand all concepts accurately, we should recall what the difference is between a subset S of a set M and a family $(s_j : j \in J)$ of elements of M. A subset $S \subseteq M$ is a set (we assume familiarity with *that* concept) such that $s \in S$ implies $s \in M$. A family $(s_j : j \in J)$ of elements of M is a function $j \mapsto s_j : J \to M$. If I have a family $(s_j : j \in J)$ then I have a set, namely $\{s_j : j \in J\}$, the image of the function. In fact for many purposes of set theory a family is even denoted by $\{s_j : j \in J\}$ which, strictly speaking, is not exact. Conversely, if I have a subset S of M then I can form a family $(s : s \in S)$ of elements of M, namely the inclusion function $s \mapsto s : S \to M$. Notice that we can have an empty family $(s_j : j \in \emptyset)$, namely the empty function $\emptyset\colon \emptyset \to M$, whose graph is the empty set, a subset of $\emptyset \times M = \emptyset$. (What we cannot have is a function $X \to \emptyset$ for for $X \neq \emptyset$! Check the definition of a function!)

Topological spaces

DEFINITION OF TOPOLOGY AND TOPOLOGICAL SPACE

Definition 1.1. A *topology* $\mathfrak{O}$ on a set X is a subset of $\mathfrak{P}(X)$ which is closed under the formation of arbitrary unions and finite intersections.

A *topological space* is a pair $(X, \mathfrak{O})$ consisting of a set X and a topology $\mathfrak{O}$ on it. If no confusion is likely to arise one also calls X a topological space. □

Let's be a bit more explicit:

A subset $\mathfrak{O} \subseteq \mathfrak{P}(X)$ is a topology iff

(i) *For any family of sets $U_j \in \mathfrak{O}$, $j \in J$, we have $\bigcup_{j \in J} U_j \in \mathfrak{O}$.*

(ii) *For any* finite *family of sets $U_j \in \mathfrak{O}$, $j \in J$, (J finite), we have $\bigcap_{j \in J} U_j \in \mathfrak{O}$.*

(iii) *$\emptyset \in \mathfrak{O}$ and $X \in \mathfrak{O}$.* □

By Exercise E1.1(ii) these statements are not independent: Proposition (iii) is a consequence of Propositions (i) and (ii).

The following set of axioms is equivalent to (i), (ii), (iii):

A subset $\mathfrak{O}$ *of* $\mathfrak{P}(X)$ *is a topology iff*

(I) *For each subset* $\mathcal{U}$ *of* $\mathfrak{O}$ *one has* $\bigcup \mathcal{U} \in \mathfrak{O}$.

(II) *For each* $U_1, U_2 \in \mathfrak{O}$ *we have* $U_1 \cap U_2 \in \mathfrak{O}$,

(III) $X \in \mathfrak{O}$.

Notation 1.2. If $(X, \mathfrak{O})$ is a topological space, then the sets $U \in \mathfrak{O}$ are called *open*. A subset A of X is called *closed*, if $X \setminus A$ is open. □

The subsets $\emptyset$ and X are both open and closed.

Example 1.3. (i) For any set X, the power set $\mathfrak{P}(X)$ is a topology, called the *discrete topology*. A space equipped with its discrete topology is called a *discrete space*.

(ii) For any set X, the set $\{\emptyset, X\}$ is a topology called the *indiscrete topology*. A space equipped with its discrete topology is called an *indiscrete space*.

(iii) For any set X, the set $\{\emptyset\} \cup \{Y \subseteq X : \operatorname{card}(X \setminus Y) < \infty\}$ is a topology, called *cofinite topology*.

(iv) A binary relation $\leq$ is called a quasiorder if it is transitive and reflexive, and it is a partial order if in addition it is antisymmetric. A *partially ordered set* or in short *poset* is a set $(X, \leq)$ endowed with a partial order.

For a subset Y in a quasiordered set $(X, \leq)$ we write $\uparrow Y \stackrel{\text{def}}{=} \{x \in X : (\exists y \in Y)\, y \leq x$; a set satisfying $\uparrow Y = Y$ is called an *upper set*. We also write $\uparrow x$ instead of $\uparrow \{x\}$.

For each quasiordered set $(X, \leq)$ the set

$$\{Y \subseteq X : \uparrow Y = Y\}$$

is a topology, called the *Alexandroff discrete topology* of the quasiordered set.

(v) A quasiordered set D is *directed* if it is not empty and for each $x, y \in D$ there is a $z \in D$ such that $x \leq z$ and $y \leq z$. A poset $(X, \leq)$ is called a *directed complete poset* or **dcpo** if every directed subset has a least upper bound. In a **dcpo** the set $\sigma(X) =$

$$\{U \subseteq X : \uparrow U = U \text{ and } (\forall D \subseteq X)\,(D \text{ is directed and } \sup D \in U) \Rightarrow D \cap U \neq \emptyset\}$$

is a topology, called the *Scott topology* of the poset. □

Example 1.4. On the set $\mathbb{R}$ of real numbers, the set

$$\mathfrak{O}(\mathbb{R}) = \{U \subseteq \mathbb{R} : (\forall u \in U)(\exists a, b \in \mathbb{R})\, a < u < b \text{ and }]a, b[\subseteq U\}$$

is a topology on $\mathbb{R}$, called the *natural topology of* $\mathbb{R}$. □

We recall from basic analysis the concept of a metric and a metric space.

Definition 1.5. A *metric* of on a set X is a function $d\colon X \times X \to \mathbb{R}$ satisfying the following conditions:

(i) $(\forall x, y \in X)\, d(x, y) \geq 0$ and $d(x, y) = 0$ iff $x = y$.

(ii) $(\forall x, y \in X)\, d(x, y) = d(y, x)$.

(iii) $(\forall x, y, z \in X)\, d(x, z) \leq d(x, y) + d(y, z)$.

Property (i) is called *positive definiteness*, Property (ii) *symmetry*, and property (iii) the *triangle inequality*.

If a set X is equipped with a metric d, then (X, d) is a *metric space*.

If $r > 0$ and $x \in X$, then $U_r(x) \stackrel{\text{def}}{=} \{u \in X : d(x, u) < r\}$ is called the *open ball of radius* r with *center* x. □

Proposition 1.6. *For a metric space* (X, d)*, the set*

$$\mathfrak{O}(X) = \{U \subseteq X : (\forall u \in U)(\exists \varepsilon > 0)\, U_\varepsilon(u) \subseteq U\}$$

is a topology. Every open ball $U_r(x)$ *belongs to* $\mathfrak{O}(X)$. □

Definition 1.7. The topology $\mathfrak{O}(X)$ of 1.6 on a metric space is called the *metric topology* for d or the *topology induced by* d. □

Thus any metric space is automatically a topological space. The natural topology of $\mathbb{R}$ is the metric topology for the metric on $\mathbb{R}$ given by $d(x, y) = |y - x|$. Given an arbitrary set, the function $d\colon X \times X \to \mathbb{R}$ such that $d(x, y) = 1$ if $x \neq y$ and $d(x, x) = 0$ is a metric whose metric topology is the discrete topology. Therefore it is called the *discrete metric*.

Proposition 1.8. *Assume that* $(X, \mathfrak{O})$ *is a topological space, and that* $Y \subseteq X$*. Then*

$$\mathfrak{O}|Y \stackrel{\text{def}}{=} \{Y \cap U : U \in \mathfrak{O}\}$$

is a topology of Y. □

Definition 1.9. The topology $\mathfrak{O}|Y$ is called the *induced topology*. The topological space $(Y, \mathfrak{O}|Y)$ is called the *subspace* of X on Y.

With the concepts introduced so far we have an immense supply of interesting topological spaces. The absolute value of complex numbers makes the complex plane $\mathbb{C}$ into a metric space via $d(u, v) = |v - u|$ and thus into a toplogical space. The space $\mathbb{S}^1 \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |z| = 1\}$ is called the *unit circle*, or the *one-sphere*. More generally, if one considers on $\mathbb{R}^n$ the *norm* $\|(x_1, \ldots, x_n)\| \stackrel{\text{def}}{=} \sqrt{x_1^2 + \ldots + x_n^2}$, then the metric space determined by the metric $d(x, y) = \|y - x\|$ is called *euclidean space*. The space $B^n \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ is called the closed *n-cell* or *unit ball* in n dimensions. The subspace $\mathbb{S}^n = \{x \in B^{n+1} : \|x\| = 1\}$ is called the *n-sphere*.

Continuous functions

Definition of continuous function

Definition 1.10. A function $f: X \to Y$ between topological spaces is called *continuous*, if $f^{-1}(U)$ is open in X for every open $U \in Y$.

The set of all continuous functions $f: X \to Y$ is often denoted by $C(X, Y)$. □

Proposition 1.11. *If $f: X \to Y$ is a continuous function between topological spaces and $A \subseteq X$ is a subspace, then $f|A: A \to Y$ is continuous.* □

Exercise E1.2. (i) Every function from a discrete space into a topological space is continuous.

(ii) Every function from a topological space into an indiscrete space is continuous.

(iii) *Let $f: (X, \leq) \to (Y, \leq)$ be a function between two* **dcpo***s. Then the following statements are equivalent:*

(a) *f is Scott continuous, i.e. is continuous with respect to the Scott topologies on X and Y.*

(b) *f preserves directed sups, i.e.* $\sup f(D) = f(\sup D)$ *for all directed subsets D of X.* □

We shall characterize continuity between metric spaces shortly.

Neighborhoods

Definition 1.12. If $(X, \mathfrak{O})$ is a topological space and $x \in X$, then a set $U \in \mathfrak{P}(X)$ is called a *neighborhood* of x iff

$$(\exists V)\, V \in \mathfrak{O} \text{ and } x \in V \subseteq U. \tag{5}$$

We write

$$\mathfrak{U}(x) = \{U \in \mathfrak{P}(X) : U \text{ is a neighborhood of } x\}. \tag{6}$$

□

Observation. *The set $\mathfrak{U}(x)$ satisfies the following conditions*

(i) $\big(\forall U \in \mathfrak{U}(x)\big)\, U \neq \emptyset$

(ii) $\big(\forall U, V \in \mathfrak{U}(x)\big)\, U \cap V \in \mathfrak{U}(x)$.

(iii) $\big(\forall U, V\big)\, \big(U \in \mathfrak{U}(x) \text{ and } U \subseteq V\big) \Rightarrow V \in \mathfrak{U}(x)$. □

This observation calls for the introduction of a new concept.

Definition 1.13. Assume that X is a set. A set $\mathfrak{F} \subseteq \mathfrak{P}(X)$ of subsets of X is called a *filter*, if it is nonempty and satisfies the following conditions

(i) $\big(\forall A \in \mathfrak{F}\big)\, A \neq \emptyset$

(ii) $\big(\forall A,\, B \in \mathfrak{F}\big)\, A \cap B \in \mathfrak{F}$.
(iii) $(\forall A,\, B)\,(A \in \mathfrak{F}$ and $A \subseteq B) \Rightarrow B \in \mathfrak{F}$.

A set $\mathfrak{B} \subseteq \mathfrak{P}(X)$ of subsets of X is called a *filter basis*, if it is nonempty and satisfies the following conditions
(i) $\big(\forall A \in \mathfrak{B}\big)\, A \neq \emptyset$,
(ii) $(\forall A,\, B \in \mathfrak{B})(\exists C \in \mathfrak{B})\, C \subseteq A \cap B$. □

Proposition 1.14. *A subset* $\mathfrak{B}$ *of* $\mathfrak{P}(X)$ *is a filter basis iff the set*

$$\mathfrak{F} \stackrel{\text{def}}{=} \{A \in \mathfrak{P}(X) : (\exists B \in \mathfrak{B})\, B \subseteq A\}$$

is a filter. □

The set of all neighborhoods of a point is a filter, the set of open neighborhoods is a filter basis.

Hausdorff characterisation of a topological space

Theorem 1.15. *Assume* $(X, \mathfrak{O})$ *to be a topological space. Then*

$$x \mapsto \mathfrak{U}(x) : X \to \mathfrak{P}\big(\mathfrak{P}(X)\big)$$

is a function satisfying the following conditions:
(i) *Each* $\mathfrak{U}(x)$ *is a filter.*
(ii) $(\forall x \in X,\, U \in \mathfrak{U}(x))\, x \in U$.
(iii) $\big(\forall x \in X, U \in \mathfrak{U}(x)\big)\big(\exists V \in \mathfrak{U}(x)\big)\,(\forall v \in V)\, U \in \mathfrak{U}(v)$.

Conversely, if $(\mathfrak{U}(x) : x \in X)$ *is a family of subsets of subsets of a set* X *satisfying* (i), (ii), (iii)*, then the collection* $\mathfrak{O}$ *of all subsets* $U \subseteq X$ *such that* $(\forall u \in U)\, U \in \mathfrak{U}(u)$ *is a unique topology on* X *such that* $\mathfrak{U}(x)$ *is exactly the neighborhood filter of* x *in the topological space* $(X, \mathfrak{O})$. □

Theorem 1.15 remains intact if (ii) and (iii) are replaced by
(II) $\big((\forall x \in X,\, U \in \mathfrak{U}(x)\big)(\exists V) x \in V \subseteq U \, and (\forall v \in V)\, V \in \mathfrak{U}(v)$.

Theorem 1.16. (Characterization of continuity of functions) *A function* $f \colon X \to Y$ *between topological spaces is continuous if and only if for each* $x \in X$ *and each* $V \in \mathfrak{U}\big(f(x)\big)$ *there is a* $U \in \mathfrak{U}(x)$ *such that* $f(U) \subseteq V$. □

Corollary 1.17. *A function* $f \colon X \to Y$ *between two metric spaces is continuous iff for each* $x \in X$ *and each* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that* $f(U_\delta(x)) \subseteq U_\varepsilon\big(f(x)\big)$.□

Expressed more explicitly, f is continuous if for each x and each positive number ε there is a positive number δ such that the relation $d_X(x, y) < \delta$ implies $d_Y\big(f(x), f(y)\big) < \varepsilon$.

This is the famous ε-δ definition of continuity between metric spaces. The topological descriptions of continuity are less technical.

On the other hand, the neighborhood concept allows us to define continuity at a point of a topological spaces:

Definition. Let X and Y be topological spaces and $x \in X$. Then a function $f: X \to Y$ is said to be *continuous at* x, if for every neighborhood $V \in \mathfrak{U}(f(x))$ there is a neighborhood $U \in \mathfrak{U}(x)$ such that $f(U) \subseteq V$. □

Clearly f is continuous if and only if it is continuous at each point $x \in X$.

Example 1.18. The two element space $\mathbf{2}$ is a topological space with respect to the discrete topology, but also with respect to the Scott topology $\sigma(\mathbf{2}) = \{\emptyset, \{1\}, \mathbf{2}\}$. Let us denote with $\mathbf{2}_\sigma$ the two element space with respect to this topology. This space is sometimes called the *Sierpinski space*.

If $(X, \mathfrak{O})$ is a topological space and A a subset of X, then the characteristic function $\chi_A: X \to \mathbf{2}$ (with the discrete topology on $\mathbf{2}$) is continuous iff A is open and closed, i.e. $A, X \setminus A \in \mathfrak{O}$.

The characteristic function $\chi_A: X \to \mathbf{2}_\sigma$ is continuous iff A is open, i.e. $A \in \mathfrak{O}$. The function

$$A \mapsto \chi_A : \mathfrak{O} \to C(X, \mathbf{2}_\sigma)$$

is a bijection. □

Exercise E1.3. Verify the assertions made in the discussion of Example 1.18.

The interior and the closure of a set

Definition 1.19. Consider a topological space $(X, \mathfrak{O})$ and $Y \subseteq X$. Define Y° or $\operatorname{int} Y$ to be the union of all open subsets $U \subseteq Y$, that is, $Y^\circ = \bigcup\{U \in \mathfrak{O} : U \subseteq Y\}$. This set is the largest open subset contained in Y and is called the *interior* of Y.

The intersection of all closed supersets $A \supseteq Y$ is the smallest closed set containing Y. It is called the *closure* of Y, written $\overline{Y}$ or $\operatorname{cl} Y$.

Proposition 1.20. *In a topological space $(X, \mathfrak{O})$ we have the following conclusions:*

(i) $Y^{\circ\circ} = Y^\circ$ and $Y \subseteq Z \subseteq X$ implies $Y^\circ \subseteq Z^\circ$.
(ii) $\overline{\overline{Y}} = \overline{Y}$ and $Y \subseteq Z \subseteq X$ implies $\overline{Y} \subseteq \overline{Z}$.
(iii) $Y^\circ = X \setminus \overline{X \setminus Y}$ and $\overline{Y} = X \setminus (X \setminus Y)^\circ$. □

Proposition 1.21. *Let $(X, \mathfrak{O})$ denote a topological space and Y a subset. Let $x \in X$. Then the following assertions are equivalent:*
(i) $x \in Y^\circ$.
(ii) $(\exists U \in \mathfrak{O})\, x \in U \subseteq Y$.
(iii) $(\exists N \in \mathfrak{U}(x))\, N \subseteq Y$.
(iv) $Y \in \mathfrak{U}(x)$.
Also, the following statements are equivalent:
(i) $x \in \overline{Y}$.

