NOTES FOR MATH 599: CONTACT GEOMETRY

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1. Introduction

1.1. Definitions and examples.

Definition 1.1. *A* contact manifold (M, ξ) *is a* $(2n+1)$*-dimensional manifold* M *equipped with a smooth maximally nonintegrable hyperplane field* $\xi \subset TM$*, i.e., locally* $\xi = \ker \alpha$*, where* α *is a 1-form which satisfies* $\alpha \wedge (d\alpha)^n \neq 0$*.* ξ *is a* contact structure *and* α *is a* contact 1-form *which locally defines* ξ*.*

Change of contact 1-form: If we multiply a local contact 1-form α by a nowhere zero function g, then

$$g\alpha \wedge (d(g\alpha))^n = g\alpha \wedge (gd\alpha + dg \wedge \alpha)^n = g^{n+1}\alpha \wedge (d\alpha)^n.$$

Hence α is a contact 1-form iff $g\alpha$ is a contact 1-form. We can make "conformal" changes to the 1-form.

Remark 1.2. *We're not interested in the specific 1-form; instead, we are interested in the hyperplane field* ξ *given by the kernel.*

Comparison with symplectic geometry: A symplectic manifold (M, ω) is a $2n$-dimensional manifold together with a closed 2-form ω satisfying $\omega^n \neq 0$, called the *symplectic form*. In symplectic geometry, conformal changes to ω (i.e., multiplying by g) would usually force $d(g\omega) \neq 0$.

HW 1. *Prove that* $\omega^n \neq 0$ *iff* ω *is* nondegenerate*, i.e., for all* $x \in M$ *and* $v \in T_xM$*, there is* $w \in T_xM$ *so that* $\omega(x)(v,w) \neq 0$*. [Observe that this is a linear algebra problem.]*

HW 2. *If* $\alpha \wedge (d\alpha)^n \neq 0$*, then* $d\alpha$ *is a nondegenerate 2-form when restricted to* ξ*.*

Hence, contact geometry is customarily viewed as the odd-dimensional sibling of symplectic geometry.

Example: The *standard contact structures* $(\mathbf{R}^{2n+1}, \xi_0)$, where $\mathbf{R}^n$ has coordinates $(x_1, y_1, \dots, x_n, y_n, z)$, and ξ_0 is given by $\alpha_0 = dz - \sum_{i=1}^n y_i dx_i$. Then $\ker \alpha_0 = \mathbf{R}\{\partial_{y_1}, \dots, \partial_{y_n}, \partial_{x_1} + y_1\partial_z, \dots, \partial_{x_n} + y_n\partial_z\}$. We verify the contact property by computing $d\alpha_0 = \sum_{i=1}^n dx_i dy_i$ and

$$\alpha_0 \wedge (d\alpha_0)^n = (dz - \sum y_i dx_i) \wedge (\sum dx_i dy_i)^n = (n!)\, dz dx_1 dy_1 \dots dx_n dy_n.$$

Our primary focus is on contact 3-manifolds. For contact 3-manifolds we will assume that our contact structures ξ on M satisfy the following:

1. M is oriented.
2. ξ is oriented, and hence given as the kernel of a global 1-form α.
3. $\alpha \wedge d\alpha > 0$, i.e., the contact structure is *positive*.

Such contact structures are often said to be *cooriented*.

HW 3. *Show that if ξ is a smooth oriented 2-plane field, then ξ can be written as the kernel of a global 1-form α.*

HW 4. *Verify that if g is a nowhere zero function, then $\alpha \wedge d\alpha > 0$ iff $(d\alpha) \wedge d(g\alpha) > 0$.*

Example: The standard contact structure in dimension 3 is $(\mathbf{R}^3, \xi_0)$, where $\mathbf{R}^3$ has coordinates (x, y, z), and ξ_0 is given by $\alpha_0 = dz - y dx$. Then $\xi_0 = \ker \alpha_0 = \mathbf{R}\{\partial_x + y\partial_z, \partial_y\}$. The orientation on ξ_0 is induced from the projection $\pi : \mathbf{R}^3 \to \mathbf{R}^2$, $(x, y, z) \mapsto (x, y)$ to the xy-plane, or, equivalently, from the normal orientation to ∂_z (in other words, (v_1, v_2) is an oriented basis for ξ_0 if (∂_z, v_1, v_2) is an oriented basis for $\mathbf{R}^3$). Moreover, we can easily verify that $\alpha_0 \wedge d\alpha_0 > 0$.

According to the standard "propeller picture" (see Figure 1), all the straight lines parallel to the y-axis are everywhere tangent to ξ_0, and the 2-planes rotate in unison along these straight lines.

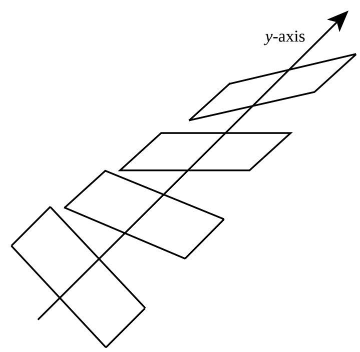

FIGURE 1. The propeller picture.

Example: (T^3, ξ_n). Here $T^3 \simeq \mathbf{R}^3/\mathbf{Z}^3$, with coordinates (x, y, z), and $n \in \mathbf{Z}^+$. Then ξ_n is given by $\alpha_n = \sin(2\pi nz)dx + \cos(2\pi nz)dy$. We have

$$\xi_n = \mathbf{R}\left\{\frac{\partial}{\partial z}, \cos(2\pi nz)\frac{\partial}{\partial x} - \sin(2\pi nz)\frac{\partial}{\partial y}\right\}.$$

This time, the circles $x = y = const$ (parallel to the z-axis) are everywhere tangent to ξ_n, and the contact structure makes n full twists along such circles.

HW 5. *Verify that (T^3, ξ_n) is a contact 3-manifold and ξ_n is a* positive *contact structure for $n > 0$. What happens to $n < 0$ and $n = 0$?*

1.2. **Pfaff's Theorem.**

Definition 1.3. *A* contact diffeomorphism *(or* contactomorphism*) $\phi : (M_1, \xi_1) \to (M_2, \xi_2)$ is a diffeomorphism such that $\phi_*\xi_1 = \xi_2$.*

Remark 1.4. *A contactomorphism usually does not preserve the contact 1-form, i.e., if α_1, α_2 are contact 1-forms for ξ_1, ξ_2, then $\phi^*\alpha_2 = g\alpha_1$ for some nowhere zero function g.*

Theorem 1.5 (Pfaff)**.** *Every contact $(2n+1)$-manifold (M,ξ) locally looks like $(\mathbf{R}^{2n+1},\xi_0)$, i.e., for all $p \in M$ there exist open sets $U \ni p$ of M and $V \ni 0$ of $\mathbf{R}^{2n+1}$ such that $\phi : (U,\xi) \to (V,\xi_0)$, $\phi(p) = 0$, is a contactomorphism.*

Pfaff's theorem essentially says that contact geometry has no local invariants. The Darboux theorem in symplectic geometry also states that there are no local invariants in symplectic geometry. (Its statement also strongly resembles the Pfaff theorem.) This contrasts with Riemannian geometry, where the curvature is a local invariant.

HW 6. *In class we will give a proof of Pfaff's theorem in dimension 3. Generalize it to higher dimensions.*

Proof. Suppose $n = 1$, i.e., we are in dimension 3. Without loss of generality we may assume that there is a local coordinate chart which maps p to $0 \in \mathbf{R}^3$ and the induced contact structure ξ is the xy-plane at 0, i.e., $\xi(0) = \ker dz$. In a neighborhood of 0 we can write the contact 1-form as $\alpha = dz + f dx + g dy$.

Now consider $y = 0$ (the xz-plane). On it, ξ restricts to the vector field $X = \partial_x - f\partial_z$. Observe that X is transverse to the z-axis $x = y = 0$. Using the fundamental theorem of ODE's, we can integrate along this vector field, starting along the z-axis. We let $\Phi(x,z)$ be the time x flow, starting at $(0,0,z)$. Hence there are new coordinates (also called (x,z)) such that the vector field is ∂_x, i.e., $f = 0$ along $y = 0$.

In other words, we have normalized the contact structure on the xz-plane. Finally, consider the restrictions of ξ to $x = const$, which give the vector field $\partial_y - g\partial_z$. We then have a vector field on all of a neighborhood of 0 in $\mathbf{R}^3$, which is transverse to $y = 0$. If we integrate along the vector field, we obtain new coordinates so that we can write $\alpha = dz + f(x,y,z)dx$ with $f(x,y,0) = 0$ and $\partial_y f < 0$ (from the contact condition). Now simply change coordinates $(x,y,z) \mapsto (x, -f(x,y,z), z)$. $\square$

2. Diffeomorphisms

2.1. Reeb vector fields. Let (M,ξ) be a contact manifold of dimension $2n+1$.

Definition 2.1. *A* contact vector field X *is a vector field whose time t flow ϕ_t is a contact diffeomorphism for all t. (In other words, $\mathcal{L}_X\alpha = g\alpha$ for some function g, where $\mathcal{L}$ is the Lie derivative.)*

Given a global contact 1-form α for ξ, we define a vector field $R=R_\alpha$ which satisfies $i_R d\alpha = 0$ and $\alpha(R)=1$. Such a vector field R is called the *Reeb vector field* corresponding to the 1-form α.

First observe that since the rank of $d\alpha$ is $2n$-dimensional, there is a unique line field which is annihilated by $d\alpha$. R is a section of this line field, with normalization $\alpha(R)=1$.

Fact: *Every vector field X on M can be uniquely written as $X=fR+Y$, where Y is a section of ξ.*

Fact: *R is a contact vector field.*

This follows from the Cartan formula: $\mathcal{L}_R\alpha = i_R d\alpha + d(\alpha(R)) = 0$.

HW 7. *Conversely, every contact vector field X transverse to ξ can be written as a Reeb vector field for some 1-form α.*

Hint: Since X is a contact vector field, $\mathcal{L}_X\alpha = i_X d\alpha + d(\alpha(X)) = g\alpha$ for some g. Write $h=\alpha(X)$. Observe that h is nowhere zero by the transversality condition. Now consider the 1-form $\frac{1}{h}\alpha$.

We will return to the study of the *dynamics* of Reeb vector fields later.

2.2. Gray's Theorem.

Theorem 2.2. *Let ξ_t, $t\in[0,1]$, be a 1-parameter family of contact structures on a closed manifold M. Then there is a 1-parameter family of diffeomorphisms ϕ_t such that $\phi_0 = id$ and $\phi_t^*\xi_t = \xi_0$.*

Gray's Theorem is analogous to Moser's Theorem in symplectic geometry. One difference is that, whereas there is a cohomological condition in Moser's Theorem, there is none in Gray's Theorem.

Proof. The point is to differentiate the formula $\phi_t^*\alpha_t = g_t\alpha_0$, where α_t is a 1-parameter family of contact forms corresponding to ξ_t. This will give us a formula in terms of vector fields X_{t_0} which are given by $X_{t_0}(\phi_{t_0}(p)) = \frac{d}{dt}|_{t=t_0}\phi_t(p)$. Integrating the vector fields gives the desired 1-parameter family of diffeomorphisms. The technique is often called the *Moser technique*.

$$\left.\frac{d}{dt}\right|_{t=t_0}\phi_t^*\alpha_t = \left.\frac{d}{dt}\right|_{t=t_0} g_t\alpha_0$$

By using the Leibniz rule, the left-hand side can be written as $\frac{d}{dt}|_{t=t_0}\phi_t^*\alpha_{t_0} + \phi_{t_0}^*\frac{d}{dt}|_{t=t_0}\alpha_t = \phi_{t_0}^*(\mathcal{L}_{X_{t_0}}\alpha_{t_0} + \frac{d\alpha}{dt}|_{t=t_0})$, whereas the right-hand side is some function times α_0. When we remove the $\phi_{t_0}^*$ (by composing with $(\phi_{t_0}^{-1})^*$), then we have an equation (after removing subscripts t_0):

$$\mathcal{L}_X\alpha = \dot\alpha + h\alpha,$$

where we are solving for a vector field X and the function h (the conformal factor). First use the Cartan formula to expand $\mathcal{L}_X\alpha = i_X d\alpha + d(i_X\alpha)$. If we set $X = fR + Y$, where R satisfies $i_R\alpha = 1$ and $i_R d\alpha = 0$, then:

$$i_Y d\alpha + df = \dot{\alpha} + g\alpha.$$

We can set f to be any function and solve for

$$i_Y d\alpha - g\alpha = \beta$$

for any β. □

HW 8. *Prove that there are unique g and i_Y that solve the equation above. (This is a linear algebra exercise.)*

Infinitesimal diffeomorphisms: Let Diff(M, ξ) be the group of contactomorphisms of (M, ξ) to itself. The proof of the following proposition is a corollary of the above Moser technique:

Proposition 2.3. T_{id}*Diff*$(M, \xi) = \{C^\infty$*-functions on* $M\}$.

Proof. If $X \in T_{id}$Diff(M, ξ), then $\mathcal{L}_X\alpha = g\alpha$. Hence $i_Y d\alpha + df = g\alpha$. If f is chosen, there is a unique Y as above. □

Remark: Diff(M, ξ) is infinite-dimensional!!

2.3. **Normal forms.** We will give another application of the Moser technique.

Definition 2.4. *A* transverse curve *is a curve γ such that $\dot{\gamma} \pitchfork \xi$ at every point of γ.*

Proposition 2.5. *Suppose γ is a closed transverse curve in (M, ξ). Then there is a neighborhood $N(\gamma)$ of γ such that $\xi|_{N(\gamma)}$ is given by*

$$\alpha_0 = dz + \frac{1}{2}(-y dx + x dy) = dz + \beta_0.$$

Proof. We may choose coordinates (z, x, y) on $N(\gamma) = S^1 \times D^2$ so that ξ is given by $\alpha = dz + \beta$, where $\beta = f dx + g dy$ and $\beta(z, 0, 0) = 0$. We interpolate between α and α_0 by setting $\alpha_t = (1-t)\alpha_0 + t\alpha = dz + (1-t)\beta_0 + t\beta_1$. Then:

HW 9. $\alpha_t \wedge d\alpha_t > 0$ *in a neighborhood of* γ.

We then use the Moser technique to solve for X_t in:

$$\mathcal{L}_{X_t}\alpha_t = \frac{d\alpha_t}{dt} + g_t\alpha = (\beta_1 - \beta_0) + g_t\alpha$$

and the result follows, by observing that we may take $g_t = 0$, $X_t = 0$ along γ. □

2.4. **Legendrian submanifolds.**

Definition 2.6. *A submanifold* $L \subset (M^{2n+1}, \xi)$ *is* Legendrian *if* $\dim L = n$ *and* $T_x L \subset \xi_x$ *for all* $x \in L$.

Claim. *Legendrian submanifolds are integral submanifolds of* ξ *of maximal dimension.*

Given a *distribution* (i.e., a subbundle ζ of TM), an *integral submanifold* N is one where $T_x N \subset \zeta_x$ for all $x \in N$.

Proof. In view of the formula:

$$d\alpha(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y]),$$

where X, Y are vector fields, we see that $[X,Y]$ is a section of ξ iff $d\alpha(X,Y) = 0$. The maximal dimensional subspace with this property is n (i.e., Lagrangians in a $2n$-dimensional symplectic vector space). $\square$

3. Legendrian knots

3.1. **Some preliminaries.** In dimension 3, we study *Legendrian knots* $L \subset (M, \xi)$. We will describe the "classical" invariants of Legendrian knots. But first some preliminaries:

Definition 3.1. *Let K be a knot. Then a* framing *of K is a choice of homotopy class of trivializations of the normal bundle νK to K. This gives an identification of a tubular neighborhood $N(K)$ with $S^1 \times D^2$.*

HW 10. *The set of framings of K can be identified with* **Z**. *(The identification is not always canonical, in the sense that sometimes there is no natural choice of the zero element.)*

The invariants that we introduce are supposed to distinguish Legendrian knots up to *Legendrian isotopy*.

Definition 3.2. *Two Legendrian knots L_0 and L_1 are* Legendrian isotopic *if there is a map $\Phi : S^1 \times [0,1] \to M$ so that $\Phi_0(S^1) = L_0$, $\Phi_1(S^1) = L_1$, and $L_t = \Phi_t(S^1)$ are (embedded) Legendrian knots. Here, $\Phi_t(x) = \Phi(x,t)$.*

HW 11. *Use Gray's Theorem to prove that two Legendrian knots L_0 and L_1 are Legendrian isotopic iff there is a 1-parameter family of contact diffeomorphisms $\phi_t : M \to M$ such that $\phi_0 = id$ and $\phi_1(L_0) = L_1$.*

Therefore, "Legendrian isotopy = contact isotopy".

We will often restrict attention to Legendrian knots in the standard contact $(\mathbf{R}^3, \xi)$, or equivalently, in the standard contact (S^3, ξ).

The standard contact structure ξ on S^3. Consider $B^4 = \{|z_1|^2 + |z_2|^2 \leq 1\} \subset \mathbf{C}^2$. Then take $S^3 = \partial B^4$. The contact structure ξ is defined as follows: for all $p \in S^3$, ξ_p is the unique complex line $\subset T_pS^3$ (the unique 2-plane invariant under the complex structure J which maps $\partial_{x_i} \mapsto \partial_{y_i}$ and $\partial_{y_i} \mapsto -\partial_{x_i}$).

HW 12. *Prove that the restriction of $\alpha = \sum_{i=1}^2 (x_i dy_i - y_i dx_i)$ from $\mathbf{C}^2$ to S^3 is a contact 1-form for (S^3, ξ), namely verify that $\alpha \wedge d\alpha > 0$ on S^3.*

Fact: The standard (S^3, ξ) with one point removed is contactomorphic to the standard $(\mathbf{R}^3, \xi)$.

3.2. **Front projection.** We now consider Legendrian knots in the standard contact $(\mathbf{R}^3, \xi)$ given by $dz - ydx = 0$. Consider the *front projection* $\pi : \mathbf{R}^3 \to \mathbf{R}^2$, where $(x, y, z) \mapsto (x, z)$. Generic Legendrian knots L (the genericity can be achieved by applying a small contact isotopy) can be projected to closed curves in $\mathbf{R}^2$ with cusps and ordinary double points but no vertical tangencies. Conversely, such a closed curve in $\mathbf{R}^2$ can be lifted to a Legendrian knot in $\mathbf{R}^3$ by setting y to be the slope of the curve at (x, z). (Observe that if $dz - ydx = 0$, then $\frac{dz}{dx} = y$.) Moreover, at an ordinary double point, the strand with more negative slope $\frac{dz}{dx}$ comes in front of the one with more positive slope.

Examples: See Figure 2 for Legendrian representatives of the unknot, the right-handed trefoil, and the figure 8.

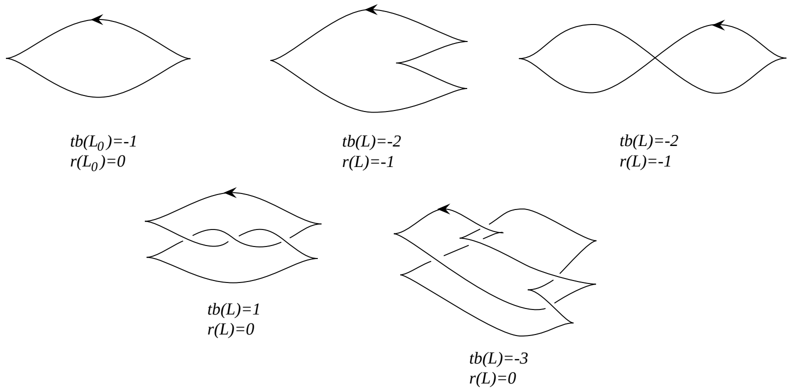

FIGURE 2. Unknots, right-handed trefoil, and figure 8 knot.

3.3. **Twisting number/Thurston-Bennequin invariant.** Our first invariant is the *relative Thurston-Bennequin invariant* $t(L,\mathcal{F})$, also known as the *twisting number*, where $\mathcal{F}$ is some fixed framing for L. Although $t(L,\mathcal{F})$ is an invariant of the *unoriented* knot L, for convenience pick one orientation of L. L has a natural framing called the *normal framing*, induced from ξ by taking $v_p \in \xi_p$ so that $(v_p, \dot{L}(p))$ form an oriented basis for ξ_p. We then define $t(L,\mathcal{F})$ to be the integer difference in the number of twists between the normal framing and $\mathcal{F}$. By convention, left twists are negative. More precisely, use $\mathcal{F}$ to identify a tubular neighborhood $N(L)$ of L with $S^1\times D^2$. We make an oriented identification $\partial(S^1\times D^2)\simeq T^2 = \mathbf{R}^2/\mathbf{Z}^2$ as follows: map $\{pt\}\times\partial D^2$ (the meridian) to $(1,0)$ and $S^1\times\{pt\}$ to $(0,1)$. Then if the closed curve on T^2 corresponding to the normal framing is $(n,1)$, then we set $t(L,\mathcal{F}) = n$. (Observe that we are essentially using the same crossing convention as in knot theory.)

