NOTES FOR MATH 535B: DIFFERENTIAL GEOMETRY

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1. Lie Groups

1.1. **Basic definitions and examples.** In this course, manifolds are assumed to be smooth, unless indicated otherwise.

Definition 1.1. *A* Lie group *is a manifold equipped with smooth maps* $\mu : G \times G \to G$ *(multiplication) and* $i : G \to G$ *(inverse) which give it the structure of a group.*

Examples: Let $M_n(K)$ be the space of $n \times n$ matrices with entries in the base field $K = \mathbf{R}$ or $\mathbf{C}$.

1. $Gl(n, K) = \{A \in M_n(K) | \det A \neq 0\}$.
2. $Gl(V) = \{\text{linear isomorphisms } V \stackrel{\sim}{\to} V\}$, where V is a vector space.
3. $Sl(n, K) = \{A \in M_n(K) | \det A = 1\}$.
4. $O(n) = \{A \in M_n(\mathbf{R}) | AA^T = id\}$.
5. $U(n) = \{A \in M_n(\mathbf{C}) | AA^* = id\}$. Here the *adjoint* A^* is $(\overline{A})^T$ (the conjugate transpose of A).
6. $SO(n) = O(n) \cap Sl(n, \mathbf{R})$.
7. $SU(n) = U(n) \cap Sl(n, \mathbf{C})$.

Example: $U(n)$. If we write $A = (a_{ij})$ and write out $AA^* = id$, then $\sum_j a_{ij}\overline{a_{kj}} = \delta_{ik}$, and hence the row vectors form a *unitary basis* for $\mathbf{C}^n$.

Example: $SU(2)$. Let us write out $AA^* = id$. Here $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$AA^* = \begin{pmatrix} a\overline{a} + b\overline{b} & a\overline{c} + b\overline{d} \\ c\overline{a} + d\overline{b} & c\overline{c} + d\overline{d} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (1)$$

In addition, we have $ad - bc = 1$. Exercise: $SU(2)$ is diffeomorphic to S^3.

Definition 1.2. *A* Lie subgroup *of* G *is a subgroup* H *which is at the same time a* submanifold *such that* H *is a Lie group with respect to this structure.*

Remark: When dealing with Lie groups, the definition of *submanifold* is the image of an injective immersion. (No properness required.)

Definition 1.3. *A* Lie group homomorphism *is a group homomorphism $\phi : G \to H$ which is also a smooth map of the underlying manifolds.*

1.2. **Representations.**

Definition 1.4. *Let V be a vector space over $K = \mathbf{R}$ or $\mathbf{C}$, and let G be a Lie group. Then a* Lie group representation *$\rho : G \to Gl(V)$ is a Lie group homomorphism, i.e., $\rho(gh) = \rho(g)\rho(h)$.*

Remark: We will assume that ρ is a finite-dimensional representation, i.e., V is a finite-dimensional vector space.

Definition 1.5. *ρ is* faithful *if ρ is 1-1. ρ is* almost faithful *if ρ has 0-dimensional kernel.*

Natural operations: Let $\rho_1 : G \to Gl(V)$ and $\rho_2 : G \to Gl(W)$ be two Lie group representations.

1. $\rho_1 \oplus \rho_2 : G \to Gl(V \oplus W)$, $g \mapsto [(v, w) \mapsto (\rho_1(g)v, \rho_2(g)w)]$.
2. $\rho_1 \otimes \rho_2 : G \to Gl(V \otimes W)$, $g \mapsto [v \otimes w \mapsto (\rho_1(g)v) \otimes (\rho_2(g)w)]$.
3. $\rho_1^* : G \to Gl(V^*)$, $g \mapsto \rho_1^*$, where $\langle(\rho_1^* g)\phi, v\rangle = \langle\phi, \rho_1(g^{-1})v\rangle$, $v \in V$, $\phi \in V^*$, and $\langle,\rangle$ is the dual pairing between V and V^*. Note the inverse is necessary to make it into a homomorphism. (This is the same as $Gl(n, K) \to Gl(n, K)$, $A \mapsto (A^{-1})^T$.)

Zen: We can pretend that every Lie group is a matrix group.

Theorem 1.6 (Ado-Iwasawa)**.** *Every connected Lie group has an almost faithful representation $\rho : G \to Gl(V)$ for some finite-dimensional vector space V.*

Proof. Omitted. We remark that the almost faithful condition is necessary due to π_1 considerations. □

2. Lie groups, day 2

2.1. **Left-invariant vector fields and 1-forms.** A Lie group G has a left action and a right action onto itself: Let $g \in G$. Then

$$L_g : G \to G, g' \mapsto gg'.$$
$$R_g : G \to G, g' \mapsto g'g.$$

There is also conjugation:

$$R_{g^{-1}} L_g : G \to G, g' \mapsto gg'g^{-1}.$$

Definition 2.1. *A vector field X (defined globally) on G is* left-invariant *if $(L_g)_* X = X$ for all $g \in G$. A 1-form ω on G is* left-invariant *if $L_g^* \omega = \omega$ for all $g \in G$.*

We denote the vector space of left-invariant vector fields by $\mathfrak{X}_G$ and the vector space of left-invariant 1-forms by Ω^1_G.

Proposition 2.2. *$\mathfrak{X}_G \simeq T_e G$ as vector spaces. Hence $\dim \mathfrak{X}_G = \dim G$.*

Proof. Let $e \in G$ be the identity. We propagate $v \in T_e G$ using L_g, $g \in G$. Recall that a tangent vector $v \in T_e G$ corresponds to an equivalence class of smooth arcs $\gamma(t)$, $t \in (-\varepsilon, \varepsilon)$, $\gamma(0) = e$. Then $(L_g)_* v$ corresponds to $g\gamma(t)$. We therefore define the vector field:

$$X_v(g) = g\gamma(t).$$

Then clearly $((L_g)_* X_v)(g') = g(g^{-1} g' \gamma(t)) = g'\gamma(t)$. Hence,

$$\dim \mathfrak{X}_G = \dim T_e G = \dim G.$$

□

Example: $O(n)$. Then $T_I O(n)$ is the set of skew-symmetric matrices. We write $\gamma(t) \in T_I O(n)$ as: $\gamma(t) = I + At$, where we do all the computations modulo t^2. Then:

$$\begin{aligned} I &= \gamma\gamma^T = (I + At)(I + A^T t) \\ &= I + (A + A^T)t. \end{aligned}$$

Hence $A = -A^T$. Since $\dim O(n) = \frac{n(n-1)}{2}$ and dim of the set of skew-symmetric matrices $= \frac{n(n-1)}{2}$, $T_I O(n)$ is indeed the set of skew-symmetric matrices. $\mathfrak{X}_{O(n)} = \{X_A | A \in$ skew-symmetric matrices$\}$, where $X_A(B) = BA$, $B \in O(n)$.

Example: $Sl(n, \mathbf{R})$. Then $T_I Sl(n, \mathbf{R}) = \{$traceless matrices$\}$.

Similarly, we have:

Proposition 2.3. *$\Omega^1_G \simeq T_e^* G$.*

Proof. Similar to the vector field case. Given $\omega_e \in T_e^* G$, propagate by setting $\omega(g) = L_{g^{-1}}^* \omega_e$.
□

Example: $G = \mathbf{R}$, the additive group of real numbers. Let x be the standard coordinate for $\mathbf{R}$, and $a \in \mathbf{R}$. If L_a is (left-)translation by a, then $\omega = dx$ is translation-invariant.

$$\Omega^1_G = \mathbf{R}\{dx\}.$$

Example: $G = \mathbf{R}^\times = \mathbf{R} - \{0\}$ viewed as a multiplicative group. We compute the left-invariant 1-forms. If $\omega = f(x)dx$, then

$$L_a^*\omega = f(ax)d(ax) = f(ax)adx = f(x)dx$$

implies $f(ax) = f(x)/a$. For example, setting $x = 1$, we have $f(a) = c/a$. Therefore,

$$\Omega^1_G = \mathbf{R}\{dx/x\}.$$

Exercise: Compute the left-invariant 1-forms of $Gl(n, \mathbf{R})$ in terms of dx_{ij}, where

$$x_{ij}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{ij}.$$

2.2. **Lie algebras.**

Definition 2.4. *A* Lie algebra $\mathfrak{g}$ *over* $K = \mathbf{R}$ *or* $\mathbf{C}$ *is a* K*-vector space together with a* Lie bracket $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *satisfying the following:*

1. $[,]$ *is bilinear,*
2. *(skew-symmetric)* $[X, Y] + [Y, X] = 0$,
3. *(Jacobi identity)* $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$.

Example: Let M be a manifold and $\mathfrak{X}(M)$ be the C^∞ vector fields on M. The Lie bracket $[X, Y]$ makes $\mathfrak{X}(M)$ into an infinite-dimensional Lie algebra.

3. Lie algebras

3.1. **Lie algebra $\mathfrak{g}$ of a Lie group G.** We now define the Lie algebra $\mathfrak{g}$ associated to a Lie group G. As a vector space, $\mathfrak{g} \simeq T_eG \simeq \mathfrak{X}_G$.

The Lie bracket on $\mathfrak{X}_G$ is inherited from that of $\mathfrak{X}(G)$ (Lie bracket of vector fields). We need to verify the following:

Lemma 3.1. $[,] : \mathfrak{X}_G \times \mathfrak{X}_G \to \mathfrak{X}_G$, *i.e., if* $X, Y \in \mathfrak{X}_G$, *then* $[X,Y] \in \mathfrak{X}_G$.

Proof. We use the fact that $\phi_*[X,Y] = [\phi_*X, \phi_*Y]$, where $\phi : M \to M$ is a diffeomorphism and $X, Y \in \mathfrak{X}(M)$. (Check this!)

Then, $(L_g)_*[X,Y] = [(L_g)_*X, (L_g)_*Y] = [X,Y]$. □

Remark: We will often write $\mathfrak{g} = Lie(G)$.

For matrix groups, i.e., $G \subset Gl(V)$, we have $\mathfrak{X}_G = \{X_A | A \in T_eG\}$, where $X_A(g) = gA$. Therefore,

$$[X_A, X_B](I) = \lim_{s,t \to 0} \frac{(I+sA)(I+tB) - (I+tB)(I+sA)}{st} = AB - BA.$$

Examples: In the following, the Lie bracket is always $[A,B] = AB - BA$.

Lie group	Lie algebra
$Gl(V)$	$\mathfrak{gl}(V) = End(V)$
$O(n)$	$\mathfrak{o}(n)$= skew-symmetric matrices
$U(n)$	$\mathfrak{u}(n)$= skew-hermitian matrices
$Sl(n, \mathbf{R})$	$\mathfrak{sl}(n, \mathbf{R})$= traceless matrices

Definition 3.2. *A* Lie subalgebra $\mathfrak{h}$ *of a Lie algebra* $\mathfrak{g}$ *is a vector subspace which is closed under* $[,]$. *A* Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ *is a bracket-preserving linear map, i.e.,* $\phi([X,Y]) = [\phi(X), \phi(Y)]$. *A* Lie algebra representation *is a Lie algebra homomorphism* $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$.

Exercise: Given a Lie group homomorphism $\phi : G \to H$, there exists a corresponding Lie algebra homomorphism $\phi_* : Lie(G) \to Lie(H)$.

Proposition 3.3. *Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. Then there exists a 1-1 correspondence between connected Lie subgroups of G and Lie subalgebras of $\mathfrak{g}$.*

Proof. Next time. □

3.2. **Adjoint representation.** We define a Lie group representation $Ad : G \to Gl(\mathfrak{g})$, where $\mathfrak{g} = Lie(G)$, as follows: Think of $\mathfrak{g} \simeq T_eG$. Then, for $a \in G$, $Ad(a) = (R_{a^{-1}} \circ L_a)_* : T_eG \to T_eG$.

We must show that $Ad(a)$ is indeed in $Gl(\mathfrak{g})$. This is immediate, since $Ad(a^{-1})$ is the inverse of $Ad(a)$.

Remark: Here we are viewing $\mathfrak{g}$ simply as a vector space.

Example: $G = Gl(n, \mathbf{R})$. $Ad(A) : T_I G \to T_I G$ is given by

$$I + tX \mapsto A(I + tX)A^{-1} = I + tAXA^{-1},$$

by viewing $X \in T_e G$ as an arc in G through I, or $X \mapsto AXA^{-1}$.

We can differentiate any Lie group homomorphism at the identity to get a Lie algebra homomorphism. Therefore, there is also an infinitesimal version of $Ad : G \to Gl(\mathfrak{g})$. On the Lie algebra level, we have:

$$ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}).$$

This is given by $Ad_*(e) : T_e G \to T_e G$.

Example: Let G be a matrix group. Then

$$Ad : G \to Gl(\mathfrak{g}),$$

$$A \mapsto [A \mapsto AXA^{-1}].$$

If we write $A = I + tY$, then:

$$I + tY \mapsto (I + tY)X(I + tY)^{-1} = (I + tY)X(I - tY) = X + t[Y, X].$$

Taking derivatives, we get $Y \mapsto [Y, X]$. Therefore,

$$ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}),$$

$$Y \mapsto [X \mapsto [Y, X]].$$

4. Day 4: Lie subalgebras and Lie subgroups

Recall the Frobenius integrability theorem:

Theorem 4.1. *Let M be an n-dimensional manifold and $\mathcal{D} \subset TM$ be a rank k distribution, i.e., a rank k subbundle of the tangent bundle. Then $\mathcal{D}$ is integrable (locally there exist coordinates $x_1, \dots, x_n$ such that $\mathcal{D} = Span\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}\}$) if and only if for all $X, Y \in \Gamma(\mathcal{D})$ we have $[X, Y] \in \Gamma(\mathcal{D})$. Here $\Gamma(\mathcal{D})$ is the set of smooth sections of $\mathcal{D}$.*

Remark: If $\mathcal{D}$ is *trivial*, i.e., is spanned by k nowhere vanishing sections $X_1, \dots, X_k$, then it is sufficient to show that $[X_i, X_j] \in \Gamma(\mathcal{D})$ for all $1 \leq i, j \leq k$.

We'll prove the following proposition from last time:

Proposition 4.2. *Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. Then there exists a 1-1 correspondence between connected Lie subgroups of G and Lie subalgebras of $\mathfrak{g}$.*

Proof. Let $\mathfrak{h}$ be a subalgebra of the Lie algebra $\mathfrak{g}$. We find the corresponding connected Lie subgroup H in steps.

Step 1: View $\mathfrak{h}$ as a subspace of $\mathfrak{g} = T_eG$. Consider the distribution $\mathcal{D}$ on G given by $\mathcal{D}_g = (L_g)_*\mathfrak{h}$, where $g \in G$. We claim that $\mathcal{D}$ is integrable. In fact, if we now view $\mathfrak{h}$ as a subspace of left-invariant vector fields on G, then $\mathfrak{h}$ being a Lie subalgebra of $\mathfrak{g}$ implies that $[X, Y] \in \mathfrak{h}$ for all $X, Y \in \mathfrak{h}$. Since left-invariant vector fields of $\mathfrak{h}$ span $\mathcal{D}$ and are closed under bracketing, $\mathcal{D}$ is integrable. If $\dim \mathfrak{h} = k$, then let H be the maximal connected integral submanifold of $\mathcal{D}$ (naturally of dimension k) which passes through e.

Step 2: We now need to show H is closed under group operations. If $a \in H$, then consider $a^{-1}H$. Since $\mathcal{D}$ is left-invariant by definition, $a^{-1}H$ is also a connected integral submanifold of $\mathcal{D}$. Now, $a^{-1}H$ contains e, since $a \in H$; by the maximality of H, we have $a^{-1}H \subset H$. This proves that H is closed under multiplication and inversion, hence is a subgroup of G. We will omit the verification that the multiplication $H \times H \to H$ is smooth. (See, for example, Warner.)

Step 3: (Uniqueness) We will now show that there is a unique connected Lie subgroup $H \subset G$ which has Lie algebra $\mathfrak{h} \subset \mathfrak{g}$. Let H' be another connected Lie subgroup with Lie algebra $\mathfrak{h}$, i.e., $T_eH' = \mathfrak{h}$. Therefore, if $h' \in H'$, then $(L_{h'})_*\mathfrak{h} = T_{h'}H'$. It follows that H' is also a connected integral submanifold of $\mathcal{D}$ through e of the same dimension. By maximality of H, clearly $H' \subset H$. Now, it is clear H and H' overlap on a neighborhood of e, i.e., there exists an open set $U \subset H \cap H'$.

Lemma 4.3. *Let G° be a connected Lie group and U any open neighborhood of $e \in U$. Then any $a \in G^\circ$ can be written as a product of elements of U.*

Proof of Lemma. A connected manifold is path-connected. There exists a path $\gamma : [0, 1] \to G^\circ$ with $\gamma(0) = e$ and $\gamma(1) = a$. The open sets $L_{\gamma(t)}(U)$, $t \in [0, 1]$, cover $\gamma(t)$, and by compactness $\exists$ a finite set $t_0 = 0 < t_1 < \cdots < t_n = 1$ for which $\gamma([t_i, t_{i+1}]) \subset L_{\gamma(t_i)}(U)$. Since $\gamma(t_{i+1}) = \gamma(t_i)a_i$, $a_i \in U$, this allows us to realize a as a product of elements in U. □

This proves that any element $h \in H$ can be written as a product of elements of $H' \supset U$, hence $H \subset H'$. Thus $H = H'$. □

5. Day 5

5.1. **Example.** The 2-torus $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ is a Lie group under addition $(x_1, y_1) + (x_2, y_2) \equiv (x_1 + x_2, y_1 + y_2) \bmod \mathbf{Z}^2$. T^2 is an *abelian* Lie group. Its Lie algebra $\mathfrak{t}^2 = \mathbf{R}^2$. Since the left-invariant vector fields on T^2 are constant vectors (a, b), the Lie bracket is trivial, i.e., $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$. (Such a Lie algebra is called *abelian.*)

Now the subalgebras of $\mathfrak{g}$ (besides 0) are the 1-dimensional Lie subalgebras $\mathfrak{h} = \mathbf{R}\{(a, b)\}$. They integrate to give $H = \mathbf{R}\{(a, b)\}/\mathbf{Z}^2$.

More generally, the n-dimensional torus $T^n = \mathbf{R}^n/\mathbf{Z}^n$ is an abelian Lie group with abelian Lie algebra $\mathfrak{t}^n = \mathbf{R}^n$ (with the trivial bracket). The Lie subalgebras are all vector subspaces of $\mathfrak{t}^n = \mathbf{R}^n$.

5.2. **Exponential mapping.** Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Define the *exponential map*

$$exp : \mathfrak{g} \to G,$$

where $v \in \mathfrak{g}$ maps to the time 1 flow of the left-invariant vector field X_v (corresponding to v) starting at e.

Let us reinterpret it in light of the correspondence between Lie subalgebras and Lie subgroups. Take $v \in \mathfrak{g}$ nonzero. Then $\mathfrak{h} = \mathbf{R}\{v\}$ is a 1-dimensional Lie subalgebra of $\mathfrak{g}$, since the bracket is trivially zero and the integrability condition is trivially met. To $\mathfrak{h}$ we can associate its Lie subgroup H, which is obtained by taking the connected 1-dimensional integral submanifold of X_v through e. Then:

$$exp : \mathfrak{h} = \mathbf{R} \to H \subset G.$$

This is a Lie group homomorphism. (There is no guarantee this map is injective: for example, see the example above.)

For *matrix groups*, we investigate the time 1 flow $\gamma : [0, 1] \to G$, $\gamma(0) = e$, where the left-invariant vector field is generated by $A \in \mathfrak{gl}(n, \mathbf{R})$. The left-invariant vector field is $X_A(g) = gA$. Therefore, we solve:

$$\gamma'(t) = \gamma(t)A,$$

and the solution is $\gamma(t) = e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{3!} + \dots$. Hence $\gamma(1) = e^A$, and the term "exponential map" is apt for matrix groups!

Remark: Here left-invariant vector fields were used in the definition of the exponential map, but the expression $exp(A) = e^A$ does not appear to depend on left vs. right .

5.3. **From Lie algebras to Lie groups.**

Theorem 5.1 (Ado)**.** *Every finite-dimensional Lie algebra $\mathfrak{g}$ is isomorphic to a subalgebra of $\mathfrak{gl}(n, \mathbf{R})$.*

Proof is beyond the scope of the course. However, this theorem implies the following:

Theorem 5.2. *For each finite-dimensional Lie algebra* $\mathfrak{g}$ *there exists a Lie group* G *whose Lie algebra is isomorphic to* $\mathfrak{g}$.

Proof. We view $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbf{R})$. $\mathfrak{gl}(n, \mathbf{R})$ is the Lie algebra of $Gl(n, \mathbf{R})$, so $\mathfrak{g}$ corresponds to some connected Lie subgroup of $Gl(n, \mathbf{R})$. □

6. Day 6: Maurer-Cartan, etc.

6.1. **The Maurer-Cartan form.** We first look at the following motivating example:

Motivating example: Let $G = Gl(n, \mathbf{R})$. We write down all the left-invariant 1-forms on G. Let $x_{ij} : Gl(n, \mathbf{R}) \to \mathbf{R}$ which gives the ij-th entry of the matrix in $Gl(n, \mathbf{R})$. Also let $x = (x_{ij})$ be the function $Gl(n, \mathbf{R}) \to M_n(\mathbf{R})$. Taking hint from $\mathbf{R}^\times = Gl(1, \mathbf{R})$ with $\Omega^1_G = \mathbf{R}\{\frac{dx}{x}\}$, we guess $\omega = x^{-1}dx$.

