NOTES FOR MATH 535A: DIFFERENTIAL GEOMETRY

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1. Day 1

1.1. Review of topology.

Definition 1.1. *A* topological space $(X, \mathcal{T})$ *consists of a set* X*, together with a collection* $\mathcal{T} = \{U_\alpha\}$ *of subsets of* X*, satisfying the following:*

1. $\emptyset \in \mathcal{T}$, $X \in \mathcal{T}$,
2. *if* $U_\alpha, U_\beta \in \mathcal{T}$*, then* $U_\alpha \cap U_\beta \in \mathcal{T}$,
3. *if* $U_\alpha \in \mathcal{T}$ *for all* $\alpha \in I$*, then* $\cup_{\alpha \in I} U_\alpha \in \mathcal{T}$*. (Here* I *is an indexing set, and is not necessarily finite.)*

$\mathcal{T}$ *is called a* topology *for* X*, and* $U_\alpha \in \mathcal{T}$ *is called an* open set *of* X*.*

Example 1: $\mathbf{R}^n = \mathbf{R} \times \mathbf{R} \times \cdots \times \mathbf{R}$ (n times) $= \{(x_1, \dots, x_n) | x_i \in \mathbf{R}, i = 1, \dots, n\}$, called *real* n*-dimensional space.*

How to define $\mathcal{T}$, the set of open sets of $\mathbf{R}^n$?

$B_y(r) = \{x \in \mathbf{R}^n || x - y| < r\}$, where $|x - y| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$.

Is $\mathcal{T}_0 = \{B_y(r) | y \in \mathbf{R}^n, r \in (0, \infty)\}$ a valid *topology* for $\mathbf{R}^n$?

No, so you must add more open sets to $\mathcal{T}_0$ to get a valid topology for $\mathbf{R}^n$. $\mathcal{T} = \{U | \forall p \in U, \exists B_p(r) \subset U\}$.

Example 2: $S^1 = \{(x, y) \in \mathbf{R}^2 | x^2 + y^2 = 1\}$. A reasonable topology on S^1 is called the *induced topology.*

Definition 1.2. *Let* $(X, \mathcal{T})$ *be a topological space and* $Y \subset X$*. Then define the* induced topology $\mathcal{T}_X = \{U \cap Y | U \in \mathcal{T}\}$*.*

Example 3: Alternative definition of S^1 as $[0, 1]/0 \sim 1$, where $[0, 1]$ is the closed interval and we are identifying 0 with 1. A reasonable topology on S^1 is called the *quotient topology.*

Definition 1.3. *Let* $(X, \mathcal{T})$ *be a topological space and* $X/\sim$ *a set obtained by identifying elements of* X *via an equivalence relation* $\sim$*. Then the* quotient topology $\mathcal{T}_\sim$ *of* $X/\sim$ *is the set of* $V \subset X/\sim$ *for which* $f^{-1}(V)$ *is open. (Alternatively, given a topological space* $(X, \mathcal{T})$

and a surjective map $f : X \twoheadrightarrow Y$, the quotient topology *on Y is the coarsest topology which makes f continuous.)*

Exercise: Show that the topologies introduced in Example 2 and Example 3 are the same. (Think about what it means for them to be the same.)

Definition 1.4. *A map $f : X \to Y$ between topological spaces is* continuous *if $f^{-1}(V) = \{x \in X | f(x) \in V\}$ is open whenever $V \subset Y$ is open.*

Exercise: Show that the inclusion $S^1 \subset \mathbf{R}^2$ is a continuous map. Show that the quotient map $[0, 1] \to S^1 = [0, 1]/\sim$ is a continuous map.

Definition 1.5. *A map $f : X \to Y$ is a* homeomorphism *is there exists an inverse $f^{-1} : Y \to X$ for which f and f^{-1} are both continuous.*

Zen of mathematics: Any "category" (world) in mathematics consists of spaces and maps between spaces.

Examples:

1. (Topological category) Topological spaces and continuous maps.
2. (Groups) Groups and homomorphisms.
3. (Linear category) Vector spaces and linear transformations.

1.2. **Topological manifolds.**

Definition 1.6. *A* topological manifold *X of* dimension *n is a topological space X together with $\mathcal{A} = \{U_\alpha\}$ of open sets (called an* atlas *of X) such that:*

1. $\cup U_\alpha = X$,
2. $\exists \phi_\alpha : U_\alpha \to \mathbf{R}^n$, *which is a homeomorphism onto its image.*
3. *(Technical condition 1) X is Hausdorff.*
4. *(Technical condition 2) X is second countable.*

Definition 1.7. *A topological space X is* Hausdorff *if for any $x \neq y \in X$ there exist open sets U_x and U_y containing x, y respectively and $U_x \cap U_y = \emptyset$.*

Definition 1.8. *A topological space $(X, \mathcal{T})$ is* second countable *if there exists a countable subcollection $\mathcal{T}_0$ of $\mathcal{T}$ and any open set $U \in \mathcal{T}$ is a union (not necessarily finite) of open sets in $\mathcal{T}_0$.*

Exercise: Show that S^1 defined above is a topological manifold.

Exercise: Give an example of a topological space X which is not a topological manifold. (You may have trouble proving that it is not a topological manifold, though.)

Observe that in the land of topological manifolds, a square and a circle are the same, i.e., they are homeomorphic! That is not the world we will explore — in other words, we seek a category (world) where squares are not the same as circles. In other words, we need derivatives!

2. Day 2

2.1. Review of linear algebra.

Definition 2.1. *A vector space V over a field $k = \mathbf{R}$ or $\mathbf{C}$ is a set V equipped with two operations $V \times V \to V$ (called addition) and $k \times V \to V$ (called scalar multiplication) s.t.*

1. *V is an* abelian group *under addition.*
   (a) *(Identity) There is a zero element 0 s.t. $0 + v = v + 0 = v$.*
   (b) *(Inverse) Given $v \in V$ there exists an element $w \in V$ s.t. $v + w = w + v = 0$.*
   (c) *(Associativity) $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$.*
   (d) *(Commutativity) $v + w = w + v$.*
2. (a) $1v = v$.
   (b) $(ab)v = a(bv)$.
   (c) $a(v + w) = av + aw$.
   (d) $(a + b)v = av + bv$.

Note: Keep in mind the Zen of mathematics — we have defined objects (vector spaces), and now we need to define maps between objects.

Definition 2.2. *A linear map $\phi : V \to W$ between vector spaces over k satisfies $\phi(v_1+v_2) = \phi(v_1) + \phi(v_2)$ $(v_1, v_2 \in V)$ and $\phi(cv) = c \cdot \phi(v)$ $(c \in k$ and $v \in W)$.*

Now, what is the equivalent of *homeomorphism* in the linear category?

Definition 2.3. *A linear map $\phi : V \to W$ is an* isomorphism *if there exists a linear map $\psi : W \to V$ such that $\phi \circ \psi = id$ and $\psi \circ \phi = id$. (We often also say ϕ is* invertible.*)*

If V and W are finite-dimensional, then we may take bases $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_m\}$ and represent ϕ as an $m \times n$ matrix A. ϕ is then invertible if and only if $m = n$ and $\det(A) \neq 0$.

Examples of vector spaces: Let $\phi : V \to W$ be a linear map of vector spaces.

1. The *kernel* $ker(\phi) = \{v \in V | \phi(v) = 0\}$ is a vector subspace of V.
2. The *image* $im(\phi) = \{\phi(v) | v \in V\}$ is a vector subspace of W.
3. Let $V \subset W$ be a subspace. Then the *quotient* $W/V = \{w + V | w \in W\}$ is a vector space. Here $w + V = \{w + v | v \in V\}$.
4. The *cokernel* $coker(\phi) = W/im(\phi)$.

2.2. Review of differentiation.

Definition 2.4. *A map $f : \mathbf{R}^n \to \mathbf{R}^m$ is* differentiable *at a point $x \in \mathbf{R}^n$ if there exists a linear map $L : \mathbf{R}^n \to \mathbf{R}^m$ satisfying*

$$\lim_{h \to 0} \frac{|f(x + h) - f(x) - L(h)|}{|h|} = 0.$$

If L exists, it is usually written as $df(x)$.

Proposition 2.5. *$df(x)$ is a linear map $\mathbf{R}^n \to \mathbf{R}^m$ satisfying*

$$df(x)(v) = \lim_{t\to 0} \frac{f(x+tv)-f(x)}{t}.$$

$df(x)(v)$ is usually called the directional derivative *of f in the direction of v.*

Let e_j be the usual basis element $(0,\dots,1,\dots,0)$ (1 in the jth position). Then $df(x)(e_j)$ is usually called the *partial derivative* $\frac{\partial f}{\partial x_j}(x)$. More explicitly, write $f=(f_1,\dots,f_m)$, where $f_i : \mathbf{R}^n \to \mathbf{R}$. Then $df(x)(e_j) = (\frac{\partial f_1}{\partial x_j}(x),\dots,\frac{\partial f_m}{\partial x_j}(x))^T$, and

$$df(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \dots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \dots & \frac{\partial f_m}{\partial x_n}(x) \end{pmatrix}$$

Note: This is often called the *Jacobian matrix.*

Shorthand: $\partial_i f = \frac{\partial f}{\partial x_i}$. Also write $\partial^\alpha f = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\dots\partial_n^{\alpha_n} f$.

Definition 2.6. *$f : \mathbf{R}^n \to \mathbf{R}^m$ is* smooth *(or C^∞) at $x \in \mathbf{R}^n$ if all partial derivatives of all orders exist at x.*

2.3. **Differentiable manifolds.**

Definition 2.7. *A* smooth manifold *is a topological manifold $(X, \{U_\alpha\})$ satisfying the following: For every $U_\alpha \cap U_\beta \neq \emptyset$, $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is a smooth map.*

"Smooth" means you can take as many derivatives as you want.

Note: When we refer to a "manifold", we mean a "smooth manifold", unless stated otherwise.

3. Day 3

3.1. Examples of smooth manifolds.

1. $\mathbf{R}^n$. Atlas is $U = \mathbf{R}^n$ itself and map is identity.
2. Any open subset U of a smooth manifold $(M, \{U_\alpha\})$. Atlas: $\{U_\alpha \cap U\}$.
3. Let M_n be the space of $n \times n$ matrices, and let $Gl(n, \mathbf{R}) = \{A \in M_n | \det(A) \neq 1\}$. $Gl(n, \mathbf{R})$ is an open subset of $M_n \simeq \mathbf{R}^{n^2}$ and is called the *general linear group* of $n \times n$ real matrices.
4. If M and N are smooth manifolds, then their *product* $M \times N$ can naturally be given the structure of a smooth manifold. Simply use $U_\alpha \times V_\beta$ where $\{U_\alpha\}$ is an atlas for M and $\{V_\beta\}$ is an atlas for N.
5. $S^1 = \{x^2 + y^2 = 1\}$ is a smooth 1-dimensional manifold. One possible atlas: $U_1 = \{y > 0\}$, $U_2 = \{y < 0\}$, $U_3 = \{x > 0\}$, $U_4 = \{x < 0\}$. Another atlas is: $U_1 = \{y \neq 1\}$ and $U_2 = \{y \neq -1\}$. Take the stereographic projection from U_1 to $y = -1$ and U_2 to $y = 1$, and compare the overlaps.
6. $S^n = \{x_1^2 + \cdots + x_n^2 = 1\} \subset \mathbf{R}^n$.
7. In dimension 2, S^2, T^2, genus g surface.

3.1.1. *A more difficult example.* $\mathbf{RP}^n = (\mathbf{R}^{n+1} - \{(0, \dots, 0)\}) / \sim$, where $(x_0, x_1, \dots, x_n) \sim (tx_0, tx_1, \dots, tx_n)$, $t \in \mathbf{R} - \{0\}$. Chart $U_0 = \{x_0 \neq 0\}$. Then $\phi_0 : U_0 \to \mathbf{R}^n$ is given by

$$(x_0, x_1, \dots, x_n) = (1, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}) \mapsto (\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}).$$

What is $\phi_i : U_0 \to \mathbf{R}^n$? What about transition maps $\phi_j \circ \phi_i^{-1}$? (Explain this in detail.) $\mathbf{RP}^n$ is called *real projective space* of dimension n.

3.1.2. *Group action.* $T^2 = \mathbf{R}^2 / \mathbf{Z}^2$. The discrete group $\mathbf{Z}^2$ acts on $\mathbf{R}^2$ by translation:

$$\mathbf{Z}^2 \times \mathbf{R}^2 \to \mathbf{R}^2$$

$$((m, n), (x, y)) \mapsto (m + x, n + y).$$

Note that for each fixed (m, n), we have a diffeomorphism $\mathbf{R}^2 \to \mathbf{R}^2$, $(x, y) \mapsto (m+x, n+y)$. $\mathbf{R}^2 / \mathbf{Z}^2$ is the set of orbits of $\mathbf{R}^2$ under the action of $\mathbf{Z}^2$. (One orbit is $(x, y) + \mathbf{Z}^2$.) Discuss the fundamental domain $[0, 1] \times [0, 1]$ and identifications of the sides.

Next time: Try to answer the question of what it means for two atlases of the same M to be "the same".

Some more zen: You can study an object (such as a manifold) either by looking at the object itself — in this case M — or by looking at the space of functions on the object. In the topological category, the space of functions would be $C^0(M)$, the space of continuous functions. What is it in the case of smooth manifolds?

4. Day 4

4.1. **Choice of atlas.** Let $(M, \mathcal{T})$ be the underlying topological space of a manifold, and $\mathcal{A}_1 = \{(U_\alpha, \phi_\alpha)\}$, $\mathcal{A}_2 = \{(V_\beta, \psi_\beta)\}$ be two atlases.

Question: When do they represent the *same* smooth manifold?

Condition 1: For any $U_\alpha \cap V_\beta \neq \emptyset$, $\phi_\alpha(U_\alpha \cap V_\beta) \stackrel{\psi_\beta \circ \phi_\alpha^{-1}}{\longrightarrow} \psi_\beta(U_\alpha \cap V_\beta)$ is a smooth map.

If Condition 1 is met, can take the union $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$. A maximal atlas A_{max} is an object which is uniquely assigned to each smooth structure.

4.2. **Function space perspective.** Recall: $C^0(M) = \{f : M \to \mathbf{R} \text{ continuous}\}$. Given a smooth manifold $(M, \mathcal{A})$, the appropriate space of functions would be $C^\infty_{\mathcal{A}}(M) = \{f : M \to \mathbf{R} \text{ smooth}\}$. $f \in C^0(M)$ is a smooth function if $f \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha) \to \mathbf{R}$ is smooth for each coordinate chart U_α of $\mathcal{A}$. This function space perspective has been especially fruitful in algebraic geometry.

An atlas $\mathcal{A}$ uniquely determines the space of smooth functions on M. When the atlas on M is understood, we write $C^\infty(M)$ for the space of smooth functions on M.

Condition 2: Two atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ give the same smooth manifold if $C^\infty_{\mathcal{A}_1}(M) = C^\infty_{\mathcal{A}_2}(M)$.

Pullback: $\phi : X \to Y$ a continuous map between topological spaces. Then there is a naturally defined *pullback map*

$$\phi^* : C^0(Y) \to C^0(X)$$

given by $f \mapsto f \circ \phi$. Note that pullback is *contravariant*, i.e., the direction is from Y to X, which is the opposite from the original map ϕ.

When $\psi : (M, \mathcal{A}) \to (M, \mathcal{A})$ is a homeomorphism, then $\psi^* : C^0(M) \to C^0(M)$. $C^\infty_{\mathcal{A}}(M) \stackrel{\sim}{\to} \psi^*(C^\infty_{\mathcal{A}}(M)))$, but in general $C^\infty_{\mathcal{A}}(M) \neq \psi^*(C^\infty_{\mathcal{A}}(M)))$.

Definition 4.1. *Two C^∞-structures $C^\infty_{\mathcal{A}_1}(M)$ and $C^\infty_{\mathcal{A}_2}(M)$ are* equivalent *if there exists a homeomorphism of M which takes $C^\infty_{\mathcal{A}_1}(M) \simeq C^\infty_{\mathcal{A}_2}(M)$.*

Amazing fact: (Due to Milnor) S^7 has several inequivalent smooth structures! (Not amazingly, S^1 has only one smooth structure.)

Open question: How many smooth structures does S^4 have?

4.3. **Smooth maps.** In the category of smooth manifolds, we need to define the appropriate maps, called *smooth maps.*

Definition 4.2. *A map $f : M \to N$ between smooth manifolds is* smooth *if $f^*(C^\infty(N)) = C^\infty(M)$.*

Problem: Give an interpretation in terms of local coordinates. Answer: Given $p \in M$, $\exists$ $U \ni p$ and $V \ni f(p)$ coordinate charts s.t. composition

$$\phi(U) \xrightarrow{\phi} U \xrightarrow{f} V \xrightarrow{\psi} \psi(V)$$

is smooth.

Upshot: Smooth maps between smooth manifolds can be "reduced" to smooth maps from $\mathbf{R}^n$ to $\mathbf{R}^m$.

4.4. **Inverse function theorem.**

Definition 4.3. *A differentiable (smooth) map $f : U \to V$ between two open sets of Euclidean space is a* diffeomorphism *if there is a differentiable (smooth) inverse $f^{-1} : V \to U$.*

Problem: Give the definition of a diffeomorphism $f : M \to N$ between two smooth manifolds.

The inverse function theorem is the most important basic theorem in differential geometry.

Theorem 4.4 (Inverse function theorem)**.** *Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a smooth map. If $df(x)$ is nonsingular (i.e., the determinant of $df(x)$ is nonzero), then df is a local diffeomorphism near x, that is there exist open sets U containing x and V containing $f(x)$ and $f|_U$ is a diffeomorphism.*

Note: f did not need to be defined on all of $\mathbf{R}^n$, only on a small open set around the origin.

The inverse function theorem says that isomorphism in the linear category implies diffeomorphism in the differentiable category. Allows us to move from "infinitesimal" to "local".

5. Day 5: Applications of the Inverse Function Theorem

Theorem 5.1 (Inverse Function Theorem)**.** *Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a smooth map. If $df(x) : \mathbf{R}^n \to \mathbf{R}^n$ is an isomorphism, then f is a local diffeomorphism near x.*

We'll treat just the local version $\mathbf{R}^n \to \mathbf{R}^n$.

Question: Adapt the inverse function theorem for a smooth map $f : M \to N$ between smooth manifolds.

5.1. **Illustrative example.** Let $f : \mathbf{R}^2 \to \mathbf{R}$, $(x, y) \mapsto x^2 + y^2 - 1$. We examine $f^{-1}(0)$. Consider the portion $x > 0$. Then form $F : \mathbf{R}^2 \to \mathbf{R}^2$, $(x, y) \mapsto (f(x, y), y)$.

$$df = \begin{pmatrix} 2x & 2y \\ 0 & 1 \end{pmatrix}.$$

Use the inverse function theorem — since $\det(df) = 2x \neq 0$, there is a local diffeomorphism between $f(x, y) = 0$ and the real line (draw picture). Hence $f^{-1}(0)$ is a smooth manifold.

Interpret slightly differently: The above example basically says that f can be used as a coordinate function in conjunction with y, provided $x > 0$.

5.2. **Submersions.**

Rank: The *rank* of a linear map $L : V \to W$ is the dimension of $im(L)$. (Recall that the *dimension* of a vector space V is the cardinality of a basis for V. If V is finite-dimensional, then $V \simeq \mathbf{R}^n$ for some n.) The *rank* of a smooth map $f : \mathbf{R}^n \to \mathbf{R}^m$ at $x \in \mathbf{R}^n$ is the dimension of $df(x)$. f has *constant rank* if the rank of $df(x)$ is constant.

Definition 5.2. *A smooth map $f : \mathbf{R}^n \to \mathbf{R}^m$ is a* submersion *if $df(x)$ is surjective for all x. (Note that this means that $n \geq m$.)*

Prototype: $f : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m$, $(x_1, \dots, x_{m+n}) \mapsto (x_1, \dots, x_m)$.

Theorem 5.3 (Implicit function theorem, submersion version)**.** *Let $f : U \subset \mathbf{R}^n \to V \subset \mathbf{R}^m$ be a submersion. Then there exists a local diffeomorphism $\phi : U' \xrightarrow{\sim} \mathbf{R}^n$, $U \supset U'$, s.t. $f \circ \phi^{-1} : \mathbf{R}^n \to \mathbf{R}^m$ is $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_m)$.*

Proof. Assume wlog that $f : 0 \mapsto 0$. Form $\phi : U \to V \times \mathbf{R}^{n-m}$ given by $(x_1, \dots, x_n) \mapsto (f_1, \dots, f_m, x_{m+1}, \dots, x_n)$. (We choose the appropriate $x_{m+1}, \dots, x_n$ so that $df(0)$ is invertible.) Then $f \circ \phi^{-1}(f_1, \dots, f_m, x_{m+1}, \dots, x_n) = (f_1, \dots, f_m)$, and F is the desired function. $\square$

Carving manifolds out of other manifolds: This shows in particular that if f is a submersion, then $f^{-1}(y)$, $y \in V$, is a manifold $\subset \mathbf{R}^n$. The easy way to prove that the circle $\{x^2 + y^2 = 1\} \subset \mathbf{R}^2$ is a manifold: Consider the map $f : \mathbf{R}^2 \to \mathbf{R}$, $f(x, y) = x^2 + y^2$. The

Jacobian $df(x,y) = (2x, 2y)$. Since x, y are never simultaneously zero, the rank of df is 1 at all points of S^1. Using the implicit function theorem, we are done.

Try to make sense of submersions for $f : M \to N$ smooth maps between smooth manifolds.

Check that rank is constant after changing coordinates? Compare the rank of $d(\psi_\alpha \circ f \circ \phi_\alpha^{-1})$ and $d(\psi_\beta \circ f \circ \phi_\beta^{-1})$, where $\phi_\alpha : U_\alpha \subset M \to \mathbf{R}^m$ and $\psi_\alpha : V_\alpha \subset N \to \mathbf{R}^n$. The invariance of rank is due to the following:

$$\begin{aligned} d(\psi_\beta \circ f \circ \phi_\beta^{-1}) &= d((\psi_\beta \circ \psi_\alpha^{-1}) \circ (\psi_\alpha \circ f \circ \phi_\alpha^{-1}) \circ (\phi_\alpha^{-1} \circ \phi_\beta)) \\ &= d(\psi_\beta \circ \psi_\alpha^{-1}) \circ d(\psi_\alpha \circ f \circ \phi_\alpha^{-1}) \circ d(\phi_\alpha^{-1} \circ \phi_\beta). \end{aligned}$$

Note that here we used the *Chain Rule.*

6. Day 6

6.1. **The Chain Rule.**

Theorem 6.1 (Chain Rule). *Let $U \subset \mathbf{R}^l$, $V \subset \mathbf{R}^m$, $W \subset \mathbf{R}^n$. Also let $f : U \to V$ and $g : U \to W$ be smooth maps. Then $g \circ f$ is smooth and $d(g \circ f)(x) = dg(f(x)) \circ df(x)$.*

Draw a picture of the maps and derivatives.