The framing $\mathcal{F}$ that we choose is often dictated by the topology. For example, if $[L] = 0 \in H_1(M;\mathbf{Z})$ (which is the case when $M = S^3$), then there is a compact surface $\Sigma\subset M$ with $\partial\Sigma = L$, i.e., a *Seifert surface*. Now Σ induces a framing $\mathcal{F}_\Sigma$, which is the normal framing to the 2-plane field $T\Sigma$ along L, and the *Thurston-Bennequin invariant* $tb(L)$ is given by:

$$tb(L) = t(L,\mathcal{F}_\Sigma).$$

HW 13. *Show that $tb(L)$ does not depend on the choice of Seifert surface Σ. (Hint: Consider $M = S^3 - N(L)$, where $N(L)$ is a neighborhood of L, and the boundary homomorphism $H_2(M,\partial M)\xrightarrow{\partial} H_1(\partial M) = \mathbf{Z}^2$.)*

Example: Consider (T^3,ξ_n) as before, with $n>0$ and coordinates (x,y,z). If $L = \{x = y = const\}$, then a convenient framing $\mathcal{F}$ is induced from tori $x = const$ (or equivalently from $y = const$). We have $t(L,\mathcal{F}) = -n$.

HW 14. *Prove that the Thurston-Bennequin invariant is equivalently the linking $lk(L,L')$, where L' is the push-off of L in either of the two directions: (a) normal to ξ or (b) in the direction of the normal framing.*

3.4. **Rotation number.** Given an oriented Legendrian knot L in $\mathbf{R}^3$, we define the *rotation number* $r(L)$ as follows: Choose a trivialization of ξ on $\mathbf{R}^3$. (Any oriented 2-dimensional vector bundle over a 1-dimensional simplicial complex is trivial; now $\mathbf{R}^3$ is homotopy equivalent to a point.) Then $r(L)$ is the winding number of $\dot{L}$ along L with respect to the trivialization.

HW 15. *Show that $r(L)$ does not depend on the choice of trivialization of ξ.*

The rotation number is easiest to understand in the xy-projection (also called the *Lagrangian projection*) $\Pi : \mathbf{R}^3 \to \mathbf{R}^2$, $(x, y, z) \mapsto (x, y)$. Since $\Pi_* \xi = \mathbf{R}^2$, we have a natural trivialization of ξ induced from the projection. With respect to this projection, $r(L)$ is simply the winding number of $\Pi(L) \subset \mathbf{R}^2$.

3.5. **Invariants in the front projection.** The Thurston-Bennequin invariant and rotation number of a Legendrian knot L can be computed in the front projection using the following formula:

$$\begin{aligned} tb(L) &= -\frac{1}{2}(\#\text{cusps}) + \#\text{positive crossings} \\ &\quad - \#\text{negative crossings}. \\ r(L) &= \frac{1}{2}(\#\text{downward cusps} - \#\text{upward cusps}) \end{aligned}$$

HW 16. *Prove the above formulas for tb and r in the front projection. (For example, the formula for tb can be obtained by using $tb(L) = lk(L, L')$ mentioned above and calculating the contribution of each cusp and crossing. r is more easily calculated by thinking of the (Lagrangian) projection to the xy-plane.)*

4. More on Legendrian knots

4.1. Legendrian Reidemeister moves.

Theorem 4.1. *Two Legendrian knots are isotopic iff their front projections are related by a sequence of* Legendrian Reidemeister moves*.*

The Legendrian Reidemeister moves are Legendrian analogs of the standard Reidemeister moves for knot projections. They are given by Figure 3, together with moves which are obtained by rotating these diagrams by 180 degrees about each coordinate axis.

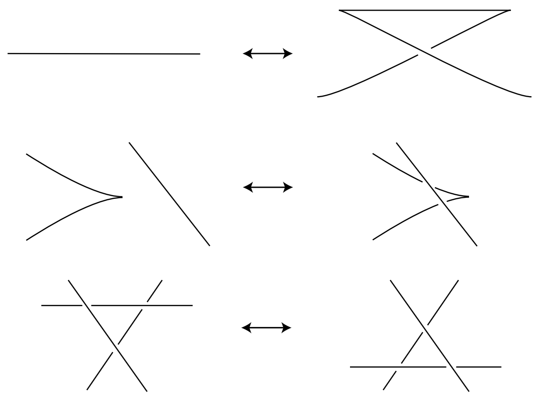

FIGURE 3. Legendrian Reidemeister moves.

HW 17. *Prove that the two Legendrian unknots in Figure 2 with $tb(L) = -2$ and $r(L) = -1$ are Legendrian isotopic by exhibiting an explicit sequence of Legendrian Reidemeister moves.*

HW 18. *Prove that a zigzag can be "passed through" a cusp, i.e., the two Legendrian arcs given in Figure 4 are Legendrian isotopic relative to their endpoints.*

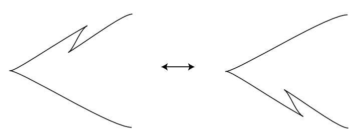

FIGURE 4. Passing a zigzag through a cusp.

4.2. C^0-approximation.

Theorem 4.2. *Any knot in any contact 3-manifold (M, ξ) can be C^0-approximated by a Legendrian knot.*

By C^0-approximation, we mean that, given a knot γ and any small $\varepsilon > 0$, there is a Legendrian knot γ_0 such that $\sup_{t\in[0,1]} |\gamma_0(t) - \gamma(t)| \leq \varepsilon$. (Suppose γ is parametrized by $t \in [0, 1]$.)

Proof. The C^0-approximation is a local result. It suffices to prove that, given an arc γ in the standard contact $(\mathbf{R}^3, \xi)$, there exists a C^0-close Legendrian approximation γ_0 of γ *relative to the endpoints* (i.e., with the endpoints fixed).

Consider $\gamma(t) = (x(t), y(t), z(t))$. Then approximate $(x(t), z(t))$ in the front projection by a zigzag curve $(x_0(t), z_0(t))$ which is C^0-close to $(x(t), z(t))$ and whose derivative is always close to $y(t)$, as in Figure 5. $\square$

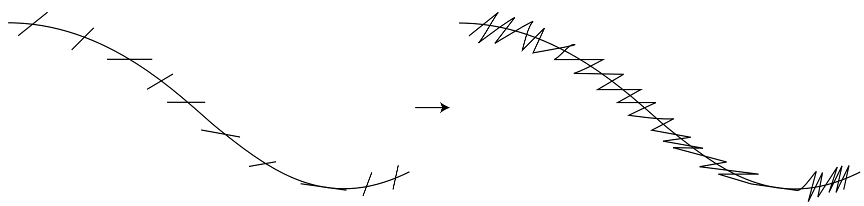

FIGURE 5. Legendrian approximation of an arc.

4.3. **Stabilization.** Given an oriented Legendrian knot L, its *positive stabilization* (resp. *negative stabilization*) $S_+(L)$ (resp. $S_-(L)$) is an operation that decreases tb by adding a zigzag in the front projection as in Figure 6.

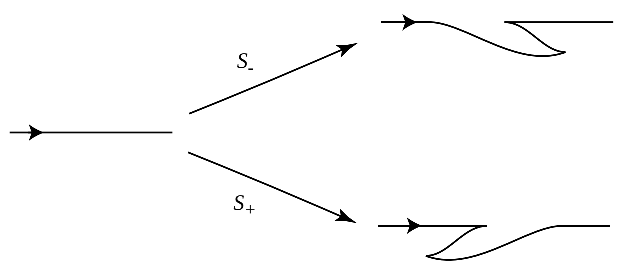

FIGURE 6. Positive and negative stabilizations.

We have $tb(S_\pm(L)) = tb(L) - 1$ and $r(S_\pm(L)) = r(L) \pm 1$.

HW 19. *Prove that the stabilization operation is well-defined (independent of the location where the zigzag is added). (Hint: Use HW 18.)*

The following theorem of Eliashberg-Fraser enumerates all the Legendrian unknots:

Theorem 4.3 (Eliashberg-Fraser)**.** *Legendrian unknots in the standard contact* $\mathbf{R}^3$ *(or* S^3*) are completely determined by* tb *and* r*.*

In fact, all the Legendrian unknots are stabilizations $S_+^{k_1} S_-^{k_2}(L_0)$ of the unique maximal tb Legendrian unknot L_0 with $tb(L_0) = -1$ and $r(L_0) = 0$, given on the left-hand side of Figure 7. The right-hand picture is $S_+^2 S_-^1(L_0)$. We will give a simple proof of Theorem 4.3 later.

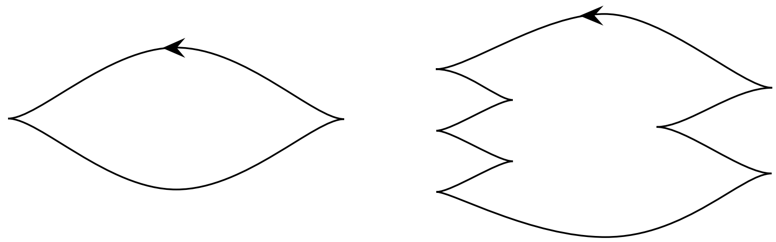

FIGURE 7. Legendrian unknots in the front projection.

For an oriented Legendrian knot in $\mathbf{R}^3$ or S^3, the topological knot type, the Thurston-Bennequin invariant, and the rotation number are called the *classical* invariants. Although Legendrian unknots are completely determined by their classical invariants according to Theorem 4.3, Legendrian knots in general are not completely classified by the classical invariants. One way of distinguishing two Legendrian knots with the same classical invariants is through *contact homology* – a topic we expect to cover later in the course.

4.4. **Transverse knots.**

Definition 4.4. *A* transverse knot *in a contact manifold (M, ξ) is a knot K that is everywhere transverse to ξ.*

Remark 4.5. *Since M and ξ are oriented, a transverse knot inherits a natural orientation.*

Given a transverse knot K in the standard $(\mathbf{R}^3, \xi)$, we may trivialize ξ, i.e., find a nowhere vanishing section s of ξ. Then the *self-linking number* $sl(K)$ is defined as $lk(K, K')$, where K' is the pushoff of K in the direction of s. (The self-linking number can also be thought of as a relative Euler/Chern class, just like the rotation number.)

HW 20. *Prove that any knot K can be C^0-approximated by a transverse knot. (Hint: one possible approach is to approximate K by a Legendrian knot L, and then approximating L by a transverse knot.)*

Remark 4.6. *Whereas a Legendrian knot has two classical invariants tb and r (besides the oriented knot type), a transverse knot only has one.*

Remark 4.7. *It turns out that transverse knot theory can be thought of as a stabilized version of Legendrian knot theory.*

4.5. **More contact structures on $\mathbf{R}^3$.** This section is meant as a preview for next week.

One of the principal tools for investigating contact structures is the study of embedded surfaces $\Sigma \subset (M, \xi)$.

Definition 4.8. *The* characteristic foliation Σ_ξ *is the singular foliation induced on Σ from ξ, where $\Sigma_\xi(p) = \xi_p \cap T_p\Sigma$. The* singular points *(or tangencies) are points $p \in \Sigma$ where $\xi_p = T_p\Sigma$.*

Keeping this in mind, we define new contact structures on $\mathbf{R}^3$:

Example: $(\mathbf{R}^3, \zeta_R)$, where $\mathbf{R}^3$ has cylindrical coordinates (r, θ, z), R is a positive real number, and ζ_R is given by $\alpha_R = \cos f_R(r) dz + r \sin f_R(r) d\theta$. Here $f_R(r)$ is a function with positive derivative satisfying $f_R(r) = r$ near $r = 0$ and $\lim_{r \to +\infty} f_R(r) = R$.

HW 21. *Show that $(\mathbf{R}^3, \xi_0) \simeq (\mathbf{R}^3, \zeta_R)$ for all $R < \frac{\pi}{2}$. (Here $\simeq$ refers to contactomorphism, and ξ_0 is the standard contact structure on $\mathbf{R}^3$.) If you are adventurous, try proving the same for $R \leq \pi$.*

However, we have the following key result of Bennequin:

Theorem 4.9 (Bennequin)**.** $(\mathbf{R}^3, \xi_0) \not\simeq (\mathbf{R}^3, \zeta_R)$ *if* $R > \pi$.

The distinguishing feature is the existence of an *overtwisted (OT) disk*, i.e., an embedded disk $D \subset (M, \xi)$ such that $\xi_p = T_p D$ at all $p \in \partial D$. A typical OT disk has a characteristic foliation given as in Figure 8. While it is not hard to see that $(\mathbf{R}^3, \zeta_R)$ has OT disks if $R > \pi$ (just look at any plane $z = const$), what Bennequin proved was that $(\mathbf{R}^3, \xi_0)$ contains no OT disks. It turns out that the existence of an OT disk is equivalent to the existence of a Legendrian unknot L with $tb(L) = 0$.

Remark 4.10. *Using the Flexibility Theorem in Section 8, one can show that if there is an OT disk in (M, ξ), there is an OT disk of the type given in Figure 8. Therefore, when we say OT disk, we will often tacitly assume (without loss of generality) that the disk is the one in Figure 8.*

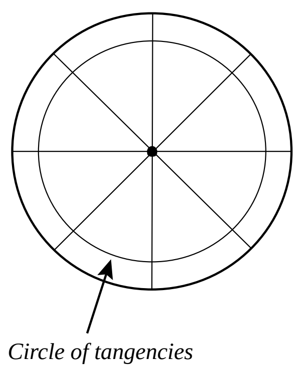

FIGURE 8. An overtwisted disk D. (Precisely speaking, the disk should end at the circle of tangencies.) The straight lines represent the singular (characteristic) foliation that $\xi \cap TD$ traces on D, and the circle is the set of points where $\xi = TD$. There is also an elliptic tangency at the center.

HW 22 (Hard)**.** *Try to prove that $(\mathbf{R}^3, \xi_0)$ has no overtwisted disks.*

It is not an exaggeration to say that modern contact geometry has its beginnings in Bennequin's theorem. There is a dichotomy in the world of contact structures, those that contain OT disks (called *overtwisted* contact structures) and those that do not (called *tight* contact structures). In view of Theorem 1.5, every contact structure is *locally tight*, and therefore the question of overtwistedness is a global one.

5. Overtwisted classification

5.1. **Homotopy classes of 2-plane fields.** Let M be a closed, oriented 3-manifold. We define $Dist(M)$ to be the set of smooth oriented 2-plane field distributions (with no integrability conditions). Then $\pi_0(Dist(M))$ is the set of homotopy classes of 2-plane fields.

Any oriented 3-manifold M is *parallelizable*, i.e., its tangent bundle TM is isomorphic to $M \times \mathbf{R}^3$ as a real vector bundle. Once such a *trivialization* is fixed, a 2-plane field ξ gives rise to a *Gauß map* $\phi_\xi : M \to Gr_2(\mathbf{R}^3)$, where $Gr_2(\mathbf{R}^3)$ is the Grassmannian of 2-planes in $\mathbf{R}^3$. If ξ is oriented and a metric is chosen on M, we choose a unit oriented normal $n(x)$ to $\xi(x)$ at every $x \in M$, and view the Gauß map as a map $\phi_\xi : M \to S^2$.

Once a trivialization of TM and a metric on M are fixed, $Dist(M)$ is in 1-1 correspondence with $Map(M, S^2)$, the set of maps from M to S^2. Since we have already described the map $\xi \mapsto \phi_\xi$, it suffices to recover a 2-plane field, given $\phi : M \to S^2$. At each point $x \in M$, simply take $(\phi(x))^\perp \subset T_xM$.

Also observing that homotopies in $Dist(M)$ are also in 1-1 correspondence with homotopies in $Map(M, S^2)$, we have the following:

Lemma 5.1. *$\pi_0(Dist(M))$ is in 1-1 correspondence with $[M, S^2]$, the homotopy classes of maps from M to S^2.*

Euler class: Let X be an m-dimensional oriented closed manifold and E a (real) oriented rank n bundle over X. Then the *Euler class* $e(E)$ of $E \to X$ is defined as follows: Take a generic section (= transverse to the zero section) s of E, and let Z be its zero set. Then $[Z] \in H_{m-n}(X; \mathbf{Z})$ and its the Poincaré dual $PD([Z]) \in H^n(X; \mathbf{Z})$ is the Euler class. In our case, ξ has rank 2, so $e(\xi) \in H^2(M; \mathbf{Z})$.

Note that we may view ξ as the pullback bundle of TS^2 under $\phi_\xi : M \to S^2$. Hence $e(\xi) = \phi_\xi^*(e(TS^2))$.

Description of $[M, S^2]$: The map $e : \pi_0(Dist(M)) \to H^2(M; \mathbf{Z})$ which maps $\xi \mapsto e(\xi)$ is surjective. Given $\alpha \in H^2(M; \mathbf{Z})$, we define its *divisibility* $d(\alpha)$ as follows: $d(\alpha) = 0$ if α is a torsion class (there is an integer $n \neq 0$ for which $n\alpha = 0$; α may be zero). If α is not torsion, then $d(\alpha)$ is the largest integer d for which $\alpha = d \cdot \beta$ for $\beta \in H^2(M; \mathbf{Z})/Torsion$. (Here the equation is viewed as an equation in $H^2(M; \mathbf{Z})/Torsion$.)

Proposition 5.2. $e^{-1}(\alpha) \simeq \mathbf{Z}/2d(\alpha)\mathbf{Z}$.

We omit the proof.

5.2. **Holonomy.** We start by briefly discussing *holonomy*. The analysis of holonomy will become important for the following extension problem:

Problem 5.3. *Suppose a contact structure ξ is defined in a neighborhood of the boundary ∂V of a solid torus V. How can one extend ξ to the interior of V?*

Consider the solid torus $V = \mathbf{R}/\mathbf{Z} \times D^2$ with coordinates $(z, (r, \theta))$ and boundary $\partial V = \mathbf{R}/\mathbf{Z} \times S^1$. (Here $D^2 = \{r \leq 1\}$.) Suppose ξ is defined in a neighborhood of ∂V subject to the following conditions:

(1) $\xi \pitchfork \partial V$,
(2) the characteristic foliation ∂V_ξ is transverse to ∂_z.

Pick a point $(r,\theta) = (1,0) \in \partial D^2$, and cut ∂V along $\mathbf{R}/\mathbf{Z} \times \{(1,0)\}$ to obtain $\mathbf{R}/\mathbf{Z} \times [0,2\pi]$ (oriented in the same way as ∂V). Then take the universal cover $\pi : \mathbf{R} \times [0,2\pi] \to \mathbf{R}/\mathbf{Z} \times [0,2\pi]$. Given $(z,0) \in \mathbf{R} \times [0,2\pi]$, there exists a unique integral curve $(\gamma_z(\theta),\theta)$, $\theta \in [0,2\pi]$, of the characteristic foliation, with $\gamma_z(0) = z$. We then define the *holonomy diffeomorphism* $hol : \mathbf{R} \to \mathbf{R}$ to be $hol(z) \stackrel{def}{=} \gamma_z(2\pi)$. ($hol$ depends on the choice of basepoint on ∂D^2.)

The holonomy diffeomorphism is an element of $\widetilde{\mathrm{Diff}^+}(\mathbf{R}/\mathbf{Z})$, the universal cover of the group of orientation-preserving diffeomorphisms of $\mathbf{R}/\mathbf{Z}$, or, equivalently, the group of periodic orientation-preserving diffeomorphisms of $\mathbf{R}$.

We say a diffeomorphism $f : \mathbf{R} \to \mathbf{R}$ is *positive* (resp. *negative*) if $f(z) > z$ (resp. $f(z) < z$) for all $z \in \mathbf{R}$. A characteristic foliation ∂V_ξ has *positive* (resp. *negative*) holonomy if hol_p is positive (resp. negative).

HW 23. *Prove that positivity/negativity is independent of the choice of basepoint on ∂D^2.*

Lemma 5.4. *Suppose ξ is defined in a neighborhood of ∂V, $\xi \pitchfork \partial V$, and $\partial V_\xi \pitchfork \partial_z$. If ∂V_ξ has negative holonomy, then there exists an extension of ξ to V transverse to ∂_z.*

Proof. Use coordinates $(z,(r,\theta))$ on $V = \mathbf{R}/\mathbf{Z} \times D^2$. The negativity of the holonomy implies that the characteristic foliation ∂V_ξ can be conjugated to the kernel of $dz + g(z,r,\theta)d\theta$, where $g(z,1,\theta) > 0$ for all $(z,1,\theta)$. We can then extend g to all of V, with the condition that $\frac{\partial g}{\partial r} > 0$ and $g(z,r,\theta) = r^2$ for r sufficiently small (this is necessary for differentiability). $\square$

HW 24. *Verify that the extension is indeed contact.*

Remark 5.5. *Observe that the extension technique consisted of finding a line field transverse to the surface and then converting the line field into a Legendrian one while forcing the contact structure to twist along the Legendrian line field in a 'propeller fashion'.*

5.3. **Existence of contact structures.** Let $Cont(M)$ be the set of smooth oriented contact 2-plane field distributions. In the 1970's, Lutz and Martinet proved the following existence theorem for contact structures in every homotopy class of 2-plane fields:

Theorem 5.6 (Lutz, Martinet)**.** *Let M be a closed oriented 3-manifold. Then*

$$\pi_0(Cont(M)) \to \pi_0(Dist(M))$$

is surjective.