(ii) $(\forall U \in \mathfrak{O})\, x \in U \Rightarrow U \cap Y \neq \emptyset$.
(iii) *Every neighborhood of x meets Y.*
(iv) *$X \setminus Y$ is not a neighborhood of x.*

Basis and subbasis of a topology

Often we shall define a topology by starting from a certain set of open sets which generate all open sets in a suitable sense.

Definition 1.22. A set $\mathfrak{B} \subseteq \mathfrak{P}(X)$ is called a *basis of a topology* if $X = \bigcup \mathfrak{B}$ and

$$(\forall A, B \in \mathfrak{B})(\forall x \in A \cap B)(\exists C \in \mathfrak{B})\, x \in C \subseteq A \cap B. \tag{7}$$ □

Proposition 1.23. *For a subset $\mathfrak{B} \subseteq \mathfrak{P}(X)$, the following conditions are equivalent:*
(i) *$\mathfrak{B}$ is a basis of a topology.*
(ii) *$\mathfrak{O} \stackrel{\text{def}}{=} \{U : (\forall u \in U)(\exists B \in \mathfrak{B})\, u \in B \subseteq U\}$ is a topology.*
(iii) *The set of all unions of sets of members of $\mathfrak{B}$ is a topology.* □

In the circumstances of 1.23 we say that $\mathfrak{B}$ *is a basis of* $\mathfrak{O}$. The discrete topology $\mathfrak{P}(X)$ of a set has a unique smallest basis, namely, $\{\{x\} : x \in X\}$.

Example 1.24. (i) Let (X, d) be a metric space. The set $\mathfrak{B}$ of all open balls $U_{1/n}(x)$, $n \in \mathbb{N}$, $x \in X$ is a basis for the metric topology $\mathfrak{O}(X)$.

(ii) The natural topology of $\mathbb{R}$ has a countable basis

$$\left\{ \left] q - \frac{1}{n}, q + \frac{1}{n} \right[: q \in \mathbb{Q},\, n \in \mathbb{N} \right\}. \tag{8}$$

Definition 1.25. One says that a topological space $(X, \mathfrak{O})$ satisfies the *First Axiom of Countability*, if every neighborhoodfilter $\mathfrak{U}(x)$ has a countable basis. It satisfies the *Second Axiom of Countability* if $\mathfrak{O}$ has a countable basis.

Exercise E1.4. Every space satisfying the Second Axiom of Countability satisfies the first axiom of countability. The discrete topology of a set satisfies the first axiom of countability; but if it fails to be countable, it does not satisfy the Second Axiom of Countability.

Every set of cardinals has a smallest element. Given this piece of information we can attach to a topological space $(X, \mathfrak{O})$ a cardinal, called its *weight* :

$$w(X) = \min\{\text{card}\, \mathfrak{B} : \mathfrak{B} \text{ is a basis of } \mathfrak{O}\}. \tag{9}$$

The weight of a topological space is countable iff it satisfies the Second Axiom of Countability.

Definition 1.26. A set $\mathfrak{B}$ of subsets of a topological space is said to be a *basis for the closed sets* if every closed subset is an intersection of subsets taken from $\mathfrak{B}$. □

The set of complements of the sets of a basis of a topology is a basis for the closed sets of this topology and vice versa.

Proposition 1.27. *Let $\mathcal{T} \subseteq \mathfrak{P}\big(\mathfrak{P}(X)\big)$ be a set of topologies. Then $\bigcap \mathcal{T} \subseteq \mathfrak{P}(X)$ is a topology.* □

By Proposition 1.27, every set $\mathfrak{M}$ of subsets of a set X is contained in a unique smallest topology $\mathfrak{O}$, called the *topology generated by* $\mathfrak{M}$. Under these circumstances, $\mathfrak{M}$ is called a *subbasis of* $\mathfrak{O}$.

Proposition 1.28. *The topology generated by a set $\mathfrak{M}$ of a set X consists of all unions of finite intersections of sets taken from $\mathfrak{M}$.* □

Definition 1.29. Let $(X, \leq)$ be a totally ordered set, i.e. a poset for which every two elements are comparable w.r.t. $\leq$. Then the set of all subsets X, $\uparrow a$, $a \in X$, and $\downarrow a$, $a \in X$ is a subbasis for the closed sets of a topology, called the *order topology* of $(X, \leq)$. □

[It should be understood that by $\downarrow a$ in a poset we mean the set of all $x \in X$ with $x \leq a$.]

Example 1.30. In $\mathbb{R}$ the set of all $]q, \infty[$, $q \in \mathbb{Q}$ and $]-\infty, q[$, $q \in \mathbb{Q}$ form a subbasis of the natural topology. □

Exercise E1.5. Show that the order topology on $\mathbb{R}$ is the natural topology of $\mathbb{R}$.

The Lower Separation Axioms

Lemma 1.31. *The relation $\preceq$ in a topological space defined by*

$$x \preceq y \text{ if and ony if } (\forall U \in \mathfrak{O})\, x \in U \Rightarrow y \in U. \tag{10}$$

is reflexive and transitive. □

We have $x \preceq y$ if every neighborhood of x is a neighborhood of y. If $(X, \leq)$ is a quasiordered set and $\mathfrak{O}$ is the Alexandroff discrete topology, then $x \preceq y$ iff $x \leq y$.

Definition 1.32. The quasiorder $\preceq$ on a topological space is called the *specialisation quasiorder.* □

While this is not relevant here, let us mention that the name arises from algebraic geometry.

Exercise E1.6. Show that on $\big(\mathbb{R}, \sigma(\mathbb{R})\big)$ with the Scott topology, one has $x \preceq y$ iff $x \leq y$. □

Notice that $\sigma(\mathbb{R})$ is bigger than the Alexandroff discrete topology on $\mathbb{R}$. So two different topologies can produce the same specialisation quasiorder.

The specialisation quasiorder with respect to the indiscrete topology is the trivial quasiorder that holds always between two elements. The specialisation order with respect to the discrete topology is equality.

Proposition 1.33. *In a topological space, the point closure $\overline{\{a\}}$ is the lower set $\downarrow a$ w.r.t. the specialisation order.* □

Definition 1.34. A topological space $(X, \mathfrak{O})$ is said to satisfy the Axiom (T_0), or is said to be a T_0*-space* if and only if the specialisation quasiorder is a partial order. Under these conditions, the topology $\mathfrak{O}$ is called a T_0*-topology.* □

Sometimes (following Alexandroff and Hopf), the Axiom (T_0) is called *Kolmogoroff's Axiom.*

The Axiom (T_0) is equivalent to each of the following statements:

- For two different elements x and y in X, there is an open set such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$.

In other words, for two different points there is an open set containing precisely one of the two points.

- The function $x \mapsto \mathfrak{U}(x): X \to \mathfrak{P}\big(\mathfrak{P}(X)\big)$ which assigns to an element its neighborhood filter is injective.

In other words: "Different points have different neighborhood filters."

Definition 1.35. The space X is said to satisfy the separation axiom (T_1) (or to be a T_1*-space*), and its topology $\mathfrak{O}$ is called a T_1*-topology*, if the specialisation quasiorder is discrete, i.e., is equality.

A topological space is a T_1 space if and only if

- every singleton subset is closed.

That is $\overline{\{a\}} = \{a\}$ for all $a \in X$.

Another equivalent formulation of the Axiom (T_1) is

- If x and y are two different points then there is an open set U containing x but not y.

Example 1.36. The cofinite topology is always a T_1-topology. □

The Alexandroff-discrete topology of a nontrivial poset is a T_0-topology but not a T_1-topology. For instance, the Sierpinski space $\mathbf{2}_\sigma$ is a T_0-space which is not a T_1-space.

The terminology for the hierarchy (T_n) of separation axioms appears to have entered the literature 1935 through the influential book by Alexandroff and Hopf in a section of the book called „**T**rennungsaxiome“ (pp. 58 ff.). Alexandroff and Hopf call the Axiom (T_1) „das erste Frechetsche Trennungsaxiom“, p. 58, 59), and they attach with the higher separation axioms the names of Hausdorff, Vietoris, and Tietze. In due time we shall face these axioms.

In Bourbaki T_0-spaces are called ≪espaces de Kolmogoroff≫ (s. §1, Ex. 2, p. 89). Alexandroff and Hopf appear to have had access to an unpublished manuscript by Kolmogoroff which appears to have dealt with quotient spaces (see Alexandroff and Hopf p. 61 and p. 619) and which is likely to have been the origin of calling (T_0) Kolomogoroff's Axiom; Alexandroff continues to refer to it under this name in later papers. Fréchet calls T_1-spaces ≪espaces accessibles≫.

Definition 1.37. The space X is said to satisfy the *Hausdorff separation axiom* (T_2) (or to be a T_2*-space*), and its topology $\mathfrak{O}$ is called a *Hausdorff topology*, respectively, T_2*-topology*, if the following condition is satisfied:

$$(T_2) \qquad (\forall x, y \in X)\, x \neq y \Rightarrow \big(\exists\, U \in \mathfrak{U}(x),\, V \in \mathfrak{U}(y)\big) U \cap V = \emptyset.$$

In other words, two different points have disjoint neighborhoods.

Exercise E1.7. Let $\mathfrak{O}_1$ and $\mathfrak{O}_2$ be two topologies on a set such that $\mathfrak{O}_1 \subseteq \mathfrak{O}_2$. If $\mathfrak{O}_1$ is a T_n-topology for $n = 0, 1, 2$, then $\mathfrak{O}_2$ is a T_n-topology.

Definition 1.38. The space X is said to be *regular*, and its topology $\mathfrak{O}$ is called a *regular topology* if the following condition is satisfied:

$$(\forall x \in X)\big(\forall U \in \mathfrak{U}(x)\big)\big(\exists A \in \mathfrak{U}(x)\big)\, \overline{A} = A \text{ and } A \subseteq U.$$

It is said to satisfy the axiom (T_3) (or to be a T_3*-space*), if it is a regular T_0-space. In other words:
(T_3) X is a T_0-space and every neighborhood filter has a basis of closed sets. □

For a T_0-space X, the axiom (T_3) is also equivalent to the following statement:

$(*)$ *For any $x \in X$ and any neighborhood $U \in \mathfrak{U}(x)$, there are open sets V and W such that $x \in V$, $V \cap W = \emptyset$, and $U \cup W = X$.*

Exercise E1.8. (a) Show that (T_3) is equivalent to (T_0) and $(*)$.

(b) Prove the following propostitions:

(i) Every metric space is regular. In particular, the natural topology of $\mathbb{R}$ is regular.

(ii) Every metric space is a Hausdorff space.

(iii) On $\mathbb{R}$ let $\mathfrak{O}_c$ be the collection of all sets $U \setminus C$ where U is open in the natural topology $\mathfrak{O}$ of $\mathbb{R}$ and C is a countable set. Then $\mathfrak{O}_c$ is a topology which is properly finer than the natural topology of $\mathbb{R}$, that is, the identity function $\mathrm{id}_{\mathbb{R}}: (\mathbb{R}, \mathfrak{O}_c) \to (\mathbb{R}, \mathfrak{O})$ is continuous, but its inverse function is not continuous. The topology $\mathfrak{O}_c$ is not regular.

A function $f: \mathbb{R} \to \mathbb{R}$ is continuous as a function $(\mathbb{R}, \mathfrak{O}_c) \to (\mathbb{R}, \mathfrak{O})$ if and only if it is continuous as a function $(\mathbb{R}, \mathfrak{O}) \to (\mathbb{R}, \mathfrak{O})$.

(iv) $(T_3) \Rightarrow (T_2) \Rightarrow (T_1) \Rightarrow (T_0)$ and $(T_0) \not\Rightarrow (T_1) \not\Rightarrow (T_2) \not\Rightarrow (T_3)$.

Quotient Spaces

An equivalence relation R on a set X is a reflexive, symmetric, and transitive relation. Recall that a binary relation is a subset of $X \times X$; in place of $(x, y) \in R$ one frequently writes $x\,R\,y$.

Every equivalence relation R on a set X gives rise to a new set X/R, the set of all equivalence classes $R(x) = \{x' \in X : (x, x') \in R\}$. Note $x \in R(x)$. If A and B are R-equivalence classes, then either $A \cap B = \emptyset$ or $A = B$. Thus X is a disjoint union of all R equivalence classes. One calls a set $\mathcal{P} \subseteq \mathfrak{P}(X)$ of subsets a *partition* of X if two different members of $\mathcal{P}$ are disjoint and $\bigcup \mathcal{P} = X$. We have seen that every equivalence relation on a set X provides us with a partition of X. Conversely, if $\mathcal{P}$ is a partition of X, then $R \stackrel{\text{def}}{=} \{(x, x') \in X \times X : (\exists A \in \mathcal{P})x,\, x' \in A\}$ is an equivalence relation whose partition is the given one. There is a bijection between equivalence relations and partitions.

The new set X/R is called the *quotient set modulo R*. The function $p_R : X \to X/R$, $p_R(x) = R(x)$ is called the *quotient map*.

One of the primary occurrences of equivalence relations is the kernel relation of a function, as follows. Let $f : X \to Y$ be a function. Define $R_f = \{(x, x') : f(x) = f(x')\}$. Then there is a bijective function $f' : X/R_f \to f(X)$ which is unabiguously defined by $f'\big(R_f(x)\big) = f(x)$. If $\mathrm{inc} : f(X) \to Y$ is the inclusion map $y \mapsto y : f(X) \to Y$ then we have the so-called *canonical decomposition* $f = \mathrm{inc} \circ f' \circ p_{R_f}$ of the given function:

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle p_{R_f}}\big\downarrow & & \big\uparrow{\scriptstyle \mathrm{inc}} \\ X/R_f & \xrightarrow[\quad f' \quad]{} & f(X). \end{array}$$

In this decomposition of f the quotient map p_{R_f} is surjective, the induced function f' is bijective, the inclusion map is injective.

The objective of this subsection is to endow the quotient space X/R of a topological space X with a topology in a natural way so that the quotient map is continuous and that, if R is the kernel relation of a continuous function, the induced bijective function $f' : X/R_f \to f(X)$ is continuous.

Lemma 1.39. *Let R be an equivalence relation on a topological space (X, R). Then there is a bijection β from the set $\mathfrak{O}_R = \{U \in \mathfrak{O} : U = \bigcup_{u \in U} R(u)\}$ and the set $\mathfrak{O}(X/R) = \big\{\{R(u) : u \in U\} : U \in \mathfrak{O}_R\big\}$, where $\beta(U) = \{R(u) : u \in U\}$.*

Both $\mathfrak{O}_R$ and $\mathfrak{O}_{X/R}$ is closed under the formation of arbitrary unions and finite intersections, and X, $\emptyset \in \mathfrak{O}_R$ and X/R, $\emptyset \in \mathfrak{O}_{X/R}$. □

The members of $\mathfrak{O}_R$ are also called the *R-saturated open sets*.

Definition 1.40. The topological space $(X/R, \mathfrak{O}_{X/R})$ is called the quotient space of X modulo R.

Proposition 1.40. *The quotient map $p_{R_f}: X \to X/R$ is continuous.*

The quotient space X/R is a T_1 space if and only if all R-equivalence classes are closed.

The quotient space X/R is a Hausdorff space if and only if for two disjoint R-classes A and B there are disjoint saturated open set U and V containing A and B, respectively. □

Definition 1.41. A function $f: X \to Y$ between topological spaces is called *open* if $f(U)$ is open for each open set U, that is, if open sets have open images.

Exercise E1.9. Show that the function $x \mapsto x^2 : \mathbb{R} \to \mathbb{R}$ fails to be open. □

Proposition 1.42. *For an equivalence relation R on a topological space X, the following statements are equivalent:*

(i) The quotient map $p_R: X \to X/R$ is open.
(ii) For each open subset U of X the *saturation* $\bigcup_{u \in U} R(u)$ is open. □

Group Actions

There is a prominent situation for which quotient maps are open.

Definition 1.43. A continuous function $f: X \to Y$ between topological spaces is called a *homeomorphism*, if it is bijective and its inverse function $f^{-1}: Y \to X$ is continuous. Two spaces X and Y are called *homeomorphic* if there exists a homeomorphism between them. □

A function $f: (X, \mathfrak{O}_X) \to (Y, \mathfrak{O}_Y)$ between topological spaces is a homeomorphism if and only if the function f implements a bijection $U \mapsto f(U) : \mathfrak{O}_X \to \mathfrak{O}_Y$.

Exercise E1.10. (i) For any topological space X, the set H of homeomorphisms $f: X \to X$ is a group.

(ii) Let G be a subgroup of H. Let us write $g{\cdot}x = g(x)$ for $g \in G$ and $x \in X$. Then the set $X/G \stackrel{\text{def}}{=} \{G{\cdot}x | x \in X\}$ is a partition of X. The corresponding equivalence relation is given by $x \sim y$ iff $(\exists g \in G)\, y = g{\cdot}x$.

(iii) We let $p: X \to X/G$ denote the quotient map defined by $p(x) = G{\cdot}x$ and endow X/G with the quotient topology. Then p is an open map. □

The set $G{\cdot}x$ is called the *orbit* of x under the action of G, or simply the *G-orbit* of x. The quotient space X/G is called the *orbit space*.