One consequence of the Chain Rule is:

Proposition 6.2. *If $f : U \to V$ is a diffeomorphism, then $df(x)$ is an isomorphism for all $x \in U$.*

Proof. Let $g : V \to U$ be the smooth inverse function. Then $g \circ f = id$. Taking derivatives, $dg \circ df = id$ as linear maps; hence $df(x)$ is an isomorphism for all x. □

6.2. **Regular values & Sard's Theorem.**

Definition 6.3. *Given $f : \mathbf{R}^n \to \mathbf{R}^m$, $y \in \mathbf{R}^m$ is said to be a* regular value *of f if for all $x \in f^{-1}(y)$, $df(x)$ is surjective. The implicit function theorem implies that $f^{-1}(y)$ is a manifold if y is a regular value. $x \in \mathbf{R}^n$ is a* critical point *of f if $df(x)$ is not surjective.*

Exercise: Prove that $S^n \subset \mathbf{R}^n$ is a manifold.

More involved example: $Sl(n, \mathbf{R}) = \{A \in M_n | \det(A) = 1\}$. Consider the determinant map $f : \mathbf{R}^{n^2} \to \mathbf{R}$, $A \mapsto \det(A)$. Also write $f : \mathbf{R}^n \times \dots \mathbf{R}^n \to \mathbf{R}$, $(a_1, \dots, a_n) \mapsto \det(a_1, \dots, a_n)$, where a_i are column vectors and $A = (a_1, \dots, a_n) = (a_{ij})$.

First we need some properties of the determinant:

1. $f(e_1, \dots, e_n) = 1$,
2. $f(a_1, \dots, c_i a_i + c_i' a_i', \dots, a_n) = c_i \cdot f(a_1, \dots, a_i, \dots, a_n) + c_i' \cdot f(a_1, \dots, a_i', \dots, a_n)$.

Property 2 is called *multilinearity.* In fact, Properties 1, 2 and 3 (alternating property) uniquely determine the determinant function.

Now compute $df(A)(B)$:

$$
\begin{aligned}
df(A)(B) &= \lim_{t\to 0} \frac{f(A+tB)-f(A)}{t} \\
&= \lim_{t\to 0} \frac{\det(a_1+tb_1,\ldots,a_n+tb_n)-\det(a_1,\ldots,a_n)}{t} \\
&= \lim_{t\to 0} \frac{\det(a_1,\ldots,a_n)+t[\det(b_1,a_2,\ldots,a_n)+\det(a_1,b_2,\ldots,a_n)}{t} \\
&\quad \frac{+\cdots+\det(a_1,\ldots,a_{n-1},b_n)]+t^2(\ldots)-\det(a_1,\ldots,a_n)}{t} \\
&= \det(b_1,a_1,\ldots,a_n)+\cdots+\det(a_1,\ldots,b_n)
\end{aligned}
$$

It's easy to show that 1 is a regular value of df. For example, take $b_1 = ca_1$, $b_i = 0$, $i \neq 1$.

Example: Zero sets of homogeneous polynomials in $\mathbf{RP}^n$. For example, in $\mathbf{RP}^2$ take $f(x_0,x_1,x_2) = x_0^3 + x_1^3 + x_2^3$. $f(tx) = t^3 f(x)$, $t \in \mathbf{R} - \{0\}$, $x \in \mathbf{R}^3$. The zero set of f is well-defined (denote by $Z(f)$). Is $Z(f)$ a submanifold?

Theorem 6.4 (Sard's Theorem)**.** *Let $f : \mathbf{R}^m \to \mathbf{R}^n$ be a smooth map. Then* almost every point *$y \in \mathbf{R}^n$ is a regular value.*

We'll make this more precise at a later date. But in the meantime:

Reality Check: What happens when $m < n$?

6.3. **Some more point-set topology.** We want to impose additional conditions to immersions to make them better-behaved. But before we do this, let's do some more point-set topology. Let X be a topological space.

1. A subset $V \subset X$ is *closed* if the complement $X - V = \{x \in X, x \notin V\}$ is open.
2. The *closure* $\overline{V}$ of a subset $V \subset X$ is the smallest closed set containing V.
3. A subset V of a topological space X is *dense* if for every open set U, $U \cap V \neq \emptyset$. In other words, $\overline{V} = X$.
4. A *compact set* V satisfies the following *finite covering property*: For any cover $\{U_\alpha\}$ of V there exists a finite subcover.
5. A metric space is compact iff every sequence has a convergent subsequence.
6. A subset of Euclidean space is compact if and only if it is closed and bounded (sits inside some ball of finite radius).
7. A map $f : X \to Y$ is *proper* if $f^{-1}(V)$ of every compact $V \subset Y$ is compact.

7. Day 7

7.1. Immersions.

Definition 7.1. *$f : \mathbf{R}^m \to \mathbf{R}^n$ is an* immersion *if $df(x)$ is injective for all x. (Note this means $n \geq m$.)*

Prototype: $f : \mathbf{R}^m \to \mathbf{R}^n$, $n \geq m$, $(x_1, \dots, x_m) \mapsto (x_1, \dots, x_m, 0, \dots, 0)$.

Theorem 7.2 (Implicit function theorem, immersion version)**.** *Let $f : U \subset \mathbf{R}^m \to V \subset \mathbf{R}^n$ be an immersion. Then there is a local diffeomorphism $\psi : V \to \mathbf{R}^n$ so that $\psi \circ f : U \to \mathbf{R}^n$ is $(x_1, \dots, x_m) \to (x_1, \dots, x_m, 0, \dots, 0)$.*

Proof. It's really the same as before. $F : U \times \mathbf{R}^{n-m} \to \mathbf{R}^n$, $(x_1, \dots, x_m, y_{m+1}, \dots, y_n) \mapsto (f_1, \dots, f_m, f_{m+1} + y_{m+1}, \dots, f_n + y_n)$. Can check to see dF is nonsingular and that $F^{-1} \circ f : U \to \mathbf{R}^n$ is given by $(x_1, \dots, x_m) \mapsto (x_1, \dots, x_m, 0, \dots, 0)$. □

Zen: The implicit function theorem tells us that under a constant rank condition we may assume that locally we can straighten our manifolds and maps and pretend we are doing linear algebra.

Examples of immersions:

1. Circle mapped to figure 8. (Not 1-1.)
2. Line mapped to figure 8.
3. Map $f : \mathbf{R} \to \mathbf{R}^2/\mathbf{Z}^2$, $t \mapsto (at, bt)$, where b/a is irrational, is an immersion. The image of f is *dense* in $\mathbf{R}^2/\mathbf{Z}^2$.

7.2. Embeddings and submanifolds.

We improve immersions $f : M \to N$ by requiring that f be:

1. 1-1.
2. Proper.

Such an f is called an *embedding*. The image of an embedding is a *submanifold*.

Explain why the pathological examples above were immersions but not embeddings.

Need to check the following:

Proposition 7.3. *The induced topology on $f(M)$ from N is the same as the topology on M, i.e., if U is an open set of M, then $f(U) \subset f(M)$ is open.*

Proof. Suppose not. Then $\exists y \in f(U)$ and a sequence $y_i \to y$ s.t. $y_i \notin f(U)$. The set $\{y_i, y\}$ is compact, so $\{x_i, x\}$ is compact by properness, where $x_i = f^{-1}(y_i)$ and $x = f^{-1}(y)$ (remember f is 1-1). There is a subsequence $x_i \to x$. Contradiction. □

8. Day 8: Tangent spaces.

8.1. **Concrete example.** Consider $S^2 = \{x^2+y^2+z^2 = 1\} \subset \mathbf{R}^3$. We compute the *tangent plane* $T_{(a,b,c)}S^2$. Draw picture.

We use the fact that S^2 is the preimage of the regular value 0 of $f : \mathbf{R}^3 \to \mathbf{R}$, $f(x, y, z) = x^2 + y^2 + z^2 - 1$.

$$df(a, b, c) = (2a, 2b, 2c).$$

The tangent directions are directions where $df(a, b, c)(x, y, z)^T = 0$. Therefore, it's, the plane through (a, b, c) parallel to $ax + by + cz = 0$, i.e., $ax + by + cz = a^2 + b^2 + c^2 = 1$.

We also define the *tangent bundle* $TS^2 = \sqcup_{p\in S^2} T_pS^2$. Here $\sqcup$ denotes disjoint union.

Question: How do you define a reasonable topology on TS^2?

Note that the tangent plane is easy to define when we think of M as a submanifold of $\mathbf{R}^n$.

8.2. **First definition.** Let M be a smooth manifold of dimension n.

Definition 8.1. *The* tangent space T_xM *to* M *at* x *is* $T_x(M) = \{$*smooth curves* $\gamma : (-\varepsilon, \varepsilon) \to M, \gamma(0) = x\}/\sim$, *where* $\gamma_1 \sim \gamma_2$ *if for all functions* $f : M \to \mathbf{R}$, $f \circ \gamma_1(t) = f \circ \gamma_2(t) + O(t^2)$.

Actually, if $x_1, \dots, x_n$ are coordinate functions near $x = 0$, it suffices to check that $x_i(\gamma_1(t)) = x_i(\gamma_2(t)) + O(t^2)$, thanks to the following theorem.

Theorem 8.2 (Taylor's Theorem). *Given* $f \in C^\infty(p)$, $f(x) = a + \sum_i a_i x_i + \sum_{i,j} a_{ij}(x)x_ix_j$, *where* a, a_i *are constants and* $a_{ij}(x)$ *are smooth functions.*

Proof. Let $g(t) = f(tx)$. Then, $g(1) - g(0) = \int_0^1 g'(t)dt$. Substituting back, we have:

$$\begin{aligned} f(x) - f(0) &= \int_0^1 \frac{d}{dt} f(tx)dt \\ &= x \cdot f'(tx) \cdot (t-1)|_{t=0}^{t=1} - \int_0^1 (t-1)x^2 \cdot f''(tx)dt \\ &= -xf'(0)(-1) - x^2 \int_0^1 (t-1)f''(tx)dt \\ &= f'(0) \cdot x + h(x) \cdot x^2 \end{aligned}$$

Here we used integration by parts with $u = f(tx)$, $v = t - 1$. □

In $\mathbf{R}^n$, it's clear that γ_1 and γ_2 are in the same equivalence class if $\gamma_1'(0) = \gamma_2'(0)$.

8.3. Second definition.

Sheaf-theoretic ideas: Let M be a smooth manifold and let $U \subset M$ be an open set. Then $C^\infty(U)$ is the set of smooth functions on U. Note $C^\infty(U)$ is an $\mathbf{R}$-algebra, i.e., it is endowed with the structure of a vector space over $\mathbf{R}$, together with a multiplication. Therefore we have operations $c \cdot f$, $f \cdot g$, $f + g$.

Let $V \subset M$ be any set. Then $C^\infty(V) = \{(f, U) | U \supset V, f : U \to \mathbf{R} \text{ smooth}\}/\sim$. Here $(f_1, U_1) \sim (f_2, U_2)$ if $\exists U \subset U_1 \cap U_2$ for which $f_1|_U = f_2|_U$. This is what we really mean by *smooth functions on* V, since we need open sets to define derivatives.

Given open sets $U_1 \subset U_2$, there exists a natural *restriction map* $\rho^{U_2}_{U_1} : C^\infty(U_2) \to C^\infty(U_1)$, $f \mapsto f|_{U_1}$. Then $C^\infty(V)$ is the *direct limit* of $C^\infty(U)$ for all U containing V.

In particular, we have $C^\infty(x)$, the *stalk* at the point x or the *germs of functions* at p.

Now, a *derivation* is an $\mathbf{R}$-linear map $X : C^\infty(x) \to \mathbf{R}$ which satisfies the Leibniz rule:

$$X(fg) = X(f) \cdot g(x) + f(x) \cdot X(g).$$

Then let T_xM be the set of derivations at x.

9. Day 9

9.1. **Second definition.** Recall that our second definition of T_pM is the set of derivations $C^\infty(p) \to \mathbf{R}$. A derivation is an $\mathbf{R}$-linear map which satisfies the Leibniz rule. Note that it did not matter whether M was a manifold — it could have been Euclidean space instead, since $C^\infty(p)$ only depends on a small neighborhood of x.

Exercise: $X(c) = 0$, $c \in \mathbf{R}$.

Examples.

1. $X_i = \frac{\partial}{\partial x_i}$. Take coordinates $(x_1, \dots, x_n)$ near $p = 0$. Then let $X(f) = \frac{\partial f}{\partial x_i}(0)$. Check: this is indeed a derivation and $\frac{\partial}{\partial x_i}(x_j) = \delta_{ij}$.
2. Given $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = p$, define $X(f) = (f \circ \gamma)'(0)$. This is the directional derivative in the direction γ. It is easy to check that two $\gamma \sim \gamma'$ give rise to the same directional derivative.

Proposition 9.1. *If M is an n-dimensional manifold, then* $\dim T_pM = n$.

Proof. Take local coordinates $x_1, \dots, x_n$ so that $p = 0$. Then let $X_i = \frac{\partial}{\partial x_i}$. Then $X_i(x_j) = \delta_{ij}$, and clearly the X_i are independent. Thus $\dim T_pM \geq n$. Now, given some derivation X, suppose $X(x_i) = b_i$. Taylor's Theorem implies that $\dim T_pM = n$, since all the quadratic terms and higher vanish for derivations. □

Proposition 9.2. *The first two definitions of T_pM are equivalent.*

Proof. Given $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = p$, one simply differentiates in that direction. $X(f) = (f \circ \gamma)'(0)$. (Check this is well-defined!) Since we already calculated $\dim T_pM = n$ for the first and second definitions, we see this map is surjective and hence an isomorphism. □

9.2. **Third definition.** Define $\mathcal{F}_p \subset C^\infty(p)$ to be germs of functions which are 0 at p. $\mathcal{F}$ is an *ideal* of $C^\infty(p)$. Let $\mathcal{F}_p^2 \subset \mathcal{F}_p$ be the ideal generated by products of elements of $\mathcal{F}_p$, i.e., consisting of elements $\sum f_i \phi_j \phi_k$, where $\phi_j, \phi_k \in \mathcal{F}_p$. Then let $T_pM = (\mathcal{F}_p/\mathcal{F}_p^2)^*$.

Equivalence of the second and third definitions: Show that $X : \mathcal{F}_p \to \mathbf{R}$ factors through $\mathcal{F}_p^2$. (Pretty easy, since it's a derivation.) Now, note that $\dim(\mathcal{F}_p/\mathcal{F}_p^2) = n$ due to Taylor's Theorem.

$\mathcal{F}_p/\mathcal{F}_p^2$ is called the *cotangent space* at p, and is denoted T_p^*M. If $f \in C^\infty(p)$, then $f - f(p) \in \mathcal{F}_p$, and is denoted $df(p)$.

9.3. **The tangent bundle.** Let $TM = \sqcup T_pM$. This is called the *tangent bundle*. We explain how to endow a smooth structure on the tangent bundle.

Consider the projection $\pi : TM \to M$, $q \in T_pM \mapsto p$. Let $U \subset M$ be an open set with coordinates $x_1, \dots, x_n$. We identify $\pi^{-1}(U) = U \times \mathbf{R}^n$. An element of T_xM is written as $\sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$. The point $q \in TM$ is given by coordinates $(x_1, \dots, x_n, a_1, \dots, a_n)$.

Check the transition functions. Let $x = (x_1, \dots, x_n)$ be coordinates on U and $y = (y_1, \dots, y_n)$ be coordinates on V. Let (x, a) be coordinates on $\pi^{-1}(U)$ and (y, b) be coordinates on $\pi^{-1}(V)$. Think of y as a function of x on $U \cap V$. Write $\frac{\partial y}{\partial x} = (\frac{\partial y_i}{\partial x_j})$. In terms of y coordinates,

$$\sum a_i \frac{\partial}{\partial x_i} = \sum a_i \frac{\partial y_j}{\partial x_i} \frac{\partial}{\partial y_j}.$$

This is easily verified by thinking of evaluation on functions. Thus, $b_j = \sum_i a_i \frac{\partial y_j}{\partial x_i}$.

Then $q \in TM$ corresponds to (x, a) or $(y, \frac{\partial y}{\partial x} a)$.

10. Day 10

10.1. **More on the tangent bundle.** Recall: (1) the projection $\pi : TM \to M$, (2) an atlas $\{U_\alpha\}$ for M gives an atlas $\{\pi^{-1}U_\alpha\}$.

Computation of the Jacobian of the transition function.

$$\begin{pmatrix} \frac{\partial y}{\partial x} & \frac{\partial y}{\partial a} \\ \frac{\partial b}{\partial x} & \frac{\partial b}{\partial a} \end{pmatrix} = \begin{pmatrix} \frac{\partial y}{\partial x} & 0 \\ \sum_k \frac{\partial^2 y_i}{\partial x_j \partial x_k} a_k & \frac{\partial y}{\partial x} \end{pmatrix}.$$

The two terms on the bottom are obtained by differentiating $b_i = \sum_k \frac{\partial y_i}{\partial x_k} a_k$.

Thus we obtain a smooth manifold TM and a C^∞-function $TM \stackrel{\pi}{\to} M$.

10.2. **The cotangent bundle.** We now "topologize" the *cotangent bundle* $T^*M = \sqcup_p T_p^*M$. Again we have a projection $\pi : T^*M \to M$. We identify $\pi^{-1}(U) \simeq U \times \mathbf{R}^n$. An element of T_p^*M is $f - f(p)$, usually denote df. In terms of the basis $dx_1, \dots, dx_n$, any df is written as $df = \sum_i \frac{\partial f}{\partial x_i} dx_i$. Therefore, an element of T_p^*M is $\sum a_i dx_i$.

Transition functions. Take $U, V \subset M$ as before, and coordinatize $\pi^{-1}(U)$ and $\pi^{-1}(V)$ by (x, a), (y, b).

Lemma 10.1.

$$dy_i = \sum_j \frac{\partial y_i}{\partial x_j} dx_j.$$

Proof. It suffices to assume that the transition map $\phi_{UV} : U \cap V \subset U \to U \cap V \subset V$ sends $0 \mapsto 0$. Then

$$dy_i = y_i(x) - y_i(0) = y_i(x) = \sum_j \frac{\partial y_i}{\partial x_j} x_j = \sum_j \frac{\partial y_i}{\partial x_j} dx_j.$$

□

We denote this more simply as $dy = \frac{\partial y}{\partial x} dx$. Then $dx = \left(\frac{\partial y}{\partial x}\right)^{-1} dy$. Hence, $\sum a_i dx_i = \sum a_i [(\frac{\partial y}{\partial x})^{-1}]_{ij} dy_j$, i.e., $(x, a) \mapsto (y, a\left(\frac{\partial y}{\partial x}\right)^{-1})$.

Exercise: Compute the Jacobian of the transition function.

10.3. **Functoriality.** Let $f : M \to N$ be a smooth map between manifolds. Then we can define two natural maps.

Contravariant functor. First observe that there is a map $C^\infty(f(p)) \stackrel{f^*}{\to} C^\infty(p)$ and also a restriction $\mathcal{F}_{f(p)} \stackrel{f^*}{\to} \mathcal{F}_p$. This allows us to define:

$$f^* : T^*_{f(p)}N \to T_p^*M,$$

$$dg \mapsto dg \circ f.$$

f^* is a *contravariant functor*.

Covariant functor. Next, we have:

$$f_* : T_pM \to T_{f(p)}M,$$

given by $X \mapsto X \circ f^*$. (Think of tangent spaces as derivations.) This makes sense:

$$C^\infty(f(p)) \xrightarrow{f^*} C^\infty(p) \xrightarrow{X} \mathbf{R}.$$

f_* is usually called the *derivative map*.

Exercise: Define the derivative map in terms of Definition 1 of the tangent space, and show the equivalence with the definition just given.

11. Day 11

11.1. **Properties of 1-forms.**

Definition 11.1. *Given the cotangent bundle* $T^*M \xrightarrow{\pi} M$*, a* 1-form *over* $U \subset M$ *is a smooth map* $s : U \to T^*M$ *such that* $\pi \circ s = id$.

Note that a 1-form simply assigns, in a smooth manner, an element of T_p^*M to a given $p \in M$. The space of 1-forms on U is denoted $\Omega^1(U)$. Note that the space of 1-forms is an $\mathbf{R}$-vector space.

1. We often write $\Omega^0(M) = C^\infty(M)$. Then there exists a map $d : \Omega^0(M) \to \Omega^1(M)$, $f \mapsto df$.

2. Given $\phi : M \to N$, there is no natural map $T^*M \to T^*N$ unless ϕ is a diffeomorphism. However, there exists a contravariant map $\phi^* : T^*N \to T^*M$. We *pull back* forms $\theta \mapsto \phi^*\theta$.

3. There exists a commutative diagram:

$$\begin{array}{ccc} \Omega^0(N) & \xrightarrow{\phi^*} & \Omega^0(M) \\ d\downarrow & \circlearrowleft & \downarrow d \\ \Omega^1(N) & \xrightarrow{\phi^*} & \Omega^1(M) \end{array},$$

i.e., $d \circ \phi^* = \phi^* \circ d$. Check this for HW by unwinding the definitions.

4. $d(fg) = fdg + gdf$. Check this for HW.

Example: $\theta = x^2dy + ydx$ on $\mathbf{R}^2$. Consider $i : \mathbf{R} \to \mathbf{R}^2$, $t \mapsto (t, 0)$. Then $i^*\theta = 0$.

11.2. **Discussion (and possible extension) of HW.**

The *orthogonal group* is $O(n) = \{A \in M_n | AA^T = I\}$.

1. $AA^T = I$ implies $\det(AA^T) = \det I \Rightarrow \det A = \pm 1$. Here we are using $\det(AB) = \det A \cdot \det B$ and $\det(A^T) = \det A$.