Sketch of Proof.

(1) Start with a 2-plane field ξ. Take a fine enough triangulation τ of M so that ξ is close to a linear foliation by planes on each 3-simplex. After subdividing if necessary, the union of any two adjacent 3-simplices can be embedded (polygonally) in $\mathbf{R}^3$ so that ξ is very close to $dz = 0$.

Claim. *There exists a triangulation τ of M and a perturbation of ξ so that ξ is transverse to the 1-simplices and the 2-simplices.*

Let Δ be a 3-simplex in $\mathbf{R}^3$. If the slope of the line between two vertices is sufficiently far from 0, then the line will be transverse to ξ (which is close to $dz = 0$). If not, and an edge $\delta \subset \Delta$ almost parallel to $z = const$ has a tangency (which we assume is generic), then apply a "crystalline" subdivision process. Namely, subdivide δ at a tangency p in $int(\delta)$ and form two 3-simplices out of Δ by cutting through by the plane P, which passes through p and the edge of Δ which does not intersect δ. Apply the subdivision to all Δ' which share the same edge δ. We can also apply this cystalline subdivision process to the 2-simplices.

(2) Homotop ξ near the 2-skeleton so it becomes contact.

HW 25. *Prove the homotopy of ξ to a contact structure successively on the 0-, 1-, and 2-simplices. (Hint: A useful technique is to foliate the neighborhood of a 3-simplex (viewed as sitting in $\mathbf{R}^3$) by parallel planes L which are (1) transverse to ξ and (2) transverse to the edges and faces of the 3-simplex. Then $\xi \cap L$ is a line field, and use Remark 5.5.)*

(3) Now we have an extension problem to the interior of each 3-simplex. We have a 3-ball whose boundary has a characteristic foliation which is C^∞-close to a horizontal one, with two singular points (one each at the north and south poles). Instead of trying to extend over the 3-ball (which is harder), we use a trick which changes the problem to an extension problem over the solid torus.

Let δ be a transverse arc from the north pole to the south pole such that $\delta \subset$ the portion of the neighborhood of the 2-skeleton which has already been made contact – recall any arc can be made into a transverse arc (relative to the endpoints), by the discussion from last time! The union of a thickening of δ and the 3-ball gives a solid torus $S^1 \times D^2$ such that the contact structure is transverse to boundary $\partial(S^1 \times D^2)$ and the 2-plane field in the interior is isotopic to the foliation by planes $pt \times D^2$.

(4) We now examine the holonomy on the boundary of the solid torus. If the holonomy is negative, we can use Lemma 5.4. Otherwise, we need to introduce extra twisting in the form of a *Lutz tube*. A Lutz tube is a contact structure on $S^1 \times D^2$ (with cylindrical coordinates (z, r, θ), where $D^2 = \{(r, \theta) | r \leq 1\}$) given by the 1-form

$$\alpha = \cos(2\pi r)dz + r\sin(2\pi r)d\theta.$$

$\square$

HW 26. *Prove that the homotopy class of the 2-plane field is preserved when a Lutz tube is inserted ("a Lutz twist is performed"). Also show that a* half Lutz twist *given by $\alpha = \cos(\pi r)dz + r\sin(\pi r)d\theta$ can change the homotopy class, but a full one leaves the homotopy type invariant.*

5.4. **Eliashberg's overtwisted classification.** The *overtwisted* classification (on closed 3-manifolds) was shown by Eliashberg to be essentially the same as the homotopy classification of 2-plane fields. (The result is quite striking, especially when contrasted with the tight classification on T^3 below.)

First define $Cont^{OT}(M) \subset Cont(M)$ to be the set of overtwisted contact structures. Now let (Δ, ζ) be an overtwisted disk, together with a contact structure ζ whose characteristic foliation Δ_ζ is the standard one from Figure 8. Then let $Cont^{OT}(M, \Delta)$ be the set of overtwisted contact structures which agree with ζ on Δ. Also let $Dist(M, \Delta)$ be the set of 2-plane fields which agree with ζ at the center (elliptic tangency) of Δ.

Theorem 5.7 (Eliashberg)**.** *Let M be a closed oriented 3-manifold. Then*

$$Cont^{OT}(M, \Delta)) \hookrightarrow Dist(M, \Delta)$$

is a weak homotopy equivalence.

A *weak homotopy equivalence* $f : X \to Y$ is a continuous map for which $f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism for all n. If X and Y are CW complexes (which is the case here), then a weak homotopy equivalence implies homotopy equivalence.

We will give a sketch of the following:

Theorem 5.8. $\pi_0(Cont^{OT}(M, \Delta)) \to \pi_0(Dist(M, \Delta))$ *is injective.*

The surjectivity of the map was the content of the Lutz-Martinet theorem. (Performing a full Lutz twist guarantees the overtwistedness of the constructed contact structure.)

Sketch of Proof. We can think of this as the 1-parameter version of the Lutz-Martinet theorem. Let ξ_s, $s \in [0, 1]$, be a family of 2-plane fields, where ξ_0, ξ_1 are overtwisted, $\xi_0 = \xi_1 = \zeta$ along the overtwisted disk Δ, and ξ_s, $s \in [0, 1]$, agree at the center tangency of Δ.

(1) First we deform this family while fixing ξ_0, ξ_1 so that the ξ_s all agree on Δ. This is simply a homotopy question: Let $\phi_{\xi_s} : \Delta \to S^2$ be the Gauß map, where the trivialization of TM is independent of s. If $c \in \Delta$ is the center of disk, then $\phi_{\xi_s}(c)$ is a fixed point p. Then the deformation of the family ξ_s near Δ is obtained by deformation retracting $\phi_{\xi_s} : \Delta \to S^2$ to a constant map that maps to p (away from $s = 0, 1$) and "reinflating" so that the new map coincides with $\phi_{\xi_0} = \phi_{\xi_1}$.

(2) The goal is to homotop ξ_s relative to ξ_0, ξ_1 so that ξ_s satisfies:

- (i) Away from a finite union of disjoint balls B_i, $i = 1, \dots, k$, (disjoint from Δ), ξ_s is contact.
- (ii) ξ_s is *almost horizontal* along ∂B_i for all i and s.

Here, a contact structure ξ is *almost horizontal* along S^2 if the characteristic foliation has exactly two tangencies (of opposite signs – let us call N the "north pole" if the orientations on ξ and TS^2 agree, and S the "south pole" if they do not) and an oriented transversal to every closed orbit moves away from S towards N.

We want to apply the homotopy step by step in a relative manner, i.e., the region that has already been made contact remains untouched (of course, a small neighborhood of the boundary of the region must be modified). We may assume that M has been subdivided into small pieces (cells, triangles, or cubes – we'll use cubes here), so that on each cube C, viewed as the unit cube in $\mathbf{R}^3$, $||\phi_{\xi_s}|| < \frac{\varepsilon}{k}$ for all s, where $k >> 0$. Here, $||\phi_{\xi_s}||$ is the supremum norm (over C) of the derivative of the Gauß map $\phi_{\xi_s} : C \to S^2$.

Observe that if S is a sphere in C whose principal curvatures are larger than $\frac{\varepsilon}{k}$, then the characteristic foliation of S is almost horizontal.

Now let S be a sphere which approximates ∂C and has principal curvatures $\geq \varepsilon$. Near the singular points, we perturb $\xi = \xi_s$ so it becomes contact (if it is not contact already). In a neighborhood $S \times [-1, 1]$, ξ is given by $\alpha = f_t dt + \beta_t$. The contact condition is:

$$f_t d\beta_t + \beta_t \wedge (df_t + \dot{\beta}_t) > 0.$$

We leave the 2-plane field untouched where it is already contact; otherwise modify $\dot{\beta}_t$ (use a bump function) away from the singular points of S so the contact condition is met, i.e., $\beta_t \wedge \dot{\beta}_t$ is sufficiently large.

Warning: This is where things get a little tricky! I have not verified the argument below to my satisfaction....

In order for the induction to work properly, we need a wide enough $S \times I$ whose image contains ∂C. I think we can do this by sacrificing $||\phi_{\xi_s}||$ a bit – now $||\phi_{\xi_s}|| < \frac{\varepsilon}{k'}$, where $k > k' >> 0$. Since we can make these modifications simultaneously on nonoverlapping C, we only need a finite number of steps. Observe that all of the extension can be done simultaneously for s in a family.

(3) For each pair B_i, B_{i+1}, take a 1-parameter family of arcs δ_t^i that connect the north pole of ∂B_i to the south pole of B_{i+1} (in the complement of the all the balls). Using the parametric version of the transverse approximation theorem, we can approximate the 1-parameter family of arcs by a 1-parameter family of transverse arcs. Finally, connect B_k to a neighborhood B_0 of the overtwisted disk Δ. Let B be the union of the B_i, together with the neighborhoods of the δ_t^i. What's left is an extension problem to the interior of B, where the characteristic foliation on ∂B consists of an almost horizontal portion and a portion that's not (coming from the overtwisted disk). The extension is an explicit construction.

HW 27. *Find an explicit model inside a neighborhood of an overtwisted disk.*

□

On the other hand, tight contact structures tend to reflect the underlying topology of the manifold, and are more difficult to understand. The goal of this course is to introduce techniques which enable us to better understand tight contact structures.

In the meantime, we list a couple of examples:

(1) S^3. Eliashberg proved that there is a unique tight contact structure up to isotopy (the standard contact structure on S^3).
(2) T^3. Giroux and Kanda independently proved that (a) every tight contact structure is isomorphic to some ξ_n and (b) $(T^3, \xi_m) \not\simeq (T^3, \xi_n)$ if $m \neq n$.

HW 28. *(Hard) Try to prove that* $(T^3, \xi_m) \not\simeq (T^3, \xi_n)$ *if* $m \neq n$.

6. Symplectic filling

A (positive) contact structure (M, ξ) is *(weakly) symplectically fillable* if there exists a compact symplectic 4-manifold (X, ω) such that:

(1) $\partial X = M$ and the boundary orientation on ∂X (induced from ω^2 on X) and the orientation on M agree, and
(2) $\omega|_\xi > 0$, i.e., if Y_1, Y_2 form an oriented basis for ξ at a point $x \in M$, then $\omega(x)(Y_1, Y_2) > 0$.

(X, ω) is said to be a *symplectic filling* of (M, ξ). (We will often simply say "fillable".)

Example: The contact structures (T^3, ξ_n), $n \in \mathbf{Z}^+$ are symplectically fillable. ξ_n is given by the 1-form $\alpha_n = \sin(2\pi nz)dx + \cos(2\pi nz)dy$, where $T^3 = \mathbf{R}^3/\mathbf{Z}^3$ has coordintes (x, y, z). Consider $\beta_t = dz + t\alpha_n$. Then we verify that β_t is contact for $t > 0$:

$$\beta_t \wedge d\beta_t = (dz + t\alpha_n) \wedge td\alpha_n = t^2\alpha_n \wedge d\alpha_n > 0.$$

The corresponding 2-plane fields ζ_t are contact for all $t > 0$ and $\zeta_t \to \xi_n$ as $t \to \infty$. Hence ξ_n and ζ_t are contact isotopic. $\zeta_0 = ker(dz)$ is the foliation by planes parallel to the xy-plane.

Now consider $T^3 = S^1 \times T^2$ as the boundary of $D^2 \times T^2$ with symplectic form $\omega = \omega_{D^2} + dxdy$, where ω_{D^2} is an area form on D^2. Then $\omega(\partial_x, \partial_y) = 1$, i.e., $\omega|_{\zeta_0} > 0$. Therefore, $\omega|_{\zeta_t} > 0$ for t small, and ξ_n is symplectically fillable.

Example: The standard (S^3, ξ) is fillable. Let S^3 be the boundary $\{|z_1|^2 + |z_2|^2 = 1\}$ of $B^4 = \{|z_1|^2 + |z_2|^2 \leq 1\} \subset \mathbf{C}^2$, where $\mathbf{C}^2$ has complex coordinates $(z_1 = x_1 + iy_1, z_2 = x_2 + iy_2)$. Let $\omega = \sum dx_i dy_i$ be the symplectic form on $\mathbf{C}^2$. The contact 1-form on M is the pullback to S^3 of the primitive $\alpha = \frac{1}{2}\sum(-y_i dx_i + x_i dy_i)$ of the symplectic form ω.

Detour: Since ω and J are *compatible*, if $X = (x_1, y_1, x_2, y_2)$ is the unit normal to S^3 at $x = (x_1, y_1, x_2, y_2)$, then $JX = (-y_1, x_1, -y_2, x_2) \in T_xS^3$ is the Reeb vector field for the 1-form α (you can verify this directly). Moreover, $\xi = (JX)^\perp \subset TS^3$. Let Y, JY form an oriented orthonormal basis for ξ (with respect to the compatible metric $\sum(dx_i \otimes dx_i + dy_i \otimes dy_i)$). Then $\alpha \wedge d\alpha = i_X d\alpha \wedge d\alpha$, and

$$(\alpha \wedge d\alpha)(JX, Y, JY) = \omega^2(X, JX, Y, JY) > 0.$$

Hence we verify the positive contact condition.

Now, we have the following:

HW 29. *Let (M^3, ξ) be the oriented boundary of (X^4, ω), and $i : M \to X$ be the inclusion map. If there is a 1-form α on M such that* $\ker\alpha = \xi$ *and $d\alpha = i^*\omega$, then $\omega|_\xi > 0$. (Such a condition is called* strong symplectic fillability.*) (Solution: Let R be the Reeb vector field for α and X_1, X_2 be an oriented basis for ξ. Then $(\alpha \wedge d\alpha)(R, X_1, X_2) = \alpha(R)d\alpha(X_1, X_2) = d\alpha(X_1, X_2) = \omega(X_1, X_2) > 0$.)*

A powerful general method for producing tight contact structures is the following theorem of Gromov and Eliashberg:

Theorem 6.1 (Gromov-Eliashberg)**.** *A symplectically fillable contact structure is tight.*

It immediately follows from the symplectic filling theorem that the standard (S^3, ξ) and the contact structures (T^3, ξ_n) are tight.

The following is an outline of the proof, with many technicalities left out....

Outline of proof. Assume to the contrary that there is an overtwisted disk Δ in (M, ξ) of the form given in Figure 8. The tangency at the center is called an *elliptic tangency*. (For more information on characteristic foliations and tangencies, see the next section.)

Let (X, ω) be a filling of (M, ξ) and let J be an almost complex structure (vector bundle map $J : TX \to TX$ so that $J^2 = -id$) on X which is *tamed* by ω, namely $\omega(X, JX) > 0$ for all tangent vectors $X \neq 0$. (Taming is a generalization of the notion of J being *compatible*, which says that $\omega(X, JY)$ is a Riemannian metric.) We also require that J leave ξ invariant. A convenient J also sends $n \mapsto R_\alpha$ and $R_\alpha \mapsto -n$, where n is the outward normal to M and R_α is a Reeb vector field for a contact 1-form α (in the case (M, ξ) is strongly fillable, α such that $d\alpha = \omega$ would be a good choice – this way, a neighborhood $[-\varepsilon, \varepsilon] \times M$ of M would agree with a *symplectization* of M, namely a symplectic form $d(e^t \alpha)$, where t is the coordinate on $[-\varepsilon, \varepsilon]$.

Step 1: (Bishop's Theorem) Let p be an elliptic tangency of a real 2-dimensional manifold S embedded in $\mathbf{C}^2$ (namely T_pS is a complex subspace of $\mathbf{C}^2$). Then there exists a family of J-holomorphic disks $\phi_s : D \to \mathbf{C}^2$, $s \in [0, \varepsilon]$, so that $\phi_s(\partial D)$ fill a neighborhood of p in S. (That is to say that $N(p) - p$ is foliated by $\phi_s(\partial D)$ and $\phi_s(D)$ are pairwise disjoint in $\mathbf{C}^2$.)

Prototype: Using complex coordinates z, w, write S near p as $z = w\overline{w} = x_1^2 + y_1^2$. Then intersect with planes $z = a$, where $a \in \mathbf{R}$ and $a \in [0, \varepsilon]$. The intersections, for $a > 0$, are circles which bound disks in $z = a$. (The filling is easy since $x_1^2 + y_1^2$ is real.) In the general case we have $z = w\overline{w} + 2\beta Re(w^2) + O(w^3)$, which is harder.

Apply Bishop's Theorem to the elliptic point of the overtwisted disk – actually, by modifying the overtwisted disk near the elliptic point so the tangency is *exactly* $z = x_1^2 + y_1^2$, we just need the prototype. To do this, we need to make sure the tamed almost complex structure J is an integrable one in a neighborhood of $p \in X$.

Step 2: Grow the family $\phi_s : D \to (X, \xi)$ so that $\phi_s(\partial D) \subset \Delta$ and are disjoint for all t.

Fact: Given a holomorphic disk $\phi_{s_0} : D \to (X, \xi)$, there exists a family $\phi_s : D \to (X, \xi)$ of holomorphic disks with $s \in (s_0 - \varepsilon, s_0 + \varepsilon)$, $\phi_s(\partial D) \subset \Delta$, and such that $\phi_s(D)$ are pairwise disjoint.

There are two things involved here. The first is an index computation (the Fredholm index of the linearized operator – the derivative map of the Cauchy-Riemann operator – is 4, but the symmetries of the disk cut the dimension of the family to desired $4 - 3 = 1$) and an argument showing that the linearized operator is a *surjective* Fredholm map. It is important to note that the surjectivity of the Fredholm map is not automatic in other contexts. The other ingredient is the positivity of intersections of holomorphic curves in a symplectic manifold of real dimension 4. The initial "germ" of the holomorphic family consists of pairwise disjoint disks; this persists even when we extend the family.

Step 3: The "J-convexity" of the boundary implies a maximal principle. Let $[-\varepsilon,\varepsilon]\times M$ be a neighborhood of M, with coordinates (t,x). Then if $\phi : D \to X$ is a J-holomorphic disk and D has coordinates (u,v), then $\phi(u,v) = (t(u,v), x(u,v))$ and one can compute that $t(u,v)$ is a *subharmonic function*, i.e, satisfies an inequality of the form $\Delta t = \frac{\partial^2 t}{\partial u^2} + \frac{\partial^2 t}{\partial v^2} \geq 0$, and moreover Δt is not identically zero. Notice that $t|_{\partial D} = 0$. Such a function satisfies the (strong) maximum principle, namely: (1) it attains the maximum only along the boundary, and (2) the derivative of t in the outward normal direction to the boundary of the disk is positive.

This implies two things: (i) no holomorphic disk ϕ_s touches M at an interior point of D and (ii) the holomorphic disks ϕ_s intersect M transversely and along ∂D. Hence, $(\phi_s)_*(T_pD)$ cannot coincide with ξ along ∂D. Since ξ is a complex plane, there can be no nontrivial intersection between the tangent plane to the disk and ξ. Therefore, the $\phi_s(\partial D)$ are transverse to the characteristic foliation Δ_ξ.

Step 4: Now apply the *Gromov compactness theorem* for the family of J-holomorphic curves $\phi_s : D \to X$ with $\phi_s(\partial D) \subset \Delta$ and $s \to s_0$. There are two types of bubbling (places $p \in D$ where the gradients $\nabla\phi_s(p)$ blow up) – bubbling in the interior or bubbling on the boundary. There can be no bubbling in the interior for index reasons (for generic tamed J away from Δ). One can also alternatively argue in the case $\omega = d\alpha$ on all of X, which is the case, for example, when $X = B^4$ and $M = S^3$: Given a holomorphic sphere $S^2 \subset X$, $\int_{S^2}\omega > 0$ by holomorphicity and taming, whereas $\int_{S^2}\omega = \int_{\partial S^2}\alpha = 0$, a contradiction. There can also be no boundary bubbling since $\phi_s(\partial D)$ is always transverse to Δ_ξ, and, as long as we stay a bounded distance away from $\partial\Delta$, there is also a minimal angle that $\phi_s(\partial D)$ can make with Δ_ξ. Hence the holomorphic curves ϕ_s converge to a holomorphic disk $\phi_{s_0} : D \to X$ which is embedded. We need to argue that ϕ_{s_0} is smooth along ∂D, but assuming that's done, we can further propagate ϕ_s to $s \in [s_0 - \varepsilon, s_0 + \varepsilon]$ by Step 2. If we assumed that ϕ_{s_0} was the last disk, then we would have a contradiction, unless ϕ_{s_0} touched $\partial\Delta$, also a contradiction by Step 3.

$\square$

Symplectic filling is a 4-dimensional way of checking whether (M,ξ) is tight. We will discuss other methods (including a purely 3-dimensional one) of proving tightness subsequently.

7. Convex surfaces

In this section, we investigate embedded surfaces Σ in the contact manifold (M, ξ). The principal notion is that of *convexity*. For the time being, ξ may be tight or overtwisted.

7.1. **Characteristic foliations.**

Before discussing convexity, we first examine how ξ traces a singular line field on an embedded surface Σ.