Check: $L_a^*x = ax$, $L_a^*(x^{-1}) = x^{-1}a^{-1}$, and $L_a^*(dx) = d(ax) = adx$, so $L_a^*\omega = x^{-1}a^{-1}adx = x^{-1}dx$.

Let's evaluate $\omega(a)(v)$, where $a \in G$ and $v \in T_aG$. Then $v = aA$, where $A \in T_eG$, i.e., $A \in \mathfrak{g}$. Hence,

$$\omega(a)(v) = x^{-1}(a)dx(a)(v) = a^{-1}dx(aA) = a^{-1}aA = A.$$

ω is a 1-form *with values in* $\mathfrak{g}$, i.e., at every point $p \in G$, it gives a map $T_pG \to \mathfrak{g}$. (Usual forms give maps $T_pG \to \mathbf{R}$.)

Given a Lie group G, there is a canonical left-invariant 1-form ω on G with values in $\mathfrak{g}$, called the *Maurer-Cartan form*, which satisfies:

$$\omega(a)(v) = (L_{a^{-1}})_*v.$$

Here $a \in G$ and $v \in T_aG$.

Remark: We can think of the Maurer-Cartan form as all the left-invariant 1-forms put into one package!

Proposition 6.1. *The Maurer-Cartan form* ω *satisfies* $d\omega + [\omega, \omega] = 0$.

By $[\omega, \omega]$ we mean $[\omega, \omega](X, Y) = [\omega(X), \omega(Y)]$.

Proof. Let $X_v, X_w \in \mathfrak{X}_G$. Then:

$$\begin{aligned} d\omega(X_v, X_w) &= X_v\omega(X_w) - X_w\omega(X_v) - \omega([X_v, X_w]) \\ &= -\omega([X_v, X_w]) \\ &= -\omega(X_{[v,w]}) \\ &= -[v, w] \\ &= -[\omega(X_v), \omega(X_w)] \end{aligned}$$

□

HW: Check that $d\omega + [\omega, \omega] = 0$ for $Gl(n, \mathbf{R})$.

6.2. **Structure equations.** Let $X_1, \dots, X_n$ be a basis for $\mathfrak{g}$ and $\theta^1, \dots, \theta^n$ be the dual basis for $\mathfrak{g}^*$, i.e., $\theta^i(X_j) = \delta_{ij}$. With respect to this basis, we can write $\omega(X) = (\theta^1(X), \dots, \theta^n(X))$.

If $[X_i, X_j] = \sum_k c_{ij}^k X_k$ (c_{ij}^k constants), then:

$$d\theta^i(X_j, X_k) = -\theta^i([X_j, X_k]) = -\theta^i(c_{jk}^l X_l) = -c_{jk}^i.$$

Therefore:

Proposition 6.2. $d\theta^i = -\frac{1}{2}\sum_{j,k} c_{jk}^i \theta^j \wedge \theta^k$.

These are called the *structure equations* of the Lie algebra.

Remark: Define the left-invariant k-forms by Ω_G^k. Then the previous proposition implies that if $\omega \in \Omega_G^1$, then $d\omega \in \Omega_G^2$.

7. Day 7

7.1. **Lie algebra cohomology.** Let G be a Lie group and $\mathfrak{g} = Lie(G)$. Also let Ω^k_G be the left-invariant k-forms on G.

Lemma 7.1. $\Omega^k_G = \wedge^k \mathfrak{g}^*$.

Proof. Let $\omega \in \Omega^k_G$. Consider $\omega(e)$, the left-invariant k-form evaluated at $e \in G$. Let ω' be the left-invariant k-form obtained by taking $\omega'(a) = (L_{a^{-1}})^*\omega(e)$. We know that $\omega - \omega'$ is zero at e and is left-invariant; hence $\omega' = \omega$. However, by construction, $\omega' \in \wedge^k \mathfrak{g}^*$. □

Consider the algebra $\Omega^*_G = \oplus_k \Omega^k_G$ of left-invariant forms on G. Then we have a chain complex:

$$0 \to \Omega^0_G \xrightarrow{d} \Omega^1_G \xrightarrow{d} \Omega^2_G \xrightarrow{d} \Omega^3_G \to \cdots \to 0.$$

or

$$0 \to \mathbf{R} \to \mathfrak{g}^* \to \wedge^2\mathfrak{g}^* \to \wedge^3\mathfrak{g}^* \to \cdots \to 0.$$

Verify the following:

1. d maps from $\wedge^i\mathfrak{g}^*$ to $\wedge^{i+1}\mathfrak{g}^*$.
2. $d^2 = 0$ is equivalent to the Jacobi identity.

$H^*(\mathfrak{g})$ is called the *Lie algebra cohomology.*

Examples:

1. $H^0(\mathfrak{g}) = \mathbf{R}$.
2. $H^1(\mathfrak{g}) = (\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])^*$. Here $[\mathfrak{g},\mathfrak{g}] = \{[X,Y] | X, Y \in \mathfrak{g}\}$.

Question: What do the higher cohomology groups mean?

Example: $\mathfrak{so}(3)$, with basis e_1, e_2, e_3. The Lie brackets are:

$$[e_1, e_2] = e_3, [e_2, e_3] = e_1, [e_3, e_1] = e_2.$$

We compute its Lie algebra cohomology.

$$0 \to \mathbf{R} \to \mathbf{R}^3 \to \mathbf{R}^3 \to \mathbf{R} \to 0.$$

Since $d\theta^i = -\frac{1}{2}c^i_{jk}\theta^j \wedge \theta^k$, we have

$$d\theta^1 = -\theta^2 \wedge \theta^3,$$
$$d\theta^2 = -\theta^3 \wedge \theta^1,$$
$$d\theta^3 = -\theta^1 \wedge \theta^2.$$

It is not difficult to see that:

$$H^0(\mathfrak{g}) = \mathbf{R},$$

$$H^1(\mathfrak{g}) = 0,$$
$$H^2(\mathfrak{g}) = 0,$$
$$H^3(\mathfrak{g}) = \mathbf{R}.$$

7.2. **Representation theory basics.** For simplicity, in this course we only consider representations of *complex Lie algebras* on *complex vector spaces*. That is to say, $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a complex linear map ($\mathfrak{g}$ and V are complex vector spaces). For example, instead of $\mathfrak{sl}(n, \mathbf{R})$, we take the complexification $\mathfrak{sl}(n, \mathbf{C}) = \mathfrak{sl}(n, \mathbf{R}) \otimes_{\mathbf{R}} \mathbf{C}$.

Definition 7.2. *Let V be a representation of $\mathfrak{g}$. A* subrepresentation *$W \subset V$ is a (complex) vector subspace W of V which is invariant under $\mathfrak{g}$. A representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is* irreducible *if V has no nontrivial subrepresentations.*

$W \subset V$ being a subrepresentation of $\mathfrak{g}$ means the following: we can take a basis $\{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ for V, where $\{v_1, \dots, v_k\}$ is a basis for W. Then with respect to this basis, $\rho(X)$, $X \in \mathfrak{g}$, acts via multiplication by

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

on the left. Here the upper left $*$ is a $k \times k$ matrix and the lower right $*$ is an $(n-k) \times (n-k)$ matrix.

Complete reducibility: Let V be a representation of a semisimple Lie algebra $\mathfrak{g}$ and $W \subset V$ a subrepresentation. Then there exists a subrepresentation W' of $\mathfrak{g}$ such that $V = W \oplus W'$.

Complete reducibility means being able to write $\rho(X)$ as:

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}.$$

Therefore, for semisimple Lie algebras it suffices to study irreducible representations. We will not define "semisimple" because it's not relevant in what we do. Instead, we will treat a particular example in detail: $\mathfrak{sl}(2, \mathbf{C})$.

8. Extended example: representations of $\mathfrak{sl}(2, \mathbf{C})$

Today's goal is to work out the irreducible representations of $\mathfrak{g} = \mathfrak{sl}(2, \mathbf{C})$. Take a basis:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then we have the equations:

$$[H, E] = 2E, [H, F] = -2F, [E, F] = H. \tag{2}$$

8.1. The adjoint representation.

We first study the adjoint representation $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. $ad : X \mapsto ad(X)$, where $ad(X) : Y \mapsto [X, Y]$.

HW: Verify that ad is a Lie algebra representation, i.e., $ad([X, Y]) = [ad(X), ad(Y)]$. Hint: this follows from the Jacobi identity.

In the expression $\mathfrak{gl}(\mathfrak{g})$, it's best to view $V = \mathfrak{g}$ as $V_{-2} \oplus V_0 \oplus V_2$, where $V_{-2} = \mathbf{C}F$, $V_0 = \mathbf{C}H$, and $V_2 = \mathbf{C}E$. The structure equations imply that all the V_i are eigenspaces of $ad(H)$, since $ad(H)(E) = [H, E] = 2E$, $ad(H)(H) = [H, H] = 0$, and $ad(H)(F) = [H, F] = -2F$.

Also note that $ad(E)$ isomorphically maps $V_{-2} \xrightarrow{\sim} V_0$, $V_0 \xrightarrow{\sim} V_2$. Similarly, $ad(F)$ isomorphically maps $V_2 \xrightarrow{\sim} V_0$, $V_0 \xrightarrow{\sim} V_{-2}$.

Lemma 8.1. *The adjoint representation is irreducible.*

Proof. Let $v \in V$. Then we can write $v = aF + bH + cE$. If $a \neq 0$, then $ad(E)(v) = aH - 2bE$ and $(ad(E))^2(v) = -2aE$. These three vectors clearly span all of V. If $a = 0$, then we need to use $ad(F)$'s as well, but the proof is similar. □

8.2. General case.

Let $\rho : \mathfrak{sl}(2, \mathbf{C}) \to \mathfrak{gl}(V)$ be a (finite-dimensional) irreducible representation. We will extensively use Equation 2. If $v \in V$ and $X \in \mathfrak{g}$, then we will write Xv to mean $\rho(X)(v)$.

Let $v \in V$ be an eigenvector of H with eigenvalue λ. (Every endomorphism of V has at least one eigenvector.)

Lemma 8.2. *If $Hv = \lambda v$, then $H(Ev) = (\lambda + 2)(Ev)$ and $H(Fv) = (\lambda - 2)(Fv)$, i.e., Ev and Fv are also eigenvectors of H with eigenvalues $\lambda + 2$ and $\lambda - 2$, respectively.*

Proof. By Equation 2,

$$H(Ev) = EHv + 2Ev = E(\lambda v) + 2Ev = (\lambda + 2)(Ev).$$

The expression for $H(Fv)$ is similar. □

Let v be the eigenvector of H with the largest eigenvalue. Such an eigenvector v is called the *highest weight vector*. Then $Ev = 0$, since Ev, if nonzero, would have a larger eigenvalue. Starting with $V_\lambda = \mathbf{C}v$, we take $V_{\lambda-2i} = \mathbf{C}F^i v$. ($F^i v$ has eigenvalue $\lambda - 2i$.) Note that $V_{\lambda-2k} = 0$ for some k. Let $W = \oplus_{i=0}^{k} V_{\lambda-2i}$.

Lemma 8.3. *W is a subrepresentation of V.*

Proof. It suffices to show that $E : W \to W$, since F and H clearly map W to itself. We have the following:

$$Ev = 0,$$
$$E(Fv) = FEv + Hv = \lambda v,$$
$$E(F^2 v) = FE(Fv) + H(Fv) = F(\lambda v) + (\lambda - 2)Fv = [(\lambda) + (\lambda - 2)]Fv.$$

In general,

$$E(F^i v) = \{(\lambda) + (\lambda - 2) + \cdots + (\lambda - 2(i-1))\} F^{i-1}v = (\lambda - i + 1)iF^{i-1}v. \tag{3}$$

□

Since V is irreducible, it follows that $V = W = \oplus_{i=0}^{k-1} V_{\lambda-2i}$. Finally, it remains to determine the possible λ:

Lemma 8.4. *If $\dim V = k$, then $\lambda = k - 1$.*

Proof. Using Equation 3, we have, for $i = k$:

$$0 = E(F^k v) = (\lambda - k + 1)kF^{k-1}v.$$

Since we know $F^{k-1}v \neq 0$ but $F^k v = 0$, we have $\lambda = k - 1$. □

Thus, we have the following theorem:

Theorem 8.5. *The irreducible representations of $\mathfrak{sl}(2, \mathbf{R})$ are parametrized by a positive integer $k \in \mathbf{Z}$. For each k, the representation $V \simeq \mathbf{C}^k$ decomposes into 1-dimensional eigenspaces V_λ of H, and $V = V_{1-k} \oplus V_{3-k} \oplus \cdots \oplus V_{k-3} \oplus V_{k-1}$.*

Exercise: Decompose the standard representation $\rho : \mathfrak{sl}(2, \mathbf{C}) \to \mathfrak{gl}(2, \mathbf{C})$ in terms of irreducible representations. Do the same for $\rho \otimes \rho$.

9. Fiber bundles

9.1. Locally trivial fiber bundles.

Definition 9.1. *A* locally trivial fiber bundle *is a quadruple* $(E, B, F, \pi : E \to B)$*, where:*

1. *the* total space E*, the* base B*, and the* fiber E *are manifolds,*
2. *there is an open cover* $\{U_\alpha\}$ *of* B *and diffeomorphisms* $\phi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times F$ *such that* ϕ_α *sends* $\pi^{-1}(p) \simeq \{p\} \times F$ *for* $p \in U_\alpha$*.*

The last condition is the same as the commutativity of the following diagram:

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\phi_\alpha} & U_\alpha \times F \\ \pi \big\downarrow & & \pi_1 \big\downarrow \\ U_\alpha & \xrightarrow{id} & U_\alpha \end{array}$$

where $\pi_1 : U_\alpha \times F \to U_\alpha$ is the first projection.

Examples:

1. (Product or trivial bundle) Consider $E = B \times F \xrightarrow{\pi} B$, where $\pi(b, f) = b$.

Definition 9.2. *A locally trivial fiber bundle* $E \xrightarrow{\pi} B$ *is* trivial *if there is a commutative diagram*

$$\begin{array}{ccc} E & \xrightarrow{\sim} & B \times F \\ \pi \big\downarrow & & \pi_1 \big\downarrow \\ B & \xrightarrow{id} & B \end{array}$$

2. (Möbius band) Take $E = ([0,1] \times [0,1])/\sim$, where s, t are coordinates of $[0,1] \times [0,1]$ and $(0, t) \sim (1, 1-t)$. We take the projection $\pi : E \to [0,1]/(0 \sim 1)$, which maps $(s,t) \mapsto s$.

3. (Vector bundle) When F, $\pi^{-1}(x)$, $\forall x \in B = M$, are vector spaces and each identification $\phi_\alpha : \pi^{-1}(x) \simeq \{x\} \times F$ is a vector space isomorphism. Examples of vector bundles are:

$$TM \to M, T^*M \to M, \wedge^k T^*M \to M.$$

Their fibers are $F \simeq T_x M$, $T_x^* M$, $\wedge^k T_x^* M$, respectively.

4. (Frame bundle) Let $B = M$, and $Fr_x(M)$, $x \in M$, be the set of ordered bases $(v_1, \dots, v_k)$ of $T_x M$. (Here $\dim M = k$.) Define the *frame bundle* $Fr(M)$ to be $\sqcup_{x \in M} Fr_x(M)$, and let $\pi : Fr(M) \to M$ be the natural projection.

We topologize $Fr(M)$ as follows: Locally on $U \subset M$, we take smooth sections $v_1, \dots, v_k$ over U which form a basis pointwise. If $(w_1(x), \dots, w_k(x))$ is another ordered basis for $T_x M$, then we can write $(w_1(x), \dots, w_k(x)) = (v_1(x), \dots, v_k(x))A$, where $A \in Gl(k, \mathbf{R})$. This allows us to identify:

$$\pi^{-1}(U) \xrightarrow{\sim} U \times Gl(k, \mathbf{R}),$$

by sending

$$(x, w_1(x), \dots, w_k(x)) \mapsto (x, A).$$

5. (Orthonormal frame bundle) Fix a Riemannian metric g on M. This is a positive definite, symmetric bilinear form $g : T_xM \times T_xM \to \mathbf{R}$ which varies smoothly with respect to x. We can define $OFr_x(M)$ to be the set of orthonormal bases $(v_1, \dots, v_k)$ of T_xM with respect to this g. (Orthonormal bases exist by the Gram-Schmidt.)

Locally, we can identify:

$$\pi^{-1}(U) \simeq U \times O(n).$$

This is because $(w_1, \dots, w_k) = (v_1, \dots, v_k)A$, where $A \in O(n)$. (Check this!)

6. (Pullbacks)

Definition 9.3. *Let $E \xrightarrow{\pi} B$ be a locally trivial fiber bundle and $B' \xrightarrow{f} B$ be a smooth map. Then the* pullback bundle *is given by: $f^{-1}(E) = \{(x, e) | f(x) = \pi(e)\} \subset B' \times E$. By definition, we have a commutative diagram*

$$\begin{array}{ccc} f^{-1}(E) & \longrightarrow & E \\ {\scriptstyle \pi'}\downarrow & & {\scriptstyle \pi}\downarrow \\ B' & \xrightarrow{f} & B \end{array}$$

It is easy to verify that $f^{-1}(E)$ is a locally trivial fiber bundle. (HW)

9.2. **Action of a Lie group on a manifold.** A Lie group G *acts on* M from the right if there exists a smooth map $M \times G \to M$ (written $(x, g) \mapsto xg$), such that $xe = x$ and $x(gh) = (xg)h$. We also write $R_g : M \to M$, $x \mapsto xg$.

Remark: R_g is a diffeomorphism, since there exists an inverse $R_{g^{-1}}$.

Remark: You can also similarly define the action of a Lie group acting on M from the left.

Example: Let $G = O(n)$, $M = \mathbf{R}^n$. Then $\mathbf{R}^n \times O(n) \to \mathbf{R}^n$ is given by $(x = (x_1, \dots, x_n), A) \mapsto xA$.

10. Day 10

10.1. **More on Lie group actions.** Recall Lie group G *acts on* M from the right if there exists a smooth map $M\times G\to M$ (written $(x,g)\mapsto xg$), such that $xe=x$ and $x(gh)=(xg)h$.

A Lie group action is *free* if $xg=x$ for some x implies that $g=e$. (This also says that $xg_1\neq xg_2$ unless $g_1=g_2$.)

Given a Lie group action $M\times G\to M$, we can take its quotient $M/G=M/\sim$, where $x\sim xg$ for all $x\in M$ and $g\in G$. M/G is naturally a topological space with the quotient topology.

Remark: M/G is usually not a manifold.

Example: $G=Gl(n,\mathbf{R})$, $M=\mathbf{R}^n$. $M\times G\to M$ given by $(x,A)\mapsto xA$. Then M/G has two elements a,b corresponding to the two orbits or $Gl(n,\mathbf{R})$: $\mathbf{R}^n-\{0\}$ and $\{0\}$. The topology of M/G is $\{\emptyset,\{a\},\{a,b\}\}$ and is not Hausdorff.

HW: Describe G/M if $G=O(n)$ and $M=\mathbf{R}^n$ as above.

Definition 10.1. *Let M and N be two manifolds with a G-action. Then the map $\phi:M\to M$ is* G-equivariant *if $\phi(xg)=\phi(x)g$ for all $x\in M$.*

10.2. **Principal G-bundles.**

Definition 10.2. *Let G be a Lie group. A (right)* principal G-bundle *is a locally trivial fiber bundle $\pi:P\to M$, where:*

1. *G acts on P* freely *on the right ($\rho:P\times G\to P$).*
2. *$M=P/G$ and $\pi:P\to P/G$ is the natural projection.*

Note: The fiber is diffeomorphic to G.

Remark: In this business, it pays to pay attention to right vs. left.

1. The Maurer-Cartan form was left-invariant.
2. Principal G-bundles are acted from the right.

Examples: 1. (Hopf map) Consider $S^{2n+1}=\{|z_0|^2+\cdots+|z_n|^2=1\}\subset\mathbf{C}^{n+1}$ with coordinates $z_0,\ldots,z_n$. Then S^1 acts on $\mathbf{C}^{n+1}$:

$$\mathbf{C}^{n+1}\times S^1\to\mathbf{C}^{n+1},$$

$$((z_0,\ldots,z_n),e^{i\theta})\mapsto(z_0e^{i\theta},\ldots,z_ne^{i\theta}).$$

Now restrict to S^{2n+1}, i.e., consider

$$S^{2n+1}\times S^1\to S^{2n+1}.$$

Then the action is free. It is easy to see that $S^{2n+1}/S^1 \simeq \mathbf{CP}^n$. Therefore, we have the principal S^1-bundle

$$\begin{array}{ccc} S^1 & \rightarrow & S^{2n+1} \\ & & \downarrow \\ & & \mathbf{CP}^n \end{array}$$

Definition 10.3. *A* product *principal* G*-bundle* $M \times G \stackrel{\pi_1}{\rightarrow} M$ *is given by the first projection and the following action of* G *on* M*:* $(x,g)g' \mapsto (x, gg')$*. A* G*-bundle* $P \stackrel{\pi}{\rightarrow} M$ *is a* trivial principal G-bundle *if there exists an equivariant diffeomorphism* $\phi : P \stackrel{\sim}{\rightarrow} M \times G$ *(i.e.,* $\phi(pg) = \phi(p)g$*) such that*

$$\begin{array}{ccc} P & \xrightarrow{\phi} & M \times G \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi_1}\downarrow \\ M & \xrightarrow{id} & M \end{array}$$

commutes.