Exercise E1.11. (i) Let X be the space $\mathbb{R}$ of real numbers with its natural topology. Every $r \in \mathbb{R}$ defines a function $T_r: X \to X$, via $T_r(x) = r + x$, the translation by r. Every such translation is a homeomorphism of $\mathbb{R}$.

(ii) Let G be the group of all homeomorphisms T_r with $r \in \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers.

Describe the orbits $G{\cdot}x$ of the action of G on X.

Describe the orbit space X/G. Show that it is homeomorphic to the one-sphere $\mathbb{S}^1$.

(iii) Now let X be as before, but take $G = \{T_r : r \in \mathbb{Q}\}$ where $\mathbb{Q}$ is the set of rational numbers. Discuss orbits and orbit space.

(iv) Test these orbit spaces for the validity of separation axioms. □

A Universal Construction

Let us consider another useful application of quotient spaces.

On any topological space X with topology $\mathfrak{O}_X$, the binary relation defined by $x \equiv y$ iff $x \leq y$ and $y \leq x$ (with respect to the specialisation quasiorder $\leq$) is an equivalence relation. The quotient space $X/{\equiv}$ endowed with its quotient topology $\mathfrak{O}_{X/\equiv}$ will be denoted by $T_0(X)$.

Proposition 1.44. *For any topological space X, the space $T_0(X)$ is a T_0-space, and if $q_X\colon X \to T_0(X) = X/{\equiv}$ denotes the quotient map which assigns to each point its equivalence class, then the function $U \mapsto q_X^{-1}(U)\colon \mathfrak{O}_{T_0(X)} \to \mathfrak{O}_X$ is a bijection. Moreover, if $f\colon X \to Y$ is any continuous function into a T_0-space, then there is a unique continuous function $f'\colon X/{\equiv} \to Y$ such that $f = f' \circ q_X$.* □

As a consequence of these remarks, for most purposes it is no restriction of generality to assume that a topological space under consideration satisfies at least the separation axiom (T_0).

The Canonical Decomposition

It is satisfying to know that the quotient topology provides the quotient space modulo a kernel relation with that topology which allows the canonical decomposition of a *continuous* function between topological spaces to work correctly.

Theorem 1.45. (The Canonical Decomposition of Continuous Functions) *Let $f\colon X \to Y$ be a continuous function between topological spaces and let*

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle p_{R_f}}\downarrow & & \uparrow{\scriptstyle \mathrm{inc}} \\ X/R_f & \xrightarrow[\quad f' \quad]{} & f(X). \end{array}$$

be its canonical decomposition, where R_f denotes the kernel relation of f. Then $f'\colon X/R_f \to f(X)$ is a continuous bijection, the quotiend map p_{R_f} is a continuous surjection, the inclusion map is an embedding, i.e., a homeomorphism onto its image. □

If the space Y is a Hausdorff space, then so is the subspace $f(X)$; then the continuous bijection f' in the canonical decomposition theorem tells us at once that the quotients space X/R_f is a Hausdorff space—whether X itself is a Hausdorf space or not.

Corollary 1.46. *If $f\colon X \to Y$ is a continuous function into a Hausdorff space, then the quotient space X/R_f is a Hausdorff space.* □

Naturally one wishes to understand when f' is a homeomorphism.

Proposition 1.47. *Let $f\colon X \to Y$ be a continuous function between topological spaces. Then the following conditions are equivalent:*
(i) *The corestriction $x \mapsto f(x)\colon X \to f(X)$ is open.*
(ii) *The quotient morphism $p_f\colon X \to X/R_f$ is open and $f'\colon X/R_f \to f(X)$ is a homeomorphism.* □

Products

Definition 1.48. Let $(X_j : j \in J)$ be a family of sets. The *cartensian product* or simply *product* of this family, written $\prod_{j\in J} X_j$, is the set of all functions $f\colon J \to \bigcup_{j\in J} X_j$ such that $(\forall j \in J)\, f(j) \in X_j$. These functions are also written $(x_j)_{j\in J}$ with $x_j = f(j)$ and are called *J-tuples.* The function $\mathrm{pr}_k\colon \prod_{j\in J} X_j \to X_k$, $\mathrm{pr}_k\big((x_j)_{j\in J}\big) = x_k$ is called the *projection* of the product onto the factor X_k. □

The following statement looks innocent, but it is an axiom:
Axiom 1.49. (Axiom of Choice) *For each set J and each family of sets $(X_j : j \in J)$ the product $\prod_{j\in J} X_j$ is not empty.* □

Proposition 1.50. *If the product $P \stackrel{\text{def}}{=} \prod_{j\in J} X_j$ is not empty, then for each $k \in J$, the projection $pr_k\colon P \to X_k$ is surjective, and there is an injection $s_k\colon X_k \to P$ such that $\mathrm{pr}_k \circ s_k = \mathrm{id}_{X_k}$.* □

Now we wish to consider families of topological spaces and to endow their products with suitable topologies. For this purpose let us consider a family $(X_j : j \in J)$ of topological spaces. Let us call a family $(U_j : j \in J)$ of open subsets U_j of X_j a *basic open subfamily*, if there is a finite subset F of J such that $U_j = X_j$ for all $j \in J \setminus F$. Thus for a basic family of open subsets only a finite number of them consists of *proper* subsets.

Lemma 1.51. *The set $\mathfrak{B}$ of all products*

$$U \stackrel{\text{def}}{=} \prod_{j\in J} U_j$$

where $(U_j : j \in J)$ *ranges through the set of all basic open subfamilies of* $(U_j : j \in J)$ *is a basis for a topology on*

$$P \stackrel{\text{def}}{=} \prod_{j \in J} X_j$$

which is closed under finite intersections. The set of all unions of members of $\mathfrak{B}$ *is a topology* $\mathfrak{O}$ *on* P. □

Definition 1.52. The topology $\mathfrak{O}$ on P is called the *product topology* or the *Tychonoff topology*. The topological space $(P, \mathfrak{O})$ is called the *product space* of the family $(X_j : j \in J)$ of topological spaces. □

Proposition 1.53. *Let* $(X_j : j \in J)$ *be a family of topological spaces and let* $P \stackrel{\text{def}}{=} \prod_{j \in J} X_j$ *the product.*

(i) *Each projection* $\mathrm{pr}_k : P \to X_k$ *is continuous and open.*

(ii) *A function* $f : X \to P$ *from a topological space into the product* P *is continuous if and only if for all* $k \in J$ *the functions* $\mathrm{pr}_k \circ f : X \to X_k$ *are continuous.* □

Proposition 1.54. *The product* $\prod_{j \in J} X_j$ *is a Hausdorff space if and only if all factors* X_j *are Hausdorff.* □

Chapter 2
Compactness

We proceed to special properties of topological spaces. From basic analysis we know that compactness is one of these.

Definition 2.1. Let $(X, \mathfrak{O})$ be a topological space. An *open cover* is a subset $\mathcal{C} \subseteq \mathfrak{O}$ such that $X = \bigcup \mathcal{C}$ or a family $(U_j : j \in J)$ of open sets $U_j \in \mathfrak{O}$ such that $X = \bigcup_{j \in J} U_j$. The cover is said to be *finite* if $\mathcal{C}$, respectively, J is finite. A subset $\mathcal{C}' \subseteq \mathcal{C}$ which is itself a cover is called a *subcover*. A subcover of an open cover $(U_j : j \in J)$ is a subfamily $(U_j : j \in K)$, $K \subseteq J$ which is itself a cover.

Definition 2.2. A topological space $(X, \mathfrak{O})$ is said to be *compact* if every open cover has a finite subcover. □

Proposition 2.3. *For a topological space $(X, \mathfrak{O})$ the following statements are equivalent:*
(i) *X is compact.*
(ii) *Every filterbasis of closed subsets has a nonempty intersection.* □

Exercise E2.1. Prove the following assertions:
(i) A closed subspace of a compact space is compact.
(ii) If X is a compact subspace of a Hausdorff space Y, then X is closed in Y.
(iii) Every finite space is compact.
(iv) In the Sierpinski space $\mathbf{2}_s$ the subset $\{1\}$ is compact but not closed.
(v) Every set is compact in the cofinite topology.
(vi) Every compact and discrete space is finite. □

Definition 2.4. An element x of a topological space is said to be an *accumulation point* or a *cluster point* of a sequence $(x_n)_{n \in \mathbb{N}}$ of X if for each $U \in \mathfrak{U}(x)$ the set $\{n \in \mathbb{N} : x_n \in U\}$ is infinite. □

A point x in a topological space is an accumulation point of the sequence $(x_n)_{n \in \mathbb{N}}$ iff for each natural number n and each $U \in \mathfrak{U}(x)$ there is an $m \geq n$ such that $x_m \in U$.

Lemma 2.5. *Assume that $(x_n)_{n \in \mathbb{N}}$ is a sequence in a topological space X. Let $\mathcal{F}$ be the set of all sets*

$$F_n \stackrel{\text{def}}{=} \overline{\{x_m : n \leq m\}} = \overline{\{x_n, x_{n+1}, x_{n+2}, \ldots\}}.$$

Then the following conclusions hold:

(i) A point $x \in X$ is an accumulation point of the sequence iff

$$x \in \overline{\{x_n, x_{n+1}, x_{n+2}, \ldots\}} \text{ for all } n \in \mathbb{N}.$$

(ii) $\mathcal{F}$ is a filter basis, and $\bigcap \mathcal{F}$ is the set of all accumulation points of $(x_n)_{n\in\mathbb{N}}$.□

Proposition 2.6. *Let X be a compact topological space. Then every sequence $(x_n)_{n\in\mathbb{N}}$ in X has at least one accumulation point.*

Proof. For a given sequence $(x_n)_{n\in\mathbb{N}}$ let $\mathcal{F}$ be the filterbasis of compact sets whose member are $F_n \stackrel{\text{def}}{=} \overline{\{x_m : n \leq m\}}$. By (i) and Proposition 2.3 we know $\bigcap \mathcal{F} \neq \emptyset$. In view of Lemma 2.5(ii), this proves the claim. □

The reverse implication is not true in general, however we shall see that it is true for metric spaces. These matters are more involved. We first establish two lemmas which are of independent interest. A topological space in which every sequence has an accumulation point is called *sequentially compact*

Lemma 2.7. (Lebesgue's Lemma) *Let (X, d) be a sequentially compact metric space and let $\mathcal{U}$ be an arbitrary open cover of X. Then there is a number $r > 0$ such that for each $x \in X$ there is a $U \in \mathcal{U}$ such that the open ball $U_r(x)$ of radius r around x is contained in U.*

Proof . Suppose that the Lemma is false; then there is an open cover $\mathcal{U}$ such that for each $m \in \mathbb{N}$ there is an $x_m \in X$ such that $U_{1/m}(x_m)$ is contained in no $U \in \mathcal{U}$. Since X is sequentially compact, the sequence $(x_m)_{m\in\mathbb{N}}$ has at least one accumulation point x. Since $X = \bigcup \mathcal{U}$ there is a $U \in \mathcal{U}$ with $x \in U$. Since U is open, there is an $s > 0$ such that $U_s(x) \subseteq U$. Now let $n \in \mathbb{N}$ be such that $2/n < s$. Then $U_{1/n}(x)$ contains at least one x_m with $m \geq n$. Then $U_{1/m}(x_m) \subseteq U_{2/n}(x) \subseteq U$, and that is a contradiction to the choice of x_m. □

A number $r > 0$ as in the conclusion of Lemma 2.7 is called a *Lebesgue number* of the cover $\mathcal{U}$.

Lemma 2.8. *Let (X, d) be a sequentially compact metric space and let $r > 0$. Then there is a finite subset $F \subseteq X$ such that for each $x \in X$ there is an element $y \in F$ with $d(x, y) < r$. That is, $\{U_r(y) : y \in Y\}$ is a cover of X.*

Proof. Suppose that the Lemma is false. Then there is a number $r > 0$ such that for each finite subset $F \subseteq X$ one finds an $x \in X$ such that $d(x, y) \geq r$ for all $y \in F$. Pick an arbitrary $x_1 \in X$ and assume that we have found elements $x_1, \ldots, x_m$ in such a fashion that $d(x_j, x_k) \geq r$ for all $j \neq k$ in $\{1, \ldots, m\}$. By hypothesis we find an $x_{m+1} \in X$ such that $d(x_j, x_{m+1}) \geq r$ for all $j = 1, \ldots, m$. Recursively we thus find a sequence $x_1, x_2, \ldots,$ in X. Since X is sequentially compact, this sequence has an accumulation point $x \in X$. By the definition of accumulation point the set $\{n \in \mathbb{N} : x_n \in U_{r/2}(x)\}$ is infinite. Thus we find two different indices $h \neq k$ in $\mathbb{N}$

such that $x_j,\, x_k \in U_{r/2}(x)$, whence $d(x_j, x_k) \le d(x_j, x) + d(x, x_k) < r/2 + r/2 = r$. This is a contradiction to the construction of $(x_n)_{n\in\mathbb{N}}$. □

Definition 2.9. A metric space (X, d) is said to be *precompact* or *totally bounded* if for each number $r > 0$ there is a finite subset $F \subseteq X$ such that $X \subseteq \bigcup_{x\in F} U_r(x)$.□

We have observed in Lemma 2.8 that every compact metric space is precompact. The space $\mathbb{Q} \cap [0, 1]$ is precompact with its natural metric but not compact.

Lemma 2.10. *Assume that X is a precompact metric space, and $(x_n)_{n\in\mathbb{N}}$ a sequence in X. Then there is an increasing sequence $(m_n)_{n\in\mathbb{N}}$ of natural numbers such that the equations $y_n \stackrel{\text{def}}{=} x_{m_n}$ define a Cauchy sequence $(y_n)_{n\in\mathbb{N}}$ in X.*

Proof . Assume that we had found a descending sequence $V_1 \supseteq V_2 \supseteq \cdots$ of subsets of X such that the diameters δ_n of V_n exist and converge to 0, and that moreover $\{m \in \mathbb{N} : x_m \in V_n\}$ is infinite for all $n \in \mathbb{N}$. Then we let $m_1 \in \mathbb{N}$ be such that $x_{m_1} \in V_1$ and assume that $m_1 < m_2 < \cdots < m_n$ have been selected so that $x_{m_k} \in V_k$ for $k = 1, \ldots, n$. Since $\{m \in \mathbb{N} : x_m \in V_{n+1}\}$ is infinite we find an $m_{n+1} > m_n$ such that $x_{m_{n+1}} \in V_{n+1}$. We set $y_n = x_{m_n}$, notice that $y_n \in V_n$ and show that $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since $V_{n+k} \subseteq V_n$ we have $d(y_n, y_{n+k}) \le \delta_n$. Thus for any $\varepsilon > 0$ we find an $N \in \mathbb{N}$ such that $n > N$ implies $\delta_n < \varepsilon$ and thus $d(y_n, y_{n+k}) < \varepsilon$ for all $k \in \mathbb{N}$. So $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

It therefore remains to construct the sets V_n. For each natural number k there is a finite number $F_k \subseteq X$ such that

$$(*)\qquad X = \bigcup_{x\in F_k} U_{1/k}(x).$$

(∗∗) If $(p_n)_{n\in\mathbb{N}}$ is a sequence in a M such that for finitely many subsets M_k, the set $\{m \in \mathbb{N} : p_m \in M_1 \cup \cdots \cup M_n\}$ is infinite, then there is at least one index k such that $\{m \in \mathbb{N} : p_m \in M_k\}$ is infinite.

From (∗) with $k = 1$ and (∗∗) we find $z_1 \in F_1$ such that $\{m \in \mathbb{N} : x_m \in U_1(z_1)\}$ is infinite. Set $V_1 = U_1(z_1)$. Assume that V_k, $k = 1, \ldots, n$ have been found such that $\{m \in \mathbb{N} : x_m \in V_n\}$ is infinite. Now $V_n \subseteq X = \bigcup_{z\in F_{n+1}} U_{1/n+1}(z)$ by (∗). Apply (∗∗) to $V_n = \bigcup_{z\in F_{n+1}} \big(V_n \cap U_{1/n+1}(z)\big)$ and find a $z_{n+1} \in F_{n+1}$ such that $\{m \in \mathbb{N} : x_m \in V_n \cap U_{1/n+1}(z_{n+1})\}$ is infinite. Set $V_{n+1} = V_n \cap U_{1/n+1}(z_{n+1})$. This completes the recursive construction of V_n with $\delta_n \le 2/n$ and thereby completes the proof of the lemma □

Recall that a metric space is said to be *complete*, if every Cauchy-sequence converges.

Theorem 2.11. *For a metric space (X, d) with the metric topology $\mathcal{O}$, the following statements are equivalent:*

(i) *$(X, \mathfrak{O})$ is compact.*

(ii) $(X, \mathfrak{O})$ *is sequentially compact.*
(iii) (X, d) *is precompact and complete.*

Proof . (i)⇒(ii): This was shown in Proposition 2.6.

(ii)⇒(iii): A sequentially compact metric space is precompact by Lemma 2.8. We verify completeness: Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. Since X is sequentially compact by (ii), this sequence has a cluster point x. We claim that $x = \lim_{n\to\infty} x_n$. Indeed let $\varepsilon > 0$. Since $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, there is an $N \in \mathbb{N}$ such that $m, n > N$ implies $d(x_m, x_n) < \varepsilon/2$. Since x is an accumulation point, there is an $m > N$ such that $d(x, x_m) < \varepsilon/2$. Thus for all $n > N$ we have $d(x, x_n) < d(x, x_m) + d(x_m, x_n) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. This proves the assertion.

