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# Fuzzy sets and sheaves. Part I
# Basic concepts

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**Abstract**

This essay shows that large parts of fuzzy set theory are actually subfields of sheaf theory, respectively, of the theory of complete $\Omega$-valued sets. Hence fuzzy set theory is closer to the mainstream in mathematics than many people would expect. Part I of this essay divided into a series of two papers presents such basic concepts as $\Omega$-valued equalities, espaces étalés, singleton monad, the change of base and the subobject classifier axiom. The application of these tools to the sheaf-theoretic foundations of fuzzy sets will appear in Part II.


*Keywords:* Category theory; $\Omega$-valued sets; Espace étalé; Singleton monad; Complete $\Omega$-valued sets; Membership functions; Geometric morphisms

## 0. Introduction

It is a remarkable fact that the historic development of fuzzy set theory (cf. [11,12,10]) proceeds completely isolated from sheaf theory. [1] Also the long lasting debate on categorical foundations of fuzzy set theory (cf. [55]) does not open the horizon for sheaf-theoretic arguments in the formulation of such fundamental notions as *membership function*, *similarity*, *fuzzy ordering*, etc. (cf. [17,56–58]).

The aim of this essay divided into a series of two papers is to explain that large parts of fuzzy set theory are actually subfields of sheaf theory. We show that fuzzy sets are *subsheaves* of constant sheaves—so-called sheaves of level cuts, fuzzy groups (cf. [9,50]) are *subsheaves of groups* of constant sheaves of groups, and stratified $\Omega$-valued topological spaces (cf. [6,31,40]) are *topological space objects* in the category of sheaves on $\Omega$. Further, intersections, unions, images and inverse images of fuzzy sets, the max–min-composition of fuzzy relations [2] are special categorical constructions in the category of sheaves. So the impression arises that in many fields of fuzzy set theory the wheel has been invented once more. Moreover, fuzzy set theorists are not able to give a proper solution of the quotient problem w.r.t. similarity relations and a proper construction of fuzzy factor groups w.r.t. normal fuzzy subgroups (cf. [7,8,14,35–37,44,46,47,49]).

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[1] A historic account on the history of sheaf theory can be found in [20].

[2] An extensive interpretation of fuzzy relations in Heyting algebra valued, hierarchic models for intuitionistic set theory has recently been given by Shimoda (cf. [52,53]).

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In order to overcome these shortcomings a certain amount of fundamental knowledge from sheaf theory is inevitable. We put this material together in *Part I*: *Basic Concepts* which is devoted to the fundamental fact that sheaves on frames $\Omega$ (cf. [33]) can be described in three different but equivalent ways: As functors $\Omega^{op} \longrightarrow \mathbf{SET}$ provided with a certain «pasting» property, as complete $\Omega$-valued sets, and in the case of spatial frames as espaces étalés with base space $pt(\Omega)$. Depending on the context we will prefer one or the other of these descriptions.

With regard to a coherent interpretation of sheaf-theoretic constructions we begin with a discussion of frames as sets of truth values (cf. Section 1). Here we emphasize that every element of a frame is not the degree, but the *domain* of truth. In particular, the bottom element represents always the *empty* domain of truth.

In Section 2 we develop the close relationship between $\Omega$-valued equalities and espaces étalés. Since the real unit interval is a spatial frame, this situation is of special interest for fuzzy set theorists. In particular, the *crisp* equality on a set $X$ gives always rise to an espace étalé being equivalent to the *constant* sheaf generated by $X$.

In Section 3 we explain the monadic background of complete $\Omega$-valued sets. Therefore this section is the place where we review some important, categorical constructions of incomplete or complete $\Omega$-valued sets with the exception of the subobject classifier axiom (cf. [39,43]) which we defer to Section 6.

In Section 4 we study sheaves from the point of view of functors. Here we first recall the concept of presheaves, and subsequently turn to the sheafification construction and to the special relationship between separated presheaves, separated $\Omega$-valued equalities and sheaves on $\Omega$ (viewed as functors). If $\Omega$ is a completely distributive lattice, then we can show that every functor $(pt(\Omega), \leqslant) \longrightarrow \mathbf{SET}$ satisfying a certain «continuity» condition can be derived from a sheaf on $\Omega$. In the case of $\Omega = [0, 1]$ this result explains the important difference between level cuts and *strict* level cuts of membership functions (cf. Theorem 4.6, Remark 4.7).

In the remaining sections of this paper we study the effect of change of base and the relationship between subobjects of complete $\Omega$-valued sets and strict and extensional membership maps. We show that every frame homomorphism $\Omega \longrightarrow \Omega'$ gives rise to a *geometric morphism* between the categories of complete $\Omega$-valued and complete $\Omega'$-valued sets. In the special case of points of [0, 1] (cf. Section 1) these constructions can be viewed as a clean treatment of the *defuzzification problem* (cf. Section 1.1.4 in Fuzzy Sets and Sheaves—Part II (cf. [30])). As a by-product of this situation we obtain that finite limits and set-indexed colimits of espaces étalés are computed *fibrewise* (cf. Remark 5.5). Further, in Section 6 we describe the *external* and *internal* identification of strict and extensional membership maps with subobjects of complete $\Omega$-valued sets. The external construction is based on the idea to describe membership maps in terms of $\Omega$-valued equalities and prototypes (cf. [10, p. 100]), while the internal one makes use of the subobject classifier diagram. Even though the external identification would be sufficient to understand fuzzy sets as subobjects of singleton spaces w.r.t. the crisp equality, we view the internal identification as a special confirmation of this important fact.

Even though this essay makes the attempt to be self-contained, we are not in the position to recall all needed categorical concepts. For this purpose we refer the reader to standard textbooks in category theory—e.g. [22,42,45,51].

## 1. Frames as sets of truth values

A complete Heyting algebra $\Omega$ is a complete lattice such that finite meets are distributive over arbitrary joins—i.e.

$$\alpha \wedge \Big(\bigvee B\Big) \quad = \quad \bigvee_{\beta \in B} (\alpha \wedge \beta).$$

Typical examples of complete Heyting algebras are complete Boolean algebras and lattices of open subsets of ordinary topological spaces. The category **FRM** consists of the following data (cf. [33]): *Objects* are complete Heyting algebras and *morphisms* are frame homomorphisms—i.e. *finite meets* and *arbitrary joins* preserving maps. In order to emphasize the use of frame homomorphisms between complete Heyting algebras, objects of **FRM** are also called *frames*.

In this essay frames will play the role of sets of truth values. Since in the relevant literature (cf. [41, pp. 48–49; 21, p. 319, p. 419, p. 419, p. 4]) the *meaning* of multiple truth values is ambivalent and has not been explained to satisfaction (cf. [2, p. 197]), we make a further attempt to give a clear, epistemological understanding of elements of frames. Referring to the tradition created by Boole (cf. [3, p. 48]) we interpret the universal upper bound $\top$ of frames as the class of "all beings", i.e. *universe*, and the universal lower bound $\bot$ as the class of "no beings", i.e. *nothing*.

Hence in a more geometric language $\top$ represents the *total domain* of TRUTH, while $\bot$ reflects the *absence* or *empty domain* of truth. In this sense, throughout this essay we understand each element of a given frame as domain of truth and maintain the traditional ontological point of view that truth is an *atomic* or *indivisible* property.

An important justification of this general geometric understanding of truth values can be given in the special context of spatial frames. For this purpose we first recall the concept of *points* of frames. As a motivation for the subsequent definition we note that in the case of ordinary topological spaces $(X, \mathcal{O})$ every element $x \in X$ induces a map $\mathcal{O} \xrightarrow{\delta_x} \{0, 1\}$ by

$$\delta_x(G) = \left\{ \begin{matrix} 1 : x \in G \\ 0 : x \notin G \end{matrix} \right\}, \quad G \in \mathcal{O}$$

which preserves obviously arbitrary joins and finite meets. Hence we introduce a *point* $p$ of a frame $\Omega$ *not* as an element of $\Omega$, but as a frame homomorphism $\Omega \xrightarrow{p} \{0, 1\}$ (cf. [33, p. 41]), and observe that there exists a bijective map between the set of all *points* $p$ of $\Omega$ and the set of all *prime elements* $\alpha \neq \top$ of $\Omega$ determined by the following relation:

$$p(\lambda) = \left\{ \begin{matrix} 1 : \lambda \nleqslant \alpha \\ 0 : \lambda \leqslant \alpha \end{matrix} \right\}, \quad \lambda \in \Omega. \tag{1.1}$$

It is not self-evident that every frame *has* points. For example, complete and atomless Boolean algebras (cf. [54, p. 28]; [15, p. 11]) do not have any point, because co-atoms do not exist in atomless Boolean algebras (cf. [15, Remark 3.12, pp. 70–71; 33, p. 4233, p. 42]).

A frame $\Omega$ is called *spatial* iff points separate elements of $\Omega$; this means that for every pair $(\alpha, \beta) \in \Omega \times \Omega$ with $\alpha \neq \beta$ there exists a point $p$ of $\Omega$ with $p(\alpha) \neq p(\beta)$. Lattices of open subsets of topological spaces are spatial frames. Moreover, continuous frames form an important class of spatial frames (cf. [15, 3.14 Theorem33, p. 31133, p. 311]). Without touching the adjunction between $\mathbf{FRM}^{op}$ and **TOP** we briefly recall that every spatial frame $\Omega$ can be identified with a (sober) topological space $(pt(\Omega), \mathcal{T}_\Omega)$. Referring to [33] the set $pt(\Omega)$ consists of all points of $\Omega$, and every element $\alpha \in \Omega$ induces a subset $\mathbb{A}_\alpha$ of $pt(\Omega)$ by

$$\mathbb{A}_\alpha = \{p \in pt(\Omega) \mid p(\alpha) = 1\}. \tag{1.2}$$

Since points preserve arbitrary joins and finite meets, it is easily seen that

$$\mathcal{T}_\Omega = \{\mathbb{A}_\alpha \mid \alpha \in \Omega\} \tag{1.3}$$

forms a topology—the so-called *canonical topology* on $pt(\Omega)$. Because of the spatiality of $\Omega$ the complete Heyting algebras $\Omega$ and $\mathcal{T}_\Omega$ are order isomorphic. In particular, every element $\alpha \in \Omega$ can be identified with the set $\mathbb{A}_\alpha$, and every element of $\mathbb{A}_\alpha$ can be understood as a *binary decider* interpreting «$\alpha$ as true». In this sense $\alpha$ is the domain in which $\alpha$ acts as truth value 1 where we make use of the standard terminology going back to G. Frege.

Since the real unit interval plays a prominent role in the fuzzy community, we recall the previous results in this special setting. Referring to (1.1) *elements* $t$ of $[0, 1[$ and *points* $p$ of the complete Heyting algebra $[0, 1]$ are equivalent objects—i.e.

$$p(\alpha) = \left\{ \begin{matrix} 1 : t < \alpha \\ 0 : \alpha \leqslant t \end{matrix} \right\}, \quad \alpha \in [0, 1]. \tag{1.4}$$

Further, $\mathbb{A}_\alpha$ can be identified with the half-open interval $[0, \alpha[$. Hence $pt([0, 1])$ is homeomorphic to $[0, 1[$ provided with the lower topology $\omega([0, 1[)$:

$$\omega([0, 1[) = \{[0, \alpha[ \,|\, \alpha \in [0, 1]\} \quad \text{(cf. [15, p. 142])}.$$

This means that the real unit interval *viewed as a frame* can be identified with the topological space $([0, 1[\,, \omega([0, 1[))$.

## 2. Ω-valued sets

Let $\Omega$ be a fixed complete Heyting algebra. A pair $(A, E)$ is called a Heyting algebra valued set or more precisely an *Ω-valued set* (cf. [13,18]) iff $A$ is a set and $A \times A \xrightarrow{E} \Omega$ is map satisfying the following axioms:

(E1) $E(a, b) = E(b, a)$ (symmetry).
(E2) $E(a, b) \wedge E(b, c) \leqslant E(a, c)$ (transitivity).

In this context $E$ is called an $\Omega$-valued equality and the value $E(a, b)$ is interpreted as the *largest domain* in which $a$ and $b$ *coincide*. In particular, $E(a, a)$ describes the domain or the extent of *existence* of $a$.

It is easily seen that the symmetry and transitivity axiom imply:

(E0) $E(a, b) \leqslant E(a, a) \wedge E(b, b)$ (strictness).

Hence Heyting algebra valued equalities fulfill the fundamental principle that equality implies existence (cf. [18, p. 274]).

An $\Omega$-valued equality $E$ is said to be *separated* iff $E$ satisfies the additional axiom:

(E3) $E(a, b) = E(a, a) = E(b, b) \Longrightarrow a = b$ (separation).

An $\Omega$-valued set $(A, E)$ is *separated*, if $E$ is a separated $\Omega$-valued equality.

We begin with some typical examples of $\Omega$-valued sets.

**Example 2.1.** (a) Let $A$ be a set. Then the *crisp equality* $E_c$ on $A$ determined by

$$E_c(a, b) = \left\{ \begin{array}{l} \top \ : a = b \\ \bot \ : a \neq b \end{array} \right\}$$

is a separated $\Omega$-valued equality on $A$.

(b) Let $\wedge$ be the binary meet operation on the underlying Heyting algebra $\Omega$. Then $(\Omega, \wedge)$ is a separated $\Omega$-valued set.

(c) Let $\longleftrightarrow$ be the bi-implication in $\Omega$—i.e. $\alpha \longleftrightarrow \beta = (\alpha \to \beta) \wedge (\beta \to \alpha)$ where

$$\alpha \to \beta = \bigvee \{\lambda \in \Omega \mid \alpha \wedge \lambda \leqslant \beta\}, \quad \alpha, \beta \in \Omega.$$

Then $(\Omega, \longleftrightarrow)$ is a separated $\Omega$-valued set.

**Example 2.2** (*Ultrametric spaces*). Let $\Omega$ be the real unit interval [0, 1], and $\varrho$ be an *ultrametric* on $A$—this is a metric satisfying the following stronger version of the triangle inequality:

$$\varrho(x, y) \leqslant \max(\varrho(x, z), \ \varrho(z, y)), \quad x, y, z \in X.$$

Then every nonincreasing map

$$\mathbb{R} \xrightarrow{f} [0, 1] \quad \text{with } f(0) = 1$$

generates a [0, 1]-valued equality on $A$ by $E = f \circ \varrho$.

Before we move to such important concepts as singleton monad, sheaves and espaces étalés, we first try to give a deeper understanding of the meaning of $\Omega$-valued sets. For this purpose we fix a *spatial* frame $\Omega$ and choose an $\Omega$-valued set $(A, E)$. Every *point* of $\Omega$ (i.e. $p \in pt(\Omega)$) determines a partial equivalence relation $\sim_p$ on $A$ as follows:

$$a \sim_p b \iff p(E(a, b)) = 1.$$

Even though $\sim_p$ is not necessarily reflexive, we can construct the set

$$\mathcal{A}_p = \{[a]_p \mid a \in A, \ p(E(a, a)) = 1\},$$

where $[a]_p$ denotes the equivalence class induced by $a \in A$ w.r.t. $\sim_p$. The aim of the following consideration is to understand the role of elements of $\mathcal{A}_p$. First we form the disjoint union

$$\mathcal{A} = \dot{\bigcup_{p \in pt(\Omega)}} \mathcal{A}_p$$

of all $\mathcal{A}_p$ and observe that there exists a map $\mathcal{A} \xrightarrow{\pi} pt(\Omega)$ defined by

$$\pi(z) = p \iff z \in \mathcal{A}_p, \quad z \in \mathcal{A}.$$

Obviously $\pi$ is an object of the comma category **SET** $\downarrow pt(\Omega)$ or a bundle over the base space $pt(\Omega)$ (cf. [18, pp. 89–90]). In particular, $\mathcal{A}_p$ is the *fibre over p* w.r.t. $\pi$, and every element of $\mathcal{A}_p$ is called a *germ at p*.

In the special case of the crisp equality the set of all germs at $p$ coincides with the underlying set $A$ and the bundle with the projection $A \times pt(\Omega) \xrightarrow{\pi} pt(\Omega)$ onto the second component.

After this brief digression we return to our train of thoughts and maintain the notation from Section 1. We now make the fundamental observation that every element $a$ of the support set $A$ of $(A, E)$ induces a map[3] $\mathbb{A}_{E(a,a)} \xrightarrow{\sigma_a} \mathcal{A}$ in the following way:

$$\sigma_a(p) = [a]_p, \quad p \in \mathbb{A}_{E(a,a)}. \tag{2.1}$$

By definition the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\quad\pi\quad} & pt(\Omega) \\ {\scriptstyle\sigma_a}\nwarrow & & \nearrow \\ & \mathbb{A}_{E(a,a)} & \end{array} \tag{2.2}$$

is commutative where $\hookrightarrow$ denotes the inclusion map. Hence $\sigma_a$ is a cross-section of $\pi$ over $\mathbb{A}_{E(a,a)}$. In particular, the largest subset of $pt(\Omega)$ in which $\sigma_a$ and $\sigma_b$ coincide has the form:

$$\mathbb{A}_{E(a,b)} = \{p \in pt(\Omega) \mid \sigma_a(p) = \sigma_b(p)\}. \tag{2.3}$$

Thus the correspondence $a \longmapsto \sigma_a$ is injective iff the underlying $\Omega$-valued equality is separated. Moreover, (2.3) is an interesting confirmation of our previous interpretation that $E(a, b)$ represents the largest domain in which «$a$ and $b$ coincide».

To sum up we have shown that in the case of separated $\Omega$-valued equalities *elements* of $A$ are *cross-sections* of $\pi$ over certain subsets $U$ of $pt(\Omega)$. We pose the question:

> Does every cross-section $\sigma$ of $\pi$ over some $U$ arise in this way—this means: does there exist an element $a \in A$ s.t. $\sigma = \sigma_a$?

We postpone the answer and return to this question when we have provided a topological wrapping for the previous situation. For this purpose we need some additional terminology:

Let $(A, E)$ be a $\Omega$-valued set. A map $A \xrightarrow{f} \Omega$ is called *E-strict and E-extensional* iff $f$ satisfies the following conditions:

(S0) $f(a) \leqslant E(a, a)$, $a \in A$ (strictness).
(S1) $f(a_1) \wedge E(a_1, a_2) \leqslant f(a_2)$, $a_1, a_2 \in A$ (extensionality).

