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Fuzzy sets and sheaves. Part II: Sheaf-theoretic foundations of fuzzy set theory with applications to algebra and topology

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Abstract

This essay shows that large parts of fuzzy set theory are actually subfields of sheaf theory, resp. of the theory of complete Ω-valued sets. Hence fuzzy set theory is closer to the mainstream in mathematics than many people would expect. Part II of this series of two papers explains the sheaf-theoretic basis of Zadeh's operations on fuzzy sets (with the exception of the complementation). Further, the quotient problem w.r.t. similarity relations and the quotient problem of fuzzy groups w.r.t. fuzzy congruence relations are solved. Finally, stratified fuzzy topologies are identified with internal topologies on constant sheaves.

Keywords: Category theory; Ω-Valued sets; Espace étalé; Singleton monad; Complete Ω-valued sets; Membership functions; Similarity relations; Quotients; Group objects; Topological space objects

0. Introduction

The purpose of this paper is to apply the results from Fuzzy Sets and Sheaves Part I (cf. [47]) to various aspects of fuzzy set theory including fuzzy groups and fuzzy topologies.

In Section 1 we solve such fundamental open problems as the clear understanding of the difference between fuzzy sets and their membership functions, the quotient w.r.t. similarity relations, the correct antisymmetry axiom for fuzzy orderings, a clean treatment of fuzzy partitions, fuzzy control maps and their defuzzification and the role of the hypergraph functor for fuzzy topologies (cf. Section 1.1). Further, we explain the categorical basis of all Zadeh's rules of combinations with the exception of the complementation of fuzzy sets (cf. Section 1.2).

In Section 2 we study mathematical structures in sheaves—e.g. groups and topological spaces. As a preparation for the formulation of axioms of topological space objects we begin with the categorical description of the power object monad in a topos (see e.g. [58,67]). Subsequently, we treat group objects in the category **CΩ-SET** of complete Ω-valued sets, resp. the category of sheaves, and obtain the important result that fuzzy groups (cf. [81]) are always subgroup objects of constant sheaves of groups. In particular, we solve the problem of constructing quotient group objects w.r.t. normal fuzzy subgroups.

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We finish this essay with a brief overview on topological space objects in **CΩ-SET** and point out their relevance for *many-valued convergence*. In this context, we show that every bounded sequence in an ultrametric space has local limit points given by local sections of the corresponding espace étalé. Further, stratified Ω-valued topologies and topological structures on singleton spaces w.r.t. the crisp equality in the sense of **CΩ-SET** are equivalent concepts. Finally, we emphasize that separated presheaves of ordinary topological spaces and Ω-probabilistic metric spaces form natural sources for topological space objects in **CΩ-SET** (cf. Remark 2.3.5(b), Example 2.3.6).

Since in this paper we frequently have to refer to Part I of this essay, we use the abbreviation FSBC for *Fuzzy Sets and Sheaves. Part I*: *Basic Concepts* (cf. [47]).

1. Sheaf-theoretic foundations of fuzzy set theory

The aim of this section is to explain the impact of such basic concepts as [0, 1]-valued sets, sheaves on [0, 1], espaces étalés with base space [0, 1[on the fundamentals of fuzzy set theory. In a first subsection we present a list of problems and their sheaf-theoretic solutions. In a second subsection we point out categorical foundations of various constructions on fuzzy sets.

1.1. Problems and solutions

1.1.1. The definition of fuzzy sets

It is surprising that the history of fuzzy set theory starts with a *vague* definition of the notion *fuzzy set*. For the sake of completeness we quote Zadeh from the year 1965 (cf. [92]):

> A fuzzy set (class) A [in a given set X] is characterized by a membership (characteristic) function $f_A(x)$ which associates with each point $x \in X$ a real number in the interval [0, 1] [or in a suitable, partially ordered set P (cf. footnote 3 in [92])], with value $f_A(x)$ at x representing the "grade of membership" of x in A.

A more recent definition of this type can be found in Gottwald's book on *many-valued logics* (cf. [30, p. 424]):

> A *fuzzy set* A is characterized by a generalized characteristic function $\mu_A : \mathcal{X} \to [0, 1]$, called *membership function* of A and defined over a *universe of discourse* $\mathcal{X}$.

It is interesting to see that in both definitions the symbol A remains undefined and appears only as an index of the membership function f_A (resp. μ_A). This situation is one of the reasons to regard fuzzy sets as [0, 1]-valued maps, and provokes the question in which sense [0, 1]-valued maps can capture the concept of *vague membership*. In the author's thesis 1973 as well as in a personal communication by I.R. Goodmann from the mid-seventies of the last century we find the observation that every map $X \xrightarrow{f} [0,1]$ determines a unique regular Borel probability measure ν_f on the ordinary power set $\mathcal{P}(X) \cong \{0,1\}^X$ (provided with the product topology w.r.t. the discrete topology on {0, 1}) s.t. for every non empty, finite subset H of X the following relation holds (cf. [35, Example 1, 29]):

$$\nu_f\left(\bigcap_{x\in H}\{A \in \mathcal{P}(X) | x \in A\}\right) = \min\{f(x) \mid x \in H\}.$$

In particular, the restriction of ν_f to the σ-algebra of all *Baire subsets* of $\mathcal{P}(X)$ coincides with the *image measure* of the Lebesgue measure on [0, 1] w.r.t two measurable maps:

$$[0,1] \xrightarrow{\Phi_f} \mathcal{P}(X), \qquad \Phi_f(\alpha) = \{x \in X \mid \alpha \leqslant f(x)\}, \quad \alpha \in [0,1], \tag{1.1}$$

$$[0,1] \xrightarrow{\Psi_f} \mathcal{P}(X), \qquad \Psi_f(\alpha) = \{x \in X \mid \alpha < f(x)\}, \quad \alpha \in [0,1]. \tag{1.2}$$

In this context the value $f(x)$ can be interpreted as the *probability* that the element x is contained in some subset of X—an idea which also appears in Menger's paper "Ensembles flous et fonctions aléatoire" 1951 (cf. [70]). Hence fuzzy sets are special types of *random sets*.

Even though this probabilistic understanding has never been accepted by Zadeh, we also have a foundation problem: since for instance probability measures on product spaces are not uniquely determined by their marginal distributions, there does not exist a sufficiently rich, categorical structure underlying the theory of random sets. This situation motivates to dismantle the previous interpretation from its probabilistic background, and to return to [0, 1]-valued maps as a concept of *generalized characteristic functions* leaving open the question *what objects* are characterized by these functions. In this context it is worthwhile to note that 1975 Negoita and Ralescu proved an important representation theorem for [0, 1]-valued maps which is also related to the maps Φ_f and Ψ_f appearing in the probabilistic representation of fuzzy sets (cf. [75]):

- For every nested family $\{A_\alpha \mid \alpha \in]0,1]\}$ satisfying the "upper continuity condition"

$$\bigcap_{\beta<\alpha} A_\beta = A_\alpha$$

  there exists a unique map $X \xrightarrow{f} [0,1]$ s.t.

$$f(a) = \sup\{\alpha \in]0,1] \mid a \in A_\alpha\}, \quad \Phi_f(\alpha) = A_\alpha, \ a \in X, \ \alpha \in]0,1].$$

- For every nested family $\{A_\alpha \mid \alpha \in]0,1]\}$ satisfying the "lower continuity condition"

$$\bigcup_{\alpha<\beta} A_\beta = A_\alpha$$

  there exists a unique map $X \xrightarrow{f} [0,1]$ s.t.

$$f(a) = \inf\{\alpha \in]0,1] \mid a \notin A_\alpha\}, \quad \Psi_f(\alpha) = A_\alpha, \ a \in X, \ \alpha \in]0,1].$$

In this *horizontal view* of [0, 1]-valued maps the values of Φ_f are called *level cuts* of f, while the values of Ψ_f are called *strict level cuts* of f (cf. [18, pp. 44–46]). It is typical for the development of fuzzy sets that in the following years it has not been made any attempt to attach a special meaning to level cuts, resp. strict level cuts of membership functions, and sometimes the choice between Φ_f and Ψ_f is quite arbitrary.

The aim of the following considerations is to close the *gap* in the definition of fuzzy sets. In particular, we continue Remark 4.7 in FSBC and give a clear meaning to the concept of *level cuts*, resp. *strict level cuts*. In a first step we clarify the role of real numbers of [0, 1]. We do not interpret elements of [0, 1] as probabilities (see our previous argumentation), and we also do not understand elements of [0, 1] as degrees of truth as Zadeh's definition of fuzzy sets might suggest, because form an ontological point of view TRUTH is an *indivisible entity*. Rather we prefer a geometric understanding of [0, 1] and view primarily elements of [0, 1] as *domains of* TRUTH (cf. Section 1 in FSBC). Since [0, 1] is a spatial frame, every domain $\alpha \in [0,1]$ can be represented by an open subset $[0,\alpha[$ in the sense of the *topological space* $([0,1[, \omega([0,1[))$ (cf. Section 1 in FSBC).

In a next step we choose a map $X \xrightarrow{f} [0,1]$, and as most people of the fuzzy community we consider the crisp equality E_c on X (cf. Example 2.1(a) in FSBC). Then f is always E_c-strict and E_c-extensional. Now we study the impact of the crisp equality on the understanding of f as a generalized characteristic function. Since in the case of spatial frames *singletons* and *local sections* are equivalent concepts (cf. Section 2 in FSBC), there exist two equivalent approaches to our investigations:

- The singleton space $\Sigma(X, E_c)$.
- The espace étalés associated with (X, E_c).

Referring to results of Section 6 in FSBC we decide to follow the first approach and begin to recall the structure of $\Sigma(X, E_c)$ in this special case. Since [0, 1] is a *chain*, every singleton s of (X, E_c) has the form:

$$\exists (x,\alpha) \in X \times [0,1]: \quad s(y) = (\alpha \cdot 1_x)(y) = \begin{Bmatrix} \alpha : y = x \\ 0 : y \neq x \end{Bmatrix}, \quad y \in X. \tag{1.3}$$

Thus singletons of (X, E_c) and "fuzzy points" in the sense of Pu and Liu (cf. [79]) are the same things (see also [50, p. 270]). Further, the [0, 1]-valued equality $\widetilde{E}_c$ on the set $S(X, E_c)$ of all singletons of (X, E_c) is given by

$$\widetilde{E_c}(\alpha \cdot 1_a, \beta \cdot 1_y) = \begin{Bmatrix} \min(\alpha, \beta) & x = y \\ \bot & x \neq y \end{Bmatrix}. \tag{1.4}$$

Then there exists an order isomorphism between the frames $P(X, E_c) = [0, 1]^X$ and $P(\Sigma(X, E_c))$. In particular, every map $f \in [0, 1]^X$ can be identified with an $\widetilde{E}_c$-strict and $\widetilde{E}_c$-extensional map in the sense of formula (6.12) in FSBC:

$$\widetilde{f}(\alpha \cdot 1_x) = \min(\alpha, f(x)), \quad \alpha \cdot 1_x \in S(X, E_c). \tag{1.5}$$

After these preparations we conclude from Section 6 in FSBC that every map

$$X \xrightarrow{\ f\ } [0, 1]$$

can be identified with a subobject

$$(U, \widetilde{E}_c|_{U \times U}) \hookrightarrow \Sigma(X, E_c)$$

of $\Sigma(X, E_c)$ in the sense of **C**[0, 1]-**SET** and vice versa (cf. Theorem 6.1 in FSBC). In particular, we have the following situation (cf. (6.1), (6.13) in FSBC):

$$U = \{\alpha \cdot 1_x \in S(X, E_c) | \alpha \leqslant f(x)\}, \tag{1.6}$$

$$f_U(x) = \widetilde{f_U}(\widetilde{x}) = \sup\{\widetilde{E}_c(1_x, \beta \cdot 1_y) | \beta \cdot 1_y \in U\}, \quad x \in X. \tag{1.7}$$

Hence the gap in the definition of fuzzy sets can be closed as follows:

> A fuzzy set (class) A [in a given set X] is a subobject of the singleton space $\Sigma(X, E_c)$ in the sense of **C**[0, 1]-**SET** and is characterized by a membership function determined by (1.7).

In this context elements of U are called *prototypes* of the membership function f_U (cf. [18, p. 101]).

In the following considerations we explain the details of the relationship defined by (1.6) and (1.7). Since complete [0, 1]-valued sets and sheaves on [0, 1] are equivalent concepts (cf. Section 4 in FSBC), we first give a sheaf-theoretic explanation. Because of (1.6) the sheaf $\mathcal{F}_U$ on [0, 1] corresponding to $(U, \widetilde{E}_c|_{U \times U})$ is given by

$$\mathcal{F}_U(\alpha) = \{\alpha \cdot 1_x \in S(X, E_c) | \alpha \leqslant f(x)\}, \quad \varrho^\alpha_\beta(\alpha \cdot 1_x) = \beta \cdot 1_x, \ \beta \leqslant \alpha.$$

In order to simplify our notation we can identify $\mathcal{F}_U(\alpha)$ with the *level cut*

$$A_\alpha = \{x \in X \mid \alpha \leqslant f(x)\},$$

and the restriction map ϱ^α_β with the inclusion map $A_\alpha \hookrightarrow A_\beta \quad (0 < \beta \leqslant \alpha)$. Hence the sheaf corresponding to $(U, \widetilde{E}_c|_{U \times U})$, coincides with the *sheaf of level cuts* determined by f (cf. Example 4.2 in FSBC). In this context the formula (1.7) attains the following form:

$$f_U(x) = \sup\{\beta \in]0, 1] \mid x \in A_\beta\}, \quad x \in X \tag{1.8}$$

and coincides with Negoita–Ralescu's formula for the reconstruction of membership functions from their level cuts.

Since the sheaf corresponding to $\Sigma(X, E_c)$ is the *constant sheaf* $\mathcal{F}_X$ on [0, 1]

$$\mathcal{F}_X(\alpha) = X, \quad \varrho^\alpha_\beta = id_X, \ 0 < \beta \leqslant \alpha,$$
$$\mathcal{F}_X(0) = \{\cdot\}, \quad \varrho^\alpha_0 = !_X, \ \beta = 0$$

and since every subsheaf of $\mathcal{F}_X$ is a sheaf of level cuts (cf. Example 4.2 in FSBC), we can define the concept of fuzzy sets form a sheaf-theoretic point of view as follows:

> A fuzzy set (class) A [in a given set X] is a subsheaf of the constant sheaf $\mathcal{F}_X$ on [0, 1] and is characterized by a membership function determined by (1.8).

Finally, we study the whole situation from the viewpoint of espaces étalés. First, we recall the construction of the espaces étalés corresponding to the crisp equality (cf. Example 2.5 in FSBC): because of $P(X, E_c) = [0, 1]^X$ we infer from formula (2.5) in FSBC that every point q of $P(X, E_c)$ can be identified with a pair $(x, t) \in X \times [0, 1[$ s.t. for all $f \in [0, 1]^X$ the following relation holds.

$$q(f) = 1 \iff t < f(a).$$

Hence the espace étalés associated with (X, E_c) is given by

$$X \times [0, 1[\xrightarrow{\pi_2} [0, 1[\ , \tag{1.9}$$

where π_2 is the projection onto the second component, and $X \times [0, 1[$ is topologized by the *standard topology* $\mathcal{T}_{\mathcal{E}}$—i.e. the product topology w.r.t. the discrete topology on X and the lower topology $\omega([0, 1[)$ on $[0, 1[$ (cf. Section 1 in FSBC). Further, we know that the sheaf corresponding to $(U, \widetilde{E}_c|_{U\times U})$, is the sheaf of level cuts determined by f. Hence the set of all *germs* at t coincides with the *strict level cut* $\{x \in X \mid t < f(x)\}$ of f w.r.t. t (cf. Remark 4.7 in FSBC), and the espace étalé $V_f \xrightarrow{\pi} [0, 1[$ corresponding to f is given by

$$V_f = \{(x, t) \in X \times [0, 1[| t < f(x)\}, \quad \pi = \pi_2|_{V_f}, \tag{1.10}$$

where V_f is an open subset of $X \times [0, 1[$ w.r.t. $\mathcal{T}_{\mathcal{E}}$.

Thus from the viewpoint of espaces étalés the definition of fuzzy sets runs as follows:

> A fuzzy set (class) A [in a given set X] is an open subset V of $X \times [0, 1[$ in the sense of $\mathcal{T}_{\mathcal{E}}$—i.e. a subspace of the espace étalé $X \times [0, 1[\xrightarrow{\pi_2} [0, 1[$ —and is characterized by a membership function determined by
>
> $$f(x) = \inf\{\alpha \in [0, 1[| \ (x, \alpha) \notin V\}.$$

To sum up we have given three equivalent definitions of fuzzy sets in which the relationship between fuzzy sets and their membership functions has been formulated externally. It is easily seen that we can internalize these constructions when we use an appropriate categorical framework. For instance, if we choose the category of complete [0, 1]-valued sets, then the internal formulation of the relationship between fuzzy sets and their membership functions is expressed by the subobject classifier axiom (cf. Theorem 6.3 in FSBC).

1.1.2. Quotients with respect to similarity relations

In mathematical taxonomy or fuzzy set theory *similarity* is understood as a *property* which individuals have by virtue of their share of common attributes. In particular, this property increases in proportion as the number of shared attributes increases. Thus, in principle *similarity relations* (cf. [93]), resp. *similarity coefficients* (cf. [52]), are *vague equivalence relations*, and the question arises what are their *quotients*. A short glance at the literature shows that usually partition trees (cf. [3,78,93]) or dually dendrograms (cf. [52]) appear as solutions. Since in both cases the fundamental concept of *quotient maps* is missing, we view both proposals as unsatisfactory. The aim of the following considerations is to present a comprehensive solution of this important problem which by the way also shows the limitation of traditional concepts in fuzzy set theory.

First, we recall the definition of similarity relations: let X be a set; a map $X \times X \xrightarrow{\mu} [0, 1]$ is called a similarity relation on X iff μ satisfies the following conditions:

(SM1) $\mu(x, x) = 1$ (Reflexivity).
(SM2) $\mu(x, y) = \mu(y, x)$ (Symmetry).
(SM3) $\min(\mu(x, y), \mu(y, z)) \leqslant \mu(x, z)$ (Transitivity).