(iii)⇒(ii): Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. By Lemma 2.10 there are natural numbers $m_1 < m_2 < \cdots$ such that $(x_{m_n})_{n\in\mathbb{N}}$ is a Cauchy sequence. Since (X, d) is complete, this sequence has a limit x. If $\varepsilon > 0$ then there is an N such that $n > N$ implies $x_{m_n} \in U_\varepsilon(x)$. Since $m_n < m_{n+1}$ we conclude that $\{m \in \mathbb{N} : x_m \in U_\varepsilon(x)\}$ is infinite. Hence x is an accumulation point of $(x_n)_{n\in\mathbb{N}}$.

We have shown that (i) and (iii) are equivalent.

(ii)⇒(i): Let $\mathcal{U}$ be an open cover. Let $r > 0$ be a Lebesgue number for this cover according to Lemma 2.7. Since X is precompact by what we know we find a finite set $F \subseteq X$ such that $X = \bigcup_{x\in X} U_r(x)$. For each $x \in F$ we find an $U_x \in \mathcal{U}$ such that $U_r(x) \subseteq U_x$ by Lemma 2.7. Then $X = \bigcup_{x\in F} U_r(x) \subseteq \bigcup_{x\in F} U_x \subseteq X$ and thus $\{U_x : x \in X\}$ is a finite subcover of $\mathcal{U}$. □

Theorem 2.11 is remarkable in so far as the three statements (i), (ii), and (iii) have very little to do with each other on the surface.

Theorem 2.11 links our general concept of compactness with elementary analysis where compactness is defined as sequential compactness.

Exercise E2.2. (i) Show that a compact subspace X of a metric space is always bounded, i.e. that there is a number R such that $d(x, y) \le R$ for all $x, y \in X$

(ii) Give an example of an unbounded metric on $\mathbb{R}$ which is compatible with the natural topology.

(iii) Show that a closed subset X of $\mathbb{R}$ which is bounded in the sense that it is contained in an interval $[a, b]$ is compact.

(iv) Prove the following result from First Year Analysis. (Theorem of Bolzano-Weierstrass).

A subset of $\mathbb{R}^n$ *is compact if an only if it is closed and bounded with respect to the norm given by* $\|(x_1, \dots, x_n)\|_\infty = \max\{|x_1|, \dots, |x_n|\}$.

(v) Show that the Theorem of Bolzano-Weierstrass holds for any norm on $\mathbb{R}^n$.□

Exercise E2.3. Use Theorem 2.11 for proving that a subset of $\mathbb{R}^n$ is compact if and only if it is closed and norm bounded. □

In this spirit, Theorem 2.11 is the “right” generalisation of the Bolzano-Weierstrass Theorem.

There are some central results concerning compact spaces which involve the Axiom of Choice. Therefore we must have an Interlude on the Axiom of Choice.

Chapter AC
An Interlude on the Axiom of Choice.

We need some concepts from order theory.

Definition AC.1. A poset $(X, \leq)$ as well as the partial order $\leq$ are said to be *inductive*, if each totally ordered subset (that is, a *chain* or a *tower* $T \subseteq X$ has an upper bound b in X (i.e. $t \leq b$ for all $b \in T$).

Example AC.2. Let V be a vector space over any field K and let $X \subseteq \mathfrak{P}(X)$ be the set of all linearly independent subsets. On X we consider the partial order $\subseteq$. If T is a totally ordered set of linearly independent subsets of V then $b \stackrel{\text{def}}{=} \bigcup T$ is a linearly independent set due to the fact, that linear independence of a set F of vectors is a property involving only finite subsets of F. Also, b contains all members of T. Hence $(X, \subseteq)$ is inductive. □

Definition AC.3. A binary relation $\leq$ is called a *well-order*, and $(X, \leq)$ is called a *well-ordered set* if $\leq$ is a total order (i.e. a partial order for which every pair of elements is comparable) such that every nonempty subset has a smallest element.□

Example AC.4. Every finite totally ordered set is well-ordered. The set $\mathbb{N}$ of natural numbrs with its natural order is well-ordered. The set $\mathbb{N} \cup \{n - \frac{1}{m} : n, m \in \mathbb{N}\}$ is well ordered with the natural order. □

We begin by formulating a couple of statement concerning sets, posets, topological spaces.

AC: The Axiom of Choice. *For every family of nonempty sets $(X_j : j \in J)$ the product $\prod_{j \in J} X_j$ is not empty.* □

ZL: Zorn's Lemma. *Every inductive set has maximal elements.* □

WOP: The Well-Ordering Principle. *Every set can be well-ordered.* □

TPT: The Tychonov Product Theorem. *Each product of compact spaces is compact.* □

The point of this interlude is to prove that these four statements are equivalent. Let us begin with a couple of simple implications

TPT⇒AC: Let $(X_j : j \in J)$. Then $Y_j = X_j \cup \{X_j\}$ is a set and $X_j \notin X_j$. The product $\prod_{j\in J} Y_j$ is not empty because it contains the element $(X_j)_{j\in J}$. Let $\mathrm{Fin}(J)$ denote the set of finite subsets of J. For each finite subset $F \in \mathrm{Fin}(J)$ we set $S_F = \prod_{j\in J} Z_j$ where

$$Z_j = \begin{cases} X_j & \text{if } j \in F, \\ Y_j & \text{if } j \in J \setminus F. \end{cases}$$

Since $X_j \neq \emptyset$ for all $j \in J$ we have $\prod_{j\in F} X_j \neq \emptyset$ since we accept the "finite" Axiom of Choice. Now we topologize Y_j by declaring $\mathfrak{O}_j = \{\emptyset, \{X_i\}, X_j\}$ to be its topology. Then X_j is a closed subset of Y_j, and Y_j is compact. Now $\mathcal{S} = \{S_F : F \in \mathrm{Fin}(J)\}$ is a filter basis of closed subsets of $P = \prod_{j\in J} Y_j$ and P is compact by TPT. Thus $\prod_{j\in J} X_j = \bigcap_{F\in\mathrm{Fin}(J)} S_F \neq \emptyset$ by 2.3. □

ZL⇒WOP: Let X be a set, pick $x_0 \in X$ and consider the set $(\mathcal{X}, \preceq)$ be the set of all well ordered subsets $(A, \leq_A)$ with x_0 as minimal element, where
$(A, \leq_A) \preceq (B, \leq_B)$ if
(1) $A \subseteq B$,
(2) $\leq_B |(A \times A) = \leq_A$, and
(3) A is an initial segment of B.
We claim that the poset $(\mathcal{X}, \preceq)$ is inductive: Let $\mathcal{T}$ be a totally ordered subset. Then We form the subset $T = \bigcup \mathcal{T}$ and define a binary relation $\leq$ on T as follows: Let $x, y \in T$. Then there is an $S \in \mathcal{X}$ containing x and y. Since S is totally ordered we have $x \leq_S y$ or $y \leq_S x$, and we set $x \leq_T y$ in the first case and $y \leq_T y$ in the second. It is readily seen that this definition is independent of the choice of S. It is seen that $S \in \mathcal{T}$ implies that S is an initial segment of T. If $\emptyset \neq A \subseteq T$, then there is an $a \in A$ and then $a \in \bigcup \mathcal{T}$; thus $a \in S$ for some $S \in \mathcal{T}$. Since S is well-ordered, $m = \min(A \cap S)$ exists. Since S is an initial segment of T we conclude $m = \min A$. Thus $(\mathcal{X}, \preceq)$ is inductive. By Zorn's Lemma ZL we find a maximal element $(M, \leq_M)$. We claim $M = X$. Suppose not. Then there is an $x \in X \setminus M$. We extend $\leq_M$ to $M' = M \cup \{x\}$ by making x bigger than all elements of M. Then $(M', \leq_{M'})$ is a well-ordered set with M as an initial segment. This contradicts the maximality of $(M, \leq_M)$. Thus $X = M$ and $(X, \leq_X)$ is well-ordered. □

WOP⇒AC: Let $(X_j : j \in J)$ be a family of nonempty sets; set $X \stackrel{\text{def}}{=} \bigcup_{j\in J} X_j$. Let $\leq$ be a well-order of X. Then $(\min X_j)_{j\in J} \in \prod_{j\in J} X_j$. □

We still have to show AC⇒ZL and ZL⇒TPT. First we shall show that AC implies ZL; then we shall conclude the interlude and prove ZL⇒TPT in the course of our discussion of compact spaces.

For a proof of AC⇒ZL we prove a Lemma of independent interest in oder theory.

Theorem AC.5. (Tarski's Fixed Point Theorem) *Let $(X, \leq)$ be a poset such that every totally ordered subset has a least upper bound. Assume that the function*

$f{:}\,X \to X$ satisfies $(\forall x \in X)\, x \leq f(x)$. Then f has a fixed point, that is, there is an $x_0 \in X$ such that $f(x_0) = x_0$.

Let us note the parallel to Banach's Fixed Point Theorem: *Let $(X, \dot{})$ be a metric space such that every Cauchy sequence converges. Assume that the function $f{:}\,X \to X$ satisfies $(\exists \lambda \in [0,1[)(\forall x, y \in X) d\big(f(x), f(y)\big) \leq \lambda d(x,y)$. Then f has a fixed point.*

That is, X satisfies some completeness hypothesis and f satisfies some contraction hypothesis. But the proofs proceed quite differently.

Yet before we prove Tarski's Fixed Point Theorem we shall show how with its aid one can use AC to prove ZL. Thus let $(X, \leq)$ be a inductive poset. Let $\mathcal{X}$ denote the set of all totally ordered subsets of X; then $(\mathcal{X}, \subseteq)$ is a poset; in which every totally ordered subset $\mathcal{T}$ has a least upper bound, namely, $T \stackrel{\rm def}{=} \bigcup \mathcal{T}$. We claim that $\mathcal{X}$ has maximal elements. If not, then for any $Y \in \mathcal{X}$ the set $\mathcal{M}_Y = \{Z \in \mathcal{X} : Y \subset Z, Y \neq Z\}$ is not empty. By the Axiom of Choice there is an element $f \in \prod_{Y \in \mathcal{X}} \mathcal{M}_Y$, that is f is a selfmap of $\mathcal{X}$ such that $Y \subset f(Y)$, $Y \neq f(Y)$, and this contradicts the Tarski Fixed Point Theorem AC.5. Thus we find a maximal chain M. Since X is inductive, M has an upper bound b. But now b is a maximal element of X because otherwise there would have to be an element $c \in X$ such that $b < c$, yielding a chain $M \cup \{c\}$ properly containing M in contradiction with the maximality of M. Therefore X has maximal elements and this is what Zorn's Lemma asserts.

Let us also notice that with the aid of ZL a proof of the Tarski Fixed Point Theorem is trivial: If X is an inductive set, then by ZL it contains a maximal element c. If now f is a self map of X with $x \leq f(x)$, then $c \leq f(c)$ implies $c = f(c)$ by the maximality of c, and thus c is a fixed point of f.

The entire point now is to prove the Tarski Fixed Point Theorem without AC or ZL.

For a proof of Tarski's Fixed Point Theorem let $(X, \leq)$ be a poset such that $\sup C$ exists for each chain $C \subseteq X$. Let us call a subset $A \subseteq X$ *closed* if for each chain $C \subseteq A$ we have $\sup C \in A$ and $f(A) \subseteq A$. The empty set is a chain and thus the set X has a smallest element $\min X = \sup \emptyset$. Moreover, if A is closed, then $\emptyset$ is a chain contained in A, and thus $\min X = \sup \emptyset \in A$.

Let $X' = \bigcap \{A \subseteq X : A \text{ is a closed subset of } X\}$. Then X' is the smallests closed subset of X. It suffices to prove the Fixed Point Theorem for X' and $f|X'$. We shall therefore assume from now on that X has no proper closed subsets. (Notice that $\emptyset$ is not closed.)

Let us call an element $x \in X$ a *roof* if $y < x$ always implies $f(y) \leq x$. We claim that any roof *decomposes* X in the sense that either $y \leq x$ or $f(x) \leq y$ for any $y \in X$. Indeed let $Z = \{y \in X : y \leq x \text{ or } f(x) \leq y\}$. We claim that Z is a closed set and leave the proof as an exercise. Since X has no proper closed subsets, this implies $Z = X$.

Next we claim that every element in X is a roof. For a proof set $D = \{y \in X : y \text{ is a roof}\}$. We claim that D is a closed subset and leave the proof as an exercise. Since X has no proper closed subsets, this proves $X = D$.

At this point it follows that X is a chain. Let $x_0 = \max X$. Then $f(x_0) \le x_0$. By hypothesis on f we have $x_0 \le f(x_0)$. Thus $f(x_0) = x_0$ and thus x_0 is the desired fixed point of f. □

Exercise EAC.1. Show that the set Z in the proof of Tarski's Fixed Point Theorem is closed.
[Hint. Let C be a chain in Z and set $z = \sup C$. Show $z \in Z$: Either $f(x) \le y$ for a $y \in C$ or ...). Secondly let $y \in Z$. Show $f(y) \in Z$: There are three cases $y \le x$ ...]

Exercise EAC.2. Show that the set D in the proof of Tarski's Fixed Point Theorem is closed.
[Hint. Use the fact that every roof decomposes X to show that D is f-invariant. If T is a totally ordered subset of D let $t = \sup T$; we must show that t is a roof; assume $y < t$. Show that there is an $s \in T$ such that $y < s$: Proof by contradition using that roofs decompose. Since s is a roof, $f(y) \le s \le t$.]

Lemma AC.6. *Let X be a set. The set Filt(X) of all filters on X, as a subset of $\mathfrak{P}\big(\mathfrak{P}(X)\big)$ is an inductive poset.* □

Definition AC.7. Any maximal element in $\mathrm{Filt}(X)$ is called an *ultrafilter.*

Proposition AC.8. (AC) *Every filter on a set is contained in an ultrafilter. Every filter basis is contained in an ultrafilter.*

Proof . Let $\mathcal{F}$ be a filter. The set of all filters containing $\mathcal{F}$ is inductive an thus by Zorn's Lemma contains maximal elements. If $\mathcal{B}$ is a filter basis, then the set $\mathcal{F}$ of all supersets of members of $\mathcal{B}$ is a filter which is contained in an ultrafilter by the preceding. □

The preceding proposition is also called the *Ultrafilter Theorem.*

Exercise EAC.3. Prove the following proposition:

Let $f\colon X \to Y$ be a surjective function and $\mathcal{U}$ an ultrafilter in X. Then $f(\mathcal{U})\}$ is an ultrafilter. □

Definition AC.9. A filter basis is called an *ultrafilter basis* if the filter of all of of its supersets is an ultrafilter.

Proposition AC.10. *The following statements are equivalent for a filter $\mathcal{F}$ on a set X:*

(i) *$\mathcal{F}$ is an ultrafilter.*
(ii) *Whenever $X = A \cup B$ and $A \cap B = \emptyset$, then $A \in \mathcal{F}$ or $B \in \mathcal{F}$.*

The following statements are equivalent for a filter basis $\mathcal{B}$ on a set X:

(I) *$\mathcal{B}$ is an ultrafilter basis.*

(II) *Whenever* $X = A \cup B$ *and* $A \cap B = \emptyset$*, then there is a* $C \in \mathcal{B}$ *such that* $C \subseteq A$ *or* $C \subseteq B$*.*

Proof. (i)⇒(ii): Assume (i) and $X = A \cup B$ and $A \cap B = \emptyset$. If the assertion fails, then $\mathcal{F}_A = \{S \subseteq X : (\exists F \in \mathcal{F})\, A \cap F \subseteq S\}$ and $\mathcal{F}_B = \{S \subseteq X : (\exists F \in \mathcal{F})\, B \cap F \subseteq S\}$ are two filters such that $\mathcal{F} \subseteq \mathcal{F}_A \cap \mathcal{F}_B$. Since $\mathcal{F}$ is an ultrafilter, $\mathcal{F}_A = \mathcal{F} = \mathcal{F}_B$;, but then $A \in \mathcal{F}$ and $B \in \mathcal{F}$ whence $\emptyset = A \cap B \in \mathcal{F}$, a contradiction.

(ii)⇒(i). Assume (ii) and consider $\mathcal{F} \subseteq \mathcal{G}$ for some filter $\mathcal{G}$ on X. Suppose that $\mathcal{G}$ is properly larger than $\mathcal{F}$; then there is a set $A \in \mathcal{G} \setminus \mathcal{F}$. Set $B = X \setminus A$. From (ii) we conclude $B \in \mathcal{F}$. But then $B \in \mathcal{G}$ and thus $\emptyset = A \cap B \in \mathcal{G}$, a contradiction.