It is interesting to see that due to the transitivity and strictness of $E$ every element $a \in A$ induces an $E$-strict and $E$-extensional map $A \xrightarrow{\widetilde{a}} \Omega$ by

$$\widetilde{a}(b) = E(a, b), \quad b \in A. \tag{2.4}$$

[3] Similar maps occur in the representation of $M$-valued equalities (cf. [29, pp. 305, p. 318]).

Further, on the set $P(A, E)$ of all $E$-strict and $E$-extensional, $\Omega$-valued maps we can introduce a partial ordering which is defined pointwisely as follows:

$$f \leqslant g \iff \forall a \in A: \ f(a) \leqslant g(a).$$

It is not difficult to show that $(P(A, E), \leqslant)$ is a complete Heyting algebra. In particular, the *extent of existence*—i.e. the map $A \xrightarrow{\mathbb{E}} \Omega$ determined by $\mathbb{E}(a) = E(a, a), \ a \in A$—is the top element in $P(A, E)$.

Since $\Omega$ is spatial, $P(A, E)$ is again spatial. In fact, if $f, g \in P(A, E)$ with $f \neq g$, then there exists $a \in A$ with $f(a) \neq f(b)$. Because of the spatiality of $\Omega$ there exists a point $p$ of $\Omega$ with $p(f(a)) \neq p(g(a))$. Hence points of $P(A, E)$ separate elements of $P(A, E)$.

**Lemma 2.3.** *There exists a bijective map between $\mathcal{A}$ and $pt(P(A, E))$.*

**Proof.** Since $\mathcal{A}$ is the disjoint union of all $\mathcal{A}_p$, we identify elements of $\mathcal{A}$ with pairs $(p, [a]_p)$ where $p$ is a point of $\Omega$ and $[a]_p$ is an element of $\mathcal{A}_p$. Then it is not difficult to see that $(p, [a]_p)$ induces a point $q$ of $P(A, E)$ as follows:

$$q(f) = p(f(a')), \quad a' \in [a]_p, \quad f \in P(A, E). \tag{2.5}$$

In particular, the extensionality of $f \in P(A, E)$ guarantees the independence of the previous definition from representatives of $[a]_p$.

(a) We show that the correspondence defined in (2.5) is injective. Therefore let $(\hat{p}, [\hat{a}]_{\hat{p}})$ be a further pair satisfying (2.5). Because of $[a]_p \in \mathcal{A}_p$ and $[\hat{a}]_{\hat{p}} \in \mathcal{A}_{\hat{p}}$ we obtain

$$p(E(a', a')) = 1, \quad \hat{p}(E(a'', a'')) = 1, \quad a' \in [a]_p, \ \hat{a}'' \in [\hat{a}]_{\hat{p}}.$$

Then we infer from (2.5):

$$p(\alpha) = p(\alpha \wedge E(a', a')) = q(\alpha \wedge \mathbb{E}) = \hat{p}(\alpha \wedge E(\hat{a}'', \hat{a}'')) = \hat{p}(\alpha), \quad \alpha \in \Omega.$$

Hence $p = \hat{p}$. Further, in the case of $f = \widetilde{a'}$ with $a' \in [a]_p$ and $\hat{a}'' \in \hat{a}_p$ we apply again (2.5) and obtain

$$1 = p(E(a', a')) = q(\widetilde{a'}) = p(E(a', \hat{a}''));$$

i.e. $[a]_p = [\hat{a}]_p$.

(b) On the other hand, let $q$ be a point of $P(A, E)$. We show that there exists a point $p$ and an equivalence class $[a]_p \in \mathcal{A}_p$ s.t. (2.5) holds. First we observe that $q$ induces a point $p$ of $\Omega$ by $p(\alpha) = q(\alpha \wedge \mathbb{E})(\alpha \in \Omega)$. Because of $\mathbb{E} = \bigvee_{a \in A} \widetilde{a}$ there exists an element $a \in A$ with $q(\widetilde{a}) = 1$. Now we apply the $E$-extensionality of $f \in P(A, E)$ and obtain

$$f \wedge \widetilde{a} = (f(a) \wedge \mathbb{E}) \wedge \widetilde{a}.$$

Hence the relation $q(f) = q(f \wedge \widetilde{a}) = q(f(a) \wedge \mathbb{E}) = p(f(a))$ follows. This means that the correspondence defined by (2.5) is surjective. $\square$

In the following considerations we will identify $\mathcal{A}$ with the set $pt(P(A, E))$ of all points of $P(A, E)$. Due to this identification the bundle $pt(P(A, E)) \xrightarrow{\pi} pt(\Omega)$ attains the following form (cf. (2.5)):

$$[\pi(q)](\alpha) = q(\alpha \wedge \mathbb{E}), \quad \alpha \in \Omega, \ q \in pt(P(A, E)). \tag{2.6}$$

Moreover, this situation is an invitation to topologize $pt(P(A, E))$ and $pt(\Omega)$ by the respective canonical topologies $\mathcal{T}_{P(A,E)}$ and $\mathcal{T}_{\Omega}$ (cf. Section 1). Because of $\pi^{-1}(\mathbb{A}_\alpha) = \mathbb{A}_{\alpha \wedge \mathbb{E}}$ the map $\pi$ is obviously continuous. But we can show something more.

**Lemma 2.4.** *$\pi$ is a local homeomorphism—this means that for every $q \in pt(P(A, E))$ there exists an open neighbourhood $U$ of $q$ and an open neighbourhood $V$ of $\pi(q)$ s.t. the restriction*

$$U \xrightarrow{\pi_{|U}} V$$

*of $\pi$ is a homeomorphism w.r.t. the respective relative topologies.*

**Proof.** We choose $q \in pt(P(A, E))$. Because of $\mathbb{E} = \bigvee_{a\in A} \widetilde{a}$ there exists an element $a \in A$ with $q(\widetilde{a}) = 1$. We show that

$$\mathbb{A}_{\widetilde{a}} \xrightarrow{\pi_{|\mathbb{A}_{\widetilde{a}}}} \mathbb{A}_{E(a,a)}$$

is a homeomorphism. Form the strictness condition (E0) we conclude

$$(E(a, a) \wedge \mathbb{E}) \wedge \widetilde{a} = \widetilde{a}.$$

Hence the range of $\pi_{|\mathbb{A}_{\widetilde{a}}}$ is contained in $\mathbb{A}_{E(a,a)}$. Now we choose two points $q_1, q_2 \in \mathbb{A}_{\widetilde{a}}$ with $\pi(q_1) = \pi(q_2)$. Because of $f \wedge \widetilde{a} = (f(a) \wedge \mathbb{E}) \wedge \widetilde{a}$ we obtain

$$q_1(f) = q_1((f(a) \wedge \mathbb{E}) \wedge \widetilde{a}) = [\pi(q_1)](f(a)) = [\pi(q_2)](f(a)) = q_2(f).$$

Hence $\pi_{|\mathbb{A}_{\widetilde{a}}}$ is injective. Further every point $p \in \mathbb{A}_{E(a,a)}$ induces a point $q \in \mathbb{A}_{\widetilde{a}}$ by

$$q(f) = p(f(a)), \quad f \in P(A, E).$$

Then we obtain: $[\pi(q)](\alpha) = q(\alpha \wedge \mathbb{E}) = p(\alpha \wedge E(a, a)) = p(\alpha)$. Hence $\pi_{|\mathbb{A}_{\widetilde{a}}}$ is surjective. Finally we observe $\mathbb{A}_f \cap \mathbb{A}_{\widetilde{a}} = \mathbb{A}_{f(a)\wedge\mathbb{E}} \cap \mathbb{A}_{\widetilde{a}}$. Then the bijectivity of $\pi_{|\mathbb{A}_{\widetilde{a}}}$ implies

$$\pi_{|\mathbb{A}_{\widetilde{a}}}(\mathbb{A}_f \cap \mathbb{A}_{\widetilde{a}}) = \mathbb{A}_{f(a)}.$$

Thus $\pi_{|\mathbb{A}_{\widetilde{a}}}$ is not only continuous, but also open. □

The previous lemma can also be expressed by the following important statement: $pt(P(A, E)) \xrightarrow{\pi} pt(\Omega)$ is an *espace étalé* (cf. [43, p. 88]). In order to emphazise the dependence on $(A, E)$ we sometimes call $\pi$ the *espace étalé associated with* $(A, E)$.

Further, it is not difficult to see that the relation

$$\sigma_a^{-1}(\mathbb{A}_f) = \mathbb{A}_{f(a)}, \quad a \in A$$

follows immediately from (2.1) and (2.5). Hence for every $a \in A$ the cross-section $\sigma_a$ of $\pi$ over $\mathbb{A}_{E(a,a)}$ is continuous w.r.t. the relative topology induced by $\mathcal{T}_\Omega$ on $\mathbb{A}_{E(a,a)}$— this means that $\sigma_a$ is *local section* of $\pi$ (cf. [18, p. 98]). Because of this observation we return to the above mentioned problem and pose the following topological question:

Does there exist local sections of $\pi$ which are not determined by elements of the underlying support set $A$?

In general, the answer is affirmative as the next example demonstrates.

**Example 2.5** (*Crisp equality*). We assume $\Omega \neq \{\bot, \top\}$ and consider the crisp equality $E_c$ on $A$. Then $P(A, E_c)$ and $\Omega^A$ coincide. Further, we identify $pt(\Omega^A)$ with $A \times pt(\Omega)$ in the sense of Lemma 2.3. Since

$$\{\mathbb{A}_{\alpha\wedge\widetilde{a}} \mid \alpha \in \Omega,\ a \in A\}$$

is a base of the canonical topology $\mathcal{T}_{\Omega^A}$, the *standard topology* $\mathcal{T}_\mathcal{E}$ on $A \times pt(\Omega)$ corresponding to $\mathcal{T}_{\Omega^A}$ is the product topology of $\mathcal{T}_\Omega$ on $pt(\Omega)$ and the discrete topology on $A$. Hence every local section $\sigma$ of the espace étalé $A \times pt(\Omega) \xrightarrow{\pi} pt(\Omega)$ with domain $\mathbb{A}_\alpha$ has the shape $\sigma(p) = (g(p), p)$ where $\mathbb{A}_\alpha \xrightarrow{g} A$ is a *locally constant map*.

Since in the case of $\sigma_a$ ($a \in A$) the corresponding map $g$ coincides with the globally defined, constant map determined by $a$, it is easily seen that there exist various local sections which are *not* induced by points of the underlying support set $A$ of $(A, E_c)$. Hence the correspondence $a \longmapsto \sigma_a$ is an injective, but *not* surjective map from $A$ to the set of all local sections of $A \times pt(\Omega) \xrightarrow{\pi} pt(\Omega)$.

Even though in general we have more local sections than elements, there *exist* $\Omega$-valued sets $(A, E)$ such that elements of $A$ and local sections of the espace étalé associated with $(A, E)$ are equivalent concepts.

**Example 2.6.** Let $(\Omega, \wedge)$ be the $\Omega$-valued set introduced in Example 2.1(b). Since every $\wedge$-strict and $\wedge$-extensional map $f \in P(\Omega, \wedge)$ has the form

$$f(\alpha) = \alpha \wedge f(\top), \quad \alpha \in \Omega,$$

there exists an order isomorphism between $P(\Omega, \wedge)$ and $\Omega$. In particular,

$$pt(\Omega) \xrightarrow{id_{pt(\Omega)}} pt(\Omega)$$

is the espace étalé corresponding to $(\Omega, \wedge)$. Moreover, every local section

$$\mathbb{A}_\alpha \xrightarrow{\sigma} pt(\Omega)$$

coincides with the inclusion map—this means: $\sigma = \sigma_\alpha$. Thus the correspondence $\alpha \longmapsto \sigma_\alpha$ is a *bijective* map from $\Omega$ to the set of all local sections of $pt(\Omega) \xrightarrow{id_{pt(\Omega)}} pt(\Omega)$.

In the following considerations we give a lattice-theoretic characterization of local sections. For this purpose we need some more terminology:

Let $(A, E)$ be an $\Omega$-valued set. An $E$-extensional map $A \xrightarrow{s} \Omega$ is called a *singleton* of $(A, E)$ iff $s$ satisfies the additional property (cf. [13])

(S2) $s(a) \wedge s(b) \leqslant E(a, b)$ $a, b \in A$ (singleton condition).

The *height* of a *singleton* $s$ is defined by $\mathbb{E}(s) = \bigvee_{a \in A} s(a)$. Obviously (S2) implies (S0)—this means that every singleton is *strict*. Further, the symmetry and transitivity axiom of $\Omega$-valued equalities imply that every map $\widetilde{a}$ $(a \in A)$ is a singleton, and the height of $\widetilde{a}$ coincides with $E(a, a)$. Finally, the set of all singletons of $(A, E)$ is denoted by $S(A, E)$.

**Proposition 2.7.** *Let $\Omega$ be a spatial frame, $(A, E)$ be an $\Omega$-valued set, and let $pt(P(A, E)) \xrightarrow{\pi} pt(\Omega)$ be the espace étalé associated with $(A, E)$.*

(a) *Every local section $\mathbb{A}_\alpha \xrightarrow{\sigma} pt(P(A, E))$ of $\pi$ induces a singleton $s$ of $(A, E)$* by

$$\mathbb{A}_{s(a)} = \sigma^{-1}(\mathbb{A}_{\widetilde{a}}), \quad a \in A. \tag{2.7}$$

(b) *For every singleton $s$ of $(A, E)$ there exists a unique local section $\mathbb{A}_{\mathbb{E}(s)} \xrightarrow{\sigma_s} pt(P(A, E))$ of $\pi$ s.t.*

$$[\sigma_s(q)](f) = q(f(a)), \quad f \in P(A, E), \;\; q \in \mathbb{A}_{s(a)}, \quad a \in A. \tag{2.8}$$

(c) *The correspondence $s \longmapsto \sigma_s$ (cf. (b)) is a bijective map from $S(A, E)$ to the set of all local sections of $\pi$.*

**Proof.** The relation (2.7) follows immediately from:

$$\mathbb{A}_{\widetilde{a}} \cap \mathbb{A}_{E(a,b) \wedge \mathbb{E}} \subseteq \mathbb{A}_{\widetilde{b}}, \quad \mathbb{A}_{\widetilde{a}} \cap \mathbb{A}_{\widetilde{b}} \subseteq \mathbb{A}_{E(a,b) \wedge \mathbb{E}}.$$

In order to show that the definition in (2.8) is independent from $a \in A$ we choose a further element $\hat{a} \in A$ with $q \in \mathbb{A}_{s(\hat{a})}$. Then the singleton condition (S2) implies

$$1 = q(s(a) \wedge s(\hat{a})) \leqslant q(E(a, \hat{a})).$$

Now we make use of the $E$-extensionality of $f \in P(A, E)$ and obtain

$$q(f(a)) = q(f(a) \wedge E(a, \hat{a})) = q(f(\hat{a}) \wedge E(a, \hat{a})) = q(f(\hat{a})).$$

Hence the uniqueness of $\sigma_s$ in Assertion (b) follows. We now show that $\sigma_s$ is a local section of $\pi$. Because of

$$\sigma_s^{-1}(\mathbb{A}_f) = \bigcup_{a \in A} \mathbb{A}_{f(a) \wedge s(a)} \tag{2.9}$$

the continuity of $\sigma_s$ is obvious. Further, we choose $q \in \mathbb{A}_{\mathbb{E}(s)}$. Then there exists an element $a \in A$ with $q(s(a)) = 1$. We conclude from the strictness of $s$:

$$[\pi(\sigma_s(q))](\alpha) = [\sigma_s(q)](\alpha \wedge \mathbb{E}) = q(\alpha \wedge E(a, a)) = q(\alpha), \quad \alpha \in \Omega.$$

Hence $\pi \cdot \sigma_s$ coincides with the inclusion map of $\mathbb{A}_{\mathbb{E}(s)}$.

Finally, we make use of the $E$-extensionality of $s$ and obtain from (2.9):

$$\sigma_s^{-1}(\mathbb{A}_{\widetilde{a}}) = \mathbb{A}_{s(a)}, \quad a \in A.$$

Thus the correspondence $s \longmapsto \sigma_s$ is injective. In order to verify the surjectivity of $s \longmapsto \sigma_s$ we consider a singleton $s$ induced by a local section $\sigma$ of $\pi$ according to (2.7) (cf. (a)). Then we obtain for $q \in \mathbb{A}_{s(a)}$ $(a \in A)$:

$$[\sigma(q)](f) = [\sigma(q)](f \wedge \widetilde{a}) = [\sigma(q)]((f(a) \wedge \mathbb{E}) \wedge \widetilde{a}) = [\pi \circ \sigma(q)](f(a)) = q(f(a)).$$

Hence $\sigma$ satisfies (2.8)—i.e. $\sigma_s = \sigma$. □

It follows from the proof of the previous proposition that the domain of local sections coincide with open subsets determined by the height of the corresponding singletons. This observation is a confirmation to interpret the *height of singletons* as their domain or *extent of existence*.

Moreover, we conclude from (2.1), (2.5) and (2.8) that for all $a \in A$ the local sections $\sigma_a$ and $\sigma_{\widetilde{a}}$ coincide. Then the problem that not every local section is determined by an element of the underlying support set can be understood as the fact (cf. Proposition 2.7(c)) that in general not every singleton of $(A, E)$ has the form $\widetilde{a}$ for an appropriate $a \in A$. Thus this situation is a motivation for the following definition (cf. [13]).

**Definition 2.8** (*Completeness*). An $\Omega$-valued set $(A, E)$ is said to be *complete* iff $E$ is separated, and for every singleton $s$ of $(A, E)$ there exists an element $a \in A$ s.t. $s = \widetilde{a}$.

The $\Omega$-valued set $(\Omega, \wedge)$ is complete. In fact, every singleton $s$ of $(\Omega, \wedge)$ has the form: $s(\lambda) = s(\top) \wedge \lambda,\ \lambda \in \Omega$. Moreover, in the case of spatial frames the previous results show that *elements* of support sets of complete $\Omega$-valued sets and *local sections* of the associated espaces étalés are *equivalent* things as illustrated for instance by Example 2.6.