Obviously, similarity relations are [0, 1]-valued equalities with *global* extent of existence—i.e. $E(x, x) = 1$ for all $x \in X$ (see also Example 2.2 in FSBC). Therefore, before we explain the quotient construction, we first consider the espace étalé associated with a given similarity relation μ. Referring to Sections 1 and 2 in FSBC we again identify $(pt([0, 1]), \mathcal{T}_{[0,1]})$ with $([0, 1[, \omega([0, 1[))$ and obtain immediately that the set $\mathcal{A}_t$ of *all germs* at $t \in [0, 1[$ is given by the ordinary quotient of X w.r.t. the equivalence relation $\sim_t$ determined by the *strict level cut* of μ at t—i.e.

$$x \sim_t y \iff t < \mu(x, y). \tag{1.11}$$

Now we proceed as in Section 2 of FSBC and form the disjoint union

$$\mathcal{A} = \dot{\bigcup_{t \in [0,1[}} \mathcal{A}_t$$

of all $\mathcal{A}_t$ and arrive at the desired espace étalé $\mathcal{A} \xrightarrow{\pi} [0,1[$. In particular, π is determined by $\pi(z) = t \iff z \in \mathcal{A}_t$, $z \in \mathcal{A}$. Further, we know from Section 1.1.1 that $X \times [0,1[\xrightarrow{\pi_2} [0,1[$ is the espace étalé associated with the *crisp* equality on X (cf. (2.5) in FSBC). Then it is not difficult to see that there exists a bundle morphism $\pi_2 \xrightarrow{\varphi} \pi$ —i.e. a local homeomorphism $X \times [0,1[\xrightarrow{\varphi} \mathcal{A}$ (cf. footnote 4 in FSBC) making the following diagram commutative:

$$\begin{array}{ccc} X \times [0,1[& \xrightarrow{\quad\varphi\quad} & \mathcal{A} \\ & {\scriptstyle\pi_2}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & [0,1[& \end{array}$$

In particular, for all $t \in [0,1[$ the fibre map φ_t coincides with the quotient map from X to $\mathcal{A}_t = X/\sim_t$. Hence we view π as a candidate for the *quotient* w.r.t. the similarity relation μ. We confirm this understanding by the observation that $\pi_2 \xrightarrow{\varphi} \pi$ is the coequalizer of its kernel pair in the category $\mathcal{E}([0,1[)$ of espaces étalés with base space $[0,1[$. In fact, we first view μ as a membership function in $X \times X$. Then the corresponding *fuzzy set* in the sense of the previous Section 1.1.1 is the espace étalé $V \xrightarrow{\pi_V} [0,1[$ where

$$V = \{(x,y,t) \in X \times X \times [0,1[\mid t < \mu(x,y)\}, \quad \pi_V(x,y,t) = t.$$

In particular, the set V_t of all germs at t coincides with the strict level cut of μ at t (cf. (1.11)). Further, we consider the following pair of bundle morphisms $\pi_V \underset{\psi}{\overset{\xi}{\rightrightarrows}} \pi_2$ $\xi(x,y,t) = (x,t)$, $\psi(x,y,t) = (y,t)$, $(x,y,t) \in V$, and denote the respective fibre maps by ξ_t and ψ_t $(t \in [0,1[)$. Then we conclude from the ordinary quotient construction in **SET** that for all $t \in [0,1[$ the diagram

$$V_t \underset{\psi_t}{\overset{\xi_t}{\rightrightarrows}} X \overset{\varphi_t}{\twoheadrightarrow} \mathcal{A}_t \tag{1.12}$$

is *exact*—this means that (ξ_t, ψ_t) is the kernel pair of φ_t, *and* φ_t is the coequalizer of (ξ_t, ψ_t). Now we make use of the following facts:

- The topology on the codomain of any surjective local homeomorphism ϑ coincides always with the final topology w.r.t. ϑ.
- The composition and the pullback of local homeomorphisms in **TOP** is again a local homeomorphism (cf. [67, p. 99]).
- Finite limits and colimits in $\mathcal{E}([0,1[)$ are computed fibrewise (cf. Remark 5.5 in FSBC).

And conclude from (1.12) that the diagram

$$\pi_V \underset{\psi}{\overset{\xi}{\rightrightarrows}} \pi_2 \overset{\varphi}{\twoheadrightarrow} \pi$$

is exact in $\mathcal{E}([0,1[)$. Hence from the mathematical point of view the espace étalé $\mathcal{A} \xrightarrow{\pi} [0,1[$ is in fact the *quotient* w.r.t. the given similarity relation μ, and the bundle morphism $\pi_2 \xrightarrow{\varphi} \pi$ is the corresponding *quotient map*.

In the following considerations we review some developments of the theory of similarity relations in the light of the previous results. Partition trees are based on the technique of *level cuts* and not *strict level cuts* (cf. [78, pp. 244–255]); hence partition trees are far from being an appropriate concept of quotients w.r.t. similarity relations. On the other hand, *similarity classes* introduced by Zadeh 1971 (cf. [93]) have a clear meaning in the framework of quotients w.r.t. similarity relations: the similarity class $S[a]$ of an element $a \in X$ w.r.t. a similarity relation μ is nothing but the singleton $\widetilde{a}$ (cf. (2.4) in FSBC and the comment preceding Proposition 2.7 in FSBC). Thus $S[a]$ can be identified with the *global, continuous section* $[0,1[\xrightarrow{\sigma_a} \mathcal{A}$ of π determined by a (cf. Proposition 2.7(b) in FSBC)—i.e.

$$\sigma_a(t) = [a]_t, \quad t \in [0,1[.$$

The only problem with Zadeh's concept of similarity classes is the fact that in general there exist more singletons than only singletons induced by elements (cf. Section 2 in FSBC). We illustrate this situation by an *incomplete*, ultrametric space (X, ϱ).

Let μ_ϱ be the similarity relation on X induced by ϱ in the sense of Example 2.2 in FSBC—e.g.

$$\mu_\varrho(x, y) = \frac{1}{1 + \varrho(x, y)}, \quad x, y \in X.$$

Further, let $(\widehat{X}, \widehat{\varrho})$ be the metrical completion of (X, ϱ). Then elements of $\widehat{X} \setminus X$ correspond unequivocally to continuous global sections of the quotient w.r.t. μ_ϱ which are *not* induced by elements of X. Thus it is not an overstatement to understand all *local sections* as *vague equivalence classes*.

Further, it is remarkable to note that the concept of *extensionality* does not show up in investigations on similarity relations by members of the fuzzy community. Since the frame of extensional, [0, 1]-valued maps play a crucial role in the construction of the respective espaces étalés (cf. Lemmas 2.3 and 2.4 in FSBC), it is not surprising that so far fuzzy set theorists are not able to compute "clean" quotients w.r.t. similarity relations.

Finally, since espaces étalés with base space [0, 1[and complete [0, 1]-valued sets are equivalent concepts (cf. Sections 2 and 4 in FSBC), we can also reformulate the solution of the quotient problem w.r.t. similarity relations as follows:

> The quotient of $\Sigma(X, E_c)$ w.r.t. a similarity relation μ is the *singleton space* $\Sigma(X, \mu)$, and the corresponding *quotient map* is the *unique extension* of the correspondence $x \longmapsto \widetilde{x}$ to $\Sigma(X, E_c)$ (cf. (3.3) in FSBC).

Obviously, $\Sigma(X, \mu)$ is in general not a subobject of some singleton space w.r.t. the crisp equality (e.g. $\Sigma(Y, E_c)$) and consequently not a fuzzy set in the sense of the previous Section 1.1.1. Hence the construction of quotients w.r.t. similarity relations takes us beyond fuzzy set theory—a situation which can also be understood as a limitation of the traditional concept of fuzzy sets.

1.1.3. Fuzzy partial orderings

In 1971 Zadeh introduced the concept of fuzzy preorders in the following way (cf. [93]): a map $X \times X \xrightarrow{p} [0,1]$ is called a *fuzzy preorder* on X iff p satisfies the following axioms:

(P1) $p(x, x) = 1, \quad x \in X$ (Reflexivity).
(P2) $\min(p(x, y), p(y, z)) \leqslant p(x, z)$ (Transitivity).

A short glance at the literature shows that these axioms have been widely accepted. Moreover, there exist various interesting examples of fuzzy preorders—e.g. *Gödel's implication* (cf. [32, p. 30]):

$$p(\alpha, \beta) = \alpha \to \beta = \begin{Bmatrix} 1 : \alpha \leqslant \beta \\ \beta : \beta < \alpha \end{Bmatrix}, \quad \alpha, \beta \in [0,1].$$

A problem arose in Zadeh's paper from 1971 when he tried to introduce a concept of fuzzy partial orderings and added the following kind of antisymmetry axiom to fuzzy preorders:

$$\min(p(x, y), p(y, x)) = 0 \quad \text{whenever} \quad x \neq y. \tag{1.13}$$

It seems that the condition (1.13) is too strong for an appropriate antisymmetry axiom, because the addition of the linearity axiom:

$$\max(p(x, y), p(y, x)) = 1, \quad x, y \in X$$

leads to the situation that p reduces to a *crisp* linear ordering—an observation made already by Chakraborty and Das 1985 (cf. [11]). In order to overcome these obstacles Bodenhofer proposed recently a formulation of an antisymmetry axiom based on a priori given similarity relation: in the special case of $* = \min$ Bodenhofer's definition of a fuzzy partial ordering runs as follows (cf. [6, Definition 3.1]): let S be a similarity relation on X. A fuzzy preorder p on X is called an fuzzy (partial) S-ordering iff p satisfies the following additional axioms:

(SP1) $S(x, y) \leqslant p(x, y), \quad x, y \in X$ (S-Reflexivity).
(SP3) $\min(p(x, y), p(y, x)) \leqslant S(x, y), \quad x, y \in X$ (S-Antisymmetry).

In the presence of (P2) it is evident that the axiom of S-reflexivity is equivalent to (P1) and the $S \otimes S$-extensionality of p—i.e.

$$\min\{S(x_2, x_1), p(x_1, y_1), S(y_1, y_2)\} \leqslant p(x_2, y_2).$$

Even though Bodenhofer is coming close to an appropriate formulation of an antisymmetry axiom, he does not obtain the final solution. One of the reasons for this situation is the fact that he ignores the *standard construction of partial orderings* which even can be internalized in any category with products and quotients w.r.t. equivalence relations. For the sake of completeness we recall here the respective steps of this construction:

- The initial object is a preorder.
- The symmetrization of the preorder.
- The construction of the quotient w.r.t. the symmetrization of the given preorder.
- The image of the preorder on the quotient (what can be called a partial ordering).

In this context it is fair to remark that Bodenhofer carries out the symmetrization (cf. [6, Theorem 3.1]), but not the quotient construction w.r.t. the symmetrized fuzzy preorder.

In the following consideration we solve the problem of specifying the "clean" antisymmetry axiom for fuzzy partial orderings. For this purpose we start with a fuzzy preorder p on X and apply subsequently the standard construction mentioned above. First, we notice that the *symmetrization* of p leads to a *similarity relation* μ

$$\mu(x, y) = \min(p(x, y), p(y, x)), \quad x, y \in X,$$

on X. Now we construct the quotient w.r.t. μ—i.e. the espace étalé

$$\mathcal{A} \xrightarrow{\ \pi\ } [0, 1[$$

associated with μ (cf. Section 1.1.2 or 2 in FSBC). In the special case of Gödel's implication the symmetrization leads to the bi-implication (cf. Example 2.1(c) in FSBC), and the fibre over t w.r.t. the associated espace étalé coincides with the set of all germs of open subsets from $\omega([0, 1[)$ at $t \in [0, 1[$ (cf. footnote 5 in FSBC). After this marginal note we return to the general case and observe that for all $t \in [0, 1[$ the *strict level cut* of p at t induces an ordinary partial ordering $\leqslant_t$ on the set $\mathcal{A}_t$ of all *germs* at t:

$$[x]_t \leqslant_t [y]_t \iff \exists x' \in [x]_t, \exists y' \in [y]_t \quad \text{s.t.} \quad t < p(x', y'). \tag{1.14}$$

In particular, on the set $\mathcal{F}(\alpha)$ of all *local sections* of π with domain $[0, \alpha[$ there exists an ordinary partial ordering $\leqslant$ defined by

$$\sigma_1 \leqslant \sigma_2 \iff \sigma_1(t) \leqslant_t \sigma_2(t), \quad t \in [0, \alpha[. \tag{1.15}$$

Hence the sheaf $\mathcal{F}$ on [0, 1] corresponding to the espace étalé π is a sheaf of *partially ordered sets*.

From the viewpoint of complete [0, 1]-valued sets the situation is as follows: the quotient of $\Sigma(X, E_c)$ w.r.t. the similarity relation μ is the singleton space $\Sigma(X, \mu)$. Since the quotient map φ is the unique extension of the correspondence $x \longmapsto \widetilde{x}$ (cf. (3.3) in FSBC), the image of p under φ is given by (cf. Theorem 3.2(ii) in FSBC)

$$\widehat{p}(s_1, s_2) = \sup\{\min(s_1(x), p(x, y), s_2(y)) | x, y \in X\}, \quad s_1, s_2 \in S(X, \mu). \tag{1.16}$$

We show that $\widehat{p}$ satisfies the following conditions:

(PP0) $\widehat{p}(s_1, s_2) \leqslant \min(\mathbb{E}(s_1), \mathbb{E}(s_2))$ (Strictness).
(PP1) $\widetilde{\mu}(s_1, s_2) \leqslant \widehat{p}(s_1, s_2)$ (Reflexivity).
(PP2) $\min(\widehat{p}(s_1, s_2), \widehat{p}(s_2, s_1)) \leqslant \widetilde{\mu}(s_1, s_2)$ (Antisymmetry).
(PP3) $\min(\widehat{p}(s_1, s_2), \widehat{p}(s_2, s_3)) \leqslant, \widehat{p}(s_1, s_3)$ (Transitivity),

where $\widetilde{\mu}$ is the canonical [0, 1]-valued equality on $S(X, \mu)$ (cf. Theorem 3.2 in FSBC). The strictness of $\widehat{p}$ is obvious. Further, (P1) implies immediately (PP1). In order to verify (PP2) we make use of the extensionality axiom (S1) and the singleton property (S2) w.r.t. μ (cf. Section 2 in FSBC) and obtain:

$$\begin{aligned}\min(\widehat{p}(s_1, s_2), \widehat{p}(s_2, s_1) &= \sup\{\min\{s_1(x), p(x, y), s_2(y), s_2(y'), p(y', x'), s_1(x')\} \mid x, x', y, y' \in X\}\\ &= \sup\{\min\{s_1(x), p(x, y), \mu(y, y'), p(y', x'), \mu(x, x'), s_2(y)\} \mid x, x', y, y' \in X\}\\ &= \sup\{\min\{s_1(x), \mu(x, y), s_2(y)\} \mid x, y \in X\} = \widetilde{\mu}(s_1, s_2).\end{aligned}$$

Finally, the verification of (PP3) uses (P2) and again the singleton axioms (S1) and (S2). We leave the details of this proof to the reader.

Since the sheaf of local sections of π is equivalent to the singleton space $\Sigma(X, \mu)$ (cf. Sections 2 and 4 in FSBC), we are interested in rediscovering the partial ordering in (1.15) on the side of singletons. First, we fix a real number $\alpha \in [0, 1]$ and consider the set $S(X, \mu, \alpha)$ of all singletons s of (X, μ) with extent α—i.e $\mathbb{E}(s) = \alpha$. Further, the correspondence $s \longmapsto \sigma_s$ is a bijective map from $S(X, \mu, \alpha)$ to the set $\mathcal{F}(\alpha)$ of all local sections of π with domain $[0, \alpha[$ (cf. Proposition 2.7 in FSBC). Then in the case of $s_1, s_2 \in S(X, \mu, \alpha)$ we conclude from (1.16) and Proposition 2.7(b) in FSBC:

$$\sigma_{s_1} \leqslant \sigma_{s_2} \text{ in the sense of } \quad (1.15) \iff \alpha = \widehat{p}(s_1, s_2).$$

Hence $\widehat{p}$ is intimately related to the sheaf of partially ordered sets corresponding to espace étalé π associated with μ.

To sum up we come to the following conclusion: a *partially ordered fuzzy set* is a sheaf of partially ordered sets on [0, 1] or a complete [0, 1]-valued set (A, E) provided with a binary map $A \times A \xrightarrow{p} [0, 1]$ satisfying the following conditions [1] :

(PP0) $p(a, b) \leqslant \min(E(a, a), E(b, b))$ (Strictness).
(PP1) $E(a, b) \leqslant p(a, b)$ (Reflexivity).
(PP2) $\min(p(a, b), p(b, a)) \leqslant E(a, b)$ (Antisymmetry).
(PP3) $\min(p(a, b), p(b, c)) \leqslant p(a, c)$ (Transitivity).

What is missing in Bodenhofer's axiom system for fuzzy partial orderings is the strictness axiom (PP0) and the environment given by complete [0, 1]-valued sets.

1.1.4. Fuzzy partitions, fuzzy control and defuzzification

A control situation comprehends a system S, an *input* universe X (set of measured values) and an *output* universe Y (set of the so-called control actions applied to the system under control). The basic task of designing a controller for a system S consists in the specification of a function associating an appropriate output value with each input value.

In many practical situation we encounter the situation that the highest degree of precision is not necessary—e.g. do descriptions of economic or technological systems always require the full concept of real numbers? The rationale behind Mamdani's fuzzy controller is the idea to base possible control algorithms on a concept of "fuzzy maps". Since every totally defined, crisp map can be viewed as a correspondence between a partition of the domain and a special partition of the codomain, a *fuzzy map* can consequently be understood as a correspondence between a "fuzzy partition" of the input and a certain "fuzzy partition" of the output universe. Therefore we first recall the concept of fuzzy partitions as developed by the author in the case of frames (cf. [39, Section 4.2]).

[1] Precursors of these axioms can be found in [48, Definition II.1]

A family $\mathcal{F} = \{f_i \mid i \in I\}$ of maps $X \xrightarrow{f_i} [0, 1]$ is called a *fuzzy partition* of X iff for every pair $(f_i, f_j) \in \mathcal{F}$ the following *disjointness* property holds

$$\sup_{x\in X} \min(f_i(x), f_j(x)) \leqslant \inf_{z\in X} f_i(z) \longleftrightarrow f_j(z), \tag{1.17}$$

where $\longleftrightarrow$ denotes the bi-implication w.r.t. Gödel's implication in [0, 1]. Referring to Theorem 2.2.2 in [39] it is not difficult to establish the subsequent facts:

- Let (X, E) be a [0, 1]-valued set, and $\mathcal{F}$ be a family of singletons s of (X, E) provided with the property $\sup_{s\in\mathcal{F}} s(x) = E(x, x),\ x \in X$. Then for all $x, y \in X$ the relation $E(x, y) = \sup_{s\in\mathcal{F}} \min(s(x), s(y))$ holds.
- Let $\mathcal{F} = \{f_i \mid i \in I\}$ be a family of maps $X \xrightarrow{f_i} [0, 1]$. Then the following assertions are equivalent:
  (a) There exists a unique [0, 1]-valued equality E on X satisfying the following conditions:
    - Every element of $\mathcal{F}$ is a singleton w.r.t. E.
    - For all $x \in X$ the relation $\sup_{f\in\mathcal{F}} f(x) = E(x, x)$ holds.
  (b) $\mathcal{F}$ satisfies the disjointness condition (1.17).

After these preparations we return to the general control problem: in a first step we apply certain (fuzzy) clustering procedures to observed process data of S which lead to a fuzzy partition $\mathcal{F}$ of the input universe X, to a fuzzy partition $\mathcal{G}$ of the output universe Y, and to a map $\mathcal{F} \xrightarrow{\Phi} \mathcal{G}$ satisfying the following coherence conditions:

(CF1) $\inf_{x\in X}\ f_1(x) \longleftrightarrow f_2(x) \leqslant \inf_{y\in Y}[\Phi(f_1)](y) \longleftrightarrow [\Phi(f_2)](y),\ f_1, f_2 \in \mathcal{F}$.
(CF2) $f(x) \leqslant \sup_{y\in Y}[\Phi(f)](y),\ x \in X$.