2. Recall $Gl(n, \mathbf{R}) = \{A \in M_n | \det A \neq 0\}$ and $Sl(n, \mathbf{R}) = \{A \in M_n | \det A = 1\}$. Thus, $O(n) \subset Gl(n, \mathbf{R})$ but $O(n) \not\subset Sl(n, \mathbf{R})$ (not quite). $O(n)$ has two connected components $\det A = 1$ and $\det A = -1$. Examples of $\det A = -1$ is $(-1, 0; 0, 1)$. The connected component with $\det A = 1$ is called $SO(n)$. Note $SO(n) = O(n) \cap Sl(n, \mathbf{R})$.

3. Show $O(n)$ is a submanifold of $Gl(n, \mathbf{R})$. Consider the map $\phi : Gl(n, \mathbf{R}) \to Sym_n$ given by $A \mapsto AA^T$, compute its derivative $d\phi(A)(B) = AB^T + BA^T$, and show $d\phi(A)$ is surjective.

4. $\dim O(n) = \dim Gl(n, \mathbf{R}) - \dim Sym_n = n^2 - n(n+1)/2 = n(n-1)/2$.

5. Show compactness.

6. Exercise: $SO(2)$. Elements are of the form $(\cos\theta, \sin\theta; -\sin\theta, \cos\theta)$. Show $SO(2)$ is diffeomorphic to S^1.

12. Day 12

12.1. **Some examples.**

Example: $S^2 \subset \mathbf{R}^3$, $S^2 = \{x_1^2 + x_2^2 + x_3^2 = 1\}$. Think of $TS^2 \subset \mathbf{R}^3 \times \mathbf{R}^3$ with coordinates $(x_1, x_2, x_3, y_1, y_2, y_3)$. At $(x_1, x_2, x_3) \in S^2$, $T_{(x_1,x_2,x_3)}S^2$ is the set of points (y_1, y_2, y_3) such that $(y_1, y_2, y_3) \cdot (x_1, x_2, x_3) = 0$. Therefore:

$$TS^2 = \{(x, y) \in \mathbf{R}^3 \times \mathbf{R}^3 | |x| = 1, x \cdot y = 0\}.$$

Think of TS^2 as follows: At each point $x \in S^2$, there is a plane (2-dimensional vector space) T_xS^2 sitting over it.

More abstract example: S^2 defined by gluing coordinate charts. Let $U = \mathbf{R}^2$ and $V = \mathbf{R}^2$, with coordinates (x_1, y_1), (x_2, y_2), respectively. Alternatively, think of $\mathbf{R}^2 = \mathbf{C}$. Take $U \cap V = \mathbf{C} - \{0\}$. The transition functions are:

$$U - \{0\} \stackrel{\phi_{UV}}{\longrightarrow} V - \{0\},$$

$$z \mapsto \frac{1}{z},$$

in terms of complex coordinates $z = x + iy$.

In terms of real coordinates, $(x, y) \mapsto (\frac{x}{x^2+y^2}, -\frac{y}{x^2+y^2})$.

S^2 has the structure of a *complex manifold.*

Definition 12.1. *A function $\phi : \mathbf{C} \to \mathbf{C}$ is* holomorphic *(or* complex analytic*) if $\frac{d\phi}{dz} = \lim_{h\to 0} \frac{\phi(z+h)-\phi(z)}{h}$ exists for all $z \in \mathbf{C}$. Note here that $h \in \mathbf{C}$. A function $\phi : \mathbf{C}^n \to \mathbf{C}^m$ is holomorphic if $\frac{\partial\phi}{\partial z_i} = \lim_{h\to 0} \frac{\phi(z_1,\dots,z_i+h,\dots,z_n)-\phi(z_1,\dots,z_n)}{h}$ exists for all $z = (z_1, \dots, z_n)$ and $i = 1, \dots n$. A complex manifold is a topological manifold with an atlas $\{U_\alpha, \phi_\alpha\}$, where $\phi_\alpha : U_\alpha \to \mathbf{C}^n$ and $\phi_\beta \circ \phi_\alpha^{-1} : \mathbf{C}^n \to \mathbf{C}^n$ is a holomorphic map.*

A *holomorphic map* $f : \mathbf{C} \to \mathbf{C}$, when viewed as a map $f : \mathbf{R}^2 \to \mathbf{R}^2$, is a smooth map. Therefore, a complex manifold is automatically a smooth manifold.

Compute the Jacobians. Rewriting as a map $\phi_{UV} : \mathbf{R}^2 - \{0\} \to \mathbf{R}^2 - \{0\}$, we compute:

$$J_{\phi_{UV}} = \begin{pmatrix} \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) & \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) \\ \frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}\right) & \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) \end{pmatrix} = \frac{1}{(x^2+y^2)^2}\begin{pmatrix} y^2 - x^2 & -2xy \\ 2xy & y^2 - x^2 \end{pmatrix}.$$

Remark: It is not a coincidence that $a_{11} = a_{22}$ and $a_{21} = -a_{12}$.

Explain that TS^2 is obtained by gluing two copies of $(\mathbf{R}^2 - \{0\}) \times \mathbf{R}^2$ together via a map which sends $(a_1, a_2)^T$ over (x_1, y_2) to $J(a_1, a_2)^T$ over (x_2, y_2).

12.2. Lie groups.

Definition 12.2. *A* Lie group G *is a smooth manifold together with smooth maps* $\mu : G \times G \to G$ *(multiplication),* $i : G \to G$ *(inverse) which endow* G *the structure of a group.*

Definition 12.3. *A* Lie subgroup $H \subset G$ *is a subgroup of* G *which is also a submanifold of* G*. A* Lie group homomorphism $\phi : H \to G$ *is a homomorphism which is also a smooth map of the underlying manifolds.*

Examples:

1. $Gl(n, \mathbf{R}) = \{A \in M_n |\det(A) \neq 0\}$. We already showed that this is a manifold. The product AB is defined by a formula which is polynomial in the matrix entries of A and B, so μ is smooth. Similarly prove that i is smooth.
2. $Sl(n, \mathbf{R}) = \{A \in M_n |\det(A) = 1\}$ is a Lie subgroup of $Gl(n, \mathbf{R})$.
3. $O(n) = \{A \in M_n | AA^T = id\}$.
4. $SO(n, \mathbf{R}) = Sl(n, \mathbf{R}) \cap O(n)$.

More invariantly, given a vector space V, define $Gl(V)$ to be the set of isomorphisms $V \to V$.

Definition 12.4. *A* Lie group representation *is a Lie group homomorphism* $\phi : G \to Gl(V)$*, for some vector space* V*.*

13. Day 13

13.1. **Vector bundles.** $T^*M \xrightarrow{\pi} M$ is an example of a *vector bundle.*

Definition 13.1. *A* (real) vector bundle *over a manifold M is a pair $(E, \pi : E \to M)$ such that:*

1. *$\pi^{-1}(p)$ has a structure of a vector space (over $\mathbf{R}$) of dimension n.*
2. *There exists a cover $\{U_\alpha\}$ of M such that $\pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times \mathbf{R}^n$ which restricts to a vector space isomorphism $\pi^{-1}(p) \xrightarrow{\sim} \mathbf{R}^n$*

n is said to be the rank *of the vector bundle. A rank 1 vector bundle is often called a* line bundle*.*

The second condition: π admits a *local trivalization.*

Definition 13.2. *A* section *of a vector bundle $\pi : E \to M$ over $U \subset M$ is a smooth map $s : U \to E$ such that $\pi \circ s = id$. A section over M is called a* global section. *The space of sections of E over U is often written $\Gamma(E, U)$. Also write $\Gamma(E)$ if $U = M$. $\Gamma(E, U)$ clearly has an $\mathbf{R}$-vector space structure.*

Sections of TM are called *vector fields.* We often write $\mathfrak{X}(M) = \Gamma(TM)$. Sections of T^*M are *1-forms.* $\Omega^1(M) = \Gamma(T^*M)$.

13.2. **Transition functions, reinterpreted.** Consider $\pi : TM \to M$ and local trivializations $\pi^{-1}(U) = U \times \mathbf{R}^n$, $\pi^{-1}(V) = V \times \mathbf{R}^n$. We have transition functions $\phi_{UV} : (U \cap V) \times \mathbf{R}^n \to (U \cap V) \times \mathbf{R}^n$ (the first a subset of $U \times \mathbf{R}^n$ and the second a subset of $V \times \mathbf{R}^n$), sending $a_i \frac{\partial}{\partial x_i} \mapsto \frac{\partial y_j}{\partial x_i} a_i \frac{\partial}{\partial y_j}$. Note that $a = (a_1, \dots, a_n)^T$ is multiplied by $\frac{\partial y}{\partial x}$. Alternatively, think of $\frac{\partial y}{\partial x} : \mathbf{R}^n \to \mathbf{R}^n$ as:

$$\Phi_{UV} : U \cap V \to Gl(n, \mathbf{R}).$$

1. For double intersections $U \cap V$, we have $\Phi_{UV} \circ \Phi_{VU} = id$.

2. For triple intersections $U_\alpha \cap U_\beta \cap U_\gamma$ with coordinates x, y, z, we have $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial x}$ (chain rule), i.e.,

$$\Phi_{U_\gamma U_\alpha} = \Phi_{U_\gamma U_\beta} \circ \Phi_{U_\beta U_\alpha}.$$

This is usually called the *cocycle condition.*

What's this cocycle condition? This cocycle condition (triple intersection property) is clearly necessary if we want to construct a vector bundle by patching together $U_\alpha \times \mathbf{R}^n$. It guarantee that the gluings that we prescribe, i.e., $\Phi_{U_\alpha U_\beta}$ from U_α to U_β, etc. are *compatible.*

On the other hand, if we can find a collection $\{\Phi_{U_\alpha U_\beta}\}$ (for all U_α, U_β), which satisfies the cocycle condition, we have a legitimate vector bundle.

Consider $\pi : T^*M \to M$. Then $\Phi_{UV} : U \cap V \to Gl(n, \mathbf{R})$ is given by $\left[\left(\frac{\partial y}{\partial x}\right)^{-1}\right]^T$. This makes it clear that the inverse and transpose are both necessary for the cocycle condition to be met.

14. Day 14

14.1. **Orientability.** Let $Gl^+(n,\mathbf{R}) \subset Gl(n,\mathbf{R})$ be the space of $n\times n$ matrices with positive determinant. ($Gl(n,\mathbf{R})$, like $O(n)$, is not connected, and has two connected components.) M is *orientable* if there exists an atlas $\{U_\alpha\}$ such that $\Phi_{U_\alpha U_\beta}: U_\alpha \cap U_\beta \rightarrow Gl(n,\mathbf{R})$ factors through $Gl^+(n,\mathbf{R})$.

Example: Recall S^2 given by gluing together $U=\mathbf{C}$ and $V=\mathbf{C}$. The transition function was:

$$\Phi_{UV}: U\cap V \rightarrow Gl(2,\mathbf{R}),$$
$$(x,y)\mapsto \frac{1}{(x^2+y^2)^2}\left(\begin{array}{cc} y^2-x^2 & -2xy \\ 2xy & y^2-x^2 \end{array}\right).$$

The determinant is positive, so S^2 is oriented.

Example: $\mathbf{RP}^2 = \mathbf{R}^3 - \{0\}/\sim$, where $x\sim tx$, $t\in \mathbf{R}-\{0\}$. This is the set of lines through the origin of $\mathbf{R}^3$. Take the unit sphere S^2, the $\mathbf{RP}^2$ is S^2 with x identified with $-x$, i.e., $\mathbf{RP}^2 = S^2/(\mathbf{Z}/2\mathbf{Z})$.

Classification of compact 2-manifolds (surfaces). The oriented ones are: S^2, T^2, surface of genus g. The nonorientable ones are: $\mathbf{RP}^2$, Klein bottle, and one for each orientable surface of genus g.

14.2. **Complex manifolds.** Let $Gl(n,\mathbf{C})$ be the space of $n\times n$ matrices with complex coefficients with nonzero determinant. How do you view $Gl(n,\mathbf{C})$ as a Lie subgroup of $Gl(2n,\mathbf{R})$? We'll do $Gl(1,\mathbf{C})\subset Gl(2,\mathbf{R})$, and leave the general case as HW. Consider $z\in Gl(1,\mathbf{C})$. We can write $z=x+iy$. $z: c\in\mathbf{C}\mapsto zc$. If we write $c=a+ib$, then

$$\left(\begin{array}{c} a \\ b\end{array}\right)\mapsto \left(\begin{array}{cc} x & -y \\ y & x\end{array}\right)\left(\begin{array}{c} a \\ b\end{array}\right).$$

Previous example: As can be seen from the transition function, for S^2, there exists a factorization:

$$\Phi_{UV}: U\cap V\rightarrow Gl(1,\mathbf{C})\rightarrow Gl(2,\mathbf{R}).$$

HW: Show that $Gl(n,\mathbf{C})\subset Gl^+(2n,\mathbf{R})$. Therefore, complex manifolds are always orientable.

14.3. **Constructing new vector bundles out of TM.** Let M be a manifold and $\{U_\alpha\}$ an atlas for M. View TM as being constructed out of $U_\alpha\times\mathbf{R}^n$ by gluing using transition functions $\Phi_{U_\alpha U_\beta}: U\cap V\rightarrow Gl(n,\mathbf{R})$. Recall $U_\alpha\times\mathbf{R}^n$ and $U_\beta\times\mathbf{R}^n$ are glued using $\Phi_{U_\alpha U_\beta}$ along the fibers, and $\Phi_{U_\alpha U_\beta}$ satisfies the *cocycle condition.*

Consider a representation $\rho: Gl(n,\mathbf{R})\rightarrow Gl(m,\mathbf{R})$.

Examples:

1. $\rho : Gl(n,\mathbf{R}) \to Gl(1,\mathbf{R}) = \mathbf{R}^{\times}$, $A \mapsto \det(A)$.
2. $\rho : Gl(n,\mathbf{R}) \to Gl(n,\mathbf{R})$, $A \mapsto BAB^{-1}$.
3. $\rho : Gl(n,\mathbf{R}) \to Gl(n,\mathbf{R})$, $A \mapsto (A^{-1})^T$.

Use ρ and glue $U_\alpha \times \mathbf{R}^m$ together using:

$$\rho \circ \Phi_{U_\alpha U_\beta} : U \cap V \to Gl(n,\mathbf{R}) \to Gl(m,\mathbf{R}).$$

Observe that the cocycle condition is satisfied since ρ is a representation. Therefore we obtain a new vector bundle $TM \times_\rho \mathbf{R}^m$, called TM *twisted by* ρ.

Note that Example 3 above is just the cotangent bundle T^*M.

Example 1 gives rise to a *line bundle* (i.e., vector bundle of rank 1), usually denoted $\bigwedge^n TM$. The *orientability* of M is equivalent to the existence of a global section $s \in \Gamma(\bigwedge^n TM, M)$ which is never zero.

15. Day 15

15.1. **Integrating 1-forms.**

Property of 1-forms: Let $f : M \to N$ and $g : L \to M$ be smooth maps between smooth manifolds, and let ω be a 1-form on N. Then $(f \circ g)^*\omega = g^* \circ (f^*\omega)$. [Exercise. Note however that the order of pulling back is reasonable.]

Let C be an embedded arc in M, i.e., it is the image of some embedding $\gamma : [c, d] \to M$. Let ω be a 1-form on M. Then we define the *integral* of ω over C to be:

$$\int_C \omega \stackrel{def}{=} \int_c^d \gamma^*\omega.$$

If t is the coordinate on $[c, d]$, then $\gamma^*\omega$ will have the form $f(t)dt$.

Lemma 15.1. *The definition does not depend on the particular parametrization $\gamma : [c, d] \to M$.*

Proof. Taking a different $\gamma_1 : [a, b] \to M$. Then there exists a diffeomorphism $g : [a, b] \to [c, d]$ such that $\gamma_1 = \gamma \circ g$. Now, $\gamma_1^*\omega = (\gamma \circ g)^*\omega = g^*(\gamma^*\omega)$, and

$$\int_c^d \gamma^*\omega = \int_c^d f(t)dt = \int_a^b f(g(s))dg(s) = \int_a^b \gamma_1^*\omega.$$

□

Now we know how to integrate 1-forms. Over the next few weeks we will define objects that we can integrate on higher-dimensional submanifolds (not just curves), called k-forms. For this we need to do quite a bit of preparation.

15.2. **Linear algebra.**

We define some notions in linear algebra. The vector spaces we are concerned with do not need to be finite-dimensional, but you may suppose they are if you want. Let V, W be vector spaces over $\mathbf{R}$.

1. (Direct sum) $V \oplus W$. As a set, $V \times W$. Addition $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$. $dim(V \oplus W) = dim(V) + dim(W)$.

2. $Hom(V, W) = \{\text{linear maps } \phi : V \to W\}$. In particular, we have $V^* = Hom(V, \mathbf{R})$. $dim(Hom(V, W)) = dim(V) \cdot dim(W)$.

3. (Tensor product) $V \otimes W$.

Informal definition. Suppose V and W are finite-dimensional, and let $\{v_1, \dots, v_m\}$, $\{w_1, \dots, w_n\}$ be bases for V and W, respectively. Then $V \otimes W$ is a vector space which has $\{v_i \otimes w_j | i = 1, \dots, m; j = 1, \dots, n\}$ as basis. Elements of $V \otimes W$ are linear combinations $\sum_{ij} a_{ij} v_i \otimes w_j$.

Definition 15.2. *Let $V_1, \dots, V_k, U$ be vector spaces. A map $\phi : V_1 \times \cdots \times V_k \to U$ is multilinear if ϕ is linear in each V_i separately, i.e., $\phi : \{v_1\} \times \cdots \times V_i \times \cdots \times \{v_k\} \to U$ is linear for each $v_j \in V_j$, $j \neq i$. If $k = 2$, we say ϕ is bilinear.*

Formal definition. $V \otimes W$ is a vector space Z together with a bilinear map $i : V \times W \to Z$, which satisfies the following *universal mapping property.* Given any bilinear map $\phi : V \times W \to U$, there exists a linear map $\tilde{\phi} : Z \to U$ such that $\phi = \tilde{\phi} \circ i$.

Actual construction. Start with the *free vector space* $F(V, W)$ generated by $V \times W$. By this we mean $F(V, W)$ consists of finite linear combinations $\sum_i a_i(v_i, w_i)$, where $(v_i, w_i) \in V \times W$, $a_i \in \mathbf{R}$, and we have relations $a_1(v_1, w_1) + a_2(v_2, w_2) = a_3(v_3, w_3)$ if and only if $v_1 = v_2$ and $w_1 = w_2$. Next let $R(V, W)$ be the vector space generated by the "bilinear relations"

$$(v_1 + v_2, w) - (v_1, w) - (v_2, w),$$
$$(v, w_1 + w_2) - (v, w_1) - (v, w_2),$$
$$(cv, w) - c(v, w),$$
$$(v, cw) - c(v, w).$$

Then the quotient space $F(V, W)/R(V, W)$ is $V \otimes W$.

For $V \otimes W$, we want finite linear combinations of things that look like $v \otimes w$. For bilinearity, we also require:

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w,$$
$$v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2,$$
$$c(v \otimes w) = (cv) \otimes w = v \otimes (cw).$$

Verification of universal mapping property. With $V \otimes W$ defined as above, let $i : V \times W \to V \otimes W$ be $(v, w) \mapsto v \otimes w$. The bilinearity of i follows from the construction of $V \otimes W$. (For example, $i(v_1 + v_2, w) = (v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w = i(v_1, w) + i(v_2, w)$.) $\tilde{\phi}$ maps $\sum_i a_i v_i \otimes w_i \mapsto \sum_i a_i \phi(v_i, w_i)$. This map is well-defined because all the elements of $R(V, W)$ get mapped to 0.

16. Day 16

16.1. **More on tensor products.** Recall the definition of the tensor product as $V \otimes W = F(V,W)/R(V,W)$ and the universal property. The universal property is useful for the following reason: If we want to construct a linear map $V \otimes W \to U$, it is equivalent to check the existence of a bilinear map $V \times V \to U$.

Dimension of $V \otimes W$. Suppose V, W are finite-dimensional. Then we claim $\dim(V \otimes W) = \dim V \cdot \dim W$. To see this, consider the map $V^* \otimes W \to Hom(V,W)$ which sends $f \otimes w \mapsto fw$, where $fw : v \mapsto f(v)w$. The universal mapping property guarantees the well-definition of this map. $\dim Hom(V,W)$ can be easily calculated to be $\dim V \cdot \dim W$. Now, it suffice to check surjectivity and injectivity. Surjectivity: let f_i be dual to a basis $\{v_1, \dots, v_m\}$ for V, i.e., $f_i(v_j) = \delta_{ij}$; also let $\{w_1, \dots, w_n\}$ be a basis for W. Then any linear map in $Hom(V,W)$ is of the form $\sum a_{ij} f_i w_j$, i.e., comes from $\sum a_{ij} f_i \otimes w_j$. Details are left for HW.

Properties of tensor products.

1. $V \otimes W \simeq W \otimes V$.
2. $(V \otimes W) \otimes U \simeq V \otimes (W \otimes U)$.

1. Worked out. It's difficult to directly get a well-defined map $V \otimes W \to W \otimes V$, so start with a bilinear map $V \times W \to W \times V \to W \otimes V$, where $(v,w) \mapsto w \otimes v$. It then lifts to a map $V \otimes W \to W \otimes V$ which sends $v \otimes w \mapsto w \otimes v$. It is easy to verify injectivity and surjectivity.

The second property ensures us that we do not need to write parentheses when we take a tensor product of several vector spaces.

Let $A : V \to V$ and $B : W \to W$ be linear maps. Then we have

$$A \oplus B : V \oplus W \to V \oplus W$$

and

$$A \otimes B : V \otimes W \to V \otimes W.$$

We denote $V^{\otimes k}$ for the k-fold tensor product of V. Then we have a representation $\rho : Gl(V) \to Gl(V^{\otimes k})$, $A \mapsto A \otimes \cdots \otimes A$. This gives us an associated vector bundle twisted by ρ.

The tensor algebra. $T(V) = \mathbf{R} \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} + \dots$. The multiplication is given by $(v_1 \otimes \cdots \otimes v_s)(v_{s+1} \otimes \cdots \otimes v_t) = v_1 \otimes \cdots \otimes v_t$.