We finish this section with a discussion of finding an appropriate morphism notion between $\Omega$-valued sets. For this purpose we restrict our interest again to spatial frames and look at bundle morphisms between espaces étalés associated with $\Omega$-valued sets as possible candidates. Since $\Omega$ and $P(A, E)$ are spatial frames, we conclude from the duality between **TOP** and **FRM** (cf. [33, Theorem 1.4] or [13, pp. 334–338]) that the espace étalé $\pi$ associated with $(A, E)$ can be identified with the frame homomorphism $\Omega \xrightarrow{h_\pi} P(A, E))$ determined by

$$h_\pi(\alpha) = \alpha \wedge \mathbb{E} \quad \text{where } \mathbb{E}(a) = E(a, a), \ a \in A.$$

Further, let $(A_1, E_1)$ and $(A_2, E_2)$ be two $\Omega$-valued sets and $\pi_1$ and $\pi_2$ be their corresponding espaces étalés. Because of the duality between **TOP** and **FRM** every continuous map $pt(P(A_1 E_1)) \xrightarrow{\varphi} pt(P(A_2, E_2))$ can be identified with a frame homomorphism $P(A_2, E_2) \xrightarrow{h_\varphi} P(A_1, E_1)$ s.t. the following relation holds:

$$\varphi^{-1}(\mathbb{A}_g) = \mathbb{A}_{h_\varphi(g)}, \quad g \in P(A_2, E_2). \tag{2.10}$$

Hence every bundle morphism[4] $\pi_1 \xrightarrow{\ \varphi\ } \pi_2$ —i.e. every continuous map $\varphi$ making the diagram

$$\begin{array}{ccc} pt(P(A_1,E_1)) & \xrightarrow{\qquad \varphi \qquad} & pt(P(A_2,E_2)) \\ & {\scriptstyle \pi_1} \searrow \quad \swarrow {\scriptstyle \pi_2} & \\ & pt(\Omega) & \end{array}$$

commutative—corresponds to the following commutative diagram in **FRM**:

$$\begin{array}{ccc} P(A_2, E_2) & \xrightarrow{\quad h_\varphi \quad} & P(A_1, E_1) \\ & {\scriptstyle h_{\pi_2}} \nwarrow \quad \nearrow {\scriptstyle h_{\pi_1}} & \\ & \Omega & \end{array} \tag{2.11}$$

Further, the universal upper bound in $P(A_2, E_2)$ is denoted by $\mathbb{E}_2$ where $\mathbb{E}_2(a) = E_2(a,a)$. Since $\{\widetilde{a}_2 \mid a_2 \in A_2\} \cup \{\alpha \wedge \mathbb{E}_2 \mid \alpha \in \Omega\}$ is a subbase of $P(A_2, E_2)$, we conclude from (2.11) that $h_\varphi$ is uniquely determined by a map $A_1 \times A_2 \xrightarrow{\ R_\varphi\ } \Omega$ in the following way:

$$[h_\varphi(\widetilde{a}_2)](a_1) = R_\varphi(a_1, a_2), \quad a_1 \in A_1, \ a_2 \in A_2. \tag{2.12}$$

Because of (2.11), (2.12) we infer from the relations

$$\widetilde{a}_2 \leqslant E_2(a_2, a_2) \wedge \mathbb{E}_2, \quad \widetilde{a}_2 \wedge E_2(a_2, b_2) \leqslant \widetilde{b}_2,$$
$$\widetilde{a}_2 \wedge \widetilde{b}_2 \leqslant E_2(a_2, b_2) \wedge \mathbb{E}_2, \quad \bigvee_{a_2 \in A_2} \widetilde{a}_2 = \mathbb{E}_2$$

that the map $R_\varphi$ satisfies the following conditions:

(F1) $E_1(b_1, a_1) \wedge R_\varphi(a_1, a_2) \wedge E_2(a_2, b_2) \leqslant R_\varphi(b_1, b_2)$.
(F2) $R_\varphi(a_1, a_2) \ \wedge \ R_\varphi(a_1, b_2) \leqslant E_2(a_2, b_2)$.
(F3) $E_1(a_1, a_1) = \bigvee_{a_2 \in A_2} R_\varphi(a_1, a_2)$.

In particular, (F2) and (F3) imply the subsequent strictness condition

(F0) $R_\varphi(a_1, a_2) \leqslant E_1(a_1, a_1) \ \wedge \ E_2(a_2, a_2), \quad a_1 \in A_1, \ a_2 \in A_2$.

On the other hand, every map $A_1 \times A_2 \xrightarrow{\ R\ } \Omega$ provided with (F1)–(F3) induces a frame homomorphism $P(A_2, E_2) \xrightarrow{\ h_R\ } P(A_1, E_1)$ by

$$h_R(g) = \bigvee_{a_2 \in A_2} R(a_1, a_2) \wedge g(a_2), \quad g \in P(A_2, E_2)$$

which makes the diagram in (2.11) commutative. Because of (2.10) the map $R$ corresponds to a bundle morphism $\pi_1 \xrightarrow{\ \varphi^R\ } \pi_2$. In particular, $\varphi^R$ is determined fibrewise as follows:

$$\varphi^R_p([a_1]_p) = [a_2]_p \quad \text{where } p(R(a_1, a_2)) = 1, \quad p \in pt(\Omega). \tag{2.13}$$

In this sense mappings satisfying (F1)–(F3) can be viewed as a lattice-theoretic description of bundle morphism between the respective espaces étalés associated with $\Omega$-valued sets.

Finally, if $(A_2, E_2)$ is *complete* we make the important observation that for every map $A_1 \times A_2 \xrightarrow{\ R\ } \Omega$ provided with (F1)–(F3) there exists a *unique* map $A_1 \xrightarrow{\ \psi\ } A_2$ s.t. the following relation holds:

$$R(a_1, a_2) = E_2(\psi(a_1), a_2), \quad a_1 \in A_1, \ a_2 \in A_2. \tag{2.14}$$

[4] It is worthwhile to note that bundle morphisms between espaces étalés are always local homeomorphisms (cf. [43, Exercise 10(b), p. 105]).

In fact, because of (F1) and (F2) for all $a_1 \in A_1$ the map $R(a_1, \_)$ is a singleton of $(A_2, E_2)$. Hence depending on $a_1 \in A_1$ the completeness of $(A_2, E_2)$ (cf. Definition 2.8) implies the unique existence of an element $b_2$ of $A_2$ denoted by $\psi(a_1)$ such that the following relation holds:

$$E_2(\psi(a_1), a_2) = E_2(b_2, a_2) = R(a_1, a_2), \quad a_2 \in A_2.$$

Because of (F3) we obtain immediately:

(m1) $E_1(a_1, a_1) = E_2(\psi(a_1), \psi(a_1))$ (invariance of existence).

Referring again to (F1) we conclude from (2.14), (m1) and the axioms of $\Omega$-valued equalities:

$$\begin{aligned} E_1(a_1, b_1) &= \bigvee_{a_2 \in A_2} E_1(a_1, b_1) \wedge R(b_1, a_2) \leqslant \bigvee_{a_2 \in A_2} R(a_1, a_2) \wedge R(b_1, a_2) \\ &= \bigvee_{a_2 \in A_2} E_2(\psi(a_1), a_2) \wedge E_2(\psi(b_1), a_2) \leqslant E_2(\psi(a_1), \psi(b_1)); \end{aligned}$$

this means that also $\psi$ fulfills the following axiom:

(m2) $E_1(a_1, b_1) \leqslant E_2(\psi(a_1), \psi(b_1))$ (preservation of equality).

On the other hand, if a map $\psi$ fulfills (m1) and (m2), then by means of (2.14) the map $\psi$ induces a map $A_1 \times A_2 \xrightarrow{R_\psi} \Omega$ provided with the properties (F1)–(F3). Thus, if $(A_2, E_2)$ is complete, then bundle morphisms $\pi_1 \xrightarrow{\varphi} \pi_2$ and maps $A_1 \xrightarrow{\psi} A_2$ satisfying (m1) and (m2) are *equivalent* things.

Returning now to our question what are the appropriate morphisms between $\Omega$-valued sets we can give two answers:

1. If we prefer to work with incomplete $\Omega$-valued sets, then «fuzzy morphisms» $R$ satisfying (F1)–(F3) form the right concept, and indeed (F1)–(F3) are precisely the morphism axioms in *Higgs' topos* (cf. [23,18]).
2. If we prefer to work with *crisp maps*, then we have to restrict our interest to complete $\Omega$-valued sets, and the right morphism axioms are given by (m1) and (m2).

In the next section we show that complete $\Omega$-valued sets are *algebras* of the *singleton monad*.

## 3. Singleton monad

In this section $\Omega$ is an arbitrary frame. First we prove that singletons are *irreducible* in the following sense.

**Lemma 3.1.** *Let $s_1$ and $s_2$ be singletons of an $\Omega$-valued set $(A, E)$ provided with the property*

$$\mathbb{E}(s_1) = \mathbb{E}(s_2), \quad s_1(a) \leqslant s_2(a), \ \ a \in A.$$

*Then $s_1$ and $s_2$ coincide—i.e. $s_1 = s_2$.*

**Proof.** Since finite meets are distributive over arbitrary joins, we obtain from (S1) and (S2):

$$s_2(a) = s_2(a) \wedge \mathbb{E}(s_1) = \bigvee_{b \in A} s_2(a) \wedge s_1(b) \wedge s_1(b) \leqslant s_1(a). \qquad \square$$

The next theorem gives a canonical construction of an $\Omega$-valued equality on the set of all singletons of a given $\Omega$-valued set.

**Theorem 3.2.** *Let $(A, E)$ be an $\Omega$-valued set and $S(A, E)$ be the set of all singletons of $(A, E)$. Then there exists a unique $\Omega$-valued equality $\widetilde{E}$ on $S(A, E)$ satisfying the following conditions*:

(i) $\mathbb{E}(s) = \widetilde{E}(s, s), \ \ s \in S(A, E)$.
(ii) $\widetilde{E}(s, \widetilde{x}) = s(x), \ \ x \in A$.

**Proof.** It is easily seen that the $\Omega$-valued equality $\widetilde{E}$ defined by

$$\widetilde{E}(s_1, s_2) = \bigvee_{a \in A} s_1(a) \wedge s_2(a)$$

fulfills the required properties. In order to verify the uniqueness of $\widetilde{E}$ let us consider a further $\Omega$-valued equality $\widehat{E}$ on $S(X, E)$ with (i) and (ii). Then the uniqueness follows from

$$\begin{aligned}\widetilde{E}(s_1, s_2) &\leqslant \widehat{E}(s_1, s_2) = \widehat{E}(s_1, s_2) \wedge \mathbb{E}(s_2)\\ &= \bigvee_{a \in A} \widehat{E}(s_1, s_2) \wedge \widehat{E}(s_2, \widetilde{a}) \wedge s_2(a) \leqslant \widetilde{E}(s_1, s_2). \qquad \square\end{aligned}$$

The $\Omega$-valued set $\Sigma(A, E) := (S(A, E), \widetilde{E})$ is said to be the *singleton space* of $(A, E)$. An immediate application of Lemma 3.1 shows that the $\Omega$-valued equality of singleton spaces is always separated. Moreover, Property (ii) in Theorem 3.2 implies

$$E(a, b) = \widetilde{E}(\widetilde{a}, \widetilde{b}), \quad a, b \in A. \tag{3.1}$$

Hence the singleton space $\Sigma(A, E)$ can be viewed as an *extension* of $(A, E)$ provided the underlying equality $E$ is separated.

In order to make the meaning of this extension more precise we need some categorical tools. For this purpose we first recall the concept of monads (cf. [42]) or algebraic theories according to the terminology proposed by Manes (cf. [45]).

Let $\mathcal{C}$ be a category with the composition denoted by $\cdot$ and $\mathbf{T}$ be a triple $(T, \eta, \circ)$ such that $|\mathcal{C}| \xrightarrow{T} |\mathcal{C}|$ is an *object function*, $\eta$ is an assignment attaching an «insertion-of-the-variables» map $A \xrightarrow{\eta_A} T(A)$ to each object $A \in |\mathcal{C}|$ and $\circ$ assigns a «clone-composition» function

$$hom(X, T(Y)) \times hom(Y, T(Z)) \xrightarrow{\circ} hom(X, T(Z))$$

to each triple $(X, Y, Z)$ of objects of $\mathcal{C}$. A triple $\mathbf{T}$ is called a *monad* or *algebraic theory* (in clone form) in $\mathcal{C}$ iff $\circ$ is associative and the following properties are valid:

$$\eta_Y \circ \varphi = \varphi, \quad \varphi \in hom(X, T(Y)),$$

$$\psi \circ (\eta_Y \cdot f) = \psi \cdot f, \quad \psi \in hom(Y, T(Z)), \quad f \in hom(X, Y).$$

The application of this categorical concept to Heyting algebra valued sets requires an appropriate categorical structure *on* Heyting algebra valued sets. Here we deviate from the way how Fourman and Scott proceed (cf. [18], see also [23]) and do not use «fuzzy» but crisp maps as morphisms. More precisely $(A, E) \xrightarrow{\varphi} (B, F)$ is a morphism from $(A, E)$ to $(B, F)$ iff $A \xrightarrow{\varphi} B$ is a map satisfying the axioms (m1) and (m2) (cf. Section 2). Obviously, $\Omega$-valued sets and morphisms in the previous sense form a category $\Omega$-**SET**. The axiom (m1) means that there exists a functor

$$\Omega\text{-}\mathbf{SET} \xrightarrow{\mathcal{F}} \mathbf{SET} \downarrow \Omega$$

which leaves morphisms invariant and acts on objects as follows:

$$\mathcal{F}(A, E) = \mathbb{E} \quad \text{where } \mathbb{E}(a) = E(a, a), \quad a \in A. \tag{3.2}$$

We conclude from Section 3 in [25] that ($\Omega$-**SET**, $\mathcal{F}$) is a topological category over **SET** $\downarrow \Omega$. Hence Theorem 21.16 in [1] implies:

**Proposition 3.3.** $\Omega$-**SET** *is a complete and cocomplete category.*

For those readers being not all too much familiar with category theory we give an explicit construct of the terminal object, products and equalizers in $\Omega$-**SET**. The *terminal object* $\mathbb{1}$ in $\Omega$-SET has the form $(\Omega, \wedge)$ (cf. Example 2.1(b)). In particular, the strictness property (E0) implies that the unique arrow

$$(A, E) \xrightarrow{\ !\ } \mathbb{1}$$

coincides with the *extent* $\mathbb{E} = \mathcal{F}(A, E)$. Further, the (categorical) *product* $((A, E) \times (B, F), \pi_A, \pi_B)$ is determined by: $(A, E) \times (B, F) = (A \boxtimes B, E \boxtimes F)$ where

$$A \boxtimes B = \{(a, b) \mid E(a, a) = F(b, b)\},$$

$$E \boxtimes F((x_1, y_1), (x_2, y_2)) = E(x_1, x_2) \wedge F(y_1, y_2),$$

$$\pi_A(a, b) = a, \quad \pi_B(a, b) = b.$$

Finally, $D = \{a \in A \mid \varphi(a) = \psi(a)\}$ is the support set of the *equalizer*

$$(D, G) \overset{d}{\hookrightarrow} (A, E) \underset{\psi}{\overset{\varphi}{\rightrightarrows}} (B, F)\ ,$$

where $G$ denotes the restriction of $E$ to $D \times D$ and $D \overset{d}{\hookrightarrow} A$ is the inclusion map.

Moreover, $\Omega$-**SET** carries a *symmetric, monoidal structure* (cf. [34,42]), and the corresponding *tensor product* $\otimes$ and *unit object* $I$ are determined as follows:

$$(A, E) \otimes (B, F) = (A \times B, E \otimes F), \quad I = (\{\cdot\}, \top),$$

where

$$E \otimes B((a_1, b_1), (a_2, b_2)) = E(a_1, a_2) \wedge F(b_1, b_2), \quad a_1, a_2 \in A, \ \ b_1, b_2 \in B.$$

and $\{\cdot\}$ is a set consisting of a single element with global extent of existence. We recall that $(\Omega\text{-}\mathbf{SET}, \otimes, I)$ is even a monoidal closed category—this means that the functor $\_ \otimes (A, E) : \Omega\text{-}\mathbf{SET} \to \Omega\text{-}\mathbf{SET}$ has a right adjoint functor $[(A, E), \_] : \Omega\text{-}\mathbf{SET} \to \Omega\text{-}\mathbf{SET}$ (cf. [25, Theorem 3.6] in the case $\wedge = *$). For this purpose we first construct the *evaluation ev* and the *internal hom-objects*. Let $hom_{\mathrm{loc}}((A, E), (B, F))$ be the set of all pairs $(\alpha, f)$ where $\alpha \in \Omega$ and $A \xrightarrow{f} B$ is a map satisfying the following *localized* version of the $\Omega$-**SET**-morphism axioms (m1) and (m2):

(m1)* $\alpha \wedge E(a, a) = F(f(a), f(a))$.
(m2)* $\alpha \wedge E(a_1, a_2) \leqslant F(f(a_1), f(a_2))$.

Then we put (see also [25, pp. 137, 138]):

- $[(A, E), (B, F)] = (hom_{\mathrm{loc}}((A, E), (B, F)), [\![\ ,\ ]\!])$ where $[\![(\alpha, f), (\beta, g)]\!] = (\alpha \wedge \beta) \wedge \bigwedge_{a \in A} (E(a, a) \to F(f(a), g(a)))$.
- $[(A, E), (B, F)] \times (A, E) \xrightarrow{ev_{(B,F)}} (B, F) : \quad ev_{(B,F)}((\alpha, f), a) = f(a)$.

Then $ev = (ev_{(B,F)})_{(B,F) \in |\Omega\text{-}\mathbf{SET}|}$ is the *counit* of this adjoint situation, and for every $\Omega$-**SET**-morphism

$$(C, G) \otimes (A, E) \xrightarrow{\varphi} (B, F)$$

there exists a *unique* $\Omega$-**SET**-morphism $(C, G) \xrightarrow{\ulcorner \varphi \urcorner} [(A, E), (B, F)]$ s.t. the following diagram is commutative:

$$\begin{array}{ccc} (C, G) \otimes (A, E) & \xrightarrow{\ulcorner \varphi \urcorner \times id_{(A,E)}} & [(A, E), (B, F)] \otimes (A, E) \\ & \searrow^{\varphi} & \downarrow{\scriptstyle ev_{(B,F)}} \\ & & (B, F) \end{array}$$

The morphism $\ulcorner \varphi \urcorner$ is called the *monoidal adjoint* of $\varphi$.