Examples: 2. The frame bundle $Fr(M) \rightarrow M$ is a principal $Gl(n, \mathbf{R})$-bundle, if $n = \dim M$. The action $Fr(M) \times Gl(n, \mathbf{R}) \rightarrow Fr(M)$ is given by:

$$((x, (v_1, \dots, v_n)), A) \mapsto (x, (v_1, \dots, v_n)A).$$

Here $x \in M$ and $v_1, \dots, v_n$ is a basis for T_xM. (Verify that the action is free.)

3. Similarly, the orthonormal frame bundle $OFr(M) \rightarrow M$ is a principal $O(n)$-bundle. (The action is free since it's already free for $Gl(n, \mathbf{R})$.)

11. Day 11

11.1. More on principal bundles.

Definition 11.1. *A* bundle map *between two principal G-bundles $P \xrightarrow{\pi} M$ and $P' \xrightarrow{\pi'} M'$ consists of a G-equivariant map $F : P \to P'$ and a map $f : M \to M'$ such that the following diagram commutes:*

$$\begin{array}{ccc} P & \xrightarrow{F} & P' \\ {\scriptstyle \pi}\big\downarrow & & {\scriptstyle \pi'}\big\downarrow \\ M & \xrightarrow{f} & M' \end{array}$$

Definition 11.2. *Two principal G-bundles $P \xrightarrow{\pi} M$ and $P' \xrightarrow{\pi'} M$ are* isomorphic *if there is a G-equivariant diffeomorphism $F : P \to P'$ satisfying:*

$$\begin{array}{ccc} P & \xrightarrow{F} & P' \\ {\scriptstyle \pi}\big\downarrow & & {\scriptstyle \pi'}\big\downarrow \\ M & \xrightarrow{id} & M \end{array}$$

Lemma 11.3. *A principal G-bundle $P \xrightarrow{\pi} M$ is trivial if and only if there exists a global cross section.*

Recall: A section s is a map $M \to P$ which satisfies $\pi \circ s = id$.

Proof. If $P \xrightarrow{\pi} M$ is trivial, then there exists an equivariant diffeomorphism $\phi : P \xrightarrow{\sim} M \times G$ and a map $M \times G \xrightarrow{\pi_1} M$ such that $\pi_1 \circ \phi = \pi$. One possible global section is $s : x \mapsto \phi^{-1}(x, e)$.

On the other hand, if there exists a global cross section $s : M \to P$, then we map $P \xrightarrow{\sim} M \times G$, $s(x)g \mapsto (x, g)$. □

Remark: This is in sharp contrast with vector bundles. Vector bundles always have at least one section – the zero section.

11.2. Cocycle description.

Lemma 11.4. *Let $P \xrightarrow{\pi} M$ be a principal G-bundle. Then there exists an open cover $\{U_\alpha\}$ of M and G-*equivariant *trivializations $\phi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times G$ such that ϕ_α maps $\pi^{-1}(x)$, $x \in M$, diffeomorphically onto $\{x\} \times G$.*

Remark: The important thing is that we now need to keep track of the group action as well as the local trivializations.

Proof. We have an identification $\pi^{-1}(U_\alpha) \simeq U_\alpha \times G$ since P is a locally trivial fiber bundle. Choose any section of $\pi^{-1}(U_\alpha)$ over U_α, i.e., a map $s : U_\alpha \to \pi^{-1}(U_\alpha)$ such that $\pi \circ s = id$. We

then construct a G-equivariant map $\phi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times G$ by setting $\phi_\alpha(s(x)g) = (x, g)$.
□

Suppose now we want to reconstruct the G-bundle P from the local trivializations ϕ_α by patching. This means that on the overlaps $U_\alpha \cap U_\beta$ we have G-equivariant maps

$$\phi_\alpha \circ (\phi_\beta)^{-1} : (U_\alpha \cap U_\beta) \times G \to (U_\alpha \cap U_\beta) \times G.$$

The G-equivariance implies that we can represent this function by

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to G,$$

i.e., $\phi_\alpha \circ (\phi_\beta)^{-1}(x, g) = (x, g_{\alpha\beta}(x)g)$. The $g_{\alpha\beta}$ satisfy the *cocycle conditions*

$$g_{\alpha\alpha} = 1,$$

$$g_{\alpha\beta} \circ g_{\beta\gamma} = g_{\alpha\gamma}.$$

The bundle can be reassembled by taking $\{g_{\alpha\beta}\}$ and patching using these transition functions.

12. Classification of principal G-bundles I

For more information, see *Fibre Bundles* by Husemöller or the appendix to *Spin Geometry* by Lawson and Michelsohn.

12.1. Preliminaries.

Proposition 12.1. *Let M be a manifold and P be a principal G-bundle over $M \times [0,1]$. Then P is isomorphic to a principal G-bundle which is a product in the $I = [0,1]$-direction, namely to: $P' \times I \stackrel{\pi\times id}{\to} M \times I$, where $P' \stackrel{\pi}{\to} M$ is a principal G-bundle.*

Proof. Note that if a G-bundle is trivial on U, then it is trivial on any subset $U' \subset U$. By applying the local triviality of the G-bundle, as well as the compactness of the interval $[0,1]$ we get the following: For any $x \in M$, there exists a small enough open set $U_x \subset M$ containing x and $t_0 = 0 < t_1 < t_2 < \cdots < t_k = 1$ such that P is trivial on each $U_x \times (t_i - \varepsilon, t_{i+1} + \varepsilon)$, where $\varepsilon > 0$ is small. (Strictly speaking, it's $(t_i - \varepsilon, t_{i+1} + \varepsilon) \cap [0,1]$ instead of $(t_i - \varepsilon, t_{i+1} + \varepsilon)$.)

We will show that $P|_{U_x \times [0,1]}$ (the restriction of P to $U_x \times [0,1]$) is trivial. For this we will do the simple example when $P_1 = P|_{U_1 = U_x \times [0,1/2]}$ and $P_2 = P|_{U_2 = U_x \times [1/2,1]}$ are trivial. Then there are sections $s_1 : U_1 \to P_1$ and $s_2 : U_2 \to P_2$. Along $U_1 \cap U_2 = U_x \times \{1/2\}$, the sections differ by a function $g_{U_2U_1} : U_x \to G$, i.e., $s_1(a, 1/2) = g_{U_2U_1}(a) \cdot s_2(a, 1/2)$, $a \in U_x$. In view of this, we can simply modify s_2 to $s_2'(a,x) = g_{U_2U_1}(a) s_2(a,x)$. This gets us the global section for $P|_{U_x \times [0,1]}$. This proves the triviality of $P|_{U_x \times [0,1]}$.

Let $\{U_\alpha\}$ be a open cover of M so that each $P|_{U_\alpha \times I}$ is trivial. We will now find a G-equivariant isomorphism $F : f^{-1}P \stackrel{\sim}{\to} P$ compatible with $f : M \times I \to M \times I$ which sends $(x,t) \mapsto (x,0)$. This is done in stages. We explain the first step. Let $V_\alpha \subset U_\alpha$ be a slightly smaller open set. Then there exists a smooth function $\phi_\alpha : M \to \mathbf{R}$ which is 1 outside of U_α, 0 on V_α, and is always nonnegative. Define $f_\alpha : M \times I \to M \times I$ by sending $(x,t) \mapsto (x, \phi_\alpha(x) t)$. We show how to obtain a compatible G-equivariant isomorphism $F_\alpha : f_\alpha^{-1}P \stackrel{\sim}{\to} P$. If $G_\alpha : P|_{U_\alpha \times I} \stackrel{\sim}{\to} (U_\alpha \times I) \times G$ is a G-equivariant (trivializing) isomorphism, then F_α is the inverse of $G_\alpha^{-1} \circ (f_\alpha \times id) \circ G_\alpha$ on $U_\alpha \times I$ and is the identity elsewhere.

Since $f_\alpha^{-1}P$ is still trivial on all the $U_\alpha \times I$, we can inductively apply f_α for all the U_α, and obtain the desired map F. □

Corollary 12.2. *Let $P \stackrel{\pi}{\to} M$ be a principal G-bundle, N be a manifold, and $\phi_0, \phi_1 : N \to M$ be homotopic maps, i.e., there exists a smooth map $\Phi : N \times [0,1] \to M$ such that $\Phi(x,i) = \phi_i(x)$, $i = 0,1$. Then $\phi_0^{-1}(P)$ is isomorphic to $\phi_1^{-1}(P)$.*

Hence homotopic maps induce isomorphic bundles after pullback.

HW: Prove that a principal G-bundle on $\mathbf{R}^n$ (or on some contractible space) is trivial.

12.2. Čech cohomology group $H^1(M;G)$.

Let M be a manifold. We say that an open cover $\{U_\alpha\}$ of an n-dimensional manifold M is a *good cover* if the following hold:

1. $U_\alpha \simeq \mathbf{R}^n$.
2. $U_{\alpha_1} \cap \cdots \cap U_{\alpha_k} \simeq \mathbf{R}^n$ or $\emptyset$.
3. If $k > n+1$, then $U_{\alpha_1} \cap \cdots \cap U_{\alpha_k} = \emptyset$ (if the open sets are distinct).

We will assume the following somewhat reasonable proposition.

Proposition 12.3. *A good cover exists on any manifold M.*

In this section we will assume our open covers are good covers.

Last time we showed that a 1-cocycle $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to G\}$ satisfying

$$g_{\alpha\alpha} = 1,$$
$$g_{\alpha\beta} \circ g_{\beta\gamma} = g_{\alpha\gamma}$$

gave rise to a principal G-bundle P.

Lemma 12.4. *Two G-bundles constructed from $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ are equivalent if and only if there exist maps $g_\alpha : U_\alpha \to G$ such that*

$$g'_{\alpha\beta} = g_\alpha^{-1} \cdot g_{\alpha\beta} \cdot g_\beta. \tag{4}$$

Proof. We will prove one direction. Let P and P' be principal G-bundles constructed from $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$, respectively, and suppose they are isomorphic. Then on U_α the isomorphism restricts to a G-equivariant map $U_\alpha \times G \to U_\alpha \times G$. This is encoded by the map $g_\alpha : U_\alpha \to G$. Equation 4 is a result of looking at the following diagram:

$$\begin{array}{ccc}
(U_\alpha \cap U_\beta) \times G & \xrightarrow{(id, g'_{\alpha\beta})} & (U_\alpha \cap U_\beta) \times G \\
{\scriptstyle (id, g_\beta)}\downarrow & & {\scriptstyle (id, g_\alpha)}\downarrow \\
(U_\alpha \cap U_\beta) \times G & \xrightarrow{(id, g_{\alpha\beta})} & (U_\alpha \cap U_\beta) \times G
\end{array}$$

□

Define the *Čech cohomology* $H^1(M; G)$ to be the collection of $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to G\}$ satisfying the cocycle condition, modulo the equivalence relation given by Equation 4. More or less by definition, the set of isomorphisms classes of principal G-bundles is in 1-1 correspondence with $H^1(M; G)$. (Note that in the usual definition of Čech cohomology, we need to take refinements of the open cover $\{U_\alpha\}$ and use a *direct limit* construction. However, if we choose a good cover, taking direct limits is not necessary.)

13. Classification of principal G-bundles II

The references to today's lecture are again *Fibre Bundles* by Husemöller or the appendix to *Spin Geometry* by Lawson and Michelsohn.

Much of this discussion will be informal, i.e., our standard of rigor is relaxed even more than usual. Most of what we discuss more properly belongs to the realm of algebraic topology and is done in the category of topological spaces.

13.1. **Homotopy.**

Our category is the category of topological spaces and continuous maps.

Definition 13.1. *Two maps $f_0, f_1 : X \to Y$ are* homotopic *if there exists a continuous map $F : X \times [0,1] \to Y$ such that $f_i(x) = F(x,i)$, $i = 0,1$. We denote by $[X,Y]$ the set of equivalence classes of (continuous) maps from X to Y, where $f_0 \sim f_1$ iff they are homotopic. Sometimes our topological spaces X have base points $x \in X$, and are called* pointed spaces *(X,x). Maps, homotopies, etc. between pointed spaces (X,x), (Y,y) are assumed to send x to y. Then the i-th homotopy group ($i > 0$) $\pi_i(X,x)$ is defined to be $[(S^i, pt), (X,x)]$. The 0-th homotopy group is the set of connected components of X.*

Definition 13.2. *A topological space X is* contractible *if X is homotopy equivalent to a point. (In this case this means that there exists a homotopy $F : X \times I \to X$, where $f_t(x) \stackrel{def}{=} F(x,t)$, $f_0 = id$, and f_1 maps X to a point in X.)*

Remark: For the spaces we are interested in, the contractibility of X is equivalent to the vanishing of all the homotopy groups $\pi_i(X,x)$ for all $i \geq 0$.

13.2. **Classifying spaces.**

Universal principal G-bundle: There exists a "universal" principal G-bundle $EG \stackrel{\pi}{\to} BG$, with the following properties:

1. Every principal G-bundle is pulled back from the universal principal G-bundle, i.e., given a principal G-bundle $P \to M$, there exists a map $M \stackrel{f}{\to} BG$ such that P is isomorphic to $f^{-1}EG$.
2. EG is contractible,
3. If we let $Isom(M,G)$ be the set of isomorphism classes of principal G-bundles, and $[M,BG]$ be the set of homotopy classes of maps from M to BG, then
$$Isom(M,G) \simeq [M,BG].$$

We say that the isomorphism classes of principal G-bundles are *classified* by homotopy classes of maps from M to BG and that BG is a *classifying space* for G.

Remark: EG and BG are infinite-dimensional topological spaces (and are certainly not (finite-dimensional) manifolds).

The classification of principal G-bundles follows from the following two theorems:

Theorem 13.3. *Let $E \to B$ be a principal G-bundle with E contractible. Then B is a classifying space for G.*

Proof. We use the following fact, which we will come back to when we do Morse theory:

Fact: Let e_i be an *i-cell* (i.e., an i-dimensional disk). Then a manifold M is obtained by starting with a disjoint union of 0-cells and inductively attaching 1-cells, then attaching 2-cells, and so on.

Let P be a principal G-bundle over M. Suppose we have constructed compatible maps $f : M' \to BG$ and G-equivariant $F : P|_{M'} \xrightarrow{\sim} f^{-1}EG$, where $M' \subset M$. We will show how to extend F to $P|_{M'\cup e_i}$, where e_i is attached onto M' by sending $\partial e_i \to M'$. We know $P|_{e_i}$ is trivial, so there is a section $s : e_i \to P|_{e_i}$. Compose $s : \partial e_i \to P|_{e_i}$ with $P|_{\partial e_i} \xrightarrow{F} f^{-1}EG \to EG$. Now, $\partial e_i = S^{i-1}$, so by $\pi_{i-1}(EG) = 0$, we have an extension of the map $\partial e_i \to EG$ to $e_i \to EG$. This gives us an extension $f : M' \cup e_i \to BG$ and $F : P|_{M'\cup e_i} \xrightarrow{\sim} f^{-1}EG$ is immediate. □

14. Day 14

14.1. More on classifying spaces.

Theorem 14.1. *For any connected Lie group G, there exists a principal G-bundle $E \to B$ with contactible total space E.*

Proof. (Due to Milnor) We will construct a contractible topological space E, together with a free G-action on E. First define the join $X * Y$ of two topological spaces X and Y as $(X \times Y \times [0,1])/\sim$, where $(x, y, 0) \sim (x, y', 0)$ and $(x, y, 1) \sim (x', y, 1)$ for all $x, x' \in X$ and $y, y' \in Y$. In other words, we squash $X \times Y \times \{0\}$ to $X \times \{pt\} \times \{0\}$ and $X \times Y \times \{1\}$ to $\{pt\} \times Y \times \{1\}$.

Fact: If $\pi_0(X) = \cdots = \pi_i(X) = 0$ and $\pi_0(Y) = 0$, then $\pi_0(X * Y) = \cdots = \pi_{i+1}(X * Y) = 0$.

We do a sequence of joins: G, $G * G$, $G * G * G$, and obtain an infinite join construction $E = G * G * G * G * \ldots$. Here we are thinking of $*_i G$ as being homeomorphic to $(*_i G) \times G \times \{0\}/\sim$ inside $*_{i+1} G = (*_i G) * G$. Using the Fact, we see that E is contractible. It is not hard to see that G acts freely on E. □

Example: When $G = S^1$, $BG = \mathbf{CP}^\infty$, $EG = S^1 * S^1 * S^1 * \ldots$ is the unit sphere S^∞ in $\mathbf{C}^\infty$. We can see that $S^1 * S^1 = S^3$ and in general $S^{2n-1} * S^1 = S^{2n+1}$. To see this, we write S^{2n+1} as the unit sphere $(x_1^2 + \cdots + x_{2n}^2) + (x_{2n+1}^2 + x_{2n+2}^2) = 1$. We split S^{2n+1} into three pieces:

1. $x_1^2 + \cdots + x_{2n}^2 = a$, $x_{2n+1}^2 + x_{2n+2}^2 = 1 - a$, $a \leq \varepsilon$.
2. $x_1^2 + \cdots + x_{2n}^2 = a$, $x_{2n+1}^2 + x_{2n+2}^2 = 1 - a$, $a \geq 1 - \varepsilon$.
3. $x_1^2 + \cdots + x_{2n}^2 = a$, $x_{2n+1}^2 + x_{2n+2}^2 = 1 - a$, $\varepsilon < a < 1 - \varepsilon$.

Here ε is a small positive number. (1) corresponds to $(S^{2n+1} \times S^1 \times [1 - \varepsilon, 1])/\sim$, (2) corresponds to $(S^{2n+1} \times S^1 \times [0, \varepsilon])/\sim$, and (3) corresponds to $S^{2n+1} \times S^1 \times (\varepsilon, 1 - \varepsilon)$.

The Hopf fibration gives the action of S^1 on S^{2n+1}, and the quotient is $\mathbf{CP}^n$. (Remark: S^n satisfies $\pi_i(S^n) = 0$ for $i < n$.)

14.2. Reduction of structure group.

Write $P(M, G)$ for the principal G-bundle over M with total space P. Then a *bundle map* $P'(M', G') \to P(M, G)$ consists of the following data: A group homomorphism $\phi : G' \to G$, a (G', G)-equivariant map $F : P' \to P$ (a (G', G)-equivariant map $P' \to P$ satisfies $F(p'g') = F(p')\phi(g')$), and a map $f : M' \to M$ which satisfy commutative diagram:

$$\begin{array}{ccc} P' & \xrightarrow{F} & P \\ {\scriptstyle \pi'}\downarrow & & {\scriptstyle \pi}\downarrow \\ M' & \xrightarrow{f} & M \end{array}$$

Definition 14.2. *Let P', P be principal bundles over M with structure groups G', G, respectively. If G' is a subgroup of G and there exists a bundle map $P'(M,G') \to P(M,G)$, then $F : P' \to P$ is said to be a* reduction *of the structure group from G to G'.*

Let us interpret the notion of reduction from the Čech perspective. Let $P(M,G)$ be a principal G-bundle. Then there exists an open cover $\{U_\alpha\}$ over M and bundle maps $\phi_\alpha : P|_{U_\alpha} \stackrel{\sim}{\to} U_\alpha \times G$. On the overlap $U_\alpha \cap U_\beta$, there exist maps

$$P|_{U_\alpha \cap U_\beta} \stackrel{\phi_\alpha}{\to} (U_\alpha \cap U_\beta) \times G,$$

$$P|_{U_\alpha \cap U_\beta} \stackrel{\phi_\beta}{\to} (U_\alpha \cap U_\beta) \times G.$$

Then $\phi_\beta \circ \phi_\alpha^{-1}(x,e) = (x, g_{\beta\alpha}(x))$, where $g_{\beta\alpha} : U_\alpha \cap U_\beta \to G$. This $\{g_{\alpha\beta}\}$ satisfies the cocycle condition as above.