16.2. **The exterior algebra.** We define $\bigwedge V$ to be $T(V)/\mathcal{I}$, where $\mathcal{I}$ is a (2-sided) ideal generated by elements of the form $v \otimes v$, i.e., elements of $\mathcal{I}$ are finite sums of terms that look like $\eta_1 \otimes v \otimes v \otimes \eta_2$, where $\eta_1, \eta_2 \in T(V)$. Elements of $\bigwedge V$ are denoted $\sum a_{i_1 \dots i_k} v_{i_1} \wedge \cdots \wedge v_{i_k}$. Then in $T(V)$ we have $v \wedge v = 0$. Also note that $(v_1 + v_2) \wedge (v_1 + v_2) = v_1 \wedge v_1 + v_1 \wedge v_2 + v_2 \wedge v_1 + v_2 \wedge v_2$. The first and last terms are zero, so $v_1 \wedge v_2 = -v_2 \wedge v_1$. Therefore, we may assume that $i_1 < \cdots < i_k$ in the expression above. $\bigwedge V$ is clearly an algebra, i.e., there is a multiplication $\omega \wedge \eta$ given elements ω, η in $\bigwedge V$.

We define $\bigwedge^k V$ to be the degree k terms of $\bigwedge V$.

Alternating multilinear forms. A multilinear form $\phi : V \times \cdots \times V \to U$ is *alternating* if $\phi(v_1, \ldots, v_i, v_{i+1}, \ldots, v_k) = -1 \cdot \phi(v_1, \ldots, v_{i+1}, v_i, \ldots v_k)$. Recall that transpositions generate the full symmetric group S_k. If $(1, \ldots, k) \mapsto (i_1, \ldots, i_k)$, and σ is the number of transpositions needed, then $\phi(v_1, \ldots, v_k) = (-1)^\sigma \phi(v_{i_1}, \ldots, v_{i_k})$.

Universal property. $\bigwedge^k V$ and $i : V \times \cdots \times V \to \bigwedge^k V$ satisfy the following. Given an alternating multilinear map $\phi : V \times \cdots \times V \to U$, there is a linear map $\tilde{\phi} : \bigwedge^k V \to U$ such that $\phi = \tilde{\phi} \circ i$.

Proposition 16.1. *Given a basis $\{e_1, \ldots, e_n\}$ for V, a basis for $\bigwedge^k V$ consists of degree k monomials $e_{i_1} \wedge \cdots \wedge e_{i_k}$ with $i_1 < \cdots < i_k$. Therefore, if $k > n$, $\dim \bigwedge^k V = 0$, and if $k \leq n$, $\dim \bigwedge^k V = \begin{pmatrix} n \\ k \end{pmatrix}$.*

If $V = \mathbf{R}^3$ with basis $\{e_1, e_2, e_3\}$, then $\bigwedge^0 V = \mathbf{R}$, $\bigwedge^1 V = \mathbf{R}\{e_1, e_2, e_3\}$, $\bigwedge^2 V = \mathbf{R}\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$, $\bigwedge^3 V = \mathbf{R}\{e_1 \wedge e_2 \wedge e^3\}$, and $\bigwedge^k V = 0$, $k > 3$.

17. Day 17

17.1. **Basis for $\bigwedge^k V$.** We'll give the proof of Proposition 16.1 in several steps.

1. If $k > n$, $\bigwedge^k V = 0$. This is clear since $e_1 \wedge \cdots \wedge e_l \wedge v$, $v = \sum_{i=1}^l a_i e_i$, is written as $\sum_{i=1}^l a_i e_1 \wedge \cdots \wedge e_k \wedge e_i = 0$.

2. If $k = n$, $\bigwedge^k V = \mathbf{R}$, and the basis given by $e_1 \wedge \cdots \wedge e_n$. Any other $e_{i_1} \wedge \cdots \wedge e_{i_n}$ with duplicate e_i's are clearly zero, and if e_i's are not duplicated $(i_1, \dots, i_n)$ is a permutation of $(1, \dots, n)$ and $e_{i-1} \wedge \cdots \wedge e_{i_n} = (-1)^{\sigma(i_1,\dots,i_n)} e_1 \wedge \cdots \wedge e_n$, where σ is the *sign function*. It remains to show that $e_1 \wedge \cdots \wedge e_n$ is nonzero! This is done by defining an alternating multilinear form $V \times \cdots \times V \to \mathbf{R}$ (n copies of V). Then by the universal property $\bigwedge^n V$ cannot be zero and hence must be $\mathbf{R}$. Details are HW.

3. For any $v_1, \dots, v_k \in V$, $v_1 \wedge \cdots \wedge v_k \neq 0$ in $\bigwedge^k V$ iff $v_1, \dots, v_k$ are linearly independent. 'Only if' is clear — it's identical to part 1. For the 'if' part, take $\sum_{i_1 < \cdots < i_k} a_{i_1,\dots,i_k} v_{i_1} \wedge \cdots \wedge v_{i_k} = 0$. For each summand, there is a unique term $v_{j_1} \wedge \cdots \wedge v_{j_{n-k}}$ which kills all the other summands and gives $\pm a_{i_1,\dots,i_k} v_1 \wedge \cdots \wedge v_n$. Hence this implies that the a's are zero.

17.2. **Tensor calculus on manifolds.** We have now constructed $V^{\otimes k}$ and $\bigwedge^k V$, given a finite-dimensional vector space V. Also note that there exist natural representations $\rho_0 : Gl(V) \to Gl(V^{\otimes k})$ and $\rho_1 : Gl(V) \to Gl(\bigwedge^k V)$.

Example: $\dim V = 2$. Basis $\{v_1, v_2\}$. $\bigwedge V$ has basis $\{1, v_1, v_2, v_1 \wedge v_2\}$. If $A : V \to V$ is linear and sends $v_i \mapsto a_{1i} v_1 + a_{2i} v_2$, $i = 1, 2$, then $A(v_1 \wedge v_2) = Av_1 \wedge Av_2 = det(A) v_1 \wedge v_2$.

Thus we can form $TM \times_{\rho_0} V^{\otimes k} = \otimes_k TM$ and $TM \times_{\rho_1} \bigwedge^k V = \bigwedge^k TM$. Also can form $\otimes_k T^*M$ and $\bigwedge^k T^*M$.

We'll focus on $\bigwedge^k T^*M$ in what follows. Sections of $\bigwedge^k T^*M$ are called k-forms and locally look like:

$$\omega = \sum_{i_1 < \cdots < i_k} f_{i_1,\dots,i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Denote by $\Omega^k(M)$ the sections of $\bigwedge^k T^*M$.

Pullback: Let $\phi : M \to N$ be a smooth map between manifolds, and ω a k-form. Then we can define the pullback $\phi^*\omega$ in a manner similar to 1-forms:

$$\phi^*\omega = \sum_{i_1 < \cdots < i_k} (f_{i_1,\dots,i_k} \circ \phi) \cdot d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_k}.$$

Check: The global well-definition, i.e., independent of choice of coordinates.

17.3. **The exterior derivative.** $d : \Omega^k \to \Omega^{k+1}$ extends $d : \Omega^0 \to \Omega^1$ as follows (in local coordinates $x_1, \dots, x_n$:

1. Recall for $f \in \Omega^0$, $df = \sum_i \frac{\partial f}{\partial x_i} dx_i$.
2. If $\omega = \sum_I f_I dx_I$, then $d\omega = \sum_I df_I dx_I$.

Here $I = (i_1, \dots, i_k)$ is an indexing set, and $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$.

Example on $\mathbf{R}^3$. Consider $\mathbf{R}^3$ with coordinates (x, y, z). Consider

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \Omega^3.$$

The first d is the *gradient*

$$d : f \mapsto df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz.$$

The second d is the *curl*

$$d : f dx + g dy + h dz \mapsto \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) dy dz + \dots.$$

The last d is the *divergence*

$$d : f_1 dy \wedge dz + f_2 dz \wedge dx + f_3 dx \wedge dy \mapsto \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \right) dx \wedge dy \wedge dz.$$

Note: We will often omit the $\wedge$.

18. Day 18

18.1. **De Rham cohomology.** This material is nicely presented in Bott & Tu.

Proposition 18.1. *d satisfies the product formula:*

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta), \tag{1}$$

where $\alpha \in \Omega^k$ and $\beta \in \Omega^l$.

Proposition 18.2. *Consider the composite $\Omega^k \xrightarrow{d} \Omega^{k+1} \xrightarrow{d} \Omega^{k+2}$. Then $d^2 = 0$.*

Proof. The proof is by induction on degree, by taking the exterior derivative of: $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^l \alpha \wedge (d\beta)$ to get

$$d^2(\alpha \wedge \beta) = (-1)^{l+1} d\alpha \wedge d\beta + (-1)^l d\alpha \wedge d\beta = 0.$$

For $d^2 : \Omega^0 \to \Omega^2$, compute:

$$d \circ df = d(\sum_i \frac{\partial f}{\partial x_i} dx_i) = \sum_{ij} \frac{\partial^2 f}{\partial x_i x_j} dx_j \wedge dx_i = 0.$$

□

Consider:

$$0 \xrightarrow{d_{-1}} \Omega^1 \xrightarrow{d_1} \Omega^2 \xrightarrow{d_2} \cdots \to \Omega^n \xrightarrow{d_n} 0, \tag{2}$$

where $n = \dim M$.

Example: $M = \mathbf{R}^3$. $d_0 = grad$, $d_1 = curl$, $d_2 = div$. Then $div(curl) = 0$, $curl(grad) = 0$.

This means that $Im(d_{k-1}) \subset ker(d_k)$.

We define the *kth de Rham cohomology of M* to be:

$$H^k_{dR}(M) \stackrel{def}{=} \ker d_k / \text{ Im } d_{k-1}.$$

Facts: The de Rham cohomology groups are diffeomorphism invariants of the manifold M, and are finite-dimensional if M is compact or admits a finite atlas.

Definition 18.3. *A sequence of vector spaces $\dots C^i \xrightarrow{d_i} C^{i+1} \xrightarrow{d_{i+1}} C^{i+2} \dots$ is said to be* exact *if Im $d_{i-1} = \ker d_i$ for all i.*

The de Rham cohomology measure the failure of Equation 2 to be *exact.*

Examples.

1. $M = \{pt\}$. Then $\Omega^0(M) = \mathbf{R}$ and $\Omega^i(M) = 0$, $i \neq 0$. We have $H^0(pt) = \mathbf{R}$ and $H^i(pt) = 0$, $i > 0$.

2. $M = \mathbf{R}$. Then $\Omega^0(M) = C^\infty(\mathbf{R})$. Also $\Omega^1(M) \simeq C^\infty(\mathbf{R})$ because every 1-form is of the form $f \cdot dx$. Now, $d : f \mapsto \frac{df}{dx}dx$, i.e., $d : C^\infty(\mathbf{R}) \to C^\infty(\mathbf{R})$ is the map $f \to f'$. $\ker d = \{\text{constant functions}\}$. Therefore, $H^0_{dR}(\mathbf{R}) = \mathbf{R}$. Next, $\mathrm{Im}\, d$ is all of $C^\infty(\mathbf{R})$, since given any f we can take its antiderivative $\int_0^x f(t)dt$. Therefore, $H^1_{dR}(\mathbf{R}) = 0$. Similarly, $H^i(\mathbf{R}) = 0$ for $i > 1$.

3. Similarly, for M a disjoint union of n copies of $\mathbf{R}$, $H^0_{dR}(M) = \mathbf{R}^n$ and $H^i_{dR}(M) = 0$, $i \geq 1$.

4. $M = S^1$. View S^1 as $\mathbf{R}/\mathbf{Z}$. $\Omega^0(S^1) = \{\text{Periodic functions on } \mathbf{R}\}$. (Here the period is 1.) $\Omega^1(S^1)$ is also the set of periodic functions on $\mathbf{R}$ by identifying $f(x)dx \mapsto f(x)$. As before, $H^0_{dR}(S^1) = \mathbf{R}$. Now, for $H^1_{dR}(S^1)$, $Im(d)$ is the space of all C^∞-functions $f(x)$ with integral $\int_0^1 f(x)dx = 0$. Thus, $H^1_{dR}(S^1) = \mathbf{R}$. We also have an *exact sequence*:

$$0 \to \mathbf{R} \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{\int} \mathbf{R} \to 0.$$

Proposition 18.4. $\dim H^0(M)$ *is the number of connected components of* M.

Proof. $df = 0$ if and only if f is a constant on each connected component. □

19. Day 19

19.1. Pullback.

Let $\phi : M \to N$ be a smooth map between manifolds. Then $d \circ \phi = \phi \circ d$. (Verify this!) This follows easily by computing in local coordinates.

Lemma 19.1. *There is an induced map $\phi^* : H^k(N) \to H^k(M)$ on the level of cohomology.*

Let $\omega \in \Omega^k(M)$. ω is said to be *closed* if $d\omega = 0$; it is *exact* if $\omega = d\eta$.

Proof. Let ω be a closed k-form on N, i.e., $\omega \in \Omega^k(N)$ satisfies $d\omega = 0$. Then, $\phi^*\omega$ satisfies $d\phi^*\omega = \phi^*(d\omega) = 0$. Now, if ω is exact, i.e., $\omega = d\eta$, then $\phi^*\omega = \phi^* d\eta = d(\phi^*\eta)$ is exact as well. □

19.2. Mayer-Vietoris sequences.

This is a method for effectively decomposing a manifold and computing its cohomology from its components.

Suppose $M = U \cap V$. Then we have

$$U \cap V \stackrel{i_U, i_V}{\longrightarrow} U \sqcup V \stackrel{i}{\to} M. \tag{3}$$

i_U and i_V are two inclusions, one into U and the other into V.

Example: $M = S^1$. $U = V = \mathbf{R}$. $U \cap V = \mathbf{R} \sqcup \mathbf{R}$.

Theorem 19.2. *We have the following long exact sequence:*

$$\begin{aligned} 0 \to H^0(M) \stackrel{i^*}{\to} H^0(U) \oplus H^0(V) & \stackrel{i_U^* - i_V^*}{\longrightarrow} H^0(U \cap V) \to \\ \to H^1(M) \stackrel{i^*}{\to} H^1(U) \oplus H^1(V) & \stackrel{i_U^* - i_V^*}{\longrightarrow} H^1(U \cap V) \to \\ & \to \dots \end{aligned}$$

Its proof will be given over the next couple of lectures, but for the time being we will apply it.

Example: Compute $H^i(S^1)$ using Mayer-Vietoris.

Observe the following: $0 \to A \to B$ exact means $A \to B$ is injective. $A \to B \to 0$ exact means $A \to B$ is surjective. Hence $0 \to A \to B \to 0$ implies isomophism.

19.3. Poincaré lemma.

Lemma 19.3 (Poincaré lemma)**.** $\omega \in \Omega^k(\mathbf{R}^n)$, $k \geq 1$, *is closed if and only if* ω *is exact.*

In other words, $H^k_{dR}(\mathbf{R}^n) = 0$ for $k \geq 1$. We will give the proof later, together with some other homotopy-theoretic properties.

19.4. Partitions of unity.

Definition 19.4. *Let* $\{U_\alpha\}$ *be an open cover of* M*. Then a collection of functions* $\{f_\alpha \geq 0\}$ *is a* partition of unity subordinate to $\{U_\alpha\}$ *if:*

1. f_α *has support inside* U_α*. Here the support of* f_α *is the closure of* $\{x \in M | f_\alpha(x) \neq 0\}$.
2. $f_\alpha \geq 0$.
3. *At every point* $x \in M$*, there is a finite subset* $\{f_1, \dots, f_k\}$ *of* $\{f_\alpha\}$ *which are nonzero, and* $\sum_{i=1}^k f_i(x) = 1$.

Proposition 19.5. *Given* $\{U_\alpha\}$ *an open cover of* M*. There exists a partition of unity subordinate to* $\{U_\alpha\}$.

Proof. The proof is done in stages.

Step 1. Consider the function $f : \mathbf{R} \to \mathbf{R}$:

$$f(x) = \begin{cases} e^{-1/x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

It is easy to show that $f \geq 0$ and f is smooth.

Step 2. Take $g : \mathbf{R} \to \mathbf{R}$ to be $g_{ab}(x) = f(x-a) \cdot f(b-x)$. (Suppose $a < b$.) Then $g(x)$ is a *bump function.*

- $g \geq 0$,
- $supp(g) = [a, b]$,
- $g > 0$ on (a, b).

Step 3. Construct a bump function on $[a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbf{R}^n$ with coordinates $(x_1, \dots, x_n)$ by letting $\phi(x) = g_{a_1 b_1}(x_1) \dots g_{a_n b_n}(x_n)$. Then ϕ is supported on $[a_1, b_1] \times \cdots \times [a_n, b_n]$ and is positive on the interior.

Step 4. We will only treat the case where M is compact. For each $p \in M$, choose an open neighborhood U_p of p of the form $(a_1, b_1) \times \cdots \times (a_n, b_n)$ whose closure is contained inside some U_α. For each U_p, construct ϕ_p as in Step 3. Now, since M is compact, there exists a finite collection of $\{p_1, \dots, p_k\}$ where $\{U_{p_i}\}$ cover M. Note that $\sum_{i=1}^k \phi_p > 0$ everywhere on M. If we let $\psi_p = \frac{\phi_p}{\sum \phi_p}$, then $\sum \psi_p = 1$. Finally, we associate to each ψ_p an open set U_α for which $U_p \subset U_\alpha$. Then ψ_α is the sum of all the ψ_p associated to U_α. □

20. Day 20: Some homological algebra

20.1. Short exact sequence.

Proposition 20.1. *Equation 3 gives the following short exact sequence.*

$$0 \to \Omega^i(M) \xrightarrow{i^*} \Omega^i(U) \oplus \Omega^i(V) \xrightarrow{i_U^* - i_V^*} \Omega^i(U \cap V) \to 0. \tag{4}$$

Proof. The only thing we need to do is prove that $i_U^* - i_V^*$ is surjective. (The rest of the exact sequence is easy.) Use partition of unity subordinate to $\{U, V\}$, ρ_U, ρ_V which adds up to $\rho_U + \rho_V = 1$. Given $\omega \in \Omega^i(U \cap V)$, consider $\rho_V \omega$ on U and $-\rho_U \omega$ on V. This works. $\square$

20.2. Short exact sequences to long exact sequences.

Getting from the short exact sequence to the long exact sequence is a purely algebraic operation.

Define a *complex* $(\mathcal{C}, d)$: $\cdots \to C^i \xrightarrow{d_i} C^{i+1} \xrightarrow{d_{i+1}} C^{i+1} \cdots \to$ to be a sequence of vector spaces and maps with $d_{i+1} \circ d_i = 0$. $(\mathcal{C}, d)$ gives rise to $H^i(\mathcal{C}) = \ker d_i / \operatorname{Im} d_{i-1}$, the ith cohomology of the complex.

A *cochain map* $\phi : \mathcal{A} \to \mathcal{B}$ is:

$$\begin{array}{ccccccccc}
\xrightarrow{d_{k-2}} & A^{k-1} & \xrightarrow{d_{k-1}} & A^k & \xrightarrow{d_k} & A^{k+1} & \xrightarrow{d_{k+1}} \\
 & \downarrow{\scriptstyle \phi_{k-1}} & & \downarrow{\scriptstyle \phi_k} & & \downarrow{\scriptstyle \phi_{k+1}} & \\
\xrightarrow{d_{k-2}} & B^{k-1} & \xrightarrow{d_{k-1}} & B^k & \xrightarrow{d_k} & B^{k+1} & \xrightarrow{d_{k+1}}
\end{array}$$

which satisfies $d_k \circ \phi_k = \phi_{k+1} \circ d_k$.

A cochain map $\phi : \mathcal{A} \to \mathcal{B}$ induces a map on cohomology:

$$\phi : H^k(\mathcal{A}) \to H^k(\mathcal{B}).$$

The verification is identical to that of the special case of de Rham.

Given an exact sequence $0 \to \mathcal{A} \xrightarrow{\phi} \mathcal{B} \xrightarrow{\psi} \mathcal{C} \to 0$, (i.e., we have collections of $0 \to A^i \to B^i \to C^i \to 0$ and all the maps are cochain maps),

$$\begin{array}{ccccccccc}
 & & \uparrow{\scriptstyle d_{k+1}} & & \uparrow{\scriptstyle d_{k+1}} & & \uparrow{\scriptstyle d_{k+1}} & & \\
0 & \longrightarrow & A^{k+1} & \xrightarrow{\phi_{k+1}} & B^{k+1} & \xrightarrow{\psi_{k+1}} & C^{k+1} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle d_k} & & \uparrow{\scriptstyle d_k} & & \uparrow{\scriptstyle d_k} & & \\
0 & \longrightarrow & A^k & \xrightarrow{\phi_k} & B^k & \xrightarrow{\psi_k} & C^k & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle d_{k-1}} & & \uparrow{\scriptstyle d_{k-1}} & & \uparrow{\scriptstyle d_{k-1}} & &
\end{array}$$

we always get a long exact sequence:

$$\begin{aligned} \cdots \longrightarrow\ & H^k(\mathcal{A}) \xrightarrow{\phi_k} H^k(\mathcal{B}) \xrightarrow{\psi_k} H^k(\mathcal{C}) \xrightarrow{\delta_k} \\ \longrightarrow\ & H^{k+1}(\mathcal{A}) \xrightarrow{\phi_{k+1}} H^{k+1}(\mathcal{B}) \xrightarrow{\psi_{k+1}} H^{k+1}(\mathcal{C}) \xrightarrow{\delta_{k+1}} \cdots \end{aligned}$$

Verification of $\operatorname{Ker}\psi_k \supset \operatorname{Im}\phi_k$**.** Suppose $[b] \in \operatorname{Im}\phi_k$. Then $b = \phi_k a + db'$, where $a \in A^k$ and $b' \in B^{k-1}$. Now, $\psi_k b = \psi_k(\phi_k a) + \psi_k(db') = d(\psi_{k-1}b')$. Therefore, $[\psi_k b] = 0 \in H^k(\mathcal{C})$.

Verification of $\operatorname{Ker}\psi_k \subset \operatorname{Im}\phi_k$**.** Suppose $[b] \in \operatorname{Ker}\psi_k$. Then $\psi_k b = dc'$, $c' \in C^{k-1}$. Next, use the fact that $B^{k-1} \to C^{k-1} \to 0$ to find $b' \in B^{k-1}$ such that $c' = \psi_{k-1}b'$. Then $\psi_k b = d(\psi_{k-1}b') = \psi_k(db')$. Hence, by the exactness, $b - db' = \phi_k(a)$ for some $a \in A^k$. Thus, $\phi_k[a] = [b]$.

Definition of $\delta_k : H^k(\mathcal{C}) \to H^{k+1}(\mathcal{A})$**.** Let $[c] \in H^k(\mathcal{C})$. Then $dc = 0$. Also we have $b \in B^k$ with $\psi_k b = c$ by the surjectivity of $B^k \to C^k$. Consider db. Since $\psi_{k+1}(db) = d(\psi_k b) = dc = 0$, there exists an $a \in A^{k+1}$ such that $\phi_{k+1}a = dc$. Let $[a] = \delta_k[c]$. Here, $da = 0$, since $\phi_{k+2}(da) = d(\phi_{k_1}a) = d(db) = 0$, and $A^{k+2} \to B^{k+2}$ is injective. We need to show that this definition is independent of the choice of c, choice of b, and choice of a. This is left for HW.