After these preparations we return to the concept of singletons of Heyting algebra valued sets and observe that

- $(A, E) \rightsquigarrow \Sigma(A, E)$ is an object function in $\Omega$-**SET**.
- $A \xrightarrow{\eta_{(A,E)}} \Sigma(A, E)$ by $\eta_{(A,E)}(x) = \widetilde{x}$ is an $\Omega$-**SET**-morphism from $(A, E)$ to $\Sigma(A, E)$ (cf. (3.1)).
- the «clone-composition» function $\circ$ is determined by

$$(A, E) \xrightarrow{\varphi} \Sigma(B, F), \quad (B, F) \xrightarrow{\psi} \Sigma(C, G),$$
$$[\psi \circ \varphi(a)](c) = \bigvee_{b \in B} [\psi(b)](c) \wedge [\varphi(a)](b), \quad a \in A.$$

Referring to Section 3 in [24] we obtain the following theorem in the special case of $* = \wedge$.

**Theorem 3.4.** $\mathbf{T}_\Sigma = (\Sigma, \eta, \circ)$ *is a monad* (*in clone form*) *in* $\Omega$-**SET**.

$\mathbf{T}_\Sigma$ is called the *singleton monad* in $\Omega$-**SET**. In particular, the multiplication $\mu$ is given by

$$[\mu_{(A,E)}(\sigma)](a) = [id_{\Sigma(A,E)} \circ id_{\Sigma(\Sigma(A,E))}(\sigma)](a) = \bigvee_{s \in S(A,E)} s(a) \wedge \sigma(s), \quad a \in A.$$

With regard to Lemma 3.1 it is not difficult to show that $\mathbf{T}_\Sigma$ is a *degenerated* monad—i.e. $\mu : \Sigma \cdot \Sigma \to \Sigma$ is a natural isomorphism. In particular, the *structure morphism* $\xi$ of any $\mathbf{T}_\Sigma$-algebra $((A, E), \xi)$ has the form $\xi = \eta^{-1}_{(A,E)}$. Therefore we make the important observation that $((A, E), \xi)$ is a $\mathbf{T}_\Sigma$-algebra iff $(A, E)$ is a *complete* $\Omega$-valued set (cf. Definition 2.8). Hence the *Eilenberg–Moore* category $\Omega$-$\mathbf{SET}^{\mathbf{T}_\Sigma}$ of $\mathbf{T}_\Sigma$-algebras is isomorphic to the full subcategory $\mathbf{C}\Omega$-**SET** of $\Omega$-**SET** consisting of complete $\Omega$-valued sets (cf. [18]). Since $\Sigma(A, E), \mu_{(A,E)})$ is the *free* $\mathbf{T}_\Sigma$-algebra generated by $(A, E)$, *singleton spaces* are always *complete* $\Omega$-valued sets. In particular, due to the universal property of free $\mathbf{T}_\Sigma$-algebras every $\Omega$-**SET**-morphism $(Y, F) \xrightarrow{\varphi} \Sigma(A, E)$ has a *unique* extension $\Sigma(Y, F) \xrightarrow{\varphi^\sharp} \Sigma(A, E)$ where $\varphi^\sharp$ is given by

$$[\varphi^\sharp(s)](a) = \bigvee_{y \in Y} s(y) \wedge [\varphi(y)](a), \quad s \in S(Y, F), \ a \in A. \tag{3.3}$$

Finally, referring again to the universal property of free algebras it is not difficult to establish the fact that the functor $\Sigma : \Omega\text{-}\mathbf{SET} \to \Omega\text{-}\mathbf{SET}^{\mathbf{T}_\Sigma}$ is left adjoint to the forgetful functor $\mathcal{U} : \Omega\text{-}\mathbf{SET}^{\mathbf{T}_\Sigma} \to \Omega\text{-}\mathbf{SET}$ (cf. [1, Proposition 20.7]); hence $\mathbf{C}\Omega$ -**SET** is a reflective subcategory of $\Omega$-**SET** and the corresponding reflector coincides with the formation of taking singleton spaces.

It is easily seen that the *Kleisli* category $\Omega$-$\mathbf{SET}_{\mathbf{T}_\Sigma}$ is nothing but *Higgs' topos* (cf. [18,23]). Since the singleton monad is degenerated, Proposition 21.3.3 in [51] implies that the Eilenberg–Moore category $\Omega$-$\mathbf{SET}^{\mathbf{T}_\Sigma}$ and the Kleisli category $\Omega$-$\mathbf{SET}_{\mathbf{T}_\Sigma}$ are equivalent. This equivalence is known as Higgs' theorem (see [13, Theorem 5.9]) and is obviously an immediate corollary of the degeneracy of the singleton monad. Thus from a categorical point of view there does not exist an essential difference between $\Omega$-$\mathbf{SET}^{\mathbf{T}_\Sigma}$ and $\Omega$-$\mathbf{SET}_{\mathbf{T}_\Sigma}$. In particular, with regard to the answers given at the end of Section 2 the equivalence between *Higgs' topos* and $\mathbf{C}\Omega$-**SET** indicates that from a categorical point of view «fuzzy morphisms» with (F1)–(F3) between not necessarily complete $\Omega$-valued sets and crisp maps with (m1) and (m2) between complete $\Omega$-valued sets are equivalent concepts. Since $\Omega$-$\mathbf{SET}^{\mathbf{T}_\Sigma}$ contains less isomorphic objects than $\Omega$-$\mathbf{SET}_{\mathbf{T}_\Sigma}$ (every $\Omega$-valued set is isomorphic to its singleton space in $\Omega$-$\mathbf{SET}_{\mathbf{T}_\Sigma}$), we prefer to work with the Eilenberg–Moore category $\Omega$-$\mathbf{SET}^{\mathbf{T}_\Sigma}$, respectively, $\mathbf{C}\Omega$-**SET**.

We finish this section with the explanation of various categorical properties of $\Omega$-$\mathbf{SET}^{\mathbf{T}_\Sigma}$. First we notice that the completeness and cocompleteness is inherited by $\Omega$-$\mathbf{SET}^{\mathbf{T}_\Sigma}$ from $\Omega$-**SET** (see also [51, Remarks 21.3.4]). In the case of spatial frames a simple description of these properties can be found in Remark 5.5. Further, we have the following lemma.

**Lemma 3.5.** *There exists a natural isomorphism* $\Theta : \Sigma \circ \otimes \to \times \circ (\Sigma \times \Sigma)$.

**Proof.** Let $(A, E)$ and $(B, F)$ be $\Omega$-valued sets. Obviously, every pair $(s_1, s_2) \in S(A, E)\boxtimes S(B, F)$ induces a singleton $\sigma$ of $(A, E) \otimes (B, F)$ by

$$\sigma(a, b) = s_1(a) \wedge s_2(b), \quad a \in A, \; b \in B.$$

On the other hand, every singleton $\sigma$ of $(A, E) \otimes (B, F)$ determines a pair of singletons $(s_1^\sigma, s_2^\sigma) \in S(A, E)\boxtimes S(B, F)$ as follows:

$$s_1^\sigma(a) = \bigvee_{b\in B} \sigma(a, b), \quad s_2^\sigma(b) = \bigvee_{a\in A} \sigma(a, b).$$

It is not difficult to show that the correspondence $\sigma \longmapsto s_1^\sigma \wedge s_2^\sigma$ is an $\Omega$-**SET**-isomorphism from $\Sigma((A, E) \otimes (B, F))$ to $\Sigma(A, E) \times \Sigma(B, F)$. The verification of the naturalness of this isomorphism is left to the reader. □

Since $\Sigma$ is left adjoint to the forgetful functor $\mathcal{U}$, we conclude Lemma 3.5 that the existence of the monoidal closed structure on $\Omega$-**Set** entails the existence of the exponentiation in $\Omega$-$\mathbf{SET}^{\mathbf{T}_\Sigma}$, resp. **C**$\Omega$-**SET**. In Section 2.1 of Fuzzy Sets and Sheaves Part II (cf. [30]), we will return to this situation.

## 4. Sheaves

In this section we view complete Heyting algebras $\Omega$ as (complete) categories, and denote the corresponding opposite categories by $\Omega^{op}$. In particular, *objects* of $\Omega^{op}$ are elements of $\Omega$ and *hom-sets* of $\Omega^{op}$ are given as follows:

$$hom(\alpha, \beta) = \left\{ \begin{array}{l} \{\cdot\} \,:\, \beta \leqslant \alpha \\ \emptyset \,:\, \beta \not\leqslant \alpha \end{array} \right\},$$

where $\{\cdot\}$ denotes an ordinary singleton.

Let **SET** be the category of ordinary sets. A *presheaf of sets* on $\Omega$ is a functor $\mathcal{F} : \Omega^{op} \to$ **SET**. In the case of $\beta \leqslant \alpha$ the morphism $\mathcal{F}(\alpha) \xrightarrow{\mathcal{F}(\cdot)} \mathcal{F}(\beta)$ is called the *restriction map* from $\mathcal{F}(\alpha)$ to $\mathcal{F}(\beta)$ and is denoted by $\varrho^\alpha_\beta$. Since $\mathcal{F}$ preserves the *identity* and the *composition*, the restriction maps satisfy always the following conditions:

$$\varrho^\alpha_\alpha = id_{\mathcal{F}(\alpha)}, \quad \varrho^\beta_\gamma \circ \varrho^\alpha_\beta = \varrho^\alpha_\gamma \quad \text{provided } \gamma \leqslant \beta \leqslant \alpha. \tag{4.1}$$

On the other hand, any $\Omega$-indexed family of sets together with a family of maps satisfying (4.1) can be regarded as a presheaf of sets on $\Omega$.

*Morphisms* between presheaves on $\Omega$ are natural transformations $\eta : \mathcal{F} \to \mathcal{G}$; this means that $\eta$ is an $\Omega$-indexed family $(\eta_\alpha)_{\alpha\in\Omega}$ of maps $\mathcal{F}(\alpha) \xrightarrow{\eta_\alpha} \mathcal{G}(\alpha)$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(\alpha) & \xrightarrow{\eta_\alpha} & \mathcal{G}(\alpha) \\ \downarrow & & \downarrow \\ \mathcal{F}(\beta) & \xrightarrow[\eta_\beta]{} & \mathcal{G}(\beta) \end{array} \qquad \beta \leqslant \alpha$$

where the left and right arrow denote the respective restriction maps. Obviously, presheaves on $\Omega$ and (presheaf-) morphisms form a category denoted by $psh(\Omega)$.

A presheaf $\mathcal{F}$ is called *separated* iff $\mathcal{F}$ satisfies the axiom:

($\mathfrak{F}$1) For every $\alpha \in \Omega$, for every pair $(a_1, a_2) \in \mathcal{F}(\alpha) \times \mathcal{F}(\alpha)$ and for every subset $\{\beta_i \mid i \in I\}$ of $\Omega$ with $\alpha = \bigvee_{i\in I} \beta_i$ the following implication holds:

$$\forall i \in I : \; \varrho^\alpha_{\beta_i}(a_1) = \varrho^\alpha_{\beta_i}(a_2) \implies a_1 = a_2.$$

Let $\mathcal{F}$ be a presheaf. A family $\{a_i \mid i \in I\}$ of elements $a_i \in \mathcal{F}(\beta_i)$ is called *compatible* iff the property $\varrho^{\beta_i}_{\beta_i\wedge\beta_j}(a_i) = \varrho^{\beta_j}_{\beta_i\wedge\beta_j}(a_j)$ holds for all $i, j \in I$. A *sheaf* on $\Omega$ is a separated presheaf $\mathcal{F}$ on $\Omega$ provided with the following important property:

($\mathfrak{F}2$) For every subset $\{\beta_i \mid i \in I\}$ of $\Omega$ and for every compatible family $\{a_i \mid i \in I\}$ of elements $a_i \in \mathcal{F}(\beta_i)$ there exists an element $a_0 \in \mathcal{F}(\alpha)$ s.t. $\alpha = \bigvee_{i \in I} \beta_i$ and for all $i \in I : \varrho^{\alpha}_{\beta_i}(a_0) = a_i$.

**Example 4.1.** Every set $X$ can be viewed as a presheaf $\mathcal{F}_X$ on $\Omega$ as follows:

$$\mathcal{F}_X(\alpha) = X, \quad \varrho^{\alpha}_{\beta} = id_X, \quad \alpha, \beta \in \Omega, \quad \beta \leqslant \alpha.$$

$\mathcal{F}_X$ is also called the *constant presheaf* generated by $X$.

The next example is related to «level cuts» which is a widespread method in fuzzy set theory (cf. [10, pp. 44–45]).

**Example 4.2** (*Unit interval*). Let $(A_\alpha)_{\alpha \in [0,1]}$ be a [0, 1]-indexed family of sets equipped with the following property:

$$A_0 = \{\cdot\}, \quad A_\alpha \subseteq A_\beta \quad \text{provided } 0 < \beta \leqslant \alpha.$$

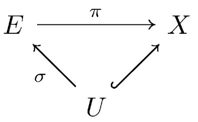
If in the case of $0 < \beta \leqslant \alpha$ we consider the respective inclusion maps as restriction maps, then the previous situation gives rise to a separated presheaf $\mathcal{F}$ on [0, 1]. Sometimes $\mathcal{F}$ is also called a *presheaf of level cuts*.

Obviously, $\mathcal{F}$ is a sheaf on [0, 1] iff $(A_\alpha)_{\alpha \in [0,1]}$ fulfills the so-called «upper continuity» condition:

$$\bigcap_{0<\beta<\alpha} A_\beta = A_\alpha.$$

**Remark 4.3** (*Sheaf of local sections*). Let $(X, \mathcal{T})$ be a topological space and **TOP** be the category of topological spaces. An object $\pi$ of the comma category **TOP** $\downarrow (X, \mathcal{T})$ is called a *fibrewise topological space* with base space $(X, \mathcal{T})$ (cf. [32]).

A *local section over U* of a fibrewise topological space $(E, \mathcal{T}_E) \xrightarrow{\pi} (X, \mathcal{T})$ is a continuous map $U \xrightarrow{\sigma} E$ s.t. the following diagram commutes:



where $U$ is an open subset of $X$ and $\hookrightarrow$ denotes the inclusion map.

We define a set-valued map $\mathcal{G}_\pi$ on $\mathcal{T}$ as follows: For every $U \in \mathcal{T}$ let $\mathcal{G}_\pi(U)$ be the set of all local sections over $U$. In particular, $\mathcal{G}_\pi(\emptyset)$ contains exactly a unique element, namely the empty map. As restriction operation we consider the usual restriction of maps. Then this situation gives rise to a sheaf on $\mathcal{T}$ which again is denoted by $\mathcal{G}_\pi$. In particular $\mathcal{G}_\pi$ is called the *sheaf of local sections* of the fibrewise topological space $\pi$.

In the following considerations we explain the relationship between presheaves on $\Omega$ and $\Omega$-valued sets. First we notice that with every separated presheaf $\mathcal{F}$ on $\Omega$ we can associate a unique separated $\Omega$-valued equality being compatible with the underlying presheaf structure in the following sense: Let $A_{\mathcal{F}}$ be the support set of $\mathcal{F}$ defined by the disjoint union of all $\mathcal{F}(\alpha)$—i.e. $A_{\mathcal{F}} = \dot{\bigcup}_{\alpha \in \Omega} \mathcal{F}(\alpha)$. Then there exists a *unique* separated $\Omega$-valued equality $E_{\mathcal{F}}$ on $A_{\mathcal{F}}$ provided with the following properties (cf. [28]):

- $\beta \wedge E_{\mathcal{F}}(a, b) = E_{\mathcal{F}}(\varrho^{\alpha}_{\alpha \wedge \beta}(a), b), \quad a \in \mathcal{F}(\alpha) \;\; b \in A_{\mathcal{F}}.$
- $E_{\mathcal{F}}(a, a) = \alpha, \;\; a \in \mathcal{F}(\alpha).$

In particular, $E_{\mathcal{F}}$ is given by (cf. [13]):

$$E_{\mathcal{F}}(a, b) = \bigvee \{\gamma \in \Omega \mid \gamma \leqslant \alpha \wedge \beta, \; \varrho^{\alpha}_{\gamma}(a) = \varrho^{\beta}_{\gamma}(b)\}, \quad a \in \mathcal{F}(\alpha), \;\; b \in \mathcal{F}(\beta). \tag{4.2}$$

**Proposition 4.4.** *Let $\mathcal{F}$ be a separated presheaf on $\Omega$, and let $(A_{\mathcal{F}}, E_{\mathcal{F}})$ be the associated separated $\Omega$-valued set. Then the following assertions are equivalent*:

(i) $(A_{\mathcal{F}}, E_{\mathcal{F}})$ *is complete*.
(ii) $\mathcal{F}$ *is a sheaf*.

**Proof.** (a) ((i) $\Longrightarrow$ (ii)) Every compatible family $\{a_i \mid i \in I\}$ of elements $a_i \in \mathcal{F}(\beta_i)$ induces a singleton $s$ of $(A_{\mathcal{F}}, E_{\mathcal{F}})$ as follows:

$$s(b) = \bigvee_{i\in I} E_{\mathcal{F}}(a_i, b), \quad b \in A_{\mathcal{F}}.$$

The extensionality of $s$ is evident. In order to verify the singleton condition it is sufficient to show:

$$E_{\mathcal{F}}(a_i, b_1) \wedge E_{\mathcal{F}}(a_j, b_2) \leqslant E_{\mathcal{F}}(b_1, b_2).$$

In fact, the compatibility of $E_{\mathcal{F}}$ with the presheaf structure implies:

$$\begin{aligned} &E_{\mathcal{F}}(a_i, b_1) \wedge E_{\mathcal{F}}(a_j, b_2) = \beta_i \wedge \beta_j \wedge E_{\mathcal{F}}(a_i, b_1) \wedge E_{\mathcal{F}}(a_j, b_2) \\ &\quad = E_{\mathcal{F}}(\varrho^{\beta_i}_{\beta_i\wedge\beta_j}(a_i), b_1) \wedge E_{\mathcal{F}}(\varrho^{\beta_j}_{\beta_i\wedge\beta_j}(a_j), b_2) \leqslant E_{\mathcal{F}}(b_1, b_2). \end{aligned}$$

Since $(A_{\mathcal{F}}, E_{\mathcal{F}})$ is complete, there exists $a_0 \in A_{\mathcal{F}}$ with $s = \widetilde{a}_0$. Because of $\mathbb{E}(s) = \alpha := \bigvee_{i\in I} \beta_i$ the element $a_0$ is contained in $\mathcal{F}(\alpha)$, and for all $i \in I$ the relation $E_{\mathcal{F}}(a_0, a_i) = \beta_i$ holds. Then the axiom ($\mathfrak{F}1$) and the definition of $E_{\mathcal{F}}$ imply $\varrho^{\alpha}_{\beta_i}(a_0) = a_i$. Hence the assertion (ii) is verified.