(I)⇔(II). Apply the preceding to $\mathcal{F} = \langle \mathcal{B} \rangle$, the filter generated by $\mathcal{B}$. □

Some filter arithmetic

Definition AC.11. Let $f: X \to Y$ be a funtion and $\mathcal{F}$ and $\mathcal{G}$ be filter bases on X and Y, respectively. Set

$$f(\mathcal{F}) = \{f(F) : F \in \mathcal{F}\} \text{ and } f^{-1}(\mathcal{G}) = \{f^{-1}(G) : G \in \mathcal{G}\}.$$ □

Proposition AC.12. (i) *$f(\mathcal{F})$ is a filterbasis, and if $\mathcal{F}$ is a filter and f is surjective, then $f(\mathcal{F})$ is a filter as well.*

(ii) *$f^{-1}(\mathcal{G})$ is a filter basis, and if $\mathcal{G}$ is a filter and f is injective, then $f^{-1}(\mathcal{G})$ is a filter.*

Proof. (i) Let $F_1, F_2 \in \mathcal{F}$; then $\mathcal{F}$ contains an F such that $F \subseteq F_1 \cap F_2$ since $\mathcal{F}$ is a filter basis. Then $f(F) \subseteq f(F_1 \cap F_2) \subseteq f(F_1) \cap f(F_2)$. Thus $f(\mathcal{F})$ is a filter basis.

Now assume that f is surjective and that $\mathcal{F}$ is a filter. Let $F \in \mathcal{F}$ and $f(F) \subseteq B$. Then $F \subseteq f^{-1}(B)$, and since $\mathcal{F}$ is a filter, $f^{-1}(B) \in \mathcal{F}$. Since f is surjective, $B = f\big(f^{-1}(B)\big) \in f(\mathcal{F})$.

(ii) Let $G_1, G_2 \in \mathcal{G}$. Then there is a $G \in \mathcal{G}$ with $G \subseteq G_1 \cap G_2$. Then $f^{-1}(G) \subseteq f^{-1}(G_1 \cap G_2) = f^{-1}(G_1) \cap f^{-1}(G_2)$. Thus $f^{-1}(\mathcal{G})$ is a filter basis.

Now assume that f is injective and that $\mathcal{G}$ is a filter. Let $G \in \mathcal{G}$ and $f^{-1}(G) \subseteq A$. Then $G \subseteq f(A) \cup \big(Y \setminus f(X)\big)$ since f is injective and $f(A) \cup \big(Y \setminus f(X)\big) \in \mathcal{G}$ since $\mathcal{G}$ is a filter. Now $A = f^{-1}\big[f(A) \cup \big(Y \setminus f(X)\big)\big]$ is in $f^{-1}(\mathcal{G})$. □

Proposition AC.13. *Let $\mathcal{U}$ be an ultrafilter basis in X and $f: X \to Y$ any function. Then $f(\mathcal{U})$ is an ultrafilter basis.*

In particular, if f is surjective and $\mathcal{U}$ is an ultrafilter, then $f(\mathcal{U})$ is an ultrafilter as well.

Proof . We prove this by using the equivalence of (I) and (II) in Proposition AC.8. So let $Y = A \cup B$, $A \cap B = \emptyset$. Then $X = f^{-1}(A) \cup f^{-1}(B)$ and $f^{-1}(A) \cap f^{-1}(B) = \emptyset$. Now, since $\mathcal{U}$ is an ultrafilter basis, by (I)$\Rightarrow$(II) in Proposition AC.8, there is a $U \in \mathcal{U}$ such that either $U \subseteq f^{-1}(A)$ or $U \subseteq f^{-1}(B)$. In the first case, $f(U) \subseteq A$, in the second case, $f(U) \subseteq B$. Thus by (II)$\Rightarrow$(I) in Proposition AC.8 we see that $f(\mathcal{U})$ is an ultrafilter basis.

Finally, AC.10(i) proves the remainder. □

Chapter 2
Compactness Continued

Now we shall show that Zorn's Lemma implies Tychonov's Product Theorem. We need the concept of convergence for filters.

Definition 2.12. We say that a filter $\mathcal{F}$ on X *converges* to $x \in X$ if $\mathfrak{U}(x) \subseteq \mathcal{F}$. A filter basis $\mathcal{B}$ converges to x if the filter generated by $\mathcal{B}$ converges to x. □

It is immediate that a filter basis $\mathcal{B}$ converges to x if for each neighborhood of x there is a member $B \in \mathcal{B}$ such that $B \subseteq U$.

A sequence $(x_n)_{n\in\mathbb{N}}$ is said to converge to x if for every neighborhood U of x there is an $N \in \mathbb{N}$ such that $x_n \in U$ for all $n > N$.

Exercise E2.4. Show that a sequence $(x_n)_{n\in\mathbb{N}}$ converges to x iff the filter basis $\mathcal{B} = \big\{\{x_n, x_{n+1}, \ldots\} : n \in \mathbb{N}\big\}$ converges to x.

Proposition 2.13. (AC) *For a topological space* $(X, \mathfrak{O})$ *the following statements are equivalent:*
(i) X *is compact.*
(ii) *Every ultrafilter converges.*

Proof . (i)⇒(ii): Let $\mathcal{U}$ be an ultrafilter. By (i) there is an x such that $x \in \overline{V}$ for all $V \in \mathcal{U}$ (see 2.3). This means that $U \cap V \neq \emptyset$ for all $U \in \mathfrak{U}(x)$ and all $V \in \mathcal{U}$. Then $\mathcal{F} \stackrel{\text{def}}{=} \{F : U \cap V \subseteq F, U \in \mathfrak{U}(x), V \in \mathcal{U}\}$ is a filter containing $\mathcal{U}$. Since $\mathcal{U}$ is maximal among all filters, we have $\mathcal{F} = \mathcal{U}$ and thus $\mathcal{U} = \mathcal{F} \subseteq \mathfrak{U}(x)$, i.e., $\mathcal{U}$ converges to x.

(ii)⇒(i). (AC) Let $\mathcal{B}$ be a filter basis of closed sets; we must show that $\bigcap \mathcal{B} \neq \emptyset$. By the Ultrafilter Theorem, the filter basis $\mathcal{B}$ is contained in an ultrafilter $\mathcal{U}$ which by (ii) converges to some element x. Let U be a neighborhood of x. Then there is a $V \in \mathcal{U}$ such that $V \subseteq U$. If $B \in \mathcal{B}$ then $\mathcal{B} \subseteq \mathcal{U}$ implies $B \in \mathcal{U}$ and thus $B \cap V \in \mathcal{U}$; in particular, $\emptyset \neq B \cap V \subseteq B \cap U$. Therefore $x \in B$ for all $B \in \mathcal{B}$. □

The next theorem will prove that AC⇒TPT.

Theorem 2.14. (AC) *The product of any family of compact spaces is compact.*

Proof . Let $(X_j : j \in J)$ be a family of compact spaces. Let $P \stackrel{\text{def}}{=} \prod_{j\in J} X_j$. If one X_j is empty, then $P = \emptyset$ and thus P is compact. Assume now that $X_j \neq \emptyset$.

We prove compactness of P by considering an ultrafilter $\mathcal{U}$ and showing that it converges.

For each $j \in J$ the projection $\mathrm{pr}_j(\mathcal{U})$ is an ultrafilter. Let $L_j \subseteq X_j$ be the set of points to which it converges. Since X_j is compact, $L_j \neq \emptyset$. By the Axiom of Choice $L \stackrel{\mathrm{def}}{=} \prod_{j\in J} L_j \neq \emptyset$. Let $(x_j)_{j\in J} \in L$.

Now let U be a neighborhood of $x \stackrel{\mathrm{def}}{=} (x_j)_{j\in J}$. We may assume that U is a basic neighborhood of the form $U = \prod_{j\in J} U_j$, where $U_j = X_j$ for all $j \in J \setminus F$ for some finite subset of J. Then we find a member $M \in \mathcal{U}$ such that $\mathrm{pr}_j(M) \subseteq U_j$ for $j \in F$ and then for all $j \in J$. Accordingly $M \subseteq \bigcap_{j\in F} \mathrm{pr}_j^{-1}(U_j) = \prod_{j\in J} U_j$. □

Notice that we have used the Axiom of Choice by applying the Ultrafilter Theorem **and** by selecting $(x_j)_{j\in J}$.

Exercise E2.5. Prove:

In a Hausdorff space, a filter $\mathcal{F}$ converges to at most one point..

Thus in a Hausdorff space a converging filter converges to exactly one point x, called the *limit point* and written $x = \lim \mathcal{F}$.

Corollary 2.15. *The product of a family of compact Hausdorff spaces is a compact Hausdorff space.* □

The Ultrafilter Theorem suffices for a proof of this theorem.

Example. (Cubes) Let $\mathbb{I}$ denote the unit interval $[0, 1]$ and $\mathbb{D}$ the complex unit disc. For each set J the products $\mathbb{I}^J$ and $\mathbb{D}^J$ are compact spaces. □

We have made good use of the concept of a filter and its convergence. In passing we mention the concept of a Cauchy-filter on a metric space. Let us first recall that in a metric space (X, d) a subset $B \subseteq X$ is *bounded* if there is a number C such that $d(b, c) \leq C$ for all $b, c \in B$. For a bounded subset B, the number $\sup\{d(b, c) : b, c \in B\}$ exists and is called the *diameter* of B. When we speak of the diameter of a subset, we imply that we assume that the subset is bounded.

Definition 2.16. A filter $\mathcal{F}$ on a metric space (X, d) is called a *Cauchy-filter* if for each $\varepsilon > 0$ it contains a set of diameter less than ε. A filter basis is a *Cauchy-filter basis* if it contains a set of diameter less than ε. □

Clearly, a filter basis is a Cauchy-filter basis if and only if the filter of all super sets of its members is a Cauchy-filter.

Exercise E2.6. Show that a sequence $(x_n)_{n\in\mathbb{N}}$ is a a Cauchy-sequence iff the filter basis of all $\{x_n, x_{n+1}, \ldots\}$, $n \in \mathbb{N}$ is a Cauchy filter basis.

Lemma 2.17. (i) *Let $\mathcal{F}$ be a Cauchy-filter in metric space. Then there is a countable Cauchy-filter basis $\mathcal{C}$, $C_1 \supseteq C_2 \supseteq \cdots$ such that the diameter of C_n is less than $\frac{1}{n}$ and $\mathcal{C} \subseteq \mathcal{F}$.*

(ii) If $\mathcal{F} \subseteq \mathcal{G}$ are two filters in a metric space such that $\mathcal{F}$ is a Cauchy-filter and $\mathcal{G}$ converges to x then $\mathcal{F}$ converges to x. □

If $\mathcal{C}$ converges, and thus the filter $\langle\mathcal{C}\rangle$ of all supersets of the C_n converges, that is, contains some neighborhood filter $\mathfrak{U}(x)$, then the given filter $\mathcal{F}$ converges. If now we select in each set C_n an element c_n, then $(c_n)_{n\in N}$ is a Cauchy-sequence. Then $\big\{\{c_n, c_{n+1}, \ldots\} : n \in \mathbb{N}\big\}$ is a Cauchy-filter basis $\mathcal{B}$ which converges iff $(c_n)_{n\in\mathbb{N}}$ converges. Moreover, $\langle\mathcal{C}\rangle \subseteq \langle\mathcal{B}\rangle$.

Proposition 2.18. *A metric space (X, d) is complete if and only if every Cauchy-filter converges.* □

For a given ε, a precompact metric space is covered by finite number of open ε-balls. Thus any ultrafilter contains one of them. Hence every ultrafilter on a precompact spaces is a Cauchy-filter. Thus on a complete precompact metric space every ulterfilter converges. This is an alternative proof that a metric space is compact iff it is complete and precompact. This approach has the potential of being generalized beyond the metric situation.

Exercise E2.7. Fill in the details of this argument.

Compact spaces and continuous functions

Proposition 2.19. *Let $f\colon X \to Y$ be a continuous surjective function of topological spaces and assume that X is compact. Then Y is compact. Let*

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle q_f}\big\downarrow & & \big\uparrow{\scriptstyle \mathrm{id}_Y} \\ X/R_f & \xrightarrow[\quad f' \quad]{} & Y \end{array}$$

be the canonial decomposition of f. If Y is a Hausdorff space, then f' is a homeomorphism. □

In short: Continuous images of compact spaces are compact, and as easy consequence we know that a bijective continuous map between Hausdorff spaces is a homeomorphism.

Corollary 2.20. *If $\mathfrak{O} \subseteq \mathfrak{O}'$ are Hausdorff topologies on a set and $\mathfrak{O}'$ is compact, then $\mathfrak{O} = O'$.* □

Among Hausdorff topologies, compact ones are minimal.

A totally ordered set is defined to be *order complete* iff every subset has a least upper bound.

If $A \subseteq X$ and $(X, \leq)$ is order complete, let L be the set of lower bounds of A. Then $\sup L = \inf A$.

If $A \subseteq X$ is closed w.r.t. the order topology, then $\sup A = \max A$ and $\inf A = \min A$.

Exercise E2.8. Prove these claims.

Lemma 2.21. *A totally ordered space $(X, \leq)$ is compact w.r.t. the order topology if and only if X is order complete.*

Proof . If $X = \emptyset$, then X is complete by default. Assume that X is compact and $A \subseteq X$. Show that $\max \overline{A} = \sup A$

Now assume that X is order complete. $\mathcal{B}$ be a filter basis of closed subsets. Let $M = \{\min B : B \in \mathcal{B}\}$. Show that $\sup M \in \bigcap \mathcal{B}$. □

Exercise E2.9. Fill in the details of the proof of Lemma 2.21.

Proposition 2.22. (Theorem of the Maximum) *Let $f \colon X \to Y$ be a continuous function from a compact space into a totally ordered space. Then f attains its maximum and it s minimum, i.e. there are elements $x, y \in X$ such that $f(x) = \max f(X)$ and $f(y) = \min f(X)$.* □

Uniform Continuity, Uniform Convergence, Equicontinuity

Compactness has substantial applications in Analysis; we sample some of them

Definition 2.23. A function $f \colon X \to Y$ between metric spaces is called *uniformly continuous*, if

$$(1) \qquad (\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in X)\, f(U_\delta(x)) \subseteq U_\varepsilon\big(f(x)\big). \qquad \square$$

Recall that f if continuous if

$$(2) \qquad (\forall \varepsilon > 0)(\forall x \in X)(\exists \delta > 0)\, f(U_\delta(x)) \subseteq U_\varepsilon\big(f(x)\big).$$

Proposition 2.24. *A continuous function $f \colon X \to Y$ from a compact metric space into a metric space is uniformly continuous.* □

[For each $\varepsilon > 0$ and each $x \in X$ and pick $d > 0$ so that $f\big(U_d(x)\big) \subseteq U_{\varepsilon/2}\big(f(x)\big)$. Find a finite subcover $\mathcal{C}$ of $\{U_d(x) : x \in X\}$ and let δ be a Lebesgue number of $\mathcal{C}$.]

Definition 2.25. Let X be a set and Y a metric space. Define *B(X,Y)* to be the set of all bounded functions, i.e. functions $f\colon X \to Y$ such that $\{d\big(f(x), f(x')\big) : x, x' \in X\}$ is a bounded subset of $\mathbb{R}$.

If X and Y are topological spaces, then *C(X,Y)* denotes the set of all continuous functions from X to Y. □

Proposition 2.25. (i) *Let X be a set and Y a metric space. Then $B(X,Y)$ is a metric space with respect to the metric $d(f,g) = \sup\{d\big(f(x), g(x)\big) : x \in X\}$*

(ii) If X is a compact topological space and Y is a metric space, then $C(X,Y) \subseteq B(X,Y)$, and $C(X,Y)$ is a closed subset. □

We say that $B(X,Y)$ carries the (metric) topology $\mathfrak{O}_{\mathrm{u}}$ of *uniform convergence.* The topology induced on $B(X,Y)$ by the product topology of Y^X is called the topology of *pointwise convergence*, denoted $\mathfrak{O}_{\mathrm{p}}$. Clearly $\mathfrak{O}_{\mathrm{p}} \subseteq \mathfrak{O}_{\mathrm{u}}$.

Let $\mathbf{F} \subseteq C(X,Y)$ be a set of functions. Can we give conditions such that $\mathbf{F}|\mathfrak{O}_{\mathrm{p}} = \mathbf{F}|\mathfrak{O}_{\mathrm{u}}$?