Proposition 14.3. *Let $H \subset G$ be a Lie subgroup. Then there exists a reduction of the structure group of $P(M,G)$ to H if and only if $P(M,G)$ admits a 1-cocycle $\{g_{\alpha\beta}\}$ where each $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ factors through H, i.e., $g_{\alpha\beta} : U_\alpha \cap U_\beta \to H \subset G$.*

15. Day 15

15.1. Examples of reduction of structure group:

1. Consider $Fr(M) \to M$ with structure group $G = Gl(n, \mathbf{R})$, where $n = \dim M$. By choosing a Riemannian metric on M, we can reduce the structure group from $G = Gl(n, \mathbf{R})$ to $G' = O(n)$. The reduction yields the orthonormal frame bundle $OFr(M) \to M$. Recall that every manifold M admits a Riemannian metric, so the reduction from $Gl(n, \mathbf{R})$ to $O(n)$ is always automatic. Conversely, a reduction

$$\begin{array}{ccc} P' & \longrightarrow & Fr(M) \\ \downarrow & & \downarrow \\ M & \xrightarrow{id} & M \end{array}$$

of the structure group to $O(n)$ specifies a Riemannian metric. (HW!)

2. Reduction of $Fr(M) \to M$ with structure group $Gl(n, \mathbf{R})$ to $Gl^+(n, \mathbf{R})$. Reducibility is equivalent to the orientability of M, and a reduction corresponds to a *choice of orientation.* If M is orientable, we can reduce from $Gl(n, \mathbf{R})$ to $SO(n)$.

3. Reduction from $Gl(n, \mathbf{R})$ to other Lie groups:
   1. $Gl(2n, \mathbf{R}) \supset Gl(n, \mathbf{C}) \leftrightarrow$ almost complex structure.
   2. $Gl(2n, \mathbf{R}) \supset U(n) \leftrightarrow$ Hermitian structure (choice of Hermitian metric).
   3. $Gl(2n, \mathbf{R}) \supset Sp(n, \mathbf{R}) \leftrightarrow$ almost symplectic structure.

Here,
$$Gl(n, \mathbf{C}) = \{A \in Gl(2n, \mathbf{R}) | AJ = JA\},$$
and
$$Sp(2n, \mathbf{R}) = \{A \in Gl(2n, \mathbf{R}) | AJA^T = J\},$$
where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

15.2. Associated vector bundles.

Consider a principal G-bundle $P(M, G)$ and a representation $\rho : G \to Gl(V)$. Then we can construct a *vector bundle* $P \times_\rho V$ *associated to* P, as follows:
$$P \times_\rho V \stackrel{def}{=} P \times V / \sim,$$
where $(pg, v) \sim (p, \rho(g)v)$. (In other words, g jumps from the left-hand side to the right-hand side.)

Locally,
$$(U_\alpha \times G \times V / \sim) \xrightarrow{\sim} U_\alpha \times V$$
is given by
$$(x, g, v) \sim (x, e, \rho(g)v) \mapsto (x, \rho(g)v).$$

If we take transition maps

$$(U_\alpha \cap U_\beta) \times G \times V/\sim \quad \to (U_\alpha \cap U_\beta) \times G \times V/\sim,$$
$$(x,e,v) \mapsto (x, g_{\alpha\beta}(x), v) \sim (x, e, \rho(g_{\alpha\beta}(x))v),$$

then the corresponding map

$$(U_\alpha \cap U_\beta) \times V \to (U_\alpha \cap U_\beta) \times V$$

is

$$(x,v) \mapsto (x, \rho(g_{\alpha\beta}(x))v).$$

This implies that $P \times_\rho V$ is constructed from $\{U_\alpha \times V\}$ via the transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$.

Remark: This construction is identical to the construction of vector bundles "twisted by ρ" from last semester.

Examples:

1. $Fr(M) \times_{\rho_0} \mathbf{R}^n = TM$, $\rho_0 : Gl(n,\mathbf{R}) \stackrel{id}{\to} Gl(n,\mathbf{R})$.
2. $Fr(M) \times_{\rho_1} \mathbf{R}^n = T^*M$, $\rho_1 : Gl(n,\mathbf{R}) \to Gl(n,\mathbf{R})$, $A \mapsto (A^{-1})^T$.
3. $Fr(M) \times_{\rho_2} \mathbf{R}^n = \wedge^k TM$, $\rho_2 : Gl(n,\mathbf{R}) \to Gl(\wedge^k \mathbf{R}^n)$.

16. Day 16: Connections on Principal Bundles

16.1. Definition.

Definition 16.1. *Let P be a principal G-bundle with projection map $\pi : P \to M$. Then a connection Γ is an assignment of a subspace $H_p \subset T_pP$ at every $p \in P$ such that*

1. $\pi_* : H_p \xrightarrow{\sim} T_{\pi(p)}M$.
2. *(G-invariance)* $(R_g)_* H_p = H_{pg}$, *where* $R_g : P \to P$ *maps* $p \mapsto pg$.
3. H_p *depends differentiably on* $p \in P$.

The tangent spaces to the fibers are called the *vertical subspaces* and the H_p are called the *horizontal subspaces*.

16.2. First interpretation.

Let $n = \dim M$. Then a connection Γ on P is a rank n distribution on P which is invariant under right action by G and is transverse to the fibers.

Let $i_p : G \to P$, $p \in P$, be the fiber map $g \mapsto pg$. This is the map which sends G isomorphically onto a fiber of P. Also let $\mathfrak{g} = Lie(G)$.

Lemma 16.2. *There exists a short exact sequence*

$$0 \to \mathfrak{g} \xrightarrow{(i_p)_*} T_pP \xrightarrow{\pi_*} T_xM \to 0,$$

and a connection Γ gives a splitting $s : T_xM \to T_pP$ so that $\pi \circ s = id$. Here $x \in M$ is $\pi(p)$ and we are viewing $\mathfrak{g} = T_eG$.

Proof. The exactness of the sequence is clear from the definition of the maps $(i_p)_*$ and π_*. The splitting is given by the inverse of the isomorphism $\pi_* : H_p \xrightarrow{\sim} T_xM$. □

16.3. Second interpretation.

Proposition 16.3. *A connection Γ is given by $\omega \in \Omega^1(P; \mathfrak{g})$ such that*

1. $\omega(p)((i_p)_*\xi) = \xi$ *for all* $\xi \in \mathfrak{g}$.
2. $R_g^*\omega = Ad(g^{-1})\omega$.

Conversely, any $\omega \in \Omega^1(P; \mathfrak{g})$ which satisfies conditions (1) and (2) gives rise to a connection.

Remark: Recall the definition of forms *with value in a vector bundle*. Then $\Omega^1(P; \mathfrak{g}) = \Gamma(T^*P \otimes \mathfrak{g})$, i.e., at $p \in P$, $\omega \in \Omega^1(P; \mathfrak{g})$ gives a map $\omega(p) : T_pP \to \mathfrak{g}$.

Remark: Such a 1-form ω is said to be a *connection 1-form*.

Proof. Consider the injection $(i_p)_* : \mathfrak{g} \to T_pP$. Define $\omega(p) : T_pP \to \mathfrak{g}$ so that the composition

$$\mathfrak{g} \xrightarrow{(i_p)_*} T_pP \xrightarrow{\omega(p)} \mathfrak{g}$$

is the identity map and $\ker \omega(p) = H_p$ This completely determines $\omega(p)$ since $(i_p)_*(\mathfrak{g}) \oplus H_p = T_pP$. Condition (1) is satisfied by the definition.

Next, we show condition (2). We compute that if $Y \in H_p$, then

$$(R_g^*\omega)(p)(Y) = \omega(pg)((R_g)_*Y) = 0,$$

since $(R_g)_*H_p = H_{pg}$. If $p \in P$, $\xi \in \mathfrak{g}$, and $pe^{t\xi} \in (i_p)_*\mathfrak{g}$, then

$$(R_g^*\omega)(p)(pe^{t\xi}) = \omega(pg)(pe^{t\xi}g) = \omega(pg)(pg(g^{-1}e^{t\xi}g)) = g^{-1}e^{t\xi}g = Ad(g^{-1})\xi.$$

This proves (2).

Now suppose $\omega \in \Omega^1(P;\mathfrak{g})$ satisfies (1) and (2). Then we set $H_p = \ker \omega(p)$. Condition (1) tells us that $\dim H_p = \dim M$ and $\pi_* : H_p \to T_xM$ is an isomorphism. The G-invariance follows from (2). □

Remark: Condition (1) says that $(i_p)^*\omega$ is the Maurer-Cartan form on G. Indeed, if $g \in G$ and an arc through g is given by $ge^{t\xi}$, $\xi \in \mathfrak{g}$, then

$$(i_p)^*\omega(g)(ge^{t\xi}) = \omega(pg)(pge^{t\xi}) = \omega(pg)((i_{pg})_*\xi) = \xi.$$

Note that $(i_p)^*\omega$ is the Maurer-Cartan form precisely because G acts on P *from the right* – you can check that it does not work if G acts on P from the left.

17. Day 17: More on Connections

17.1. **Existence.** Consider the trivial G-bundle $P = M \times G$. If $\mathfrak{g} = Lie(G)$, then we define the *trivial* or *flat connection* on $M \times G$ by setting $\omega = \pi_2^* \omega_{MC}$, where $\pi_2 : M \times G \to G$ is the second projection and ω_{MC} is the Maurer-Cartan form on G. Note that $H_{(x,g)} = T_x M \times \{0\} \subset T_x M \times T_g G$.

Proposition 17.1. *Let $P(M, G)$ be a principal bundle. Then there exists a connection on P.*

Proof. The proof is identical to the proof of the existence of connections on a vector bundle and very similar to the existence of a Riemannian metric. Let $\{U_\alpha\}$ be an open cover of M such that $P|_{U_\alpha}$ is trivial. On each $P|_{U_\alpha}$ construct the flat connection ω_α. If $\{f_\alpha\}$ is a partition of unity subordinate to $\{U_\alpha\}$, then we set $\omega = \sum_\alpha f_\alpha \omega_\alpha$.

We verify that ω satisfies the properties of a connection 1-form. For condition (1) of Proposition 16.3,

$$\omega(p)(pe^{t\xi}) = \sum_\alpha f_\alpha(x)\omega_\alpha(p)(pe^{t\xi}) = \sum_\alpha f_\alpha(x)\xi = \xi,$$

where $p \in P$, $\pi(p) = x \in M$, $\xi \in \mathfrak{g}$, and $pe^{t\xi} \in (i_p)_* \mathfrak{g}$. Condition (2) is similar and is left for HW. □

17.2. **The space of connections.** In the proof of existence, the crucial fact that was used was that if ω, ω' are connection 1-forms on P, then $t\omega + (1-t)\omega'$ is also a connection 1-form. This shows that:

Lemma 17.2. *The space $\mathcal{A}(P)$ of all (smooth) connections on $P(M, G)$ is an affine subspace of $\Omega^1(P; \mathfrak{g})$.*

Remark: An affine space has the defining property that the line through any two elements ω and ω' is contained in the space.

We will give a more precise characterization of the space of connection 1-forms. But first we define the following: Let $P(M, G)$ be a principal bundle and $Ad : G \to Gl(\mathfrak{g})$ be the adjoint representation. Then the vector bundle associated to Ad is $Ad(P) = P \times_{Ad} \mathfrak{g}$. Recall that $(pg, e^{t\xi}) \sim (p, Ad(g)e^{t\xi}) = (p, ge^{t\xi}g^{-1})$ by the definition of the associated vector bundle. Here, $p \in P$, $g \in G$, $\xi \in \mathfrak{g}$.

Proposition 17.3. *If we fix a connection ω_0 on P, then there is an identification $\mathcal{A}(P) \tilde{\to} \Omega^1(M; Ad(P))$.*

Proof. Given a connection 1-form ω on P, we take the difference $\omega - \omega_0$. Then

$$(\omega - \omega_0)(p)(pe^{t\xi}) = \xi - \xi = 0,$$

by condition (1) of Proposition 16.3. Here $\xi \in \mathfrak{g}$. Moreover,

$$R_g^*(\omega - \omega_0) = Ad(g^{-1})(\omega - \omega_0),$$

by condition (2). We will interpret this in terms of $Ad(P)$.

Given $x \in M$, we lift $X \in T_xM$ to $\widetilde{X} \in H_p \subset T_pP$, where $p \in \pi^{-1}(x)$ and H_p is the horizontal subspace for ω_0. (This means that $\pi_*\widetilde{X} = X$.) Then we have the assignment:

$$X \mapsto (p, (\omega - \omega_0)(p)(\widetilde{X}(p))) = (p, \omega(p)(\widetilde{X}(p))),$$

since $\widetilde{X}$ is horizontal for ω_0. Now,

$$(pg, \omega(pg)(\widetilde{X}(pg))) = (pg, R_g^*\omega(p)(\widetilde{X}(p))) = (pg, Ad(g^{-1})\omega(p)(\widetilde{X}(p))) \sim (p, \omega(p)(\widetilde{X}(p))).$$

Note that here we used the fact that $(R_g)_*\widetilde{X}(p) = \widetilde{X}(pg)$.

Conversely, it is easy to verify that if $A \in \Omega^1(M; Ad(P))$, then $\omega_0 + A$ is a connection 1-form. (HW!) $\square$

Remark: A connection 1-form is a form on the total space P. However, *the difference* of two connections can be viewed as a 1-form on the base M (with values in the vector bundle $Ad(P)$).

Example: Suppose $G = S^1$. Identify its Lie algebra $\mathfrak{g}$ with $i\mathbf{R}$ by thinking of $e^{i\theta} \in S^1$ and $i\theta \in i\mathbf{R}$. Consider the adjoint representation $Ad : S^1 \to Gl(\mathfrak{g}) = Gl(i\mathbf{R})$, where

$$e^{i\theta} \mapsto (e^{itv} \mapsto e^{i\theta}e^{itv}e^{-i\theta} = e^{itv}).$$

Hence Ad maps to the identity in $Gl(i\mathbf{R})$, and $Ad(P) = M \times i\mathbf{R}$. Therefore, connections on the principal S^1-bundle P are in a 1-1 correspondence with $\Omega^1(M; Ad(P)) = \Omega^1(M; i\mathbf{R}) = i\Omega^1(M; \mathbf{R})$.

18. Holonomy

18.1. **Parallel translation.** Let P be a principal G-bundle over M with projection $\pi : P \to M$ and connection $\Gamma = \{H_p | \ p \in P\}$. We have the following obvious fact:

Lemma 18.1. *Let X be a vector field on M. Then there exists a unique* horizontal lift $\widetilde{X}$ *of X to P. By this we mean a vector field $\widetilde{X}$ on P such that $\widetilde{X}(p) \in H_p$ and $\pi_*(\widetilde{X}(p)) = X(\pi(p))$.*

Let $\gamma : [0,1] \to M$ be a smooth arc on M with $\gamma(0) = x$, $\gamma(1) = y$, and let p be any point in $\pi^{-1}(x)$. Then we can pull back the bundle P to $\gamma^{-1}(P)$ and pull back the connection to $\gamma^{-1}(\Gamma)$.[1] If t is the coordinate on $[0,1]$, then we set $X = \frac{\partial}{\partial t}$. Then there exists a lift $\widetilde{X}$ of X on $\gamma^{-1}(P)$. By integrating along $\widetilde{X}$ and mapping back to P, we obtain a lift $\widetilde{\gamma}_p : [0,1] \to P$ of γ, starting at $p \in \pi^{-1}(x)$. Moreover, we obtain the following *parallel translation* or *holonomy map along* γ

$$h(\gamma) : \pi^{-1}(x) \xrightarrow{\sim} \pi^{-1}(y),$$
$$p \mapsto \widetilde{\gamma}_p(1).$$

Note that $h(\gamma)$ is G-equivariant.

Let $P(M, x, y)$ be the set of smooth paths γ on M from x to y. Then parallel translation gives a map

$$P(M, x, y) \to \mathrm{Diff}^G(\pi^{-1}(x), \pi^{-1}(y)),$$

where $\mathrm{Diff}^G(\pi^{-1}(x), \pi^{-1}(y))$ stands for the G-equivariant diffeomorphisms from $\pi^{-1}(x)$ to $\pi^{-1}(y)$.

When $x = y$, $P(M, x, y)$ is written as $\Omega(M, x)$ and is called the *loop space* with base point x. Parallel translation gives a homomorphism

$$h_x : \Omega(M, x) \to Aut(\pi^{-1}(x)),$$

where $Aut(\pi^{-1}(x))$ is the set of G-equivariant automorphisms of $\pi^{-1}(x)$. h is called the *holonomy representation* of the monoid $\Omega(M, x)$. It is not hard to see that $Aut(\pi^{-1}(x)) \simeq G$, i.e., where one point $p \in \pi^{-1}(x)$ goes determines the rest.

18.2. **Flat connections.**

Definition 18.2. *A connection Γ on P is* flat *if for each $x \in M$ the holonomy representation $h_x : \Omega(M, x) \to G$ is locally constant.*

By *locally constant* we mean the following: Let $\gamma_s(t) : [0,1] \to M$, $\gamma_s(0) = \gamma_s(1) = x$, be a 1-parameter family of loops based at x. Then $h_x(\gamma_0) = h_x(\gamma_\varepsilon)$ for small ε. A moment's reflection tells us that this implies that $h_x(\gamma_0) = h_x(\gamma_s)$ for all s, i.e., that the value of h_x is constant on each homotopy class of loops based at x. This implies the following proposition:

[1] The easiest way to do this is to pull back the connection 1-form. It is left for HW to verify that the pulled back 1-form is indeed a connection 1-form.

Proposition 18.3. *A flat connection* Γ *gives rise to a holonomy representation*

$$h_x : \pi_1(M,x) \to G.$$

Remark: If M is connected, then it suffices to check that h_x is locally constant at one point $x \in M$. A connection on P is flat if and only if, for all $x \in M$, the holonomy map $h_x : \Omega(M,x) \to Aut(\pi^{-1}(x))$ maps $\gamma \mapsto id$ for sufficiently small homotopically trivial loops γ. (HW!)

19. Curvature

Let $P(M,G)$ be a principal bundle and $\omega \in \Omega^1(P;\mathfrak{g})$ be a connection 1-form. Then the *curvature* of ω is given by:

$$\widetilde{F}_\omega = d\omega + [\omega,\omega].$$

Therefore, $\widetilde{F}_\omega \in \Omega^2(P;\mathfrak{g})$. Recall that the Maurer-Cartan form ω_{MC} on G satisfies the equation

$$d\omega_{MC} + [\omega_{MC},\omega_{MC}] = 0.$$

Roughly speaking, the quadratic term in ω is designed to factor out the contributions in the fiber direction.

Proposition 19.1. *The curvature form $\widetilde{F}_\omega$ satisfies the following:*

1. $R_g^*\widetilde{F}_\omega = Ad(g^{-1})\widetilde{F}_\omega$.
2. $\widetilde{F}_\omega(p)(Y_1,Y_2) = 0$ *if Y_1 is vertical. (Note we need only one of the two tangent vectors to be vertical.)*

Proof. (1) We compute:

$$\begin{aligned} R_g^*\widetilde{F}_\omega &= R_g^*(d\omega + [\omega,\omega]) \\ &= d(R_g^*\omega) + [R_g^*\omega, R_g^*\omega] \\ &= d(Ad(g^{-1})\omega) + [Ad(g^{-1})\omega, Ad(g^{-1})\omega] \\ &= Ad(g^{-1})(d\omega + [\omega,\omega]) \\ &= Ad(g^{-1})\widetilde{F}_\omega \end{aligned}$$

As for (2), if Y_1 and Y_2 are both vertical, then $\widetilde{F}_\omega(p)(Y_1,Y_2) = 0$ follows immediately from the fact that $\widetilde{F}_\omega$ coincides with the Maurer-Cartan form on each fiber.

Next suppose Y_1 is vertical and Y_2 is horizontal. Thinking of tangent vectors as equivalence classes of arcs, if $Y_1 = pe^{t\xi}$ with $\xi \in \mathfrak{g}$, then extend Y_1 to a vertical vector field near p so that $Y_1(q) = qe^{t\xi}$. Also extend Y_2 to an R_g-invariant vector field near p. Then

$$\begin{aligned} \widetilde{F}_\omega(p)(Y_1,Y_2) &= d\omega(Y_1,Y_2) + [\omega(Y_1),\omega(Y_2)] \\ &= Y_1\omega(Y_2) - Y_2\omega(Y_1) - \omega([Y_1,Y_2]) \\ &= -\omega([Y_1,Y_2]). \end{aligned}$$

Here we note that $\omega(Y_2) = 0$ and $\omega(Y_1)$ has the constant value ξ near p. We leave the verification that $[Y_1,Y_2] = 0$ for HW. ☐

Corollary 19.2. *The curvature form $\widetilde{F}_\omega \in \Omega^2(P;\mathfrak{g})$ pushes down to a form $F_\omega \in \Omega^2(M;Ad(P))$.*

Proof. We define $F_\omega(x)(X_1,X_2)$, with $x \in M$ and $X_i \in T_xM$, to be $\widetilde{F}_\omega(p)(\widetilde{X}_1,\widetilde{X}_2)$, where $p \in \pi^{-1}(x)$ and $\widetilde{X}_i \in H_p$ is a lift of $X_i \in T_xM$. The definition is independent of the lift, and is dependent on the choice of p only up to Ad. Therefore, F_ω is a 2-form on M with values in $Ad(P)$. □

Remark: The form F_ω on M is the one that is usually called the *curvature 2-form* of the connection P.