Definition 7.1. *The* characteristic foliation Σ_ξ *is the singular foliation induced on* Σ *from* ξ*, where* $\Sigma_\xi(p) = \xi_p \cap T_p\Sigma$*. The* singular points *(or tangencies) are points* $p \in \Sigma$ *where* $\xi_p = T_p\Sigma$*.*

Lemma 7.2. *For a* C^∞*-generic closed oriented surface* $\Sigma \subset (M, \xi)$*, its characteristic foliation* Σ_ξ *is of* Morse-Smale type*, i.e., satisfies the following:*

1. *the singularities and closed orbits are hyperbolic in the dynamical systems sense,*
2. *there are no saddle-saddle connections, and*
3. *every point* $p \in \Sigma$ *limits to some isolated singularity or closed orbit in forward time and likewise in backward time.*

See below for a more detailed description and diagrams of the singularities and closed orbits.

Proof. First observe that a C^∞-small perturbation of ξ is still contact.

HW 30. *Show that if* α *is a contact 1-form, then* $\alpha + \beta$ *is contact if* β *is sufficiently small in the* C^1*-topology.*

By a theorem of Peixoto, a C^∞-generic vector field on a closed oriented surface Σ is of Morse-Smale type. Hence, given a contact 1-form α for ξ, we may choose a perturbation β to be compactly supported near Σ, so that the characteristic foliation induced by ξ' (from $\alpha + \beta$) on Σ is Morse-Smale. Now use Gray's theorem to construct an isotopy ϕ_t so that $\phi_0 = id$, $\phi_1(\xi') = \xi$, and the isotopy is compactly supported near Σ. Hence a characteristic foliation Σ'_ξ on a surface Σ' which is C^∞-close to Σ is Morse-Smale. □

Remark 7.3. *A Morse-Smale vector field is a generalization of the (generic) gradient vector field of a Morse function.*

Description of singularities and orbits: There are two types of isolated singularities which are hyperbolic (in the dynamical systems sense): *elliptic* and *hyperbolic* (not in the dynamical systems sense). Choose coordinates (x, y) on Σ and let the origin be the singular point. If we write $\alpha = dz + f dx + g dy$, then $X = g\frac{\partial}{\partial x} - f\frac{\partial}{\partial y}$ is a vector field for the characteristic foliation near the origin. If the determinant of the matrix

$$\begin{pmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ -\frac{\partial f}{\partial x} & -\frac{\partial f}{\partial y} \end{pmatrix}$$

is positive (resp. negative), then the singular point is elliptic (resp. hyperbolic). An example of an elliptic singularity is $\alpha = dz + (xdy - ydx)$, and an example of a hyperbolic singularity is $\alpha = dz + (2xdy + ydx)$.

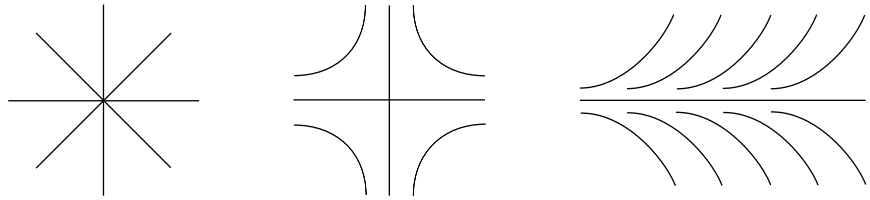

FIGURE 9. An elliptic singularity, a hyperbolic singularity, and a hyperbolic closed orbit. For the hyperbolic orbit, we are assuming that the left and the right are identified so we get an actual closed orbit.

We now define a *hyperbolic closed orbit*: Let γ be a closed orbit and $S^1 \times [-\varepsilon, \varepsilon]$ be an annular neighborhood (with $\gamma = S^1 \times \{0\}$) so that $\theta \times [-\varepsilon, \varepsilon]$ is transverse to γ for all $\theta \in S^1$. If we fix a section $\theta = 0$, then the *return map* $\Phi : [-\varepsilon, \varepsilon] \dashrightarrow [-\varepsilon, \varepsilon]$ (defined on a neighborhood of $0 \in [-\varepsilon, \varepsilon]$) maps $s \mapsto t$, where $(0, t)$ is the subsequent point that the orbit through $(0, s)$ (inside $S^1 \times [-\varepsilon, \varepsilon]$) intersects $\theta = 0$. (If the orbit leaves $S^1 \times [-\varepsilon, \varepsilon]$, the $\Phi(s)$ is not defined.) We say γ is *hyperbolic* if $d\Phi(0)$ is not the identity.

Signs. Assume Σ and ξ are both oriented. Then a singular point p is *positive* (resp. *negative*) if $T_p\Sigma$ and ξ_p have the same orientation (resp. opposite orientations).

Claim. *The characteristic foliation Σ_ξ is oriented.*

We use the convention that positive elliptic points are sources and negative elliptic points are sinks. If p is a nonsingular point of a leaf L of the characteristic foliation, then we choose $v \in T_pL$ so that (v, n) is an oriented basis for $T_p\Sigma$. Here $n \in T_p\Sigma$ is an oriented normal vector to ξ_p.

Examples of characteristic foliations:

(1) Consider $S^2 = \{x^2 + y^2 + z^2 = 1\} \subset (\mathbf{R}^3, \zeta_{\pi/2})$. Then S^2 will have two singular points, the positive elliptic point $(0, 0, 1)$ and the negative elliptic point $(0, 0, -1)$, and the leaves spiral downward from $(0, 0, 1)$ to $(0, 0, -1)$.
(2) An example of an overtwisted disk D is one which has a positive elliptic point at the center and radial leaves emanating from the center, such that ∂D is a circle of singularities. Often in the literature one sees overtwisted disks whose boundary is transverse to ξ and whose leaves emanating from the center spiral towards the limit cycle ∂D. (Strictly speaking, such a D with a limit cycle is not an OT disk according to our definition, but can easily be modified to fit our definition.)

The importance of the characteristic foliation Σ_ξ comes from the following proposition:

Proposition 7.4. *Let ξ_0 and ξ_1 be two contact structures which induce the same characteristic foliation on an oriented surface Σ. Then there is an isotopy φ_t, $t \in [0, 1]$, rel Σ, with $\varphi_0 = id$ and $(\varphi_1)_*\xi_0 = \xi_1$.*

Proof. Let $\Sigma\times[-\varepsilon,\varepsilon]$ be a neighborhood of $\Sigma=\Sigma\times\{0\}$, and write the 1-forms α_i for ξ_i ($i=0,1$) as $\alpha_i=f_i dt+\beta_i$, where f_i is a function, $\Sigma\times[-\varepsilon,\varepsilon]$ has coordinates (x,t), $\beta_i(x,t)$ is a 1-form on Σ for fixed t, and f_i, β_i depend on t.

We would like to say that since $\Sigma_{\xi_0}=\Sigma_{\xi_1}$, we may take $\beta_0=u\beta_1$ along $t=0$, where u is nowhere vanishing funciton. This is certainly the case away from the common zero sets of β_0 and β_1, but I'm a little uneasy near the zero set, so we'll do the following instead, which would ensure that $\beta_0=u\beta_1$: Without loss of generality we may assume that $f_i=1$ in a neighborhood of $Z=\{\beta_0=0\}=\{\beta_1=0\}$ (since f_i cannot vanish on Z). We can then interpolate by taking $\alpha_s=(1-s)\alpha_0+s\alpha_1$, which takes the form $dt+\beta_s$ (with $\beta_s=(1-s)\beta_0+s\beta_1$) near Z. We then use the Moser technique – solve for Y (with only a Σ-component) in the equation:

$$\mathcal{L}_Y\alpha_s=i_Y d\beta_s+d(\alpha_s(Y))=\frac{d\alpha_s}{ds}=\beta_1-\beta_0.$$

If Y satisfies $i_Y d\beta_s=\beta_1-\beta_0$, then $d\beta_s(Y,Y')=0$ for any Y' in the characteristic foliation; since $d\beta_s$ is an area form, Y must be tangent to the characteristic foliation. By making this local modification, we may take $\beta_0=\beta_1$ near Z.

Once $\beta_0=u\beta_1$, we may divide α_0 by u and assume that $\beta_0=\beta_1=\beta$ along $t=0$. We can easily verify that the interpolation $\alpha_s=(1-s)\alpha_0+s\alpha_1$, $s\in[0,1]$, is contact. [$\alpha_i\wedge d\alpha_i=(f_i dt+\beta_i)\wedge(df_i dt+d\beta_i)=dt(\beta_i\wedge df_i+f_i d\beta_i)>0$. Along Σ, $\alpha_i\wedge d\alpha_i=dt(\beta\wedge df_i+f_i d\beta)>0$, which is additive.]

We again(!) use the Moser technique, while ensuring that $\phi_s(\Sigma)=\Sigma$ under the isotopy ϕ_s. Solve for X_s in the equation:

$$\mathcal{L}_{X_s}\alpha_s=i_{X_s}d\alpha_s+d(\alpha_s(X_s))=\frac{d\alpha_s}{ds}+\phi_s\alpha_s. \tag{1}$$

Here we are free to choose ϕ_s. Splitting $X_s=g_sR_s+Y_s$ in the proof of Gray's theorem, where R_s is the Reeb vector field for α_s and $Y_s\in\xi_s$, we take $g_s=0$, and then solve for Y_s (there is a unique solution) in:

$$i_{Y_s}d\alpha_s=\frac{d\alpha_s}{ds}+\phi_s\alpha_s.$$

From this formula we see that, if X' is a vector tangent to the characteristic foliation, then $d\alpha_s(Y_s,X')=0$. Hence, $Y_s\in\xi_s\cap T\Sigma$. This proves the proposition. □

7.2. **Convexity.** The notion of a *convex surface*, introduced by Giroux and extended to the case of a compact surface with Legendrian boundary by Kanda, is the key ingredient in the cut-and-paste theory of contact structures.

Definition 7.5. *A properly embedded oriented surface Σ is* convex *if there exists a contact vector field $v\pitchfork\Sigma$.*

Remark 7.6. *We assume that our convex surfaces are either* closed *or* compact with Legendrian boundary.

If $\Sigma=\Sigma\times\{0\}$ is convex, then there is an invariant neighborhood $\Sigma\times[-\varepsilon,\varepsilon]\subset M$ where ∂_t is a contact vector field. (Here t is the coordinate for $[-\varepsilon,\varepsilon]$.) Hence the contact structure remains invariant under translations by t. We usually assume that v agrees with the normal orientation to Σ.

HW 31. *Prove that if there is a contact vector field v transverse to Σ, defined* only on a neighborhood of Σ, *then there is a contact vector field $\tilde{v}$ transverse to Σ and defined on all of M.*

HW 32. *Prove that if Σ is convex, then there is an* **R**-*invariant neighborhood $\Sigma \times \mathbf{R}$ of Σ so that the contact structure is the kernel of a 1-form $\alpha = fdt + \beta$, where t is the coordinate for* **R**, *β is a 1-form on Σ, and the function f and β have no t-dependence.*

7.3. **Properties of convex surfaces.**

Definition 7.7. *Given a convex surface Σ and a contact vector field v transverse to Σ, let $\Gamma_\Sigma \stackrel{def}{=} \{x \in \Sigma | v(x) \in \xi_x\}$ be its* dividing set. *Write $\#\Gamma_\Sigma$ for the number of connected components of Γ_Σ.*

Remark 7.8. *We may think of Γ_Σ as the set of points where $\xi \perp \Sigma$, where $\perp$ is measured with respect to v.*

If $\Sigma \times \mathbf{R}$ is the invariant neighborhood of a convex surface Σ and $\alpha = fdt + \beta$ as in HW 32, then Γ_Σ (with respect to the contact vector field ∂_t) is the zero set of f.

1. Γ_Σ *is a multicurve, i.e., a properly embedded (smooth) 1-manifold, possibly disconnected and possibly with boundary.*

Proof. Writing $\alpha = fdt + \beta$, the contact condition $\alpha \wedge d\alpha > 0$ can be written as:

$$(fdt + \beta) \wedge (dfdt + d\beta) = \beta df dt + fdtd\beta > 0. \tag{2}$$

If $f = 0$, then $\beta df dt > 0$, implying $df > 0$. Hence 0 is a regular value of f, and is an embedded 1-manifold. □

2. $\Gamma \pitchfork \Sigma_\xi$.

Proof. By Equation 2, $\beta \wedge df > 0$. Since $\ker \beta$ gives the characteristic foliation and $\ker df$ is tangent to the dividing set, we have $\Gamma \pitchfork \Sigma_\xi$. (In the cas Σ has Legendrian boundary, this ensures that Γ_Σ meets the boundary transversely, i.e., is properly embedded.) □

3. *The isotopy class of Γ_Σ does not depend on the choice of v.*

Proof. The dividing set $\Gamma_\Sigma(v)$ corresponding to the transverse contact vector field v is properly embedded by Property 1. Suppose, for $i = 0, 1$, v_i is a contact vector field transverse to Σ, i.e., $v_i = g_i\partial_t + Y_i$, where g_i is a positive function and Y_i is in the Σ-direction.

HW 33. *Prove that the space of contact vector fields forms a vector space.*

Therefore, we can interpolate between v_0 and v_1 by taking $v_s = (1-s)v_0 + sv_1$, which is a contact vector field, and for which $(1-s)g_0 + sg_1$ is positive. The corresponding $\Gamma_\Sigma(v_s)$ gives an isotopy between $\Gamma_\Sigma(v_0)$ and $\Gamma_\Sigma(v_1)$. □

4. *Write* $\Sigma \setminus \Gamma_\Sigma = R_+(\Gamma_\Sigma) \sqcup R_-(\Gamma_\Sigma)$, *where* $R_+(\Gamma_\Sigma) \subset \Sigma$ *(resp.* $R_-(\Gamma_\Sigma)$*) is the set of points* x *where the normal orientation to* Σ *given by* $v(x)$ *agrees with (resp. is opposite to) the normal orientation to* ξ_x. *Then as we cross* Γ_Σ *(once, transversely), we move from* $R_\pm$ *to* $R_\mp$.

Proof. We are assuming that the normal orientation to Σ is given by ∂_t, and the orientation of ξ is given by $\alpha = f dt + \beta$. Since 0 is a regular value of f, the sign of f changes as we cross a dividing curve. Let $R_+(\Gamma_\Sigma)$ be the set of points of Σ where $f > 0$; likewise, let $R_-(\Gamma_\Sigma)$ be the set where $f < 0$. Then $\alpha(\partial_t) > 0$ on R_+ and $\alpha(\partial_t) < 0$ on R_-, which implies the claim.

□

5. Γ_Σ *is nonempty.*

Proof. Suppose on the contrary that f is never zero. Then we can divide $\alpha = f dt + \beta$ by f and rewrite the contact 1-form as $\alpha' = dt + \beta'$ with $d\beta' > 0$. Now,

$$\int_\Sigma d\alpha' = \int_\Sigma d\beta' > 0,$$

whereas

$$\int_\Sigma d\alpha' = \int_{\partial\Sigma} \alpha' = 0$$

if $\partial\Sigma = \emptyset$ or Legendrian. □

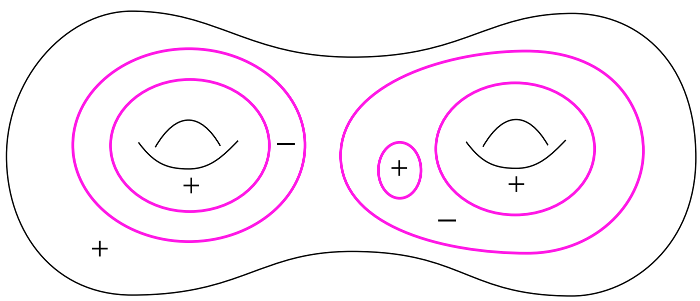

FIGURE 10. A sample dividing set.

8. Properties of convex surfaces

8.1. **Genericity.**

Proposition 8.1. *A C^∞-generic closed embedded surface Σ is convex.*

The same is almost true for compact surfaces with Legendrian boundary, but more care is needed along the boundary.

Proof. A C^∞-generic closed embedded surface has Morse-Smale characteristic foliation. We prove that an embedded surface Σ with a Morse-Smale characteristic foliation is convex. Since two contact structures defined in a neighborhood of Σ are isotopic iff they induce the same characteristic foliation $\mathcal{F}$ on Σ, it suffices to construct a contact structure ξ on $\Sigma \times \mathbf{R}$ with the given characteristic foliation $\mathcal{F}$ on Σ, so that Σ is manifestely convex. In other words, with coordinates (x,t)on $\Sigma \times \mathbf{R}$, we are looking for a contact form $\alpha = fdt + \beta$ where f and β do not depend on t.

Construct a region U_+ as the union of the following: (i) a small disk around each positive singular point, (ii) a small annulus around each closed orbit which is a source, and (iii) a band about each orbit δ which flows into a positive hyperbolic point (i.e., the *stable submanifold*). Observe that δ emanated from a positive elliptic point or a source closed orbit, since there are no saddle-saddle connections. After a small perturbation if necessary, we assume that ∂U_+ is transverse to $\mathcal{F}$. Similarly construct the negative region U_-. Since every closed orbit or singularity is contained in one of the $U_\pm$ and every orbit must come from U_+ and end on U_- (by the Morse-Smale condition), the complement $\Sigma - U_+ - U_-$ is a union of annuli $A = S^1 \times [-1,1]$, which are foliated by leaves which we may take to be $\{pt\} \times [-1,1]$, after a diffeomorphism of A.

Near a positive singular point, the original contact structure ξ' can be written as the kernel of $\alpha' = dt + \beta'$, with $d_2\beta' > 0$ (d_2 refers to derivative in the Σ-direction). We then use $\beta = \beta'(x,0)$. Also take f to be a large positive constant C in a neighborhood of positive elliptic points. Take f to be a small positive constant c in a neighborhood of the positive hyperbolic points.

HW 34. *Prove that the characteristic foliation $\mathcal{F}$, near a closed orbit γ, can be written as the kernel of a 1-form β on Σ with $d\beta > 0$. (Hint: Take a vector field X (in a neighborhood of γ) which is positively transverse to both Σ and ξ'. If t is a coordinate obtained by integrating X, then ξ' is given by $\alpha' = dt + \beta'$ with $d_2\beta' > 0$. Use this $\beta'(x,0)$.)*

In a similar manner, take $f = C$ near a source close orbit, and β be the 1-form in the HW.

We now extend $\alpha = fdt + \beta$ to a neighborhood $B = [-\varepsilon, \varepsilon] \times [0,1]$ of a subarc δ_0 of δ (with coordinates (r,s)), where $\delta_0 = \{r = 0\}$, ∂_s is an oriented vector field for $\mathcal{F}$, and α is already defined on $s = 0$ and $s = 1$. Hence $\alpha = fdt - g(r,s)dr$, and $\alpha/f = dt - g(r,s)/fdr$. Observe that since $\alpha(\partial_r) > 0$ (∂_r is a positively transverse to ξ), $g(r,s)/f > 0$. By taking $C >> c$, $g(r,0)/C < g(r,1)/c$, and we may extend $g(r,s)/f$ so that $\frac{\partial g(r,s)/f}{\partial s} > 0$. Define α similarly on U_-, except that $f < 0$.

It remains to extend across the annulus $A = S^1 \times [-1,1]$ with coordinates (θ, s) and characteristic foliation oriented by ∂_θ. If we write $\alpha = fdt + hd\theta$, then $\alpha/h = f/hdt + \theta$ (h is nonvanishing). On $s = -1$, $f > 0$ and $h < 0$, hence $f/h < 0$; on $s = 1$, $f < 0$ and $h < 0$, hence $f/h > 0$. Extending f/h so that $\frac{\partial f/h}{\partial s} > 0$, we are done. ∎

Remark 8.2. *If Σ is an embedded surface with Legendrian boundary, then for Σ to be perturbable into a convex surface, we first require that each component L of $\partial\Sigma$ have nonpositive twisting number $t(L, \mathcal{F}_\Sigma)$ (with respect to the framing $\mathcal{F}_\Sigma$ induced from Σ). This follows from Lemma 8.7. If each $t(L, \mathcal{F}_\Sigma) \leq 0$, then there is a C^0-small perturbation near $\partial\Sigma$ (fixing $\partial\Sigma$), followed by a C^∞-small perturbation away from $\partial\Sigma$ which makes Σ convex. The proof is similar to the proof given in the closed case – the extra ingredient is a normal form analysis near $\partial\Sigma$.*

8.2. **The Flexibility Theorem.** The usefulness of the dividing set Γ_Σ comes from the following:

Theorem 8.3 (Giroux's Flexibility Theorem)**.** *Assume Σ is convex with characteristic foliation Σ_ξ, contact vector field v, and dividing set Γ_Σ. Let $\mathcal{F}$ be another singular foliation on Σ which is* adapted to Γ_Σ *(i.e., there is a contact structure ξ' in a neighborhood of Σ such that $\Sigma_{\xi'} = \mathcal{F}$ and Γ_Σ is also a dividing set for ξ'). Then there is an isotopy φ_s, $s \in [0,1]$, of Σ in (M, ξ) such that:*

(1) $\varphi_0 = id$ *and* $\varphi_s|_{\Gamma_\Sigma} = id$ *for all* s.
(2) $\varphi_s(\Sigma) \pitchfork v$ *for all* s.
(3) $\varphi_1(\Sigma)$ *has characteristic foliation* $\mathcal{F}$.