Definition 2.26. A set $\mathbf{F}$ of functions $X \to Y$ between topological spaces is said to be *equicontinuous* if

$$(3)\qquad (\forall x \in X)\big(\forall V \in \mathfrak{U}\big(f(x)\big)\big)\big(\exists U_{x,V} \in \mathfrak{U}(x)\big)(\forall f \in \mathbf{F})\, f(U_{x,V}) \subseteq V. \qquad \square$$

The statement that all functions in $\mathbf{F}$ are continuous reads as follows:

$$(4)\qquad (\forall f \in \mathbf{F})(\forall x \in X)\big(\forall V \in \mathfrak{U}\big(f(x)\big)\big)\big(\exists U_{f,x,V} \in \mathfrak{U}(x)\big)\, f(U_{f,x,V}) \subseteq V.$$

Proposition 2.27. *Let X be a compact space, Y a metric space and $\mathbf{F}$ an equicontinuous set of functions $X \to Y$. Then $\mathbf{F}|\mathfrak{O}_{\mathrm{p}} = \mathbf{F}|\mathfrak{O}_{\mathrm{u}}$, that is the topologies of pointwise and of uniform convergence agree on $\mathbf{F}$.*

Proof. Let $f \in \mathbf{F}$, and $\varepsilon > 0$. We must find a $\delta > 0$ and $E \subseteq X$ finite such that

$$(5)\qquad (\forall g \in \mathbf{F})\big[(\forall e \in E)\, d\big(f(e), g(e)\big) < \delta\big] \Rightarrow (\forall x \in X)\, d\big(f(x), g(x)\big) < \varepsilon.$$

For each $x \in X$ we find an open neighborhood V_x of x in X such that $(\forall f \in \mathbf{F})\, f(V_x) \subseteq U_{\varepsilon/3}\big(f(x)\big)$ Since X is compact, there is a finite set $E \subseteq X$ such that $X = \bigcup_{e \in E} V_e$. Set $\delta = \varepsilon/3$ and assume that $g \in \mathbf{F}$ satisfies $d\big(f(e), g(e)\big) < \delta$ for $e \in E$. Now let $x \in X$ arbitrary. Then there is an $e \in E$ such that $x \in V_e$. Accordingly, $d\big(f(x), g(x)\big) \le d\big(f(x), f(e)\big) + d\big(f(e), g(e)\big) + d\big(g(e), g(x)\big) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$ □

Now let us say that a set $\mathbf{F}$ of functions $X \to Y$ from a set to a metric space is *pointwise relatively compact* if the set $\mathbf{F}(x) \stackrel{\text{def}}{=} \{f(x) : f \in \mathbf{F}\}$ has a compact closure in Y for each $x \in X$. As a corollary of the previous proposition we get

Corollary 2.28. *Let X be a compact space, Y a metric space and* $\mathbf{F}$ *an equicontinuous pointwise relatively compact set of functions $X \to Y$. Then $\mathbf{F}|\mathfrak{O}_\mathrm{u}$, that is,* $\mathbf{F}$ *with the (metric) topology of uniform convergence is precompact.*

Proof . For $x \in X$ the set $K_x = \overline{\{f(x) : f \in \mathbf{F}\}}$ is compact. Then the space $P \overset{\mathrm{def}}{=} \prod_{x\in X} K_x$ is compact by the Tychonoff Product Theorem. Now $\mathbf{F}|\mathfrak{O}_\mathrm{u} = \mathbf{F}|\mathfrak{O}_\mathrm{p}$ is a subspace of the compact space P and is therefore a precompact metric space. □

Lemma 2.29. *If* $\mathbf{F}$ *is an equicontinuous subset of $B(X, Y)$ for some topological space X and a metric space Y, then the closure $\overline{\mathbf{F}}$ of* $\mathbf{F}$ *in Y^X is equicontinuous. As a consequence. the closures of $\mathcal{F}$ in the topologies of uniform convergence and that of pointwise convergensce agree.*

Proof . Let $\varepsilon > 0$ and $x \in X$. Find a $U \in \mathfrak{U}(c)$ such that $f(U) \in \mathfrak{U}_{\varepsilon/3}\big(f(x)\big)$ for all f. Now let g be in the closure of $\mathcal{F}$ with respect to the pointwise topology and let $u \in U$. Then there is an $f \in \mathcal{F}$ such that $d\big(f(u), g(u)\big) < \varepsilon/3$ and $d\big(f(x), g(x)\big) < \varepsilon/3$. Now $d\big(g(u), g(x)\big) \le d\big(g(u), f(u)\big) + d\big(f(u), f(x)\big) + d\big(f(x), g(x)\big) < 3\cdot\frac{\varepsilon}{3} = \varepsilon$. This proves the first claim. Now let $\mathcal{G}$ be the closure of $\mathcal{F}$ with respect to the uniform topology in $C(X, Y)$. Then $\mathcal{G} \subseteq \overline{\mathcal{F}}$. By Proposition 2.27, $\overline{\mathcal{F}}|\mathcal{O}_u = \overline{\mathcal{F}}|\mathcal{O}_p$. Therefore, $\mathcal{G} = \overline{\mathcal{F}}$. □

Lemma 2.30. *If* $\mathbf{F}$ *is a compact subset of $\big(C(X, Y), \mathcal{O}_u|C(X, Y)\big)$, then $\mathcal{F}$ is equicontinuous.*

Proof . Exercise. □

Exercise E2.10. Prove Lemma 2.30. [Hint. Use the fact that $(f, x) \mapsto f(x) : C(X, Y) \times X \to Y$ is continuous. Let $\varepsilon > 0$ and $x \in X$ be given. For each $g \in \mathcal{F}$ find a neighborhood $\mathcal{W}_g$ of g in $\mathcal{F}$ and a neighborhood U_g of x in X such that $\mathcal{W}_g(U_g) \subseteq U_{\varepsilon/2}\big(g(x)\big)$. Use compactness of $\mathcal{F}$ to find a finite set $\mathcal{E} \subseteq \mathcal{F}$ such that $\mathcal{F} = \bigcup_{g\in\mathcal{E}} \mathcal{W}_{\}}$. Set $U = \bigcap_{g\in\mathcal{E}}$. Then U is a neighborhood of x in X. Show tht for eavery $f \in \mathcal{F}$ and every $u \in U$ we get $d\big(f(u), f(x)\big) < \varepsilon$.]

These pieces of information, taken together lead to the following theorem which plays an important role in analysis.

Theorem 2.31. (Ascoli Theorem) *Let X be a compact space and Y a metric space. Let* $\mathbf{F}$ *be a set of bounded functions $X \to Y$. We endow* $\mathbf{F}$ *with the metric topology of uniform convergence. Then the following statements are equivalent:*

(i) $\mathbf{F}$ *is compact.*

(ii) $\mathbf{F}$ *is equicontinuous, pointwise relatively compact, and closed.* □

Under the circumstances of an equicontinuous set closedness of $\mathbf{F}$ *means closedness in either $B(X, Y)$ with respect to uniform convergence or in Y^X with respect to the product topology.*

The Ascoli Theorem has variants which generalize what is said in 2.30, but they are not different in principle. The Ascoli Theorem is the only way to verify that a space of continuous functions is compact.

Example. Let $(E, \|\cdot\|)$ be a Banach space. Let I be a compact real interval, $K \geq 0$ a nonnegative number, and let $\mathcal{F}_K \subseteq C(I, E)$ be the set of of all differentiable functions such that $\|f'(t)\| \leq K$ for all $f \in \mathcal{F}$. Then $\mathcal{F}_K$ is equicontinuous.

Let $a = \min I$, $b = \min I$, and let $x_0 \in E$. Define $\mathcal{F}_{K,x_0}$ to be the set of all $f \in \mathcal{F}_K$ satisfying $f(a) = x_0$. Then $f(I) \subseteq B_{K(b-a)}\big(f(t)\big) \subseteq B_{2K(b-a)}(x_0)$.

If $\dim E < \infty$ then $B_{2K(b-a)}(x_0)$ is compact, and thus by the Ascoli Theorem, $\mathcal{F}_{K,x_0}$ is compact.

This permits very quickly a proof of a basic theorem in the theory of ordinary differential equations stating the eqistence of local solutions of the initial value problem $\dot{u}(t) = f\big(t, u(t)\big)$, $f(t_0) = x_0$.

Chapter 3
Connectivity

We proceed to further special properties of topological spaces. From basic analysis we know that, next to compactness, connectivity is another important property of topological spaces.

A subset S of a topological space X is called *open-closed* or *clopen* if it is at the same time open and closed. The empty set and the whole space are clopen. We shall say that S is a *proper clopen subset* if is is a clopen subset which is neither $\emptyset$ nor X.

[The adjective “clopen” is artificial. It is convenient, but stylistically it is far from being a brilliant creation.]

Definition 3.1. A topological space $(X, \mathfrak{O})$ is said to be *disconnected*, if it has a proper clopen subset. Otherwise it is called *connected.* □

Exercise E3.1. Let $(X, \leq)$ be a totally ordered set and consider the order topology on it. Prove:

If X has a nonempty subset Y which has an upper bound but does not have a least upper bound, then X is disconnected.

If X contains two elements $a < b$ such that $X = {\downarrow}a \cup {\uparrow}b$, then X is disconnected.

If S is a proper clopen subset of X then ${\downarrow}S$ is clopen.

A subset $\{a, b\} \subseteq X$ such that $a < b$ and $X = {\downarrow}a \cup {\uparrow}b$ is called a *gap.*

Proposition 3.2. *A totally ordered set is connected in the order topology, if and only if every nonempty subset with an upper bound has a least upper bound and no gaps exist.* □

Corollary 3.3. *A set of real numbers is connected in the induced topology if and only if it is an interval.* □

There is a small subtlety here. The subset $X \overset{\text{def}}{=} [0, 1] \cup]3, 4]$ is disconnected in the induced topology but is connected in its own order topology.

Recall our convention $\mathbb{I} = [0, 1]$.

Definition 3.4. A topological space is called *arcwise connected* or *path-connected* if for all $(x, y) \in X \times X$ there is a $\gamma \in C(\mathbb{I}, X)$ such that $\gamma(0) = x$ and $\gamma(1) = y$. □

Proposition 3.5. *An arcwise connected space is connected.* □

Exercise E3.2. Set $\mathbb{R}^+ = \{r \in \mathbb{R} : 0 \le r\}$. In $\mathbb{R} \times \mathbb{C}$ consider the following subspace

$$S \stackrel{\text{def}}{=} \{(x, z) : \big((\exists r \in \mathbb{R}^+)\, x = e^{-r},\ z = e^{2\pi i r}\big) \text{ or } x = 0,\ |z| = 1\}.$$

Draw a sketch of this set. Prove that it is connected but not arcwise connected.

Prove that $\mathbb{R} \times \mathbb{C}$ has a continuous commutative and associative multiplication given by

$$(r, c)(r', c') = (rr', cc'), \quad (r, c),\ (r', c') \in \mathbb{R} \times \mathbb{C}.$$

A topological space with a continuous associative multiplication is called a *topological semigroup*. If it has an identity, one also calls it a *topological monoid*.

Show that S is a compact subset satisfying $SS \subseteq S$. Thus S is a compact topological monoid.

Does it contain a subset which is a topological monoid and a group?

Theorem 3.6. *Let $f\colon X \to Y$ be a continuous surjective function between topological spaces. If X is connected, then Y is connected. If X is arcwise connected, then Y is arcwise connected.* □

One may express this result in the form: *A continuous image of a connected space is connected; a continuus image of an arcwise connected space is arcwise connected.*

Corollary 3.7. *A continuous image of a compact connected space is compact and connected.* □

Corollary 3.8. *A continuous image of a real interval is arcwise connected. A continuous image of a compact interval is compact and connected.*

Corollary 3.9. (The Intermediate Value Theorem of Real Calculus) *Let $f\colon [a, b] \to \mathbb{R}$ be a continuous function and $f(a) \le y \le f(b)$ or $f(b) \le y \le f(a)$. Then there is an $x \in [a, b]$ such that $y = f(x)$.* □

The Intermediate Value Theorem gives us a solution x of the equation $y = f(x)$ for given y.

Corollary 3.10. *A continuous self-map of $[0, 1]$ has a fixed point.* □

Lemma 3.11. *If Y is a connected subspace of a topological space X, then the closure $\overline{Y}$ is connected as well.* □

Proposition 3.12. *Let X be a topological space. The relation R given by*

$$R = \{(x, y) \in X \times X : (\exists Y)\, Y \text{ is a connected subspace of } X \text{ and } x, y \in Y\}$$

is an equivalence relation with closed cosets. □

Definition 3.13. The equivalence relation R of Lemma 3.12 is called the *connectivity relation*, and its equivalence classes are called the *connected components* or *components* of the space X.

Exercise E3.3. Prove the following analog of Proposition 3.13:
Let X be a topological space. Recall that a curve from p to q in a topological space X is a continuous function $f\colon \mathbb{I} \to X$, $\mathbb{I} = [0,1]$ such that $f(0) = p$ and $f(1) = q$. The relation R on X given by

$$R = \{(x, y) \in X \times X : \text{ there is a curve from } x \text{ to } y\}$$

is an equivalence relation.

Give an example of a space such that the equivalence classes of this relation fail to be closed.

Each connected component of a space is the intersection of its open neighborhoods: Indeed, if $y \notin R(x)$ then $R(y) \cap R(x) = \emptyset$, and thus $R(x)$ is the intersection of the open sets $X \setminus R(y)$, $y \in X \setminus R(x)$.

Definition 3.14. A topological space in which all components are singletons is called *totally disconnected*.

Exercise E3.4. (i) Show that every discrete space is totally disconnected.

(ii) Show that the space of rational numbers, the space of irrational numbers, the Cantor set are all totally disconnected but nondiscrete spaces.

Theorem 3.15. (i) *If A is a connected subspace of a space X and $\{B_j : j \in J\}$ is a family of connected subspaces of a topological space X such that $A \cap B_j \neq \emptyset$ for all $j \in J$, then $A \cup \bigcup_{j\in J} B_j$ is connected.*

(ii) *Let $\{X_j : j \in J\}$ be a family of topological spaces and let $X \stackrel{\text{def}}{=} \prod_{j\in J} X_j$ be its product. If all X_j are connected, respectively, arcwise connected, then X is connected, respectively, arcwise connected.*

(iii) *For any family of topological spaces $\{X_j : j \in J\}$, if R is the connectivity relation of $X \stackrel{\text{def}}{=} \prod_{j\in J} X_j$ and R_j the connectivity relation of X_j for $j \in J$, then*

$$R = \{\big((x_j)_{j\in J}, (y_j)_{j\in J}\big) : (\forall j \in J)\, (x_j, y_j) \in R_j\}.$$

Equivalently, $R\big((x_j)_{j\in J}\big) = \prod_{j\in J} R(x_j)$ for all $j \in J$.

(iv) *If all X_j are totally disconnected, then X is totally disconnected.* □

The proof of (ii) is easy for arc connectivity, but is less obvious for connectivity. (See Exercise Sheet Nr. 10.)

Lemma 3.16. *If X is a space such that for each pair $x, y \in X$ of different points there is a clopen subset U with $x \in U$ and $y \notin U$, then X is totally disconnected.*□

Proposition 3.17. *Let R be the connectivity relation on X. Then X/R is totally disconnected T_1-space.*

Proof . If U and V are open and $U \cup V = X$ and $U \cap V = \emptyset$, then any R-class is entirely contained in either U or V. Hence U and V are R-saturated, i.e. are unions of R-equivalence classes. Thus by the definition of the quotient topology, the sets U/R and V/R are open; morover, $X/R = (U/R) \cup (V/R)$ and $(U/R) \cap (V/R) = \emptyset$. Suppose that C is a component of X/R. Then we consider $X' = \bigcup C$ (i.e., $X' = q_R^{-1}(C)$ where $q_R\colon X \to X/R$ is the quotient map. Then $R' \stackrel{\text{def}}{=} R \cap (X' \times X')$ is the connectivity relation of X' and $C = X'/R'$. By replacing X by X' and renaming, if necessary, let us assume that X/R is connected. We claim that X/R is singleton, i.e. that X is connected. So let $X = U \cup V$, $U \cap V = \emptyset$ for open subsets U and V of X. By what we have seen this implies $X/R = (U/R) \cup (V/R)$ and $(U/R) \cap (V/R) = \emptyset$. Since X/R is connected, one of U/R or V/R is empty. Hence one of U and V is empty, showing that X is connected.

This shows that X/R is totally disconnected. Since all connected components $R(x)$ are closed by 3.12, the singletons in X/R are closed by the definition of the quotient topology. Hence X/R satisfies the Frechet separation axiom T_1. □

Let $X = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ with the topology induced from that of of $\mathbb{R}$ and let $Y = \{0, 1\}$ with the discrete topology. Let ρ the equivalence relation on $X \times Y$ whose cosets are $\frac{1}{n} \times Y$, $n \in \mathbb{N}$ and $\{(0, 0)\}$ and $\{(0, 1)\}$. Then $T \stackrel{\text{def}}{=} (X \times Y)/\rho$ is a totally disconnected compact T_1-space which is not Hausdorff. Each equivalence class of ρ is closed, but $\rho \subseteq T \times T$ is not closed.

Proposition 3.18. *Any continuous function $f\colon X \to Y$ into a totally disconnected space factors through $q_R\colon X \to X/R$ where R is the connectivity relation on X. That is, there is a continuous function $\varphi\colon X/R \to Y$ such that $f = \varphi \circ q_R$.*

Proof . If $x \in X$ then the image $f\big(R(x)\big)$ of the component $R(x)$ of x is connected by 3.6. On the other hand, as a subspace of the totally disconnected space Y it is totally disconnected. Hence it is singleton. Set $\varphi\big(R(x)\big) = f(x)$. If V is open in Y, then $q_R^{-1}\big(\varphi^{-1}(V)\big) = f^{-1}(U)$ is an open R-saturated set. But then $\varphi^{-1}(U) = q_R\big(f^{-1}(U)\big)$ is open by the definition of the quotient topology. Thus φ is continuous. □

Corollary 3.19. *For a topological space X the following conditions are equivalent:*

(i) *X is a connected.*

(ii) *All continuous functions $f\colon X \to Y$ into a totally disconnected space are constant.*

Proof . Let R be the connectivity relation on X and $q_R\colon X \to X/R$ the quotient map.