The map Φ is usually interpreted as a rule base of IF–THEN–RULES—i.e.

If $x(\in X)$ is $f(\in \mathcal{F})$, then $y(\in Y)$ is $\Phi(f)$.

In this context the axiom (CF1) represents a kind of consistency of the given rule base, while (CF2) means that for every input value there exists at least an output value.

In a second step we aggregate these informations according to *Mamdani's* method (cf. [76, pp. 38–46]) and obtain a map $X \times Y \xrightarrow{R} [0, 1]$ determined by

$$R(x, y) = \sup_{f\in\mathcal{F}} \min(f(x), [\Phi(f)](y)), \quad x, y \in X \times Y. \tag{1.18}$$

In the following considerations we show that R is in fact a "fuzzy morphism" in the sense of Section 2 in FSBC—i.e. a bundle morphism between certain espaces étalés.

First, let E and F be the respective [0, 1]-valued equalities on X and Y determined by the fuzzy partitions $\mathcal{F}$ of X and $\mathcal{G}$ of Y. Then

$$pt(P(X, E)) \xrightarrow{\pi_E} [0, 1[, \qquad pt(P(Y, F)) \xrightarrow{\pi_F} [0, 1[$$

are the respective espaces étalés associated with E and F (cf. Section 2 in FSBC). In order to show that R induces a bundle morphism

$$\pi_E \xrightarrow{\varphi^R} \pi_F$$

it is sufficient to observe that R fulfills the axioms (F1)–(F3) from Section 2 in FSBC. The axiom (F1) follows immediately from the extensionality of all members of $\mathcal{F}$, resp. of $\mathcal{G}$. In order to prove (F2) we make use of (1.17), (CF1) and obtain

$$\begin{aligned}\min(R(x, y_1), R(x, y_2)) &= \sup_{f,f'\in\mathcal{F}} \min\{f(x), f'(x), [\Phi(f)](y_1), [\Phi(f')](y_2)\}\\ &\leqslant \sup_{f,f'\in\mathcal{F}} \min\left\{\left(\inf_{z\in Y}([\Phi(f)](z) \longleftrightarrow [\Phi(f')](z))\right), [\Phi(f)](y_1), [\Phi(f')](y_2)\right\}\\ &\leqslant \sup_{f\in\mathcal{F}} \min([\Phi(f)](y_1), [\Phi(f)](y_2)) \leqslant F(y_1, y_2).\end{aligned}$$

Finally, (F3) is an immediate corollary of (CF2). Hence R is in fact a "fuzzy morphism", in the sense of Section 2 in FSBC, and the fibre maps φ_t^R of the associated bundle morphism φ^R are determined by (cf. (2.13) in FSBC)

$$\varphi_t^R([x]_t) = [y]_t \quad \text{where } t < R(x, y), \quad t \in [0, 1[. \tag{1.19}$$

With regard to the implementation of IF–THEN-RULES we emphasize that the meaning of the previous relation (1.19) can be expressed as follows: for every input value x there exists a member f of the fuzzy partition $\mathcal{F}$ s.t.

$$\text{if } x \text{ is } f \text{ at } t, \text{ then } y \text{ is } \Phi(f) \text{ at } t \text{ where } t \in [0, 1[. \tag{1.20}$$

Finally, we turn to one of the major problems in fuzzy control—the *defuzzification problem*. A short glance at the literature shows that all algorithms in this context (e.g. centre of gravity method/centroid defuzzification, mean of maximum method (cf. [76,59,17])) are ad hoc and do not pay attention to the fact that DEFUZZIFICATION means the *change of base* from [0, 1] back to $2 = \{0, 1\}$. Since Mamdani's approach to fuzzy control uses only the frame-theoretic structure of the real unit interval, the defuzzification of [0, 1] is nothing but a frame homomorphism $[0, 1] \xrightarrow{p} 2$ — i.e. a *point p* of [0, 1] (cf. Section 1 in FSBC). Here it is worthwhile to note that points of [0, 1] and elements of [0, 1[are equivalent concepts (cf. (1.4) in FSBC).

Referring to Section 5 in FSBC we know that every point of the underlying frame—i.e. every element $t \in [0, 1[$— induces a geometric morphism $\mathfrak{F}_t$ from the category $\mathcal{E}([0, 1[)$ of espaces étalés with base space [0, 1[to **SET**. In this context the image of an espace étalé $pt(P(X, E)) \xrightarrow{\pi_E} [0, 1[$ under $\mathfrak{F}_t$ coincides with the fibre over t w.r.t. π_E (cf. Remark 5.5 in FSBC). In particular, if the underlying equality E on X is generated by a fuzzy partition $\mathcal{F}$ of X, then the fibre over t w.r.t. π_E is given by

$$\mathcal{A}_t = \{[x]_t \mid \exists x \in X, \ \exists f \in \mathcal{F} : t < f(x)\}. \tag{1.21}$$

If $pt(P(Y, F)) \xrightarrow{\pi_F} [0, 1[$ is a further espace étalé and $\pi_E \xrightarrow{\varphi} \pi_F$ is a bundle morphism, then the image of φ under $\mathfrak{F}_t$ coincides with the corresponding *fibre map* φ_t between the respective fibres over t. Thus we conclude from (1.19) and (1.21) that the relation (1.20) describes the *defuzzification* of Mamdani's "control relation" R at t. In this context the formulation

if "x is f at t", then "y is $\Phi(f)$ at t"

has the mathematical translation

if $t < f(x)$, then $t < [\Phi(f)](y)$

which means that the class $[x]_t$ is mapped to the class $[y]_t$.

1.1.5. Fuzzy topologies and the hypergraph functor

In this subsection we show that the espace étalé associated with the crisp equality plays also a fundamental role in fuzzy topology. In particular, we briefly sketch the relationship between fuzzy topological spaces and fibrewise topological spaces (cf. [49,51]).

From a frame-theoretic point of view we can formulate the concept of fuzzy topologies as follows: a *fuzzy topology* on a set X is simply a *subframe* τ of $[0, 1]^X$ (see also Definition 2.3.2(b)). Hence we view fuzzy topologies on X as subsets of $[0, 1]^X$ which are closed under arbitrary joins and finite meets in the sense of the frame $[0, 1]^X$. Further, let $X \times [0, 1[\xrightarrow{\pi_2} [0, 1[$ be the espace étalé associated with the crisp equality on X (cf. Example 2.5 in FSBC). In particular, the standard topology $\mathcal{T}_{\mathcal{E}}$ on $X \times [0, 1[$ is the product topology w.r.t. the discrete topology on X and the lower topology $\omega([0, 1[)$ on [0, 1[. Then we know from Section 1.1.1 that every element $f \in [0, 1]^X$ can be understood as an open subset

$$V_f = \{(x, t) \in X \times [0, 1[\mid t < f(x)\}$$

of $X \times [0, 1[$ w.r.t. $\mathcal{T}_{\mathcal{E}}$. Therefore every fuzzy topology τ on X can be identified with an ordinary topology $\mathcal{E}(\tau)$ on $X \times [0, 1[$ which is coarser than $\mathcal{T}_{\mathcal{E}}$. A short glance at the literature shows that this identification is the essential part of the *hypergraph functor* from the category of fuzzy topological spaces to the category of ordinary topological space (cf. [80,42]). Moreover, if τ satisfies Lowen's constant condition—this means that all constant maps from X to [0, 1] are contained in τ (cf. [64]), then the projection $X \times [0, 1[\xrightarrow{\pi_2} [0, 1[$ onto the second component is continuous w.r.t. $\mathcal{E}(\tau)$. Hence every fuzzy topology τ on X satisfying the constant condition gives rise to a *fibrewise topological space* $(X \times [0,1[, \mathcal{E}(\tau)) \xrightarrow{\pi_2} ([0,1[, \omega([0,1[))$. Finally, we can consider the sheaf $\mathcal{G}_{\pi_2}$ of local sections of π_2 w.r.t. $\mathcal{E}(\tau)$ on $[0, 1]$ (cf. Remark 4.3 in FSBC), and observe that τ induces a *topology* on $\mathcal{G}_{\pi_2}$ in the *sense* of the *category of sheaves* on $[0, 1]$. Further details of this situation are explained in Example 2.3.5(a)).

1.2. *Categorical basis for Zadeh's rules of combination of membership functions*

With the exception of the "complementation of fuzzy sets" we show that all rules of combination of membership functions introduced by Zadeh 1965 [92] have a categorical basis in the sense of the category **C**[0, 1]-**SET** of complete [0, 1]-valued sets, resp. in the sense of the category $\mathcal{E}([0,1[)$ of espaces étalés with base space $([0,1[, \omega([0,1[))$.

1.2.1. *Intersection and unions of fuzzy sets*

Let X be a universe of discourse and E_c be the *crisp* equality on X. We know already that $[0,1]^X$ is order isomorphic to $P(\Sigma(X, E_c))$ and consequently order isomorphic to the class of all subobjects of the singleton space $\Sigma(X, E_c)$ (cf. Section 6 in FSBC). Since fuzzy sets in X are subobjects of $\Sigma(X, E_c)$, we obtain that the pointwisely defined *infimum* (resp. *supremum*) of membership functions viewed as elements of $[0,1]^X$ corresponds to the categorical construction of *intersection* (resp. *union*) of subobjects of $\Sigma(X, E_c)$ in the sense of **C**[0, 1]-**SET** (cf. [33]).

Since **C**[0, 1]-**SET** and $\mathcal{E}([0,1[)$ are equivalent categories, we can also look at the previous situation from the view-point of espaces étalés. Obviously the singleton space $\Sigma(X, E_c)$ is replaced by the espace étalé $X \times [0,1[\xrightarrow{\pi_2} [0,1[$, and subobjects of π_2 can be identified with open subsets of $X \times [0,1[$ w.r.t. the standard topology $\mathcal{T}_{\mathcal{E}}$ (cf. Example 2.5 in FSBC, Sections 1.1.1 and 1.1.5). Then finite intersections (resp. arbitrary unions) of subobjects of π_2 in the sense of $\mathcal{E}([0,1[)$ correspond to finite set-theoretical intersections (resp. arbitrary set-theoretical unions) of *open subsets* of $X \times [0,1[$. In particular, finite intersections (resp. arbitrary unions) of subobjects of π_2 are computed fibrewise as finite intersections (resp. arbitrary unions) of sets (cf. Remark 5.5 in FSBC).

Referring to Theorem 6.3 in FSBC there exists also an *internal point of view* looking at the relationship between membership functions and subobjects of the singleton space $\Sigma(X, E_c)$. For this purpose we internalize membership functions $X \xrightarrow{f} [0,1]$ as characteristic morphisms $\Sigma(X, E_c) \xrightarrow{\chi_{\tilde{f}}} (R_\Omega, E_\Omega)$ (cf. Section 6 in FSBC) and the binary minimum operation as *truth arrow* in **C**[0, 1]-**SET**. First, we form the product $(R_\Omega, E_\Omega) \times (R_\Omega, E_\Omega)$ in **C**[0, 1]-**SET** (cf. Section 3 in FSBC) and consider the following subobject of $(R_\Omega, E_\Omega) \times (R_\Omega, E_\Omega)$:

$$\mathbb{1} = ([0,1], \min) \xrightarrow{\langle t,t \rangle} (R_\Omega, E_\Omega) \times (R_\Omega, E_\Omega) \quad , \tag{1.22}$$

where t denotes the *arrow true* (cf. (6.6) in FSBC). Now we apply Theorem 6.3 in FSBC and classify the monomorphism $\langle t, t \rangle$ by the characteristic morphism

$$(R_\Omega, E_\Omega) \times (R_\Omega, E_\Omega) \xrightarrow{\chi_\wedge} (R_\Omega, E_\Omega).$$

In particular, the subsequent diagram is a pullback square (cf. (6.7) in FSBC)

$$\begin{array}{ccc} \mathbb{1} & \xrightarrow{\quad ! \quad} & \mathbb{1} \\ {\scriptstyle \langle t,t \rangle} \big\downarrow & & \big\downarrow {\scriptstyle t} \\ (R_\Omega, E_\Omega) \times (R_\Omega, E_\Omega) & \underset{\chi_\wedge}{\dashrightarrow} & (R_\Omega, E_\Omega) \end{array} \tag{1.23}$$

Now we apply (6.8) in FSBC and obtain

$$\begin{aligned} f_\wedge((\alpha, \lambda_1), (\alpha, \lambda_2)) &= \sup_{\beta \in [0,1]} \min(E_\Omega((\alpha, \lambda_1), (\beta, \beta)), E_\Omega((\alpha, \lambda_2), (\beta, \beta))) \\ &= \min(\lambda_1, \lambda_2). \end{aligned}$$

Hence $\chi_\wedge$ is in fact the internalized version of the binary minimum operation, and is called the *conjunction* in the sense of **C**[0, 1]-**SET**. After these categorical preparations we consider the categorical intersection of two fuzzy sets—i.e.

of two subobjects $(U_1, F_1) \overset{\varphi_1}{\rightarrowtail} \Sigma(X, E_c)$ and $(U_1, F_1) \overset{\varphi_1}{\rightarrowtail} \Sigma(X, E_c)$ — which is given by the following pullback square:

$$\begin{array}{ccc}
(D, G) & \dashrightarrow & (U_2, F_2) \\
\downarrow & \overset{\psi}{\searrow} & \downarrow \varphi_2 \\
(U_1, F_1) & \underset{\varphi_1}{\rightarrowtail} & \Sigma(X, E_c)
\end{array}$$

Further, let χ_{φ_1} and χ_{φ_2} be the characteristic morphisms of φ_1 and φ_2. We show that $\chi_\wedge \circ \langle \chi_{\varphi_1}, \chi_{\varphi_2} \rangle$ is the characteristic morphism of the intersection determined by the monomorphism $(D, G) \overset{\psi}{\rightarrowtail} \Sigma(X, E_c)$. First, it is not difficult to verify that the diagram

$$\begin{array}{ccc}
(D, G) & \xrightarrow{\quad ! \quad} & \mathbb{1} \\
\psi \downarrow & & \downarrow \langle t, t \rangle \\
\Sigma(X, E_c) & \underset{\langle \chi_{\varphi_1}, \chi_{\varphi_2} \rangle}{\longrightarrow} & (R_\Omega, E_\Omega) \times (R_\Omega, E_\Omega)
\end{array}$$

is a pullback square. Because of the *Pullback Lemma* (cf. [28, p. 67; 83, Proposition 7.8.4, p. 59]) the outer rectangle of the diagram

$$\begin{array}{ccccc}
(D, G) & \xrightarrow{\quad ! \quad} & \mathbb{1} & \xrightarrow{\quad ! \quad} & \mathbb{1} \\
\psi \downarrow & & \downarrow \langle t, t \rangle & & \downarrow t \\
\Sigma(X, E_c) & \underset{\langle \chi_{\varphi_1}, \chi_{\varphi_2} \rangle}{\longrightarrow} & (R_\Omega, E_\Omega) \times (R_\Omega, E_\Omega) & \underset{\chi_\wedge}{\longrightarrow} & (R_\Omega, E_\Omega)
\end{array}$$

is also a pullback square. Hence Theorem 6.3 in FSBC implies that $\chi_\wedge \circ \langle \chi_{\varphi_1}, \chi_{\varphi_2} \rangle$ is the characteristic morphism of the intersection $(D, G) \overset{\psi}{\rightarrowtail} \Sigma(X, E_c)$.

In a similar way we can deal with the internal discussion of the union of subobjects in **C**[0, 1]-**SET** where the internalization of the binary maximum operation leads to the *disjunction* in **C**[0, 1]-**SET**. Since these constructions are more extensive than in the case of intersections (e.g. the (*epi–mono*)-factorization property in **C**[0, 1]-**SET** is needed), we refer here the reader to standard textbooks on topos theory (cf. [28,53]).

We finish this subsection with some critical remarks on the compositionality of membership functions described in terms of similarity. We begin with a quotation of a passage from a survey article by Dubois, Ostasiewicz and Prade (cf. [18, pp. 100–101]):

> Namely, let A be a subset of prototypes of F and define the membership function of the fuzzy set F in terms of similarity to a prototype as follows:
>
> $$F(u) = \max_{u' \in A} S(u, u').$$
>
> ....One might be tempted to use fuzzy set-theoretic operations to combine such fuzzy sets, requiring the usual compositional assumptions. Unfortunately such calculus cannot be compositional. Namely given B the set of prototypes of another fuzzy set G, it is easy to see that we cannot in general accept that $F \cap G = \min(F(\cdot), G(\cdot))$

nor any connective different from min. ... So the max–min theory of fuzzy sets cannot precisely account for this particular similarity-based semantics of F.

With regard to Section 6 in FSBC and the previous results in the special case of crisp equalities the last sentence in the previous quotation sounds somehow strange. The fundamental problem is not the max–min theory of fuzzy sets, but Dubois' and his co-author's understanding of "prototypes" of fuzzy sets, resp. membership functions. Obviously, the formula in the previous quotation coincides with (6.1) in FSBC, but in contrast to the assumptions of Theorem 6.1 in FSBC the *completeness* of the [0, 1]-valued set (X, S) determined by the similarity relation S is missing as a prerequisite to the environment studied by Dubois and his co-authors. As explained in Sections 2 and 3 of FSBC *completeness* can easily be added to the situation given by any similarity relation S—viz. by the transition to the singleton space $\Sigma(X, S)$ (cf. Section 3 in FSBC) or equivalently to the espace étalé $pt(P(X, S)) \xrightarrow{\pi} [0, 1[$ associated with S (cf. Section 2 in FSBC). In this context the set U of all *prototypes* of a S-extensional membership function f is the set of all *singletons* s of (X, S) *contained* in f (cf. (6.13) in FSBC) or equivalently the set of all local sections of π *factorizing through* the open subset $\mathbb{A}_f$ of $pt(P(X, S))$. Since Theorem 6.1 in FSBC forces us to view the set U as support set of a subobject of $\Sigma(X, S)$ and *not* of $\Sigma(X, E_c)$, we have to replace the crisp equality by the similarity relation S in our previous considerations. It is interesting to see that the previous results obtained above in that rather special case of a crisp equality remain valid in the more general context given by similarity relations and even in the case of arbitrary [0, 1]-valued equalities. Hence, if we move to the environment determined by the singleton space $\Sigma(X, S)$ (resp. the espace étalé associated with S), then the problem mentioned by Dubois and his co-authors disappears. This means that the problem does not consist in the max–min operations for the combination of membership functions, but in the traditional concept of fuzzy sets (in a universe X) as subobjects of $\Sigma(X, E_c)$ (cf. Section 1.1.1). Therefore we can view the previous quotation as a motivation to go beyond the traditional limits of fuzzy set theory and to consider also subobjects of singleton spaces as fuzzy sets. In particular, this *new* understanding of *fuzzy sets* implies that fuzzy sets are closed under quotients w.r.t. similarity relations (cf. Section 1.1.2).