Interpretation of the curvature: Suppose $X_1, X_2 \in T_xM$. Lift X_1, X_2 to horizontal vector fields Y_1, Y_2 in a neighborhood of $p \in P$. Then we compute:

$$F_\omega(x)(X_1, X_2) = \widetilde{F}_\omega(p)(Y_1, Y_2) = \omega(p)([Y_1, Y_2]). \tag{5}$$

Therefore, F measures the failure of H_p to be integrable.

20. Day 20

20.1. Curvature and flatness.

Theorem 20.1. *Let $P \xrightarrow{\pi} M$ be a principal G-bundle and ω a connection 1-form. Then the following are equivalent:*

1. *ω is flat, i.e., the holonomy is locally constant).*
2. *$F_\omega = 0$.*
3. *$\ker \omega$ is integrable, i.e., are tangent spaces to leaves of a foliation of P.*
4. *There exists an open cover $\{U_\alpha\}$ of M such that ω on $P|_{U_\alpha}$ is the trivial connection.*

Proof. (1) $\Rightarrow$ (2). (2) is just the infinitesimal version of (1), in view of Equation 5.

(2) $\Rightarrow$ (3). Identify $\mathfrak{g} \simeq \mathbf{R}^n$ by taking a basis $(e_1, \dots, e_n)$ for $\mathfrak{g}$. Then $[e_i, e_j] = \sum_k c_{ij}^k e_k$. We rewrite the equation

$$\widetilde{F}_\omega = d\omega + [\omega, \omega] = 0$$

with respect to this basis. Letting $\omega = \sum_i \omega_i e_i$, we have $d\omega = \sum_i d\omega_i \cdot e_i$ and

$$[\omega, \omega] = \left[\sum_i \omega_i e_i, \sum_j \omega_j e_j\right] = \frac{1}{2} \sum_{ij} \omega_i \wedge \omega_j \cdot [e_i, e_j] = \frac{1}{2} \sum_{ij} c_{ij}^k \omega_i \wedge \omega_j \cdot e_k.$$

Therefore, we have

$$\begin{pmatrix} d\omega_1 \\ \vdots \\ d\omega_n \end{pmatrix} = \Omega \cdot \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_n \end{pmatrix},$$

for some matrix Ω of 1-forms. This is an involutive system, and hence $\ker \omega$ is integrable.

(3) $\Rightarrow$ (4). Since $\ker \omega$ is integrable, for sufficiently small $U_\alpha \subset M$, there exists a section $s : U_\alpha \to P|_{U_\alpha}$ such that $s(U_\alpha)$ is horizontal everywhere. Using the G-equivariance, we can write $P|_{U_\alpha} = U_\alpha \times G$, foliated by integral submanifolds $U_\alpha \times \{0\}$ of ω.

(4) $\Rightarrow$ (1). If ω is the trivial (flat) connection, then the holonomy around any sufficiently small loop is the identity. (This is equivalent to the holonomy being locally constant, by the discussion from Day 18.) □

20.2. Local coordinates.

We will give local expressions for connection 1-forms and curvature 2-forms. Start with a trivial principal G-bundle $P = U \times G \xrightarrow{\pi} U$. The given trivialization gives rise to a connection 1-form $\omega_0 = \pi_2^* \omega_{MC}$, where $U \times G \xrightarrow{\pi_2} G$ is the second projection and ω_{MC} is the Maurer-Cartan form on G.

If ω is a connection 1-form on P, then we already explained that the difference $\omega - \omega_0$ can be realized as an element $A \in \Omega^1(U; Ad(P))$. Now, since $Ad(P) = U \times G \times_{Ad} \mathfrak{g} \simeq U \times \mathfrak{g}$, we can view $A \in \Omega^1(U; \mathfrak{g})$. Equivalently, we have the following:

Lemma 20.2. *Let $i : U \to U \times G$ be the inclusion $x \mapsto (x, e)$. Then $A = i^*(\omega - \omega_0)$, when A is viewed as an element of $\Omega^1(U; \mathfrak{g})$.*

Lemma 20.3. *$F_\omega = dA + [A, A]$, when both are viewed as elements of $\Omega^2(U; \mathfrak{g})$.*

Both lemmas are left for HW!

We will now compute the change in A brought about by a change in the trivialization. Any bundle automorphism $\psi : U \times G \xrightarrow{\sim} U \times G$ can be written as a map $(x, g) \mapsto (x, \phi(x)g)$, for some map $\phi : U \to G$. (Recall that bundle automorphisms satisfy $\pi = \pi \circ \psi$.) Conversely, any $\phi : U \to G$ gives rise to a bundle automorphism.

Lemma 20.4. *Under the bundle automorphism, $A(x) \mapsto \phi(x)^{-1}A(x)\phi(x) + \phi(x)^{-1}d\phi(x)$.*

Proof. We compare A, which is obtained by pushing down $\omega - \omega_0$, to A', which is obtained by pushing down $\psi^*\omega - \omega_0$. Given $(x, e) \in P$ and $(X, \xi) \in T_{(x,g)}P$, we have:

$$\begin{aligned}
(\psi^*\omega - \omega_0)(x, e)(X, \xi) &= \psi^*\omega(x, e)(X, 0) \\
&= \omega(x, \phi(x))(X, d\phi(x)(X)) \\
&= \omega(x, \phi(x))(X, \phi(x)(\phi(x)^{-1}d\phi(x)(X))) \\
&= R^*_{\phi(x)}\omega(x, e)(X, 0) + \phi(x)^{-1}\phi(x + tX) \\
&= Ad(\phi(x)^{-1})\omega(x, e)(X, 0) + \phi(x)^{-1}d\phi(x)(X) \\
&= \phi(x)^{-1}A(x)(X)\phi(x) + \phi(x)^{-1}d\phi(x)(X)
\end{aligned}$$

Note that

$$(\omega - \omega_0)(x, e)(X, \xi) = \omega(x, e)(X, 0) = A(x)(X).$$

□

21. Day 21: Covariant Differentiation

21.1. **Covariant differentiation on associated bundles.** Let $P \xrightarrow{\pi} M$ be a principal G-bundle, $\rho : G \to Gl(V)$ be a representation of G, and let $E = P \times_\rho V$ be the associated vector bundle. We explain how a connection ω on P induces covariant differentiation ∇ on the associated $E = P \times_\rho V$.

Recall the following definition:

Definition 21.1. *A covariant derivative ∇ on a vector bundle E over M assigns to every vector field $X \in \mathfrak{X}(M)$ a differential operator $\nabla_X : \Gamma(E) \to \Gamma(E)$ where*
1. *$\nabla_X s$ is $\mathbf{R}$-linear in s,*
2. *$C^\infty(M)$-linear in X, and*
3. *$\nabla_X(fs) = (Xf)s + f\nabla_X s$.*

Let $x \in M$, $X \in T_xM$, and $x(t) \in M$ be an arc with $x(0) = x$ and $\dot{x}(0) = X$. Pick $p \in \pi^{-1}(x)$. Then there is a unique horizontal lift $\tilde{x}(t)$ of $x(t)$ to P, where $\tilde{x}(0) = p$. We declare that, given any $v \in V$, the section $s(t) = (\tilde{x}(t), v)$ of E over the arc $x(t)$ is covariant constant, i.e., $\nabla_X s = 0$.

In general, the horizontal space $H_p \subset T_pP$ induces a horizontal space $\widetilde{H}_{(p,v)} \subset T_{(p,v)}E$ as follows: Given $v \in V$, consider the map $\eta_v : P \to E$ which sends $p \mapsto (p, v)$. Then $\widetilde{H}_{(p,v)} = (\eta_v)_* H_p$.

HW: Prove that $(\eta_v)_* H_p = (\eta_{\rho(g^{-1})v})_* H_{pg}$. In other words, the definition of $\widetilde{H}_{(p,v)}$ does not depend on the representative (p, v) for an element in $P \times_\rho V$.

Therefore, we have the decomposition $T_{(p,v)}E = \widetilde{H}_{(p,v)} \oplus V$, where V is identified with the tangent space to the fiber of E at (p, v). Let $T_{(p,v)}E \xrightarrow{\pi_V} V$ be the projection onto the V summand. If $s \in \Gamma(E)$, then we define

$$\nabla_X s(x) = (p, \pi_V \circ ds(x)(X)),$$

where $ds(x) : T_xM \to T_{s(x)}E$.

21.2. **Local coordinates.** We will write the covariant derivative in local coordinates. We take $P = U \times G \xrightarrow{\pi} U$ and $A \in \Omega^1(U; \mathfrak{g})$ the difference $\omega - \omega_0$ between the fixed flat connection ω_0 and a connection ω. Recall that $A = i^*\omega$, where $i : U \to U \times G$, $x \mapsto (x, e)$.

Claim. $H_{(x,e)} = \{(X, -A(x)(X)) | X \in T_xU\}$.

The claim is immediate by evaluating $\omega(x, e)(X, -A(x)(X)) = 0$. Now, $E = U \times G \times_\rho V \xrightarrow{\sim} U \times V$, where $(x, g, v) \mapsto (x, \rho(g)v)$. Then we have:

Claim. $\widetilde{H}_{(x,s(x))} = \{(X, -\rho_*(A(x)(X))s(x)) | X \in T_xU\}$.

This follows from writing elements $T_{(x,s(x))}E$ instead as $(X, -A(x)(X), s(x))$. Consider the section $s \in \Gamma(E; U)$, viewed as a function $U \to V$. By the definition of $\nabla_X s(x)$, we take $ds(x)(X)$ and project to the V summand. But then, this is equivalent to writing:

Proposition 21.2. $\nabla_X s(x) = ds(x)(X) + \rho_*(A(x)(X))s(x)$.

This is often written $\nabla = d + A$ for short. Check that ∇_X is indeed a covariant derivative (HW)!

22. Day 22

22.1. **Covariant differentiation of forms of higher degree.** There is a natural extension of $\nabla : \Omega^0(M;E) \to \Omega^1(M;E)$ to $\nabla : \Omega^i(M;E) \to \Omega^{i+1}(M;E)$, defined as follows:

1. $\nabla\eta$ is $\mathbf{R}$-linear in η.
2. If $s \in \Omega^0(M;E)$ and $\eta \in \Omega^k(M;\mathbf{R})$, then

$$\nabla s \otimes \eta = (\nabla s) \wedge \eta + s \otimes d\eta.$$

We now have a sequence:

$$0 \to \Omega^0(M;E) \xrightarrow{\nabla_A} \Omega^1(M;E) \xrightarrow{\nabla_A} \Omega^2(M;E) \to \dots.$$

Last semester, we defined the *curvature* of the covariant derivative to be

$$\nabla_A \circ \nabla_A : \Omega^0(M;E) \to \Omega^2(M;E).$$

We then have the following:

Proposition 22.1. *In local coordinates, $\rho_*(F_A) = \nabla_A \circ \nabla_A$, where $\rho : G \to Gl(V)$ is the Lie group representation and ρ_* is the corresponding Lie algebra representation.*

Proof.

$$\begin{aligned}(\nabla_A \circ \nabla_A)s &= (d + \rho_* A) \circ (d + \rho_* A)s \\ &= d(ds + (\rho_* A)s) + \rho_* A \wedge (ds + (\rho_* A)s) \\ &= d((\rho_* A)s) + (\rho_* A)ds + (\rho_* A \wedge \rho_* A)s \\ &= \rho_*(dA + A \wedge A)s.\end{aligned}$$

Here, $(\rho_* A \wedge \rho_* A)(X_1, X_2) \stackrel{def}{=} (\rho_* A)(X_1) \cdot (\rho_* A)(X_2) - (\rho_* A)(X_2) \cdot (\rho_* A)(X_1)$, where $\cdot$ represents matrix multiplication (or composition of endomorphisms). It is immediate that $\rho_* A \wedge \rho_* A = [\rho_* A, \rho_* A]$. □

Remark: It looks as though $A \mapsto F_A = dA + [A, A]$ is given by the map

$$\Omega^1(M; Ad(P)) \xrightarrow{\nabla_A} \Omega^2(M; Ad(P)),$$

i.e., $F_A = \nabla_A A$. Unfortunately, if we write $A = \sum_i e_i \otimes A_i$, then

$$\begin{aligned}\nabla A &= \sum_i \nabla(e_i \otimes A_i) \\ &= \sum_i \{(\nabla e_i) \wedge A_i + e_i \otimes dA_i\} \\ &= \sum_{ij} [e_j, e_i] A_j \wedge A_i + \sum_i e_i \otimes dA_i \\ &= dA + 2[A, A].\end{aligned}$$

We are off by a factor of 2.

Proposition 22.2 (Bianchi identity)**.** $\nabla F_\omega = 0$.

Proof. HW. Note that $F_\omega = dA + [A, A]$ locally. You should be careful of the definition $[A, A](x)(X_1, X_2) = [A(x)(X_1), A(x)(X_2)]$. Probably the surest proof is if you wrote out $A = \sum_i e_i \otimes A_i$, where $\{e_1, \dots, e_n\}$ is a basis of sections for $E|_U = U \times V$ and $A_i \in \Omega^1(U; \mathbf{R})$. □

22.2. **Gauge group.** We denote by $\mathcal{G}(P)$ the space of C^∞-bundle automorphisms $\psi : P \overset{\sim}{\to} P$ such that $\pi = \pi \circ \psi$. $\mathcal{G}(P)$ is called the *gauge group* of P.

Lemma 22.3. *$\mathcal{G}(P)$ is equivalently the set of maps $\phi : P \to G$ for which $\phi(pg) = g^{-1}\phi(p)g$.*

This is easy and is left for HW. Taking the infinitesimal version, we have:

Proposition 22.4. *The tangent space to $\mathcal{G}(P)$ at $\phi \in \mathcal{G}(P)$ is $\Omega^0(M; Ad(P))$, i.e., $T_\phi\mathcal{G}(P) = \Omega^0(M; Ad(P))$.*

Proof. By "tangent space" we mean the space of suitable sections $\frac{d}{dt}\phi_t|_{t=0}$, where $\phi_0 = \phi$ and ϕ_t is a 1-parameter family of maps $P \to G$ satisfying $\phi_t(pg) = g^{-1}\phi_t(p)g$. Evaluating at $p \in P$, we have $\frac{d}{dt}\phi_t(p)|_{t=0}$, and this gives a map $\alpha : P \to \mathfrak{g}$ which satisfies $\alpha(pg) = Ad(g^{-1})\alpha(p)$. The section $x \in M \mapsto (p, \alpha(p))$, $p \in \pi^{-1}(x)$, is an element of $\Omega^0(M; Ad(P))$. □

We already showed that, locally, if we view ϕ as a function $U \to G$, then

$$A \mapsto \phi^{-1}A\phi + \phi^{-1}d\phi.$$

Warning: Note that by ϕ^{-1} we mean the function $U \to G$, $x \mapsto (\phi(x))^{-1}$. We will use this somewhat confusing notation to avoid more cumbersome notation.

We also have:

Proposition 22.5. *$F_A \mapsto Ad(\phi^{-1})F_A$.*

Proof. We will first give a proof, assuming that G is a matrix group.

$$\begin{aligned}
dA' + [A', A'] &= d(\phi^{-1}A\phi + \phi^{-1}d\phi) + [\phi^{-1}A\phi + \phi^{-1}d\phi, \phi^{-1}A\phi + \phi^{-1}d\phi] \\
&= (d\phi^{-1}) \wedge A\phi + \phi^{-1} \cdot dA \cdot \phi - \phi^{-1} \cdot A \wedge d\phi + d\phi^{-1} \wedge d\phi \\
&\quad + \phi^{-1} \cdot [A, A] \cdot \phi + [\phi^{-1}A\phi, \phi^{-1}d\phi] + [\phi^{-1}d\phi, \phi^{-1}A\phi] + [\phi^{-1}d\phi, \phi^{-1}d\phi] \\
&= Ad(\phi^{-1})(dA + [A, A]) \\
&\quad + (-\phi^{-1} \cdot d\phi \cdot \phi^{-1} \wedge A\phi - \phi^{-1} \cdot A \wedge d\phi + [\phi^{-1}A\phi, \phi^{-1}d\phi] + [\phi^{-1}d\phi, \phi^{-1}A\phi]) \\
&\quad + (-\phi^{-1} \cdot d\phi \cdot \phi^{-1} \wedge d\phi + [\phi^{-1}d\phi, \phi^{-1}d\phi]) \\
&= Ad(\phi^{-1})F_A
\end{aligned}$$

Here we use $d(\phi^{-1}) = -\phi^{-1} \cdot d\phi \cdot \phi^{-1}$ which arises from differentiating $\phi^{-1}\phi = e$. Also note that $[\eta_1, \eta_2](X_1, X_2) \overset{def}{=} [\eta_1(X_1), \eta_2(X_2)]$.

The proof in the general case follows along exactly the same lines. We just need to suitably interpret the expression $d(Ad(\phi^{-1}))$ as in:

$$d(Ad(\phi^{-1})A) = d(Ad(\phi^{-1})) \wedge A + Ad(\phi^{-1})dA.$$

Note that $Ad(\phi^{-1})$ is the composition $U \overset{\phi^{-1}}{\to} G \overset{Ad}{\to} Gl(\mathfrak{g})$. Therefore, $d(Ad(\phi^{-1}))$ is the composition of the two derivatives. Now, $T_xU \to T_{\phi^{-1}(x)}G$ clearly maps $X \mapsto d\phi^{-1}(x)(X)$. On the other hand, $T_gG \to \mathfrak{gl}(\mathfrak{g})$ maps $ge^{t\xi} \mapsto (\zeta \mapsto g[\xi, \zeta]g^{-1})$. Hence we can write:

$$\begin{aligned} d(Ad(\phi^{-1}))A &= Ad(\phi^{-1})([\phi \cdot d\phi^{-1}, A] + [A, \phi \cdot d\phi^{-1}]) \\ &= [-\phi^{-1}d\phi, Ad(\phi^{-1})A] + [Ad(\phi^{-1})A, -\phi^{-1}d\phi]. \end{aligned}$$

Note that the appearance of two term on the right-hand side is due to the fact that $[,]$ is not skew-symmetric by definition. $\square$

23. Chern-Weil theory

Goal: Assign invariants of principal G-bundles $P \to M$ or associated vector bundles E using the space of connections $\mathcal{A}(P)$.

Given a connection $A \in \Omega^1(M; Ad(P))$, we have the curvature $F_A \in \Omega^2(M; Ad(P))$. For today, assume that $\mathfrak{g}$ is a matrix Lie algebra.

One way of constructing real-valued forms is by taking traces:

$$tr(F_A^k) = tr(F_A \wedge F_A \wedge \cdots \wedge F_A) \in \Omega^{2k}(M; \mathbf{R}).$$

Here, if $\omega_i \in \Omega^1(M; \mathfrak{g})$, then we set

$$(\omega_1 \wedge \cdots \wedge \omega_k)(X_1, \dots, X_k) = \sum_{\text{permutations } \sigma} (-1)^\sigma \omega_1(X_{\sigma(1)}) \cdots \omega_k(X_{\sigma(k)}),$$

where the product on the right-hand side is matrix multiplication.

Theorem 23.1. *Let $A, A' \in \Omega^1(M; Ad(P))$ be two connection 1-forms. Then*

1. *$tr(F_A^k)$ is closed.*
2. *$tr(F_A^k) - tr(F_{A'}^k) = d\eta$.*

In other words, $[tr(F^k)] \in H^{2k}(M; \mathbf{R})$ is well-defined and independent of the choice of $A \in \mathcal{A}(P)$.

Proof. (1) First note that $tr((\phi^{-1} F_A \phi)^k) = tr(F_A^k)$ since the trace is invariant under conjugation. This means that a change of gauge does not alter $tr(F_A^k)$. Hence we may work locally!