(b) ((ii) $\Longrightarrow$ (i)) Let $s$ be a singleton of $(A_{\mathcal{F}}, E_{\mathcal{F}})$. We put $\alpha = \mathbb{E}(s)$ and $\beta_a = E_{\mathcal{F}}(a, a) (a \in A_{\mathcal{F}})$. Then the singleton condition implies that

$$\{\varrho^{\beta_a}_{s(a)}(a) \mid a \in A_{\mathcal{F}}\}$$

is a compatible family. Since $\mathcal{F}$ is a sheaf, there exists an element $a_0 \in \mathcal{F}(\alpha)$ s.t. $\varrho^{\alpha}_{s(a)}(a_0) = \varrho^{\beta_a}_{s(a)}(a)$. Now we invoke again the definition of $E_{\mathcal{F}}$ and obtain: $s(a) \leqslant E_{\mathcal{F}}(a_0, a)$. Finally, we make use of $\mathbb{E}(s) = \alpha = E_{\mathcal{F}}(a_0, a_0)$ and infer form Lemma 3.1: $\widetilde{a}_0 = s$. Hence the assertion (i) is verified. □

It is not difficult to show that every complete $\Omega$-valued set $(A, E)$ determines a sheaf $\mathcal{F}$ as follows:

$$\mathcal{F}(\alpha) = \{a \in A \mid E(a, a) = \alpha\}, \qquad \mathcal{F}(\alpha) \xrightarrow{\varrho^{\alpha}_{\beta}} \mathcal{F}(\beta) \quad \text{by}$$
$$E(\varrho^{\alpha}_{\beta}(a), b) = \beta \wedge E(a, b), \quad a \in \mathcal{F}(\alpha), \ b \in A, \ \beta \leqslant \alpha.$$

Hence Proposition 4.4 and the previous observation lead to the following statement: Complete $\Omega$-valued sets and sheaves on $\Omega$ are equivalent concepts (cf. [13, Theorem 4.13]).

We continue our considerations with the investigation of the relationship between sheaves and espaces étalés. First we recall that every presheaf on a *spatial* frame $\Omega$ generates an espace étalé. For this purpose we identify $\Omega$ with the topological space $(pt(\Omega), \mathcal{T}_{\Omega})$ (cf. Section 1). Then we fix a point $p$ of $\Omega$ and define a subset $\mathbb{V}_p$ of $\Omega$ by

$$\mathbb{V}_p \quad = \quad \{\alpha \in \Omega \mid p(\alpha) = 1\}.$$

Obviously $\mathbb{V}_p$ is order-isomorphic to the set of all open neighbourhoods of $p$. Since $p$ preserves finite meets, the set $\mathbb{V}_p$ is directed downward. Because of (4.1) it is easily seen that $(\mathcal{F}(\alpha), \varrho^{\alpha}_{\beta})_{\alpha\in\mathbb{V}_p}$ is an *inductive system of sets*. Hence the inductive limit of $(\mathcal{F}(\alpha), \varrho^{\alpha}_{\beta})_{\alpha\in\mathbb{V}_p}$ exists—i.e.

$$\widetilde{\mathcal{F}}(p) \quad = \quad \text{ind.} \lim_{\alpha\in\mathbb{V}_p} \mathcal{F}(\alpha), \tag{4.3}$$

and the canonical map from $\mathcal{F}(\alpha)$ to $\widetilde{\mathcal{F}}(p)$ is denoted by $\varrho^{\alpha}_{p}$ (cf. [5, pp. 202–211]). In particular, elements of $\widetilde{\mathcal{F}}(p)$ are called *germs of* $\mathcal{F}$ at $p$.

Further, let $\widetilde{\mathcal{F}}$ be the *disjoint union* of $\widetilde{\mathcal{F}}(p)$ where $p$ is varying in $pt(\Omega)$; and let $\widetilde{\mathcal{F}}(p) \xhookrightarrow{\imath_p} \widetilde{\mathcal{F}}$ be the respective inclusion map. Then every $a \in \mathcal{F}(\alpha)$ induces a map $\mathbb{A}_\alpha \xrightarrow{\sigma_a} \widetilde{\mathcal{F}}$ by

$$\sigma_a(p) = \imath_p(\varrho_p^\alpha(a)), \quad p \in \mathbb{A}_\alpha. \tag{4.4}$$

On $\widetilde{\mathcal{F}}$ we consider the finest ordinary topology $\mathcal{T}_{\widetilde{\mathcal{F}}}$ such that all maps $\sigma_a$ are continuous where $a$ is varying in $\mathcal{F}(\alpha)$ and $\alpha$ in $\Omega$. Further, we introduce a map $\widetilde{\mathcal{F}} \xrightarrow{\pi} pt(\Omega)$ by

$$\pi(e) = p \iff e \in \imath_p(\widetilde{\mathcal{F}}(p)). \tag{4.5}$$

It can be shown that $\pi$ is a local homeomorphism (cf. [16, p. 111]). Hence $\pi$ is an espace étalé.

We observe that for every element $a \in \mathcal{F}(\alpha)$ the map $\sigma_a$ is a local section over $\mathbb{A}_\alpha$. In particular, the construction of inductive limits imply the following relation:

$$\sigma_{a|\mathbb{A}_\beta}(p) = \imath_p(\varrho_p^\alpha(a)) = \imath_p(\varrho_p^\beta(\varrho_\beta^\alpha(a))) = \sigma_{\varrho_\beta^\alpha(a)}(p),$$

where $p \in \mathbb{A}_\beta$ and $\beta \leqslant \alpha$. Thus this situation gives rise to a presheaf-morphism $\eta = (\eta_\alpha)_{\alpha\in\Omega}$ from $\mathcal{F}$ to the sheaf $\mathcal{G}_\pi$ of local sections of $\pi$ determined by $\eta_\alpha(a) = \sigma_a$. Even though this construction can be completed to an adjoint situation (where $\eta$ plays the part of the unit), we restrict ourselves to quote only the following facts (cf. [16, II.1.2]):

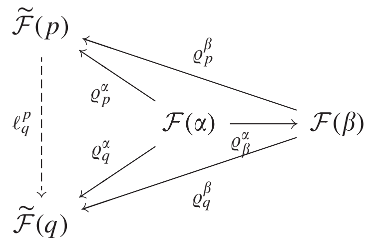
- All components $\eta_\alpha$ of $\eta$ are injective iff $\mathcal{F}$ is a separated presheaf.
- All components $\eta_\alpha$ of $\eta$ are bijective iff $\mathcal{F}$ is a sheaf.

Hence every sheaf on a spatial frame can be understood as a sheaf of local sections of an appropriate espace étalé. Or more precisely: Sheaves on spatial frames $\Omega$ and espaces étalés with base space $(pt(\Omega), \mathcal{T}_\Omega)$ are equivalent concepts.

In the following considerations we disclose the role of the *specialization order* on $pt(\Omega)$ in the light of the previous constructions. First we recall its definition:

$$p \leqslant q \iff \forall \alpha \in \Omega : \quad p(\alpha) \leqslant q(\alpha).$$

Then the universal arrow $\widetilde{\mathcal{F}}(p) \xrightarrow{\ell_q^p} \widetilde{\mathcal{F}}(q)$ of the inductive limit $\widetilde{\mathcal{F}}(p)$ is determined by the commutativity of the following diagram:




where $p \leqslant q, \ \beta \leqslant \alpha$. (4.6)

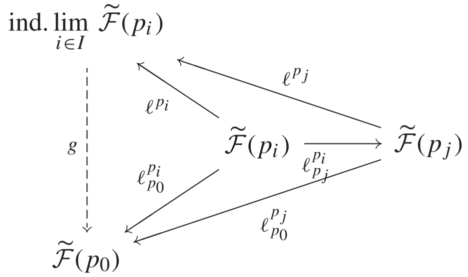
If we view $(pt(\Omega), \leqslant)$ as a category, then we conclude from (4.6) that the assignment $p \longmapsto \widetilde{\mathcal{F}}(p)$ constitutes a functor form $(pt(\Omega), \leqslant)$ to **SET**. The next lemma points out a « continuity» property of this functor.

**Lemma 4.5.** *Let $\mathcal{P} = \{p_i \mid i \in I\}$ be a directed subset of points w.r.t. the specialization order. Then $p_0$ defined by*

$$p_0(\alpha) = \bigvee_{i\in I} p_i(\alpha), \quad \alpha \in \Omega$$

*is a point of $\Omega$, and the universal arrow* $\text{ind.}\lim_{i\in I} \widetilde{\mathcal{F}}(p_i) \xrightarrow{g} \widetilde{\mathcal{F}}(p_0)$ *determined by $(\ell_{p_0}^{p_i})_{i\in I}$ is a bijective map.*

**Proof.** It is easily seen that $p_0$ is a point. Further, the map $g$ is determined by the commutativity of the subsequent diagram:




where $\quad p_i \leqslant p_j$. (4.7)

In order to show the surjectivity of $g$ we choose a germ $[a]$ at $p_0$—i.e. $[a] \in \widetilde{\mathcal{F}}(p_0)$. Then there exists $\alpha \in \Omega$ and an element $a \in \mathcal{F}(\alpha)$ s.t. $p_0(\alpha) = 1$ and $\varrho^{\alpha}_{p_0}(a) = [a]$. In particular, the definition of $p_0$ implies that there exists a point $p_i \in \mathcal{P}$ with $p_i(\alpha) = 1$. Now we invoke the definition of $\ell^{p_i}_{p_0}$ and obtain: $[a] = \ell^{p_i}_{p_0}(\varrho^{\alpha}_{p_i}(a))$. Hence the relation $g(\ell^{p_i}(\varrho^{\alpha}_{p_i}(a))) = [a]$ follows.
Further, we choose $[\![a]\!], [\![b]\!] \in \text{ind}\lim_{i\in I} \widetilde{\mathcal{F}}(p_i)$ with $g([\![a]\!]) = g([\![b]\!])$. Then there exist points $p_i, p_j \in \mathcal{P}$ and $[a] \in \widetilde{\mathcal{F}}(p_i)$, $[b] \in \widetilde{\mathcal{F}}(p_j)$ s.t. $\ell^{p_i}([a]) = [\![a]\!]$ and $\ell^{p_j}([b]) = [\![b]\!]$. In order to verify the injectivity of $g$ we have to show that there exists a point $p_l \in \mathcal{P}$ with $p_i \leqslant p_l,\ p_j \leqslant p_l,\ \ell^{p_i}_{p_l}([a]) = \ell^{p_j}_{p_l}([b])$. For this purpose we choose $\alpha, \beta \in \Omega,\ a \in \mathcal{F}(\alpha),\ b \in \mathcal{F}(\beta)$ with $\varrho^{\alpha}_{p_i}(a) = [a]$ and $\varrho^{\beta}_{p_j}(b) = [b]$. Because of

$$\varrho^{\alpha}_{p_0}(a) = \ell^{p_i}_{p_0}([a]) = g([\![a]\!]) = g([\![b]\!]) = \ell^{p_j}_{p_0}([b]) = \varrho^{\beta}_{p_0}(b)$$

there exists $\gamma \in \Omega$ with $p_0(\gamma) = 1,\ \gamma \leqslant \alpha \wedge \beta$ and $c := \varrho^{\alpha}_{\gamma}(a) = \varrho^{\beta}_{\gamma}(b)$. Referring again to the definition of $p_0$ we can choose a point $p_k \in \mathcal{P}$ with $p_k(\gamma) = 1$. Since $\mathcal{P}$ is directed there exists a point $p_l \in \mathcal{P}$ s.t. $p_i \leqslant p_l,\ p_j \leqslant p_l$ and $p_k \leqslant p_l$. Then we conclude from the previous constructions:

$$\ell^{p_i}_{p_l}([a]) = \varrho^{\alpha}_{p_l}(a) = \varrho^{\gamma}_{p_l}(c) = \varrho^{\beta}_{p_l}(b) = \ell^{p_j}_{p_l}([b]). \qquad \square$$

Motivated by the previous lemma we ask the following question: Can every functor $(pt(\Omega), \leqslant) \xrightarrow{\mathcal{G}} \mathbf{SET}$ provided with the «continuity» property from Lemma 4.5 be derived from a sheaf $\mathcal{F}$ on $\Omega$ in such a way that for all $p \in pt(\Omega)$ the set $\mathcal{G}(p)$ coincides with the set $\widetilde{\mathcal{F}}(p)$ of all germs at $p$?

In the case of completely distributive lattices we have a positive answer as the next theorem demonstrates.

**Theorem 4.6.** *Let* $(pt(\Omega), \leqslant) \xrightarrow{\mathcal{G}} \mathbf{SET}$ *be a functor provided with the «continuity» condition from Lemma* 4.5. *If* $\Omega$ *is a completely distributive lattice, then there exists a sheaf* $\mathcal{F}$ *on* $\Omega$ *and a natural isomorphism* $\widetilde{\mathcal{F}} \xrightarrow{\eta} \mathcal{G}$.

**Proof.** (a) Let $\ll^{op}$ be the *dual way below* relation (cf. [15])—i.e. $\alpha \ll^{op} \beta$ iff for all $D \subseteq \Omega$ being directed downward the relation $\bigwedge D \leqslant \beta$ implies the existence of a $\gamma \in D$ with $\gamma \leqslant \alpha$. Then for every $\lambda \in \Omega$ we define a subset $F(\lambda)$ of $\Omega$ by

$$F(\lambda) = \{\alpha \in \Omega \mid \alpha \text{ is prime},\ \alpha \ll^{op} \lambda\}.$$

Since $\Omega$ is completely distributive, we conclude from 3.15 Theorem in [15] that for all $\lambda$ the relation $\bigwedge F(\lambda) = \lambda$ holds. Moreover, if $\alpha \in \Omega$ is prime, then $F(\alpha)$ is even directed downward. Indeed, let us consider two elements $\beta_1,\ \beta_2 \in F(\alpha)$. If we assume

$$\bigwedge\{\gamma \in F(\alpha) | \gamma \leqslant \beta_1\} \nleqslant \alpha,$$

then the prime property of $\alpha$ implies:

$$\bigwedge\{\gamma \in F(\alpha) | \gamma \nleqslant \beta_1\} = \alpha.$$

Since $\beta_1$ is chosen from $F(\alpha)$, we conclude from the prime property of $\beta_1$ that there exists a $\gamma \in F(\alpha)$ with $\gamma \leqslant \beta_1$ *and* $\gamma \nleqslant \beta_1$ which is absurd. Hence the prime property of $\alpha$ implies:

$$\bigwedge \{\gamma \in F(\alpha) | \gamma \leqslant \beta_1\} = \alpha.$$

Since $\beta_2$ is also an element of $F(\alpha)$, there exists an element $\gamma \in F(\alpha)$ with $\gamma \leqslant \beta_1$ *and* $\gamma \leqslant \beta_2$.

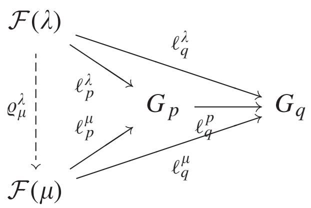
(b) Let $(pt(\Omega), \leqslant) \xrightarrow{\mathcal{G}} \mathbf{SET}$ be a functor. Then we put

$$G_p = \mathcal{G}(p), \quad \ell_q^p = \mathcal{G}(\cdot) \quad \text{where } p \leqslant q, \quad p, q \in pt(\Omega).$$

Further, we identify every $\lambda \in \Omega$ with the open subset $\mathbb{A}_\lambda$ of $pt(\Omega)$ (cf. Section 1) and consider the following projective (resp. inverse) limit (cf. [5]):

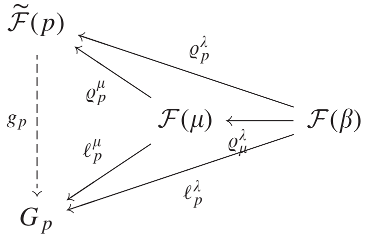
$$\mathcal{F}(\lambda) = \text{proj.} \lim_{p \in \mathbb{A}_\lambda} G_p = \left\{ (a_p)_{p \in \mathbb{A}_\alpha} \in \prod_{p \in \mathbb{A}_\lambda} G_p \,\middle|\, \ell_q^p(a_p) = a_q \text{ whenever } p \leqslant q \right\}.$$

Obviously, the canonical map $\mathcal{F}(\lambda) \xrightarrow{\ell_p^\lambda} G_p$ coincides with the restriction of the projection $\prod_{p' \in \mathbb{A}_\lambda} G_{p'} \xrightarrow{\pi_p} G_p$ to $\mathcal{F}(\lambda)$. Moreover, the restriction map $\varrho_\mu^\lambda$ is determined by the following diagram:




where $p \leqslant q, \ \mu \leqslant \lambda$.

Because of $\varrho_\mu^\lambda((a_p)_{p \in \mathbb{A}_\lambda}) = (a_p)_{p \in \mathbb{A}_\mu}$ $(\mu \leqslant \lambda)$ it is not difficult to show that $\mathcal{F}$ is a sheaf on $\Omega$. Further, let $\widetilde{F}(p) = \text{ind.} \lim_{\lambda \in \mathbb{V}_p} \mathcal{F}(\lambda)$ be the set of all germs at $p$ w.r.t. $\mathcal{F}$, and $\varrho_p^\lambda$ be the corresponding canonical map. Then $g_p$ makes the following diagram commutative:




where $\mu \leqslant \lambda$.

It is not difficult to show that $\eta = (g_p)_{p \in pt(\Omega)}$ is a natural transformation from $\widetilde{\mathcal{F}}$ to $\mathcal{G}$.