1.2.2. Images and inverse images of fuzzy sets

First, we consider the situation that $X \xrightarrow{\vartheta} Y$ is an ordinary map, and g is a membership function of a fuzzy set in Y—i.e. a subobject of $\Sigma(Y, E_c)$ determined by a monomorphism $(U, F) \rightarrowtail^{\varphi} \Sigma(Y, E_c)$ (cf. Section 1.1.1). Since singleton spaces are *free* $\mathbf{T}_\Sigma$-algebra (see Section 3 in FSBC), we obtain from the universal property of free $\mathbf{T}_\Sigma$-algebras that every map $X \xrightarrow{\vartheta} Y$ can be identified with a $\mathbf{C\Omega}$-$\mathbf{SET}$-morphism $\Sigma(X, E_c) \xrightarrow{\Sigma(\vartheta)} \Sigma(Y, E_c)$. In particular, $\Sigma(\vartheta) = (\eta_{(X,E_c)} \cdot \vartheta)^\sharp$ is given by

$$[\Sigma(\vartheta)](\alpha \cdot 1_{x_0}) = \alpha \cdot 1_{\vartheta(x_0)}, \quad \alpha \cdot 1_{x_0} \in S(X, E_c).$$

According to standard textbooks on category theory (cf. [33, p. 139]) the inverse image of $(U, F) \rightarrowtail^{\varphi} \Sigma(Y, E_c)$ under $\Sigma(\vartheta)$ is a subobject

$$(V, G) \xrightarrow{\Sigma(\vartheta)^{-1}(\varphi)} (X, E_c)$$

of $\Sigma(X, E_c)$ determined by the following pullback square

$$\begin{array}{ccc} (V, G) & \dashrightarrow & (U, F) \\ {\scriptstyle \Sigma(\vartheta)^{-1}(\varphi)} \downarrow & & \downarrow {\scriptstyle \varphi} \\ \Sigma(X, E_c) & \xrightarrow{\Sigma(\vartheta)} & \Sigma(Y, E_c) \end{array}$$

Now χ_φ denotes the characteristic morphism of φ—i.e.

$$\chi_\varphi(\alpha \cdot 1_y) = (\alpha, \min(g(y), \alpha)), \quad \alpha \cdot 1_y \in S(Y, E_c).$$

Then we apply again the *Pullback Lemma* and obtain that the outer rectangle of the diagram

$$\begin{array}{ccccc} (V,G) & \dashrightarrow & (U,F) & \xrightarrow{\;!\;} & \mathbb{1} \\ {\scriptstyle \Sigma(\vartheta)^{-1}(\varphi)}\downarrow & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle t} \\ \Sigma(X,E_c) & \xrightarrow[\Sigma(\vartheta)]{} & \Sigma(Y,E_c) & \xrightarrow[\chi_\varphi]{} & (R_\Omega, E_\Omega) \end{array}$$

is a pullback square. Hence $\chi_\varphi \cdot \Sigma(\vartheta)$ is the characteristic morphism of $\Sigma(\vartheta)^{-1}(\varphi)$ and the corresponding membership map coincides with $\widetilde{g \cdot \vartheta}$ (see also (6.12) in FSBC). This means that Zadeh's concept of "inverse images of fuzzy sets" (cf. in [92, p. 346]) is the *inverse image* of subobjects in the sense of **C**[0, 1]-**SET**.

Referring to [28, p. 320] the *image* of a subobject $(V,G) \overset{\psi}{\rightarrowtail} \Sigma(X,E_c)$ (i.e. a fuzzy set in *X*) under $\Sigma(\vartheta)$ is defined as the image of $\Sigma(\vartheta) \cdot \psi$ which is determined by the monomorphism $(U,F) \overset{\xi}{\rightarrowtail} \Sigma(Y,E_c)$ of the (*epi–mono*)-decomposition of $\Sigma(\vartheta) \cdot \psi$ (cf. Remark 5.5 in FSBC). If now f is the membership function corresponding to $(V,G) \overset{\psi}{\rightarrowtail} \Sigma(X,E_c)$, then $(U,F) \overset{\xi}{\rightarrowtail} \Sigma(Y,E_c)$ is given as follows (cf. (5.12), (613) in FSBC):

$$U = \{\beta \cdot 1_y \in S(Y,E_c) \,|\, \exists x \in X : \vartheta(x) = y, \beta \leqslant f(x)\}, \quad F = \widetilde{E_c}|_{U \times U},$$

where ξ denotes the inclusion map. Finally, we classify the monomorphism $(U,F) \overset{\xi}{\rightarrowtail} \Sigma(Y,E_c)$ by the corresponding characteristic morphism

$$\Sigma(Y,E_c) \xrightarrow{\chi_\xi} (R_\Omega, E_\Omega)$$

(cf. Theorem 6.3 in FSBC) and obtain from (6.8) in FSBC:

$$\chi_\xi(\beta \cdot 1_y) = (\beta, \widetilde{g}(\beta \cdot 1_y)), \quad \widetilde{g}(\beta \cdot 1_y) = \min(\beta, \sup\{f(x) \mid \vartheta(x) = y\}).$$

Hence Zadeh's "image of membership functions" (cf. [92, p. 346]) coincides with those membership functions corresponding to the image of fuzzy sets in the categorical sense of **C**[0, 1]-**SET**.

1.2.3. Composition of fuzzy relations

First, we recall that a *fuzzy relation between X and Y* is a fuzzy set in $X \times Y$—i.e. subobject of $\Sigma(X \times Y, E_c)$ [2] (cf. Section 1.1.1). Since $\Sigma(X \times Y, E_c)$ and $\Sigma(X,E_c) \times \Sigma(Y,E_c)$ are isomorphic in the sense of **C**[0, 1]-**SET**, fuzzy relations between *X* and *Y* are subobjects of the product $\Sigma(X,E_c) \times \Sigma(Y,E_c)$, and form consequently a pair of **C**[0, 1]-**SET**-morphisms with common domain

$$(R,F) \xrightarrow{\psi} \Sigma(X,E_c), \qquad (R,F) \xrightarrow{\xi} \Sigma(Y,E_c)$$

s.t. $(R,F) \overset{\langle\psi,\xi\rangle}{\rightarrowtail} \Sigma(X,E_c) \times \Sigma(Y,E_c)$ is a **C**[0, 1]-monomorphism. Hence *fuzzy relations* between *X* and *Y* and *relations* between the singleton spaces $\Sigma(X,E_c)$ and $\Sigma(Y,E_c)$ *in the sense* of **C**[0, 1]-**SET** (cf. [58, Definition 5.10, p. 93]) are equivalent concepts. Further, we recall that the *composition* of relations

$$(R,F) \overset{\langle\varphi,\psi\rangle}{\rightarrowtail} \Sigma(X,E_c) \times \Sigma(Y,E_c), \qquad (S,G) \overset{\langle\zeta,\vartheta\rangle}{\rightarrowtail} \Sigma(Y,E_c) \times \Sigma(Z,E_c)$$

is defined as follows (cf. [58, p. 83, 93]): first, we form the following pullback square

$$\begin{array}{ccc} (T,H) & \overset{\delta}{\dashrightarrow} & (S,G) \\ {\scriptstyle \gamma}\,\vdots & & \downarrow{\scriptstyle \zeta} \\ (R,F) & \xrightarrow[\psi]{} & \Sigma(Y,E_c) \end{array}$$

[2] In Zadeh's definition of fuzzy relation the ordinary sets *X* and *Y* coincide (cf. [92, p. 345]).

and subsequently we apply the (*epimono*)-factorization property to $\langle \varphi \cdot \gamma, \vartheta \cdot \delta \rangle$ (cf. Remark 5.5 in FSBC):

$$\begin{array}{ccc} (T, H) & \xrightarrow{\langle \varphi\cdot\gamma, \vartheta\cdot\delta \rangle} & \Sigma(X, E_c) \times \Sigma(Z, E_c) \\ & \searrow \qquad \nearrow {\scriptstyle \langle \zeta, \vartheta \rangle \circ \langle \varphi, \psi \rangle} & \\ & (C, I) & \end{array}$$

The monomorphism $\langle \zeta, \vartheta \rangle \circ \langle \varphi, \psi \rangle$ is called the *composition* of $\langle \varphi, \psi \rangle$ and $\langle \zeta, \vartheta \rangle$.

After these preparations we consider a fuzzy relation between X and Y with the membership function $X \times Y \xrightarrow{r_1} [0,1]$ and a fuzzy relation between Y and Z with the membership function $Y \times Z \xrightarrow{r_2} [0, 1]$. Then the support sets R, S, T are given as follows:

$$R = \{(\alpha \cdot 1_x, \alpha \cdot 1_y) | \alpha \leqslant r_1(x, y)\}, \quad S = \{(\alpha \cdot 1_y, \alpha \cdot 1_z) | \alpha \leqslant r_2(y, z)\},$$
$$T = \{((\alpha \cdot 1_x, \alpha \cdot 1_y), (\alpha \cdot 1_y, \alpha \cdot 1_z)) | \alpha \leqslant \min(r_1(x, y), r_2(y, z))\}.$$

Now we make use of (5.12) in FSBC and obtain

$$C = \left\{ (\alpha \cdot 1_x, \alpha \cdot 1_z) | \alpha \leqslant \sup_{y \in Y} \min(r_1(x, y), r_2(y, z)) \right\}.$$

Then we conclude from Theorem 6.3 in FSBC and (6.8) in FSBC that the membership function $X \times Z \xrightarrow{r_3} [0, 1]$ of the composition of both relations has the following form:

$$r_3(x, z) = \sup_{y \in Y} \min(r_1(x, y), r_2(y, z)), \quad x \in X, \ z \in Z.$$

Hence Zadeh's "max–min-composition" of membership functions of fuzzy relations is equivalent to the composition of relations in the sense of **C**[0, 1]-**SET** (cf. [92, p. 346; 18, p. 71]).

Finally, if we take the equivalent view of espaces étalés, then it is not difficult to see that the composition of relations in $\mathcal{E}([0, 1[)$ is constructed fibrewise as ordinary composition of relations in **SET** (see also [3, p. 304]). The details of this situation are left to the reader.

We finish this section with some general comments on the mathematical tools for fuzzy set theory:

Comment: Section 1.1.1 explains three different, but equivalent ways in which fuzzy sets can be understood as *special* complete [0, 1]-valued sets, as *special* sheaves on [0, 1] (the so-called sheaves of level cuts) or as *special* espaces étalés whose fibres are given by strict level cuts. But Sections 1.1.2–1.1.4 show that various problems arising quite naturally in fuzzy set theory (e.g. quotients w.r.t. similarity relations, antisymmetry axiom for fuzzy partial orderings, fuzzy control maps and their defuzzification) can only be treated properly when the range of fuzzy sets is enlarged to the whole field of complete [0, 1]-valued sets, resp. sheaves on [0, 1] or espaces étalés with base space [0, 1[. This approach is confirmed by the compositionality of membership functions described in terms of similarity (cf. Section 1.2.1). Further, intersections, unions, images and inverse images of fuzzy sets and the max–min-compositions of fuzzy relations are special cases of general categorical constructions.

Finally, we emphasize that all fundamental results of this section remain valid, if we replace the real unit interval [0, 1] by arbitrary frames Ω.

2. Mathematical structures in sheaves

In this section we incorporate two basic mathematical structures from fuzzy set theory, namely fuzzy groups and fuzzy topological spaces, into the larger context given by the category of sheaves on Ω. Since sheaves on Ω, resp. complete Ω-valued sets form a topos, we first recall some important facts from topos theory.

2.1. Power object monad in a topos

Let $\mathcal{C}$ be a topos (cf. [28]); this means that $\mathcal{C}$ is a finitely complete and cocomplete category having exponentiation and a subobject classifier—i.e. there exists an object $\mathbb{S}$ and an arrow $\mathbb{1} \xrightarrow{t} \mathbb{S}$ s.t. for every monomorphism $U \overset{\varphi}{\rightarrowtail} X$ there exists a *unique* arrow $X \xrightarrow{\chi} \mathbb{S}$ completing the following diagram to a pullback square:

$$\begin{array}{ccc} U & \xrightarrow{\;!\;} & \mathbb{1} \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle t} \\ X & \xrightarrow[\chi]{} & \mathbb{S} \end{array}$$

It is well known that every topos has the (*epi*, *mono*)-factorization property (cf. [28, p. 114]). In particular, a subobject classifier is unique up to an isomorphism (cf. [28, p. 81]).

At the end of Section 3 and in Section 6 in FSBC we have seen that the category **CΩ-SET** of complete Ω-valued sets satisfies the previous, categorical axioms and forms consequently a topos. In particular, Theorem 6.3 in FSBC implies that the complete Ω-valued set (R_Ω, E_Ω) is the subobject classifier of **CΩ-SET**.

In the following considerations we show that every topos permits the construction of a *power object monad*. We begin with a definition: let A be an object of $\mathcal{C}$; a pair $(S(\varepsilon), \Pi(A))$ is a called the *power object* of A (cf. [53,28]) iff $\Pi(A)$ is an object of $\mathcal{C}$ and $S(\varepsilon)$ is a subobject of $\Pi(A) \times A$ determined by the $\mathcal{C}$-monomorphism $\varepsilon_A \overset{\varepsilon_A}{\rightarrowtail} \Pi(A) \times A$ such that for any $\mathcal{C}$-monomorphism $R \overset{r}{\rightarrowtail} B \times A$ there exists a unique arrow $B \xrightarrow{\varphi_r} \Pi(A)$ making the following diagram into a pullback square:

$$\begin{array}{ccc} R & \overset{r}{\rightarrowtail} & B \times A \\ \big\downarrow & & \big\downarrow{\scriptstyle \varphi_r \times id_A} \\ \varepsilon_A & \underset{\varepsilon_A}{\rightarrowtail} & \Pi(A) \times A \end{array} \tag{2.1}$$

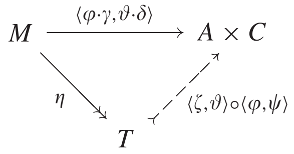
Obviously power objects are unique up to an isomorphism. Since $\mathcal{C}$ has exponentiation, the power object of an object A can be determined as follows (cf. [28, Theorem 1, p. 104]):

- $\Pi(A)$ is given by the internal hom-object $[A, \mathbb{S}]$.
- The monomorphism ε_A is the pullback of the arrow true t along the evaluation arrow $[A, \mathbb{S}] \times A \xrightarrow{ev} \mathbb{S}$.

A *relation* between $\mathcal{C}$-objects A and B is a pair of arrows $R \xrightarrow{\varphi} A$ and $R \xrightarrow{\psi} B$ such that the universal arrow $R \overset{\langle\varphi,\psi\rangle}{\rightarrowtail} A \times B$ is monic (see also Section 1.2.3). Because of (2.1) every relation $\langle\varphi, \psi\rangle$ can be identified with an arrow $A \xrightarrow{\theta} \Pi(B)$ —i.e. the *exponential adjoint* of its *characteristic morphism* $\chi_{\langle\varphi,\psi\rangle}$. The *composition of* relations $D \xrightarrow{\langle\varphi,\psi\rangle} A \times B$ and $K \xrightarrow{\langle\zeta,\vartheta\rangle} B \times C$ can be defined as follows (cf. [58, pp. 86, 93]; see also Section 1.2.3): first, we form the pullback (ψ, ζ, B) and obtain

$$\begin{array}{ccccccc} & & & M & & & \\ & & {\scriptstyle\gamma}\swarrow & & \searrow{\scriptstyle\delta} & & \\ & D & & & & K & \\ {\scriptstyle\varphi}\swarrow & & \searrow{\scriptstyle\psi} & & {\scriptstyle\zeta}\swarrow & & \searrow{\scriptstyle\vartheta} \\ A & & & B & & & C \end{array}$$

Then we apply the epi–mono–factorization property of $\mathcal{C}$ to $\langle \varphi \cdot \gamma, \vartheta \cdot \delta \rangle$:

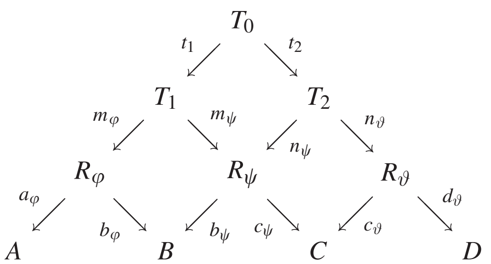


The monomorphism $\langle \zeta, \vartheta \rangle \circ \langle \varphi, \psi \rangle$ is called *the composition of* $\langle \varphi, \psi \rangle$ *and* $\langle \zeta, \vartheta \rangle$.

After these preparations we can introduce the *power object monad* in $\mathcal{C}$ (in clone form) as follows:

- $A \rightsquigarrow \Pi(A)$ is the object function.
- The "insertion-of-the-variables" map η_A coincides with the exponential adjoint of the characteristic morphism of the diagonal $A \xrightarrow{\langle id_A, id_A \rangle} A \times A$.
- The "clone-composition" function $\circledcirc$ is determined by the composition $\circ$ of relations—i.e. if $ev_{(R_\Omega, E_\Omega)} \cdot (\Theta_\varphi \times id_B)$ is the characteristic morphism of $R \xrightarrow{\varphi} A \times B$ and $ev_{(R_\Omega, E_\Omega)} \cdot (\Theta_\psi \times id_C)$ is the characteristic morphism of $T \xrightarrow{\psi} B \times C$, then $\Theta_\psi \circledcirc \Theta_\varphi$ is the exponential adjoint of the characteristic morphism of $\psi \circ \varphi$.