(i)⇒(ii): Every continuous function $f\colon X \to Y$ into a totally disconnected space Y factors through $q_R\colon X \to X/R$ by 3.18. But since X is connected by (i), X/R is singleton, and thus f is constant.

(ii)⇒(i): $q\colon X \to X/R$ is a continuous surjective function into a totally disconnected space by 3.17; since such a function is constant by (ii), X/R is singleton. i.e. X is connected. □

We saw that a connected component C of a space X does have clopen neighborhoods. It is not true in general that C is the intersection of all of its clopen neighborhoods.

Proposition 3.20. *For an arbitrary topological space X with connectivity relation R, the following conditions are equivalent:*

(i) *Every component is the intersection of its clopen neighborhoods.*

(ii) *X/R is a totally disconnected Hausdorff space in which every singleton is the intersection of its clopen neighborhoods.* □

The best situation prevails for compact spaces. We discuss this now; but we need a bit of preparation.

Lemma 3.21. (A. D. Wallace's Lemma) *Let A be a compact subspace of X and B a compact subspace of Y, and assume that there is an open subset U of $X \times Y$ containing $A \times B$. Then there are open neighborhoods V of A in X and W of B in Y such that $V \times W \subseteq U$.* □

[See Exercise Sheet no 9, Exercise 1 with hints.]

Lemma 3.22. (Normality Lemma) *Let A and B be two disjoint compact subsets of a Hausdorff space X. Then there are two disjoint open neighborhoods of A and B, respectively.*

[See Exercise Sheet no 9, Exercise 2 with hints.]

In fact the Normality Lemma shows that A and B have disjoint *closed* neighborhoods: Let U and V be open neighborhoods of A and B, respectively. Then $\overline{U} \cap V = \emptyset$ since $X \setminus V$ is a closed set containing U. Now apply the Normality Lemma to $\overline{U}$ and B and find disjoint open sets P and Q such that $\overline{U} \subseteq P$ and $B \subseteq Q$. Now $\overline{Q} \cap \overline{U} = \emptyset$. Hence $\overline{U}$ and $\overline{Q}$ are two disjoint closed neighborhoods of A and B, respectively.

Lemma 3.23. (Filter Basis Lemma). *Let $\mathfrak{B}$ be a filter basis of closed subsets in a space and assume that $\mathfrak{B}$ has a a compact member B. If U is an open set containing $\bigcap \mathfrak{B}$, then there is a $C \in \mathfrak{B}$ such that $C \subseteq U$.*

In particular, a filter basis of closed sets in a compact Hausdorff space converges to x iff $\bigcap \mathfrak{B} = \{x\}$.

[Hint. Suppose not, then $\{C \setminus U : C \in \mathfrak{B}\}$ is a filter basis of closed sets, whose members are eventually contained in the compact space B, hence there is an element $b \in \bigcap_{C\in\mathfrak{B}} C \setminus U$. Then $b \in (\bigcap \mathfrak{B}) \setminus U = \emptyset$, a contradiction.]

The Filter Basis Lemma allows us to formulate 3.20 for compact Hausdorff spaces in a sharper form

Proposition 3.20′. *For a compact Hausdorff space X with connectivity relation R the following conditions are equivalent:*

(i) *Every component has a basis of clopen neighborhoods.*

(ii) *X/R is a totally disconnected Hausdorff space in which every singleton is the intersection of its clopen neighborhoods.* □

If U is a clopen subset of a space X, then U and $X \setminus U$ are the classes of an equivalence relation with open cosets. The intersection of any family of equivalence classes is an equivalence class; a finite collection of open closed sets thus gives rise to a finite decomposition of the space into finitely many clospen sets. An equivalence relation with clopen classes on a compact space has finitely many classes.

Definition 3.24. An equivalence relation R on a topological space is *open* if R is open as a subset of $X \times X$.

Remark 3.25. The connectivity relation is contained in all open equivalence relations. □

Proposition 3.26. *Let R be an equivalence relation on a space X. Then the following conditions are equivalent:*

(i) *All equivalence classes are open.*

(ii) *R is open in $X \times X$.*

(iii) *The quotient space X/R is discrete.*

(iv) *All components are clopen.*

Proof. (i)⇔(ii): For every equivalence relation R we have $R = \bigcup_{x\in X} R(x) \times R(x)$. If each $R(x)$ is open in X, then each $R(x) \times R(x)$ is open in $X \times X$ and vice versa.

(i)⇒(iii): If $R(x)$ is open in X, then by the definition of the quotient topology, the singleton set $\{R(x)\}$ is open in X/R.

(iii)⇒(iv): In a discrete space every subset is clopen, so $\{R(x)\}$ is clopen in X/R and thus $R(x)$ is clopen by the continuity of the quotient map.

(iv)⇒(i) is trivial. □

Notice that for a compact space X, the component space X/R is a compact totally disconnected Hausdorff space regardless of any separation property of X.

Lemma 3.27. *On a topological space X, the following conditions are equivalent:*

(i) *The connectivity relation is the intersection of all open equivalence relation.*

(ii) *Every component is the intersection of its clopen neighborhoods.*

Proof . Exercise. □

[Hint for (ii)⇒(i): If U is a clopen subset of X, then U and $X \setminus U$ are the two classes of an open equivalence relation $_U$. If $(x, y) \notin R$, let U be clopen neighborhood of $R(a)$ not containing b (by (ii)). Then $(a, b) \notin R_U$.]

If $A, B \subseteq X \times X$ are binary relations on X, then

$$A \circ B \stackrel{\text{def}}{=} \{(x, z) : (\exists y \in X)\, (x, y) \in A \text{ and } (y, z) \in B\}.$$

Note that $A \circ A \subseteq A$ means that A is transitive.

Exercise E3.5. Show that
on a compact Hausdorff space the relation product $A \circ B$ of two closed binary relations is closed.

A space X is a Hausdorff space iff the diagonal is closed. Then by the Normality Lemma, Δ has a basis of closed neighborhoods.

Exercise E3.6. Show that
on a compact Hausdorff space every neighborhood U of the diagonal Δ of $X \times X$ contains a neighborhood W of Δ such that $W \circ W \subseteq U$.

[Hint. Suppose that U is an open member of $\mathfrak{U}(\Delta)$, the set of neighborhoods of the diagonal Δ in $X \times X$ such that $W \circ W \not\subseteq U$ for all $W \in \mathfrak{U}(\Delta)$. Then $\{(W \circ W) \setminus U : W = \overline{W} \in \mathfrak{U}(\Delta)\}$ is a filter basis of closed sets on the compact space $(X \times X) \setminus U$. Let (x, y) be in the intersection of this filterbasis. Then, on the one hand, $(x, y) \in \Delta$, i.e. $x = y$ and one the other $(x, y) \notin U$.]

Theorem 3.28. *Let X by a compact Hausdorff space. Then every component has a neighborhood basis of clopen subsets.*

Proof . Let X by a compact Hausdorff space. Then every component has a neighborhood basis of clopen subsets.

Proof . Let U be a neighborhood of the diagonal Δ in $X \times X$. By replacing U by $\{(u, v) : (u, v), (v, u) \in U\}$ if necessary, we may assume that U is symmetric. We define R_U to be the set of all pairs (x, y) such that there is a finite sequence $x_0 = x, x_1, \ldots, x_n = y$ such that $(x_{j-1}, x_j) \in U$; we shall call such a sequence a U-chain. Then R_U is reflexive, symmetric, and transitive. Hence R_U is an equivalence relation. Write $U(x) = \{u \in X : (x, u) \in U\}$. Then $U(x)$ is a neighborhood of x. Since $U(x') \subseteq R_U(x)$ for each $x' \in R_U(x)$, the relation R_U is open and therefore closed as the complement of all other equivalence classes. Let S be the intersection of the clopen equivalence relations R_U as U ranges through the filterbasis $\mathfrak{U}_s(\Delta)$ of symmetric neighborhoods of Δ. Then S is an equivalence relation and S is closed in $X \times X$. Then every pair of elements in C is R_U-equivalent for all $U \in \mathfrak{U}_s(\Delta)$. Let R denote the connectivity relation on X. Set $C = S(x)$. The component $R(x)$ of x is contained in C. We aim to show that C is connected. Then $C = R(x)$ for all x and thus $R = S$. So $R(x) = \bigcap_{U \in \mathfrak{U}_s(\Delta)} R_U(x)$, and then, by the Filter Basis

Lemma, the sets $R_U(x)$ form a basis of the neighborhoods of $C = R(x)$. This will complete the proof.

Now suppose that C is not connected. Then $C = C_1\dot{\cup}C_2$ with the disjoint nonempty closed subsets of C. We claim that there is an open symmetric neighborhood $U \in \mathfrak{U}(\Delta)$ of the diagonal Δ in $X \times X$ such that the set $U(C_1) \cap C_2$ is empty. [It suffices to show that every open neighborhood W of a compact subset K of X contains one of the form $U(K)$. Proof by contradiction: If not, then for all open neighborhoods U of the diagonal in $X\times X$, $U(K)\cap(X\setminus W)$ is not empty and the collection of sets $U(K)\cap(X\setminus W)$ is a filterbasis on the compact space $X\setminus W$. Let z be in the intersection of the closures of the sets in this filterbasis. Since X is Hausdorff, the diagonal is closed in $X \times X$ and by the Normality Lemma is the intersection of its closed neighborhoods. Thus z in the intersection of all $U(K)$ for all closed U and this is K. Thus $z \in K \setminus W = \emptyset$, a contradiction!]

Recall that for two subsets $A, B \subseteq X\times X$ we set $A\circ B = \{x, z) \in X\times X : (\exists y \in X)\ (x, y) \in A$ and $(y, z) \in B\}$. Now assume that W is an open neighborhood of the diagonal such that $W\circ W\circ W \subseteq U$. and set $D = X \setminus \big(W(C_1)\cup W(C_2)\big)$. Now let $V \in \mathfrak{U}(\Delta)$, $V \subseteq W$. By replacing V by $\{(u, v) : (u, v), (v, u) \in V\}$ if necessary, we may assume that V is symmetric.

If $x \in C_1$ and $c_2 \in C_2$, then $(x, c_2) \in R_V$ since $C \in R_V(x)$. Now any V-chain $x = x_0, x_1, \ldots, x_n = c_2$ has at least one element in D. Thus $R_V(x) \cap D \neq \emptyset$. Thus the $R_V(x) \cap D$ form a filterbasis on the compact space D. Let y be in its intersection. Then $y \in \bigcap_{V\in\mathfrak{U}_s(\Delta)} R_V(x) = C$ and $y \in D$, whence $y \in C \cap D = \emptyset$: a contradiction. This shows that C is connected as asserted and completes the proof. □

The preceding theorem shows that
the connectivity relation R is the intersection of open equivalence relations.

In Theorem 2.28, compactness is sufficient, but it is not necessary.

Exercise E3.7. Show that in the space $\mathbb{Q}$ in its order topology every point has a basis of clopen neighborhoods. □

Chapter 4
Topological groups

Definition 4.1. A topological space X is called *homogeneous* if for $(x, y) \in X \times X$ there is a homeomorphism $f: X \to X$ such that $f(x) = y$.

Recall that a group G is said to *act* on a set X if there is a function $(g, x) \mapsto g{\cdot}x: G \times X$ such that $1{\cdot}x = x$ for all x and $g{\cdot}(h{\cdot}x) = (gh){\cdot}x$ for all $g, h \in G$ and $x \in X$. A group always acts upon itself by each of the following operations:

(i) $g{\cdot}x = gx$ (left multiplication),

(ii) $g{\cdot}x = xg^{-1}$, (right multiplication),

(iii) $g{\cdot}x = gxg^{-1}$, (conjugation).

We say that G acts transitively, if the action has only one orbit, i.e. $X = G{\cdot}x$ for one and then any $x \in X$. The action of a group on itself by multiplication is transitive.

Now we can say that X is homogeneous if the group of all homeomorphisms of X operates transitively on X.

Proposition 4.2. *Let G be a group acting on a topological space X such that the function $x \mapsto g{\cdot}x: X \to X$ is continuous for each $g \in G$. If G acts transitively, then X is homogeneous.*

Lemma 4.3. *Assume that every point x of a space X has a closed neighborhood U which contains an open neighborhood V of x such that for each $v \in V$ there is a homeomorphism f of U such that $f(x) = v$ and f leaves the boundary of U pointwise fixed.*

Then each orbit of the homeomorphism group G of X is open. In particular, if X is connected, then X is homogeneous.

Proof. Assume $x \in G{\cdot}y$. Let U and V be as in the statement of the Lemma, and consider $v \in V$. The function $\varphi: X \to X$ defined by

$$\varphi(z) = \begin{cases} f(z) & \text{if } z \in U, \\ z & \text{if } z \in X \setminus U. \end{cases}$$

Then φ is a homeomorphism such that $\varphi(x) = v$. Hence if $\gamma \in G$ is such that $\gamma(y) = x$, then $(\varphi \circ \gamma)(y) = \varphi\big(\gamma(y)\big) = \varphi(x) = v$. Hence $V \in G{\cdot}y$. Thus $G{\cdot}y$ is open. □

Lemma 4.4. *For $0 \le t < 1$ let $\tau(t) = (1 - t^2)^{-1/2}$. Let U denote the open unit ball $\{x : \|x\| < 1\}$, for the euclidean norm given by $\|x\|^2 = \sum_{m=1}^{n} x_m^2$ on $\mathbb{R}^n$. For a vector $v \in U$, define $\varphi(u) = \tau(\|u\|){\cdot}u$ and $f: U \to U$ by $f(u) = \varphi^{-1}(\varphi(u) + \varphi(v))$.*

Then f is a homeomorphism of U such that $f(0) = v$ and that f is the restriction of a homeomorphism $F: \mathbb{R}^n \to \mathbb{R}^n$, which fixes all vectors w with $\|w\| \geq 1$. □

Definition 4.5. A *topological manifold* is a topological space each point of which has an open neighborhood which is homeomorphic to $\mathbb{R}^n$ for some n. □

Exercise E4.2. Show that
(i) *every open subset of $\mathbb{R}^n$ is a topological manifold.*
(ii) *Every sphere $\mathbb{S}^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}$ is a compact topological manifold.*
(iii) *Every finite product of topological manifolds is a topological manifold.* □

In particular, a torus $\mathbb{S}^1 \times \mathbb{S}^1$ is a topological manifold. Note that every discrete space is a topological manifold.

Proposition 4.6. *A connected topological manifold is homogeneous.* □

Definition 4.7. A *topological group G* is a group endowed with a topology such that multiplication $(x, y) \mapsto xy: G \times G \to G$ and inversion are continuous. □

Since inversion $x \mapsto x^{-1}$ is an involution (i.e. satisfies $(x^{-1})^{-1} = x$), it is clearly a homeomorphism of G onto itself.

Exercise E4.3. Show that *a group G with a topology is a topological group if and only if the following function is continuous. $(x, y) \mapsto xy^{-1}: G \times G \to G$.* □

Proposition 4.8. *The space underlying a topological group is homogeneous.* □

Exercise E4.4. Show that the compact unit interval $\mathbb{I} = [0, 1]$ cannot be the underlying space of a topological group.

Examples 4.9. (i) Every group is a topological group when equipped with the discrete topology.

(ii) Every group is a topological group when equipped with the indiscrete topology.

(iii) $\mathbb{R}$ is a topological group with respect to addition. Also, $\mathbb{R} \setminus \{0\}$ is a topological group with respect to multiplication.

(iv) More generally, the additive group of $\mathbb{R}^n$ is a commutative topological group.

(v) Also more generally: Let $\mathbb{K}$ denote one of the fields $\mathbb{R}$, $\mathbb{C}$ or the division ring $\mathbb{H}$ of quaternions with the absolute value $|\cdot|$ in each case. Let $\mathbb{S}^n$, $n = 0, 1, 3$ denote the set $\{x \in \mathbb{K} : |x| = 1\}$ and $\mathbb{R}^< = \{x \in \mathbb{R} : 0 < x\}$. Then $\mathbb{R}^<$ and $\mathbb{K} \setminus \{0\}$ are topological groups under multiplication. The function

$$x \mapsto \left(|x|, \frac{x}{|x|}\right) : \mathbb{K} \setminus \{0\} \to \mathbb{R}^< \times \mathbb{S}^n$$

is an isomorphism of groups and a homeomorphism of topological spaces.

(vi) The groups $\mathrm{GL}(n, \mathbb{K})$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ of invertible real or complex matrices are topological groups. □

Proposition 4.10. (i) *If H is a subgroup of a topological group G, then H is a topological group in the induced topology.*

(ii) *If $\{G_j : j \in J\}$ is a family of topological groups, then $G \stackrel{\text{def}}{=} \prod_{j \in J} G_j$ is a topological group.*

(iii) *If N is a normal subgroup of a topological group G, then the quotient group G/N is a topological group with respect to the quotient topology.* □

Proposition 4.11. *The closure of a subgroup is a subgroup, the closure of a normal subgroup is a normal subgroup.* □

Morphisms of topological groups

Definition 4.12. A *morphism of topological groups* $f\colon G \to H$ is a continuous homomorphism between topological groups.

Proposition 4.13. (a) *A homomorphism of groups $f\colon G \to H$ between topological groups is a morphism if and only if it is continuous at* 1.