HW: Verify the rest of the exactness.

21. Integration

Let M be an n-dimensional manifold and $\omega \in \Omega^n(M)$. We will try to make sense of $\int_M \omega$.

21.1. **Orientation.**

Recall: M is orientable if there exists an open cover $\{U_\alpha\}$ and $\phi_\alpha : U_\alpha \to \mathbf{R}^n$ where the Jacobians $J_{\alpha\beta}$ of $\phi_\beta \circ \phi_\alpha^{-1}$ have positive determinant.

Proposition 21.1. *M is orientable if and only if there exists a nowhere zero n-form ω on M.*

Proof. Suppose M is orientable. Take a partition of unity $\{f_\alpha\}$ subordinate to U_α. Let $x_1, \dots, x_n$ be the coordinates on U_α. Construct $\omega_\alpha = f_\alpha x_1 \wedge \dots \wedge dx_n$. ω_α is a smooth n-form on M with support contained in U_α. Let $\omega = \sum_\alpha \omega_\alpha$. This is nowhere zero, since any $(\phi_\alpha \circ \phi_\beta^{-1})^* \omega_\beta = f_\beta \circ (\phi_\alpha \circ \phi_\beta^{-1}) \det(J_{\beta\alpha}) dx_1 \wedge \dots \wedge dx_n$. The key point here is that $\det(J_{\beta\alpha})$ are positive, so $f_\beta \circ (\phi_\alpha \circ \phi_\beta^{-1}) \det(J_{\beta\alpha}) \geq 0$. At any point $p \in M$, at least one f_α is positive, and the ω_α are *additive*, so ω is nowhere zero on M.

On the other hand, suppose there exists a nowhere zero n-form ω on M. Given U_α, we choose coordinates $x_1, \dots, x_n$ so that $dx_1 \wedge \dots \wedge dx_n$ is a positive function times ω. Once we do this, clearly $J_{\alpha\beta}$ has positive determinant. □

Since the n-form ω is nowhere zero, what this says is that $\bigwedge^n T^*M$ is isomorphic to $M \times \mathbf{R}$ as a vector bundle, i.e., is a *trivial* vector bundle.

On a connected manifold M, any two nowhere zero n-forms ω and ω' differ by a function, i.e., $\exists$ a positive (or negative) function f s.t. $\omega = f\omega'$. We have two equivalence classes, if we set $\omega \sim \omega'$ whenever f is a positive function. Each equivalence class is called an *orientation* of M.

The standard orientation on $\mathbf{R}^n$ is $dx_1 \wedge \dots \wedge dx_n$.

Equivalent definition of orientation. The set $Fr(V)$ of ordered bases (or *frames*) of a finite-dimensional vector space V of dimension n is diffeomorphic to $Gl(V)$ (albeit not naturally): Fix an ordered basis $(v_1, \dots, v_n)$. Then any other basis $(w_1, \dots, w_n)$ can be written as $(Av_1, \dots, Av_n)$, $A \in Gl(V)$. Therefore, there is a bijection $Fr(V) \simeq Gl(V)$, and we induce a smooth structure on $Fr(V)$ from $Gl(V)$. (Note however that there is a distinguished point $id \in Gl(V)$ but no distinguished basis in $Fr(V)$.) Since $Gl(V)$ has two connected components, $Fr(V)$ has two components, and each component is called an *orientation* for V. An *orientation* for M is a choice of orientation for each T_pM which is smooth in $p \in M$. [We can construct the *frame bundle* $Fr(M) = \sqcup_p Fr(T_pM)$ by topologizing as follows. Locally near p, identify its neighborhood with $\mathbf{R}^n$ and $\sqcup_{p \in \mathbf{R}^n} Fr(T_p\mathbf{R}^n) = Fr(\mathbf{R}^n) \times \mathbf{R}^n$. The frame bundle is a *fiber bundle* over M whose fibers are diffeomorphic to $Gl(V)$.]

21.2. **Change of variables formula.**

Let U, $V \subset \mathbf{R}^n$ be open sets with coordinates $(x_1, \dots, x_n)$, $(y_1, \dots, y_n)$, and $\phi : U \to V$ a diffeomorphism. Then:

$$\int_V f(y)dy_1 \dots dy_n = \int_U f(\phi(x)) \left| \frac{\partial \phi}{\partial x} \right| dx_1 \dots dx_n.$$

In light of the change of variables formula, $\int_M \omega$ makes sense only when M is orientable, since the change of variables for an n-form does not have the absolute value. At any rate, n-forms have the wonderful property of having the correct transformation property (modulo sign) under diffeomorphisms.

21.3. **Definition of the integral.** We define:

$$\int_M \omega = \sum_\alpha \int_{U_\alpha} f_\alpha \omega,$$

where $\{f_\alpha\}$ is a partition of unity subordinate to $\{U_\alpha\}$.

HW: Check that the definition of $\int_M \omega$ does not depend on the choice of U_α as well as the choice of $\{f_\alpha\}$.

22. Stokes' Theorem

22.1. **Manifolds with boundary.** We enlarge the class of manifolds by allowing ones "with boundary". These are locally modeled on the half-plane $\mathbf{H}^n = \{x_1 \leq 0\} \subset \mathbf{R}^n$.

Definition 22.1. *A Hausdorff, second countable topological space is a* manifold with boundary *if there exists an atlas* $\{(U_\alpha, \phi_\alpha)\}$*, where* $\phi_\alpha : U_\alpha \to \mathbf{H}^n$ *is a homeomorphism onto its image and the transition functions* $\phi_\alpha \circ \phi_\beta^{-1}$ *are smooth. The boundary of* M*, denoted* ∂M*, is the set of points of* M *which lie on the boundary of some half-plane* $\mathbf{H}^n$ *under some map* ϕ_α*. Equivalently, it is the non-interior points of* M*.* ∂M *is an* $(n-1)$*-dimensional manifold.*

Example: The n-dimensional unit ball $B^n = \{(x_1, \dots, x_n) \in \mathbf{R}^n | x_1^2 + \dots + x_n^2 \leq 1\}$. $\partial B^n = S^{n-1}$.

Proposition 22.2. *If* M *is an orientable manifold with boundary, then* ∂M *is an orientable manifold.*

Proof. Let $\{U_\alpha\}$ be an oriented atlas for M. Then we take an atlas $\{V_\alpha\}$ for ∂M as follows. Let $V_\alpha = U_\alpha \cap \{x_1 = 0\} \subset \mathbf{H}^n = \{x_1 \leq 0\}$. (Note that if any $p \in M$ is mapped to $\partial \mathbf{H}^n$ under a coordinate chart, then p cannot be mapped to the interior of $\mathbf{H}^n$ under any other coordinate chart.) If $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n})$ is an oriented basis for M on U_α, then let $(\frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n})$ be an oriented basis for ∂M. This works because an outward normal vector $\frac{\partial}{\partial x_1}$ will go to another outward normal vector under a change of coordinates. □

22.2. **Stokes' Theorem.**

Theorem 22.3 (Stokes' Theorem)**.** *Let* ω *be an* $(n-1)$*-form on a manifold with boundary* M *of dimension* n*. Then* $\int_M d\omega = \int_{\partial M} \omega$*.*

Remark: ∂ happily switches places (jumps up or jumps down).

Zen: The significance of Stokes' Theorem is that a topological operation ∂ is related to an analytic operation d.

Proof. Take an open cover $\{U_\alpha\}$ where U_α is diffeomorphic to (i) $(0,1) \times \dots \times (0,1)$ (U_α does not intersect ∂M) or (ii) $(0,1] \times (0,1) \times \dots \times (0,1)$ ($U_\alpha \cap \partial M = \{x_1 = 1\}$). Let $\{f_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. By linearity, it clearly suffices to compute $\int_{U_\alpha} d(f_\alpha \omega) = \int_{\partial M \cap U_\alpha} f_\alpha \omega$, i.e., assume ω is supported on one U_α.

We will treat the $n = 2$ case. The generalization is straightforward. Let ω be an $(n-1)$-form of type (ii). Then on $[0,1] \times [0,1]$ we can write $\omega = f_1 dx_1 + f_2 dx_2$.

$$\int_{\partial M} \omega = \int_{\partial M} f_1 dx_1 + f_2 dx_2 = \int_0^1 f_2(1, x_2) dx_2.$$

On the other hand,

$$\begin{aligned}
\int_M d\omega &= \int_0^1 \int_0^1 \left(-\frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1} \right) dx_1 dx_2 \\
&= \int_0^1 \left(\int_0^1 \frac{\partial f_2}{\partial x_1} dx_1 \right) dx_2 + \int_0^1 \left(\int_0^1 -\frac{\partial f_1}{\partial x_2} dx_2 \right) dx_1 \\
&= \int_0^1 (f_2(1, x_2) - f_2(0, x_2)) dx_2 + \int_0^1 (f_1(x_1, 0) - f_1(x_1, 1)) dx_1 \\
&= \int_0^1 f_2(1, x_2) dx_2
\end{aligned}$$

Try to see that $n > 2$ also work in the same way. □

Example: (Green's Theorem) Let $\Omega \subset \mathbf{R}^2$ be a compact domain with smooth boundary, i.e., Ω is a 2-dimensional manifold with boundary $\partial\Omega = \gamma$. Then

$$\int_\gamma f dx + g dy = \int_\Omega \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx dy.$$

Example: Consider ω defined on $\mathbf{R}^2 - \{0\}$:

$$\omega(x, y) = \left(\frac{-y}{x^2 + y^2} \right) dx + \left(\frac{x}{x^2 + y^2} \right) dy.$$

Let $C = \{x^2 + y^2 = R^2\}$. Then $x = R\cos\theta$, $y = R\sin\theta$, and we compute

$$\int_C \omega = 2\pi.$$

It is easy to show that $d\omega = 0$.

Claim: ω is not exact! In fact, if $\omega = d\eta$, then

$$0 = \int_{\partial C} \eta = \int_C d\eta = 2\pi,$$

a contradiction.

23. Applications of Stokes' Theorem

23.1. The Divergence Theorem.

Theorem 23.1. *Let $\Omega \subset \mathbf{R}^3$ be a compact domain with smooth boundary. Let $F = (F_1, F_2, F_3)$ be a vector field on Ω. Then*

$$\int_\Omega \operatorname{div} F \; dxdydz = \int_{\partial\Omega} \langle n, F\rangle dA,$$

where $n = (n_1, n_2, n_3)$ is the unit *outward normal to $\partial\Omega$,*

$$dA = n_1 dy \wedge dz + n_2 dz \wedge dx + n_3 dx \wedge dy,$$

and $\langle\cdot\rangle$ is the standard inner product.

Let $\omega = F_1 dydz + F_2 dzdx + F_3 dxdy$. Then $d\omega = (\operatorname{div} F) dxdydz$. It remains to see why $\int_{\partial\Omega} \omega = \int_{\partial\Omega} \langle n, F\rangle dA$.

Evaluating forms. We explain what it means to take $\omega(v_1, \dots, v_k)$, where ω is a k-form and v_i are tangent vectors. Let V be a finite-dimensional vector space. There exists a map:

$$(\wedge^k V^*) \times (V \times \dots \times V) \to \mathbf{R}$$

$$(f_1 \wedge \dots \wedge f_k, (v_1, \dots, v_k)) \mapsto \sum (-1)^\sigma f_1(v_{i_1}) \dots f_k(v_{i_k}),$$

where the sum ranges over all permutations of $(1, \dots, k)$ and σ is the number of transpositions required for the transposition $(1, \dots, k) \mapsto (i_1, \dots, i_k)$. Note that this alternating sum is necessary for the well-definition of the map.

Example. Let $\omega = F_1 dydz + F_2 dzdx + F_3 dxdy$. Then $\omega\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right) = -F_2$.

Interior product. We can define the interior product as follows: $i_v : \bigwedge^k V^* \to \bigwedge^{k-1} V^*$, $i_v \omega = \omega(v, \cdot, \cdot, \dots, \cdot)$. (Insert v into the first slot to get a $(k-1)$-form.)

Example. On $\mathbf{R}^3$, let $\eta = dxdydz$. Also let n be the unit normal vector to $\partial\Omega$. Then, along $\partial\Omega$ we can define $i_n \eta = n_1 dydz + n_2 dzdx + n_3 dxdy$.

Why is this dA? At any point of $p \in \partial\Omega$, take tangent vectors v_1, v_2 of $\partial\Omega$ so that n, v_1, v_2 is an oriented orthonormal basis. Then the *area form dA* should evaluate to 1 on v_1, v_2. Since $\eta(n, v_1, v_2) = 1$ (since η is just the determinant), we see that $dA = i_n \eta$.

Explanation of $\langle n, F\rangle dA = F_1 dydz + F_2 dzdx + F_3 dxdy$. Also note that $i_F \eta = F_1 dydz + \dots$. But now, $i_F \eta(v_1, v_2) = \eta(F, v_1, v_2) = \eta(\langle n, F\rangle n, v_1, v_2) = \langle n, F\rangle dA$ (by Gram-Schmidt).

23.2. Evaluating cohomology classes.

Let M be a compact, oriented manifold (without boundary) of dimension n.

Proposition 23.2. *There exists a well-defined, nonzero map $\int : H^n(M) \to \mathbf{R}$.*

Proof. Given $\omega \in \Omega^n(M)$, we map $\omega \mapsto \int_M \omega$. Note that every n-form ω is closed. To show the map is defined on the level of cohomology, let ω be an exact form, i.e., $\omega = d\eta$. Then

$$\int_M \omega = \int_M d\eta = \int_{\partial M} \eta = 0.$$

Next we prove the nontriviality of $\int$: if ω is an *orientation form* (ω is nowhere zero), then $\int_M \omega > 0$ or < 0, since on each coordinate chart ω is some positive funtion times $dx_1 \dots dx_n$. □

The proposition shows that $\dim H^n(M) \geq 1$. In fact, we have the following:

Theorem 23.3. $H^n(M) \simeq \mathbf{R}$.

We omit the proof.

Example. $M = S^n$. Then $H^i(S^n) = \mathbf{R}$ for $i = 0$ or n and $= 0$ for all other i.

24. Day 24

24.1. **Evaluating cohomology classes.** Let $\phi : M^m \to N^n$ be a smooth map between compact oriented manifolds of dimensions m and n, respectively, and $\phi^* : H^k(N) \to H^k(M)$ the induced map on cohomology. Let $\omega \in \Omega^m(N)$ be a closed m-form. Suppose $\int_M \phi^*\omega \neq 0$. Then $\phi^*\omega$ represents a nonzero element in $H^m(M)$. This implies that $[\omega]$ is a nonzero cohomology class in $H^m(N)$.

Example. On $\mathbf{R}^2 - \{0\}$, consider the closed 1-form

$$\omega(x,y) = \left(\frac{-y}{x^2+y^2}\right) dx + \left(\frac{x}{x^2+y^2}\right) dy.$$

We computed $\int_{S^1} \phi^*\omega = 2\pi$, where $\phi : S^1 \to \mathbf{R}^2 - \{0\}$ mapped $\theta \mapsto (R\cos\theta, R\sin\theta)$. Since $[\phi^*\omega]$ is a nonzero cohomology class in $H^1(S^1)$, so is $[\omega] \in H^1(\mathbf{R}^2 - \{0\})$.

Two maps $\phi_0, \phi_1 : M \to N$ are *(smoothly) homotopic* if there exists a map $\Phi : M \times [0,1] \to N$ where $\Phi(x,t) = \phi_t(x)$.

Proposition 24.1. *If $\phi_0, \phi_1 : M \to N$ are homotopic and $\omega \in \Omega^k(N)$, $k = \dim M$, is closed, then $\int_M \phi_0^*\omega = \int_M \phi_1^*\omega$.*

Proof.

$$\int_M \phi_1^*\omega - \int_M \phi_0^*\omega = \int_{\partial(M\times[0,1])} \Phi^*\omega = \int_M d(\Phi^*\omega) = \int_M \Phi^*(d\omega) = 0,$$

since ω is closed. □

Example, cont'd. On $N = \mathbf{R}^2 - \{0\}$. Since ω is a closed 1-form on N, $\int_C \omega = \int_{C'} \omega$ if C and C' are homotopic. That's why the integral did not depend on the radius R of the circle.

24.2. **Definition of degree.** This material can be found in Guillemin & Pollack.

Let $\phi : M \to N$ be a smooth map between oriented compact n-manifolds M and N. Let $y \in N$ be a regular value of ϕ. (Recall $y \in N$ is a *regular value* if, for all $x \in \phi^{-1}(y)$, $df(x)$ is surjective. $y \in N$ which is not a regular value is a *critical value.*)

Claim: $\phi^{-1}(y)$ consists of a finite number of preimages $x_1, \dots, x_k$.

Proof. Suppose there is an infinite number of preimages. By the compactness of M, there must be an accumulation point $x = \lim_{i\to\infty} x_i$, which itself must also be in $\phi^{-1}(y)$. However, for every $x \in \phi^{-1}(y)$ there exists an open set U_x which maps diffeomorphically onto an open set around y. Therefore, x could not have been the limit of $x_i \in \phi^{-1}(y)$. □

The claim implies that for a small enough open set V_y containing y, $\phi^{-1}(V_y)$ is a finite disjoint union of open sets $U_{x_1}, \dots, U_{x_k}$, each of which is diffeomorphic to V_y.

Definition 24.2. *The* degree *of a mapping $\phi : M \to N$ is the sum of orientation numbers ± 1 for each x_i in the preimage of a regular value y. Here the sign is $+1$ if the map from a neighborhood of x_i to a neighborhood of y is orientation-preserving and -1 otherwise.*

Regular values of ϕ do exist:

Theorem 24.3 (Sard)**.** *Let $\phi : M \to N$ be a smooth map. Then the set of critical values of ϕ has measure zero.*

A set $S \subset N$ has *measure zero* if $\{U_i\}_{i=1}^{\infty}$ is a countable atlas and, for each $U_i \subset \mathbf{R}^n$ and $\varepsilon > 0$, $U_i \cap S$ can be covered by a countable union of rectangles $[a_1, b_1] \times \cdots \times [a_n, b_n]$ with total volume ε. This actually implies that S itself can be covered by a countable union of rectangles with total volume ε: For U_i, take rectangles so that the total volume $= \varepsilon \left(\frac{1}{2}\right)^i$. Adding up over all the U_i, we get $\varepsilon \left(\frac{1}{2} + \frac{1}{4} + \dots\right) = \varepsilon$.

Consequently, the set of regular values of ϕ is dense in N.

The proof of Sard's Theorem will be given next time. We conclude with the following theorem, which will be explained in a couple of lectures.

Theorem 24.4 (Degree Theorem)**.** *The degree of a mapping $\phi : M \to N$ is well-defined.*

25. Proof of Sard's Theorem

The proof closely follows that of Milnor, *Topology from the Differentiable Viewpoint.*

Recall the statement of Sard's Theorem.

Theorem 25.1 (Sard)**.** *Let $f : M \to N$ be a smooth map. Then the set of critical values of f has measure zero.*

By our discussion from last time, suffices to prove Sard's Theorem in the following local situation.

Theorem 25.2. *Let $f : \mathbf{R}^m \to \mathbf{R}^n$ be a smooth map. If we set $C = \{x | \operatorname{rank} df(x) < n\}$, then $f(C)$ has measure zero in $\mathbf{R}^n$.*

Remark 1. The "measure" in the term *measure zero* refers to the Lebesgue measure μ.

Remark 2. Open subsets of $\mathbf{R}^n$ have nonzero Lebesgue measure.

Proof. We will prove the theorem for $n = 1$, i.e., $f : \mathbf{R}^m \to \mathbf{R}$. The general case is similar.

Define the following subsets of $\mathbf{R}^m$:

$$\begin{aligned} C_1 &= \left\{ x \in \mathbf{R}^m \,\middle|\, \frac{\partial f}{\partial x_i} = 0, \forall i \right\}, \\ C_k &= \{x \in \mathbf{R}^m | \text{ all partials of } f \text{ up to and including order } k \text{ vanish at } x\}. \end{aligned}$$

Then clearly $C = C_1 \supset C_2 \supset C_3 \dots$.

Strategy.

1. Show $f(C_1 - C_2)$ has measure zero.
2. Show $f(C_k - C_{k+1})$ has measure zero.
3. For k large enough $(k \geq n)$, $f(C_k)$ has measure zero.

Step 1. Let $x \in C_1 - C_2$. We want to show that there exists a neighborhood V of x for which $(C_1 - C_2) \cap V$ has measure zero. (This suffices because if we can cover $C_1 - C_2$ with countably many such V's, the total measure of $C_1 - C_2$ is zero, as seen from the argument used last time, right after the statement of Theorem 24.3.) Here, $\frac{\partial f}{\partial x_1} = \dots = \frac{\partial f}{\partial x_m} = 0$, but some $\frac{\partial^2 f}{\partial x_i \partial x_j} \neq 0$. Without loss of generality assume $\frac{\partial^2 f}{\partial x_1 \partial x_2} \neq 0$. Then consider the map $h : V \ni x \to \mathbf{R}^m$, $(x_1, \dots, x_m) \mapsto (\frac{\partial f}{\partial x_1}, x_2, \dots, x_m)$. Near x, $h : V \to V'$ is a local diffeomorphism, as can be seen easily by computing the Jacobian. Clearly, the critical values of $f : V \to \mathbf{R}$ are the same as the critical values of $f \circ h^{-1} : V' \to \mathbf{R}$, but if $(\tilde{x}_1, \dots, \tilde{x}_m)$ are coordinates of V', then the critical values of $f \circ h^{-1}$ are the same as the critical values of $f \circ h^{-1} : \{\tilde{x}_1 = 0\} \to \mathbf{R}$. We can then induct on the dimension m.

Note. Under a diffeomorphism, the Lebesgue measure μ changes by a positive smooth function f.

Step 2. Similar to Step 1.

Step 3. Let $0 \in C_k$, $k \geq n$. Suppose $f : [-\delta, \delta] \times \dots [-\delta, \delta] \to \mathbf{R}$. Then Taylor's theorem (with remainder) gives us:

$$f(x + h) = f(x) + R(x; h),$$

where $|R(x; h) \leq C|h|^{k+1}$, for all $x \in C_k \cap [-\delta, \delta]^m$ and $x + h \in [-\delta, \delta]^m$.