The key identity in our computations is the Bianchi identity, written as follows when $\mathfrak{g}$ is a matrix Lie algebra:

$$\nabla F_A = dF_A + A \wedge F_A - F_A \wedge A = 0.$$

We now compute:

$$\begin{aligned} d(tr(F_A^k)) &= tr(dF_A \wedge F_A \wedge \cdots \wedge F_A) + tr(F_A \wedge dF_A \wedge F_A \wedge \cdots \wedge F_A) + \dots \\ &= k \cdot tr(dF_A \wedge F_A \wedge \cdots \wedge F_A) \\ &= k \cdot tr((-A \wedge F_A + F_A \wedge A) \wedge F_A \wedge \cdots \wedge F_A) \\ &= 0. \end{aligned}$$

(2) It suffices to prove that

$$\frac{d}{dt} tr(F_{A_t}^k)|_{t=0} = d(k \cdot tr(\dot{A} \wedge F_A \wedge \cdots \wedge F_A)),$$

where A_t is a family of connection 1-forms, $A_0 = A$, and $\frac{d}{dt}A_t|_{t=0} = \dot{A}$. On the one hand we have:

$$\begin{aligned} \frac{d}{dt}tr(F_{A_t}^k) &= tr\left(\left(\frac{d}{dt}F_{A_t}\right)\wedge F_A\wedge\cdots\wedge F_A\right) + tr\left(F_A\wedge\left(\frac{d}{dt}F_{A_t}\right)\wedge F_A\wedge\cdots\wedge F_A\right)+\dots \\ &= k\cdot tr((d\dot{A}+\dot{A}\wedge A+A\wedge\dot{A})\wedge F_A\wedge\cdots\wedge F_A) \end{aligned}$$

On the other hand:

$$\begin{aligned} d(tr(\dot{A}\wedge F_A\wedge\cdots\wedge F_A)) &= tr(d\dot{A}\wedge F_A\wedge\cdots\wedge F_A) - tr(\dot{A}\wedge dF_A\wedge F_A\wedge\cdots\wedge F_A) \\ &\quad - tr(\dot{A}\wedge F_A\wedge dF_A\wedge\cdots\wedge F_A) - \dots \end{aligned}$$

Now,

$$tr(\dot{A}\wedge dF_A\wedge F_A\wedge\cdots\wedge F_A) = tr(\dot{A}\wedge(-A\wedge F_A+F_A\wedge A)\wedge F_A\wedge\cdots\wedge F_A),$$
$$tr(\dot{A}\wedge F_A\wedge dF_A\wedge\cdots\wedge F_A) = tr(\dot{A}\wedge F_A\wedge(-A\wedge F_A+F_A\wedge A)\wedge\cdots\wedge F_A),$$

and, after cancellations, we have:

$$\begin{aligned} d(tr(\dot{A}\wedge F_A\wedge\cdots\wedge F_A)) &= tr(d\dot{A}\wedge F_A\wedge\cdots\wedge F_A) + tr(\dot{A}\wedge A\wedge F_A\wedge\cdots\wedge F_A) \\ &\quad - tr(\dot{A}\wedge F_A\wedge\cdots\wedge F_A\wedge A) \end{aligned}$$

By using the invariance of trace under cyclic permutation of the matrices, we obtain the desired equality. □

The cohomology classes are usually packaged as follows: Let $ch(P) = [tr(e^F)] \in H^*(M;\mathbf{R})$ the the *Chern character*. The ring generated by the components of the Chern character $ch(P)$ is called the *Pontryagin ring*. These are *characteristic classes* of the principal bundle P.

Example: Suppose $G = S^1$. Then $\mathfrak{g} = i\mathbf{R}$. Now, locally $F_A = dA + A\wedge A = dA$, since $\mathfrak{g}$ is abelian. In this case F_A is already a 1-form with values in $\mathbf{R}$, so we do not need to take traces. Clearly, F_A is locally exact, i.e., is closed. We take $[\frac{i}{2\pi}F_A] \in H^2(M;\mathbf{R})$. This is called the *first Chern class* $c_1(P)$.

24. Hodge Theory Preliminaries

Much of the material over the next several lectures is taken from Warner, *Foundations of differentiable manifolds and Lie groups.*

24.1. **Linear algebra.** Let V be a vector space over $\mathbf{R}$ of dimension n. Then a bilinear form $\phi : V \times V \to \mathbf{R}$,is *nondegenerate* if for any $v \in V$ nonzero, there exists $w \in V$ so that $\phi(v, w) \neq 0$. This is equivalent to the induced map $V \to V^*$, $v \mapsto (w \mapsto \phi(v, w))$, being an isomorphism.

From now on, V is equipped with a positive definite inner product $\langle \cdot, \cdot \rangle$ and a choice of orientation $e_1 \wedge \cdots \wedge e_n$.

Definition 24.1. *The* star operator $* : \wedge^k V \to \wedge^{n-k} V$ *is given as follows: Let $e_1, \dots, e_n$ be an oriented orthonormal basis for V. Then*

$$*(e_{i_1} \wedge \cdots \wedge e_{i_k}) = (-1)^\sigma e_{j_1} \wedge \cdots \wedge e_{j_{n-k}},$$

where $\sigma = (i_1, \dots, i_k, j_1, \dots, j_{n-k})$ and $(-1)^\sigma$ is the sign of σ (i.e., the number of transpositions required from $(1, \dots, n)$ to σ). Since $*$ is defined on a basis of $\wedge^k V$, it extends linearly to all of $\wedge^k V$.

In particular,

$$*1 = e_1 \wedge \cdots \wedge e_n,$$
$$*(e_1 \wedge \cdots \wedge e_n) = 1,$$
$$*(e_1 \wedge \cdots \wedge e_k) = e_{k+1} \wedge \cdots \wedge e_n.$$

Proposition 24.2. *$*$ is independent of the choice of oriented orthonormal basis. In other words, if $e'_1, \dots, e'_n$ is another oriented orthonormal basis, then*

$$*(e'_{i_1} \wedge \cdots \wedge e'_{i_k}) = (-1)^\sigma e'_{j_1} \wedge \cdots \wedge e'_{j_{n-k}}.$$

Remark: This proposition is remarkably hard to prove if one tries to prove it by brute force.

Proof. First define a linear map

$$\wedge^k V \times \wedge^k V \to \mathbf{R},$$
$$(v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k) \mapsto \det(\langle v_i, w_j \rangle),$$

where $(\langle v_i, w_j \rangle)$ is a $k \times k$ matrix with entries $\langle v_i, w_j \rangle$. (HW: Show that this is well-defined!) This linear map is the extension of the inner product $\langle , \rangle$ on V to $\wedge^k V$. This gives rise to a map $L : \wedge^k V \to (\wedge^k V)^*$ which sends $v_1 \wedge \cdots \wedge v_k$ to $L(v_1 \wedge \cdots \wedge v_k)$ which maps $w_1 \wedge \cdots \wedge w_k \mapsto \det(\langle v_i, w_j \rangle)$.

More concretely, $e_1 \wedge \cdots \wedge e_k \mapsto L(e_1 \wedge \cdots \wedge e_k)$, which sends $e_{i_1} \wedge \cdots \wedge e_{i_k} \mapsto 0$ if $\{i_1, \dots, i_k\} \neq \{1, \dots, k\}$, and $e_{i_1} \wedge \cdots \wedge e_{i_k} \mapsto (-1)^\sigma e_1 \wedge \cdots \wedge e_k$, if $\{i_1, \dots, i_k\} = \{1, \dots, k\}$. Here $\sigma = (i_1, \dots, i_k)$.

Now, there is a natural isomorphism $\phi : \wedge^{n-k} V \xrightarrow{\sim} (\wedge^k V)^*$ which comes from:

$$\wedge^k V \times \wedge^{n-k} V \to \wedge^n V \simeq \mathbf{R},$$

$$(\omega, \eta) \mapsto \omega \wedge \eta,$$

and the identification with $\mathbf{R}$ comes from the choice of an orientation. Here ϕ maps $\eta \mapsto (\omega \mapsto \eta \wedge \omega)$. The fact that the pairing $\omega \wedge \eta$ is *nondegenerate* ensures that ϕ is an isomorphism. Observe that $\phi(e_{k+1} \wedge \cdots \wedge e_n)$ is the same map as $L(e_1 \wedge \cdots \wedge e_k)$.

Finally, it is easy to see that $* = \phi^{-1} \circ L$. □

Remark: The previous proposition can be restated as saying that $*$ commutes with all $A \in SO(n)$.

The proof of the following lemma is left for HW.

Lemma 24.3. $** = (-1)^{n(n-k)} id$.

24.2. **The d^*-operator.** Let (M, g) be an oriented Riemannian manifold of dimension n. Then there exists a volume form $dvol(x) = e_1 \wedge \cdots \wedge e_n$, where $e_1, \dots, e_n$ is an oriented orthonormal basis for T_x and the Riemannian metric identifies $T_xM \simeq T_x^*M$, $v \mapsto (w \mapsto \langle v, w \rangle)$.

We define the operator $d^* : \Omega^k(M) \to \Omega^{k-1}(M)$ to be $(-1)^{n(k+1)+1} * d *$. Notice that while $d : \Omega^k(M) \to \Omega^{k+1}(M)$ raises the degree, the operator d^* lowers the degree.

We quickly verify that $d^* \circ d^* = 0$: $d^* \circ d^* = \pm(*d*)(*d*) = \pm * d^2 * = 0$, using $d^2 = 0$ and $** = \pm 1$. This means that we have a chain complex which goes in the opposite direction!

$$\dots \Omega^k(M) \stackrel{d^*}{\leftarrow} \Omega^{k+1}(M) \stackrel{d^*}{\leftarrow} \Omega^{k+2}(M) \dots.$$

What we want to investigate is the relationship between the two chain complexes d and d^*.

25. Hodge Theory, Day II

25.1. **The inner product on $\Omega^k(M)$.** Given $\alpha, \beta \in \Omega^k(M)$, we define an inner product

$$(,) : \Omega^k(M) \times \Omega^k(M) \to \mathbf{R},$$

$$(\alpha, \beta) = \int_M \alpha \wedge *\beta = \int_M \langle \alpha, \beta \rangle dvol.$$

Here $\langle \alpha, \beta \rangle$ is the pointwise inner product $\langle , \rangle : \wedge^k V \times \wedge^k V \to \mathbf{R}$ given in the proof of Proposition 24.2.

Proposition 25.1. $(,)$ *is positive definite and symmetric.*

Proof. It is easy to see that $(,)$ is positive definite: $(\omega, \omega) = \int_M \langle \omega, \omega \rangle dvol \geq 0$ and $= 0$ if and only if $\omega \geq 0$ (since ω is continuous, for example).

Next to show $(,)$ is symmetric, we first verify on the vector space level that

$$\langle \omega, \eta \rangle = \langle *\omega, *\eta \rangle.$$

(This is easy to see by taking an oriented orthonormal basis $e_1, \dots, e_n$ and computing $\langle e_{i_1} \wedge \dots \wedge e_{i_k}, e_{j_1} \wedge \dots \wedge e_{j_k} \rangle$.) Then:

$$\begin{aligned} (\omega, \eta) &= \int_M \langle \omega, \eta \rangle dvol = \int_M \langle *\omega, *\eta \rangle dvol \\ &= \int_M *\omega \wedge * * \eta = (-1)^{k(n-k)} \int_M *\omega \wedge \eta \\ &= (-1)^{k(n-k)}(-1)^{k(n-k)} \int_M \eta \wedge *\omega = (\eta, \omega) \end{aligned}$$

□

Remark: $(,)$ gives rise to a norm $\|\omega\| = \sqrt{(\omega, \omega)}$.

Remark: The inner product on $\Omega^k(M)$ gives it the structure of a *pre-Hilbert space.* If we complete the pre-Hilbert space, we obtain $L^2(\Gamma(\wedge^k T^* M))$, the L^2-sections of $\wedge^k T^* M$.

Proposition 25.2. d^* *is the formal adjoint of* d *in the* L^2*-sense, i.e.,* $(d\alpha, \beta) = (\alpha, d^*\beta)$. *Here* α *is a* k*-form and* β *is a* $(k+1)$*-form.*

Proof.

$$\begin{aligned}
(d\alpha, \beta) &= \int_M (d\alpha) \wedge *\beta \\
&= \int_M d(\alpha \wedge *\beta) - (-1)^k \alpha \wedge d * \beta \\
&= (-1)^{k+1} \int_M \alpha \wedge d * \beta \\
&= (-1)^{k+1}(-1)^{(n-k)k} \int_M \alpha \wedge *(*d * \beta) \\
&= (-1)^{nk+1}(-1)^{nk+1} \int_M \alpha \wedge *(d^* \beta) \\
&= (\alpha, d^*\beta)
\end{aligned}$$

□

25.2. **The Laplacian.**

Definition 25.3. *The* Laplacian *or* Laplace-Beltrami operator $\Delta : \Omega^k(M) \to \Omega^k(M)$ *is given by* $\Delta = dd^* + d^*d$*. A form* ω *is* harmonic *if* $\Delta\omega = 0$.

The Laplacian Δ on functions on $\mathbf{R}^n$ is the operator $\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$. Functions which satisfy $\Delta f = 0$ are said to be *harmonic*. Harmonic functions appear everywhere in mathematics. For example, if we view $\mathbf{C} = \mathbf{R}^2$, then the real and imaginary parts of a holomorphic function are harmonic.

Remark: If $M = \mathbf{R}^n$ and g is the flat metric, then we can compute that $\Delta(f\,dx_1 \wedge \cdots \wedge dx_k) = -(\Delta f) dx_1 \wedge \cdots \wedge dx_k$. (HW!)

Proposition 25.4. *The Laplacian* Δ *satisfies the following:*

1. $\Delta * = * \Delta$,
2. $\Delta d = d\Delta$,
3. $\Delta d^* = d^* \Delta$,
4. Δ *is self-adjoint, i.e.,* $(\Delta\omega, \eta) = (\omega, \Delta\eta)$.

Proof. These are all easy exercises. We'll do (1): Given a k-form ω,

$$\begin{aligned}
\Delta * \omega &= (dd^* + d^*d) * \omega = (-1)^{n(n-k+1)+1} d * d * * \omega + (-1)^{n(n-k)+1} * d * d * \omega, \\
&= (-1)^{k+1} d * d\omega + (-1)^{nk+n+1} * d * d * \omega \\
*\Delta\omega &= *(dd^* + d^*d)\omega = (-1)^{n(k+1)+1} * d * d * \omega + (-1)^{nk+1} * * d * d\omega \\
&= (-1)^{nk+n+1} * d * d * \omega + (-1)^{k+1} d * d\omega
\end{aligned}$$

□

We also have the following key proposition:

Proposition 25.5. $\Delta\omega = 0$ *if and only if* $d\omega = 0$ *and* $d^*\omega = 0$.

Proof. One direction is easy. We will do the other direction. Suppose $\Delta\omega = 0$. Then $(\Delta\omega, \omega) = (d\omega, d\omega) + (d^*\omega, d^*\omega)$. Both terms on the right-hand side are nonnegative and are zero iff $d\omega = 0$ and $d^*\omega = 0$. □

25.3. **Motivation for the Hodge Theorem.** Consider the set of closed forms $\mathcal{Z}^k(M) \subset \Omega^k(M)$. Then is there a "canonical" representative of a cohomology class which minimizes the norm, i.e., how do we minimize $\|\omega + d\eta\|$, where ω is closed and we are ranging over all $d\eta$?

For a minimum ω,

$$\frac{d}{dt}\|\omega + td\eta\|^2 = 0.$$

Differentiating, $(\omega, d\eta) = 0$, for all $\eta \in \Omega^{k-1}(M)$, or, equivalently, $(\eta, d^*\omega) = 0$. This would indicate that $d^*\omega = 0$. Thus, we would be obtaining a "canonical" representative for $H^k(M; \mathbf{R})$ by taking $d\omega = 0$, $d^*\omega = 0$, or, equivalently, $\Delta\omega = 0$.

26. Hodge Theory, Day III

26.1. **The Hodge Decomposition Theorem.** Denote by $\mathcal{H}^k$ the space of *harmonic* k-forms on M. Recall $\Delta\omega = 0$ is equivalent to $d\omega = d^*\omega = 0$. Then we have the following:

Theorem 26.1 (Hodge Decomposition)**.** *There exists an orthogonal decomposition*

$$\begin{aligned}\Omega^k(M) &= \Delta\Omega^k(M) \oplus \mathcal{H}^k \\ &= dd^*\Omega^k(M) \oplus d^*d\Omega^k(M) \oplus \mathcal{H}^k \\ &= d\Omega^{k-1}(M) \oplus d^*\Omega^{k+1}(M) \oplus \mathcal{H}^k.\end{aligned}$$

Here the orthogonality is with respect to the L^2-inner product $(,)$ on $\Omega^k(M)$.

Remark: Even though Ω^k is only a pre-Hilbert space, orthogonal decompositions make sense. Let V be a pre-Hilbert space with inner product $(,)$. Then $V = W_1 \oplus W_2$ means the following: any $v \in V$ can be written as $w_1 + w_2$, where $w_i \in W_i$ and $(w_1, w_2) = 0$. Then automatically w_1 and w_2 are unique. In fact, if $w_1 + w_2 = 0$, then $(w_1 + w_2, w_1) = \|w_1\|^2 = 0$, and $w_1 = 0$ by the positive definiteness. Similarly $w_2 = 0$.

Hence, the content of the Hodge Decomposition Theorem is that every $\omega \in \Omega^k(M)$ can be written as $\omega = \Delta\xi + \eta$, where $\eta \in \mathcal{H}^k$.

It then follows that $\omega = d(d^*\xi) + d^*(d\xi) + \eta$. We can check that $d\Omega^{k-1} \perp d^*\Omega^{k+1}$: if $\alpha \in \Omega^{k-1}$ and $\beta \in \Omega^{k+1}$, then $(d\alpha, d^*\beta) = (dd\alpha, \beta) = 0$. Similarly, $d\Omega^{k-1} \perp \mathcal{H}^k$ and $d^*\Omega^{k+1} \perp \mathcal{H}^k$. Therefore, the first line of Theorem 26.1 implies the third line.

Let us assume the Hodge Decomposition Theorem for the time being. We then present some consequences. Let $\Pi : \Omega^k(M) \to \mathcal{H}^k$ be the orthogonal projection onto $\mathcal{H}^k$.

Theorem 26.2. $\mathcal{H}^k \simeq H^k(M; \mathbf{R})$.

Proof. Consider $[\omega] \in H^k(M; \mathbf{R})$. $[\omega]$ is the set of k-forms $\omega + d\alpha$, $\alpha \in \Omega^{k-1}$. It is easy to see that the set of closed forms is L^2-orthogonal to $d^*\Omega^{k+1}$: if α is a closed k-form and $\beta \in \Omega^{k+1}$, then $(\alpha, d^*\beta) = (d\alpha, \beta) = 0$. Hence, $\omega \in d\Omega^{k-1} \oplus \mathcal{H}^k$. Therefore $\Pi(\omega + d\alpha)$ is independent of the choice of $d\alpha$. Hence, we have a well-defined map $H^k(M; \mathbf{R}) \to \mathcal{H}^k$, $[\omega] \mapsto \Pi(\omega)$. Of course there is an inverse map $\mathcal{H}^k \to H^k(M; \mathbf{R})$, $\omega \mapsto [\omega]$. This proves the theorem. □

Theorem 26.3 (Poincaré Duality)**.** $H^{n-k}(M; \mathbf{R}) \simeq (H^k(M; \mathbf{R}))^*$.

Proof. By identifying $H^k(M; \mathbf{R}) \simeq \mathcal{H}^k$, we have the following commutative diagram:

$$\begin{array}{ccc} H^k(M;\mathbf{R}) \times H^{n-k}(M;\mathbf{R}) & \xrightarrow{\overline{\phi}} & \mathbf{R} \\ \downarrow & & \downarrow \\ \mathcal{H}^k \times \mathcal{H}^{n-k} & \xrightarrow{\phi} & \mathbf{R} \end{array}$$

where

$$\overline{\phi}([\omega],[\eta]) = \int_M \omega \wedge \eta$$

(show it does not depend on the choice of representatives), and

$$\phi(\omega,\eta) = \int_M \omega \wedge \eta.$$

For ϕ, ω, η are harmonic representatives. Now, if ω is harmonic, then so is $*\omega$ (check!). This means that $\phi(\omega, *\omega) = \|\omega\|^2$, and hence ϕ is *nondegenerate*. This implies that $\mathcal{H}^{n-k} \simeq (\mathcal{H}^k)^*$. □

26.2. **Examples.**

Example: Let $M = S^1 = \mathbf{R}/\mathbf{Z}$ with coordinate x and $g = dx \otimes dx$. Then $\Delta f = -\frac{\partial^2}{\partial x^2} f$, since g is the standard flat metric when lifted to $\mathbf{R}$. Although solutions of $\Delta f = 0$ are $f = cx + d$, if we require the solution to be smooth on S^1, then we must have $f = d$. Therefore, we have $\mathcal{H}^0 = \mathbf{R}$ and $\mathcal{H}^1 = \mathbf{R}\{dx\}$. Moreover, the *eigenfunctions* of Δ are $f_n(x) = \cos 2\pi nx$ and $g_n(x) = \sin 2\pi nx$. One of the fundamental facts about Fourier series is that every continuous function can be written as $c_0 + \sum_{n=1}^{\infty}(c_n \cos 2\pi nx + d_n \sin 2\pi nx)$.

Example: Let $M = T^n = \mathbf{R}^n/\mathbf{Z}^n$ with coordinates $x_1, \dots, x_n$ and the flat metric $g = \sum_i dx_i \otimes dx_i$. If f is a function, then $\Delta f = d^* df$, so if $\Delta f = 0$, then $(\Delta f, f) = (df, df) = 0$, which implies $df = 0$, and hence f is constant. Since Δ on forms is simply Δ on each coefficient of $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$,

$$\mathcal{H}^k \simeq \wedge^k \mathbf{R}^n.$$

27. Sobolev spaces

Over the next several lectures we will prepare the background for the proof of the Hodge Decomposition Theorem. For a while our ambient manifold will be $T^n = \mathbf{R}^n/\mathbf{Z}^n$.

27.1. **Fourier series.** Let $L^2(T^n)$ be the $\mathbf{C}$-valued L^2-functions on T^n, namely functions f which satisfy $\int_{T^n} |f|^2 dx < \infty$. Here $dx = dx_1 \dots dx_n$. $L^2(T^n)$ is a Hilbert space with inner product

$$(f,g)_{L^2} = \int_{T^n} f\overline{g}dx.$$

Also define $l^2 = L^2(\mathbf{Z}^n)$ to be the set of functions $u : \mathbf{Z}^n \to \mathbf{C}$ satisfying $\sum_{\xi\in\mathbf{Z}^n} |u(\xi)|^2 < \infty$. l^2 is a Hilbert space with inner product

$$(u,v)_{l^2} = \sum_{\xi\in\mathbf{Z}^n} u(\xi)\overline{v(\xi)}.$$

Check that $L^2(T^n)$ and l^2 are *complete*!