(b) The following conditions are equivalent:
 (i) f is open.
 (ii) For each $U \in \mathfrak{U}(1)$ the image $f(U)$ has a nonempty interior.
 (iii) There is a basis of identity neighborhood U such that $f(U)$ has a nonempty interior.
 (iv) there is a basis of identity neighborhoods U of G such that $f(U)$ is an identity neighborhood of H.
 (v) For all $U \in \mathfrak{U}_G(1)$ we have $f(U) \in \mathfrak{U}_H(1)$.

(c) For any normal subgroup N of G the quotient morphism $q\colon G \to G/N$ is continuous and open. □

[Hint for (ii)⇒(iii): Let $U \in \mathfrak{U}_G(1)$. We may assume that U is open. Find $V \in \mathfrak{U}_G(1)$ open such that $VV^{-1} \subseteq U$. Let $V' \in \mathfrak{U}_G(1)$ be such that $\operatorname{int} f(V') \neq \emptyset$. Then $W \stackrel{\text{def}}{=} \operatorname{int} f(V) \neq \emptyset$. Let $v \in V$ be such that $f(v) \in W$. Then $1 = ww^{-1} \in Ww^{-1}$ that is, Ww^{-1} is an open neighborhood of 1 in H. Now $V'' \stackrel{\text{def}}{=} V \cap f^{-1}(W)$ is an open neighborhood of v such that $f(V'') = W$ and hence $U' \stackrel{\text{def}}{=} V''v^{-1}$ is an open neighborhood of 1 in G contained in $VV^{-1} \subseteq U$ such that $f(U') = Ww^{-1} \in \mathfrak{U}_H(1)$.]

Proposition 4.14. (Canonical decomposition) *A morphism of topological groups* $f: G \to H$ *with kernel* $N = \ker f$ *decomposes canonically in the form*

$$\begin{array}{ccc} G & \xrightarrow{\quad f \quad} & H \\ {\scriptstyle q}\downarrow & & \uparrow{\scriptstyle \mathrm{j}} \\ G/N & \xrightarrow[\quad f' \quad]{} & f(G), \end{array}$$

where $q: G \to N$ *is the quotient morphism given by* $q(g) = gN$, $j: f(G) \to H$ *is the inclusion morphism, and* $f': G/N \to f(G)$ *is the bijective morphism of toplogical groups given by* $f'(gN) = f(g)$.

The morphism is open if and only if $f(G)$ is open in H and f' is an isomorphism of topological groups, i.e. is continuous and open. □

As we shall see shortly, the filter $\mathfrak{U} = \mathfrak{U}(1)$ of all identity neighborhoods is a very useful tool in topological group theory. We shall begin to use it now.

Lemma 4.15. (The Closure Lemma) *Let* A *be a subset of a topological group. Then* $\overline{A} = \bigcap_{U \in \mathfrak{U}} AU$.

Proof. If $U \in \mathfrak{U}$ and $x \in \overline{A}$, then xU^{-1} is a neighborhood of x, and thus there is an $a \in A \cap xU^{-1}$. Write $a = xu^{-1}$ for some $u \in U$. Accordingly, $x = au \in AU$.

Conversely, assume that $x \in \bigcap_{U \in \mathfrak{U}} AU$, and let V by a neighborhood of x. We claim that $A \cap V \neq \emptyset$, thus showing $x \in \overline{A}$. Now $U \overset{\text{def}}{=} V^{-1}x \in \mathfrak{U}$ is an identity neighborhood, and thus $x \in AU$, say $x = au$ with $a \in A$ and $u \in U$. Then $a = xu^{-1} \in xU^{-1} = xx^{-1}V = V$ and so $a \in A \cap V$ as asserted. □

Corollary 4.16. $\overline{\{1\}} = \bigcap \mathfrak{U}$, *and this is a closed normal subgroup contained in every open set meeting* $\overline{\{1\}}$.

Proof. The first part follows from the Closure Lemma and the fact that closures of normal subgroups are normal subgroups. Let U be open and $U \cap \overline{\{1\}} \neq \emptyset$. Then $1 \in U$. Thus $\overline{\{1\}} \subseteq U$ by the first part. □

Separation Axioms in topological groups

Theorem 4.17. *In a topological group* G, *every neighborhood filter of a point has a basis of closed neighborhoods, and the following conditions are equivalent equivalent:*

(i) G *is a* T_0*–space.*
(ii) $\{1\}$ *is closed.*
(iii) G *is a* T_1*–space.*
(iv) G *is a regular Hausdorff space, i.e. a* T_3*–space.*

Proof. If $U \in \mathfrak{U}$, then by the continuity of multiplication there is a $V \in \mathfrak{U}$ such that $VV \subseteq U$. By the Closure Lemma, $\overline{V} \subseteq VV \subseteq U$. Thus $\mathfrak{U}(1)$ has a basis of closed sets, and since G is homogeneous, every neighborhoodfilter has a basis of closed neighborhoods. However a T_0–space in which every point has a neighborhood basis of closed neighborhoods is a T_3–space (see 1.38). This (i) implies (iv), and $(T_3)\Rightarrow(T_2)\Rightarrow(T_1)$. □

There aslo pedestrian proofs of the individual implications: (i)⇒(ii): Let $x \neq 1$. By (i) there is an open set U containing exactly one of 1 or x. if $1 \in U$ then $x \notin U$. Now $1 \in U^{-1}$ and thus $x \in U^{-1}x$; thus $U^{-1}x$ is an open neighborhood of x which does not contain 1, for if it did, then $1 = u^{-1}x$ for some $u \in U$, and then $x = u \in U$. Thus every element $x \neq 1$ has an open neighborhood missing 1, and thus (ii) is proved.

(ii)⇒(iii): This follows from the homogeneity of G.

(iii)⇒(iv): Assume $x \neq y$ in G. Then $1 \neq xy^{-1}$. By (iii), $G \setminus \{xy^{-1}\} \in \mathfrak{U}$, and by continuity of $(g,h) \mapsto g^{-1}h$ there is a $V \in \mathfrak{U}$ such that $V^{-1}V \subseteq G \setminus \{xy^{-1}\}$. If $g = Vx \cap Vy$, then $g = vx = wy$ with $v, w \in V$, whence $xy^{-1} = v^{-1}w \in V^{-1}V \subseteq G \setminus \{xy^{-1}\}$, a contradiction. Thus Vx and Vy are two disjoint neighborhoods of x and y, respectively.

(iv)⇒(i): Trivial.

Corollary 4.18. *A quotient group of a topological group G modulo a normal subgroup N is a Hausdorff group if and only if N is closed.* □

Corollary 4.19. *For every topological group G, the factor group $G/\overline{\{1\}}$ is a Hausdorff group and for each continuous homomorphism $f: G \to H$ into a Hausdorff group there is a unique morphism $f': G/\overline{\{1\}} \to H$ such that $f = f'q$ with the quotient morphism $q: G \to G/\overline{\{1\}}$.* □

Proposition 4.20. *Let G be a topological group, and U an open subset. Set $N = \overline{\{1\}}$. Then $UN = U$. Every open set is the union of its N cosets.*

Proof. By 4.16, N is contained in every open set U meeting N. Let $x \in UN$. Then $x = un$ with $u \in U$ and $n \in N$. Then $n = u^{-1}x \subseteq U^{-1}x$. Thus $1 \in N \subseteq U^{-1}x$ and therefore $1 \in U^{-1}x$, i.e., $x \in U$. □

Corollary 4.21. *Let $\big(G, \mathfrak{O}(G)\big)$ be any topological group and let $q: G \to \overline{\{1\}}$ be the quotient morphism of G onto the Hausdorff topological group $\big(G/\overline{\{1\}}, \mathfrak{O}(G/\overline{\{1\}})\big)$. Then $U \mapsto q^{-1}(U): \mathfrak{O}(G/\overline{\{1\}}) \to \mathfrak{O}(G)$ is a $\bigcap - \bigcup$-preserving bijection.* □

The Identity Component

Definition 4.22. For a topological group G let G_0 denote the connected component of the identity, short *the identity component*. Similarly let G_a denote the arc component of the identity, the *identity arc component*. □

Definition 4.23. A subgroup H of a topological group G is called *characteristic* if it is invariant under all automorphisms of G, i.e., all continuous and continuously inverible group homomorphisms. It is called *fully characteristic* if it is invariant under all (continuous!) endomorphisms. □

Every fully characteristic subgroup is characteristic. The inner automorphisms $x \mapsto gxg^{-1}: G \to G$ are continuous and continuously invertible. Hence every characteristic subgroup is invariant under all inner automorphisms, i.e. is normal.

Proposition 4.24. *The identity component G_0 and the identity arc component G_a of any topological group G are fully characteristic subgroups of G. The identity component G_0 is closed. The factor group G/G_0 is a totally disconnected Hausdorff topological group.* □

Exercise E4.5. Prove
(i) *If H is an open subgroup of a topological group G, then H is closed and $G_0 \subseteq H$. The quotient space $G/H = \{gH : g \in G\}$ is discrete.*
(ii) *If G is a locally connected topological group, then G_0 is open and G/G_0 is discrete.*
(iii) *If G is a locally connected topological group and $f: G \to H$ is an open morphism of topological groups, then the identity component G_0 of G is mapped onto the identity component of H.*

Lemma 4.25. *If G is a compact topological group acting (continuously) on a topological space X and if $G{\cdot}x = \{x\}$, then for any open set U containing x the set $V \stackrel{\text{def}}{=} \bigcap_{g\in G} g{\cdot}U$ is open.* □

Lemma 4.26. *Assume that G is a topological group acting (continuously) on a topological space X. Let K be a compact subset of G and A a closed subset of X. Then $K{\cdot}A$ is closed in X.*

Proof . Let $y \in \overline{K{\cdot}A}$. Then for every $U \in \mathfrak{U}(y)$ we have $U \cap K{\cdot}A \neq \emptyset$, and thus $K_U = \{g \in K : (\exists a \in A)\, g{\cdot}a \in U\} \neq \emptyset$. The collection $\{K_U : U \in \mathfrak{U}(y)\}$ is a filterbasis on the compact space K and thus we find a $h \in \bigcap_{U\in\mathfrak{U}(y)} K \cap \overline{K_U}$. Thus for any $j \stackrel{\text{def}}{=} (U,V) \in \mathfrak{U}(y)\times\mathfrak{U}(h)$ the set $F_j \stackrel{\text{def}}{=} K_U \cap V$ is not empty and contained in V. For $g \in F_j$ there is a $b \in A$ such that $g{\cdot}b \in U$. Thus $b \in g^{-1}{\cdot}U$. Hence the set of all $A \cap F_j^{-1}{\cdot}U \subseteq A \cap V^{-1}{\cdot}U$, as $j = (U,V)$ ranges through $\mathfrak{U}(y) \times \mathfrak{U}(h)$ is a filter basis converging to $a \stackrel{\text{def}}{=} h^{-1}{\cdot}y$ by the continuity of the action. Since A is closed we have $a \in A$. Thus $y = h{\cdot}a \in K{\cdot}A$. □

Proposition 4.27. *If K is a compact subset and A is a closed subset of a topological group, then KA and AK are closed subsets.* □

Example 4.28. Let G be a nonsingleton group and equip it with the indiscrete topology. Then $K = \{1\}$ is a compact subset which is not closed. The only nonempty closed subset A of G is G. Then $KA = G$ is closed. □

Lemma 4.29. *Assume that G is a topological group acting (continuously) on a topological space X such that $G{\cdot}x = \{x\}$. Then for any open set U containing x and every compact subset K of G, the set $V \stackrel{\text{def}}{=} \bigcap_{g\in K} g{\cdot}U$ is open. Thus if G itself is a compact group, then x has arbitrarily small invariant neighborhoods.*

Proof. Let $A \stackrel{\text{def}}{=} X \setminus U$. Then A is closed and $K{\cdot}A = \bigcup_{g\in K} g{\cdot}A = \bigcup_{g\in K} g{\cdot}(X \setminus U) = bigcup_{g\in K}(X \setminus g{\cdot}U) = X \setminus \bigcap_{g\in K} g{\cdot}U = X \setminus V$. Since $K{\cdot}A$ is closed by 4.29, its complement V is open. □

If U is a clopen neighborhood of the identity in a topological group G, and if $A = G \setminus U$, then $\bigcap_{V\in\mathfrak{U}(1)} AV = A$ by the Closure Lemma. If U is compact, then $U \cap AV = \emptyset$ for all sufficiently small $V \in \mathfrak{U}(1)$. For if not, then the filter basis of all $\overline{U \cap AV}$, $V \in \mathfrak{U}(1)$ has an element $u \in U$ in its intersection and every neighborhood of u meets AV for all V. For each $V \in \mathfrak{U}(1)$ we find a $W \in U(1)$ with $WW \subseteq V$ and then have $uW \cap AW^{-1} \neq \emptyset$, i.e. there is an $a_V \in A$ and there are elements $w_1, w_2 \in W$ such that $uw_1 = a_V w_2^{-1}$, i.e. that $a_V = uw_1w_2 \in uV$. Then $A \cap uV$ is a filter basis converging to u. Since A is closed, $u \in A$ contrary to $U \cap A = \emptyset$.

Proposition 4.30. *Assume that U is a compact open identity neighborhood in a topological group. Then there is a compact open subgroup H contained in U, in fact $UH = U$.*

Proof. Again set $A = G \setminus U$. Find a symmetric identity neighborhood $V = V^{-1}$ such that $U \cap AV = \emptyset$. Then $UV \cap A = \emptyset$, i.e. $UV \subseteq U$. By induction, $UV^n \subseteq U$ where $V^n = \underbrace{V \cdots V}_{n \text{ times}}$. Set $H = \bigcup_{n=1}^{\infty} V^n$. Then H is an open subgroup and $UH = U$. □

Theorem 4.31. *Let G be a compact totally disconnected group. Then for any identity neighborhood U there is a compact open normal subgroup N with $N \subseteq U$.*

Proof. Since G is totally disconnected, the component of 1 is $\{1\}$. By Theorem 3.28, the filter $\mathfrak{U}(1)$ of identity neighborhoods has a basis of clopen neighborhoods U. By 4.33, every such U contains an open subgroup H such that $UH = U$. By 4.32, $N \stackrel{\text{def}}{=} \bigcap_{g\in G} gHg^{-1}$ is open. Also, N is invariant under all inner automorphisms. □

One also expresses this fact by saying that a compact totally disconnected group G has arbitrarily small compact open normal subgroups N. For each of these, the factor group is finite and discrete. Thus we might say that G is approximated by the finite subgroups G/N. Therefore compact totally disconnected groups are also called profinite groups. They occur in the Galois theory of infinite field extensions.

Exercise E4.6. Let $\{G_j : j \in J\}$ be a family of finite groups and form the totally disconnected compact group $G = \prod_{j\in J} G_j$. Identify a neighborhood basis of 1 consisting of open normal subgroups. □

Example 4.32. Let p be a natural number, $p \geq 2$, for instance a prime number. In the compact totally disconnected group $P \stackrel{\text{def}}{=} \prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}$ consider the closed subgroup $\mathbb{Z}_p$ of all N-tuples $(z_n + p^n\mathbb{Z})_{n\in\mathbb{N}}$ such that $z_{n+1} - z_n \in p^n\mathbb{Z}$.

Then $\mathbb{Z}_p$ is a compact totally disconnected abelian group with a basis of identity neighborhoods $\{p^n\mathbb{Z}_p : n \in \mathbb{N}\}$. The subgroup of all $(z + p^n)_{n\in\mathbb{N}}$, $z \in \mathbb{Z}$ is algebraically isomorphic to $\mathbb{Z}$ and is dense in $\mathbb{Z}_p$. Thus $\mathbb{Z}_p$ is a "compactification" of $\mathbb{Z}$. Elements are close to zero if they are divisible by large powers of p.

The group $\mathbb{Z}_p$ is called the *group of p-adic integers*.

The additive group P is a ring under componentwise multiplication. The subgroup $\mathbb{Z}_p$ is closed under multiplication. Thus $\mathbb{Z}_p$ is in fact a compact ring with a continuous multiplication, containing $\mathbb{Z}$ as a dense subring.

The underlying topological space of $\mathbb{Z}_p$ is homeomorphic to the Cantor set.

There is an interesting application of connnectivity.

Theorem 4.33. *Let G be a connected topological group and N a totally disconnected normal subgroup. Then N is central, that is,*

$$(\forall g \in G,\, n \in N)\, gn = ng.$$

Proof . Let $n \in N$. The continuous function $g \mapsto gng^{-1}n^{-1}: G \to N$ maps a connected space into a totally disconnected space, and the image contains 1. Then this function is constant and takes the value 1. □

Exercise E4.7. Prove the following result:

Let X be an arbitrary set and $T \cong (\mathbb{R}/\mathbb{Z})^X$ a torus which is contained as a normal subgroup in a connected topological group G. Then T is central, that is, all of its elements commute with all elements of G.

[Hint. Consider in T the subgroup S of all elements of finite order. Every automorphism of T maps S into itself, and thus S is normal in G. But S is contained in $(\mathbb{Q}\mathbb{Z})^X$ and this is a totally disconnected subgroup. Hence S is totally disconnected. By 4.36, S is central. Also, S is dense in T. Conclude that T is central.] □

Index