(c) In order to verify the bijectivity of $g_p$ we identify *prime* elements $\alpha \in \Omega \setminus \{\top\}$ with *points* of $\Omega$ and vice versa (cf. (1.1) in Section 1)—i.e.

$$p(\lambda) = p_\alpha(\lambda) = \begin{Bmatrix} 1 : \lambda \nleqslant \alpha \\ 0 : \lambda \leqslant \alpha \end{Bmatrix}, \quad \lambda \in \Omega.$$

Then we conclude from the previous part (a) that $\{p_\beta \mid \beta \in F(\alpha)\}$ is directed and $\bigvee \{p_\beta \mid \beta \in F(\alpha)\} = p_\alpha$. Further, we choose $[a], [b] \in \widetilde{\mathcal{F}}(p_\alpha)$ with $g_{p_\alpha}([a]) = g_{p_\alpha}([b])$. Hence there exists an element $\lambda \in \Omega$ and elements $a, b \in \mathcal{F}(\lambda)$ with $\varrho_{p_\alpha}^\lambda(a) = [a], \ \varrho_{p_\alpha}^\lambda(b) = [b], \ \ell_{p_\alpha}^\lambda(a) = \ell_{p_\alpha}^\lambda(b)$. Since $G_{p_\alpha}$ is the inductive limit of $\{G_{p_\beta} \mid \beta \in F(\alpha)\}$ and $\{p_\beta \mid$

$\beta \in F(\alpha)\}$ is directed, there exists an element $\beta \in F(\alpha)$ s.t. $c := \ell^{\lambda}_{p_\beta}(a) = \ell^{\lambda}_{p_\beta}(b)$ and $\ell^{p_\beta}_{p_\alpha}(c) = g_{p_\alpha}([a]) = g_{p_\alpha}([b])$. Now we define the following element $\mu$ of $\Omega$:

$$\mu = \bigwedge\{\lambda' \in \Omega | \lambda' \not\leqslant \beta\}.$$

Because of $\beta \in F(\alpha)$ we obtain: $\mu \not\leqslant \alpha$. Hence $p_\alpha(\mu) = 1$—i.e. $\mu \in \mathbb{V}_{p_\alpha}$, and for all $p \in \mathbb{A}_\mu$ the relation $p_\beta \leqslant p$ follows. In particular, for all $p \in \mathbb{A}_\mu$ the relation $\ell^{p_\beta}_{p}(c) = \ell^{\lambda}_{p}(a) = \ell^{\lambda}_{p}(b)$ holds—i.e. $\varrho^{\lambda}_{\mu}(a) = \varrho^{\lambda}_{\mu}(b)$. Hence $[a] = [b]$ follows, and the injectivity of $g_{p_\alpha}$ is verified.

In order to prove the surjectivity of $g_{p_\alpha}$ we choose an element $d \in G_{p_\alpha}$ and invoke again the fact that $G_{p_\alpha}$ is the inductive limit of $\{G_{p_\beta} \mid \beta \in F(\alpha)\}$. Hence there exists an element $\beta \in F(\alpha)$ and an element $c \in G_{p_\beta}$ with $d = \ell^{p_\beta}_{p_\alpha}(c)$. Now we enter into the previous construction and obtain an element $a \in \mathcal{F}(\mu)$ determined by

$$a = (\ell^{p_\beta}_{p}(c))_{p \in \mathbb{A}_\mu}.$$

Hence the relation $d = \ell^{p_\beta}_{p_\alpha}(c) = \ell^{\mu}_{p_\alpha}(a) = g_{p_\alpha}(\varrho^{\mu}_{p_\alpha}(a))$ holds. □

Since the real unit interval is a complete chain and therefore completely distributive, we would like to explain the previous situations in the context given by Example 4.2.

**Remark 4.7** (*Sheaf of level cuts*). Let $\mathcal{F}$ be a sheaf of level cuts. Then there exists a $[0, 1]$-indexed family $\{A_\alpha \mid \alpha \in [0, 1]\}$ of sets $A_\alpha$ s.t.

$$A_0 = \{\cdot\}, \quad \bigcap_{0<\beta<\alpha} A_\beta = A_\alpha, \quad \alpha \in ]0, 1] \quad \text{(cf. Example 4.2).}$$

Further, let $([0, 1[, \omega([0, 1[))$ be the topological space representing the frame $\Omega = [0, 1]$ (cf. Section 1). Since the restriction maps of $\mathcal{F}$ are set-inclusion maps, the set $\widetilde{\mathcal{F}}(t)$ of all *germs* at $t \in [0, 1[$ has the following form:

$$\widetilde{\mathcal{F}}(t) = \bigcup_{\alpha \in ]t,1]} A_\alpha. \tag{4.8}$$

Hence the espace étalé $\widetilde{\mathcal{F}} \xrightarrow{\ \pi\ } [0, 1[$ is given by

$$\widetilde{\mathcal{F}} = \bigcup_{t \in [0,1[} \widetilde{\mathcal{F}}(t) \times \{t\} = \{(a, t) \mid t < \alpha,\ a \in A_\alpha\}, \quad \pi(a, t) = t,$$

and

$$\{\{a\} \times [0, \beta[ \,|\, \beta < \alpha,\ a \in A_\alpha\}$$

is a base of the topology on $\widetilde{\mathcal{F}}$. In particular, for every local section $\sigma$ of $\pi$ with domain $[0, \alpha[(0 < \alpha)$ there exists a unique element $a \in A_\alpha$ s.t.

$$\sigma = \sigma_a, \quad \sigma_a(t) = (a, t), \quad t \in [0, \alpha[.$$

In this context the «continuity» property from Lemma 4.5 attains the form

$$\widetilde{\mathcal{F}}(t) = \bigcup_{t<t'} \widetilde{\mathcal{F}}(t'), \quad t \in [0, 1[. \tag{4.9}$$

Moreover Theorem 4.6 can be put in concrete terms as follows: Since the specialization order on $[0, 1[$ is dual to the usual ordering $\leqslant$ on $[0, 1[$, every $[0, 1[$-indexed family $\{F_t \mid t \in [0, 1[\,\}$ of sets $F_t$ satisfying (4.9) defines a functor $([0, 1[, \leqslant^{op}) \xrightarrow{\ \mathcal{G}\ } \mathbf{SET}$ by

$$\mathcal{G}(t) = F_t, \qquad F_t \xhookrightarrow{\ \ell^t_s\ } F_s, \quad s \leqslant t,$$

where $\ell_s^t$ denotes the set-inclusion map. Because of (4.9) $\mathcal{G}$ fulfills the «continuity» condition from Lemma 4.5. Then the projective limit construction in Part (b) of the proof of Theorem 4.6 leads to a sheaf $\mathcal{F}$ of level cuts:

$$A_0 = \{\cdot\}, \quad \mathcal{F}(\alpha) = A_\alpha = \bigcap_{t<\alpha} F_t, \quad \alpha \in ]0,1].$$

It follows immediately from (4.9) that the set $\widetilde{F}(t)$ of all germs at $t$ coincides with $F_t$ $(t \in [0,1[)$.

Finally, in the case of arbitrary spatial frames we notice that the espace étalé induced by the *constant* presheaf $\mathcal{F}_A$ (cf. Example 4.1) coincides with the espace étalé associated with the *crisp* equality on $A$ (cf. Example 2.5).

## 5. Change of base

Let $\Omega$ and $\Omega'$ be frames. Further, let $\Omega \xrightarrow{h} \Omega'$ be a frame homomorphism (i.e. a finite meets and arbitrary joins preserving map (cf. [33])), and $\mathcal{F}$ be a sheaf on $\Omega'$. Then the *composition* $\mathcal{F} \circ h$ is a sheaf on $\Omega$ (cf. [13, 6.4]). Since sheaves and complete Heyting algebra valued sets are equivalent concepts, we would like to see the effect of this construction on complete Heyting algebra sets.

Let $(A, E)$ be a complete $\Omega'$-valued set, and let $\mathbb{E}$ be the extent of existence of $(A, E)$—i.e. $\mathbb{E}(a) = E(a,a)$, $a \in A$. According to Section 4 the sheaf $\mathcal{F}$ corresponding to $(A, E)$ is given by

$$\mathcal{F}(\alpha) = \{a \in A \mid E(a,a) = \alpha\}, \qquad \mathcal{F}(\alpha) \xrightarrow{\varrho_\beta^\alpha} \mathcal{F}(\beta) \quad \text{by}$$
$$E(\varrho_\beta^\alpha(a),\ b) = \beta \wedge E(a,b), \quad a \in \mathcal{F}(\alpha),\ \ b \in A,\ \ \beta \leqslant \alpha.$$

Then the support set $A^*$ of $\mathcal{F} \circ h$ has the following form:

$$A^* = \bigcup_{\alpha\in\Omega} \mathcal{F}(h(\alpha)) \times \{\alpha\} = \{(a,\alpha) \in A \times \Omega \mid \mathbb{E}(a) = h(\alpha)\}.$$

Hence $A^*$ is determined by the subsequent pullback square in **SET**:

$$\begin{array}{ccc} A^* & \overset{\mathbb{E}^*}{\dashrightarrow} & \Omega \\ {\scriptstyle k}\,\big\downarrow & & \big\downarrow\,{\scriptstyle h} \\ A & \underset{\mathbb{E}}{\longrightarrow} & \Omega' \end{array} \tag{5.1}$$

Further, the $\Omega$-valued equality $E^*$ on $A^*$ corresponding to the sheaf $\mathcal{F} \circ h$ is given by (cf. (4.2)):

$$E^*((a,\alpha),\ (b,\beta)) = \bigvee\{\gamma \in \Omega \mid \gamma \leqslant \alpha \wedge \beta, \quad \varrho_{h(\gamma)}^{h(\alpha)}(a) = \varrho_{h(\gamma)}^{h(\beta)}(b)\}.$$

Now let $\Omega' \xrightarrow{h_*} \Omega$ be the right adjoint map associated with $h$—i.e.

$$h_*(\alpha') = \bigvee\{\alpha \in \Omega \,\big|\, h(\alpha) \leqslant \alpha'\}, \quad \alpha' \in \Omega'.$$

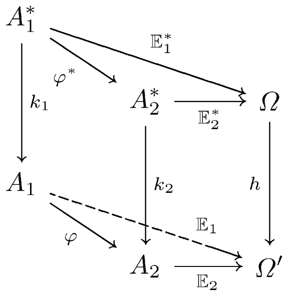
Then $E^*$ can be reformulate as follows:

$$E^*((a,\alpha),\ (b,\beta)) = \mathbb{E}^*(a,\alpha) \wedge \mathbb{E}^*(b,\beta) \wedge h_*(E(k(a,\alpha), k(b,\beta))). \tag{5.2}$$

Since $\mathcal{F} \circ h$ is a sheaf, $(A^*, E^*)$ is obviously a complete $\Omega$-valued set. In particular, we conclude from (5.1) and $\mathbb{E}^*(a,\alpha) \leqslant h_*(h(\mathbb{E}^*(a,\alpha)))$ that the *extent of existence* of $(A^*, E^*)$ coincides with $\mathbb{E}^*$.

A short glance at the previous construction shows that the derivation of the formulas (5.1) and (5.2) does not require the completeness (resp. the full account of sheaf-theoretic properties); this means that the previous construction works already for arbitrary Heyting algebra valued sets.

The aim of the following considerations is to complete this construction to a functor $\Theta_{h_*} : \Omega'\text{-}\mathbf{SET} \to \Omega\text{-}\mathbf{SET}$. Obviously on objects $\Theta_{h_*}$ is determined by: $\Theta_{h_*}(A, E) = (A^*, E^*)$. In order to define the action of $\Theta_{h_*}$ on morphisms we consider an $\Omega'$-**SET**-morphism $(A_1, E_1) \xrightarrow{\varphi} (A_2, E_2)$ and conclude from the universal property of the diagram in (5.1) that there exists an $\Omega$-**SET**-morphism $(A_1^*, E_1^*) \xrightarrow{\varphi^*} (A_2^*, E_2^*)$ making the following diagram commutative:




(5.3)

By virtue of the universal property of pullbacks the correspondence $\varphi \longmapsto \varphi^*$ defined by (5.3) is functorial. Thus we put: $\Theta_{h_*}(\varphi) = \varphi^*$.

As an immediate corollary from the introductory remarks we obtain the simple fact that the range of the restriction of $\Theta_{h_*}$ to $\mathbf{C}\Omega'$-**SET** is contained in $\mathbf{C}\Omega$-**SET**. We show now that $\Theta_{h_*}$ has a left adjoint functor. For this purpose we introduce a functor $\Theta_h : \Omega\text{-}\mathbf{SET} \to \Omega'\text{-}\mathbf{SET}$ by

$$\Theta_h(A, E) = (A, h \circ E), \quad \Theta(\varphi) = \varphi, \tag{5.4}$$

and consider a natural transformation $\eta : id_{\Omega\text{-}\mathbf{SET}} \to \Theta_{h_*} \circ \Theta_h$ whose components are determined as follows:

$$(A, E) \xrightarrow{\eta_{(A,E)}} (A^*, (h \circ E)^*), \qquad \eta_{(A,E)} = (a, \mathbb{E}(a)), \ \ a \in A. \tag{5.5}$$

Because (5.2) and $id_\Omega \leqslant h_* \circ h$ the map $\eta_{(A,E)}$ is indeed an $\Omega$-**SET**-morphism. Moreover, the commutativity of the diagram in (5.3) entails the commutativity of

$$\begin{array}{ccc} (A_1, E_1) & \xrightarrow{\eta_{(A_1,E_1)}} & (A_1^*, (h \circ E_1)^*) \\ \varphi \big\downarrow & & \big\downarrow \varphi^* \\ (A_2, E_2) & \xrightarrow{\eta_{(A_2,E_2)}} & (A^*, (h \circ E_2)^*) \end{array}$$

Hence we can view $\eta$ as a candidate for the unit of a possible adjoint situation $\Theta_h \dashv \Theta_{h_*}$ provided we can verify the following universal property:

For every $\Omega$-**SET**-morphism $(A_1, E_1) \xrightarrow{\varphi} (A_2^*, E_2^*)$ there exists a *unique* $\Omega'$-**SET**-morphism $(A_1, h \circ E_1) \xrightarrow{\ulcorner \varphi \urcorner} (A_2, E_2)$ making the diagram

$$\begin{array}{ccc} (A_1, E_1) & \xrightarrow{\eta_{(A_1,E_1)}} & (A_1^*, (h \circ E_1)^*) \\ \varphi \big\downarrow & \swarrow {\scriptstyle \Theta_{h_*}(\ulcorner \varphi \urcorner)} & \\ (A_2^*, E_2^*) & & \end{array} \tag{5.6}$$

commutative.

If we assume the commutativity of (5.6) and make use of $k_1 \circ \eta_{(A_1,E_1)} = id_{A_1}$ and of the notations from (5.1), then we obtain

$$k_2 \circ \varphi = k_2 \circ \Theta_{h_*}(\ulcorner\varphi\urcorner) \circ \eta_{(A_1,E_1)} = \ulcorner\varphi\urcorner \circ k_1 \circ \eta_{(A_1,E_1)} = \ulcorner\varphi\urcorner.$$

Hence $\ulcorner\varphi\urcorner$ is uniquely determined by (5.6).

If we now consider an $\Omega$-**SET**-morphism $(A_1, E_1) \xrightarrow{\varphi} (A_2^*, E_2^*)$, then we put $\ulcorner\varphi\urcorner := k_2 \circ \varphi$. The commutativity of (5.1) implies

$$h \circ \mathbb{E}_1(a) = h \circ \mathbb{E}_2^*(\varphi(a)) = \mathbb{E}_2(k_2 \circ \varphi(a)).$$

Hence $\ulcorner\varphi\urcorner$ fulfills the morphism axiom (m1). In order to verify the morphism axiom (m2) we refer to (5.2) and make use of the property $h \circ h_* \leqslant id_{\Omega'}$:

$$\begin{aligned} h(E(a,b)) &\leqslant h(E_2^*(\varphi(a),\varphi(b))) \leqslant h(h_*(E_2(\ulcorner\varphi\urcorner(a), \ulcorner\varphi\urcorner(b)))) \\ &\leqslant E_2(\ulcorner\varphi\urcorner(a), \ulcorner\varphi\urcorner(b)). \end{aligned}$$

Thus $\ulcorner\varphi\urcorner$ is an $\Omega'$-**SET**-morphism. Finally we observe:

$$\begin{aligned} &\mathbb{E}_2^*(\varphi(a)) = \mathbb{E}_1(a) = \mathbb{E}_2^*([\Theta_{h_*}(\ulcorner\varphi\urcorner)](a, \mathbb{E}_1(a))), \quad a \in A_1, \\ &k_2 \circ \varphi(a) = \ulcorner\varphi\urcorner(a) = k_2 \circ ([\Theta_{h_*}(\ulcorner\varphi\urcorner)](a, \mathbb{E}_1(a))). \end{aligned}$$

Since $\langle k_2, \mathbb{E}_2^* \rangle$ is injective, we obtain $\varphi = \Theta_{h_*}(\ulcorner\varphi\urcorner) \circ \eta_{(A_1,E_1)}$. Therewith the universal property of the diagram in (5.6) is verified. To sum up we have proved the following theorem.

**Theorem 5.1.** *The functor $\Theta_h$ is left adjoint to $\Theta_{h_*}$.*

Since **C**$\Omega'$-**SET** is a reflexive subcategory of $\Omega'$-**SET** (cf. Section 3), we obtain immediately the following corollary.

**Corollary 5.2.** *Let $\Sigma$ be the reflector for* **C**$\Omega'$-**SET**. *Then the functor $\Sigma \circ \Theta_h$ is left adjoint to the restriction of $\Theta_{h_*}$ to* **C**$\Omega'$-**SET**.