Theorem 27.1 (Parseval's Theorem)**.** *The map*

$$\hat{}: L^2(T^n) \to l^2,$$
$$f(x) \mapsto \hat{f}(\xi) = \int_{T^n} f(x)e^{-2\pi i x\cdot\xi}dx$$

is a unitary *isomorphism of Hilbert spaces, i.e.,* $\|f\|_{L^2} = \|\hat{f}\|_{l^2}$.

Note that $\hat{f}(\xi)$ are the *Fourier coefficients* of f.

Proof. First observe that $\{e^{2\pi i x\cdot\xi}\}_{\xi\in\mathbf{Z}^n}$ is an orthonormal set – it is easy to verify that

$$(e^{2\pi i x\cdot\xi}, e^{2\pi i x\cdot\xi'})_{L^2} = \int_{T^n} e^{2\pi i x\cdot(\xi-\xi')}dx = \delta_{\xi,\xi'},$$

where $\delta_{\xi,\xi'} = 0$ if $\xi \neq \xi'$ and $\delta_{\xi,\xi} = 1$.

We claim that any $f \in C^0(T^n)$ can be written uniquely as:

$$f(x) = \sum_{\xi\in\mathbf{Z}^n} \hat{f}(\xi)e^{2\pi i x\cdot\xi},$$

with $\hat{f} \in l^2$. By the Stone-Weierstraß theorem, there exists a sequence $P_N(x)$ of trigonometric polynomials (finite linear combination of $e^{2\pi i x\cdot\xi}$) which converges to $f(x)$ in the C^0 norm. Hence it follows that $\|f - P_N\|_{L^2} < \varepsilon$ for large N. Let $f_n = \sum_{|\xi|\leq n} \hat{f}(\xi)e^{2\pi i x\cdot\xi}$. (Here $|\xi| = \sqrt{\xi\cdot\xi}$.) Then,

$$\|f - f_n\|^2_{L^2} = (f - f_n, f - f_n)_{L^2} = \|f\|^2_{L^2} - \|f_n\|^2_{L^2},$$

by noting that $(f, f_n)_{L^2} = \|f_n\|^2_{L^2}$. Also,

$$\|f - P_N\|^2_{L^2} = \|f|^2_{L^2} + (-f_n, P_N)_{L^2} - (P_N, f_n)_{L^2} + \|P_N\|^2_{L^2} \tag{6}$$

by noting that $(f_n, P_N)_{L^2} = (f, P_N)_{L^2}$, provided n is sufficiently large. Now, since $0 \leq \|f_n - P_N\|_{L^2}$, it follows that $\|f - f_n\|_{L^2} \leq \|f - P_N\|_{L^2} \leq \varepsilon$. Therefore, $f_n \to f$ in L^2, and $\|f_n\|^2_{L^2} = \sum_{|\xi|\leq n} |\hat{f}(x)|^2 \to \|f\|^2_{L^2}$. This proves $\|f\|_{L^2} = \|\hat{f}\|_{l^2}$ for C^0-functions.

Now, since the C^0-functions are dense in $L^2(T^n)$, it suffices to approximate $f \in L^2(T^n)$ by $h \in C^0$. Take $\|f - h\|_{L^2} < \varepsilon$, and let f_n and h_n be the Fourier series up to $|\xi| \leq n$. Then, for large n, $\|h - h_n\|_{L^2} < \varepsilon$. Also,

$$\|f_n - h_n\|^2_{L^2} = (f_n - h_n, f - h)_{L^2} \leq \|f_n - h_n\|_{L^2}\|f - h\|_{L^2},$$

by the Cauchy-Schwarz inequality, which implies $\|f_n - h_n\|_{L^2} \leq \|f - h\|_{L^2}$. Hence,

$$\|f - f_n\|_{L^2} \leq \|f - h\|_{L^2} + \|h - h_n\|_{L^2} + \|h_n - f_n\|_{L^2} \leq 3\varepsilon,$$

and $f_n \to f$ in L^2. □

Therefore, the Fourier series allows us to switch between $L^2(T^n)$ and l^2.

27.2. **Sobolev spaces.** Let $Map(\mathbf{Z}^n, \mathbf{C})$ be the set of functions from $\mathbf{Z}^n$ to $\mathbf{C}$. Then we define the *Sobolev space* $H_s(T^n)$ (or simply H_s) to be the subset of $Map(\mathbf{Z}^n, \mathbf{C})$ consisting of $u : \mathbf{Z}^n \to \mathbf{C}$ for which

$$\sum_{\xi \in \mathbf{Z}^n} (1 + |\xi|^2)^s |u(\xi)|^2 < \infty.$$

The *inner product on* H_s is given by

$$(u, v)_s = \sum_{\xi \in \mathbf{Z}^n} (1 + |\xi|^2)^s u(\xi)\overline{v(\xi)},$$

and the corresponding Sobolev s-norm is written $\|u\|_s$. Just as l^2 is a Hilbert space, H_s is also a Hilbert space (with different weights). (Check completeness!)

Remark: $H_0 = l^2$.

28. Properties of Sobolev spaces

Much of the discussion can be found in Folland, *Introduction to Partial Differential Equations* (although they use Fourier transforms instead of Fourier series), or Griffiths-Harris, *Principles of Algebraic Geometry.*

28.1. **Alternate definition.** First let us introduce the following terminology:

1. $\alpha = (\alpha_1, \dots, \alpha_n)$, where $\alpha_i \geq 0$.
2. $|\alpha| = \alpha_1 + \cdots + \alpha_n$.
3. $\partial_j = \frac{\partial}{\partial x_j}$.
4. $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$.
5. $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}$.

Claim. *Let $f \in C^\infty(T^n)$. Then $\hat{f}$ satisfies $\sum_{\xi \in \mathbf{Z}^n} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 < \infty$ for all s.*

Proof. Since $f \in C^\infty(T^n)$, $\partial^\alpha f \in C^\infty(T^n)$ for all α. This implies, by the compactness of T^n, that $\partial^\alpha f \in L^2(T^n)$. Using the isomorphism $L^2(T^n) \simeq l^2$, we compute $\|\partial^\alpha f\|_{L^2}$ by taking $\|\widehat{\partial^\alpha f}\|_{l^2}$.

Integrating by parts, we obtain:

$$\widehat{\partial_j f} = \int_{T^n} (\partial_j f) e^{2\pi i x \cdot \xi} dx = -\int f(\partial_j e^{2\pi i x \cdot \xi}) = -\int 2\pi i \xi_j f e^{2\pi i x \cdot \xi} dx = -2\pi i \xi_j \hat{f}.$$

In other words, under the Fourier transform, ∂_j gets taken to (up to a constant factor) multiplication by ξ_j. Similarly, $\widehat{\partial^\alpha f} = (-2\pi i)^{|\alpha|} \xi^\alpha \hat{f}$. Then $\|\widehat{\partial^\alpha f}\|_{l^2} = (2\pi)^{|\alpha|} \sum_\xi |\xi^{2\alpha}| |\hat{f}(\xi)|^2 < \infty$. Taking a suitable combination, we obtain $\sum_{\xi \in \mathbf{Z}^n} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 < \infty$ for all positive integers s. □

Notation: To avoid repeatedly writing constants, we define $D_j = \frac{-\partial_j}{2\pi i}$ and $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$.

Definition 28.1. *The Sobolev space $H_s(T^n)$ is the completion of $C^\infty(T^n)$ with respect to the norm $\|f\|_s = \|\widehat{\Lambda^s f}\|_{l^2}$, where*

$$\widehat{\Lambda^s f}(\xi) \stackrel{def}{=} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2.$$

This definition makes sense, because $f \in C^\infty(T^n) \Rightarrow \|f\|_s < \infty$ by the Claim.

We now explain the weights $(1 + |\xi|^2)^s$.

Lemma 28.2. *For a positive integer s, the Sobolev norm $\|f\|_s$ is equivalent to $\sum_{|\alpha| \leq s} \|D^\alpha f\|_{L^2}$.*

Proof. Equivalence of norms means the existence of $c, C > 0$ such that

$$c\|f\|_s \leq \sum_{|\alpha| \leq s} \|D^\alpha f\|_{L^2} \leq C\|f\|_s.$$

Indeed,

$$\begin{aligned} \|f\|_s^2 &= \sum_\xi (1+|\xi|^2)^s |\hat{f}(\xi)|^2 \\ &\leq C\sum_\xi \sum_{|\alpha|\leq s} |\xi^\alpha|^2 |\hat{f}(\xi)|^2 \\ &= C\sum_{|\alpha|\leq s} \|D^\alpha f\|_0^2 \\ &\leq C(\sum_{|\alpha|\leq s} \|D^\alpha f\|_0)^2 \end{aligned}$$

On the other hand,

$$\begin{aligned} \|D^\alpha f\|_{L^2}^2 &= \sum_\xi |\xi^\alpha|^2 |f(\xi)|^2 \\ &\leq C\sum_\xi |1+|\xi|^2|^s |f(\xi)|^2 \\ &= C\|f(\xi)\|^2 \end{aligned}$$

Therefore, if s is a positive integer, then $f \in H_s$ is equivalent to $D^\alpha f \in L^2(T^n)$ for all derivatives up to order s. $\square$

29. Properties of Sobolev spaces

29.1. **Properties of Sobolev spaces.**

1. If $t < s$, then the inclusion $H_s \subset H_t$ is continuous, i.e., $\|f\|_t \leq \|f\|_s$.

2. Consider the map $\Lambda^t : H_s \to H_{s-t}$ given (in terms of Fourier series) by $\hat{f}(\xi) \mapsto (1+|\xi|^2)^{t/2}\hat{f}(\xi)$. Then $\|f\|_s = \|\Lambda^t f\|_{s-t}$ (i.e., Λ^t is a unitary isomorphism).

3. H_{-s} is naturally isomorphic to the dual of H_s via the pairing
$$\langle f, g\rangle = (\Lambda^s f, \Lambda^{-s} g),$$
where $f \in H_s$ and $g \in H_{-s}$. This easily follows from the Cauchy-Schwarz inequality.

4. $D^\alpha : H_s \to H_{s-|\alpha|}$ is bounded, i.e., $\|D^\alpha f\|_{s-|\alpha|} \leq \|f\|_s$. This is because
$$(1+|\xi|^2)^{s-|\alpha|}|\xi^\alpha|^2|\hat{f}(\xi)|^2 \leq (1+|\xi|^2)^s|\hat{f}(\xi)|^2.$$

5. If $f \in C^k(T^n)$, then $f \in H_k(T^n)$. That $\|f\|_k \leq C\|f\|_{C^k}$ follows immediately from the proof of the Claim from last time.

29.2. **Sobolev Embedding Theorem.**

The converse of (5) is the following theorem:

Theorem 29.1 (Sobolev Embedding)**.** *If $s > k + \frac{n}{2}$, then $H_s \subset C^k$.*

Proof. We will first show that $H_s \subset C^0$ if $s > \frac{n}{2}$. Given $f \in H_s$, we show that the partial sums $f_N = \sum_{|\xi|\leq N} \hat{f}(\xi)e^{2\pi i\xi\cdot x}$ converges to f in the C^0-norm. For this we look at the remainder:
$$\begin{aligned}\left|\sum_{|\xi|>N}\hat{f}(\xi)e^{2\pi i\xi\cdot x}\right| &\leq \sum_{|\xi|>N}|\hat{f}(\xi)| \\ &= \sum_{|\xi|>N}|(1+|\xi|^2)^{s/2}\hat{f}(\xi)|\cdot|(1+|\xi|^2)^{-s/2}| \\ &= \sqrt{\sum_{|\xi|>N}(1+|\xi|^2)^s|\hat{f}(\xi)|^2}\sqrt{\sum_{|\xi|>N}\frac{1}{(1+|\xi|^2)^s}} \\ &\leq \|f\|_s\sqrt{\sum_{\xi}\frac{1}{(1+|\xi|^2)^s}}.\end{aligned}$$

Therefore, it remains to bound $\sum_\xi \frac{1}{(1+|\xi|^2)^s}$. Using the integral test, the sum converges if and only if $\int_{\mathbf{R}^n}\frac{1}{(1+|x|^2)^s}$ converges. Now,
$$\int_{\mathbf{R}^n}(1+|x|^2)^{-s} = \omega_n\int_0^\infty (1+r^2)^{-s}r^{n-1}dr,$$

where r is the radial coordinate and ω_n is the volume of the unit $S^{n-1} \subset \mathbf{R}^n$. If $s > \frac{n}{2}$, then the integral on the right-hand side converges since the integrand will have degree $-2s + (n-1) < -1$ in r.

In general, the proof works in the same way – by bounding

$$\left| \sum_{|\xi|>N} \xi^\alpha \hat{f}(\xi) e^{2\pi i \xi \cdot x} \right|,$$

with $|\alpha| \leq k$, the corresponding series converges when $s > k + \frac{n}{2}$. □

This implies that if $f \in H_s$ for all s, then $f \in C^\infty$.

Hence, the systems of seminorms

$$\cdots \leq \|f\|_s \leq \|f\|_{s+1} \leq \ldots$$

and

$$\cdots \leq \|f\|_{C^k} \leq \|f\|_{C^{k+1}} \leq \ldots$$

give rise to equivalent topologies on C^∞.

29.3. **Rellich lemma.**

Lemma 29.2 (Rellich Lemma)**.** *If $t < s$, then the inclusion map $H_s \to H_t$ is compact, i.e., if a sequence f_i which is bounded with respect to $\|\cdot\|_s$ admits a convergent subsequence with respect to $\|\cdot\|_t$.*

Proof. Let f_i be a bounded sequence in H_s; without loss of generality, $\|f_i\|_s \leq 1$. Then we write:

$$\begin{aligned} \|f_i - f_j\|_t &= \sum_\xi (1+|\xi|^2)^t |\hat{f}_i(\xi) - \hat{f}_j(\xi)|^2 \\ &= \sum_\xi (1+|\xi|^2)^s |\hat{f}_i(\xi) - \hat{f}_j(\xi)|^2 (1+|\xi|^2)^{t-s} \\ &= \sum_{|\xi|>N} + \sum_{|\xi|\leq N} (1+|\xi|^2)^s |\hat{f}_i(\xi) - \hat{f}_j(\xi)|^2 (1+|\xi|^2)^{t-s}. \end{aligned}$$

For the $|\xi| > N$ terms, provided N is large, we can bound the sum by $\|f_i - f_j\|_s^2 (N^2+1)^{t-s}$, which can be made arbitrarily small. Next, the $|\xi| \leq N$ terms are finite in number, and we bound each term in the sum by taking a subsequence of f_i for which $|\widehat{f}_i(\xi) - \widehat{f}_j(\xi)| \to 0$. This is possible because $(1+|\xi|^2)^t |\widehat{f}_i(\xi) - \widehat{f}_j(\xi)|^2$ is bounded above by 1 for each ξ and hence $|\widehat{f}_i(\xi) - \widehat{f}_j(\xi)|$ is bounded. This proves the existence of a Cauchy subsequence of f_i with respect to the $\|\cdot\|_t$-norm. □

30. Elliptic Differential Operators

Today our ambient manifold M is not necessarily a torus.

30.1. **Definitions.** A *linear differential operator* $L : C^\infty(\mathbf{R}^n) \to C^\infty(\mathbf{R}^n)$ is of the form

$$L = \sum_{|\alpha| \le m} a^\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha},$$

with $a^\alpha(x) \in C^\infty(\mathbf{R}^n)$. We say L has *order* m if $a^\alpha \neq 0$ for some $|\alpha| = m$.

Define the *symbol* of the differential operator L:

$$\sigma(L, \xi) = \sum_{|\alpha|=m} a^\alpha(x)\xi^\alpha.$$

Then L is *elliptic* at $x \in \mathbf{R}^n$ if $\sigma(L,\xi)$ is nonzero at x for all real $\xi \neq 0$.

Examples:

1. $L = \frac{\partial}{\partial x}$ on $\mathbf{R}$. $\sigma(L,\xi) = \xi_x$, so L is elliptic.
2. $L = \Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ on $\mathbf{R}^n$ is elliptic, since $\sigma(L,\xi) = \xi_1^2 + \dots \xi_n^2 = 0$ iff $\xi = 0$.
3. $L = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$ on $\mathbf{R}^2$. There are nonzero real solutions to $\sigma(L,\xi) = \xi_x^2 - \xi_y^2 = 0$, so L is not elliptic. L is a *hyperbolic* differential operator.
4. $L = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_{n-1}^2}$ on $\mathbf{R}^n$ is not elliptic.

30.2. **Generalization.** Let M be an n-dimensional manifold and E and F be vector bundles over M of rank e and f, respectively. Then a *linear differential operator* $L : \Gamma(E) \to \Gamma(F)$ of order m is given locally by

$$L = \sum_{|\alpha| \le m} A^\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha} : C^\infty(\mathbf{R}^n, \mathbf{R}^e) \to C^\infty(\mathbf{R}^n, \mathbf{R}^f),$$

where $A^\alpha(x)$ is a $e \times f$-matrix-valued function (or alternatively $A^\alpha(x) \in Hom(E_x, F_x)$). The symbol is

$$\sigma(L, \xi) = \sum_{|\alpha|=m} A^\alpha(x)\xi^\alpha.$$

We say L is *elliptic* at x if $\sigma(L,\xi)$ is invertible at x for all $\xi \neq 0$ with real coefficients. In particular, this implies that $rk(E) = rk(F)$.

A more intrinsic interpretation of the symbol: The symbol $\sigma(L)$ is a function which for each $\xi \in T_x^*M$ defines a linear map $\sigma(L,\xi) : E_x \to F_x$.

Claim. *Let $x \in M$, $\xi \in T_x^*M$, and $v \in E_x$. Choose f in a neighborhood of x such that $f(x) = 0$ and $df(x) = \xi$, and choose a section of E near x such that $s(x) = v$. Then $\sigma(L,\xi)(v) = L(\frac{f^m}{m!}s)|_x \in E_x$.*

HW: Check that $L(\frac{f^m}{m!}s)$ is independent of the choices of f and s.

Examples: Using the claim, it is easy to see the following:

1. If $d : \Omega^k(M) \to \Omega^{k+1}(M)$, then
$$\sigma(d,\xi) : \wedge^k T_x^* M \to \wedge^{k+1} T_x^* M,$$
$$\omega_0 \mapsto \xi \wedge \omega_0.$$
In fact, if $\omega(x) = \omega_0$, then
$$\sigma(d,\xi)(\omega_0) = d(f\omega)(x) = (df \wedge \omega + f d\omega)(x) = \xi \wedge \omega_0.$$
Let $e(\xi)$ denote the operation of wedging by ξ.
2. $\sigma(d^*,\xi) = \pm * e(\xi) *$.
3. If Δ is the Laplace-Beltrami operator, then
$$\sigma(\Delta,\xi) = \pm 2((-1)^n e(\xi) * e(\xi) * + * e(\xi) * e(\xi)).$$
HW: Show that this indeed simplifies to $\sigma(\Delta,\xi) = \pm 2|\xi|^2$. Therefore, Δ is elliptic.
4. Consider $L = \frac{\partial}{\partial \overline{z}} : \mathbf{C} \to \mathbf{C}$, given by:
$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$
where we have identified $\mathbf{C} \simeq \mathbf{R}^2$ via $z \mapsto (x = Re(z), y = Im(z))$. $\sigma(L,\xi)$ is an isomorphism for all $\xi \neq 0$, so L is elliptic.

30.3. **Elliptic complexes.** We can further generalize the notion of ellipticity. We say a complex
$$\mathcal{E} : 0 \to \Gamma(E_0) \xrightarrow{D} \Gamma(E_1) \xrightarrow{D} \dots ,$$
with $D^2 = 0$ is *elliptic* if for each $\xi \in T_x^* M$, the associated symbol sequence
$$0 \longrightarrow E_{0,x} \xrightarrow{\sigma(D,\xi)} E_{1,x} \xrightarrow{\sigma(D,\xi)} \dots$$
is an exact sequence of vector spaces.

HW: Verify that the de Rham complex
$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \dots$$
is an elliptic complex.

If we write $\Box = DD^* + D^*D$, then Hodge theory works for $\mathcal{E}$, i.e.,
$$H^k(\mathcal{E}) = \mathcal{H}^k(\mathcal{E}),$$
where the right-hand side is the set of solutions to $\Box : \Gamma(E_k) \to \Gamma(E_k)$.

31. The Basic Estimate

We return to the torus T^n. The goal today is to prove the following theorem:

Theorem 31.1 (Basic Estimate). *Let $L = \sum_{|\alpha|\leq l} a^\alpha(x)\frac{\partial}{\partial x^\alpha}$ be a linear elliptic operator of order l on T^n with $a^\alpha(x) \in C^\infty(T^n)$, and let s be an integer. Then, given $f \in H_{s+l}$, we have*

$$\|f\|_{s+l} \leq C(\|Lf\|_s + \|f\|_s),$$

where C does not depend on f.