We subdivide $[0, 1]^m$ into cubes of length δ. Then there are roughly $\frac{1}{\delta^m}$ cubes. Consider one such cube Q which nontrivially intersects C_k. Then its volume is δ^m, whereas its image has length on the order of magnitude of δ^{k+1} from Taylor's theorem. Adding up the total volume of the image, we have $\frac{1}{\delta^m}\delta^{k+1}$, which can be made arbitrarily small by choosing δ small. □

26. Degree

Recall the definition of degree: Let $\phi : M \to N$ be a smooth map between compact, oriented manifolds (without boundary) of dimension n. By Sard's Theorem, there exist (a full measure's worth of) regular values of ϕ. Let $y \in N$ be a regular value, and $x_1, \dots, x_k$ be the preimages of y. Then $deg(\phi)$ is $\sum_{i=1}^k \pm 1$, where the contribution is $+1$ when ϕ is orientation-preserving near x_i and -1 is otherwise.

We will explain why the degree is well-defined.

26.1. Cohomological interpretation.

Theorem 26.1. *If M is an oriented, compact n-manifold (without boundary), then $\int : H^n(M) \to \mathbf{R}$ is an isomorphism.*

The proof will be given in the following section, but for the time being let us use this to reinterpret the degree. $\phi : M \to N$ induces the map $\phi^* : H^n(N) \to H^n(M)$. Then we have the commutative diagram:

$$\begin{array}{ccc} H^n(N) & \xrightarrow{\phi^*} & H^n(M) \\ \int\downarrow & & \int\downarrow \\ \mathbf{R} & \xrightarrow{c} & \mathbf{R} \end{array}$$

where the map $\mathbf{R} \to \mathbf{R}$ is multiplication by some real number c.

Proposition 26.2. $\deg \phi$ *satisfies*

$$\int_M \phi^*\omega = \deg \phi \int_N \omega. \tag{5}$$

Therefore $\deg \phi$ is the constant of multiplication c.

Proof. Once we can prove Equation 5 for a suitable ω of our choice, the proposition follows. Take ω to be supported on V_y with positive integral. Then $\int_M \phi^*\omega$ will be the sum of $\int_{U_{x_i}} \phi^*\omega$. Noting that ϕ is a diffeomorphism from U_{x_i} to V_y, we have $\int_{U_{x_i}} \phi^*\omega = \pm \int_{V_y} \omega$, depending on whether the orientations agree or not. This proves Equation 5. □

26.2. Proof of Theorem 26.1.

We have already shown that $\int : H^(M) \to \mathbf{R}$ is well-defined. It suffices to show that $\ker \int$ consists of exact n-forms. Let ω be an n-form with zero integral. Let $\{U_i\}$ be a cover of M which is finite and has the property that every U_i is diffeomorphic to $\mathbf{R}^n$. Take a partition of unity $\{f_\alpha\}$ subordinate to a good cover. Then we can split ω into the sum $\sum_i \omega_i$, where ω_i is supported inside U_i. Note that $\int_{U_i} \omega_i$ may not be zero.

Lemma 26.3. *If ω is an n-form with compact support and zero integral inside $\mathbf{R}^n$, then $\omega = d\eta$, where η has compact support.*

Proof. We will prove this for $n = 2$. Then $\omega = f(x,y)dxdy$. Define $g(x) = \int_{-\infty}^{\infty} f(x,y)dy$. By Fubini's theorem and the hypothesis that $\int \omega = 0$, we have $\int_{-\infty}^{\infty} g(x)dx = 0$. Define $G(x,y) = \varepsilon(y)g(x)$, where $\varepsilon(x)$ is a bump function with total area 1. Then write:

$$\eta(x,y) = -\left(\int_{-\infty}^{y} [f(x,t) - G(x,t)]dt\right) dx + \left(\int_{-\infty}^{x} G(t,y)dt\right) dy.$$

Clearly, $d\eta = [f(x,y) - G(x,y)]dxdy + G(x,y)dxdy$ and η has compact support. □

What this means is that we can replace ω_i by a cohomologically equivalent n-form which is supported on a small neighborhood of a point $x_i \in M$, i.e., we may assume that ω_i is a bump n-form. The total volume of the ω_i is still zero. Now, engulf all the x_i in an open set $U \subset M$ which is diffeomorphic to $\mathbf{R}^n$ so that ω is compactly supported in U and has total area zero. We use the lemma again to complete the proof of Theorem 26.1.

27. Lie derivatives

27.1. **Lie derivatives.** First we define the *interior product* on the linear algebra level. $i_v : \bigwedge^k V^* \to \bigwedge^{k-1} V^*$, $v \in V$, is given by:

$$f_1 \wedge \cdots \wedge f_k \mapsto \sum_l (-1)^{l+1} f_1 \wedge \ldots f_l(v) \cdots \wedge f_k.$$

Check this is well-defined!

Let X be a vector field on M. Then the *interior product* $i_X : \Omega^k(M) \to \Omega^{k-1}(M)$ satisfies the following properties:

1. For 1-forms ω, $i_X(\omega) = \omega(X)$.
2. In general, we obtain the relation:

$$i_X(\alpha \wedge \beta) = (i_X \alpha) \wedge \beta + (-1)^{deg(\alpha)} \alpha \wedge i_X \beta.$$

Note: we define $i_X : \Omega^0(M) \to \Omega^{-1}(M)$ as the zero map.

Now define $\mathcal{L}_X = d \circ i_X + i_X \circ d : \Omega^k(M) \to \Omega^k(M)$. $\mathcal{L}_X$ is called the *Lie derivative* with respect to X.

Proposition 27.1.

1. *If* $f \in \Omega^0(M)$, *then* $\mathcal{L}_X f = d(i_X f) + i_X(df) = df(X) = X(f)$. *Hence,* $\mathcal{L} : \Omega^0(M) \to \Omega^0(M)$ *satisfies the Leibniz rule.*
2. $\mathcal{L}_X(d\omega) = d(\mathcal{L}_X \omega)$.
3. $\mathcal{L}_X : \Omega^k(M) \to \Omega^k(M)$ *satisfies* $\mathcal{L}_X(\alpha \wedge \beta) = \mathcal{L}_X(\alpha) \wedge \beta + \alpha \wedge \mathcal{L}_X(\beta)$, *i.e., the Leibniz rule.*

The proof is a simple computation, and is left for HW.

Hence, $L : \Omega^k(M) \to \Omega^k(M)$ naturally extends the derivation $X : \Omega^0(M) \to \Omega^0(M)$. (We will usually call anything that satisfies the Leibniz rule a "derivation".)

The following is also a source of derivations $\Omega^k \to \Omega^k$: Let $\phi_t : M \to M$ be a *1-parameter family of diffeomorphisms*, i.e., there exists $\Phi : M \times [0,1] \to M$ smooth such that $\phi_t(\cdot) \stackrel{def}{=} \Phi(\cdot, t)$, $t \in [0,1]$, is a diffeomorphism. Assume in addition that $\phi_0 = id$. Then

$$\frac{d}{dt} \phi_t^* \omega|_{t=0}$$

is a derivation (verification is easy). If $f \in \Omega^0(M)$, then

$$\frac{d}{dt} \phi_t^* f|_{t=0} = \frac{d}{dt} f(\phi_t)|_{t=0} = df(X_0) = X_0(f),$$

where X_0 is the vector field which corresponds to ϕ_t (think in terms of the first definition of the tangent space: at every $x \in M$, we have an arc $\phi_t(x)$, $t \in [-\varepsilon, \varepsilon]$).

We have the following proposition:

Proposition 27.2 (Cartan formula). *Every* $\frac{d}{dt}\phi_t^*|_{t=0} : \Omega^k \to \Omega^k$ *is given by* $d \circ i_X + i_X \circ d$.

Proof. It suffices to check the following:

- $\frac{d}{dt}\phi_t^*$ and $\mathcal{L}_X$ both satisfy the Leibniz rule. (Already verified!)
- $\frac{d}{dt}\phi_t^*$ and $\mathcal{L}_X$ agree on $\Omega^0(M)$. (Yes, they are both vector fields.)
- d commutes with $\frac{d}{dt}\phi_t^*$ and with $\mathcal{L}_X$.

The above three properties allow us to do an induction on degree. □

28. Homotopy properties

28.1. Homotopy properties of de Rham cohomology.

Proposition 28.1. *Let $\phi_t : M \to M$, $t \in [0,1]$, be a 1-parameter family of diffeomorphisms. Then ϕ_t^* induce the same map $H^k(M) \to H^k(M)$ for all $t \in [0,1]$.*

Note. If $\phi_0 = id$, and we write X_0 as the vector field on M given by $\phi_t(x) : [-\varepsilon, \varepsilon] \to M$ at the point x, then $\frac{d}{dt}\phi_t^*\omega|_{t=0} = (d \circ i_{X_0} + i_{X_0} \circ d)\omega$. We can generalize this as follows: Let X_{t_0} be the vector field on M where the arc $\phi_t(x) : [t_0 - \varepsilon, t_0 + \varepsilon] \to M$ is assigned at the point $\phi_{t_0}(x)$ (note NOT at x). Then

$$\frac{d}{dt}\phi_t^*\omega|_{t=t_0} = \phi_{t_0}^*(d \circ i_{X_{t_0}} + i_{X_{t_0}} \circ d)\omega.$$

Proof. Consider a closed k-form ω on M. Then $\frac{d}{dt}\phi_t^*\omega|_{t=t_0} = \phi_{t_0}^*(d \circ i_{X_{t_0}} + i_{X_{t_0}} \circ d)\omega = d(\phi_{t_0}^* i_{X_{t_0}}\omega)$. Therefore it is exact. Now, $\phi_t^*\omega - \omega = \int_0^t \frac{d}{dt}\phi_s^*\omega|_{t=s}dt$, and the difference is exact as well. (This is evident by thinking of the integral as a limit of Riemann sums.) □

Next, we say two maps $\phi_0, \phi_1 : M \to N$ are *(smoothly) homotopic* if there exists a smooth map $\Phi : M \times [0,1] \to N$ with $\phi_t(\cdot) = \Phi(\cdot, t)$, $t = 0, 1$. ϕ_t is said to be the *homotopy* from ϕ_0 to ϕ_1.

Proposition 28.2 (Homotopy invariance)**.** *Suppose $\phi_t : M \to N$ is a homotopy, $t \in [0,1]$. Then $\phi_t^* : H^k(N) \to H^k(M)$ is independent of t.*

Proof. Consider $\Phi : M \times \mathbf{R} \to N$. (It is easy to extend $\Phi : M \times [0,1] \to N$ to $\Phi : M \times \mathbf{R} \to N$.) Then for $\omega \in \Omega^k(M)$ closed, consider $\Omega = \Phi^*\omega$. We have inclusions $i_t : M \to M \times \mathbf{R}$, $x \mapsto (x,t)$, and clearly $\phi_t^*\omega = i_t^*\Omega$. Now take a diffeomorphism $\Psi_t : M \times \mathbf{R} \to M \times \mathbf{R}$, $(x,s) \mapsto (x, s+t)$. Since $i_t = \Psi_t \circ i_0$, $i_t^* = i_0^* \circ \Psi_t^*$. By the previous proposition, Ψ_t^* is independent of t. Hence so are i_t and ultimately ϕ_t. □

28.2. Homotopy equivalence.

We say $\phi : M \to N$ is a *homotopy equivalence* if there exists $\psi : N \to M$ such that $\phi \circ \psi : N \to N$ and $\psi \circ \phi : M \to M$ are homotopic to $id : N \to N$ and $id : M \to M$. Using Proposition 28.2, it is easy to show:

Proposition 28.3 (Homotopy equivalence)**.** *A homotopy equivalence $\phi : M \to N$ induces an isomophism $\phi^* : H^k(N) \to H^k(M)$.*

Proof. This is because $\phi^* \circ \psi^* = id$ (by homotopy invariance) and $\psi^* \circ \phi^* = id$. This proves that ϕ^* and ψ^* are left and right inverses (as linear maps) and are isomorphisms. □

Corollary 28.4 (Poincaré lemma)**.** $H^k_{dR}(\mathbf{R}^n) = 0$ *if* $k > 0$.

Proof. We will show that $\mathbf{R}^n$ is homotopy equivalent to $\mathbf{R}^0 = \{pt\}$. Consider maps $\phi : \mathbf{R}^n \to \mathbf{R}^0$, $(x_1, \ldots, x_n) \mapsto 0$, and $\psi : \mathbf{R}^0 \to \mathbf{R}^n$, $0 \mapsto (0, \ldots, 0)$. Clearly, $\phi \circ \psi : \mathbf{R}^0 \to \mathbf{R}^0$, $0 \mapsto 0$, is the identity map. Next, $\psi \circ \phi : \mathbf{R}^n \to \mathbf{R}^n$, $(x_1, \ldots, x_n) \mapsto 0$ is homotopic to the identity map. In fact, consider $F : \mathbf{R}^n \times [0,1] \to \mathbf{R}^n$, $((x_1, \ldots, x_n), t) \mapsto (tx_1, \ldots, tx_n)$. □

Example. Consider a band $S^1 \times (-1,1)$. It is homotopy equivalent to S^1. We have maps $\phi : S^1 \times (-1,1) \to S^1$, $(\theta, t) \mapsto \theta$ and $\psi : S^1 \to S^1 \times (-1,1)$, $\theta \mapsto (\theta, 0)$. $\phi \circ \psi : S^1 \to S^1$ is id. $\psi \circ \phi : S^1 \times (-1,1) \to S^1 \times (-1,1)$ is $(\theta, t) \mapsto (\theta, 0)$ is homotopic to id. In fact, take $F : S^1 \times (-1,1) \times [0,1] \to S^1 \times (-1,1)$, $(\theta, t, s) \mapsto (\theta, ts)$. Therefore, we have:

$$H^k(S^1 \times (-1,1)) \simeq H^k(S^1)$$

.

Example. Similarly, $H^k(M \times \mathbf{R}^n) \simeq H^k(M)$. More generally, if E is a vector bundle over M, then $H^k(E) \simeq H^k(M)$.

28.3. **Extended example: surface of genus g.** Consider a surface Σ of genus g. If you remove a disk from Σ, you are left with a bouquet of $2g$ bands. You can now use Mayer-Vietoris with U a disk and V a bouquet of $2g$ bands.

28.4. **Euler characteristic.** Let M be an n-dimensional manifold. Then we define the Euler characteristic of M to be:

$$\chi(M) = \sum_{i=0}^{n} (-1)^i \dim H^i_{dR}(M).$$

Examples.

1. $\chi(\mathbf{R}^n) = 1$.
2. $\chi(S^2) = 1 + 0 + 1 = 2$.
3. $\chi(T^2) = 1 - 2 + 1 = 0$.
4. $\chi(\text{genus } g \text{ surface}) = 2 - 2g$.

Note. For compact surfaces, the Euler characteristic is given by the classical formula $V - E + F$, where V is the number of vertices, E is the number of edges, and F is the number of faces of a polyhedron representing the surface.

Proposition 28.5. *If $M = U \cup V$, then $\chi(M) = \chi(U) + \chi(V) - \chi(U \cap V)$.*

Proof. Use the Mayer-Vietoris sequence and add up the dimensions. □

29. Vector fields

Recall a *vector field* on $U \subset M$ is a section of TM defined over U.

29.1. **Lie brackets.**

Given two vector fields X and Y on M viewed as derivations, we can define its Lie bracket $[X,Y] = XY - YX$, i.e., for $f \in C^\infty(M)$, $[X,Y](f) = X(Yf) - Y(Xf)$.

Proposition 29.1. *$[X,Y]$ is also a derivation, hence is a vector field.*

Proof. This is a local computation. Take $X = \sum_i a_i \frac{\partial}{\partial x_i}$ and $Y = \sum_j b_j \frac{\partial}{\partial x_j}$. Then:

$$\begin{aligned}
[X,Y](f) &= \sum_i a_i \frac{\partial}{\partial x_i}\left(\sum_j b_j \frac{\partial f}{\partial x_j}\right) - \sum_j b_j \frac{\partial}{\partial x_j}\left(\sum_i a_i \frac{\partial f}{\partial x_i}\right) \\
&= \sum_{ij} a_i \frac{\partial b_j}{\partial x_i}\frac{\partial f}{\partial x_j} - \sum_{ij} b_j \frac{\partial a_i}{\partial x_j}\frac{\partial f}{\partial x_i}
\end{aligned}$$

In other words,

$$\left[\sum_i a_i \frac{\partial}{\partial x_i}, \sum_j b_j \frac{\partial}{\partial x_j}\right] = \sum_{ij}\left(a_j \frac{\partial b_i}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j}\right)\frac{\partial}{\partial x_i}$$

□

Properties of Lie brackets.

1. (Anticommutativity) $[X,Y] = -[Y,X]$.
2. (Jacobi identity) $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$.
3. $[fX, gY] = fg[X,Y] + fX(g)Y - gY(f)X$.

These properties are easy to verify, and are left as exercises.

29.2. **Fundamental Theorem of Ordinary Differential Equations.**

Theorem 29.2. *Given a vector field X on a manifold M and $p \in M$, there exist an open set $U \ni p$, $\varepsilon > 0$, and a smooth map $\Phi : U \times (-\varepsilon, \varepsilon) \to M$ such that if we set $\phi_t(x) = \Phi(x,t)$, $x \in U$, then $\phi_t(x) = x$ and $\phi_t(x)$ is an arc through x whose tangent vector at t is $X(\phi_t(x))$.*

Locally: take coordinates $x_1, \dots, x_n$ and $x = 0$. If $X = \sum_{i=1}^n a_i(x) \frac{\partial}{\partial x_i}$ and we write $x(t) = \phi_t(0)$, then

$$\frac{dx}{dt}(t) = (a_1(x(t)), \dots, a_n(x(t))).$$

We omit the proof of this theorem.

Definition 29.3. *A curve $\gamma : (a,b) \to M$ is an* integral curve *of X if $\frac{d\gamma}{dt} = X(\gamma(t))$.*

Remarks.

1. $\phi_t(x)$ are integral curves of X. If $\gamma : (-\delta, \delta) \to M$ is another integral curve of X with $\gamma(0) = \phi_0(x)$, then $\gamma(t) = \phi_t(x)$ on $(-\varepsilon, \varepsilon) \cap (-\delta, \delta)$. Therefore, the flow $\Phi : U \times (-\varepsilon, \varepsilon) \to M$ is unique on the domain of definition.
2. If M is compact (without boundary) and X is a vector field on M, then there exists a global flow $\Phi : M \times \mathbf{R} \to M$ with $\phi_0 = id$. (This is because if M is compact, we may choose ε to work for all the open sets U containing p, since we may assume there is a finite number of such open sets.)
3. However, if M is not compact, then there are vector fields X which do not admit global flows.
4. $\phi_s \circ \phi_t = \phi_{s+t}$, and $\phi_t^{-1} = \phi_{-t}$. In particular, on M compact, ϕ_t, $t \in [0, 1]$, forms a 1-parameter *group* of diffeomorphisms.

Example. On $\mathbf{R} - \{0\}$ consider $X = \frac{\partial}{\partial x}$. X, when integrated on $\mathbf{R}$, clearly gives $\Phi : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, $(x, t) \mapsto x + t$. However, when $\{0\}$ is removed, no matter how small an ε you take, there is no $\Phi : \mathbf{R} \times (-\varepsilon, \varepsilon) \to \mathbf{R}$.

Corollary 29.4. *Suppose $X(p) \neq 0$. Then there exists a coordinate system near p such that $X = \frac{\partial}{\partial x}$.*

Proof. Choose a smooth surface Σ which is transverse to X. Now take $\psi : \Sigma \times (-\varepsilon, \varepsilon) \to M$ given by Φ restricted to Σ. Since Σ is transverse to X, ψ is a diffeomorphism near p by the inverse function theorem. In the coordinate system $\Sigma \times (-\varepsilon, \varepsilon)$, X is clearly $\frac{\partial}{\partial t}$. □

30. Vector fields and Lie derivatives

30.1. **Pullback.**

Proposition 30.1. *Let $f : M \to N$ be a smooth map and $\omega \in \Omega^k(N)$. Then we have:*

$$f^*\omega(x)(X_1, \dots, X_k) = \omega(f(x))(f_*X_1, \dots, f_*X_k).$$

Proof. It suffices to show this for 1-forms dg. Then:

$$(f^*dg)(X) = d(g \circ f)(X) = X(g \circ f) = f_*X(g),$$

by definition of the pushforward of X. If we write this in coordinates, then

$$dg = \sum_i \frac{\partial g}{\partial y_i} dy_i$$

and

$$f^*dg = \sum_{ij} \frac{\partial g}{\partial y_i} \frac{\partial y_i}{\partial x_j} dx_j,$$

so

$$f^*dg(\frac{\partial}{\partial x_j}) = \sum_i \frac{\partial g}{\partial y_i} \frac{\partial y_i}{\partial x_j},$$

whereas

$$dg(f_* \frac{\partial}{\partial x_j}) = \sum_i \frac{\partial g}{\partial y_i} dy_i (\sum_l \frac{\partial y_l}{\partial x_j} \frac{\partial}{\partial y_l}) = \sum_i \frac{\partial g}{\partial y_i} \frac{\partial y_i}{\partial x_j}.$$

□

30.2. **Lie derivatives.**

Let X be a vector field on M. Then there exist a local or global flow $\Phi : M \times (-\varepsilon, \varepsilon) \to M$, $\phi_t(x) = \Phi(x,t)$, such that $\phi_0(x) = x$. We defined the *Lie derivative* $\mathcal{L}_X$ on forms ω as:

$$\mathcal{L}_X \omega = \frac{d}{dt} \phi_t^* \omega |_{t=0}.$$

Lie derivatives can be defined on vector fields Y as well:

$$\mathcal{L}_X Y = \frac{d}{dt} (\phi_{-t})_* Y |_{t=0}.$$

Here, vector fields cannot usually be pulled back, but for a diffeomorphism ϕ, there is a suitable substitute, namely $(\phi^{-1})_*$.

Ultimately, it is easy to see that $\mathcal{L}_X$ can be defined on any tensor of the type $\bigwedge^k T^*M \otimes \bigwedge^l TM$.

Properties of $\mathcal{L}_X$.

1. $\mathcal{L}_X f = Xf$.
2. $\mathcal{L}_X \omega = (d \circ i_X + i_X \circ d)\omega$

3. $\mathcal{L}_X(\omega(X_1,\ldots,X_k)) = (\mathcal{L}_X\omega)(X_1,\ldots,X_k) + \sum_i \omega(X_1,\ldots,\mathcal{L}_X X_i,\ldots,X_k)$.
4. $\mathcal{L}_X Y = [X,Y]$.