**Proposition 5.3.** *The restriction of the functor $\Sigma \circ \Theta_h$ to* **C**$\Omega$-**SET** *preserves finite limits.*

**Proof.** Since frame homomorphisms preserve universal upper bounds, it is easily seen that $\Sigma \circ \Theta_h|_{\mathbf{C}\Omega\text{-}\mathbf{SET}}$ preserves the terminal object. If $(A, E)$ and $(B, F)$ are *complete* $\Omega$-valued sets, then it is not difficult to prove the existence of an isomorphism between the singleton spaces $\Sigma(\Theta_h(A, E) \otimes \Theta_h(B, F))$ and $\Sigma(\Theta_h(A, E) \times \Theta_h(B, F))$. Since $\Theta_h$ preserves nonempty finite products, we conclude from Lemma 3.5 that $\Sigma \circ \Theta_h|_{\mathbf{C}\Omega\text{-}\mathbf{SET}}$ also preserves nonempty finite products. Further, we verify the preservation of equalizers. First, we notice that the support set of equalizers in **C**$\Omega$-**SET** can be computed at the level of **SET**. Then an equalizer diagram in **C**$\Omega$-**SET**

$$(D, G) \overset{\delta}{\rightarrowtail} (A, E) \underset{\psi}{\overset{\varphi}{\rightrightarrows}} (B, F)$$

can be read as follows: $D$ is a subset of $A$, $\delta$ denotes the inclusion map from $D$ to $A$, $G$ is the restriction of $E$ to $D \times D$, and the set $D$ itself is given as follows (cf. (4.2)):

$$D = \{\varrho_{F(\varphi(a),\psi(a))}^{\mathbb{E}(a)}(a) \mid a \in A\}.$$

The aim of the following considerations is to show that

$$\Sigma(D, h \circ G) \xrightarrow{\Sigma(\delta)} \Sigma(A, h \circ E) \underset{\Sigma(\psi)}{\overset{\Sigma(\varphi)}{\rightrightarrows}} \Sigma(B, h \circ F) \tag{5.7}$$

is an equalizer diagram in **C**$\Omega'$-**SET**. From the definition of the monad $\mathbf{T}_\Sigma$ (cf. Section 3) we conclude that $\Sigma(\varphi)$ has the form:

$$[\Sigma(\varphi)](s)(b) = \bigvee_{a \in A} s(a) \wedge h \circ F(\varphi(a), b), \quad b \in B, \ s \in S(A, h \circ E).$$

The same situation applies to $\Sigma(\psi)$. Now we choose a singleton $s$ of $(A, h \circ E)$ with $[\Sigma(\varphi)](s) = [\Sigma(\psi)](s)$. Then the morphism axiom (m2) implies:

$$s(a) \leqslant h \circ F(\varphi(a), \psi(a)), \quad a \in A. \tag{5.8}$$

Since $E$ is compatible with the sheaf-structure, we obtain from (5.8) and the strictness of $s$:

$$s(a) \leqslant h(E(a, a) \wedge F(\varphi(a), \psi(a))) = h \circ E(a, \ \varrho^{\mathbb{E}(a)}_{F(\varphi(a),\psi(a))}(a)).$$

If $s_{|D}$ is the restriction of $s$ to $D$, then the previous formula implies:

$$\bigvee_{d \in D} s_{|D}(d) \wedge h \circ E(d, a) = s(a), \quad a \in A;$$

this means $[\Sigma(\delta)](s_{|D}) = s$. Hence $(\Sigma(D, h \circ G), \Sigma(\delta))$ is the equalizer of $\Sigma(\varphi)$ and $\Sigma(\psi)$ in (5.7). □

We can summarize the assertions from Corollary 5.2 and Proposition 5.3 in the following statement (cf. [13, 6.10 Theorem, 9.3 and 9.4 Examples; 18, pp. 464]):

The adjoint pair $\Sigma \circ \Theta_h|_{\mathbf{C}\Omega\text{-}\mathbf{SET}} \dashv \Theta_{h_*}|_{\mathbf{C}\Omega'\text{-}\mathbf{SET}}$ is a *geometric morphism* from **C**$\Omega$-**SET** to **C**$\Omega'$-**SET**.

On the other hand every geometric morphism from **C**$\Omega$-**SET** to **C**$\Omega'$-**SET** is uniquely induced by a frame homomorphism $\Omega \xrightarrow{h} \Omega'$ up to an isomorphism (cf. [33, p. 175]).

We finish this section with three important examples.

**Example 5.4** (*Booleanization*). Let $\Omega$ be a frame and $S(\Omega)$ be the Stone space of all prime filters of $\Omega$. Then the Boolean algebra of all Borel subsets of $S(\Omega)$ modulo subsets of first category is a complete Boolean algebra $\mathbb{B}(\Omega)$ (cf. [54, p. 75]). In particular, $\mathbb{B}(\Omega)$ is called the *Booleanization* of $\Omega$. Since in any frame finite meets are distributive over arbitrary joins, there exists a frame monomorphism $\Omega \overset{m}{\rightarrowtail} \mathbb{B}(\Omega)$ sending every element $\alpha \in \Omega$ to the equivalence class determined by the open set of all prime filters $P$ with $\alpha \in P$. Then the geometric morphism $\Sigma \circ \Theta_m|_{\mathbf{C}\Omega\text{-}\mathbf{SET}} \dashv \Theta_{m_*}|_{\mathbf{C}\mathbb{B}(\Omega)\text{-}\mathbf{SET}}$ from **C**$\Omega$-**SET** to **C**$\mathbb{B}(\Omega)$-**SET** can be understood as a «measure» of the distance between intuitionistic and classical logic.

A second class of examples of geometric morphisms is furnished by points of frames.

**Remark 5.5** (*Fibres of espaces étalés*). Let $\Omega$ be a spatial frame and $\mathcal{E}(pt(\Omega))$ be the category of espaces étalé viewed as full subcategory of the comma category **TOP** $\downarrow pt(\Omega)$. Further, let $p$ be a point of $\Omega$—i.e. a frame homomorphism from $\Omega$ to $2 = \{0, 1\}$. Since we neglect the unique element with zero extent of existence, we identify **C2-SET** with the category **SET** of ordinary sets. Hence the restriction of $\Sigma \circ \Theta_p$ to **C**$\Omega$-**SET** is a functor from **C**$\Omega$-**SET** to **SET**.

The aim of the following considerations is to explain the role of the functors $\Sigma \circ \Theta_p|_{\mathbf{C}\Omega\text{-}\mathbf{SET}}$ and $\Theta_{p_*}|_{\mathbf{SET}}$ in some standard constructions in sheaf theory. First, for every ordinary set $A$ the sheaf $\mathcal{F}$ on $\Omega$ corresponding to the complete $\Omega$-valued set $\Theta_{p_*}(A)$ is the so-called «skyscraper» sheaf $Sky_p(A)$ (cf. [43, p. 93]). On the other hand, let $(A, E)$ be a complete $\Omega$-valued set and $pt(P(A, E)) \xrightarrow{\pi} pt(\Omega)$ be the espace étalé corresponding to $(A, E)$ (cf. Section 2). Since the reflector $\Sigma$ : 2-**SET** $\to$ **C**2-**SET** = **SET** means the formation of quotients w.r.t. partial equivalence relations, we conclude from the definition of $\Sigma \circ \Theta_p$ and the construction of $\pi$ in Section 2 that the image $\Sigma(A, p \circ E)$ of $(A, E)$ under $\Sigma \circ \Theta_p$ coincides with the fibre $\mathcal{A}_p$ over $p$ w.r.t. $\pi$. Moreover, the adjoint pair

$$\Sigma \circ \Theta_p|_{\mathbf{C}\Omega\text{-}\mathbf{SET}} \dashv \Theta_{p_*}|_{\mathbf{SET}}$$

is a geometric morphism from **C**$\Omega$-**SET** to **SET**. Hence *for all points $p$ of $\Omega$* the restriction of the functor $\Sigma \circ \Theta_p$ to **C**$\Omega$-**SET** preserves finite limits and set-indexed colimits. Taking into account the equivalence between **C**$\Omega$-**SET** and $\mathcal{E}(pt(\Omega))$ (see also [43, Corollary 3, p. 90]) we obtain immediately from the previous observations that finite limits and set-indexed colimits in $\mathcal{E}(pt(\Omega))$ are constructed fibrewise (resp. vertically) as in **SET**. In particular, morphisms between espaces étalés are monic (resp. epic) iff they are fibrewise injective (resp. surjective) which also implies that the (*epi*, *mono*)-*factorization* in $\mathcal{E}(pt(\Omega))$ is constructed fibrewise. By retranslation of these results to **C**$\Omega$-**SET** we obtain the following situation:

A morphism $(A, E) \xrightarrow{\varphi} (B, E)$ in **C$\Omega$-SET** is fibrewise injective iff for *all* points $p$ of $\Omega$ and for all $a_1, a_2 \in A$ the relation

$$p \circ E(a_1, a_2) = p \circ F(\varphi(a_1), \varphi(a_1))$$

holds. Since $\Omega$ is spatial, we obtain $E(a_1, a_2) = F(\varphi(a_1), \varphi(a_2))$. Thus because of Axiom (m2) a morphism $\varphi$ is monic in **C$\Omega$-SET** iff $\varphi$ satisfies the following property

$$F(\varphi(a_1), \varphi(a_2)) \leqslant E(a_1, a_2), \quad a_1, a_2 \in A. \tag{5.9}$$

A similar strategy leads to a characterization of epimorphisms in **C$\Omega$-SET**. First, we notice that $\varphi$ is fibrewise surjective iff for all points $p$ of $\Omega$ the following condition is satisfied

$$p \circ F(b, b) = \bigvee_{a \in A} p \circ F(\varphi(a), b), \quad b \in B. \tag{5.10}$$

Since $p$ preserves arbitrary joins and $\Omega$ is spatial, the previous relation is equivalent to

$$F(b, b) = \bigvee_{a \in A} F(\varphi(a), b), \quad b \in B. \tag{5.11}$$

Hence $\varphi$ is epic in **C$\Omega$-SET** iff $\varphi$ satisfies (5.11). Finally, we conclude from (5.11) (resp. (5.10)) that the (*epi*, *mono*)-factorization of $\varphi$ is given as follows:

$$\begin{array}{ccc} (A, E) & \xrightarrow{\quad\varphi\quad} & (B, F) \\ & {\scriptstyle\eta}\searrow \quad \nearrow & \\ & (C, G) & \end{array} \tag{5.12}$$

where $C = \{b \in B \mid F(b, b) = \bigvee_{a \in A} F(\varphi(a), b)\}$, $G = F|_{C \times C}$ and $\hookrightarrow$ is the inclusion map. In particular, the subobject $(C, G) \hookrightarrow (B, F)$ is called the *image* of $\varphi$.

Finally, we remark that the description of the (*epi*, *mono*)-factorization as well as the characterizations of mono- and epimorphisms in **C$\Omega$-SET** given for the moment in the case of spatial frames remain valid for arbitrary frames.

As the third example we consider the unique embedding $2 = \{0, 1\} \overset{\iota}{\hookrightarrow} \Omega$. In this context the geometric morphism

$$\Sigma \circ \Theta_\iota|_{\mathbf{SET}} \dashv \Theta_{\iota_*}|_{\mathbf{C\Omega\text{-}SET}}$$

has a very simple meaning: For every set $X$ the restriction of $\Sigma \circ \Theta_\iota$ to **SET** assigns the singleton space $\Sigma(X, E_c)$ to $X$ w.r.t. the crisp equality. In the special case of spatial frames this means the sheaf generated by the constant presheaf $\mathcal{F}_X$ (cf. Examples 2.5 and 4.1). On the other hand, the restriction of the functor $\Theta_{\iota_*}$ to **C$\Omega$-SET** assigns the set of global sections to each complete $\Omega$-valued set (resp. sheaf on $\Omega$). Finally, it follows from the properties of geometric morphisms that the functor $\Sigma \circ \Theta_\iota\big|_{\mathbf{SET}} : \mathbf{SET} \to \mathbf{C\Omega\text{-}SET}$ preserves finite limits and set-indexed colimits.

## 6. Strict and extensional maps and subobjects of CΩ-SET

Let $(A, E)$ be a not necessarily complete $\Omega$-valued set. Obviously every ordinary subset $U$ of $A$ induces an $E$-strict and $E$-extensional map $A \xrightarrow{f_U} \Omega$ by

$$f_U(a) = \bigvee_{u \in U} E(a, u). \tag{6.1}$$

If $(A, E)$ is separated and $(U, E\big|_{U \times U})$ is complete, then we obtain

$$U = \{a \in A \mid f_U(a) = E(a, a)\}. \tag{6.2}$$

In fact, if $f_U(a) = E(a, a)$, then the completeness of $(U, E\big|_{U \times U})$ implies the existence of $b \in U$ s.t. $E(a, a) = E(b, b)$ and $E(a, u) = E(b, u),\ u \in U$. Now we invoke the transitivity axiom of $\Omega$-valued equalities and obtain

$$E(a, a) = E(b, b) = \bigvee_{u \in U} E(a, u) \wedge E(b, u) = E(a, b).$$

Since $E$ is separated, we have $a = b$—i.e. $a \in U$.

The relations (6.1) and (6.2) motivate the following question:

Does for every $E$-strict and $E$-extensional map $f$ exist a subset $U$ of $A$ s.t. $(U, E\big|_{U \times U})$ is complete and $f = f_U$ ?

In the case of complete $\Omega$-valued sets the answer is affirmative as the next theorem shows. But we can prove something more!

**Theorem 6.1.** *Let $(A, E)$ be a complete $\Omega$-valued set, and $f$ be an $E$-strict and $E$-extensional map. Then there exists a* unique *ordinary subset $U$ of $A$ provided with the following properties*:

(i) $(U, E|_{U \times U})$ *is complete.*
(ii) $f(a) = f_U(a) = \bigvee_{u \in U} E(a, u)$.

**Proof.** The uniqueness follows immediately from (6.2) and Assertion (ii). In order to verify the existence we introduce a subset $U$ of $A$ by: $U = \{a \in A \mid f(a) = E(a, a)\}$, and consider a singleton $s$ of $(U, E|_{U \times U})$. Then $s$ can be extended to a singleton $\widehat{s}$ of $(A, E)$ as follows:

$$\widehat{s}(a) = \bigvee_{u \in U} s(u) \wedge E(u, a), \quad a \in A.$$

Since $(A, E)$ is complete, there exists a unique element $a_0 \in A$ s.t. $\widetilde{a}_0 = \widehat{s}$. In particular, $s(u) = E(a_0, u),\ u \in U$. Now we use the $E$-extensionality of $f$ and obtain

$$E(a_0, a_0) \geqslant f(a_0) \geqslant \bigvee_{u \in U} f(u) \wedge E(u, a_0) = \bigvee_{u \in U} E(u, a_0) = E(a_0, a_0).$$

Hence $a_0 \in U$, and the completeness of $(U, E\big|_{U \times U})$ is verified. Further, the inequality $f_U \leqslant f$ follows immediately from the $E$-extensionality of $f$. On the other hand, we fix $a \in A$ and infer from the completeness of $(A, E)$ that there exists a unique element $b \in A$ s.t.

$$E(b, b) = f(a), \quad E(b, c) = f(a) \wedge E(a, c), \quad c \in A.$$

Obviously $f(a) = E(b, a)$ and the $E$-strictness and $E$-extensionality of $f$ implies: $f(b) = f(a) = E(b, b)$—i.e. $b \in U$. In particular, $f(a) \leqslant f_U(a)$. Hence the relation $f = f_U$ is verified. □

Let $(A, E)$ be a complete $\Omega$-valued set. Since every subobject of $(A, E)$ in the sense of **CΩ-SET** has a representative of the form

$$(U, E\big|_{U \times U}) \hookrightarrow (A, E)$$

where $U$ is a subset of $A$ and $(U, E\big|_{U\times U})$ is complete, we can summarize the previous considerations in the following statement:

> There exists a bijective map between $P(A, E)$ and the set of all subobjects of $(A, E)$. Referring to Assertion (ii) in Theorem 6.1 it is evident that this bijection is even an order-isomorphism.

The aim of the following considerations is to internalize the previous constructions in **C**$\Omega$-**SET**. First we study the problem of internalizing strict and extensional maps as $\Omega$-**SET**-morphisms. For this purpose we fix a not necessarily complete, $\Omega$-valued set $(A, E)$ and an $E$-strict and $E$-extensional map $A \xrightarrow{f} \Omega$. If we choose the separated $\Omega$-valued set $(\Omega, \longleftrightarrow)$ (cf. Example 2.1(c)) as codomain for $f$, then we see immediately that the $E$-extensionality of $f$ is equivalent to the morphism axiom (m2). Unfortunately, $f$ does not satisfy the morphism axiom (m1) with regard to $(\Omega, \longleftrightarrow)$.

On the other hand, we can try to internalize $f$ as a «fuzzy morphism» between the $\Omega$-valued sets $(A, E)$ and $(\Omega, \longleftrightarrow)$. We put:

$$R_f(a, \lambda) = E(a, a) \wedge (f(a) \longleftrightarrow \lambda), \quad a \in A, \ \lambda \in \Omega. \tag{6.3}$$

Because of the $E$-extensionality of $f$ the map $R_f$ satisfies the conditions (F1)–(F3). Hence, in the case of spatial frames $f$ can be viewed as a bundle morphism between the respective espaces étalés associated with $(A, E)$ and $(\Omega, \longleftrightarrow)$ [5] (cf. Section 2). Moreover, the $E$-strictness of $f$ implies $R(a, \top) = f(a)$—this means that the correspondence $f \longmapsto R_f$ is *injective*. Thus we view $R_f$ as an *internalization* of $f$.

In what follows we identify $R_f$ with a map—i.e. with a morphism in $\Omega$-**SET**. If we replace $\Omega$-valued sets by their singleton spaces, then we first note that every «fuzzy morphism» $A \times B \xrightarrow{R} \Omega$ with (F1)–(F3) can be identified with an $\Omega$-**SET**-morphism $(A, E) \xrightarrow{\varphi_R} \Sigma(B, F)$ where $[\varphi_R(a)](b) = R(a, b), \ a \in A, \ b \in B$. Thus the internalized version $R_f$ of $f$ can be viewed as an $\Omega$-**SET**-morphism with the codomain $\Sigma(\Omega, \longleftrightarrow)$. In this context we ask the following question:

> Does every $\Omega$-**SET**-morphism $(A, E) \xrightarrow{\varphi} \Sigma(\Omega, \longleftrightarrow)$ comes from an $E$-strict and $E$-extensional map?

In order to give a comprehensive answer to this question we begin with a characterization of the singleton space of $(\Omega, \longleftrightarrow)$. Because of Lemma 3.1 it is not difficult to see that every singleton $s$ of $(\Omega, \longleftrightarrow)$ has the following form:

$$s(\lambda) = \mathbb{E}(s) \wedge (s(\top) \longleftrightarrow \lambda), \quad \lambda \in \Omega. \tag{6.4}$$

Further, we construct an $\Omega$-valued set $(R_\Omega, E_\Omega)$ by

$$R_\Omega = \{(\alpha, \lambda) \mid \lambda \leqslant \alpha\}, \quad E_\Omega((\alpha, \lambda), (\beta, \mu)) = \alpha \wedge \beta \wedge (\lambda \longleftrightarrow \mu).$$

**Proposition 6.2.** *The correspondence $s \longmapsto (\mathbb{E}(s), s(\top))$ is an $\Omega$-**SET**-isomorphism from $\Sigma(\Omega, \longleftrightarrow)$ to $(R_\Omega, E_\Omega)$.*

**Proof.** The assertion follows immediately from the definition of $E_\Omega$ and the $\Omega$-valued equality on the singleton space. □

Now we are in the position to conclude from (6.4) and Proposition 6.2 that every $E$-strict and $E$-extensional map $f$ can be internalized as an $\Omega$-**SET**-morphism $(A, E) \xrightarrow{\chi_f} (R_\Omega, E_\Omega)$ where $\chi_f$ is given by

$$\chi_f(a) = (E(a, a), \ f(a)), \quad a \in A. \tag{6.5}$$

On the other hand, let $(A, E) \xrightarrow{\chi} (R_\Omega, E_\Omega)$ be an arbitrary $\Omega$-**SET**-morphism. Then the composition of $\chi$ with the projection onto the first component coincides with the extent $\mathbb{E}$ of existence, while the composition of $\chi$ with the projection onto the second component leads to an $E$-strict and $E$-extensional map $f$ s.t. $\chi = \chi_f$. Hence the answer of the previous question is affirmative and can be summarized in the statement that there exists a bijective map between the hom-set $hom_{\Omega\text{-}\mathbf{SET}}((A, E), (R_\Omega, E_\Omega))$ and the frame $P(A, E)$ (cf. Section 2).