In essence, Γ_Σ encodes ALL of the essential contact-topological information in a neighborhood of Σ. Therefore, having discussed characteristic foliations in Section 7.1, we may proceed to discard them and simply remember the dividing set.

Proof. On $\Sigma \times \mathbf{R}$, ξ and ξ' are given by $\alpha = f dt + \beta$ and $\alpha' = f' dt + \beta'$, where they have a common dividing set $f = f' = 0$. After a small isotopy near Γ_Σ, we may take the two characteristic foliations to agree on a neighborhood of Γ_Σ and also the contact structures to agree there. (Prove this!) Away from Γ_Σ (i.e., on each component of $R_\pm$), we divide by f and assume $f = \pm 1$. Say we are on R_+ and $f = 1$. Then β and β' are area forms on R_+, and can be interpolated by taking $\beta_s = (1-s)\beta + s\beta'$. Let $\alpha_s = dt + \beta_s$. We now use the Moser technique and solve for a vector field $X = g\partial_t + Y$ (Y in the Σ-direction) in the equation:

$$\mathcal{L}_X \alpha_s = \frac{d\alpha_s}{ds}.$$

$$i_X d\beta_s + d(\alpha_s(X)) = \beta' - \beta.$$

$$i_Y d\beta_s + d(g + \beta_s(Y)) = \beta' - \beta.$$

A reasonable solution is to split into two equations:

$$i_Y d\beta_s = \beta' - \beta, \; g = -\beta_s(Y).$$

Solve for Y in the first, and then g is determined by this choice of Y.

The key observation is that X does not have any t-dependence. Hence the corresponding isotopy $\varphi_s(\Sigma)$ is transverse to ∂_t for all s. $\square$

Examples on T^2: There are two common characteristic foliations on T^2.

(1) *Nonsingular Morse-Smale*. This is when the characteristic foliation is nonsingular and has exactly $2n$ closed orbits, n of which are sources (repelling periodic orbits) and the other n are sinks (attracting periodic orbits). Γ_{T^2} consists of $2n$ closed curves parallel to the closed orbits. Each dividing curve lies inbetween two periodic orbits.
(2) *Standard form*. An example is $x = const$ inside (T^3, ξ_n). The torus is fibered by closed Legendrian fibers, called *ruling curves*, and the singular set consists of $2n$ closed curves, called *Legendrian divides*. The $2n$ curves of Γ_{T^2} lie between the Legendrian divides.

HW 35. *Find an explicit example of a T^2 inside a contact manifold with nonsingular Morse-Smale characteristic foliation.*

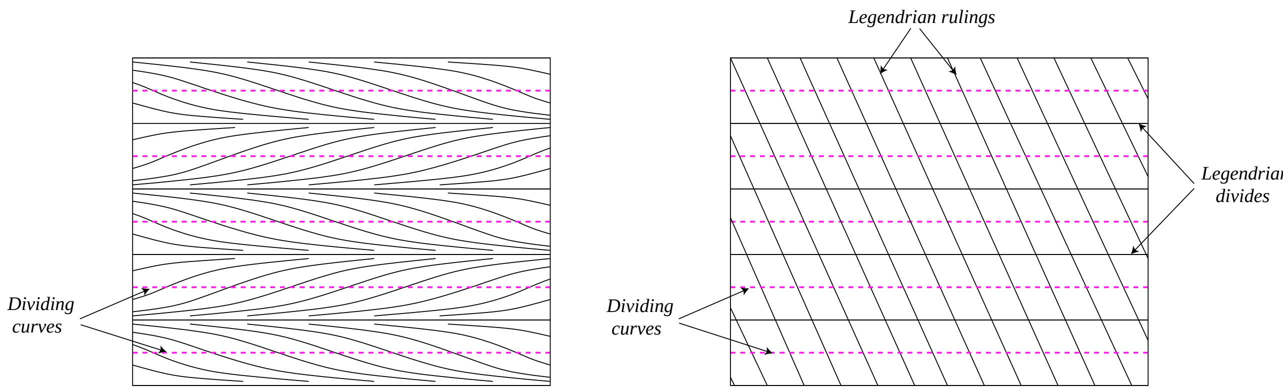

FIGURE 11. The left-hand side is a torus with nonsingular Morse-Smale characteristic foliation. The right-hand side is a torus in standard form. Here the sides are identified and the top and bottom are identified.

What the Flexibility Theorem tells us is that it is easy to switch between the two types of characteristic foliations – nonsingular Morse-Smale and standard form.

8.3. **Legendrian Realization.** The following consequence of the Flexibility Theorem is a crucial ingredient in the cut-and-paste theory of contact structures.

Proposition 8.4 (Legendrian Realization Principle, abbreviated LeRP)**.** *Let Σ be a convex surface and C be a multicurve on Σ. Assume $C \pitchfork \Gamma_\Sigma$ and C is* nonisolating, *i.e., each connected component of $\Sigma \setminus C$ nontrivially intersects Γ_Σ. Then there is an isotopy (as in the Flexibility Theorem) such that $\varphi_1(C)$ is Legendrian.*

Proof. In view of the Flexibility Theorem, it suffices to construct a characteristic foliation which has C as Legendrian curve. The proof of Proposition 8.1 applies here as well – we take normal forms near each singularity, and connect the neighborhoods of singularities of the same sign with bands about the stable submanifolds, while constructing the contact forms at the same time. Typically C will pass through several singular points of the characteristic foliation.

Consider a component $\Sigma_0 \subset R_+$ of $\Sigma \setminus (\Gamma_\Sigma \cup C)$. Denote by $\partial_-\Sigma_0$ the union of boundary components of $\partial\Sigma_0$ that nontrivially intersect Γ_Σ. $\partial_-\Sigma_0$ is designed to provide an escape route for

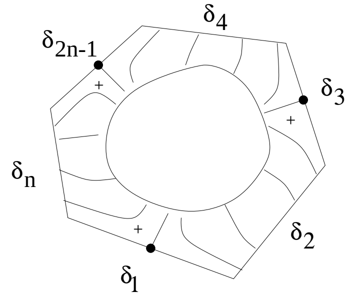

FIGURE 12. Characteristic foliation on $\gamma \times I$.

the flows whose sources are the positive elliptic points (which will be constructed in the interior) of Σ_0 or closed orbits of Morse-Smale type.

If γ is a component of $\partial_-\Sigma_0$, then either $\gamma \subset \Gamma_\Sigma$ or $\gamma \cap C \neq \emptyset$. Suppose we are in the latter case. Then $\gamma = \delta_1 \cup \delta_2 \cup \cdots \cup \delta_{2k}$, where the δ_i are in counterclockwise order and the endpoint of δ_j is the initial point of δ_{j+1}. Here δ_{2i-1}, $i = 1, \cdots, k$, are subarcs of C and δ_{2i}, $i = 1, \cdots, k$, are subarcs of Γ_Σ. (See Figure 12.) Now construct $\mathcal{F}$ so that the subarcs $\delta_{2i-1} \subset C$ become Legendrian, with a single positive hyperbolic point in the interior of the arc. Then we extend $\mathcal{F}$ to $\gamma \times [-1, 1]$ with $\gamma = \gamma \times \{0\}$ so that $\mathcal{F}$ is as in Figure 12. Renaming $\gamma = \gamma \times \{1\}$, we may assume that $\mathcal{F}$ is transverse to and flows out of γ. Hence we may assume, for the practical purpose of constructing characteristic foliations, that $\partial_-\Sigma_0 \subset \Gamma_\Sigma$.

Now consider the closed components of C that do not intersect Γ_Σ. Extend $\mathcal{F}$ so they become closed orbits of Morse-Smale type (and hence Legendrian). If $\partial\Sigma_0$ has no closed components of C, then introduce a positive elliptic point in the interior of Σ_0. It remains to extend $\mathcal{F}$ on a subsurface of Σ_0 with both positive and negative boundary components (by a positive (resp. negative) component γ we mean that $\mathcal{F} \pitchfork \gamma$ and the flow of enters (resp. exits) the region). By filling in appropriate positive hyperbolic points (modeled for example on the gradient flow of a Morse function), we may extend $\mathcal{F}$ to all of Σ_0. $\square$

Remark 8.5. *C may have extraneous intersections with Γ_Σ, i.e., the actual number of intersections $\#(C \cap \Gamma_\Sigma)$ is allowed to be larger than the geometric intersection number.*

Remark 8.6. *By modifying the proof, we may even take C to be a nonseparating* graph.

Lemma 8.7. *Prove that, if C is a closed Legendrian curve on the convex surface Σ, then the twisting number $t(C, \mathcal{F}_\Sigma)$ relative to the framing $\mathcal{F}_\Sigma$ from Σ is $-\frac{1}{2}\#(C \cap \Gamma_\Sigma)$. Here $\#(\cdot)$ represents cardinality, not geometric intersection.*

HW 36. *Prove the lemma.*

9. THURSTON-BENNEQUIN INEQUALITY

9.1. **Giroux's Criterion.** Now we present the criterion for determining when a convex surface has a tight neighborhood.

Proposition 9.1 (Giroux's Criterion)**.** *A convex surface $\Sigma \neq S^2$ has a tight neighborhood if and only if Γ_Σ has no homotopically trivial dividing curves. If $\Sigma = S^2$, then there is a tight neighborhood if and only if $\#\Gamma_\Sigma = 1$.*

HW 37. *Prove that if Γ_Σ has a homotopically trivial dividing curve, then there exists an overtwisted disk in a neighborhood of Σ, provided we are not in the situation where $\Sigma = S^2$ and $\#\Gamma_\Sigma = 1$. (Hint: Use LeRP. When Γ_Σ has no other components besides the homotopically trivial curve and Σ is not a sphere, then we need to use a trick to increase the number of dividing curves, so we can use LeRP. The trick consists of taking a nonseparating closed curve γ which does not intersect Γ_Σ, using LeRP, and then applying a folding operation in a neighborhood of γ.)*

Proof. The "only if" direction in Giroux's Criterion follows from HW 37. The "if" direction follows from constructing an explicit model inside a tight 3-ball or gluing. We will explain the former strategy today.

First observe that a convex S^2 with $\#\Gamma_{S^2} = 1$ has a tight neighborhood. This is because (i) tight contact structures exist, and (ii) any S^2 in a tight contact (M, ξ) must have $\#\Gamma_{S^2} = 1$, since otherwise there will be an overtwisted disk. By the Flexibility Theorem, any two convex S^2 with $\#\Gamma_{S^2} = 1$ will have contactomorphic neighborhoods.

Suppose Γ_Σ has no homotopically trivial dividing curves and assume to the contrary that there is an overtwisted disk Δ. (On a (compact) surface, an embedded homotopically trivial closed curve must always bound a disk.) The plan is to lift to the universal cover $\tilde{\Sigma}$, where $\Gamma_{\tilde{\Sigma}}$ also has no homotopically trivial dividing curves. By uniformization, $\tilde{\Sigma} = \mathbf{R}^2$ if Σ is closed; otherwise, $int(\tilde{\Sigma}) = \mathbf{R}^2$. In either case, there is an exhaustion of $\tilde{\Sigma}$ by convex disks $D_1 \subset D_2 \subset \dots$, $\cup_{i=1}^\infty D_i = \tilde{\Sigma}$. There's a little subtlety here: *without extra work we cannot necessarily choose the D_i to have Legendrian boundary*. Initially we can take the D_i so that a finite number of them exhaust Δ – however, when we use LeRP to Legendrian realize the ∂D_i, LeRP can potentially carry ∂D_i a long distance, and the new sequence $D_1' \subset D_2' \subset \dots$ may not exhaust Δ. Therefore the D_i are convex but not necessarily with Legendrian boundary; after perturbing ∂D_i, we may assume that Γ_{D_i} is properly embedded.

We now prove the D_i are tight by embedding them in S^2 with $\#\Gamma_{S^2} = 1$. Note that Γ_{D_i} is a union of arcs from D_i to itself. Successively extend $E_0 \stackrel{def}{=} D_i$ in a larger disk E_1 where Γ_{E_1} is obtained by attaching two adjacent endpoints of Γ_{E_0} by an arc δ outside E_0 (and ensuring that there are no closed dividing curves which are formed on E_1). See Figure 13. Eventually $\#\Gamma_{E_j} = 1$, at which point we cap off to obtain S^2 with $\#\Gamma_{S^2} = 1$.

We will now say a few words about the extension process. It is easy to extend the contact 1-form $f dt + \beta$ (with the same notation as before and f, β independent of t) to a neighborhood of δ so that $f = 0$ along δ. It remains to extend $\beta' = \beta/f$ from the boundary of a disk D to its interior. To

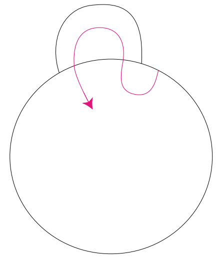

FIGURE 13. Adding the arc δ and extending the disk.

do this, take an area form ω on D which agrees with $d\beta'$ in a neighborhood of ∂D and such that $\int_D \omega = \int_{\partial D} \beta'$.

HW 38. *Prove that there is an extension of β' so that $d\beta' = \omega$ on D. (Hint: at some point you need to use a version of the Poincaré lemma for compactly supported forms.)*

$\square$

Examples of dividing sets: Suppose that (M, ξ) is tight. If $\Sigma = S^2$ is a convex surface in (M, ξ), then Γ_Σ is unique up to isotopy, consisting on one (homotopically trivial) circle. If $\Sigma = T^2$ is convex, then it consists of $2n$ parallel, homotopically essential curves. Therefore Γ_{T^2} is determined by $\#\Gamma_{T^2}$ and the slope, once a trivialization $T^2 \simeq \mathbf{R}^2/\mathbf{Z}^2$ is fixed.

9.2. **Euler class.** Given a contact manifold (M, ξ) and a closed oriented embedded surface Σ, we explain how to compute the Euler class of ξ evaluated on Σ, provided Σ_ξ is generic, i.e., consists of isolated elliptic and hyperbolic tangencies. $\langle e(\xi), \Sigma\rangle$ equals $e(i^*\xi)$, where $i : \Sigma \to M$ is the inclusion. Also recall that the Euler class of $i^*\xi$ is computed by taking a transverse section s of $i^*\xi$ and counting the number of intersection points (*with sign*). We take a vector field Y directing the characteristic foliation to be the transverse section.

Take local coordinates near a singular point $p \in \Sigma$ so that $p \mapsto 0$ in $\mathbf{R}^2$. ξ near p projects to $\mathbf{R}^2$ with the same sign if the tangency is positive and the opposite sign if the tangency is negative. We now consider a Gauß map $\phi : S^1 \to S^1 = \xi/\mathbf{R}^+$ around each tangency. Given $\theta \in S^1$, $\phi(\theta) = Y(\varepsilon \cos\theta, \varepsilon \sin\theta)$, viewed as an element of $\xi/\mathbf{R}^+$. Hence the index contribution of each type of singularity is as follows:

(1) Positive elliptic +1.
(2) Positive hyperbolic -1.
(3) Negative elliptic -1.
(4) Negative hyperbolic +1.

Contrast this with the *Euler characteristic*, where we are computing $\langle e(T\Sigma), \Sigma\rangle$, and elliptics contribute $+1$ and hyperbolics contribute -1. (This is the case for example for a gradient trajectory for a Morse function.)

9.3. **Thurston-Bennequin inequality.** Using Giroux's Criterion and our Euler class calculation, we prove a fundmental inequality for embedded surfaces Σ in tight contact manifolds (M, ξ).

The proof is a consequence of the following:

Lemma 9.2. $\langle e(\xi), \Sigma\rangle = \chi(R_+(\Gamma_\Sigma)) - \chi(R_-(\Gamma_\Sigma))$.

We often abbreviate $\chi_\pm = \chi(R_\pm)$.

Proof. We compute $\langle e(\xi), \Sigma\rangle$ by choosing a suitable characteristic foliation using the Flexibility Theorem. For R_+, take an upward gradient flow X and model the characteristic foliation on this upward gradient flow. Then the contributions of the singular points toward ξ is the same as that towards $T\Sigma$; hence the contribution is χ_+. Similarly R_- contributes χ_-. □

Theorem 9.3 (Thurston-Bennequin inequality)**.** *Let Σ be an embedded surface in a tight (M, ξ).*

(1) *If $\Sigma \neq S^2$ is closed, then $|\langle e(\xi), \Sigma\rangle| \leq -\chi(\Sigma) = 2g(\Sigma) - 2$.*
(2) *If $\partial\Sigma$ is a Legendrian knot L, then $tb(L) \pm r(L) \leq -\chi(\Sigma) = 2g(\Sigma) - 1$. Here Σ is any Seifert surface for L, and the genus $g(\Sigma)$ of a surface with boundary is the genus of the closed surface obtained by capping with disks.*

These inequalities are simple manifestations of the fact that a convex surface $\neq S^2$ cannot have homotopically trivial dividing curves.

Proof. (1) Observe that the only surface (with boundary) with positive Euler characteristic is a disk. If $\Sigma = S^2$, then $|\langle e(\xi), \Sigma\rangle| = 0$. Otherwise, no components of $R_\pm$ can be disks, by Giroux's Criterion. This means that $\chi_+ - \chi_- \leq -\chi_+ - \chi_- = -\chi(\Sigma)$. Similarly, $\chi_- - \chi_+ \leq -\chi(\Sigma)$. The inequality then follows from Lemma 9.2.

(2) We will prove the result for $tb(L) + r(L)$. The case for $tb(L) - r(L)$ is similar. Without loss of generality we may stabilize L to L' so that $tb(L') \leq 0$. Here, if we take positive stabilizations, then $tb(L) + r(L) = tb(L') + r(L')$.

Fact: Given a Legendrian knot L in (S^3, ξ) standard with $tb(L) \leq 0$, there exists a convex surface Σ which spans L.

HW 39. *Prove the assertion above.*

Now observe that $r(L) = \chi_+ - \chi_-$ and $tb(L) + \chi_+ + \chi_- = \chi(\Sigma)$.

HW 40. *Prove $r(L) = \chi_+ - \chi_-$ by interpreting the winding number of TL in term of the winding around the singularities. (Hint: This requires an understanding of how the singular points of Σ are distributed along L – there are* half-elliptic points *and* half-hyperbolic points *which will be described later. Take a vector field Y which directs the characteristic foliation of Σ and compute $\chi_+ - \chi_-$ as before. Now there is a difference between Y and $\dot{L}$, which needs to be analyzed.)*

The second equation is due to the fact that there are $-tb(L)$ many arcs of Γ_Σ from the boundary to itself and when you attach components of Σ_+ to components of Σ_- you are gluing along one such arc, which contributes -1 to the Euler characteristic.

Also observe that $tb(L) \leq -\chi_+$ since there can be at most $|tb(L)|$ many disk components of Σ_+. There are no closed curves of Γ_Σ bounding disks, so the disk components must use up arcs of Γ_Σ.

Finally,

$$tb(L) + r(L) = tb(L) + (\chi_+ - \chi_-) \leq tb(L) + (-2tb(L) - \chi_+ - \chi_-) = -\chi(\Sigma).$$

$\square$

Remark 9.4. *The Thurston-Bennequin inequality is an instance of*

$$|\langle \mathfrak{s}, \Sigma \rangle| \leq 2g(\Sigma) - 2,$$

whenever $\mathfrak{s}$ is a privileged class in $H^2(M; \mathbf{Z})$. $\mathfrak{s}$ is privileged, if $\mathfrak{s}$ is the Euler class of a taut foliation, a tight contact structure, or a Spin^c-structure with nonzero (Heegaard, monopole, etc.) Floer homology $HF(M, \mathfrak{s})$. It is also a 3-dimensional "shadow" of a phenomenon in 4-dimensions called the "adjunction inequality".

Example: If L is the unknot, then its Seifert surface Σ is a disk. Then $tb(L) \pm r(L) \leq 2g(\Sigma) - 1 = -1$. Hence this proves that $tb(L) \leq -1$ for the unknot.

HW 41. *Prove that, if there is a Legendrian knot with $r(L) = -k$, then there is another Legendrian knot in the same oriented topological type and with the same tb so that $r = k$. (Hint: try a symmetry in the front projection.)*

Example: If L is a trefoil (either left- or right-handed), then $tb(L) \pm r(L) \leq 1$. The Thurston-Bennequin inequality turns out to be exact for the right-handed trefoil (the maximum tb representative has $tb(L) = 1$ and $r(L) = 0$), whereas the inequality is inexact for the left-handed trefoil (the maximum tb is -6).

10. BYPASSES

In this section, we introduce the other chief ingredient in the cut-and-paste theory of tight contact structures: the *bypass*. As a surface is isotoped inside the ambient tight contact manifold (M, ξ), the dividing set changes in discrete units, and the fundamental unit of change is effected by the bypass.