(1), (2) are already proven. (3) is left for homework. Use Proposition 30.1 above. We will do (4), assuming (1), (2), (3). We compute:

$$\begin{aligned} X(Y(f)) &= \mathcal{L}_X(Yf) \\ &= \mathcal{L}_X(df(Y)) \\ &= (\mathcal{L}_X df)(Y) + df(\mathcal{L}_X Y) \\ &= (d\circ\mathcal{L}_X f)Y + df(\mathcal{L}_X Y) \\ &= d(X(f))Y + (\mathcal{L}_X Y)(f) \\ &= Y(X(f)) + (\mathcal{L}_X Y)(f). \end{aligned}$$

Therefore, $(\mathcal{L}_X Y)f = X(Y(f)) - Y(X(f)) = [X,Y](f)$.

30.3. **Interpretation of $\mathcal{L}_X Y = [X,Y]$.** As before, X, Y may not have global flows, but for simplicity let us assume they do. Let $\phi_s : M \to M$, $s \in \mathbf{R}$, be the 1-parameter group of diffeomorphisms generated by X and $\psi_t : M \to M$, $t \in \mathbf{R}$, be the 1-parameter group of diffeomorphisms generated by Y. Noting that $Y(x) = \lim_{t\to 0} \frac{\psi_t(x)-x}{t}$, we have

$$\begin{aligned} \mathcal{L}_X Y(x) &= \lim_{s\to 0} \frac{((\phi_{-s})_* Y)(x) - Y(x)}{s} \\ &= \lim_{s,t\to 0} \frac{(\phi_{-s}\circ\psi_t\circ\phi_s(x) - x) - (\psi_t(x) - x)}{st} \\ &= \lim_{s,t\to 0} \frac{\phi_{-s}\circ\psi_t\circ\phi_s(x) - \psi_t(x)}{st} \\ &= \lim_{s,t\to 0} \psi_t\left(\frac{\psi_t^{-1}\circ\phi_s^{-1}\circ\psi_t\circ\phi_s(x) - x}{st}\right) \\ &= \lim_{s,t\to 0} \frac{\psi_t^{-1}\circ\phi_s^{-1}\circ\psi_t\circ\phi_s(x) - x}{st} \end{aligned}$$

Hence, the Lie bracket $[X,Y]$ measures the infinitesimal discrepancy when you flow s units along X, t units along Y, $-s$ units along X and finally $-t$ units along Y.

31. Day 31

31.1. Relationship between d and $[,]$.

Proposition 31.1. *Consider $\theta \in \Omega^1(M)$ and $X, Y \in \mathfrak{X}(M)$. Then $d\theta(X,Y) = X\theta(Y) - Y\theta(X) - \theta([X,Y])$.*

Proof.

$$\begin{aligned} d\theta(X,Y) &= i_Y i_X d\theta = i_Y(\mathcal{L}_X - d\circ i_X)\theta \\ &= i_Y(\mathcal{L}_X\theta - d(\theta(X))) \\ &= (\mathcal{L}_X\theta)Y - Y\theta(X) \\ &= X\theta(Y) - Y\theta(X) - \theta([X,Y]), \end{aligned}$$

by using

$$\mathcal{L}_X(\theta(Y)) = (\mathcal{L}_X\theta)Y + \theta([X,Y]).$$

□

More generally we have, for $\Omega^k(M)$, $X_1, \dots, X_{k+1} \in \mathfrak{X}(M)$:

$$\begin{aligned} d\omega(X_1,\dots,X_{k+1}) &= \sum_i (-1)^{i+1} X_i(\omega(X_1,\dots,\widehat{X_i},\dots,X_{k+1})) \\ &\quad + \sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],X_1,\dots,\widehat{X_i},\dots,\widehat{X_j},\dots,X_{k+1}). \end{aligned}$$

Here $\widehat{X_i}$ means omit the term with X_i. The proof is for HW.

31.2. Distributions.

Recall that if X is a vector field with $X(p) \neq 0$, then locally near p there exists an open set with coordinates $(x_1, \dots, x_n)$ where $X = \frac{\partial}{\partial x_1}$. Can we generalize this? If X, Y are two vector fields which span a 2-dimensional subspace of TM at p, then near p span(X,Y) assigns a 2-plane field at every x in a neighborhood of p.

Definition 31.2. *Let M be an n-dimensional manifold.*

1. *A k-dimensional* distribution *$\mathcal{D}$ is a smooth choice of a k-dimensional subspace of T_pM at every point $p \in M$. By a* smooth choice *we mean there exist k linearly independent vector fields $X_1, \dots, X_k$ which span $\mathcal{D}_x$ locally near p.*
2. *An* integral submanifold *N of M is a submanifold where $T_pN \subset \mathcal{D}_x$ at every $p \in N$.* $\dim N$ *is not necessarily* $\dim \mathcal{D}$*, but* $\dim N \leq \dim \mathcal{D}$*.*
3. *$\mathcal{D}$ is an* integrable distribution *if there is a coordinate system $\{x_1, \dots, x_n\}$ near every $p \in M$ such that $\mathcal{D} = \mathrm{Span}\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}\}$. Equivalently, $\mathcal{D}$ is* integrable *if there locally exist functions $f_1, \dots, f_{n-k}$ such that $\{f_1 = const, \dots, f_{n-k} = const\}$ are integral submanifolds of $\mathcal{D}$ and the f_i are independent (i.e., $df_1 \wedge \cdots \wedge df_{n-k} \neq 0$.*

Dually we can define a k-dimensional distribution on M (of dimension n) locally by prescribing $n-k$ linearly independent 1-forms $\omega_1, \ldots, \omega_{n-k}$.

Example: On $\mathbf{R}^3$, let $\omega = dz$. Then $\mathcal{D} = \ker \omega = \text{Span}\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$. The *integral surfaces* (surfaces everywhere tangent to $\mathcal{D}$) are $z = const$. $\mathcal{D}$ is an integrable 2-plane field distribution.

Example: On $\mathbf{R}^3$, consider $\omega = dz + (xdy - ydx)$. Then $\mathcal{D} = \text{Ker}\,\omega = \text{Span}\{x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\}$. $\mathcal{D}$ is called a *contact distribution*, and is not integrable.

For 2-plane fields in $\mathbf{R}^3$ integrability amounts to: Can you find a function f such that $f = const$ are everywhere tangent to $\mathcal{D}$?

First calculate $\omega \wedge d\omega = 2dxdydz \neq 0$. Then $\mathcal{D}$ is not integrable for the following reason: If $\mathcal{D} = \mathbf{R}\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\}$, then ω is of the form fdx_3. Now, $d\omega = df dx_3$ and $\omega \wedge d\omega = f dx_3 \wedge df \wedge dx_3 = 0$. This gives a contradiction. Therefore, the contact 2-plane field distribution is not integrable.

32. Frobenius' Theorem

Let M be an n-dimensional manifold. A distribution $\mathcal{D}$ of rank k is a rank k subbundle of TM. Locally, $\mathcal{D}$ is defined as the span of independent vector fields $X_1, \dots, X_k$ or as the kernel of independent 1-forms $\omega_1, \dots, \omega_{n-k}$.

Theorem 32.1 (Frobenius' Theorem)**.** *A distribution $\mathcal{D} \subset TM$ of rank k is integrable if and only if for all $X, Y \in \Gamma(\mathcal{D})$, $[X, Y] \in \Gamma(\mathcal{D})$.*

32.1. **Proof of Frobenius' Theorem.** Suppose $\mathcal{D} \subset TM$ is integrable. Then there exist coordinates $x_1, \dots, x_n$ so that $\mathcal{D} = \mathrm{Span}\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}\}$. Hence $X = \sum_{i=1}^k a_i(x)\frac{\partial}{\partial x_i}$ and $Y = \sum_{j=1}^k b_j(x)\frac{\partial}{\partial x_j}$, and

$$[X, Y] = \sum_{i=1}^k \sum_{j=1}^k \left(a_i \frac{\partial b_j}{\partial x_i}\frac{\partial}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j}\frac{\partial}{\partial x_i} \right) \in \Gamma(\mathcal{D}).$$

Suppose for all $X, Y \in \Gamma(\mathcal{D})$, $[X, Y] \in \Gamma(\mathcal{D})$. We will find coordinates $x_1, \dots, x_n$ so that $\mathcal{D} = \mathrm{Span}\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}\}$. Note that all our computations are local, so we restrict to $M = \mathbf{R}^n$. We will first do a slightly easier situation.

Proposition 32.2. *Let $X_1, \dots, X_k$ be independent vector fields with $\mathcal{D} = \mathrm{Span}\{X_1, \dots, X_k\}$, if $[X_i, X_j] = 0$ for all i, j, then $\mathcal{D}$ is integrable.*

Proof. We will deal with the case where $\dim \mathcal{D} = 2$ and $M = \mathbf{R}^3$. Suppose $[X, Y] = 0$. Using the fundamental theorem of ODE's, we can write $X = \frac{\partial}{\partial x_1}$. Then $Y = \sum_{i=1}^3 b_i \frac{\partial}{\partial x_i}$, and $[X, Y] = 0$ implies that $\frac{\partial b_i}{\partial x_1} = 0$, i.e., $b_i = b_i(x_2, x_3)$ (there is no dependence on x_1). Now take $Y' = Y - b_1 X = b_2(x_2, x_3)\frac{\partial}{\partial x_2} + b_3(x_2, x_3)\frac{\partial}{\partial x_3}$. If we project to $\mathbf{R}^2$ with coordinates x_2, x_3, then Y' can be integrated to $\frac{\partial}{\partial x_2'}$, after a possible change of coordinates. Therefore, $\mathcal{D} = \mathrm{Span}\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2'}\}$. □

Still assuming $\dim \mathcal{D} = 2$ and $M = \mathbf{R}^3$, suppose $[X, Y] = AX + BY$. Without loss of generality, $X = \frac{\partial}{\partial x_1}$ and $Y = b_2 \frac{\partial}{\partial x_2} + b_3 \frac{\partial}{\partial x_3}$. Then,

$$[X, Y] = \frac{\partial b_2}{\partial x_1}\frac{\partial}{\partial x_2} + \frac{\partial b_3}{\partial x_1}\frac{\partial}{\partial x_3} = A\frac{\partial}{\partial x_1} + Bb_2\frac{\partial}{\partial x_2} + Bb_3\frac{\partial}{\partial x_3}.$$

This implies: $A = 0$, $\frac{\partial b_2}{\partial x_1} = Bb_2$, $\frac{\partial b_3}{\partial x_1} = Bb_3$. Hence,

$$b_2 = f(x_2, x_3)e^{\int_{t=0}^{t=x_1} B(t, x_2, x_3)dt}, b_3 = g(x_2, x_3)e^{\int_{t=0}^{t=x_1} B(t, x_2, x_3)dt}.$$

Therefore, $Y = e^{\int B}(f(x_2, x_3)\frac{\partial}{\partial x_2} + g(x_2, x_3)\frac{\partial}{\partial x_3})$, and by rescaling Y we get $Y' = f(x_2, x_3)\frac{\partial}{\partial x_2} + g(x_2, x_3)\frac{\partial}{\partial x_3}$. As before, now Y' can be integrated to give $\frac{\partial}{\partial x_2'}$.

HW: Write out a general proof.

32.2. **Restatement in terms of forms.** If $\mathcal{D}$ has rank k on M of dimension n, then dually there exist 1-forms $\omega_1, \dots, \omega_{n-k}$ such that $\mathcal{D} = \{\omega_1 = \cdots = \omega_{n-k} = 0\}$.

Proposition 32.3. *$\mathcal{D}$ is integrable if and only if $d\omega_i = \sum_{j=1}^{n-k} \theta_{ij} \wedge \omega_j$, where θ_{ij} are 1-forms.*

Proof. We use the identity

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]). \tag{6}$$

First suppose $d\omega_i = \sum_{j=1}^{n-k} \theta_{ij} \wedge \omega_j$. Then for sections X, Y of $\mathcal{D}$,

$$d\omega_i(X, Y) = X\omega_i(Y) - Y\omega_i(X) - \omega_i([X, Y]) = -\omega_i([X, Y]).$$

On the other hand, $d\omega_i(X, Y) = 0$. Hence $\omega_i([X, Y]) = 0$ for all i, which implies that $[X, Y]$ is a section of $\mathcal{D}$.

Next, suppose $\mathcal{D}$ is integrable. Complete $\omega_1, \dots, \omega_{n-k}$ into a basis by adding $\eta_1, \dots, \eta_k$. Then

$$d\omega_i = \sum_{i<j} a_{ij}\omega_i \wedge \omega_j + \sum_{i=1}^{k} \sum_{j=1}^{n-k} b_{ij}\eta_i \wedge \omega_j + \sum_{i<j} c_{ij}\eta_i \wedge \eta_j.$$

Using Equation 6, for X, Y sections of $\mathcal{D}$, $d\omega_i(X, Y) = 0$. Taking $X_1, \dots, X_k$ dual to $\eta_1, \dots, \eta_k$, we find that $d\omega_i(X_r, X_s) = c_{rs}$ (or $-c_{sr}$). This proves that all the c_{ij} are zero. □

33. Connections

33.1. **Definition.** Let E be a rank k vector bundle over M and let s be a section of E. s may be local (i.e., in $\Gamma(E,U)$) or global (i.e., in $\Gamma(E,M)$). Also let X be a vector field. We want to differentiate s at $p \in M$ in the direction of $X(p) \in T_pM$.

Definition 33.1. *A* connection *or* covariant derivative ∇ *assigns to every vector field* $X \in \mathfrak{X}(M)$ *a differential operator* $\nabla_X : \Gamma(E) \to \Gamma(E)$ *which satisfies:*

1. $\nabla_X s$ *is* $\mathbf{R}$*-linear in* s*, i.e.,* $\nabla_X(c_1 s_1 + c_2 s_2) = c_1 \nabla_X s_1 + c_2 \nabla_X s_2$ *if* $c_1, c_2 \in \mathbf{R}$.
2. $\nabla_X s$ *is* $C^\infty(M)$*-linear in* X*, i.e.,* $\nabla_{fX+gY} s = f\nabla_X s + g\nabla_Y s$.
3. *(Leibniz rule)* $\nabla_X(fs) = (Xf)s + f\nabla_X s$.

Note: The definition of connection is tensorial in X (condition (2)), so $(\nabla_X s)(p)$ depends on s *near* p but only on X *at* p.

33.2. **Flat connections.** We will now present the first example of a connection.

A vector bundle E of rank k is said to be *trivial* or *parallelizable* if there exist sections $s_1, \dots, s_k \in \Gamma(E,M)$ which span E_p at every $p \in M$. Although not every vector bundle is parallelizable, locally every vector bundle is trivial since $E|_U \simeq U \times \mathbf{R}^k$. We will now construct connections on trivial bundles.

Write any section s as $s = \sum_i f_i s_i$, where $f_i \in C^\infty(M)$. Then define

$$\nabla_X s = \sum_i (Xf_i)s_i = (Xf_1)s_1 + \cdots + (Xf_k)s_k \in \Gamma(E).$$

This connection is usually called a *flat connection*.

HW: Check that this satisfies the axioms of a connection.

Note that $\nabla_X s_i = 0$ for all $X \in \mathfrak{X}(M)$. Sections s satisfying such a property are said to be *covariant constant*.

Important remark: We can define a connection ∇ for each trivialization $E|_U = U \times \mathbf{R}^k$, and there is nothing canonical about the connection ∇ above. (It depends on the choice of trivialization.) The space of connections is a large space (to be made more precise later).

Proposition 33.2. *Any two covariant constant frames* $s_1, \dots, s_k$ *and* $\overline{s_1}, \dots, \overline{s_k}$ *differ by an element of* $Gl(k, \mathbf{R})$.

Proof. Let $\overline{s_1}, \dots, \overline{s_k}$ be another covariant constant frame, i.e., $\nabla_X \overline{s_i} = 0$. Since we can write

$$\overline{s_i} = \sum_j f_{ij} s_j,$$

with $f_{ij} \in C^\infty(M)$, we have:

$$\begin{aligned} 0 = \nabla_X \overline{s_i} &= \sum_j \nabla_X(f_{ij} s_j) \\ &= \sum_j [(X f_{ij}) s_j + f_{ij} \nabla_X s_j] \\ &= \sum_j (X f_{ij}) s_j. \end{aligned}$$

This proves that $X f_{ij} = 0$ for all X and hence $f_{ij} = const.$ □

Therefore, a flat connection determines a covariant constant frame $\{s_1, \ldots, s_k\}$ up to an element of $Gl(k, \mathbf{R})$.

34. More on connections

34.1. **Preliminaries on vector bundles.** Let E be a vector bundle over M and $\phi : N \to M$ be a smooth map. Then we can define the *pullback bundle* $\phi^{-1}E$ over N as follows:

1. The fiber $(\phi^{-1}E)_n$ over $n \in N$ is the fiber $E_{\phi(n)}$ over $\phi(n) \in M$.
2. There exist sufficiently small open sets $V \subset N$, so that $\phi(V) \subset U$ and $\varphi_U : E|_U \stackrel{\sim}{\to} U \times \mathbf{R}^k$. The trivialization $\phi^{-1}E|_V \simeq V \times \mathbf{R}^k$ is induced from this.

Next, if E and F are vector bundles over M, then we can define $E \oplus F$ as follows:

1. The fiber $(E \oplus F)_m$ over $m \in M$ is $E_m \oplus F_m$.
2. Take $U \subset M$ small enough so that $E|_U \stackrel{\sim}{\to} U \times \mathbf{R}^k$ and $F|_U \stackrel{\sim}{\to} U \times \mathbf{R}^l$. Then we get $(E \oplus F)|_U \simeq U \times (\mathbf{R}^k \oplus \mathbf{R}^l)$.

$E \otimes F$ is defined similarly.

34.2. **Existence.** Let M be an n-dimensional manifold and E be a rank k vector bundle over M. Recall a connection ∇ is a way of differentiating sections of E in the direction of a vector field X.

$$\nabla_X : \Gamma(E) \to \Gamma(E),$$

$$\nabla_X(fs) = (Xf)s + f\nabla_X s.$$

Definition 34.1. *A connection ∇ on E is* flat *if there exists an open cover $\{U_\alpha\}$ of M such that $E|_{U_\alpha}$ admits a covariant constant frame $s_1, \dots, s_k$.*

Proposition 34.2. *Connections exist on any vector bundle E over M.*

Note that if E is parallelizable we have already defined connections globally on E. The key point is to pass from local to global when E is not globally trivial.

Let ∇' and ∇'' be two connections on $E|_U$. Let us see whether $\nabla' + \nabla''$ is a connection.

$$\begin{aligned} (\nabla'_X + \nabla''_X)(fs) &= \nabla'_X(fs) + \nabla''_X(fs) \\ &= (Xf)s + f\nabla'_X s + (Xf)s + f\nabla''_X s \\ &= 2(Xf)s + f(\nabla'_X + \nabla''_X)s. \end{aligned}$$

This is not quite a connection, since $2(Xf)$ should be Xf instead. However, a simple modification presents itself:

Lemma 34.3. *Suppose $\lambda_1, \lambda_2 \in C^\infty(U)$ satisfies $\lambda_1 + \lambda_2 = 1$. Then $\lambda_1\nabla' + \lambda_2\nabla''$ is a connection on $E|_U$.*

Proof. HW. □

Proof of Proposition 34.2. Let $\{U_\alpha\}$ be an open cover such that $E|_{U_\alpha}$ is trivial. Let ∇^α be a flat connection on $E|_{U_\alpha}$ associated to some trivialization. Next let $\{f_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. Then form $\sum_\alpha f_\alpha \nabla^\alpha$. By the previous lemma, the Leibniz rule is satisfied. □

Remark: Although each of the pieces ∇^α is flat before patching, the patching destroys flatness. There is no guarantee that (even locally) there exist sections $s_1, \dots, s_k$ which are covariant constant. In fact, for a generic connection, there is not even a single covariant constant section. One way of measuring the failure of the existence of covariant constant sections is the *curvature*.

34.3. **The space of connections.** Given two connections ∇ and ∇', we compute their difference:

$$(\nabla_X - \nabla'_X)(fs) = f(\nabla_X - \nabla'_X)s.$$

Therefore, the difference of two connections is *tensorial* in s.

Locally, take sections $s_1, \dots, s_k$ (not necessarily covariant constant). Then $(\nabla_X - \nabla'_X)s_i = \sum_j a_{ij} s_j$, where (a_{ij}) is a $k \times k$ matrix of functions. In other words, $\nabla - \nabla'$ can be represented by a matrix $A = (A_{ij})$ of 1-forms A_{ij}. Here $a_{ij} = A_{ij}(X)$. Hence, locally it makes sense to write:

$$\nabla = d + A.$$

Here $s = \sum f_i s_i$ corresponds to $(f_1, \dots, f_k)^T$ and more precisely

$$\nabla(f_1, \dots, f_k)^T = d(f_1, \dots, f_k)^T + A(f_1, \dots, f_k)^T.$$

Globally, $\nabla - \nabla'$ is a section of $T^*M \otimes End(E)$. Here $End(E) = Hom(E, E)$. The space of such sections is often written as $\Omega^1(End(E))$ and a section is called a "1-form with values in $End(E)$". This proves:

Proposition 34.4. *The space of connections on E is an affine space $\Omega^1(End(E))$.*

Remark: We view $\Omega^1(End(E))$ not as a vector space (which has a preferred zero element) but rather as an affine space, which is the same thing except for the lack of a preferred zero element.

35. Curvature

Let $E \to M$ be a rank r vector bundle and ∇ be a connection on E.

Definition 35.1. *The* curvature R_∇ *(or simply R) of a connection ∇ is given by:*

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]},$$

or

$$R(X,Y)s = [\nabla_X, \nabla_Y]s - \nabla_{[X,Y]}s.$$

Proposition 35.2.