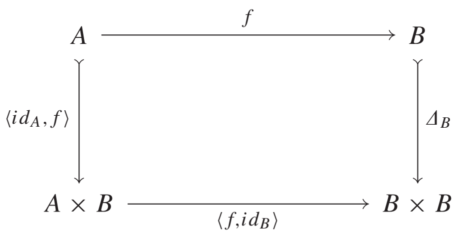
In fact, the *associativity* of the clone-composition function follows from the epimono–factorization property, the *Pullback Theorem* (cf. [58, Theorem 1.14, p. 20; 28, Fact 1, p. 115]) and the following diagram:



where all squares are pullbacks. Since the diagonal Δ_B of B acts as unity w.r.t. the composition $\circ$ of relations, $\eta_B \circledcirc \varphi = \varphi$ follows immediately. Further, let us consider the following $\mathcal{C}$-morphisms:

$$A \xrightarrow{f} B, \qquad B \xrightarrow{\varphi} \Pi(C) \ .$$

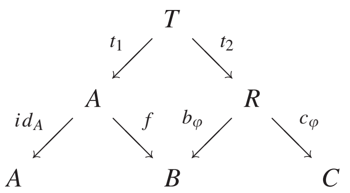
It is easily seen that the diagram



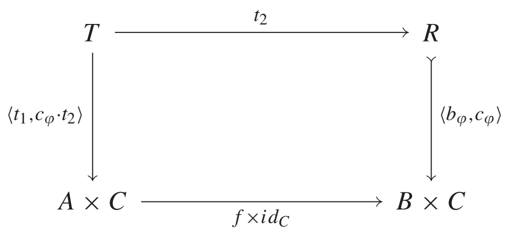
is a pullback square. Because of the *Pullback Lemma* the characteristic morphism χ_1 of $\langle id_A, f \rangle$ is given by

$$\chi_1 = \chi_{\Delta_B} \cdot (f \times id_B), \tag{2.2}$$

where χ_{Δ_B} denotes the characteristic morphism of Δ_B. Hence the exponential adjoint $\ulcorner \chi_1 \urcorner$ coincides with $\eta_B \cdot f$; this means that the relation $\langle id_A, f \rangle$ corresponds to $\eta_B \cdot f$. Further, let $\langle b_\varphi, c_\varphi \rangle$ be the relation corresponding to φ. Then the composition of $\langle id_A, f\rangle$ with $\langle b_\varphi, c_\varphi\rangle$ leads to the following situation:



where the square is a pullback. Hence it is not difficult to show that the subsequent diagram



is also a pullback square. Because of the *Pullback Lemma* the characteristic morphism χ_2 of $\langle t_1, c_\varphi \cdot t_2\rangle$ coincides with

$$ev \cdot (\varphi \times id_C) \cdot (f \times id_C)\,. \tag{2.3}$$

Since by definition $\varphi \circledcirc (\eta_B \cdot f)$ is the exponential adjoint of χ_2, we obtain from (2.3)

$$\varphi \circledcirc (\eta_B \cdot f) = \varphi \cdot f.$$

To sum up we have proved the following important theorem:

Theorem 2.1.1. *The triple* $(\Pi, \eta, \circledcirc)$ *is a monad* (*in clone form* (*cf.* [69])) *in* $\mathcal{C}$*—the so-called* power object monad.

Since $(\Pi, \eta, \circledcirc)$ is a monad, it is well known that the object function Π can be completed to an endofunctor of $\mathcal{C}$—the so-called *power object functor* which is also denoted by Π. In particular, for every $\mathcal{C}$-morphism $A \xrightarrow{\varphi} B$ the action of Π on φ is determined by:

$$\Pi(\varphi) = (\eta_B \cdot \varphi) \circledcirc id_{\Pi(A)}. \tag{2.4}$$

2.2. Sheaves of groups and fuzzy subgroups

Group objects and group homomorphisms in finitely complete categories are well known concepts (cf. [66, pp. 75–76; 83, Chapter 11]). Here we prepare the construction of group objects in the category of complete Ω-valued sets (resp. sheaves of groups on Ω) by starting first from monoids in the sense of the category Ω-**SET** of (not necessarily complete) Ω-valued sets. Since Ω-**SET** is a monoidal category (cf. Section 3 in FSBC), we introduce monoids in Ω-**SET** as follows (cf. [66,69]):

Let $\otimes$ be the tensor product, and I be the unit object specified in Section 3 in FSBC. Further, let (A, E) be an Ω-valued set and

$$(A, E) \otimes (A, E) \xrightarrow{m} (A, E), \qquad I \xrightarrow{e} (A, E)$$

be Ω-**SET**-morphisms. Then the triple $((A,E),m,e)$ is called a *monoid* in Ω-**SET** iff the following diagrams are commutative:

$$\text{(I)}\quad \begin{array}{ccc} ((A,E)\otimes(A,E))\otimes(A,E) & \xrightarrow{\;a\;} & (A,E)\otimes((A,E)\otimes(A,E)) \\ \Big\downarrow{\scriptstyle m\otimes id_{(A,E)}} & & \Big\downarrow{\scriptstyle id_{(A,E)}\otimes m} \\ & & (A,E)\otimes(A,E) \\ & & \Big\downarrow{\scriptstyle m} \\ (A,E)\otimes(A,E) & \xrightarrow[\;m\;]{} & (A,E) \end{array}$$

(Associativity)

$$\text{(II)}\quad \begin{array}{ccccc} I\otimes(A,E) & \xrightarrow{e\otimes id_{(A,E)}} & (A,E)\otimes(A,E) & \xleftarrow{id_{(A,E)}\otimes e} & (A,E)\otimes I \\ & \searrow{\scriptstyle \ell} & \Big\downarrow{\scriptstyle m} & \swarrow{\scriptstyle r} & \\ & & (A,E) & & \end{array}$$

(Existence of Unity)

where a, ℓ and r are components of natural isomorphisms determined by

$$a((x,y),z)=(x,(y,z)),\quad \ell(\cdot,x)=x,\quad r(x,\cdot)=x,\quad x,y,z\in A.$$

In this context m is called the *multiplication* and e the *unit morphism*. A monoid $((A,E),m,e)$ in Ω-SET is said to be *commutative* iff m fulfills the additional property expressed by the commutativity of the diagram

$$\text{(III)}\quad \begin{array}{ccc} (A,E)\otimes(A,E) & \xrightarrow{\;c\;} & (A,E)\otimes(A,E) \\ & \searrow{\scriptstyle m} \quad \swarrow{\scriptstyle m} & \\ & (A,E) & \end{array}$$

where c is the concerning component of the symmetry of $(\Omega\text{-}\mathbf{SET},\otimes,I)$ determined by $c(x,y)=(y,x)$.

An *external* description of monoids in Ω-**SET** can be given as follows: (A,E) is an Ω-valued set, e is an element of the support set A, and $A\times A\xrightarrow{\;m\;}A$ is an ordinary map s.t. the subsequent conditions are valid

(G0) $E(m(a,b),m(a,b))=E(a,a)\wedge E(b,b)$ (Strictness).
(G1) $E(a_1,a_2)\wedge E(b_1,b_2)\leqslant E(m(a_1,b_1),m(a_2,b_2))$ (Congruence).
(G2) (A,m,e) is a monoid in the sense of **SET**.
(G3) $E(e,e)=\top$ (Global Existence of Unity).

Finally, a commutative monoid $((A,E),m,e)$ in Ω-**SET** satisfies the *cancellation law* iff the property

(C) $E(m(a_1,b),m(a_2,b))\leqslant E(a_1,a_2),\quad a_1,a_2,b\in A$

holds.

We begin with an example playing a significant role in the theory of Ω-probabilistic metric spaces (cf. Remark 2.3.6).

Example 2.2.1 (*Ω-valued non-negative real numbers*). Let $\mathbb{R}^+$ be the set of all non-negative, real numbers. A map $\mathbb{R}^+\xrightarrow{\;F\;}\Omega$ is called an *Ω-valued, non-negative, real number* iff F satisfies the following properties

$$F(0)=\bot,\quad F(r)=\bigvee_{r'<r}F(r'),\quad 0<r\in\mathbb{R}^+. \tag{2.5}$$

The *extent of existence* of an Ω-valued, non-negative, real number F is defined by

$$\mathbb{E}(F) = \bigvee_{n \in \mathbb{N}} F(n).$$

Then $\Delta^+(\Omega)$ denotes the set of all Ω-valued, non-negative, real numbers, while $\mathcal{D}^+(\Omega)$ comprises only all global, Ω-valued, non-negative, real numbers—i.e. $\mathcal{D}^+(\Omega) = \{F \in \Delta^+(\Omega) \mid \mathbb{E}(F) = \top\}$.

In the case of spatial frames, it can be shown that global, Ω-valued, non-negative, real numbers and upper semicontinuous maps from $pt(\Omega)$ to $\mathbb{R}^+$ are equivalent concepts. This observation justifies the above chosen terminology.

Further, we introduce a sheaf $\mathcal{F}$ of "non-negative, real numbers" on Ω as follows:

$$\mathcal{F}(\alpha) = \{F \in \Delta^+(\Omega) \mid \mathbb{E}(F) = \alpha\}, \quad \varrho_\beta^\alpha(F) = \beta \wedge F, \ \beta \leqslant \alpha.$$

Then the complete Ω-valued set corresponding to $\mathcal{F}$ (cf. Section 4 in FSBC) has the form $(\Delta^+(\Omega), E_{\Delta^+})$ where E_{Δ^+} is given by

$$E_{\Delta^+}(F, G) = \bigvee \{\alpha \in \Omega | \alpha \leqslant \mathbb{E}(F) \wedge \mathbb{E}(G), \forall\, r \in \mathbb{R}^+ : \alpha \wedge F(r) = \alpha \wedge G(r)\}.$$

After these preparations we introduce an algebraic structure on $(\Delta^+(\Omega), E_{\Delta^+})$ in the following way:

$$\tau_\wedge(F, G)(z) = \bigvee_{x+y=z} F(x) \wedge G(y), \quad z \in \mathbb{R}^+,$$

$$\varepsilon_0(x) = \left\{ \begin{array}{l} \top \ : 0 < x \\ \bot \ : 0 = x \end{array} \right\}, \quad x \in \mathbb{R}^+.$$

Then it is not difficult to show that $((\Delta^+(\Omega), E_{\Delta^+}), \tau_\wedge, \varepsilon_0)$ is a commutative monoid in Ω-**SET**. Finally, referring to the Booleanization of Ω (cf. Example 5.4 in FSBC) we deduce the validity of the *cancellation law* in $((\Delta^+(\Omega), E_{\Delta^+}), \tau_\wedge, \varepsilon_0)$ from the fact that $\mathcal{D}^+(\mathbb{B}(\Omega))$ is the positive cone of an order complete vector lattice (cf. [77]).

In the next remark we study monoids in Ω-**SET** from the view point of espaces étalés.

Remark 2.2.2 (*Spatial frames*). Let Ω be a spatial frame, and $((A, E), m, e)$ be a monoid in Ω-**SET**. Further, let $pt(P(A, E)) \xrightarrow{\pi} pt(\Omega)$ be the espace étalé associated with E. We maintain the notation from Section 2 in FSBC and conclude from the axioms (G0)–(G3) that on every fibre $\mathcal{A}_p$ over $p \in pt(\Omega)$ w.r.t. π there exists the structure of a monoid in the sense of **SET** defined by

$$[a]_p *_p [b]_p = [m(a, b)]_p, \quad [a]_p *_p [e]_p = [e]_p *_p [a]_p = [a]_p, \quad [a]_p, [b]_p \in \mathcal{A}_p.$$

Further, let $\mathcal{G}_\pi$ the sheaf of local sections of π (cf. Remark 4.3 in FSBC), and $*_\alpha$ be the pointwisely defined multiplication

$$(\sigma_1 *_\alpha \sigma_2)(p) = \sigma_1(p) *_p \sigma_2(p), \quad \sigma_1, \sigma_2 \in \mathcal{G}_\pi(\alpha), \ p \in \mathbb{A}_\alpha.$$

Because of

$$(\sigma_1 *_\alpha \sigma_2)^{-1}(\mathbb{A}_{\tilde{c}}) = \bigcup_{a,b \in A} (\sigma_1^{-1}(\mathbb{A}_{\tilde{a}}) \cap \sigma_2^{-1}(\mathbb{A}_{\tilde{b}}) \cap \mathbb{A}_{E(m(a,b),c)}). \tag{2.6}$$

$\sigma_1 *_\alpha \sigma_2$ is again a local section of π; this means that $\mathcal{G}_\pi(\alpha)$ is closed w.r.t. $*_\alpha$. Hence $(\mathcal{G}_\pi(\alpha), *_\alpha)$ is also an ordinary monoid, and $\mathcal{G}_\pi$ is a sheaf of monoids on Ω.

Now we require that every fibre $\mathcal{A}_p$ is a *group* w.r.t. $*_p$ in the sense of **SET**. Hence there exists a map $\mathcal{A}_p \xrightarrow{\ell_p} \mathcal{A}_p$ s.t. $[a]_p *_p \ell_p([a]_p) = [e]_p$. Further, we require that all ℓ_p are fibre maps of a bundle endomorphism $\pi \xrightarrow{\ell} \pi$. Then the monoid $(\mathcal{G}_\pi(\alpha), *_\alpha)$ is a group (in the sense of **SET**), and the inverse of a local section is given as follows:

$$\sigma^{-1} = \ell \circ \sigma, \quad \sigma \in \mathcal{G}_\pi(\alpha).$$

In particular, $\mathcal{G}_\pi$ is now a *sheaf of groups*. Hence we view the previous requirements as those axioms making a monoid into a *group-like object* in the sense of Ω-**SET**. Therefore we finish this remark with the translation of these requirements to axioms which only use those data given by the monoid $((A, E), m, e)$. First, the bundle endomorphism $\pi \xrightarrow{\ell} \pi$ can be identified with a "fuzzy morphism"—i.e. a map $A \times A \xrightarrow{R} \Omega$ provided with the axioms (F1)–(F3) (cf. Section 2 in FSBC) s.t. for all $p \in pt(\Omega)$ the relation $\varphi_p^R = \ell_p$ (cf. (2.13) in FSBC) holds. In this context the requirement $[a]_p *_p \ell_p([a]_p) = [e]_p$ is equivalent to

$$R(a, b) = E(e, m(a, b)), \quad a, b \in A. \tag{2.7}$$

Because of (2.7) the congruence axiom (G1) implies immediately the extensionality of R—i.e. (F1). Further, under the assumption of (2.7) the conditions (F2) and (F3) are equivalent to the following axioms:

(G4) $E(e, m(a, b)) \wedge E(e, m(a, c)) \leqslant E(b, c)$.
(G5) $E(a, a) = \bigvee_{b \in A} E(e, m(a, b))$.

Finally, it can be shown that the axioms (G0)—(G5) imply the symmetry property

$$E(e, m(a, b)) = E(e, m(b, a)), \quad a, b \in A \tag{2.8}$$

which reflects the standard fact from group theory that right inverse elements are also left inverse.

Because of the previous remark we introduce the following terminology: a monoid $((A, E), m, e)$ in Ω-**SET** is said to be a *group-like object* iff $((A, E), m, e)$ satisfies the additional axioms (G4) and (G5).

In general, group-like objects in Ω-**SET** are not group objects in Ω-**SET** as the next examples demonstrates.

Example 2.2.3 (*Crisp equality*). Let Ω be a frame with $\Omega \neq \{\bot, \top\}$. Further, let (X, m, e) be an ordinary group (i.e. a group object in **SET**) and E_c be the crisp equality on X. Then $((X, E_c), m, e)$ is a group-like object in Ω-**SET**, but not a group object in Ω-**SET**, because there does *not* exist any Ω-**SET**-morphism from the terminal object $\mathbb{1}$ to (X, E_c).

The next remark shows that commutative monoids in Ω-**SET** satisfying the cancellation law form a natural source for group-like objects in Ω-**SET**.

Remark 2.2.4 (*Cancellation law*). Let $((A, E), m, e)$ be a commutative monoid in Ω-**SET** provided with the cancellation law. Then we define a map $(A \times A) \times (A \times A) \xrightarrow{r} \Omega$ by

$$r((a_1, a_2), (b_1, b_2)) = E(m(a_1, b_2), m(b_1, a_2)), \quad a_1, a_2, b_1, b_2 \in A.$$

Obviously r is symmetric. The transitivity of r follows from the cancellation law and the commutativity of m. Hence $(A \times A, r)$ is an Ω-valued set. Further, we introduce an algebraic structure on $(A \times A, r)$ as follows:

$$\widehat{e} = (e, e), \quad \widehat{m}((a_1, a_2), (b_1, b_2)) = (m(a_1, b_1), m(a_2, b_2)).$$

It is a matter of routine to show that $((A \times A, r), \widehat{m}, \widehat{e})$ is a commutative monoid in Ω-**SET**. Using again the cancellation law we obtain

$$\begin{aligned} r(\widehat{e}, \widehat{m}((a_1, a_2), (b_1, b_2))) \wedge r(\widehat{e}, \widehat{m}((a_1, a_2), (\bar{b}_1, \bar{b}_2))) \\ = E(m(a_1, b_1), m(a_2, b_2)) \wedge E(b_2, b_2) \wedge E(\bar{b}_2, \bar{b}_2) \wedge E(m(a_1, \bar{b}_1), m(a_2, \bar{b}_2)) \\ \leqslant E(m(a_1, m(b_1, \bar{b}_2)), m(a_2, m(b_2, \bar{b}_2))) \wedge E(m(a_2, m(\bar{b}_2, b_2)), m(a_1, m(\bar{b}_1, b_2))) \\ \leqslant E(m(a_1, m(b_1, \bar{b}_2)), m(a_1, m(\bar{b}_1, b_2))) \leqslant r((b_1, b_2), (\bar{b}_1, \bar{b}_2)). \end{aligned}$$

Hence (G4) is verified. Finally, (G5) follows from

$$\begin{aligned} r(\widehat{e}, \widehat{m}((a_1, a_2), (a_2, a_1))) &= E(m(a_1, a_2), m(a_2, a_1)) = E(m(a_1, a_2), m(a_1, a_2)) \\ &= r((a_1, a_2), (a_1, a_2)). \end{aligned}$$

To sum up we have shown that $((A \times A, r), \widehat{m}, \widehat{e})$ is a group-like object in Ω-**SET**.

An application of this result to the situation of Example 2.2.1 leads to a possible concept of "Ω-valued real numbers" which in general seems to be unrelated to the Dedekind real number object in the topos **C**Ω-**SET** of *complete* Ω-valued sets (cf. [28,67]).

The next theorem explains a natural method of constructing group objects in the category **CΩ-SET**.

Theorem 2.2.5. *Let* $(A, E), m, e)$ *be a group-like object in* Ω-**SET**. *Then on the singleton space* $\Sigma(A, E)$ *there exists a group structure* $(\widetilde{m}, \widetilde{e}, \widetilde{\ell})$ *in the sense of* **CΩ-SET** *determined by the following operations*:

$$[\widetilde{m}(s_1, s_2)](c) = \bigvee_{a,b\in A} s_1(a) \wedge s_2(b) \wedge E(m(a, b), c), \quad s_1, s_2 \in S(A, E),$$
$$\widetilde{e}(x) = E(e, x), \quad x \in A,$$
$$[\widetilde{\ell}(s)](b) = \bigvee_{a\in A} s(a) \wedge E(e, m(a, b)), \quad s \in S(A, E).$$

Proof. Since $\Sigma(X, E)$ is the free $\mathbf{T}_\Sigma$-algebra generated by (A, E), we conclude from Lemma 3.5 in FSBC that the verification of the assertion is a matter of routine. □

As an immediate corollary of Example 2.2.3 and Theorem 2.2.5 we obtain that every group G in the sense of **SET** generates a group structure on the singleton space $\Sigma(G, E_c)$ in the sense of **CΩ-SET**. Further, every increasing sequence of seminorms on a given vector space viewed as a presheaf of vector spaces on the lower topology $\omega(\mathbb{N})$ of the natural numbers (cf. [43, Example 5.1.1]) generates a group object in **C**$\omega(\mathbb{N})$-**SET**.

The next remark shows that group objects in **CΩ-SET** and group-like objects in Ω-**SET** with underlying *complete* Ω-valued sets are equivalent concepts.