10.1. **Convex surfaces with Legendrian boundary and edge-rounding.**

Since it becomes important in what follows, we will make a few remarks about compact convex surfaces $\Sigma \subset (M, \xi)$ with Legendrian boundary. For a convex surface Σ to exist, we need $t(L, \mathcal{F}_\Sigma) \leq 0$ for each connected component L of $\partial\Sigma$. Suppose $t(L, \mathcal{F}_\Sigma) < 0$; the case $t(L, \mathcal{F}_\Sigma) = 0$ should be treated slightly differently. Using a normal form theorem for L, we can write its neighborhood as $N(L) = S^1 \times D^2 = \mathbf{R}/\mathbf{Z} \times \{x^2 + y^2 \leq \varepsilon\}$ (with coordinates z, x, y) and the contact 1-form as $\alpha = \sin(2\pi nz)dx + \cos(2\pi nz)dy$, where $n = -t(L, \mathcal{F}_\Sigma)$ and $L = \{x = y = 0\}$. Then, after a C^0-small perturbation of Σ, we can let $\Sigma \cap N(L) = \{y = 0, x \geq 0\}$. (We will often tacitly assume that if $t(L, \mathcal{F}_\Sigma) < 0$, then a convex surface with boundary component L will have such a *collared Legendrian boundary*, which consists of parallel Legendrian ruling curves.) Next, away from the collar, we perturb the characteristic foliation to create half-elliptic or half-hyperbolic singularities, given in Figure 14, and then apply the techniques in Section 8 to make Σ convex, after a C^∞-small perturbation away from the collar.

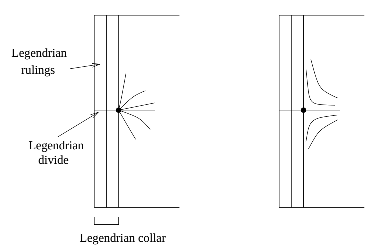

FIGURE 14. Half-elliptic point and half-hyperbolic point.

Now suppose we have two convex surfaces Σ_1 and Σ_2 with collared Legendrian boundary, which intersect transversely along a common boundary Legendrian curve L with $t(L, \mathcal{F}_{\Sigma_1}) = t(L, \mathcal{F}_{\Sigma_2}) = -n$. With $N(L)$ as in the above paragraph, we take $\Sigma_1 \cap N(L) = \{x = 0, y \geq 0\}$ and $\Sigma_2 \cap N(L) = \{y = 0, x \geq 0\}$. The following explains how to round $\Sigma_1 \cup \Sigma_2$ along the common edge L.

Lemma 10.1 (Edge-rounding)**.** *The surface $\Sigma = ((\Sigma_1 \cup \Sigma_2) \setminus \{x^2 + y^2 \leq \delta^2\}) \cup (\{(x-\delta)^2 + (y-\delta)^2 = \delta^2\} \cap \{x^2 + y^2 \leq \delta^2\})$, where $\delta < \varepsilon$, is a convex surface and the dividing curve $z = \frac{k}{2n}$ on Σ_1 connects to the dividing curve $z = \frac{k}{2n} - \frac{1}{4n}$ on Σ_2, where $k = 0, \cdots, 2n-1$.*

Refer to Figure 15 for an illustration.

Proof. Take the transverse contact vector field for Σ_1 to be $\frac{\partial}{\partial x}$ on $N(L)$ and the transverse contact vector field for Σ_2 to be $\frac{\partial}{\partial y}$ on $N(L)$. Then the transverse contact vector field for $\{(x-\delta)^2 + (y -$

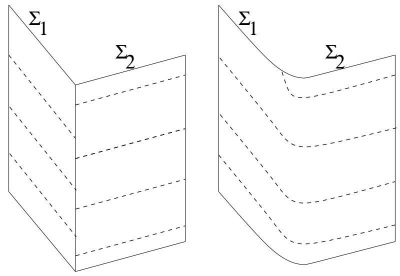

FIGURE 15. Edge rounding. Dotted lines are dividing curves.

$\delta)^2 = \delta^2\} \cap N_\delta$ is the inward-pointing radial vector $-\frac{\partial}{\partial r}$ for the circle $\{(x-\delta)^2 + (y-\delta)^2 = \delta^2\}$. □

10.2. **Definition and examples.**

Definition 10.2. *Let Σ be a convex surface and α be a Legendrian arc in Σ which intersects Γ_Σ in three points p_1, p_2, p_3, where p_1 and p_3 are endpoints of α. A* bypass half-disk *is a convex half-disk D with Legendrian boundary, where $D \cap \Sigma = \alpha$, $D \pitchfork \Sigma$, and $tb(\partial D) = -1$. α is called the* arc of attachment *of the bypass, and D is said to be a bypass* along *α or Σ.*

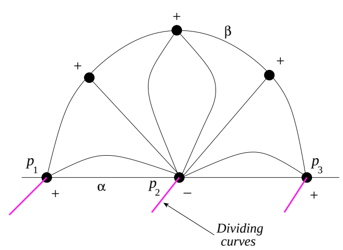

FIGURE 16. A bypass.

Remark 10.3. *A bypass can be thought of as half of an overtwisted disk.*

Remark 10.4. *Most bypasses do not come for free. Finding a bypass is equivalent to raising the twisting number (or Thurston-Bennequin invariant) by 1. Although it is easy to lower the twisting number by attaching "zigzags" in a front projection, raising the twisting number is usually a nontrivial operation.*

Lemma 10.5 (Bypass Attachment Lemma)**.** *Let D be a bypass for Σ. If Σ is isotoped across D, then we obtain a new convex surface Σ' whose dividing set is obtained from Γ_Σ via the move in Figure 17.*

Note that this is reasonable because a bypass attachment increases the twisting number along the arc of attachment by 1.

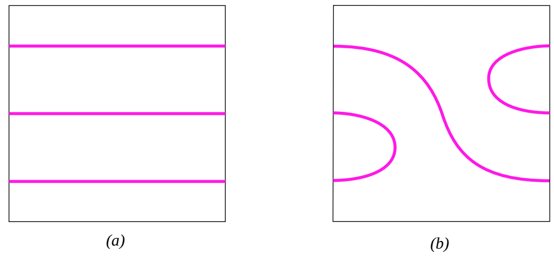

FIGURE 17. The effect of attaching a bypass from the front. Γ_Σ is (a) and $\Gamma_{\Sigma'}$ is (b).

Proof. Observe that the dividing set of an overtwisted disk consists of one closed curve which encircles the center singularity. Similarly, the dividing set of a bypass D consists of one arc which begins and ends on the arc of attachment α, while encircling the half-elliptic singularity at the center. The edge-rounding lemma tells us that the two bottom arcs on the left-hand side of Figure 17 must be connected and the two upper arcs on the right-hand side must be connected. It remains to connect the two remaining arcs: the upper left and the lower right. (Now, the argument above needs some careful fleshing out, but is correct in spirit....) $\square$

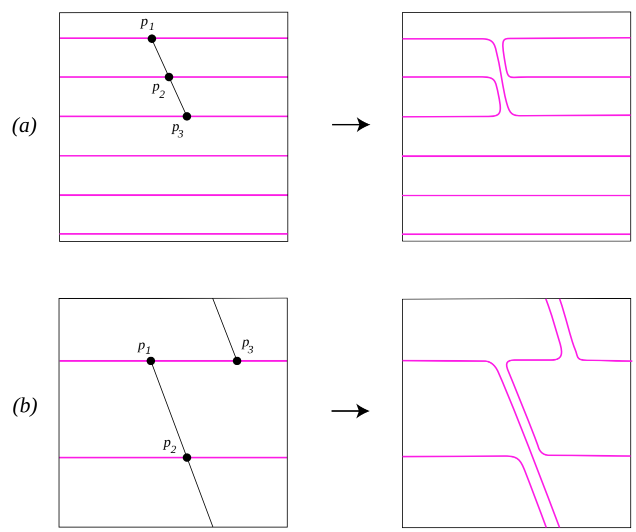

FIGURE 18. Possible bypasses on tori.

Example: T^2. Let us enumerate the possible bypass attachments – see Figure 18. (a) is the case where $\#\Gamma_{T^2} = 2n > 2$, and the bypass reduced $\#\Gamma$ by two, while keeping the slope fixed. (b) is the case where $\#\Gamma_{T^2} = 2$, and the slope is modified. In addition, there also are trivial and disallowed moves, which are moves locally given in Figure 19. It turns out that the trivial move always exists inside a tight contact manifold, whereas the disallowed move can never exist inside a tight contact manifold.

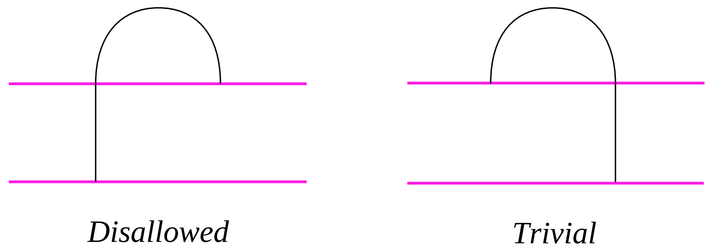

FIGURE 19. A disallowed bypass attachment and a trivial bypass attachment.

HW 42. *Is there a bypass attachment which increases $\#\Gamma$?*

10.3. **Intrinsic interpretation on the Farey tessellation.** Observe that, in case (b), the bypass move is equivalent to performing a *positive Dehn twist* along a particular curve. We can therefore reformulate this bypass move and give an *intrinsic interpretation* in terms of the Farey tessellation of the hyperbolic unit disk **H** (Figure 20). The set of vertices of the Farey tessellation is $\mathbf{Q} \cup \{\infty\}$ on $\partial\mathbf{H}$. (More precisely, fix a fractional linear transformation f from the upper half-plane model of hyperbolic space to the unit disk model **H**. Then the set of vertices is the image of $\mathbf{Q} \cup \{\infty\}$ under f.) There is a unique edge between $\frac{p}{q}$ and $\frac{p'}{q'}$ if and only if the corresponding shortest integer vectors form an integral basis for $\mathbf{Z}^2$. (The edge is usually taken to be a geodesic in **H**.)

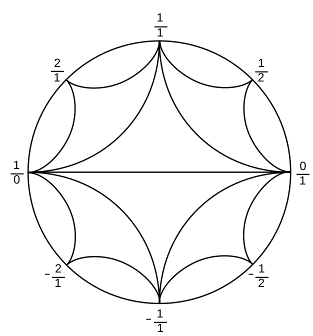

FIGURE 20. The Farey tessellation. The spacing between vertices are not drawn to scale.

Proposition 10.6. *Let $s = slope(\Gamma_{T^2})$. If a bypass is attached along a closed Legendrian curve of slope s', then the resulting slope s'' is obtained as follows: Let $(s', s) \subset \partial\mathbf{H}$ be the counterclockwise interval from s' to s. Then s'' is the point on (s', s) which is closest to s' and has an edge to s.*

See Figure 21 for an illustration.

HW 43. *Prove Proposition 10.6.*

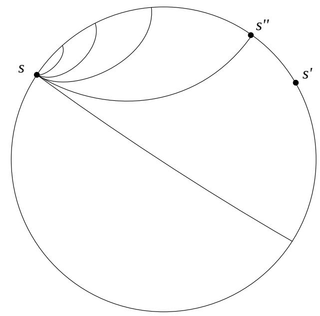

FIGURE 21. Intrinsic interpretation of the bypass attachment.

10.4. **Finding bypasses.** Bypasses would be quite useless if they were difficult to find. In this section we show that bypasses can often be found by examining the next step in the Haken hierarchy.

Let M be a closed manifold and $\Sigma \subset M$ be a closed surface. Consider $M \setminus \Sigma$. Let $S \subset M \setminus \Sigma$ be an incompressible surface with nonempty boundary, for example the next cutting surface in the Haken hierarchy. Under mild conditions on ∂S, we can take S to be a convex surface with nonempty Legendrian boundary.

Definition 10.7. *An arc component δ of Γ_S is* ∂-parallel *if $\Gamma_S \setminus \delta$ has a disk component D with $\Gamma_S \cap int(D) = \emptyset$.*

Lemma 10.8. *Suppose that Γ_S has a ∂-parallel component and either $S \neq D^2$ or else if $S = D^2$ then $tb(\partial S) < -1$. Then there exists a bypass along ∂S and hence along Σ.*

Proof. Draw an arc $\delta' \subset S$ so that δ' cuts off a half-disk with only the ∂-parallel arc δ on it. The condition on S is needed to ensure that we can use LeRP to find a Legendrian arc δ''. The half-disk cut off by δ'' (and containing a copy of δ) is the bypass for Σ. □

Corollary 10.9. *Let $S = D^2$ be a convex disk with Legendrian boundary so that $tb(\partial S) < -1$. Then there exists a bypass along ∂S.*

Corollary 10.9 follows from Lemma 10.8, by observing that all components of Γ_{D^2} cut off half-disks of D^2 and that a ∂-parallel component is simply an outermost arc of Γ_{D^2}.

Remark 10.10. *Corollary 10.9 does not work when $tb(\partial D) = -1$.*

Similarly, we can prove the following:

Corollary 10.11 (Imbalance Principle)**.** *Let $S = S^1 \times [0,1]$ be a convex annulus. If $t(S^1 \times \{1\}, \mathcal{F}_S) < t(S^1 \times \{0\}, \mathcal{F}_S)$, then there is a ∂-parallel arc and hence a bypass along $S^1 \times \{1\}$. Here $\mathcal{F}_S$ is the framing induced from the surface S.*

Figure 22 gives an example of a convex annulus with $t(S^1 \times \{1\}, \mathcal{F}_S) < t(S^1 \times \{0\}, \mathcal{F}_S)$. There is necessarily a bypass along $S^1 \times \{1\}$.

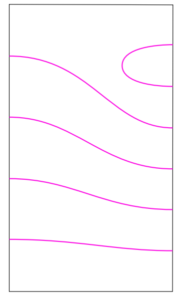

FIGURE 22. One possible dividing set for the annulus. Here the top and the bottom are identified.

11. TIGHT CONTACT STRUCTURES ON THE 3-BALL

11.1. **Equivalence of bifurcations and bypasses.** Let Σ be a closed oriented surface and $M = \Sigma \times [0,1]$. Denote by Σ_t, $t \in [0,1]$, the slice $\Sigma \times \{t\}$. We study the *film* consisting of the characteristic foliations $(\Sigma_t)_\xi$ for all t, induced by a contact structure ξ on M. The following is due to Giroux:

Theorem 11.1. *Let ξ_0 be a contact structure on $M = \Sigma \times [0,1]$ for which Σ_0 and Σ_1 are convex. Then ξ_0 is isotopic to ξ relative to the boundary so that the F_t are convex on an open dense set U of $[0,1]$ and the complement $[0,1] - U$ is a disjoint union of closed sets K_0 and K_1, where the characteristic foliations $(\Sigma_t)_\xi$ satisfy the Poincaré-Bendixson property and:*

1. *If $t_0 \in K_0$, then $(\Sigma_{t_0})_\xi$ has a single "retrogradient" saddle-saddle connection, i.e., an orbit from a negative hyperbolic point to a positive hyperbolic point.*
2. *If $t_0 \in K_1$, then $(\Sigma_{t_0})_\xi$ has a degenerate closed orbit. For $t < t_0$ (say), the closed orbit vanishes, and for $t > t_0$ we have two closed orbits of Morse-Smale type.*

A vector field satisfies the *Poincaré-Bendixson property* if every limit set of every half-orbit (orbit in forward time or backward time) is one of the following: (i) a singularity, (ii) a closed orbit, or (iii) a finite union of singularities and orbits connecting the singularities.

Remark 11.2. *K_0 can be taken to be finite. I do not know whether K_1 can be made finite, although that is highly probable.*

Remark 11.3. *If $\Sigma = S^2$, and ξ is tight, then K_1 is empty, and there are only finitely many $0 < t_0 < t_1 < \cdots < t_k < 1$, where Σ_{t_i} is not convex.*

Now, a retrogradient orbit at time t_0 acts as a switch: at time $t < t_0$, it connects to one negative elliptic point, and for $t > t_0$ it connects to a different elliptic point. One can see that this is equivalent to a bypass attachment. (The degenerate orbit also corresponds to a bypass attachment for increasing $\#\Gamma$ by folding.)

11.2. **Classification of tight contact structures on $S^2 \times [0,1]$.** Recall that if S^2 is a convex surface in a tight contact 3-manifold (M, ξ), then the only possiblity for Γ_{S^2} is one closed circle. Also, the only possible bypasses that can exist on S^2 are the trivial ones. Using this observation and the equivalence of a trivial bypass with a retrogradient saddle-saddle connection, we prove the following theorem:

Theorem 11.4 (Eliashberg)**.** *Let $M = S^2 \times [0,1]$ and fix a contact structure ξ_0 in a neighborhood of ∂M so that $\#\Gamma_{S^2_t} = 1$ for $t = 0, 1$. Then there is a unique tight contact structure on M up to isotopy, relative to ∂M.*

Proof. In view of the above discussion, it suffices to prove that $S^2 \times [0,1]$ with one bifurcation (namely, a trivial bypass move) is isotopic relative to the boundary to $S^2 \times [0,1]$ with no bifurcations (i.e., an invariant neighborhood of a convex S^2). This can be done by finding an explicit model: Consider the standard contact $(\mathbf{R}^3, \xi)$ given by $dz - ydx = 0$. The pieces we put together are (i) a transverse arc (whose thickened neighborhood has precisely two singularities, one positive elliptic and the other negative elliptic, and no closed orbits) and (ii) a Y (whose thickened neighborhood has two positive ellliptics, one negative elliptic, and a positive hyperbolic) or an upside-down Y. Let G be the graph in **R**^{3} obtained by taking a Y and an upside down Y above it, and glue in a transverse arc from the upper left vertex of Y to the lower left vertex of the upside-down Y. Its thickened neighborhood has an ustable trajectory emanating from a negative hyperbolic point which may or may not end at the positive hyperbolic point – this can be controlled by modifying the *thickness* of the neighborhood. (We can model the retrogradient switch simply by changing the thickness!) Let $N_0(G)$ be the neighborhood of G before the switch and $S_0 = \partial N_0(G)$. Also let $N_1(G) \subset N_0(G)$ be the slightly smaller neighborhood after the switch and $S_1 = \partial N_1(G)$. There is a transverse arc G' so that $N_0(G)$ can be dug out of a neighborhood $N_0(G')$ of G' and $N_0(G') - N_0(G)$ is foliated by convex surfaces; similarly, there is a smaller neighborhood $N_1(G')$ such that $N_1(G) - N_1(G')$ is also foliated by convex surfaces. It remains to replace the foliation of $N_0(G') - N_1(G')$ by another foliation by S^2's, where all the S^2's are convex (instead of all but one)! This is straightforward.

□

Using this, we can also classify tight contact structures on the 3-ball B^3, **R**^{3}, S^3, and $S^1 \times S^2$, up to isotopy.

Theorem 11.5. *There is a unique tight contact structure on (each of) B^3 (rel boundary),* **R**^{3}*, S^3, and $S^1 \times S^2$, up to isotopy.*

Proof. B^3 can be broken up into a standard 3-ball (by Pfaff's theorem) and $S^1 \times [0,1]$, which is unique relative to the boundary by the above theorem. Similarly, S^3 can be broken up into two standard B^3 and $S^1 \times [0,1]$, each of which is unique relative to the boundary.

Exhaust **R**^{3} by concentric 3-balls $B_1 \subset B_2 \dots$. Without loss of generality, take ∂B_i to be convex – tightness implies that Γ_{S^2} is always S^1. Apply the Flexibility Theorem to normaliize ξ on ∂B_i, and use the fact that $S^1 \times [0,1]$ has a unique tight contact structure up to isotopy relative to the boundary.

Finally, cut $S^1 \times S^2$ along a convex $\{pt\} \times S^2$, and use the classification on $S^2 \times [0,1]$. □

11.3. **The standard neighborhood of a Legendrian curve.**

Consider a (closed) Legendrian curve L with $t(L,\mathcal{F}) = -n < 0$, $n \in$ **Z**$^+$. (Pick some framing $\mathcal{F}$ for which the twisting number is negative.) Then a *standard neighborhood* $S^1 \times D^2 =$ **R**/**Z** $\times \{x^2 + y^2 \leq \varepsilon\}$ (with coordinates z, x, y) of the Legendrian curve $L = S^1 \times \{(0,0)\}$ is given by

$$\alpha = \sin(2\pi n z)dx + \cos(2\pi n z)dy,$$

and satisfies the following:

(1) $T^2 = \partial(S^1 \times D^2)$ is convex.
(2) $\#\Gamma_{T^2} = 2$.
(3) slope$(\Gamma_{T^2}) = -\frac{1}{n}$, if the meridian has zero slope and the longitude given by $x = y = const$ has slope ∞.

The following is due to Kanda and Makar-Limanov.

Proposition 11.6 (Kanda, Makar-Limanov)**.** *Given a solid torus $S^1 \times D^2$ and boundary conditions (1), (2), (3), there exists a unique tight contact structure on $S^1 \times D^2$ up to isotopy rel boundary, provided we have fixed a characteristic foliation $\mathcal{F}$ adapted to $\Gamma_{\partial(S^1\times D^2)=T^2}$.*

Remark 11.7. *The precise characteristic foliation is irrelevant in view of Giroux Flexibility.*

Proof.