1. *$R(X,Y)s$ is tensorial ($C^\infty(M)$-bilinear) in X, Y, and s.*
2. *$R(X,Y) = -R(Y,X)$.*

Proof. (2) is easy. For (1), we will prove that $R(X,Y)$ is tensorial in s and leave the verification for X and Y as HW.

$$\begin{aligned} R(X,Y)(fs) &= (\nabla_X\nabla_Y - \nabla_Y\nabla_X)(fs) - \nabla_{[X,Y]}(fs) \\ &= \nabla_X((Yf)s + f\nabla_Y s) - \nabla_Y((Xf)s + f\nabla_X s) - (([X,Y]f)s + f\nabla_{[X,Y]}s) \\ &= fR(X,Y)s \end{aligned}$$

□

Proposition 35.3. *The flat connection $\nabla_X s = \sum (Xf_k)s_k$ has $R = 0$. (Here $s_1, \dots, s_r$ trivializes $E|_U$ and $s = \sum f_k s_k$.)*

Proof. We use $X = \frac{\partial}{\partial x_i}$, $Y = \frac{\partial}{\partial x_j}$. Since $R(X,Y)$ is tensorial, it suffices to compute it for our choices.

$$\begin{aligned} R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)s &= \sum_k \left(\nabla_{\frac{\partial}{\partial x_i}}\nabla_{\frac{\partial}{\partial x_j}} - \nabla_{\frac{\partial}{\partial x_j}}\nabla_{\frac{\partial}{\partial x_i}}\right) f_k s_k \\ &= \sum_l \left[\nabla_{\frac{\partial}{\partial x_i}}\left(\frac{\partial f_k}{\partial x_j}s_k\right) - \nabla_{\frac{\partial}{\partial x_j}}\left(\frac{\partial f_k}{\partial x_i}s_k\right)\right] \\ &= \sum \left(\frac{\partial^2 f_k}{\partial x_i \partial x_j} - \frac{\partial^2 f_k}{\partial x_j \partial x_i}\right) s_k = 0. \end{aligned}$$

□

35.1. Interpretations of curvature.

Think of ∇ as $d + A$ in local coordinates if necessary. We have a sequence:

$$\Omega^0(E) \xrightarrow{\nabla} \Omega^1(E) \xrightarrow{\nabla} \Omega^2(E) \to \dots.$$

The first map is covariant differentiation (interpreted slightly differently). It turns out that this sequence is not a chain complex, i.e., $\nabla \circ \nabla \neq 0$ usually. In fact the obstruction to this being a chain complex is the curvature. Let us locally write:

$$\nabla \circ \nabla s = (d + A)(d + A)s = (d^2 + Ad + dA + A \wedge A)s = (dA + A \wedge A)s.$$

Proposition 35.4. *$R = dA + A \wedge A$, i.e., $R(X, Y)s = (dA + A \wedge A)(X, Y)s$.*

Proof. It suffices to prove the proposition for $X = \frac{\partial}{\partial x_i}$, $Y = \frac{\partial}{\partial x_j}$, and $s = s_k$, where $s_1, \dots, s_r$ is a local frame for $E|_U$. A is an $r \times r$ matrix of 1-forms $(A^t_{ij} dx_t)$. (We will use the Einstein summation convention – if the same index appears twice we assume it is summed over this index.) Then we compute:

$$\nabla_{\frac{\partial}{\partial x_i}} \nabla_{\frac{\partial}{\partial x_j}} s_k = \nabla_{\frac{\partial}{\partial x_i}} (s_m A^j_{mk}) = s_m \frac{\partial A^j_{mk}}{\partial x_i} + s_n A^i_{nm} A^j_{mk} \tag{7}$$

The computation of the rest is left for HW. □

36. Riemannian metrics, Levi-Civita

36.1. **Leftovers from last time.** Last time we defined the curvature R_∇ of a connection ∇. Locally, if ∇ is given by $d+A$, then $R = dA + A\wedge A$.

Theorem 36.1. *∇ is a flat connection if and only if $R_\nabla \equiv 0$.*

We have already shown the easy direction: If ∇ is flat, then $R_\nabla \equiv 0$. The other direction will be omitted for now (probably will be given next semester), since a "good proof" will take us a bit far afield. Our only comment is that $R = dA + A\wedge A = 0$ or $dA = A\wedge A$ looks a lot like the Frobenius integrability condition given in terms of forms....

Corollary 36.2. *Let E be a rank r vector bundle over $\mathbf{R}$ and ∇ be a connection on E. Then ∇ is flat.*

Proof. This is because all 2-forms on $\mathbf{R}$ are zero. ☐

Remark: There are lots of connections which are not flat, since it is easy to find A so that $dA + A\wedge A \neq 0$. (The easiest case is when the rank of E is 1.)

36.2. **Riemannian metrics.**

Definition 36.3. *A* Riemannian metric *$\langle , \rangle$ or g on M is a positive definite symmetric bilinear form (or inner product) $g(x) : T_xM \times T_xM \to \mathbf{R}$ which is smooth in $x \in M$.*

Recall: A bilinear form $\langle , \rangle : V\times V \to \mathbf{R}$ is *positive definite* if $\langle v, v\rangle \geq 0$ and $\langle v, v\rangle = 0$ iff $v = 0$. $\langle , \rangle$ is *symmetric* if $\langle v, w\rangle = \langle w, v\rangle$.

Example: On $\mathbf{R}^n$ take the *standard Euclidean metric* $g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \delta_{ij}$. This is usually written as $g = \sum_i dx_i \otimes dx_i$. Any other Riemannian metric on $\mathbf{R}^n$ can be written as $g(x) = \sum_{ij} g_{ij}(x) dx_i \otimes dx_j$, where $g_{ij}(x) = g_{ji}(x)$.

Proposition 36.4. *Every manifold M admits a Riemannian metric.*

Proof. Let $\{U_\alpha\}$ be an open cover so that $U_\alpha \simeq \mathbf{R}^n$. On each U_α, we take the standard Euclidean metric g_α. Now let $\{f_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. Then $\sum_\alpha f_\alpha g_\alpha$ is the desired metric. ☐

The pair (M, g) of a manifold M together with a Riemannian metric g on M is called a *Riemannian manifold.*

Let $i : N \to (M, g)$ be an embedding or immersion. Then the *induced Riemannian metric* i^*g on N is defined as follows:

$$i^*g(x)(v, w) = g(i(x))(i_*v, i_*w),$$

where $v, w \in T_xN$. The injectivity of i_* is required for the positive definiteness.

36.3. **Levi-Civita connections.** Connections on $TM \to M$ have extra structure because X and Y are the same type of object in the expression $\nabla_X Y$. In fact, we can define the *torsion*:

$$\mathcal{T}_\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

Proposition 36.5. *$\mathcal{T}_\nabla(X,Y)$ is tensorial in X and Y.*

This is an easy exercise and is left for HW. (Note that the notion of torsion does not depend at all on the Riemannian metric.) We say ∇ is *torsion-free* if $\mathcal{T}_\nabla = 0$.

Definition 36.6. *∇ is* compatible *with g if $X\langle Y,Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$. Here $X,Y,Z \in \mathfrak{X}(M)$.*

Theorem 36.7. *Let (M,g) be a Riemannian manifold. Then there exists a unique torsion-free connection which is compatible with g.*

Proof. For any vector fields X,Y,Z, we have:

$$X\langle Y,Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle, \tag{8}$$

$$Y\langle X,Z\rangle = \langle \nabla_Y X, Z\rangle + \langle X, \nabla_Y Z\rangle, \tag{9}$$

$$Z\langle X,Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle. \tag{10}$$

Taking (8) + (9) − (10), we get:

$$2\langle \nabla_X Y, Z\rangle + \langle [Y,X], Z\rangle + \langle [X,Z], Y\rangle + \langle [Y,Z], X\rangle = X\langle Y,Z\rangle + Y\langle X,Z\rangle - Z\langle X,Y\rangle, \tag{11}$$

and solving for $\langle \nabla_X Y, Z\rangle$, we get:

$$\langle \nabla_X Y, Z\rangle = \frac{1}{2}(\langle [X,Y], Z\rangle + \langle [Z,X], Y\rangle + \langle [Z,Y], X\rangle + X\langle Y,Z\rangle + Y\langle X,Z\rangle - Z\langle X,Y\rangle). \tag{12}$$

It is clear that the values of Equation 12 determine ∇. It remains to show that Equation 12 indeed defines a connection which satisfies the torsion-free and compatibility conditions. It is clear that $\langle \nabla_X Y, Z\rangle = \langle \nabla_Y X, Z\rangle$ (torsion-free) and Equation 8 can be recovered from Equation 12. The details are left for HW. □

The unique torsion-free, compatible connection is called the *Levi-Civita connection* for (M,g).

37. Shape operator

Let Σ be a surface embedded in the standard Euclidean $(\mathbf{R}^3, g)$, and let $\overline{g}$ be the induced metric on Σ. We will denote the Levi-Civita connection on $(\mathbf{R}^3, g)$ by ∇ and the Levi-Civita connection on $(\Sigma, \overline{g})$ by $\overline{\nabla}$.

Claim: ∇ satisfies $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = 0$.

The verification is easy. The claim implies that $\nabla_X Y$ is simply $\frac{d}{dt} Y(\gamma(t))|_{t=0}$, where $\gamma(t)$ is the arc representing X at a given point.

What we will do today is valid for hypersurfaces ($(n-1)$-dimensional submanifolds) M inside (N, g) of dimension n, but we will restrict our attention to $N = \mathbf{R}^3$ for simplicity.

Definition 37.1. *Let X, Y be vector fields of $\mathbf{R}^3$ which are tangent to Σ, and let N be the* unit *normal vector field to Σ inside $\mathbf{R}^3$. The* shape operator *is $S(X, Y) = \langle \nabla_X Y, N\rangle$. In other words, $S(X, Y)$ is the projection in the N-direction of $\nabla_X Y$.*

Proposition 37.2. *$S(X, Y)$ is tensorial in X, Y and is symmetric.*

Proof. $S(X, Y) = S(Y, X)$ follows from the torsion-free condition and the fact that $[X, Y]$ is a tangent to Σ. Now,

$$S(fX, Y) = \langle \nabla_{fX} Y, N\rangle = \langle f\nabla_X Y, N\rangle = fS(X, Y).$$

Tensoriality in Y is immediate from the symmetric condition. □

Remark: The shape operator is usually called the *second fundamental form* in classical differential geometry and measures how curved a surface is. (In case you are curious what the *first fundamental form* is, it's simply the induced Riemannian metric.)

Also observe that $S(X, Y) = \langle \nabla_X Y, N\rangle = \langle \nabla_X N, Y\rangle$, by using the fact that $\langle Y, N\rangle = 0$ (since N is a normal vector and Y is tangent to Σ).

37.1. Induced connection vs. Levi-Civita.

If $X, Y \in \mathfrak{X}(M)$, we can write:

$$\nabla_X Y = \nabla^h_X Y + S(X, Y)N,$$

where $\nabla^h_X Y$ denotes the projection of $\nabla_X Y$ onto $T\Sigma$.

Proposition 37.3. *$\nabla^h = \overline{\nabla}$, i.e., ∇^h is the Levi-Civita connection of (Σ, g).*

Proof. We have defined $\nabla^h_X Y = \nabla_X Y - S(X, Y)N$. It is easy to verify that ∇^h satisfies the properties of a connection on Σ.

∇^h is torsion-free:

$$\begin{aligned} \nabla^h_X Y - \nabla^h_Y X &= (\nabla_X Y - S(X, Y)N) - (\nabla_Y X - S(Y, X)N) \\ &= \nabla_X Y - \nabla_Y X \\ &= [X, Y] \end{aligned}$$

∇^h is compatible with $\overline{g}$:

$$\begin{aligned} X\langle Y, Z\rangle &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \\ &= \langle \nabla^h_X Y, Z\rangle + \langle Y, \nabla^h_X Z\rangle, \end{aligned}$$

since $\langle N, X\rangle = 0$ for any vector field N on Σ. □

It seems miraculous that somehow the induced connection is a Levi-Civita connection. Classically, the induced covariant derivative came first, and Levi-Civita came as an abstraction of the covariant derivative.

38. Gauss' Theorema Egregium

Let $(\Sigma, \overline{g})$ be a 2-dimensional Riemannian submanifold of the standard Euclidean $(\mathbf{R}^3, g)$. The shape operator is a symmetric bilinear form:

$$S : T_x\Sigma \times T_x\Sigma \to \mathbf{R},$$
$$S(X,Y) = \langle \nabla_X Y, N\rangle$$

where N is a unit normal to Σ, X, Y are vectors in $T_x\Sigma$ which are extended to an arbitrary vector field tangent to Σ, and ∇ is the Levi-Civita connection for $(\mathbf{R}^3, g)$. We can represent $S(x)$, $x \in \Sigma$, as a matrix by taking an orthonormal basis $\{e_1, e_2\}$ at $T_x\Sigma$ and taking the entries $S(e_i, e_j)$. The trace of this matrix is called the *mean curvature* and the determinant is called the *scalar curvature* or the *Gaußian curvature*.

Denote by ∇ the Levi-Civita connection for g and $\overline{\nabla}$ the Levi-Civita connection for $\overline{g}$. Also write $R = R_\nabla$ and $\overline{R}$ for $R_{\overline{\nabla}}$.

Theorem 38.1 (Gauß' Theorema Egregium). *If X, Y are vector fields on Σ, then*

$$\langle \overline{R}(X,Y)Y, X\rangle = S(X,X)S(Y,Y) - S(X,Y)^2.$$

What this says is that the right-hand side, an extrinsic quantity (depends on the embedding into 3-space) is equal to te left-hand side, an intrinsic quantity (only depends on the Riemannian metric $\overline{g}$ and not on the particular embedding into $\mathbf{R}^3$). Therefore, the scalar curvature is expressed purely in terms of the curvature of the induced metric.

Proof. Let N be the unit normal vector to Σ.

$$\begin{aligned}
\langle \overline{\nabla}_X \overline{\nabla}_Y Y, X\rangle &= X\langle \overline{\nabla}_Y Y, X\rangle - \langle \overline{\nabla}_Y Y, \overline{\nabla}_X X\rangle \\
&= X\langle \nabla_Y Y - S(Y,Y)N, X\rangle - \langle \nabla_Y Y - S(Y,Y)N, \nabla_X X - S(X,X)N\rangle \\
&= X\langle \nabla_Y Y, X\rangle - \langle \nabla_Y Y, \nabla_X X\rangle + \langle S(X,X)N, \nabla_Y Y\rangle \\
&= \langle \nabla_X \nabla_Y Y, X\rangle + S(X,X)S(Y,Y).
\end{aligned}$$

Similarly,

$$\langle \overline{\nabla}_Y \overline{\nabla}_X Y, X\rangle = \langle \nabla_Y \nabla_X Y, X\rangle - S(X,Y)^2,$$
$$\langle \overline{\nabla}_{[X,Y]} Y, X\rangle = \langle \nabla_{[X,Y]} Y, X\rangle.$$

Finally,

$$\begin{aligned}
\langle \overline{R}(X,Y)Y, X\rangle &= \langle R(X,Y)Y, X\rangle + S(X,X)S(Y,Y) - S(X,Y)^2 \\
&= S(X,X)S(Y,Y) - S(X,Y)^2.
\end{aligned}$$

□

39. Euler class

39.1. **Compatible connections.** Let E be a rank k vector bundle over a manifold M. A *fiber metric* is a family of positive definite inner products $\langle , \rangle_x : E_x \times E_x \to \mathbf{R}$ which varies smoothly with respect to $x \in M$. A connection ∇ is *compatible* with $\langle , \rangle$ if $X\langle s_1, s_2\rangle = \langle \nabla_X s_1, s_2\rangle + \langle s_1, \nabla_X s_2\rangle$, for all vector fields X and $s_1, s_2 \in \Gamma(E)$.

Remark: We can view the Riemannian metric g on M as a fiber metric of $TM \to M$. When we think of TM as a vector bundle over M, we forget the fact that TM was derived from M.

Let $U \subset M$ be an open set over which E is trivializable, and let $\{s_1, s_2, \dots, s_k\}$ be an orthonormal frame of E over U. An orthonormal frame can be obtained by starting from some frame of E over U and applying the Gram-Schmidt orthogonalization process.

With respect to $\{s_1, \dots, s_k\}$ we can write $\nabla = d + A$, where A is a $k \times k$ matrix with entries which are 1-forms on U.

Lemma 39.1. *A is a skew-symmetric matrix, i.e., $A^T = -A$.*

Proof. If we write $A = (A^k_{ij} dx_k)$, then we have $\nabla_{\frac{\partial}{\partial x_k}} s_j = s_i A^k_{ij}$.

$$\frac{\partial}{\partial x_k}\langle s_i, s_j\rangle = \langle \nabla_{\frac{\partial}{\partial x_k}} s_i, s_j\rangle + \langle s_i, \nabla_{\frac{\partial}{\partial x_k}} s_j\rangle,$$

so we have

$$A^k_{ij} = -A^k_{ij}.$$

□

Lemma 39.2. *Let $\{s'_1, \dots, s'_k\}$ be another orthonormal frame for E over U. If $g : U \to SO(k)$ is the transformation sending coordinates with respect to s_i to coordinates with respect to s'_i (by left multiplication), then the connection matrix transforms as: $A \mapsto g^{-1}dg + g^{-1}Ag$.*

Proof.

$$\begin{aligned} g^{-1}(d + A)g &= g^{-1}dg + g^{-1}gd + g^{-1}Ag \\ &= d + (g^{-1}dg + g^{-1}Ag). \end{aligned}$$

You may want to check that if A is skew-symmetric and g is orthogonal, then $g^{-1}dg + g^{-1}Ag$ is also skew-symmetric. □

39.2. **Rank 2 case.** Suppose from now on that E has rank 2 over M of arbitrary dimension. Then A_U (the connection matrix over U with respect to some trivialization) is given by

$$A_U = \begin{pmatrix} 0 & A_{21} \\ -A_{21} & 0 \end{pmatrix}.$$

Then the curvature matrix R_U is

$$R_U = dA_U + A_U \wedge A_U = \begin{pmatrix} 0 & \omega_U \\ -\omega_U & 0 \end{pmatrix},$$

where ω_U is the 2-form dA_{21}.

Theorem 39.3. *There is a global closed 2-form ω which coincides with ω_U on each open set U. Hence a connection ∇ on E gives rise to an element $[\omega] \in H^2_{dR}(M)$. This cohomology class is independent of the choice of connection ∇ compatible with $\langle , \rangle$, and hence is an invariant of the vector bundle E. It is called the* Euler class *of E and is denoted $e(E)$.*

Proof. We need to show that on overlaps $U \cap V$, $\omega_U = \omega_V$. If $g : U \cap V \to SO(2)$ is the orthogonal transformation taking from U to V, then we compute R with respect to the connection 1-form $g^{-1}dg + g^{-1}A_U g$. It is not hard to see that we still get

$$R = \begin{pmatrix} 0 & \omega_U \\ -\omega_U & 0 \end{pmatrix}.$$

Now, two different connections ∇ and ∇' have difference in $\Omega^1(End(E))$. (Moreover, they have values in 2×2 skew-symmetric matrices.) It is not hard to see that if we pick out the upper right hand corner of the matrix on each local coordinate chart U, then they coincide and yield a global 1-form α, and the difference between R_∇ and $R_{\nabla'}$ will be the exact form $d\alpha$. □

Example: For the Levi-Civita connection $\overline{\nabla}$ on a surface $(\Sigma, \overline{g}) \hookrightarrow (\mathbf{R}^3, g)$, we have, locally,

$$R_U = \begin{pmatrix} 0 & \kappa\theta_1 \wedge \theta_2 \\ -\kappa\theta_1 \wedge \theta_2 & 0 \end{pmatrix},$$

where κ is the scalar curvature, $\{e_1, e_2\}$ is an orthonormal frame, and $\{\theta_1, \theta_2\}$ is dual to the frame (called the *dual coframe*). (The fact that κ is the scalar curvature is the content of the Theorema Egregium!)

39.3. **The Gauß-Bonnet Theorem.** Let (M, g) be an oriented Riemannian manifold of dimension n. Then there exists a naturally defined volume form ω which has the following property: At $x \in M$, let $e_1, \dots, e_n$ be an oriented orthonormal basis for $T_x M$. Then $\omega(x)(e_1, \dots, e_n) = 1$. If we change the choice of orthonormal basis by multiplying by $A \in SO(n)$, then we have a change of $det(A)$, which is still 1. Therefore, ω is well-defined.

For surfaces (Σ, g), we have an area form dA.

Theorem 39.4 (Gauß-Bonnet)**.** *Let Σ be a compact submanifold of Euclidean space $(\mathbf{R}^3, g)$. Then, for one of the orientations of Σ,*

$$\int_\Sigma \kappa dA = 2\pi\chi(\Sigma).$$

Here κ is the scalar curvature, dA is the area form for $\overline{g}$ induced from $(\mathbf{R}^3, g)$, and $\chi(\Sigma)$ is the Euler characteristic of Σ.

The *Euler characteristic* of a compact manifold M of dimension n is:

$$\chi(M) = \sum_{i=0}^{n} (-1)^i \dim H^i_{dR}(M).$$

Note that a compact surface Σ (without boundary) of genus g has $\chi(\Sigma) = 2 - 2g$.

Proof. Notice that κdA is simply $\kappa\theta_1 \wedge \theta_2$ above in the Example, and hence the Euler class is $e(TM) = [\kappa dA]$. In order to evaluate $\int_\Sigma \kappa dA$, we therefore need to compute $\int_\Sigma \omega$ for the connection of our choice on $T\Sigma$ compatible with g, by using Theorem 39.3.

In what follows we will frequently identify $SO(2)$ with the unit circle $S^1 = \{e^{i\theta} | \theta \in [0, 2\pi]\}$ in $\mathbf{C}$ via

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \leftrightarrow e^{i\theta}.$$

We will do a sample computation in the case of the sphere S^2. Let S^2 be the union of two regions $U = \{|z| \leq 1\}$ and $V = \{|w| \leq 1\}$ identified via $z = 1/w$ along their boundaries. Here z, w are complex coordinates. (Note that U and V are not open sets, but it doesn't really matter....) If we trivialize $T\Sigma$ on U and V using the natural trivalization from $T\mathbf{C}$, then the gluing map $g : U \cap V \to SO(2)$ is given by $\theta \mapsto e^{2i\theta}$. If we set A_V to be identically zero, then $A_U = g^{-1}dg + g^{-1}A_V g = g^{-1}dg = \begin{pmatrix} 0 & 2d\theta \\ -2d\theta & 0 \end{pmatrix}$ along ∂U (after transforming via g). No matter how we extend A_U to the interior of U, we have the following by Stokes' Theorem:

$$\int_U \omega_U = \int_{\partial U} 2d\theta = 4\pi = 2\pi\chi(S^2).$$

Now let Σ be a compact surface of genus g (without boundary). Then we can remove g annuli $S^1 \times [0,1]$ from Σ so that Σ becomes a disk Σ' with $2g-1$ holes. We make A flat on the annuli, and see what this induces on Σ'. A computation similar to the one above gives the desired formula. (Check this!!) $\square$