Remark 2.2.6. On every complete Ω-valued set (A, E) there exists a right exterior operation $A \times \Omega \xrightarrow{\upharpoonright} A$ determined by $\lambda \wedge E(a, b) = E(a\upharpoonright\lambda, b)$. Further, let $((A, E), m, e, \ell)$ be a group object in **CΩ-SET**. In particular,

$$(A, E) \times (A, E) \xrightarrow{m} (A, E), \qquad \mathbb{1} \xrightarrow{e} (A, E), \qquad (A, E) \xrightarrow{\ell} (A, E)$$

are **CΩ-SET**-morphisms where m is the multiplication, e is the unit morphism, and ℓ is the inversion in (A, E). Then m can be extended to

$$(A, E) \otimes (A, E) \xrightarrow{m^*} (A, E)$$

by

$$m^*(a, b) = m(a\upharpoonright E(b, b), b\upharpoonright E(a, a)), \quad a, b \in A. \tag{2.9}$$

It is not difficult to show that $((A, E), m^*, e(\top))$ is a monoid in Ω-**SET**. In order to prove that $((A, E), m^*, e(\top))$ is even a group-like object in Ω-**SET** it is sufficient to verify the following relation:

$$E(e(\top), m^*(a, b)) = E(\ell(a), b), \quad a, b \in A. \tag{2.10}$$

Because of $m(a, \ell(a)) = m(\ell(a), a) = e(E(a, a))$ and $e(\top)\upharpoonright\alpha = e(\alpha)$ we obtain

$$E(e(\top), m(a, \ell(a))) = E(e(\top), m(\ell(a), a)) = E(a, a). \tag{2.11}$$

Obviously (2.11) and (G1) imply

$$E(a, a) \wedge E(b, b) \leqslant E(b, m^*(m(\ell(a), a), b)). \tag{2.12}$$

Finally, we use (G0), (G1), (2.11) and (2.12) in the following estimation:

$$\begin{aligned} E(e(\top), m^*(a, b)) &= E(e(\top), m^*(a, b)) \wedge E(a, a) \wedge E(b, b) \\ &\leqslant E(\ell(a), m^*(m(\ell(a), a), b)) \wedge E(a, a) \wedge E(b, b) \\ &\leqslant E(\ell(a), b) = E(\ell(a), b) \wedge E(a, a) \\ &\leqslant E(m(a, \ell(a)), m^*(a, b)) \wedge E(a, a) \leqslant E(e(\top), m^*(a, b)). \end{aligned}$$

On the other hand, let $((A, E), m^*, e^*)$ be a group-like object in Ω-**SET** s.t. the underlying Ω-valued set is *complete*. Because of (G1), (G4) and (G5) there exists an **C**Ω-**SET**-endomorphism $(A, E) \xrightarrow{\ell} (A, E)$ satisfying the following relation:

$$E(e^*, m^*(a, b)) = E(\ell(a), b), \quad a, b \in A. \tag{2.13}$$

Further, e^* induces a morphism $\mathbb{1} \xrightarrow{e} (A, E)$ by $e(\alpha) = e^* \lceil \alpha,\ \alpha \in \Omega$. Now, let m be the restriction of m^* to the categorical product $(A, E) \times (A, E)$. Since (A, E) is separated, we infer from (2.13): $e(E(a, a)) = m(a, \ell(a))$. Hence it is not difficult to show that $((A, E), m, e, \ell)$ is indeed a group object in **C**Ω-**SET**.

Finally, we use again the fact that (A, E) is separated and obtain from (G0) and (G1):

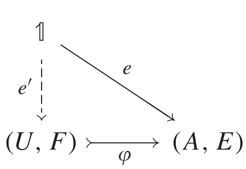
$$m^*(a, b) = m^*(a \lceil E(b, b), b \lceil E(a, a)), \quad a, b \in A.$$

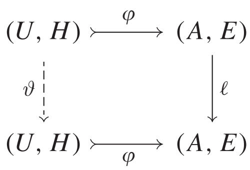
This means that the previous correspondence between group objects and group-like objects with underlying complete Ω-valued sets is indeed bijective.

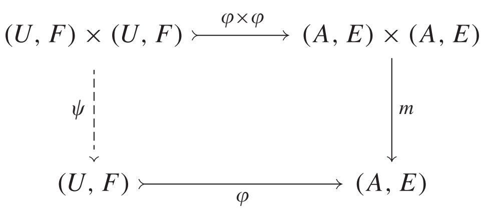
In the following considerations we characterize subgroups of group objects in **C**Ω-**SET** and explain the quotient construction w.r.t. subgroups in **C**Ω-**SET**. We begin with a definition: let $((A, E), m, e, \ell)$ be a group object in **C**Ω-**SET**, and $(U, F) \overset{\varphi}{\rightarrowtail} (A, E)$ be a subobject of (A, E). $((U, H), \varphi)$ is called a *subgroup object* of $((A, E), m, e, \ell)$ iff the following conditions are satisfied:

(SG0) e factors though φ.
(SG1) $m \cdot (\varphi \times \varphi)$ factors through φ.
(SG2) ℓ factors through φ.

Obviously the conditions (SG0)–(SG2) mean the commutativity of the following diagrams:







In particular, every subgroup object $((U, F), \varphi)$ is again a group object and the **C**Ω-**SET**-monomorphism φ turns into a group homomorphism.

A characterization of subgroup objects by their characteristic morphisms is given in the next theorem.

Theorem 2.2.7. *Let* $((A, E), m, e, \ell)$ *be a group object in* **C**Ω-**SET**. *Further, let* $(U, H) \overset{\varphi}{\rightarrowtail} (A, E)$ *be a subobject of* (A, E), *and let* $A \xrightarrow{\mu_\varphi} \Omega$ *be the corresponding E-strict and E-extensional membership map. Then the following assertions are equivalent*:

(i) $((U, H), \varphi)$ *is a subgroup object of* $((A, E), m, e, \ell)$.
(ii) μ_φ *satisfies the subsequent conditions for all* $a, b \in A$

$$\mu_\varphi(e(\top)) = \top, \quad \mu_\varphi(a) \leqslant \mu_\varphi(\ell(a)), \quad \mu_\varphi(a) \wedge \mu_\varphi(b) \leqslant \mu_\varphi(m(a, b)).$$

Proof. First, we make use of the formulas (6.2) and (6.8) in FSBC and obtain

$$\mu_\varphi(a) = \bigvee_{u\in U} E(a, \varphi(u)), \quad a \in A. \tag{2.14}$$

$$\varphi(U) = \{a \in A | \mu_\varphi(a) = E(a, a)\}. \tag{2.15}$$

(a) (i) $\Rightarrow$ (ii): The property $\mu_\varphi(e(\top)) = \top$ follows immediately from (G3), (SG0) and (2.14). Because of (SG2) the morphism φ factors through ℓ. Hence there exists a map $U \xrightarrow{\vartheta} U$ with $\varphi \cdot \vartheta = \ell \cdot \varphi$, and the relation

$$\begin{aligned}\mu_\varphi(\ell(a)) &\geqslant \bigvee_{u\in U} E(\ell(a), \varphi\cdot\vartheta(u)) = \bigvee_{u\in U} E(\ell(a), \ell(\varphi(u)))\\ &\geqslant \bigvee_{u\in U} E(a, \varphi(u)) = \mu_\varphi(a)\end{aligned}$$

follows from (2.14). Finally, let $D = U\boxtimes U$ be the support set of the product $(U, F) \times (U, F)$. Since $m \cdot (\varphi\times\varphi)$ factors through φ (cf. (SG1)), there exists a map $D \xrightarrow{\psi} U$ with $\varphi\cdot\psi = m\cdot(\varphi\times\varphi)$. Then we obtain from (2.14) and (G1):

$$\begin{aligned}\mu_\varphi(m(a,b)) &\geqslant \bigvee_{(u,v)\in D} E(m(a,b), \varphi(\psi(u,v)))\\ &= \bigvee_{(u,v)\in D} E(m(a,b), m(\varphi(u),\varphi(v)))\\ &\geqslant \bigvee_{(u,v)\in D} E(a,\varphi(u)) \wedge E(b,\varphi(v))\\ &= \bigvee_{(u,v)\in U\times U} E(a, \varphi(u)\rceil E(v,v)) \wedge E(b, \varphi(v)\rceil E(u,u))\\ &= \bigvee_{(u,v)\in U\times U} E(a,\varphi(u)) \wedge E(b,\varphi(v)) = \mu_\varphi(a)\wedge\mu_\varphi(b).\end{aligned}$$

(b) (ii) $\Rightarrow$ (i): Since φ is a **CΩ-SET**-monomorphism, the assertion (i) follows immediately from (ii) and relation (2.15). □

A subgroup object $(U, H) \overset{\varphi}{\rightarrowtail} (A, E)$ of $((A, E), m, e, \ell)$ is said to be *normal* iff the corresponding membership map μ_φ satisfies the condition

$$E(a,a) \wedge \mu_\varphi(x) \quad\leqslant\quad \mu_\varphi(m^*(m^*(a,x), \ell(a))), \quad a, x \in A, \tag{2.16}$$

where m^* is determined by (2.9).

The aim of the following considerations is to construct the *quotient* of a group object $((A, E), m, e, \ell)$ w.r.t. a normal subgroup $((U, H), \varphi)$—i.e. a group object $((B, F), n, d, \jmath)$ such that the following diagram:

$$(U, H) \overset{\varphi}{\rightarrowtail} (A, E) \overset{\pi}{\twoheadrightarrow} (B, F) \tag{2.17}$$

is *exact*; this means that φ is the *kernel* of π, and π is the *cokernel* of φ (see also [33, p. 105]). First, let μ_φ be the E-strict and E-extensional membership map corresponding to the subgroup object $((U, H), \varphi)$ (cf. Theorem 2.2.7). Then ℓ, μ_φ and m^* determined by (2.9) induce a map $A\times A \xrightarrow{r} \Omega$ as follows:

$$r(a,b) = \mu_\varphi(m^*(a, \ell(b))), \quad a, b \in A. \tag{2.18}$$

We show that r is an Ω-valued equality on A. Because of Assertion (ii) of Theorem 2.2.7 we obtain

$$r(a,b)=\mu_\varphi(m^*(a,\ell(b)))\leqslant\mu_\varphi(\ell(m^*(a,\ell(b))))=\mu_\varphi(m^*(b,\ell(a)))=r(b,a),$$

$$\begin{aligned}r(a,b)\wedge r(b,c)&\leqslant\mu_\varphi(m^*(m^*(a,\ell(b)),m^*(b,\ell(c))))\\&=\mu_\varphi(m^*(m^*(a,e(\top)\rceil E(b,b)),\ell(c)))\\&\leqslant\mu_\varphi(m^*(a,\ell(c))=r(a,c).\end{aligned}$$

Further, we infer from the property $\mu_\varphi(e(\top))=\top$ and the E-extensionality of μ_φ:

$$r(a,a)=E(a,a)=r(\ell(a),\ell(a)),\quad a\in A. \tag{2.19}$$

Now we use again the extensionality of μ_φ and obtain

$$\begin{aligned}E(a,b)&=E(a,b)\wedge E(\ell(b),\ell(b))\wedge r(b,b)\\&\leqslant E(m^*(a,\ell(b)),m^*(b,\ell(b)))\wedge\mu_\varphi(m^*(b,\ell(b)))\\&\leqslant\mu_\varphi(m^*(a,\ell(b)))=r(a,b).\end{aligned}$$

Hence we have verified the following relation:

$$E(a,a)=r(a,a),\quad E(a,b)\leqslant r(a,b),\quad a,b\in A. \tag{2.20}$$

After these preparations we show that $((A,r),m^*,e(\top))$ is a group-like object in Ω-**SET**. The strictness axiom (G0) follows immediately from (2.19). In order to prove (G1) we use the normality of $((U,F),\varphi)$ and again the assertion (ii) of Theorem 2.2.7:

$$\begin{aligned}r(a_1,a_2)\wedge r(b_1,b_2)&=r(a_1,a_2)\wedge r(b_1,b_2)\wedge E(\ell(a_1),\ell(a_1))\\&\leqslant\mu_\varphi(m^*(m^*(\ell(a_1),a_1),m^*(\ell(a_2),a_1)))\wedge r(b_1,b_2)\\&=\mu_\varphi(m^*(\ell(a_2),a_1))\wedge\mu_\varphi(m^*(b_1,\ell(b_2)))\\&\leqslant E(a_2,a_2)\wedge\mu_\varphi(m^*(m^*(\ell(a_2),m^*(a_1,b_1)),\ell(b_2)))\\&\leqslant\mu_\varphi(m^*(m^*(a_2,\ell(a_2)),m^*(m^*(a_1,b_1),m^*(\ell(b_2),\ell(a_2)))))\\&=\mu_\varphi(m^*(m^*(a_1,b_1),\ell(m^*(a_2,b_2))))\\&=r(m^*(a_1,b_1),m^*(a_2,b_2)).\end{aligned}$$

The axioms (G2) and (G3) are evident. Further, we conclude from $m(a,\ell(a))=m(\ell(a),a)=e(\top)\rceil E(a,a)$ and $\mu_\varphi(e(\top))=\top$:

$$r(e(\top),m(a,\ell(a)))=r(e(\top),m(\ell(a),a))=r(a,a).$$

Hence we can use the argumentation from Remark 2.2.6 and obtain:

$$r(e(\top),m^*(a,b))=r(\ell(a),b),\quad a,b\in A.$$

Obviously, the previous relation implies (G4), (G5) and moreover that ℓ is an Ω-**SET**-endomorphism of (A,r). Thus $((A,r),m,e(\top))$ is a group-like object in Ω-**SET**. Now we apply Theorem 2.2.5 and obtain that $(\Sigma(A,r),\widetilde{m},\widetilde{e(\top)},\widetilde{\ell})$ is a group object in **C**Ω-**SET** and $(A,E)\xrightarrow{\varkappa}\Sigma(A,r)$ defined by

$$[\pi(a)](b)\quad=\quad r(a,b),\quad a,b\in A$$

is the *quotient morphism* in the sense of **C**Ω-**SET** (cf. (2.20)).

If Ω is spatial, then the situation is as follows: let $pt(P(A,E))\xrightarrow{\pi}pt(\Omega)$ be the espace étalé associated with E (cf. Section 2 in FSBC). Since μ_φ is the membership map of the subobject $(U,F)\overset{\varphi}{\rightarrowtail}(A,E)$, the espace étalé associated with F can be identified with the open subset $\mathbb{A}_{\mu_\varphi}$ of $pt(P(A, E))$. This situation leads to the following commutative diagram

$$\begin{array}{ccc} \mathbb{A}_{\mu_\varphi} & \overset{\iota}{\hookrightarrow} & pt(P(A,E)) \\ & \searrow^{\pi_U} & \downarrow \pi \\ & & pt(\Omega) \end{array}$$

where ι denotes the inclusion map and $\mathbb{A}_{\mu_\varphi}$ is given by (cf. Section 2 in FSBC)

$$\mathbb{A}_{\mu_\varphi} = \{q \in pt(P(A, E)) | q(\mu_\varphi) = 1\}.$$

In particular, if we identify points q of $P(A, E)$ with equivalence class $[a]_p$ induced by E (cf. Lemma 2.3 in FSBC), then the fibre $\mathcal{U}_p$ over p w.r.t. π_U has the following form:

$$\mathcal{U}_p = \{[a]_p \in \mathcal{A}_p \mid p(\mu_\varphi(a)) = 1\}, \quad p \in pt(\Omega).$$

Because of Remarks 2.2.2, 2.2.6 and Property (2.16) all fibres $\mathcal{A}_p$ w.r.t. π are ordinary groups and all fibres $\mathcal{U}_p$ w.r.t. π_U are normal subgroups of the respective $\mathcal{A}_p$'s. Hence we can compute *fibrewise* the ordinary quotient group $\mathcal{A}_p/\mathcal{U}_p$. We show that $\mathcal{A}_p/\mathcal{U}_p$ is isomorphic to the fibre $\mathcal{B}_p$ over p w.r.t. the espace étalé $pt(P(A,r)) \xrightarrow{\pi_r} pt(\Omega)$ associated with r. For this purpose let A^p be the set of all $a \in A$ with $p(E(a, a)) = 1$. Because of (2.20) there exists an epimorphism $\mathcal{A}_p \overset{\varkappa_p}{\twoheadrightarrow} \mathcal{B}_p$ s.t. the following diagram is commutative:

$$\begin{array}{ccc} A^p & \xrightarrow{id_{A^p}} & A^p \\ \downarrow & & \downarrow \\ \mathcal{A}_p & \underset{\varkappa_p}{\dashrightarrow} & \mathcal{B}_p \end{array}$$

where the vertical arrows denote the respective quotient maps. Obviously $\varkappa_p$ is the fibre map corresponding to $\varkappa$ at p. Further, we conclude from (2.14) and (2.18) that the kernel of the epimorphism $\varkappa_p$ coincides with $\mathcal{U}_p$. Hence $\mathcal{B}_p$ and $\mathcal{A}_p/\mathcal{U}_p$ are isomorphic.

Since finite limits and colimits in $\mathcal{E}(pt(\Omega)$ are computed fibrewise (cf. Remark 5.5 in FSBC), the diagram

$$\pi_U \overset{\iota}{\hookrightarrow} \pi \overset{\varkappa}{\twoheadrightarrow} \pi_r$$

is exact. Without proof we only mention here that this result holds also in the case of non-spatial frames, if we replace $\mathcal{E}(pt(\Omega))$ by **C**Ω-**SET** and the espace étalé π_r by the singleton space $\Sigma(A, r)$.

We finish this section with some comments on the relationship between fuzzy groups and group objects in **C**[0, 1]-**SET**. First, we begin with a definition (cf. [81]): let $(G, \cdot, e)$ be an ordinary group (i.e. a group object in the sense of **SET**). A membership map $\mu : G \to [0, 1]$ is called a *fuzzy group* in G iff μ satisfies the following conditions [3] (cf. [60,61]):

(FSG1) $\min(\mu(a), \mu(b)) \leqslant \mu(a \cdot b)$.
(FSG2) $\mu(e) = 1, \ \mu(a) \leqslant \mu(a^{-1})$.

Since membership functions can be understood in three different, but equivalent ways (cf. Section 1.1.1), we have three different, but equivalent understandings of fuzzy groups.

[3] Here we have added the condition $\mu(e) = 1$ to Rosenfeld's original axioms of a fuzzy group (cf. [81]); otherwise we can also replace [0, 1] by the complete Heyting algebra $[0, \mu(e)]$.

First, we identify μ with the map $S(G, E_c) \xrightarrow{\widetilde{\mu}} [0,1]$ (cf. (1.5))

$$\widetilde{\mu}(\alpha \cdot 1_x) = \min(\alpha, \mu(x)), \quad \alpha \cdot 1_x \in S(G, E_c)$$

and consider the canonical group structure on $\Sigma(G, E_c) = (S(G, E_c), \widetilde{E_c}))$ generated by $(G, \cdot, e)$ in the sense of Theorem 2.2.5. Then the axioms of fuzzy groups are equivalent to those conditions specified in Assertion (ii) of Theorem 2.2.7. Hence the subobject of $\Sigma(G, E_c)$—i.e. the fuzzy set in G—corresponding to the membership function μ is a *subgroup object* of $(\Sigma(G, E_c), \widetilde{\cdot}, \widetilde{e}, \widetilde{\ell})$ in the sense of **C**[0, 1]-**SET**.