[5] The fibre over $p$ of the espace étalé associated with $(\Omega, \longleftrightarrow)$ consists of all *germs* of *open* subsets of $pt(\Omega)$ at $p$ (cf. [4, pp. 65–68]).

Since the terminal object $\mathbb{1} = (\Omega, \wedge)$ is isomorphic to the singleton space $\Sigma(\{\cdot\}, \top)$ where $\{\cdot\}$ is a set consisting of a single element with total extent of existence, we can introduce the *arrow true* $\mathbb{1} \xrightarrow{t} (R_\Omega, E_\Omega)$ as the *unique extension* of the correspondence $\cdot \longmapsto \top$—i.e.

$$t(\alpha) = (\alpha, \alpha), \quad \alpha \in \Omega. \tag{6.6}$$

Further, let $(A, E)$ be a complete $\Omega$-valued set and $f$ be a $E$-strict and $E$-extensional map. Using the fact that pullbacks in **C$\Omega$-SET** are computed at the level of **SET** we obtain immediately that

$$(U, E\big|_{U\times U}) \hookrightarrow (A, E) \qquad \text{with } U = \{a \in A \mid f(a) = E(a, a)\}$$

is the pullback of $t$ along $\chi_f$. On the other hand the following theorem holds.

**Theorem 6.3.** *Let* $(U, F) \overset{\varphi}{\rightarrowtail} (A, E)$ *be a* **C$\Omega$-SET**-*monomorphism. Then there exists a* unique **C$\Omega$-SET**-*morphism* $(A, E) \xrightarrow{\chi} (R_\Omega, E_\Omega)$ *such that the diagram*

$$\begin{array}{ccc} (U, F) & \xrightarrow{\ !\ } & \mathbb{1} \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle t} \\ (A, E) & \underset{\chi}{\dashrightarrow} & (R_\Omega, E_\Omega) \end{array} \tag{6.7}$$

*is a pullback square*.

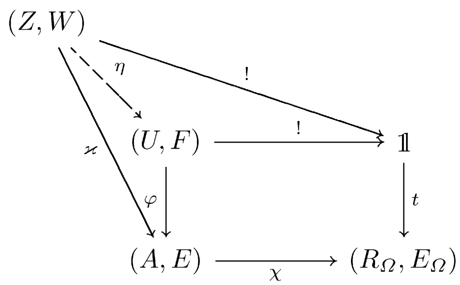
**Proof.** (a) (Existence) Let $f$ be an $E$-strict and $E$-extensional map defined by

$$f(a) = \bigvee_{u\in U} E(a, \varphi(u)), \quad a \in A. \tag{6.8}$$

Obviously $\chi_f$ makes the diagram (6.7) commutative. In order to verify the universal property of pullbacks we consider a further complete $\Omega$-valued set $(Z, W)$ and a pair of arrows $(Z, W) \xrightarrow{\varkappa} (A, E)$ , $(Z, W) \xrightarrow{\ !\ } \mathbb{1}$ with $\chi_f \circ \varkappa = t \circ !$. In particular, for all $z \in Z$ we have $f(\varkappa(z)) = W(z, z) = E(\varkappa(z), \varkappa(z))$. Since $\varphi$ is a **C$\Omega$-SET**-monomorphism (cf. (5.9) in Remark 5.5), we obtain that for every $z \in Z$ there exists a unique element $\eta(z)$ of $U$ s.t.

$$E(\varphi(\eta(z)), \varphi(u)) = F(\eta(z), u) = E(\varkappa(z), \varphi(u)), \quad z \in Z, \ \ u \in U.$$

In particular, $\eta$ makes the following diagram commutative:




Since $\varphi$ is monic, the uniqueness of $\eta$ is evident.

(b) (Uniqueness) Let $(A, E) \xrightarrow{\chi_{\hat{f}}} (R_\Omega, E_\Omega)$ be a **C$\Omega$-SET**-morphism s.t.

$$\begin{array}{ccc} (U, F) & \xrightarrow{\ !\ } & \mathbb{1} \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle t} \\ (A, E) & \underset{\chi_{\hat{f}}}{\dashrightarrow} & (R_\Omega, E_\Omega) \end{array} \tag{6.9}$$

is a pullback square. The commutativity of the previous diagram implies $f \leqslant \widehat{f}$. On the other hand, we choose an element $a \in A$ and consider the singleton space $\Sigma(\{\cdot\}, \widehat{f}(a))$ where $\{\cdot\}$ is a set consisting of a single element with extent of existence $\widehat{f}(a)$. Since $(A, E)$ is complete, we conclude from the universal property of free algebras (cf. Section 3) that there exists a unique arrow $\Sigma(\{\cdot\}, \widehat{f}(a)) \xrightarrow{\iota} (A, E)$ determined by the following property:

$$E(\iota(\cdot), b) = \widehat{f}(a) \wedge E(a, b), \quad b \in A$$

where we have identified $\tilde{\cdot}$ with $\cdot$. Hence the relation

$$E(\iota(\cdot), \iota(\cdot)) = \widehat{f}(a) = E(\iota(\cdot), a) = \widehat{f}(\iota(\cdot)) \tag{6.10}$$

follows (cf. Proof of Theorem 6.1); this means that the following diagram is commutative:

$$\begin{array}{ccc} \Sigma(\{\cdot\}, \widehat{f}(a)) & \xrightarrow{!} & \mathbb{1} \\ {\scriptstyle\iota}\downarrow & & \downarrow{\scriptstyle t} \\ (A, E) & \xrightarrow[\chi_{\widehat{f}}]{} & (R_\Omega, E_\Omega) \end{array}$$

Now we invoke the universal property of (6.9) and obtain an element $u_0 \in U$ with $\varphi(u_0) = \iota(\cdot)$ Then (6.10) implies: $\widehat{f}(a) = E\big(a, \varphi(u_0)\big) \leqslant f(a)$. Hence $\widehat{f}$ and $f$ coincide. $\square$

The representation of monomorphisms specified in the previous theorem is also known as the *subobject classifier axiom* in **C$\Omega$-SET** (cf. [18; p. 81; 43, pp. 31–34]). In this context elements of the hom-set $hom_{\Omega\text{-}\mathbf{SET}}((A, E), (R_\Omega, E_\Omega))$ are called *characteristic morphisms*. Since characteristic morphisms are the internalized version of strict and extensional maps, elements of the frame $P(A, E)$ are sometimes called *membership maps*—a terminology which will be justified by Section 1.1.1. in Fuzzy Sets and Sheaves Part II (cf. [30]).

Finally, we return to the case of *incomplete* $\Omega$-valued sets. Then there exists a order isomorphism between $P(A, E)$ and $P(\Sigma(A, E))$. In particular, for every $E$-strict and $E$-extensional map $A \xrightarrow{f} \Omega$ there exists a *unique* $\widetilde{E}$-strict and $\widetilde{E}$-extensional map $S(A, E) \xrightarrow{\widetilde{f}} \Omega$ satisfying the condition

$$\widetilde{f}(\widetilde{a}) = f(a), \quad a \in A. \tag{6.11}$$

Obviously $\widetilde{f}$ has the following form:

$$\widetilde{f}(s) = \bigvee_{a \in A} s(a) \wedge f(a), \quad s \in S(A, E). \tag{6.12}$$

Subsequently, we apply the previous techniques to $\widetilde{f}$ and obtain that the support set of the subobject corresponding to $\widetilde{f}$ is given by

$$U = \{s \in S(A, E) \mid s \leqslant f\}. \tag{6.13}$$

In particular, $U$ coincides with the set of all singletons «contained in» f.

## 7. Concluding remark

In this paper we have explained such important concepts as $\Omega$-valued sets, espaces étalés, singleton monad and sheaves. In particular, the specification of their mutual relations can be regarded as an attempt to teach a larger audience a deeper understanding of sheaf theory. The application of these basic tools to the sheaf-theoretic foundation of fuzzy set theory including its applications to algebra and topology will appear in a forthcoming paper under the title: Fuzzy Sets and Sheaves—Part II (cf. [30]).

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## References

[1] J. Adámek, H. Herrlich, G.E. Strecker, Abstract and Concrete Categories, Wiley, New York, 1990.
[2] T. Bilgiç, B. Türkşen, Measurement of membership functions: theoretical and empirical work, in: D. Dubois, H. Prade (Eds.), Fundamentals of Fuzzy Sets, Kluwer Academic Publishers, Boston, London, Dordrecht, 2000, pp. 195–230.
[3] G. Boole, An Investigation into The Laws of Thought, Macmillan, London, 1854 (Reprinted by Dover Publications, New York, 1958).
[4] N. Bourbaki, Elements of Mathematics, General Topology, Part 1, Addison-Wesley, Reading, MA (A translation of Éléments de Mathématique, Topologie Générale, Hermann, Paris, 1966).
[5] N. Bourbaki, Elements of Mathematics, Theory of Sets, Hermann, Addison-Wesley, Reading, MA (A translation of Éléments de Mathématique, Théorie des Ensembles, Hermann, Paris, 1968).
[6] C.L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24 (1968) 182–190.
[7] D.-G. Chen, W.-X. Gu, Product structure of the fuzzy factor groups, Fuzzy Sets and Systems 60 (1993) 229–232.
[8] D.G. Chen, W.-X. Gu, Generating fuzzy factor groups and the fundamental theorem of isomorphism, Fuzzy Sets and Systems 82 (1996) 357–360.
[9] P.S. Das, Fuzzy groups and level groups, J. Math. Anal. Appl. 84 (1981) 264–269.
[10] D. Dubois, W. Ostasiewicz, H. Prade, Fuzzy sets: history and basic notions, in: D. Dubois, H. Prade (Eds.), Fundamentals of Fuzzy Sets, Kluwer Academic Publishers, Boston, London, Dordrecht, 2000, pp. 21–124.
[11] D. Dubois, H. Prade, Fuzzy Sets and Systems—Theory and Applications, Academic Press, New York, 1980.
[12] D. Dubois, H. Prade (Eds.), Fundamentals of Fuzzy Sets, Kluwer Academic Publishers, Boston, London, Dordrecht.
[13] M.P. Fourman, D.S. Scott, Sheaves and logic, in: M.P. Fourman, C.J. Mulvey, D.S. Scott (Eds.), Applications of Sheaves, Lecture Notes in Mathematics, vol. 753, Springer, Berlin, Heidelberg, New York, 1979, pp. 302–401.
[14] M. Garzon, G.C. Muganda, Free fuzzy groups and fuzzy group presentations, Fuzzy Sets and Systems 48 (1992) 249–255.
[15] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, D.S. Scott, A Compendium of Continuous Lattices, Springer, Berlin, Heidelberg, New York, 1980.
[16] R. Godement, Topologie Algébrique et Théorie des Faisceaux, Hermann, Paris, 1964.
[17] J.A. Goguen, *L*-Fuzzy sets, J. Math. Anal. Appl. 18 (1967) 145–174.
[18] R. Goldblatt, Topoi—The Categorical Analysis of Logic, revised ed., Elsevier Science Publishers B.V., Amsterdam, London, New York, 1984.
[19] S. Gottwald, A Treatise on Many-Valued Logics, Research Studies Press, Baldock, Hertfordshire, 2001.
[20] J.W. Gray, Fragments of the history of sheaf theory, in: M.P. Fourman, C.J. Mulvey, D.S. Scott (Eds.), Applications of Sheaves, Lecture Notes in Mathematics, vol. 753, Springer, Berlin, Heidelberg, New York, 1979, pp. 1–79.
[21] P. Hájek, Metamathematics of Fuzzy Logic, Trends in Logic, vol. 4, Kluwer Academic Publishers, Dordrecht, Boston, 1998.
[22] H. Herrlich, G.E. Strecker, Category Theory, Allyn and Bacon, Boston, 1973.
[23] D. Higgs, A category approach to Boolean-valued set theory, Waterloo, 1973, unpublished.
[24] U. Höhle, *M*-valued sets and sheaves over integral commutative *cl*-monoids, in: S.E. Rodabaugh et al. (Eds.), Applications of Category Theory to Fuzzy Subsets, Kluwer Academic Publishers, Dordrecht, Boston, London, 1992, pp. 33–72.
[25] U. Höhle, Presheaves over *GL*-monoids, in: U. Höhle, E.P. Klement (Eds.), Non-classical Logics and their Applications to Fuzzy Subsets, Kluwer Academic Publishers, Dordrecht, Boston, London, 1995, pp. 127–157.
[28] U. Höhle, Separated presheaves of normed spaces and $\Omega$-valued norms, Internat. J. Gen. Systems 32 (2003) 385–414.
[29] U. Höhle, Many-valued equalities and their representations, in: E.P. Klement, R. Mesiar (Eds.), Logical, Algebraic, and Probabilistic Aspects of Triangular Norms, Elsevier, Amsterdam, Boston, Heidelberg, 2005, pp. 301–319.
[30] U. Höhle, Fuzzy sets and sheaves. Part II: Sheaf-theoretic foundations of fuzzy set theory with applications to algebra and topology, Fuzzy Sets and Systems (2007), to appear.
[31] U. Höhle, A.P. Šostak, Axiomatic foundations of fixed-basis fuzzy topologies, in: U. Höhle, S.E. Rodabaugh (Eds.), Mathematics of Fuzzy Sets—Logic, Topology and Measure Theory, Kluwer Academic Publishers, Boston, Dordrecht, London, 1999, pp. 123–272.
[32] I.M. James, Fibrewise Topology, Cambridge University Press, Cambridge, 1989.
[33] P.T. Johnstone, Stone Spaces, Cambridge University Press, Cambridge, London, New York, 1982.
[34] G.M. Kelly, Basic Concepts of Enriched Category Theory, London Mathematical Society Lecture Note Series, vol. 64, Cambridge University Press, Cambridge, 1982.
[35] J.P. Kim, D.R. Bae, Fuzzy congruences in groups, Fuzzy Sets and Systems 85 (1997) 115–120.
[36] I.J. Kumar, P.K. Saxena, P. Yadav, Fuzzy normal subgroups and fuzzy quotients, Fuzzy Sets and Systems 46 (1992) 121–132.
[37] S. Kundu, Membership functions for a fuzzy group from similarity relations, Fuzzy Sets and Systems 101 (1999) 391–402.
[38] F.W. Lawvere, An elementary theory of the category of sets, Proc. Nat. Acad. Sci., USA 52 (1964) 1506–1511.
[39] F.W. Lawvere, Quantifiers and sheaves, Actes des Congrès International des Mathématiques, Nice 1970, vol. I, Gauthier-Villars, Paris, 1971 pp. 329–334.
[40] R. Lowen, Fuzzy topological spaces and fuzzy compactness, J. Math. Anal. Appl. 56 (1976) 621–633.

[41] J. Łukasiewicz, Selected Works, in: L. Borkowski (Ed.), North-Holland, Amsterdam, 1970.
[42] S. MacLane, Categories for the Working Mathematician, second ed., Springer, New York, Berlin, Heidelberg, 1998.
[43] S. MacLane, I. Moerdijk, Sheaves in Geometry and Logic, Universitext, Springer, New York, Berlin, Heidelberg, 1992.
[44] D.S. Malik, J.N. Mordeson, Fuzzy relations on rings and groups, Fuzzy Sets and Systems 43 (1991) 117–123.
[45] E.G. Manes, Algebraic Theories, Springer, New York, Heidelberg, Berlin, 1976.
[46] N.P. Mukherjee, P. Bhattacharya, Fuzzy normal subgroups and fuzzy cosets, Inform. Sci. 34 (1984) 225–239.
[47] V. Murali, Fuzzy congruence relations, Fuzzy Sets and Systems 41 (1991) 359–369.
[48] C.V. Negoita, D.A. Ralescu, Representation theorems for fuzzy concepts, Kybernetes 4 (1975) 169–174.
[49] S. Ovchinnikov, An introduction to fuzzy relations, in: D. Dubois, H. Prade (Eds.), Fundamentals of Fuzzy Sets, Kluwer Academic Publishers, Boston, London, Dordrecht, 2000, pp. 233–259.
[50] A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl. 35 (1971) 512–517.
[51] H. Schubert, Categories, Springer, Berlin, Heidelberg, New York, 1972.
[52] M. Shimoda, A natural interpretation of fuzzy sets and fuzzy relations, Fuzzy Sets and Systems 128 (2002) 135–147.
[53] M. Shimoda, A natural interpretation of fuzzy mappings, Fuzzy Sets and Systems 138 (2003) 67–82.
[54] R. Sikorski, Boolean algebras, Ergebnisse der Mathematik und ihrer Grenzgebiete 25, third ed., Springer, Berlin, Heidelberg, New York, 1969.
[55] O. Wyler, Fuzzy logic and categories of fuzzy sets, in: U. Höhle, E.P. Klement (Eds.), Non-classical Logics and their Applications to Fuzzy Subsets, Kluwer Academic Publishers, Dordrecht, Boston, London, 1995, pp. 235–268.
[56] L.A. Zadeh, Fuzzy sets, Inform. and Control 8 (1965) 338–353.
[57] L.A. Zadeh, Similarity relations and fuzzy orderings, Inform. Sci. 3 (1971) 177–200.
[58] L.A. Zadeh, Calculus of fuzzy restrictions, in: L.A. Zadeh, K.S. Fu, K. Tanaka, M. Shimura (Eds.), Fuzzy Sets and their Applications to Cognitive and Decision Processes, Academic Press, New York, 1975, pp. 1–39.