(1) Let $L \subset T^2$ be a curve which bounds the meridian D. Using LeRP, realize it as a Legendrian curve with $tb(L) = -1$.
(2) Using the genericity of convex surfaces, realize the surface D with $\partial D = L$ as a convex surface with Legendrian boundary. Since $tb(L) = -1$, there is only one possibility for Γ_D, up to isotopy.
(3) Next, using Giroux Flexibility, fix some characteristic foliation on D adapted to Γ_D. Note that any two tight contact structures on $S^1 \times D$ with boundary condition $\mathcal{F}$ can be isotoped to agree on $T^2 \cup D$.
(4) The rest is a 3-ball B^3. Use Eliashberg's uniqueness theorem for tight contact structures on B^3.

$\square$

12. GLUING

Let us start by asking the following question:

Question 12.1. *Let Σ be a convex surface in (M,ξ). If $(M\setminus\Sigma,\xi|_{M\setminus\Sigma})$ is tight, then is (M,ξ) tight?*

Answer: This is usually not true. Our goal is to understand to what extent it is true. For example, is it possible that an OT disk is split into two bypasses along Σ?

12.1. Basic examples with trivial state transitions.

Example A: (Colin, Makar-Limanov) Suppose $\Sigma = S^2$. If $(M\setminus\Sigma,\xi|_{M\setminus\Sigma})$ is tight, then (M,ξ) is tight.

Proof. Recall that there is only one possibility for Γ_{S^2} inside a tight contact manifold. We argue by contradiction. Suppose there is an OT disk $D\subset M$. A priori, the OT disk D can intersect Σ in a very complicated manner. We obtain a contradiction as follows:

(1) Isotop Σ to Σ' so that $\Sigma'\cap D=\emptyset$.
(2) *Discretize* the isotopy
$$\Sigma_0=\Sigma\to\Sigma_1\to\cdots\to\Sigma_n=\Sigma',$$
so that Σ_i and Σ_{i+1} cobound a $\Sigma\times[0,1]$.
(3) Using the identification of a bifurcation and a bypass for a family $\Sigma\times\{t\}$ in $\Sigma\times[0,1]$, we reduce to the case where each step $\Sigma_i\to\Sigma_{i+1}$ is obtained by attaching a *bypass*.
(4) If $(M\setminus\Sigma_i,\xi|_{M\setminus\Sigma_i})$ is tight, then $\Gamma_{\Sigma_i}=\Gamma_{\Sigma_{i+1}}=S^1$ and the bypass must be *trivial*. Hence,
$$(M\setminus\Sigma_i,\xi|_{M\setminus\Sigma_i})\simeq(M\setminus\Sigma_{i+1},\xi|_{M\setminus\Sigma_{i+1}}).$$

We have proved inductively that $(M\setminus\Sigma',\xi|_{(M\setminus\Sigma')})$ is tight, a contradiction.

□

More generally, one can prove:

Theorem 12.2 (Colin)**.** *If $M=M_1\#M_2$, then*
$$\textit{Tight}(M)\simeq\textit{Tight}(M_1)\times\textit{Tight}(M_2).$$

Here Tight(M) is the set of isotopy classes of tight contact structures on M, i.e., π_0 of the space of tight contact 2-plane fields.

Example B: (Colin) If $\Sigma=D^2$ and Γ_Σ is *∂-parallel*, then $(M\setminus\Sigma,\xi|_{M\setminus\Sigma})$ tight $\Rightarrow$ (M,ξ) tight.

Definition 12.3. *A dividing set Γ_Σ is* ∂-parallel *if all its dividing curves are ∂-parallel arcs.*

Example C: (Colin) Let Σ be an incompressible surface with $\partial\Sigma\neq\emptyset$. If Γ_Σ is ∂-parallel and $(M\setminus\Sigma,\xi|_{M\setminus\Sigma})$ is universally tight, then (M,ξ) is universally tight.

An *incompressible surface* Σ is an embedded surface for which $\pi_1(\Sigma)$ injects into $\pi_1(M)$. We will also use the fact that, given any $\gamma \neq 0 \in \pi_1(M)$, there is a large finite cover $\widetilde{M}$ of M so that $\gamma \notin \pi_1(\widetilde{M})$ (this is called *residual finiteness*).

Proof. In this case, the two types of bypasses that can exist are either trivial or $\#\Gamma$-increasing (i.e., analogous to a folding). If the arc of attachment for a $\#\Gamma$-increasing bypass wraps around $\gamma \in \pi_1(\Sigma)$, we pass to a large finite cover $\widetilde{M}$ for which its lift $\tilde{\gamma}$ is no longer a closed curve. Then the bypass becomes a trivial one. $\square$

Question 12.4. *In Example C, does $(M \setminus \Sigma, \xi|_{M\setminus\Sigma})$ tight imply (M, ξ) tight? In other words, can universal tightness be avoided?*

All of the above examples can be characterized by the fact that the *state transitions* are trivial. However, to create more interesting examples, we need to "traverse all states".

12.2. **More complicated example.** Let H be a handlebody of genus g and $D_1, \dots, D_g$ be compressing disks so that $H \setminus (D_1 \sqcup \cdots \sqcup D_g) = B^3$. Fix $\Gamma_{\partial H}$ (and a compatible characteristic foliation). Note that we need $tb(D_i) \leq -1$, since otherwise we can find an OT disk using LeRP.

Let $\mathcal{C}$ be the *configuration space*, i.e., the set of all possible $C = (\Gamma_{D_1}, \dots, \Gamma_{D_g})$, where each Γ_{D_i} has no closed curves. The cardinality of $\mathcal{C}$ is *finite*. If we cut H along $\Sigma = D_1 \cup \cdots \cup D_g$, then we obtain a 3-ball with corners. Given a configuration C, we can round the corners, as previously described. Now, if $\Gamma_{\partial(H\setminus\Sigma)} = S^1$ after rounding, then C is said to be *potentially allowable*.

State transitions: The smallest unit of isotopy (in the contact world) is a bypass attachment. Therefore we examine the effect of one bypass attachment onto D_i. First we need to ascertain whether a candidate bypass exists.

Criterion for existence of state transition: The candidate bypass exists if and only if attaching the bypass from the interior of $B^3 = H \setminus \Sigma$ does not increase $\#\Gamma_{\partial B^3}$.

We construct a graph Γ with $\mathcal{C}$ as the vertices. We assign an edge from $(\Gamma_{D_1}, \dots, \Gamma_{D_i}, \dots, \Gamma_{D_g})$ to $(\Gamma_{D_1}, \dots, \Gamma_{D_i'}, \dots, \Gamma_{D_g})$ if there is a state transition $D_i \to D_i'$ given by a single bypass move. Note that the bypass may be from either side of D_i. Then we have:

Theorem 12.5. *Tight$(H, \Gamma_{\partial H})$ is in 1-1 correspondence with the connected components of Γ, all of whose vertices C are potentially allowable.*

HW 44. *Explain why Tight$(H, \Gamma_{\partial H})$ is finite.*

Remark 12.6. *Since $\mathcal{C}$ is a finite graph, in theory we can compute Tight$(H, \Gamma_{\partial H})$ for any handlebody H with a fixed boundary $\Gamma_{\partial H}$. Tanya Cofer, a (former) graduate student at the University of Georgia, has programmed this for $g = 1$, and the experiment agrees with the theoretical number from Theorem 14.1, in case $\#\Gamma_{\partial H} = 2$ and the slope is $-\frac{p}{q}$ with $p \leq 10$.*

HW 45. *Using the state transition technique, analyze tight contact structures on $S^1 \times D^2$, where Γ_{T^2}, $T^2 = \partial(S^1 \times D^2)$, satisfies the following:*

(1) $\#\Gamma_{T^2} = 2$ *and slope*$(\Gamma_{T^2}) = -2$.

(2) $\#\Gamma_{T^2} = 2$ *and slope*$(\Gamma_{T^2}) = -3$.
(3) $\#\Gamma_{T^2} = 4$ *and slope*$(\Gamma_{T^2}) = \infty$.

Here the slope of the meridian is 0 *and the slope of some preferred longitude is* ∞.

13. Sutured manifolds and convex decompositions

13.1. **Taut foliations.** Foliations are the other type of locally homogeneous 2-plane field distributions on 3-manifolds. The following table is a brief list of analogous objects from both worlds (note that the analogies are not precise):

Foliations	Contact Structures
$\alpha \wedge d\alpha = 0$ integrable	$\alpha \wedge d\alpha > 0$ nonintegrable
$\alpha = dz$ Frobenius	$\alpha = dz - ydx$ Pfaff
Reeb components	Overtwisted disks
Taut	Tight

A (rank 2 or codimension 1) *foliation* ξ is an integrable 2-plane field distribution, i.e., locally given as the kernel of a 1-form α with $\alpha \wedge d\alpha = 0$. According to Frobenius' theorem, ξ can locally be written as the kernel of $\alpha = dz$. The world of foliations also breaks up into the topologically significant *taut* foliations, and the foliations with *generalized Reeb components*, which exist on every 3-manifold. A generalized Reeb component is a compact submanifold $N \subset M$ whose boundary ∂N is a union of torus leaves, and such that there are no transversal arcs which begin and end on ∂N. The primary example of a generalized Reeb component is a *Reeb component*, i.e., a foliation of the solid torus $S^1 \times D^2$ whose boundary $S^1 \times S^1$ is a leaf and whose interior is foliated by planes as in Figure 23.

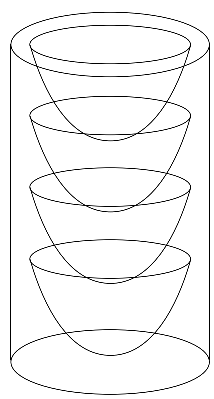

FIGURE 23. A Reeb component. Here the top and bottom are identified.

Equivalent conditions for a foliation $\mathcal{F}$ to be taut are:

(1) Through each leaf L of $\mathcal{F}$, there is a closed transversal curve δ.
(2) There is a closed 2-form ω such that $\omega|_{\mathcal{F}} > 0$.
(3) There are no generalized Reeb components.

13.2. **Gabai's sutured manifold theory.**

Definition 13.1. *A* sutured manifold (M, γ) *consists of the following data:*

1. M *is a compact, oriented, irreducible 3-manifold with corners,*
2. γ *is a disjoint union of annuli (sutures) on* ∂M*,*
3. $\partial\gamma$ *is the union of corners of* ∂M*, and*
4. γ *divides* ∂M *into positive and negative regions, whose sign changes every time* γ *is crossed. We write* $\partial M \setminus \gamma = R_+ \sqcup R_-$*. (We will also write* $R_\pm(\gamma)$ *if the sutured manifold is ambiguous.)*

In addition, we assume that each component of M *has nonempty boundary and each component of* ∂M *has nonempty intersection with* γ*. Here, a 3-manifold* M *is* irreducible *if every embedded 2-sphere* S^2 *bounds a 3-ball* B^3*.*

Note that our definition of a sutured manifold, chosen to simplify the exposition in this paper, is slightly different from that of Gabai.

In our diagrams, sutures will be represented by closed curves (think of very thin annuli).

Definition 13.2. *Let* S *be a compact oriented surface with connected components* $S_1, \dots, S_n$*. The* Thurston norm *of* S *is:*

$$x(S) = \sum_{i \text{ such that } \chi(S_i)<0} |\chi(S_i)|.$$

Definition 13.3. *A sutured manifold* (M, γ) *is* taut *if*

1. $R_\pm$ *minimize the Thurston norm in* $H_2(M, \gamma)$*,*
2. $R_\pm$ *are incompressible in* M*, and*
3. *no components of* $R_\pm$ *are disks, unless* $(M, \gamma) = (B^3, S^1 \times I)$*.*

Here, a surface $S \subset M$ *is* incompressible *if for every embedded disk* $D \subset M$ *with* $D \cap S = \partial D$*, there is a disk* $D' \subset S$ *such that* $\partial D = \partial D'$*.*

Roughly speaking, (M, γ) is taut if $R_\pm$ attain the minimum genus amongst all the embedded representatives in the relative homology class $H_2(M, \gamma)$. In particular, $x(R_+) = x(R_-)$. (1) implies (2) except when $R_\pm$ has disk components; (2) prohibits disk components of $R_\pm$ except when $(M, \gamma) = (B^3, S^1 \times I)$ and hence (3) is redundant. Observe the similarities between the definition of a taut sutured manifold and Giroux's criterion for neighborhood of a convex surface to be tight.

Examples: (1) Let M be a genus 2 handlebody with 4 sutures, represented by 2 parallel nonseparating curves, and 2 other parallel nonseparating curves (not parallel to the first two). Suppose R_+ consists of two annuli and R_- is a sphere with 4 holes. Then $x(R_+) = 0$ and $x(R_-) = 2$, which implies that (M, γ) is not taut.

(2) $(M, \gamma) = (B^3, S^1 \times I)$ is taut.

(3) Let $M = B^3$ and γ consist of three concentric circles on S^2. Then (B^3, γ) is not taut because of the compressibility of the middle annular components of $S^2 \setminus \gamma$.

Sutured manifold splittings. We now explain how to split a sutured manifold (M,γ) along an oriented surface S to obtain a new sutured manifold (M',γ').

Definition 13.4. *Let S be an oriented, properly embedded surface in (M,γ) which intersects γ transversely. Then a* sutured manifold splitting $(M,\gamma) \overset{S}{\rightsquigarrow} (M',\gamma')$ *is given as follows (see Figure 24 for an illustration): Define $M' = M \setminus S$, and let S_+ (resp. S_-) be the copy of S on $\partial M'$ where the orientation inherited from S and the outward normal agree (are opposite). Then set $R_\pm(\gamma') = (R_\pm(\gamma)\backslash S) \cup S_\pm$. The new suture γ' forms the boundary between the regions $R_+(\gamma')$ and $R_-(\gamma')$.*

According to Gabai, you simply paint the positive region red, the negative region blue, and the new sutures are purple!

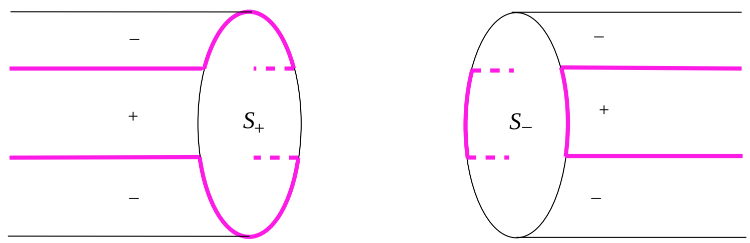

FIGURE 24

A sutured manifold (M,γ) is *decomposable*, if there is a sequence of sutured manifold splittings:

$$(M,\gamma) \overset{S_1}{\rightsquigarrow} (M_1,\gamma_1) \overset{S_2}{\rightsquigarrow} \cdots \overset{S_n}{\rightsquigarrow} (M_n,\gamma_n) = \sqcup(B^3, S^1\times I).$$

The following theorems are due to Gabai. The first is an analog of the Haken decomposition theorem.

Theorem 13.5 (Gabai)**.**

(1) *(Decomposition) If (M,γ) is taut, then it is decomposable.*
(2) *(Reconstruction) Given a sutured manifold decomposition, we can backtrack and construct a taut foliation which is carried by (M,γ).*

13.3. **The Gabai-Eliashberg-Thurston theorem.** We now explain the original proof of the following important theorem:

Theorem 13.6 (Gabai-Eliashberg-Thurston)**.** *Let M be a closed, oriented, irreducible 3-manifold with $H_2(M;\mathbf{Z}) \neq 0$. Then M carries a tight contact structure.*

We first explain the original proof, and present an alternative in the next section.

Proof.

Step 1: Let Σ be a closed surface which is a Thurston-norm-minimizing representative of a nonzero class of $H_2(M;\mathbf{Z})$. Then split $(M,\emptyset) \overset{\Sigma}{\rightsquigarrow} (M\setminus\Sigma,\emptyset)$. Now apply Gabai's decomposition theorem to construct a sutured manifold decomposition:

$$(M,\emptyset) \overset{\Sigma}{\rightsquigarrow} (M\setminus\Sigma,\emptyset) \rightsquigarrow \dots \rightsquigarrow \sqcup(B^3, S^1\times I).$$

Step 2: Use the reconstruction theorem to construct a transversely oriented taut foliation $\mathcal{F}$ from the sutured manifold hierarchy. In the reconstruction process, the $R_\pm$ become subsets of leaves of the taut foliation – the orientations of $\mathcal{F}$ and R_+ agree and the orientations of $\mathcal{F}$ and R_- are opposite. (In the inductive process, a foliation carried by (M,γ) is taut if there is either a closed transversal curve or a transversal arc with endpoints on R_+ and R_- through each leaf.)

Step 3: The following is the key theorem which allows us to transfer information from foliation theory to contact geometry.

Theorem 13.7 (Eliashberg-Thurston)**.** *Let M be a closed, oriented 3-manifold $\neq S^1\times S^2$. Then every foliation admits a C^0-small perturbation into positive and negative contact structures $\xi_\pm$.*

Step 4: (Symplectic filling) If the foliation $\mathcal{F}$ is taut, then there is a closed 2-form ω with $\omega_{\mathcal{F}}>0$. Then we construct a symplectic filling $M\times[-\delta,\delta]$ with symplectic form $\Omega=\omega+\varepsilon d(e^t\alpha)$, where α is a 1-form which defines $\mathcal{F}$, t is the coordinate for $[-\delta,\delta]$, and ε and δ are small. Take the contact manifolds on the boundary to be $(M\times\{\delta\},\xi_+)\sqcup(M\times\{-\delta\},\xi_-)$. [For symplectic filling, we can possibly have several boundary components, provided all of them are convex.] Now, fillable contact structures are tight. $\square$

13.4. **Convex decomposition theory.** We have the following theorem which gives the equivalence between tightness and tautness in the case of a manifold with boundary:

Theorem 13.8 (Kazez-Matić-Honda)**.** *Let (M,γ) be a sutured manifold. Then the following are equivalent:*

(1) (M,γ) *is taut.*
(2) (M,γ) *carries a taut foliation.*
(3) (M,Γ) *carries a universally tight contact structure.*
(4) (M,Γ) *carries a tight contact structure.*

Here, Γ is obtained from γ by collapsing the annuli to their core curves.

A contact structure ξ on M is *carried by* (M,Γ) if ∂M is a convex surface for ξ with dividing set Γ. A transversely oriented foliation ξ on M is *carried by* (M,γ) if there exists a thickening of Γ to a union $\gamma\subset\partial M$ of annuli, so that $\partial M\setminus\gamma$ is a union of leaves of ξ, ξ is transverse to γ, and the orientations of $R_\pm$ and ξ agree. (Strictly speaking, in this case M is a manifold with corners.) A tight contact structure is *universally tight* if it remains tight when pulled back to the universal cover of M.

Now, in the contact category, we choose a dividing set Γ_S so that every component of Γ_S is ∂-*parallel*, i.e., cuts off a half-disk of S which does not intersect any other component of Γ_S. Such a dividing set Γ_S is also called ∂-*parallel*.

If there is an invariant contact structure defined in a neighborhood of ∂M with dividing set $\Gamma = \Gamma_{\partial M}$, then by an application of LeRP, we may take ∂S to be Legendrian. (There are some exceptional cases, but we will not worry about them here.) Extend the contact structure to be an invariant contact structure in a neighborhood of S with ∂-parallel dividing set Γ_S. Now, if we cut M along S, we obtain a manifold with corners. To smooth the corners, we apply *edge-rounding*. This is given in Figures 25 and 26. Figure 25 gives the surface S before rounding, and Figure 26 after rounding. Notice that we may think of S as a lid of a jar, and the edge-rounding operation as twisting to close the jar.

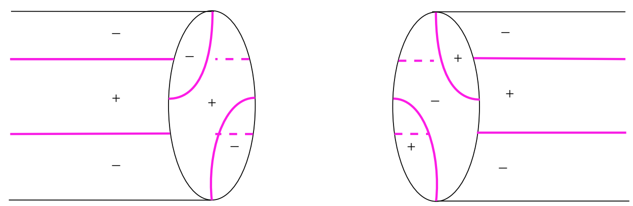

FIGURE 25

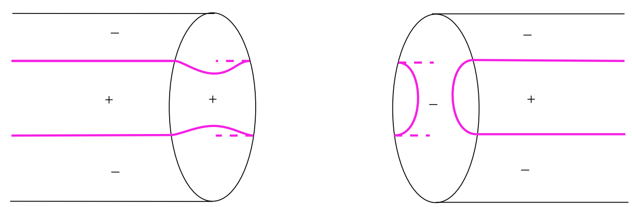

FIGURE 26

Observe that the dividing set in Figure 26 is isotopic to the sutures in Figure 24. Therefore, given a sutured manifold splitting $(M, \gamma) \overset{S}{\rightsquigarrow} (M', \gamma')$, there is a corresponding convex splitting $(M, \Gamma) \overset{(S,\Gamma_S)}{\rightsquigarrow} (M', \Gamma')$, with a ∂-parallel dividing set Γ_S. Using the decomposition theorem of Gabai, if (M, γ) is taut, then there exists a convex decomposition:

$$(M, \Gamma) \overset{(S_1,\Gamma_{S_1})}{\rightsquigarrow} (M_1, \Gamma_1) \overset{(S_2,\Gamma_{S_2})}{\rightsquigarrow} \cdots \overset{(S_n,\Gamma_{S_n})}{\rightsquigarrow} (M_n, \Gamma_n) = \sqcup (B^3, S^1).$$

We now work backwards, starting with Eliashberg's classification of tight contact structures on the 3-ball:

Theorem 13.9 (Eliashberg)**.** *Fix a characteristic foliation $\mathcal{F}$ adapted to $\Gamma_{\partial B^3} = S^1$. Then there is a unique tight contact structure on B^3 up to isotopy relative to ∂B^3.*

We now apply Colin's gluing theorem from last time to inductively build a universally tight contact structure carried by (M,Γ).