Second, we understand membership functions as sheaves of level cuts on [0, 1] (cf. Example 4.2 in FSBC)—i.e. as subsheaves of constant sheaves (cf. Section 1.1.1). In this context the axioms of fuzzy groups are equivalent to the fact that all level cuts are subgroups (of G) (see also [16]) ; this means that the sheaf of level cuts is a *sheaf of groups on* [0, 1].

Finally, we can consider the espace étalé $G \times [0,1[\xrightarrow{\pi_2} [0,1[$ associated with the crisp equality on G (cf. Example 2.5 in FSBC) and identify the membership function μ with the open subset V_μ determined by (cf. Section 1.1.1)

$$V_\mu = \{(x, \alpha) \in G \times [0,1[\mid \alpha < \mu(x)\}.$$

This situation leads to the following commutative diagram:

$$\begin{array}{ccc} V_\mu & \overset{\iota}{\hookrightarrow} & G \times [0,1[\\ & \searrow^{\pi_\mu} & \downarrow \pi_2 \\ & & [0,1[\end{array}$$

where ι denotes the inclusion map. Then the axioms of fuzzy groups are equivalent to the fact that all *fibres* $\mathcal{U}_t$ *over* t *w.r.t.* π_μ

$$\mathcal{U}_t = \{x \in G \mid t < \mu(x)\}$$

are *subgroups* of G.

Further, we find the notion of a *normal* fuzzy subgroup in the literature (cf. [60,71])—these are fuzzy groups μ in G satisfying the additional property:

(FSG3) $\mu(a) \leqslant \mu(b \cdot a \cdot b^{-1}), \ a, b \in G.$

In this context we note that the construction of the quotient w.r.t. normal fuzzy (sub)groups is a major problem in the literature on fuzzy sets [13, p. 230; 14; 24, p. 253; 56; 60, p. 125; 61; 74]. Even though the result that every normal fuzzy (sub)group μ induces a translation invariant similarity relation r in the sense of (2.18) is known (cf. [61, Theorem 2, Corollary 1]), the inability of computing the quotient w.r.t. r persists.

As demonstrated above the *quotient* w.r.t. a translation invariant similarity relation r is *simply* the espace étalé associated with r or the singleton space $\Sigma(A, r)$ to which the appropriate group structure is added in the sense of respective categories (see Section 1.1.2 and Theorem 2.2.5).

2.3. Topological space objects in CΩ-SET and Ω-valued topologies

First, we recall that in any topos the "internal formation of (arbitrary) unions" coincides with the multiplication of the power object monad (cf. Section 2.1, [58, Remark 5.24], see also [88]). In **CΩ-SET** the situation is as follows: let (A, E) be a complete Ω-valued set. On $P(A, E)$ we introduce an Ω-valued equality by

$$[\![f, g]\!] = \bigwedge_{a \in A} ((f(a) \longleftrightarrow g(a)), \quad f, g \in P(A, E), \tag{2.21}$$

where $\longleftrightarrow$ denotes the bi-implication (cf. Example 2.1(c) in FSBC). Further, let $\mathfrak{P}(A, E)$ be the set of all pairs $(\alpha, f) \in \Omega \times P(A, E)$ provided with the property

$$\bigvee_{a \in A} f(a) \leqslant \alpha,$$

and E_P be the Ω-valued equality on $\mathfrak{P}(A, E)$ define by

$$E_P((\alpha, f), (\beta, g)) = \alpha \wedge \beta \wedge [\![f, g]\!]. \tag{2.22}$$

Then it is not difficult to show that $(\mathfrak{P}(A, E), E_P)$ is complete and isomorphic to the singleton space $\Sigma(P(A, E), [\![,]\!])$. Now we refer to the monoidal structure on Ω-**SET** (cf. Section 3 in FSBC) and make the important observation that there exists a bijective map between $hom_{\mathrm{loc}}((A, E), (R_\Omega, E_\Omega))$ and $\mathfrak{P}(A, E)$. This bijection is given by the identification of characteristic morphisms $(A, E) \xrightarrow{\chi} (R_\Omega, E_\Omega)$ with E-strict and E-extensional membership maps $A \xrightarrow{f} \Omega$. Further, we observe

$$\begin{aligned}[\![(\alpha, \chi_1), (\beta, \chi_2)]\!] &= \alpha \wedge \beta \wedge \left(\bigwedge_{a \in A} (E(a, a) \to E_\Omega(\chi_1(a), \chi_2(a))) \right) \\ &= \alpha \wedge \beta \wedge \left(\bigwedge_{a \in A} (f_1(a) \longleftrightarrow f_2(a)) \right) = E_P((\alpha, f_1), (\beta, f_2)).\end{aligned}$$

Then Section 3 in FSBC and Section 2.1 show that

$$\Pi(A, E) := (\mathfrak{P}(A, E), E_P)$$

is the *power object* of (A, E), and

$$\Pi(A, E) \times (A, E) \xrightarrow{ev} (R_\Omega, E_\Omega), \qquad ev((E(a, a), f), a) := (E(a, a), f(a)) \tag{2.23}$$

is the *evaluation arrow*. Since the *power sheaf* corresponding to $\Pi(A, E)$ is *flabby*,[4] E_P-strict and E_P-extensional membership maps $\mathfrak{P}(A, E) \xrightarrow{F} \Omega$ are already uniquely determined by their values at those elements of $\mathfrak{P}(A, E)$ having *global* extent of existence. In particular, the following relation holds:

$$F(\alpha, f) = \alpha \wedge F(\top, f), \quad (\alpha, f) \in \mathfrak{P}(A, E). \tag{2.24}$$

Now we return to the definition of the *multiplication* μ of the power object monad in **C**Ω-**SET**. First, we notice that the (A, E)-component $\mu_{(A,E)}$ of μ is given by

$$\mu_{(A,E)} = id_{\Pi(A,E)} \circledcirc id_{\Pi(\Pi(A,E))},$$

where $\circledcirc$ is the clone-composition function of the power object monad. Since the identity of the power object $\Pi(A, E)$ is the exponential adjoint of the evaluation arrow ev, and ev itself is the characteristic morphism of the relation

$$\varepsilon_{(A,E)} \rightarrowtail^{\varepsilon_{(A,E)}} \Pi(A, E) \times (A, E),$$

we conclude from the definition of the clone-composition $\circledcirc$ that $\mu_{(A,E)}$ is the exponential adjoint of the characteristic morphism corresponding to the *composition* $\varepsilon_{(A,E)} \circ \varepsilon_{\Pi(A,E)}$ of the *relations* $\varepsilon_{\Pi(A,E)}$ and $\varepsilon_{(A,E)}$. Because of Theorem 2.1.1 and (2.24) the morphism $\Pi(\Pi(A, E)) \xrightarrow{\mu_{(A,E)}} \Pi(A, E)$ is determined by

$$\mu_{(A,E)}(\alpha, F) = (\alpha, \nu_{(A,E)}),$$

where

$$[\nu_{(A,E)}(\alpha, F)](a) = \bigvee_{f \in P(A,E)} f(a) \wedge F(\top, f), \quad a \in A. \tag{2.25}$$

[4] This means that all restriction maps are surjective.

Further, for every **CΩ-SET**-morphism $(A, E) \xrightarrow{\varphi} (B, F)$ the morphism $\Pi(A, E) \xrightarrow{\Pi(\varphi)} \Pi(B, F)$ (cf. (2.4)) has the following form:

$$[\Pi(\varphi)](\alpha, f) = (\alpha, \varphi(f)),$$

where

$$[\varphi(f)](b) = \bigvee_{a \in A} f(a) \wedge F(\varphi(a), b)\,, \quad b \in B. \tag{2.26}$$

A subobject $(T, F) \overset{\varphi}{\rightarrowtail} \Pi(A, E)$ of the power object $\Pi(A, E)$ is said to be *closed under the formation of unions* iff the arrow $\mu_{(A,E)} \cdot \Pi(\varphi)$ factors through φ—i.e. the commutativity of the diagram

$$\begin{array}{ccc} \Pi(T, F) & \xrightarrow{\Pi(\varphi)} & \Pi(\Pi(A, E)) \\ \Big\downarrow & & \Big\downarrow \mu_{(A,E)} \\ (T, F) & \underset{\varphi}{\rightarrowtail} & \Pi(A, E) \end{array} \tag{2.27}$$

It is not difficult to show that every subobject of the power object being closed under the formation of unions is necessarily flabby (cf. footnote 4).

We extend the conjunction to power objects. First, we recall that the *conjunction on* (R_Ω, E_Ω) is the characteristic morphism $\chi_\wedge$ of $\mathbb{1} \overset{\langle t,t\rangle}{\rightarrowtail} (R_\Omega, E_\Omega)$ where t is the *arrow true* (cf. Section 6 in FSBC). In particular, the membership map $\mu_\wedge$ corresponding to $\chi_\wedge$ is given as follows (see also Section 1.2.1):

$$\mu_\wedge(\alpha, \beta_1), (\alpha, \beta_2)) = (\alpha, \beta_1 \wedge \beta_2).$$

Further, let (A, E) be a complete Ω-valued set. Since **CΩ-SET** is Cartesian closed, there exists a unique arrow $\Pi(A, E) \times \Pi(A, E) \xrightarrow{\sqcap} \Pi(A, E)$ such that the following diagram is commutative:

$$\begin{array}{ccc} (\Pi(A, E) \times \Pi(A, E)) \times (A, E) & \xrightarrow{\sqcap \times id_{(A,E)}} & \Pi(A, E) \times (A, E) \\ \Big\downarrow \langle\langle \pi_1 \times \pi_3\rangle, \langle \pi_2, \pi_3\rangle\rangle & & \\ (\Pi(A, E) \times (A, E)) \times (\Pi(A, E) \times (A, E)) & & \Big\downarrow ev \\ \Big\downarrow ev \times ev & & \\ (R_\Omega, E_\Omega) \times (R_\Omega, E_\Omega) & \xrightarrow[\chi_\wedge]{} & (R_\Omega, E_\Omega) \end{array}$$

The arrow $\sqcap$ is also called the *binary infimum operation* on $\Pi(A, E)$.

Finally, let $\mathbb{1} \xrightarrow{\ulcorner \chi_{id_{(A,E)}} \urcorner} \Pi(A, E)$ be the exponential adjoint of the characteristic morphism $\chi_{id_{(A,E)}}$ of $(A, E) \xrightarrow{id_{(A,E)}} (A, E)$.

After these preparations we are in the position to define a *topology on* (A, E) as a subobject $(T, F) \overset{\varphi}{\rightarrowtail} \Pi(A, E)$ of the power object of (A, E) satisfying the following axioms (cf. [88], see also [58, pp. 105, 111]):

($\mathfrak{T}$1) $\ulcorner \chi_{id_{(A,E)}} \urcorner$ factors through $(T, F) \overset{\varphi}{\rightarrowtail} \Pi(A, E)$.

($\mathfrak{T}$2) $\sqcap \circ (\varphi \times \varphi)$ factors through $(T, F) \overset{\varphi}{\rightarrowtail} \Pi(A, E)$.

($\mathfrak{T}$3) $(T, F) \overset{\varphi}{\rightarrowtail} \Pi(A, E)$ is closed under the formation of unions.

The next theorem gives an external characterization of topologies.

Theorem 2.3.1. *Let* $(T, F) \overset{\varphi}{\rightarrowtail} \Pi(A, E)$ *be a flabby subobject of the power object of a complete* Ω*-valued set* (A, E). *Further, let* τ_φ *be a subset of* $P(A, E)$ *defined by*

$$\tau_\varphi = \{g \in P(A, E) | (\top, g) \in \varphi(T))\}. \tag{2.28}$$

Then $((T, F), \varphi)$ *is a topology on* (A, E) *iff* τ_φ *satisfies the following conditions*:

(O1) $\mathbb{E} \in \tau_\varphi$ *where* $\mathbb{E}(a) = E(a, a)$ *for all* $a \in A$.
(O2) *If* $g_1, g_2 \in \tau_\varphi$, *then* $g_1 \wedge g_2 \in \tau_\varphi$.
(O3) *If* $\{g_i \mid i \in I\}$ *is an arbitrary subset of* τ_φ, *then* $\bigvee_{i \in I} g_i \in \tau_\varphi$.
(O4) *For every* $\alpha \in \Omega$ *the map* $\alpha \wedge \mathbb{E}$ *is an element of* τ_φ.

Proof. It is easily seen that $\mathbb{E}$ is the membership map corresponding to the characteristic morphism of the identity of (A, E). Hence ($\mathfrak{T}1$) and (O1) are equivalent. Since the binary infimum in the Heyting algebra $P(A, E)$ is given by

$$(f_1 \wedge f_2)(a) = f_1(a) \wedge f_2(a) \quad \text{for all } a \in A,$$

the equivalence of ($\mathfrak{T}2$) and (O2) is trivial. In order to show that ($\mathfrak{T}3$) implies (O3) and (O4) we proceed as follows: let $\{g_i \mid i \in I\}$ be a non-empty subset of τ_φ. Then we define a characteristic morphism $(T, F) \xrightarrow{\chi} (R_\Omega, E_\Omega)$ by

$$\chi(k) = (F(k, k), \mu(k)) \quad \text{where } \mu(k) = \bigvee_{i \in I} E_P(\varphi(k), (\top, g_i)), \quad k \in T,$$

and consider the name $\ulcorner \chi \urcorner$ (cf. [28, p. 78]) of χ. Now we make use of (2.25) and (2.26) and observe that the map g_0 determined by

$$(\top, g_0) = (\mu_{(A,E)} \cdot \Pi(\varphi) \cdot \ulcorner \chi \urcorner)(\top)$$

has the following form:

$$\begin{aligned} g_0(a) &= \bigvee_{f \in P(A,E)} f(a) \wedge \left(\bigvee_{k \in T} \mu(k) \wedge E_P(\varphi(k), (\top, f)) \right) \\ &= \bigvee_{f \in P(A,E)} f(a) \wedge \left(\bigvee_{k \in T} \left(\bigvee_{i \in I} E_P(\varphi(k), (\top, g_i)) \wedge E_P(\varphi(k), (\top, f)) \right) \right) \\ &= \bigvee_{f \in P(A,E)} f(a) \wedge \left(\bigvee_{i \in I} E_P((\top, g_i), (\top, f)) \right) = \bigvee_{i \in I} g_i(a). \end{aligned}$$

Since $((T, F), \varphi)$ is closed under the formation of unions, we obtain that g_0 is an element of τ_φ. Hence (O3) is verified.

Further, let us consider a characteristic morphism $(T, F) \xrightarrow{\chi_\alpha} (R_\Omega, E_\Omega)$ defined by

$$\chi_\alpha(F(k, k), \mu_\alpha(k)) \quad \text{where } \mu_\alpha(k) = \alpha \wedge E_P(\varphi(k), (\top, \mathbb{E})), \quad k \in T.$$

Then it follows from the previous considerations that the relation

$$\mu_{(A,E)} \cdot \Pi(\varphi) \cdot \ulcorner \chi_\alpha \urcorner(\top) = (\top, \alpha \wedge \mathbb{E})$$

holds. Hence the implication ($\mathfrak{T}3$) $\Rightarrow$ (O4) is verified.

Now we prove that (O3) and (O4) implies ($\mathfrak{T}3$). Since power objects are flabby, it is sufficient to show that for every $h \in P(T, F)$ there exists a $g_0 \in \tau_\varphi$ provided with the following property: $\mu_{(A,E)} \cdot \Pi(\varphi)(\top, h) = (\top, g_0)$. Since (T, F) is flabby, we obtain from (2.25) and (2.26):

$$[\nu_{(A,E)}(\top, \varphi(h))](a) = \bigvee_{f \in P(A,E)} f(a) \wedge \left(\bigvee_{k \in T} h(k) \wedge E_P(\varphi(k), (\top, f)) \right)$$

$$= \bigvee_{\substack{\{k\in T \mid \\ F(k,k)=\top\}}} h(k) \wedge \left(\bigvee_{f\in P(A,E)} E_P(\varphi(k),(\top,f)) \wedge f(a) \right)$$
$$= \bigvee_{g\in\tau_\varphi} g(a) \wedge h(\varphi^{-1}(\top,g)).$$

Because of (O3) and (O4) the map $g_0 := v_{(A,E)}(\top, \varphi(h))$ is an element of τ_φ. Hence $\mu_{(A,E)} \cdot \Pi(\varphi)$ factors through $(T,F) \overset{\varphi}{\rightarrowtail} \Pi(A,E)$. $\Box$

For the following considerations we need some more terminology:

Definition 2.3.2. (a) Let (A,E) be an Ω-valued set which is not necessarily complete. A subset τ of $P(A,E)$ is called a *topology on* (A,E) iff τ satisfies the axioms (O1)–(O4) (cf. [41, p. 351]).

(b) Let X be an arbitrary, ordinary set. A subset τ of Ω^X is called an Ω*-valued topology* (or short Ω-topology) on X iff τ fulfills the conditions (O1)–(O3) (cf. [49, p. 153]). An Ω-valued topology τ on X is said to be *stratified* iff τ contains all constant maps from X to Ω—i.e. τ is a topology on (X, E_c). In the special case of $\Omega = [0,1]$ Ω-topologies are also called *fuzzy topologies* (cf. [12]).

If (A,E) is a *complete* Ω-valued set and τ is a topology on (A,E), then

$$T_\tau = \{(\alpha, \alpha \wedge g) \mid \alpha \in \Omega, g \in \tau\}$$

is a topology on (A,E) in the sense of **CΩ-SET**. Hence we can reformulate Theorem 2.3.1 as follows:

Let (A,E) be an arbitrary (not necessarily complete) Ω-valued set. Then topologies on (A,E) and topologies on the singleton space $\Sigma(X,E)$ in the sense of **CΩ-SET** are equivalent concepts. [5] In the special case of crisp equalities this means that *stratified Ω-valued topologies* on X are equivalent to *topologies* on $\Sigma(X,E_c)$ in the sense of **CΩ-SET**.

Remark 2.3.3 (*Spatial frames*). Let Ω be a spatial frame and (A,E) be a (not necessarily complete) Ω-valued set. Further let $pt(P(A,E)) \xrightarrow{\pi} pt(\Omega)$ be the espace étalé associated with E (cf. Section 2 in FSBC). Since the axioms (O1)–(O3) mean that τ is a subframe of $P(A,E)$, we obtain that τ can be identified with an ordinary topology

$$\mathcal{T}_\tau = \{\mathbb{A}_f \mid f \in \tau\}$$

which is coarser than the canonical topology $\mathcal{T}_{P(A,E)}$ on $pt(P(A,E))$ (cf. Section 1 in FSBC). Because of (O4) the "projection" π remains continuous w.r.t. $\mathcal{T}_\tau$. Hence every topology τ on (A,E) gives rise to a *fibrewise topological space* $(pt(P(A,E)), \mathcal{T}_\tau) \xrightarrow{\pi} (pt(\Omega), \mathcal{T}_\Omega)$. In the special case of the crisp equality E_c the espace étalé associated with E_c has the form $A \times pt(\Omega) \xrightarrow{\pi_2} pt(\Omega)$ (cf. Example 2.5 in FSBC), and $\mathcal{T}_\tau$ coincides with the topology $\mathcal{E}(\tau)$ assigned to τ by the hypergraph functor (see also Section 1.1.5 in then case of $\Omega = [0,1]$).

In the next remark we characterize topologies by certain extensional membership maps on power objects.

Remark 2.3.4 (*Membership maps of topologies*). Let (A,E) be a complete Ω-valued set. Since the power object $\Pi(A,E)$ is always flabby, membership maps of subobjects of $\Pi(A,E)$ are already uniquely determined on the set $P(A,E)$ of all global elements of the support set of $\Pi(A,E)$ (cf. (2.24)). Hence subobjects of $\Pi(A,E)$ and $[\![\,,\,]\!]$-extensional maps

$$P(A,E) \xrightarrow{v} \Omega$$

are *equivalent concepts*.

[5] In the case of Higgs' topos this result has already been obtained by the author in [36].

Now we fix an arbitrary subobject $(T, F) \overset{\varphi}{\rightarrowtail} \Pi(A, E)$ of $\Pi(A, E)$. Then $((T, F), \varphi)$ is a *topology* on (A, E) iff the corresponding membership map

$$P(A, E) \xrightarrow{\ v\ } \Omega$$

satisfies the following conditions:

(o1) $v(\mathbb{E}) = \top$.
(o2) $v(g_1) = v(g_2) = \top \Longrightarrow v(g_1 \wedge g_2) = \top$.
(o3) $A \subseteq \{g \in P(A, E) \mid v(g) = \top\} \Longrightarrow v(\bigvee A) = \top$.
(o4) $v(v(g) \wedge g) = \top$ for all $g \in P(A, E)$.

In particular, we conclude from the $[\![\,,\,]\!]$-extensionality of v that the conditions (o2)—(o4) imply the properties

(o2′) $v(f_1) \wedge v(f_2) \leqslant v(f_1 \wedge f_2),\ \ f_1, f_2 \in P(A, E)$.
(o3′) $\bigwedge_{i \in I} v(f_i) \leqslant v(\bigvee_{i \in I} f_i),\ \{f_i \mid i \in I\} \subseteq P(A, E)$.

Hence *membership maps* of *topologies* on $\Sigma(X, E_c)$ in the sense of **CΩ-SET** are *Ω-fuzzy topologies* on X in the sense of Šostak, but not vice versa (cf. [86,87]; see also [49, Definition 2.1] in the case of $\otimes = \wedge$).

We continue our exposé with a discussion on continuous **C**Ω-SET-morphisms. A pair $((A, E), ((T, F), \varphi))$ is called a *topological space object* in **CΩ-SET** iff (A, E) is a complete Ω-valued set and $((T, F), \varphi)$ is a topology on (A, E). Further, let $((A, E), ((T, F), \varphi))$ and $((B, G), ((S, H), \psi))$ be topological space objects. A **CΩ-SET**-morphism $(A, E) \xrightarrow{\ \Theta\ } (B, G)$ is said to be *continuous* iff the following diagram is commutative:

$$\begin{array}{ccc} \Pi(B, G) & \xrightarrow{\ \widehat{\Theta}\ } & \Pi(A, E) \\ {\scriptstyle \psi}\uparrow & & \uparrow{\scriptstyle \varphi} \\ (S, H) & \dashrightarrow & (T, F) \end{array}$$

where $\widehat{\Theta}$ is the exponential adjoint of $ev \cdot (id_{\Pi(B,G)} \times \Theta)$. If we identify topologies in the sense of **CΩ-SET** with topologies on Ω-valued sets—i.e. $((T, F), \varphi)$ with τ_φ and $((S, H), \psi)$ with τ_ψ (cf. Theorem 2.3.1), then continuity means precisely the following condition:

$$g \in \tau_\psi \quad \Longrightarrow \quad g \cdot \Theta \in \tau_\varphi \, .$$

Hence the category of stratified Ω-valued topological spaces (cf. [49, Section 5.1]) is isomorphic to a full subcategory of the category of topological spaces objects in **CΩ-SET**. In this sense the theory of stratified Ω-valued topological spaces is part of sheaf-theoretic investigations of internal topological space objects in the category of sheaves on Ω.

We close this subsection with a list of examples explaining the importance of topologies on Ω-valued sets.

Example 2.3.5. (a) Let $(E, \mathcal{T}_E) \xrightarrow{\ \pi\ } (X, \mathcal{T})$ a fibrewise topological space (cf. [51]), $\mathcal{G}_\pi$ be the sheaf of local sections of π (cf. Remark 4.3 in FSBC), and let (A_π, E_π) be the complete $\mathcal{T}$-valued set corresponding to $\mathcal{G}_\pi$ (cf. Section 4 in FSBC). Further, every $G \in \mathcal{T}_E$ induces a membership map $\mathbb{A}_G \in P(A_\pi, E_\pi)$ as follows:

$$\mathbb{A}_G(s) = s^{-1}(G), \quad s \in A_\pi.$$

Then $\tau = \{\mathbb{A}_G \mid G \in \mathcal{T}_E\}$ is a topology on (A_π, E_π) (cf. [21, Example 8.11]).

(b) Let **TOP** be the category of ordinary topological spaces and Ω be a frame. A functor $\mathcal{F} : \Omega^{\mathrm{op}} \to$ **TOP** is called a *separated presheaf of topological spaces* on Ω iff the composition of $\mathcal{F}$ with the forgetful functor is a separated presheaf of sets (see also [31]). Then we first associate an Ω-valued set $(A_\mathcal{F}, E_\mathcal{F})$ with every separated presheaf $\mathcal{F}$ of topological spaces (cf. Section 4 in FSBC)

$$A_{\mathcal{F}} = \dot{\bigcup_{\alpha \in \Omega}} \mathcal{F}(\alpha), \quad \mathbb{E}(a) = \alpha \text{ iff } a \in \mathcal{F}(\alpha),$$
$$E_{\mathcal{F}}(a,b) = \bigvee \{\lambda \in \Omega \mid \lambda \leqslant \mathbb{E}(a) \wedge \mathbb{E}(b), a{\upharpoonright}\lambda = b{\upharpoonright}\lambda\},$$

and subsequently topologize $(A_{\mathcal{F}}, E_{\mathcal{F}})$ as follows (cf. [89]): we consider the set $\mathfrak{T}$ of all subsets $G \subseteq A_{\mathcal{F}}$ provided with the following properties:

- $G_\alpha = \{a \in G \mid \mathbb{E}(a) = \alpha\}$ is an open subset of $\mathcal{F}(\alpha)$ w.r.t. the given topology on $\mathcal{F}(\alpha)$ for all $\alpha \in \Omega$.
- $\bigvee_{c \in G} E_{\mathcal{F}}(a,c) = \mathbb{E}(a) \Longrightarrow a \in G$.

It is easily seen that $\mathfrak{T}$ is closed under finite set-theoretical intersections. In particular, for all $\alpha \in \Omega$ the set $A_{\underline{\alpha}} = \{a \in A_{\mathcal{F}} \mid \mathbb{E}(a) \leqslant \alpha\}$ is an element of $\mathfrak{T}$. Obviously, every $G \in \mathfrak{T}$ induces a membership map $\mathbb{P}_G \in P((A_{\mathcal{F}}, E_{\mathcal{F}}))$ by

$$\mathbb{P}_G(a) = \bigvee_{c \in G} E_{\mathcal{F}}(a,c), \quad a \in A_{\mathcal{F}}.$$

Since $\mathcal{F}$ is a separated presheaf, we obtain

$$\mathbb{P}_{G_1}(a) \wedge \mathbb{P}_{G_2}(a) = \mathbb{P}_{G_1 \cap G_2}(a), \quad \mathbb{P}_{A_{\underline{\alpha}}}(a) = \mathbb{E}(a) \wedge \alpha, \ \ a \in A_{\mathcal{F}}.$$

Hence $\{\mathbb{P}_G \mid G \in \mathfrak{T}\}$ forms a base for a *topology* τ on $(A_{\mathcal{F}}, E_{\mathcal{F}})$. In particular, every element $g \in \tau$ has the following representation: there exists a subset $\{G_i \mid i \in I\} \subseteq \mathfrak{T}$ s.t. $g = \bigvee_{i \in I} \mathbb{P}_{G_i}$.

Example 2.3.6 (*Ω-probabilistic metric spaces*). On the set $\mathcal{D}^+(\Omega)$ of all global, Ω-valued, non-negative real numbers (cf. Example 2.2.1) we introduce an ordinary partial ordering $\preccurlyeq$ by

$$F \preccurlyeq G \iff \forall\, r \in \mathbb{R}^+ : G(r) \leqslant F(r).$$

Further, let A be a set. A map $A \times A \xrightarrow{\ \mathfrak{F}\ } \mathcal{D}^+(\Omega)$ is called an *Ω-probabilistic metric* on A iff $\mathfrak{F}$ satisfies the following axioms:

(PM1) $\mathfrak{F}(a,b) = \varepsilon_0 \Longleftrightarrow a = b$.
(PM2) $\mathfrak{F}(a,b) = \mathfrak{F}(b,a)$, $a, b \in A$ (Symmetry).
(PM3) $\mathfrak{F}(a,c) \preccurlyeq \tau_\wedge(\mathfrak{F}(a,b), \mathfrak{F}(b,c))$ (Triangle Inequality).

If $\mathfrak{F}$ is an Ω-probabilistic metric on A, then $(A, \mathfrak{F})$ is called an *Ω-probabilistic metric space*. In particular, [0, 1]-probabilistic metric spaces form a special class of *Menger spaces* (cf. [84, Definition 8.1.4, p. 125]).

Now we make the important observation that every Ω-probabilistic metric $\mathfrak{F}$ on A induces an Ω-valued equality $E_{\mathfrak{F}}$ on A in the following way:

$$E_{\mathfrak{F}}(a,b) = \bigwedge_{m \in \mathbb{N}} [\mathfrak{F}(a,b)]\left(\frac{1}{m}\right), \quad a, b \in A.$$

As an immediate corollary from the triangle inequality (PM3) we obtain

$$E_{\mathfrak{F}}(a,b) \wedge [\mathfrak{F}(b,c)]\left(r - \frac{1}{m}\right) \leqslant [\mathfrak{F}(a,c)]\left(r - \frac{1}{2m}\right), \quad 1 < m \cdot r.$$

Hence the "left-continuity" of Ω-valued non-negative real numbers (cf. (2.5)) implies the important property

$$E_{\mathfrak{F}}(a,b) \wedge [\mathfrak{F}(b,c)](r) \leqslant [\mathfrak{F}(a,c)](r), \quad r \in \mathbb{R}^+.$$

Because of the previous relation it is easily seen that $\mathfrak{F}$ can be lifted to an Ω-**SET**-morphism $(A, E_{\mathfrak{F}}) \otimes (A, E_{\mathfrak{F}}) \xrightarrow{\ \mathfrak{F}\ } (\Delta^+(\Omega), E_{\Delta^+})$ (cf. Example 2.2.1).

After these preparations we are now in the position to explain that every Ω-probabilistic metric $\mathfrak{F}$ on A induces a *topology* on $(A, E_{\mathfrak{F}})$. For this purpose let $\tau_{\mathfrak{F}}$ be the set of all membership maps $g \in P(A, E_{\mathfrak{F}})$ provided with the following property:

For every element $a \in A$ there exists a map $\varkappa : \mathbb{N} \to \Omega$ satisfying the subsequent conditions:

(i) $\bigvee_{m\in\mathbb{N}} \varkappa(m) = g(a)$.
(ii) $\varkappa(m) \wedge [\mathfrak{F}(a,b)](1/m) \leqslant g(b) \;\; \forall\, b \in A$.

It is not difficult to show that $\tau_{\mathfrak{F}}$ is in fact a topology on $(A, E_{\mathfrak{F}})$. In particular, if we fix $a \in A$ and $r \in \mathbb{R}^+ \setminus \{0\}$, then the map

$$A \xrightarrow{B_{(a,r)}} \Omega, \qquad B_{(a,r)}(b) = [\mathfrak{F}(a,b)](r), \;\; b \in A$$

is an element of $\tau_{\mathfrak{F}}$. Indeed, for every $b \in A$ we can define a map $\mathbb{N} \xrightarrow{\varkappa} \Omega$ by

$$\varkappa(m) = \left\{ \begin{array}{ll} [\mathfrak{F}(a,b)]\left(r - \dfrac{1}{m}\right), & \dfrac{1}{m} \leqslant r, \\ \bot, & r < \dfrac{1}{m} \end{array} \right\}$$

and conclude from the triangle inequality (PM3) and the "left-continuity" of Ω-valued non-negative numbers that $\varkappa$ satisfies the conditions (i) and (ii) w.r.t. $B_{(a,r)}$. In this context $B_{(a,r)}$ is called the *Ω-valued open ball with center a and radius r*. Obviously, the collection of all open Ω-valued balls $B_{(a,1/m)}$ forms a "base" of $\tau_{\mathfrak{F}}$.

The following example explains an application of the *convergence theory* associated with topological space objects in **CΩ-SET**. First, we recall that a sequence $(x_n)_{n\in\mathbb{N}}$ in a complete Ω-valued set (A, E) is convergent to $x_0 \in A$ (or is a *limit point* of $(x_n)_{n\in\mathbb{N}}$) w.r.t. a topology τ on (A, E) iff *for all* $g \in \tau$ the following relation holds:

$$g(x_0) \leqslant \bigvee_{n\in\mathbb{N}} \left(\bigwedge_{n\leqslant k} g(x_k) \right). \tag{2.29}$$

Example 2.3.7. (a) Let (X, ϱ) be an ultrametric space and E_ϱ be the $[0, 1]$-valued equality determined by ϱ in the sense of Example 2.2 in FSBC—e.g.

$$E_\varrho(x, y) = \frac{1}{1 + \varrho(x, y)}, \quad x, y \in X.$$

Further, let $\Sigma(X, E_\varrho)$ be the singleton space corresponding to (X, E_ϱ). We show that every *bounded* sequence $(x_n)_{n\in\mathbb{N}}$ in (X, ϱ) has *local limit points*—this means that there exists a singleton of (X, E_ϱ) (i.e. a local section of the espace étalé associated with (X, E_ϱ)) s.t. s is a limit point of the sequence $(\widetilde{x_n})_{n\in\mathbb{N}}$ w.r.t. the discrete topology $\tau_d = P(\Sigma(X, E_\varrho))$ on $\Sigma(X, E_\varrho)$.

Since $(x_n)_{n\in\mathbb{N}}$ is bounded, it follows from the construction of E_ϱ that for all $z \in X$ the following relation holds:

$$0 < \alpha_z := \liminf_{n\to\infty} E_\varrho(x_n, z).$$

Hence for every $z \in X$ we can introduce a singleton s_z of (X, E_ϱ) by

$$s_z(a) = \min(E_\varrho(a, z), \alpha_z), \quad a \in X.$$

Further, every element of $P(\Sigma(X, E_\varrho))$ has the form $\widetilde{f}$ with $f \in P(X, E_\varrho)$ (cf. (6.12) in FSBC). Then the E_ϱ-extensionality of f implies

$$\widetilde{f}(s_z) = \min(f(z), \alpha_z) \leqslant \sup_{n\in\mathbb{N}} \left(\bigwedge_{n\leqslant k} \widetilde{f}(\widetilde{x_k}) \right).$$

Hence the condition (2.29) is verified.

(b) Let ([0, 1], $\mathcal{M}$, λ) be the Lebesgue measure space and $\mathbb{B}([0, 1])$ be the associated probability σ-algebra (cf. [41, Section 5.3.2]). It is well known that $\mathbb{B}([0, 1])$ is a complete atomless Boolean algebra. Further, it is also well known that pointwise λ-almost everywhere convergence is *not topological* in the usual sense (cf. [22,41, Section 5.1]). But on the space $L^0([0, 1])$ of all λ-almost everywhere defined, real valued, random variables there exists a stratified $\mathbb{B}([0, 1])$-valued topology τ_0 s.t. pointwise λ-almost everywhere convergence is topological in the sense of (2.29) (cf. [41, Theorem 5.3.2.7]). The fact that $L^0([0, 1])$ can be viewed as the real number object in $\mathbf{C}\mathbb{B}([0, 1])$**-SET** and τ_0 can be understood as the internalized topology of the usual topology on the standard real line is another aspect of the same situation.

3. Epilogue

The purpose of this exposé was to explain the close ties between fuzzy set theory and sheaf theory. Moreover, it is self-evident that the real unit interval has not only an order structure making sheaf-theoretic arguments possible, but also a rich algebraic structure represented by the usual product, Łukasiewicz' arithmetic conjunction (cf. [23,65]) or by various kinds of *triangular norms* (cf. [57]). Since 1979, this observation has led to a replacement of the binary meet operation by an appropriate semigroup operation $*$ on the underlying frame. As a consequence we can find a wide-spread application of residuated lattices, resp. quantales (cf. [82]), in fuzzy set theory. Typical examples of these developments are the *max-star*-composition of fuzzy relations (cf. [18, p. 71; 78, Section 4.6; 7, 30, pp. 438–470]) and the concept of fuzzy groups in the sense of Anthony and Sherwood (cf. [2]).

In order to obtain also a sound and coherent mathematical basis for these concepts it is clear that these topics require a refinement of sheaf theory. This goal can be accomplished by introducing the concept of *M-valued sets* and the corresponding *singleton monad* in the category of *M*-valued sets (cf. [37,38,40]) where *M* is usually a *GL*-monoid—e.g. a complete *MV*-algebra (cf. [4]) or a continuous triangular norm. By virtue of the *non-idempotency* of the underlying semigroup operation it is interesting to observe that the singleton monad in this setting is *not degenerated*—this means that the corresponding Eilenberg–Moore category and the Kleisli category are *not* necessarily equivalent. Since the *structure map* of algebras induce a kind of restriction map—indeed much material of sheaf theory depends on the existence of restriction (see [90]), algebras w.r.t. the singleton monad in the category of *M*-valued sets form an appropriate non-idempotent generalization of sheaves. In this context it is fairly remarkable to see that Łukasiewicz' negation can be internalized as a truth arrow (cf. [40]). In this sense Zadeh's complement of fuzzy sets (cf. Section 1.2) admits also a categorical interpretation. Further, the result in Example 2.3.7(a) remains valid for arbitrary metric spaces (cf. [44]). Even though there does not exist an espace étalé, the singleton space of a metric space plays here the crucial role.

To sum up we believe that the singleton monad w.r.t. an *GL*-monoid or more general w.r.t. quantales with involutions (cf. [72,46]) is an appropriate basis for a categorical understanding of various aspects in fuzzy set theory or more general of non-commutative geometry (cf. [15]).

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