Theorem 13.10 (Colin)**.** *Let Σ be an incompressible surface with $\partial\Sigma \neq \emptyset$. If Γ_Σ is ∂-parallel and $(M \setminus \Sigma, \xi|_{M\setminus\Sigma})$ is universally tight, then (M,ξ) is also universally tight.*

This also gives an alternate proof of the Gabai-Eliashberg-Thurston theorem (with the exception of the last gluing along a closed surface – that requires a bit more work!).

14. Classification of tight contact structures on lens spaces

As an illustration of the technology introduced in the previous two sections, we give a complete classification of tight contact structures on the lens spaces $L(p,q)$.

14.1. Lens spaces.

Let $p > q > 0$ be relatively prime integers. The *lens space* $L(p,q)$ is obtained by gluing $V_1 = S^1 \times D^2$ and $V_2 = S^1 \times D^2$ together via $A : \partial V_2 \xrightarrow{\sim} \partial V_1$, where $A = \begin{pmatrix} -q & q' \\ p & -p' \end{pmatrix} \in -SL(2, \mathbf{Z})$. Here we are making an oriented identification $\partial V_i \simeq \mathbf{R}^2/\mathbf{Z}^2$, where the meridian of V_i is mapped to $\pm(1,0)$, and some chosen longitude is mapped to $\pm(0,1)$.

Continued fractions: Let $-\frac{p}{q}$ have a continued fraction expansion

$$-\frac{p}{q} = r_0 - \frac{1}{r_1 - \frac{1}{r_2 \cdots - \frac{1}{r_k}}},$$

where $r_i \leq -2$.

Example: $-\frac{14}{5} = -3 - \frac{1}{-5}$. We write $-\frac{14}{5} \leftrightarrow (-3,-5)$.

Theorem 14.1 (Giroux, Honda)**.** *On $L(p,q)$, there are exactly $|(r_0+1)(r_1+1)\dots(r_k+1)|$ tight contact structures up to isotopy. They are all* holomorphically fillable.

A surgery presentation for $L(p,q)$ is given as follows:

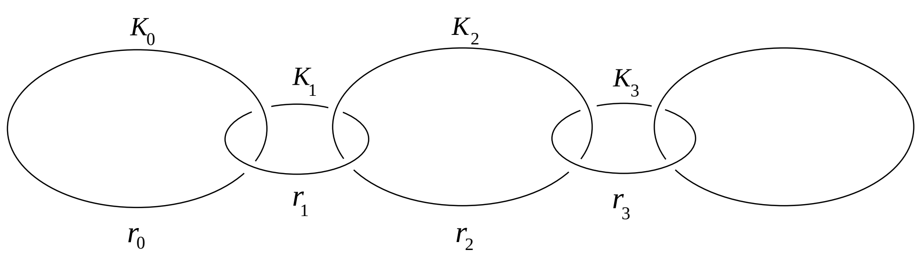

Figure 27

Legendrian surgery: Given a Legendrian knot $K = K_0$ or link $L = \sqcup_{i=0}^k K_i$ in a contact manifold (M,ξ), we can perform a surgery along the K_i with coefficient $tb(K_i) - 1$. At the 4-dimensional level, if $M = S^3$, then we start with a Stein domain B^4 with $\partial B^4 = S^3$, and attach 2-handles in a way which makes the resulting 4-manifold X^4 a Stein domain (and in particular symplectic). The resulting contact 3-manifold (M',ξ') with $\partial X = M'$ is said to be *holomorphically fillable*. Similarly, if (M,ξ) is symplectically fillable, then (M',ξ') obtained by Legendrian surgery is also symplectically fillable. The Stein construction was done by Eliashberg and the symplectic construction by Weinstein.

Suppose K_i is a Legendrian unknot with $tb(K_i) = r_i + 1$ and $r(K_i) =$ one of $r_i + 2, r_i + 4, \dots, -(r_i + 2)$. There are precisely $|r_i + 1|$ choices for the rotation number $r(K_i)$. (In fact, these are all the Legendrian unknots with $tb(K_i) = r_i + 1$ by Theorem 4.3.)

HW 46. *Show that the* $|r_0 + 1||r_1 + 1| \dots |r_k + 1|$ *holomorphically fillable contact structures are distinct.*

Therefore, we have the lower bound:

$$\#\text{Tight}(L(p,q)) \geq |(r_0 + 1)(r_1 + 1) \dots (r_k + 1)|. \tag{3}$$

Here Tight(M) refers to the set of isotopy classes of tight contact structures on M. In order to prove Theorem 14.1, it remains to show the reverse inequality.

14.2. **Solid tori.** We now consider tight contact structures on the solid torus $S^1 \times D^2$ with the following conditions on the boundary $T = S^1 \times D^2$:

1. $\#\Gamma_T = 2$.
2. slope$(\Gamma_T) = -\frac{p}{q}$, where $-\infty < -\frac{p}{q} \leq -1$. (After performing Dehn twists, we can normalize the slope as such.)
3. The fixed characteristic foliation $\mathcal{F}$ is adapted to Γ_T.

Theorem 14.2. *There are exactly* $|(r_0 + 1)(r_1 + 1) \dots (r_{k-1} + 1) r_k|$ *tight contact structures on* $S^1 \times D^2$ *with this boundary condition.*

Step 1: In this step we factor $S^1 \times D^2$ into a union of $T^2 \times I$ layers and a standard neighborhood of a Legendrian curve isotopic to the core curve of $S^1 \times D^2$. Assume $-\frac{p}{q} < -1$, since $-\frac{p}{q} = -1$ has already been treated.

Let D be a meridional disk with ∂D Legendrian and $tb(\partial D) = -p < -1$. Then by Lemma 10.9 there is at least one bypass along ∂D. Attach the bypass to T from the interior and apply the Bypass Attachment Lemma. We obtain a convex torus T' isotopic to T, such that T and T' cobound a $T^2 \times I$. Denote slope$(\Gamma_{T'}) = -\frac{p'}{q'}$.

HW 47. *If* $-\frac{p}{q} \leftrightarrow (r_0, r_1, \dots, r_{k-1}, r_k)$*, then* $-\frac{p'}{q'} \leftrightarrow (r_0, r_1, \dots, r_{k-1}, r_k + 1)$.

We successively peel off $T^2 \times I$ layers according to the Farey tessellation. The sequence of slopes is given by the continued fraction expansion, or, equivalently, by the shortest sequence of counterclockwise arcs in the Farey tessellation from $-\frac{p}{q}$ to -1. Once slope -1 is reached, $S^1 \times D^2$ with boundary slope -1 is the standard neighborhood of a Legendrian core curve with twisting number -1 (with respect to the fibration induced from the S^1-fibers $S^1 \times \{pt\}$).

Step 2: (Analysis of each $T^2 \times I$ layer)

Fact: Consider $T^2 \times [0,1]$ with convex boundary conditions $\#\Gamma_0 = \#\Gamma_1 = 2$, $s_0 = \infty$, and $s_1 = 0$. Here we write $\Gamma_i = \Gamma_{T^2 \times \{i\}}$ and $s_i = \text{slope}(\Gamma_i)$. (More invariantly, the shortest integers corresponding s_0, s_1 form an integral basis for $\mathbf{Z}^2$.) Then there are exactly two tight contact structures (up to isotopy rel boundary) which are *minimally twisting*, i.e., every convex torus T' isotopic to $T^2 \times \{i\}$ has slope$(\Gamma_{T'})$ in the interval $(0, +\infty)$. They are distinguished by the Poincaré duals of the *relative half-Euler class*, which are computed to be $\pm((1,0)-(0,1)) \in H_1(T^2\times[0,1];\mathbf{Z})$. We call these $T^2\times[0,1]$ layers *basic slices*.

The proof of the fact will be omitted, but one of the key elements in the proof is the following lemma:

HW 48. *Prove, using the Imbalance Principle, that for any tight contact structure on $T^2\times[0,1]$ with boundary slopes $s_0\neq s_1$ and any rational slope s in the interval (s_1,s_0), there exists a convex surface $T'\subset T^2\times[0,1]$, which is parallel to $T^2\times\{pt\}$ and has slope s. Here, if $s_0<s_1$, (s_1,s_0) means $(s_1,+\infty]\cup[-\infty,s_0)$.*

Step 3: (Shuffling) Consider the example of the solid torus where $-\frac{p}{q}=-\frac{14}{5}$. We have the following factorization:

$$\begin{array}{ccc}
-\frac{14}{5} & \leftrightarrow & (-3,-5)\\
-\frac{11}{4} & \leftrightarrow & (-3,-4)\\
-\frac{8}{3} & \leftrightarrow & (-3,-3)\\
-\frac{5}{2} & \leftrightarrow & (-3,-2)\\
-2 & \leftrightarrow & (-3,-1)=(-2)\\
-1 & \leftrightarrow & (-1)
\end{array}$$

We group the basic slices into *continued fraction blocks*. Each block consists of all the slopes whose continued fraction representations are of the same length. In the example, we have two blocks: slope $-\frac{14}{5}$ to -2, and slope -2 to -1. All the relative half-Euler classes of the basic slices in the first block are $\pm(-1,3)$; for the second block, they are $\pm(0,1)$. Therefore, a naive upper bound for the number of tight contact structures would be 2 to the power $\#(\text{basic slices})$.

A closer inspection however reveals that we may *shuffle* basic slices which are in the same continued fraction block. More precisely, if $T^2\times[0,2]$ admits a factoring into basic slices $T^2\times[0,1]$ and $T^2\times[1,2]$ with relative half-Euler classes (a,b) and $-(a,b)$, then it also admits a factoring into basic slices where the relative half-Euler classes are $-(a,b)$ and (a,b), i.e., the order is reversed.

Shuffling is (more or less) equivalent to the following proposition:

Lemma 14.3. *Let L be a Legendrian knot. Then $S_+S_-(L)=S_-S_+(L)$.*

HW 49. *Prove Lemma 14.3. (Observe that the ambient contact manifold is irrelevant and that the commutation can be done in a standard tubular neighborhood of L.)*

Returning to the example at hand, the first continued fraction block has at most $|-5|=4+1$ tight contact structures (distinguished by the relative half-Euler class), and the second has at most $|-3+1|=2$ tight contact structures. We compute $\#\text{Tight}\leq 2\cdot 5$.

In general, for the solid torus with slope $-\frac{p}{q}\leftrightarrow(r_0,r_1,\dots,r_k)$ we have:

$$\#\text{Tight}\leq|(r_0+1)(r_1+1)\dots(r_{k-1}+1)r_k|. \tag{4}$$

14.3. **Completion of the proof of Theorems 14.1 and 14.2.** We prove the following, which instantaneously completes the proof of both theorems.

$$\#\text{Tight}(L(p,q))\leq|(r_0+1)(r_1+1)\dots(r_k+1)|. \tag{5}$$

Recall that on ∂V_1, the meridian of V_2 has slope $-\frac{p}{q} \leftrightarrow (r_0, r_1, \ldots, r_{k-1}, r_k)$. First, take a Legendrian curve γ isotopic to the core curve of V_2 with largest twisting number. (Such a Legendrian curve exists, since any closed curve admits a C^0-small approximation by a Legendrian curve; the upper bound exists by the Thurston-Bennequin inequality.) We may assume V_2 is the standard neighborhood of γ; the tight contact structure on V_2 is then unique up to isotopy. Next, slope$(\Gamma_{\partial V_1}) = -\frac{p'}{q'} \leftrightarrow (r_0, \ldots, r_{k-1}, r_k + 1)$, and we have already computed the upper bound for $\#$Tight(V_2) to be $|(r_0 + 1) \ldots (r_{k-1} + 1)(r_k + 1)|$ by Equation 4. This completes the proof of Equation 5 and hence of Theorems 14.1 and 14.2.

Question 14.4. *Give a complete classification of tight contact structures on $T^2 \times [0, 1]$ when $\#\Gamma_{T^2\times\{i\}} > 2$, $i = 0, 1$. (The general answer is not yet known.)*

15. Open book decompositions

Today we explain the relationship contact structures and open books. Although we will only consider the 3-dimensional situation, there exist higher-dimensional extensions of this theory.

15.1. **Open books and fibered links.**

Definition 15.1. *An* open book decomposition *of a closed, oriented 3-manifold M is a pair (K, θ), where:*

1. *K is a link in M;*
2. *$\theta : M \setminus K \to S^1$ is a fibration which coincides with the angular coordinate of D^2 in a neighborhood $K \times D^2$ of $K = K \times \{0\}$.*

Such a link K is called a fibered link, *since its complement fibers over S^1.*

Alternatively, M can be written as follows: Let ϕ be a diffeomorphism of F to itself which is identity along ∂F. Then $M = ((F \times [0,1])/\sim) \cup (K \times D^2)$, where $(x, 1) \sim (\phi(x), 0)$, and $\partial F \times \{t\}$ is identified with $K \times \{r = 1, \theta = 2\pi t\}$. [We will not make the distinction between the fiber F in the "relative mapping torus" description in this paragraph and the fiber of θ. Either will also be called a *page* of the open book.]

An open book decomposition induces a special type of *Heegaard decomposition*, i.e., a decomposition of M into two genus g handlebodies H_1 and H_2. Let $H_1 = F \times [0, 1/2]$ and $H_2 = M - H_1 = (F \times [1/2, 1]) \cup (K \times D^2) \simeq F \times [1/2, 1]$.

Detecting fibered knots: We view $M \setminus N(K)$ as a sutured manifold with torus suture $\partial N(K)$. Splitting $M \setminus N(K)$ along $F \times \{0\}$ and $F \times \{1/2\}$, we obtain $H_1 = F \times [0, 1/2]$ and $H_2 = F \times [1/2, 1]$, with sutures $\gamma_{\partial H_1} = (\partial F) \times [0, 1/2]$ and $\gamma_{\partial H_2} = (\partial F) \times [1/2, 1]$.

Definition 15.2. *A sutured handlebody H of genus g is* disk decomposable *if there is a collection of embedded disks $D_1, \dots, D_g$ for which $\gamma_{\partial H}$ intersects ∂D_i in two arcs, where the intersection is essential and transverse, so that the resulting sutured manifold is $D^2 \times I$ with suture $\partial D^2 \times I$. Here $\#$ denotes the cardinality.*

Observe that $F \times I$ with sutures $(\partial F) \times I$ admits a disk decomposition: Let δ be an essential arc on F. Then $\delta \times I$ intersects γ in exactly two arcs. Successively splitting along such $\delta \times I$'s, we eventually obtain $D^2 \times I$ with suture $(\partial D^2) \times I$. In fact, the converse is also true: disk decomposability of a sutured handlebody H indicates that $H = F \times I$ with sutures $(\partial F) \times I$. (Check this!) Therefore we have the following proposition:

Proposition 15.3. *Let $M \setminus N(K) = H_1 \cup H_2$, where H_1 is a neighborhood of a Seifert surface and H_2 is the complement of H_1. If H_2 admits a disk decomposition, then K is a fibered link, with the Seifert surface as a fiber.*

Observe that $H_1 = F \times I$ is already disk decomposable.

We will often blur the distinction between sutured manifolds and convex surfaces, and think of M itself as decomposed into $H_1 \cup H_2$ with dividing set $\Gamma_{\partial H_1} = \Gamma_{\partial H_2}$.

Examples: The unknot, the Hopf link (two unknots linked once), the trefoil, and the figure 8 knot are all examples of fibered knots/links. The knot with the smallest number of crossings which is not fibered is the 5_2 knot (see Rolfson's knot tables). To see any of these links K are fibered, take a Seifert surface F for K, obtained using *Seifert's algorithm*. Let H_1 be a thickening of K, and take $\gamma_{\partial H_1}$ to be K. H_2 is its complement. It is easy to check the disk decomposability of H_2.

15.2. **Contact structures adapted to open books.**

Definition 15.4. *A contact structure ξ on M is* adapted to the open book (K, θ) *if there is a contact 1-form α which:*

1. *induces a symplectic form $d\alpha$ on each fiber F of θ;*
2. *K is transverse to ξ, and the orientation on K given by α is the same as the boundary orientation induced from F coming from the symplectic structure.*

Proposition 15.5 (Thurston-Winkelnkemper)**.** *Given an open book decomposition of M, there exists a contact structure ξ which is adapted to the open book.*

Example: The unknot K is a fibered link (knot) in S^3, with fiber $F = D^2$. On $S^1 \times D^2$ with coordinates (t, x, y), take the primitive $\beta = -\frac{1}{2}(ydx - xdy)$ for the area form $d\beta$ on D^2, and set $\alpha = dt + \beta$. This can be easily extended to the standard tight contact structure on S^3 in a neighborhood of K. Similarly, the standard tight contact structure on S^3 is also adapted to the Hopf link K (link of two unknots oriented so that the linking number is 1). [More precisely, take the 1-form $\alpha = r_1^2 d\theta_1 + r_2^2 d\theta_2$ where $z_j = r_j e^{i\theta_j}$. Then the unknot is given by $r_2 = 0$ and $\pi(r_1, \theta_1, r_2, \theta_2) = \theta_2$. The Hopf link is given by $r_1 r_2 = 0$ with $\pi = \theta_1 + \theta_2$.]

Proof. Let $\phi : F \to F$ be the monodromy of the open book, and let β be a 1-form on F so that $d\beta$ is an area form and β is transverse to ∂F with the proper orientation. We want to modify $dt + \beta$ so that it is invariant when we glue using the monodromy map ϕ to construct the mapping torus. To this end, consider $\beta_t = (1-t)\beta + t\phi^*\beta$ for $t \in [0, 1]$. Then for $\varepsilon > 0$ small, $\alpha = dt + \varepsilon\beta_t$ is contact on $F \times [0, 1]$:

$$d\alpha = \varepsilon((1-t)d\beta + t\phi^*(d\beta)) + \varepsilon dt \wedge (-\beta + \phi^*\beta),$$

and the $\alpha \wedge d\alpha$ terms that we do not like are of order ε^2. By construction, $dt + \varepsilon\beta_t$ is invariant under the gluing. □

Moreover, the following is true:

Proposition 15.6. *Two contact structures that are supported by the same open book are isotopic.*

HW 50. *Prove this! (Hint: this is an application of the Moser technique. Some care is needed near the binding K.)*

15.3. **Proof of Giroux's theorem.** We prove the converse:

Theorem 15.7 (Giroux)**.** *Every contact structure (M, ξ) is supported by some open book decomposition. Moreover, two supporting open books for (M, ξ) become the same after* stabilization.

Definition 15.8. *Given a diffeomorphism $\phi : F \to F$ which is the identity on the boundary, a* stabilization *is given by $\phi' : F' \to F'$, where F' is obtained from F by attaching a 1-handle H onto ∂F and $\phi' = \phi \circ \phi_\gamma$, where ϕ_γ is a positive (or right-handed) Dehn twist about a closed curve γ which intersects the cocore of H at one point.*

Proof of existence of open book. Let (M, ξ) be a contact structure – tight or overtwisted. Let τ be a triangulation of M so that each 3-simplex is contained in a standard tight $\mathbf{R}^3$ (by Pfaff's theorem). By using the C^0-approximation theorem by Legendrian curves, we may take the 1-skeleton to be Legendrian. (For safety, make sure each edge twists sufficiently by adding stabilizing.) Next make each face S convex – this is possible since $tb(\partial S) << 0$. Now use the Legendrian realization principle to find (properly embedded) Legendrian graphs in S which subdivide S into polygons P with $tb(\partial P) = -1$. Note that $tb(\partial P) = -1$ means that P contains precisely one dividing curve (an arc) of Γ_S. By adding the Legendrian graphs to the 1-skeleton, we obtain a *cell decomposition* of M. Let H_1 be a neighborhood of the Legendrian graph and H_2 be its complement. $(H_1, \Gamma_{\partial H_1})$ is easily seen to be disk decomposable. $(H_2, \Gamma_{\partial H_1})$, on the other hand, is disk decomposable with decompositions P with $tb(\partial P) = -1$! In view of Proposition 15.3, we obtain an open book decomposition for (M, ξ). $\square$

Corollary 15.9. *A contact structure ξ on M is holomorphically fillable iff ξ is supported by an open book whose monodromy is a product of positive Dehn twists.*