We may intuitively think of the purpose of the Basic Estimate as follows:

1. Apart from the term $\|f\|_s$ on the right-hand side, the estimate says that L^{-1} is a bounded operator.
2. If $Lf = 0$, then $\|f\|_{s+l} \leq C\|f\|_s$. Roughly speaking, this says that if $f \in H_s$, then $f \in H_{s+l}$. Hence the Basic Estimate allows us to 'improve' on the smoothness.

These ideas will be made more precise in later lectures.

31.1. Preliminaries.

We gather some lemmas which is needed in the proof of the Basic Estimate.

Lemma 31.2. *Suppose $a \in C^\infty(T^n)$. Then for all integers s, given $f \in C^\infty(T^n)$, we have:*

$$\|af\|_s \leq C|a|_{C^0}\|f\|_s + C'\|f\|_{s-1},$$

where C does not depend on a but C' does. Here C and C' do not depend on f.

Proof. Suppose s is a positive integer. Then

$$\|af\|_s \leq C\sum_{|\alpha|\leq s}\|D^\alpha(af)\|_0,$$

using the equivalence of the norms $\|\cdot\|_s$ and $\sum_{|\alpha|\leq s}\|D^\alpha\cdot\|_0$. Now, since

$$D^\alpha(af) = a(D^\alpha f) + \text{ expression involving lower order derivatives of } f,$$

we have

$$\|af\|_s \leq C|a|_{C^0}\sum_{|\alpha|=s}\|D^\alpha f\|_0 + C'\sum_{|\alpha|<s}\|D^\alpha f\|_0,$$

which proves the lemma for s positive.

Now, if s is negative integer, by using the operator Λ^s we try to get back to the case already considered.

$$\begin{aligned}\|af\|_s^2 &= (a\Lambda^{-2s}\Lambda^{2s}f, \Lambda^{2s}af)_0\\ &= (\Lambda^{-2s}(a\Lambda^{2s}f), \Lambda^{2s}af)_0 - \left(\left(\sum_{|\alpha|<-2s}a^\alpha D^\alpha\right)\Lambda^{2s}f, \Lambda^{2s}af\right)_0,\end{aligned}$$

by switching the order of integration. Next,

$$\begin{aligned}(\Lambda^{-2s}(a\Lambda^{2s}f),\Lambda^{2s}af)_0 &= (a\Lambda^{2s}f,\Lambda^{2s}af)_{-s}\\ &\leq \|a\Lambda^{2s}f\|_{-s}\|\Lambda^{2s}af\|_{-s}\\ &\leq (C|a|_{C^0}\|f\|_s + C'\|f\|_{s-1})\|af\|_s.\end{aligned}$$

On the other hand, for $|\alpha| < -2s$,

$$\begin{aligned}(a^\alpha D^\alpha(\Lambda^{2s}f),\Lambda^{2s}af)_0 &\leq \|a^\alpha D^\alpha(\Lambda^{2s}f)\|_s\|\Lambda^{2s}af\|_{-s}\\ &\leq C'\|\Lambda^{2s}f\|_{s+|\alpha|}\|af\|_s\\ &\leq C'\|\Lambda^{2s}f\|_{-s-1}\|af\|_s\\ &\leq C'\|f\|_{s-1}\|af\|_s.\end{aligned}$$

Dividing both sides by $\|af\|_s$ gives the desired inequality. □

Corollary 31.3. *Let $a \in C^\infty(T^n)$. Then the multiplication $f \mapsto af$ is a bounded map $H_s \to H_s$.*

Lemma 31.4 (Peter-Paul Inequality)**.** *If $r < s < t$, then $\|f\|_s \leq \varepsilon\|f\|_t + C\|f\|_r$ for all $f \in H_t$. Here ε is arbitrarily small.*

Proof. Follows from the inequality:

$$(1+|\xi|^2)^{s-r} \leq \varepsilon(1+|\xi|^2)^{t-r} + C,$$

provided we are allowed to take C very large. □

31.2. **Proof of the Basic Estimate.** We will do this in steps.

31.2.1. $L = \Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$. Then

$$\begin{aligned}(1+|\xi|^2)^{s+2}|\hat{f}(\xi)|^2 &= (1+|\xi|^2)^s(1+2|\xi|^2+|\xi|^4)|\hat{f}(\xi)|^2\\ &= C((1+|\xi|^2)^s|\hat{f}(\xi)|^2 + (1+|\xi|^2)^s|\xi|^4|\hat{f}(\xi)|^2).\end{aligned}$$

Summing over ξ, we obtain

$$\|f\|_{s+2}^2 \leq C(\|Lf\|_s^2 + \|f\|_s^2) \leq C(\|Lf\|_s + \|f\|_s)^2.$$

This implies that:

$$\|f\|_{s+2} \leq C(\|Lf\|_s + \|f\|_s).$$

31.2.2. *L is elliptic with constant coefficients and without lower order terms.* In other words, $L = \sum_{|\alpha|=l} a^\alpha \frac{\partial}{\partial x^\alpha}$ with a^α constant. Then the symbol $\sigma(L,\xi) = \sum_{|\alpha|=l} a^\alpha \xi^\alpha$ is a homogeneous polynomial in $\xi_1, \dots, \xi_n$ and is never zero, provided $\xi \neq 0$. Hence, on the $(n-1)$-sphere $|\xi| = 1$, $\sigma(L,\xi)$ is bounded below by some $C > 0$ by compactness. The homogeneity of $\sigma(L,\xi)$ implies that:

$$|\sigma(L,\xi)| > C|\xi|^l$$

for all $\xi \neq 0$ in $\mathbf{R}^n$. Therefore we have:

$$\begin{aligned} (1+|\xi|^2)^{s+l} &\leq (1+|\xi|^2)^s(1+c_1|\xi|^2+c_2|\xi|^4+\cdots+|\xi|^{2l}) \\ &\leq C(1+|\xi|^2)^s(1+|\xi|^{2l}) \leq C(1+|\xi|^2)^2(1+|\sigma(L,\xi)|^2). \end{aligned}$$

Finally, multiplying by $|\hat{f}(\xi)|^2$ and summing over ξ gives us the desired inequality.

31.2.3. *General L; f with small support.* We will prove the Basic Estimate for f with small support near $x_0 \in T^n$. Then we 'freeze' L at x_0 to obtain $L_0 = \sum a^\alpha(x_0)\frac{\partial}{\partial x^\alpha}$. Hence $L - L_0$ has highest order coefficients which are bounded by a small ε. Now, we have:

$$\begin{aligned} \|f\|_{s+l} &\leq C(\|L_0 f\|_s + \|f\|_s) && (7) \\ &\leq C(\|Lf\|_s + \|(L-L_0)f\|_s + \|f\|_s). && (8) \end{aligned}$$

Using Lemma 31.2,

$$\|(L-L_0)f\|_s \leq \varepsilon\|f\|_{s+l} + C\|f\|_{s+l-1},$$

where ε can be made arbitrarily small. Using the Peter-Paul inequality to 'tame' $\|f\|_{s+l-1} \leq \varepsilon\|f\|_{s+l} + C\|f\|_s$ and moving $2C\varepsilon\|f\|_{s+l}$ to the left-hand side of Equation 7, we obtain the desired inequality.

31.2.4. *General L and f.* Cover T^n with small balls U_i centered about $x_1, \dots, x_k$. Choose a partition of unity $\{\phi_i\}$ subordinate to U_i. Then:

$$\begin{aligned} \|f\|_{s+l} &= \|\sum_i \phi_i f\|_{s+l} \leq \sum_i \|\phi_i f\|_{s+l} \\ &\leq C\sum_i(\|L(\phi_i f)\|_s + \|\phi_i f\|_s) \\ &\leq C\sum_i(\|\phi_i L(f)\|_s + \|[L,\phi_i]f\|_s + \|\phi_i f\|_s). \end{aligned}$$

Now, using Lemma 31.2, the fact that $[L,\phi_i]$ is a differential operator of order $l-1$, and the Peter-Paul inequality, we obtain

$$\|f\|_{s+l} \leq C(\|Lf\|_s + \|f\|_s).$$

32. Regularity

We are still working on the torus T^n. Let L be an elliptic differential operator $C^\infty(T^n) \to C^\infty(T^n)$ of order l. Recall the Basic Estimate: given $f \in H_{s+l}$, we have:

$$\|f\|_{s+l} \leq C(\|Lf\|_s + \|f\|_s).$$

Today we prove one consequence, namely the following:

Theorem 32.1 (Elliptic Regularity). *If $g \in H_t$ and $f \in H_s$ satisfies $Lf = g$, then $f \in H_{t+l}$.*

Usually we start with $f \in H_s$ with $s < t + l$ and *bootstrap* up to $f \in H_{t+l}$. In particular,

Corollary 32.2. *If $f \in H_s$ satisfies $Lf = 0$, then $f \in C^\infty$.*

Proof. Since $0 \in C^\infty$, it is in all H_s. Therefore, $f \in H_{s+l}$ for all s, and is in C^∞ by the Sobolev Embedding Theorem. □

32.1. **Difference quotients.** In order to 'improve' the differentiability of a function, we use the method of *difference quotients*. Given $f \in H_s$, define the difference quotient

$$\Delta_h f = \frac{f(x+h) - f(x)}{|h|},$$

where $h \in T^n$ is nonzero. The difference quotient is intended to be a substitute for actually taking derivatives.

We compute

$$\widehat{f(x+h)}(\xi) = \int_{T^n} f(x+h)e^{-2\pi\xi\cdot x}dx = e^{2\pi i\xi\cdot h}\hat{f}(\xi).$$

This show that the difference quotient $\Delta_h f$ is also in H_s. Now,

$$\widehat{\Delta_h f}(\xi) = \frac{e^{2\pi i\xi\cdot h} - 1}{|h|}\hat{f}(\xi).$$

Lemma 32.3. $\|\Delta_h f\|_s \leq C\|f\|_{s+1}$, *where C does not depend on h.*

Proof. Follows from:

$$\begin{aligned} \left|\frac{e^{2\pi i\xi\cdot h} - 1}{|h|}\right|^2 &= \frac{(\cos(2\pi\xi\cdot h) - 1)^2 + \sin^2(2\pi\xi\cdot h)}{|h|^2} \\ &= \frac{2 - 2\cos(2\pi\xi\cdot h)}{|h|^2} = \frac{4\sin^2(\pi\xi\cdot h)}{|h|^2} \\ &= 4|\pi\xi|^2 \leq C|\xi|^2 \end{aligned}$$

□

Lemma 32.4. *Suppose $f \in H_s$. If for all $h \neq 0$, $\|\Delta_h f\|_s \leq C$ (C independent of h), then $f \in H_{s+1}$.*

Proof. Let $h = (0, \ldots, h_i, \ldots, 0)$, where h_i is in the ith position. Then, as $h_i \to 0$, we have

$$\left|\frac{e^{2\pi i\xi\cdot h} - 1}{|h|}\right|^2 \to \frac{|2\pi\xi_i h_i|^2}{|h_i|^2} = 4\pi^2|\xi_i|^2.$$

Therefore, since

$$\sum_{|\xi|\leq N} (1+|\xi|^2)^s \left|\frac{e^{2\pi i\xi\cdot h} - 1}{|h|}\right|^2 |\hat{f}(\xi)|^2 \leq C,$$

we obtain

$$\sum_{|\xi|\leq N} (1+|\xi|^2)^s |\xi_i|^2 |\hat{f}(\xi)|^2 \leq C',$$

and finally

$$\sum_{|\xi|\leq N} (1+|\xi|^2)^{s+1} |\hat{f}(\xi)|^2 \leq C' + \|f\|_s^2.$$

This implies that the partial sums converge to f in H_{s+1} □

32.2. **Proof of regularity.** We now prove the regularity theorem. It will be shown that if $f \in H_s$ and $g = Lf \in H_{s-l+1}$, then $f \in H_{s+1}$.

$$\|\Delta_h f\|_s \leq C(\|L(\Delta_h f)\|_{s-l} + \|\Delta_h f\|_{s-l}) \tag{9}$$

$$\leq C\left(\|\Delta_h(Lf)\|_{s-l} + \left\|\frac{L(x+h)-L(x)}{|h|}f(x+h)\right\|_{s-l} + \|\Delta_h f\|_{s-l}\right), \tag{10}$$

by noting that

$$\begin{aligned}\Delta_h(Lf)(x) &= \frac{(Lf)(x+h)-(Lf)(x)}{|h|} \\ &= \frac{L(x+h)f(x+h) - L(x)f(x+h)}{|h|} + L(\Delta_h f).\end{aligned}$$

Now, $\frac{L(x+h)-L(x)}{|h|}$ is uniformly bounded, so

$$\left\|\frac{L(x+h)-L(x)}{|h|}f(x+h)\right\|_{s-l} \leq C'\|f(x+h)\|_s = C'\|f\|_s. \tag{11}$$

Also, by Lemma 32.3,

$$\|\Delta_h(Lf)\|_{s-l} \leq C'\|Lf\|_{s-l+1}, \tag{12}$$

$$\|\Delta_h f\|_{s-l} \leq C'\|f\|_{s-l+1} \leq C'\|f\|_s. \tag{13}$$

Combining Equations 11, 12, and 13 with Equation 9, we obtain an upper bound for $\|\Delta_h f\|_s$ which is independent of h. Finally, by Lemma 32.4, $f \in H_{s+1}$.

33. Globalization

At this point we remark that everything we did so far for $H_s(T^n)$ applies to vector-valued functions on the torus, i.e., to $H_s(T^n, \mathbf{C}^r)$. (Try to go back and check some of the lemmas or theorems!)

Today we deal with an arbitrary elliptic differential operator $L : \Gamma(M; E) \to \Gamma(M; F)$, where M is a compact manifold without boundary, and E, F are vector bundles over M of (the same) rank r. For explicit details, you may go see the Appendix on linear elliptic PDE's in the book *Complex Manifolds and Deformation of Complex Structures* by Kodaira.

33.1. **Definition of $H_s(M; E)$.** We define the Sobolev spaces of sections of a vector bundle. The definition will depend on several noncanonical choices. First pick a fiberwise inner product $\langle , \rangle$ on E. Then $E \to M$ admits a local trivialization $U \times \mathbf{C}^r \to U$ (first projection) where $\langle , \rangle$ becomes the standard Hermitian inner product on $\mathbf{C}^r$. Cover M with finitely many sufficiently small balls U_i which admit the local trivializations, and let $\{\phi_i\}$ be a partition of unity subordinate to $\{U_i\}$. Take a diffeomorphism $\Psi_i : U_i \to T^n$ onto its image, i.e., *graft* U_i into T^n. (For convenience we assume that Ψ_i extends to a diffeomorphism of the closure closure $\overline{U}_i$ onto its image in T^n.) $\phi_i\eta$ has support inside U_i, and we denote by $(\phi_i\eta) \circ \Psi_i^{-1}$ the map $T^n \to \mathbf{C}^r$ which equals $(\phi_i\eta) \circ \Psi_i^{-1}$ on $\Psi(U_i)$ and is zero on $T^n - \Psi(U_i)$. We define a global section η of $E \to M$ to be in $H_s(M; E)$ if $(\phi_i\eta) \circ \Psi_i^{-1} \in H_s(T^n, \mathbf{C}^n)$ for all i.

Moreover, since $\eta = \sum_i \phi_i\eta$, we define:

$$\|\eta\|_s = \sum_i \|(\phi_i\eta) \circ \Psi_i^{-1}\|_s.$$

Remark: The Sobolev s-norm depends on the choices made above: (1) the fiberwise inner product, (2) the open cover and partition of unity, and (3) the grafting diffeomorphism Ψ_i. However, as we will see, two Sobolev s-norms $\|\eta\|_s$ and $\|\eta\|'_s$ obtained from different choices of (1), (2), and (3), are *equivalent*, i.e., there exist nonzero constants c, C such that $c\|\eta\|_s \leq \|\eta\|'_s \leq C\|\eta\|_s$.

(1) Choice of inner product. Two inner products on $U_i \times \mathbf{C}^r$ differ by a positive definite Hermitian matrix-valued function A on U_i (i.e., the fiberwise inner products are $\langle \cdot, \cdot \rangle$ vs. $\langle \cdot, A \cdot \rangle$). Now, $\|(\phi_i\eta) \circ \Psi_i^{-1}\|'_s$ and $\|(\phi_i\eta) \circ \Psi_i^{-1}\|_s$ (which differ only by A and have the same (2) and (3)) are equivalent for the same reason that multiplication by a smooth function a is a bounded map $H_s \to H_s$.

(2) Choice of open cover and partition of unity. Consider partitions of unity $\{\phi_i\}$ on $\{U_i\}$ and $\{\psi_j\}$ on $\{V_j\}$. We show the equivalence of $\sum_i \|\phi_i\eta\|_s$ and $\sum_j \|\psi_j\eta\|_s$. Here we write $\phi_i\eta$ to stand for $(\phi_i\eta) \circ \Psi^{-1}$, etc. We have:

$$\|\phi_i\eta\|_s = \|\sum_j \phi_i\psi_j\eta\|_s \leq \sum_{i,j} \|\phi_i\psi_j\eta\|_s \leq C\|\phi_i\eta\|_s,$$

where the last inequality follows from the boundedness of multiplication by a smooth function. This shows that the norms given by $\{\phi_i\}, \{U_i\}$ and $\{\phi_i\psi_j\}, \{U_i \cap V_j\}$ are equivalent. The norms for ψ_j and $\phi_i\psi_j$ are also equivalent, so $\sum_i \|\phi_i\eta\|_s$ and $\sum_j \|\psi_j\eta\|_s$ are equivalent.

(3) Choice of diffeomorphism Ψ_i. Two diffeomorphisms $\Psi_i, \Psi'_i : U_i \to T^n$ differ by a diffeomorphism $\Phi : T^n \to T^n$ such that $\Phi \circ \Psi_i = \Psi'_i$. Then

$$\int_{T^n} |(\phi_i\eta) \circ (\Psi'_i)^{-1}|^2 dx = \int_{T^n} |(\phi_i\eta) \circ \Psi_i^{-1} \circ \Phi^{-1}|^2 dx,$$

and comparing with

$$\int_{T^n} |(\phi_i\eta) \circ \Psi_i^{-1} \circ \Phi^{-1}|^2 |d\Phi^{-1}| dx = \int_{T^n} |(\phi_i\eta) \circ \Psi_i^{-1}|^2 dx,$$

we see that the two norms $\|(\phi_i\eta) \circ \Psi_i^{-1}\|_s$ and $\|(\phi_i\eta) \circ (\Psi'_i)^{-1}\|_s$ are equivalent.

33.2. **Properties of $H_s(M; E)$.** All the standard properties of Sobolev spaces (for example the Sobolev embedding theorem, the Rellich lemma, and difference quotients) carry over to the situation of sections of vector bundles. Here we will derive the global version of the Basic Estimate. Let L be an elliptic differential operator of order l. Then

$$\|\eta\|_{s+l} \leq C(\|L\eta\|_s + \|\eta\|_s).$$

Writing $\phi_i\eta$ to stand for $(\phi_i\eta) \circ \Psi^{-1}$, we compute:

$$\begin{aligned}
\|\eta\|_{s+l} &= \sum \|\phi_i\eta\|_{s+l} \\
&\leq C\sum(\|L(\phi_i\eta)\|_s + \|\phi_i\eta\|_s) \\
&\leq C\sum(\|\phi(L\eta)\|_s + \|[L, \phi_i]\eta\|_s + \|\phi_i\eta\|_s) \\
&\leq C(\|L\eta\|_s + \|\eta\|_{s+l-1} + \|\eta\|_s),
\end{aligned}$$

where the commutator $[L, \phi_i]$ is a differential operator of order $l - 1$. (Check this!) Now, using the Peter-Paul inequality, $\|\eta\|_{s+l-1} \leq \varepsilon\|\eta\|_{s+l} + C'\|\eta\|_s$, and choosing ε sufficiently small, we can bring $C\varepsilon\|\eta\|_{s+l}$ to the left-hand side to obtain the desired Basic Estimate.

34. Elliptic Operators are Fredholm

A bounded linear operator $L : V \to W$ between Banach spaces is *Fredholm* if $ker(L)$ is finite-dimensional and $Im(L)$ is a closed subspace of finite codimension. Today's lecture is devoted to the proof of the following theorem:

Theorem 34.1. *Let s be a nonnegative integer and let L be an elliptic differential operator $\Gamma(M;E) \to \Gamma(M;F)$ of order l. Then $L : H_{s+l} \to H_s$ is a Fredholm operator.*

HW: Verify that Theorem 34.1 implies the Hodge Decomposition Thereom.

Note that L is bounded from the standard properties of Sobolev spaces. Recall the Basic Estimate:

$$\|\eta\|_{s+l} \leq C(\|L\eta\|_s + \|\eta\|_s).$$

Also recall the following:

Theorem 34.2 (Regularity)**.** *If $\eta \in H_s$ for some s and $L\eta = 0$, then $L \in C^\infty$.*

We now prove the first part of the Fredholm theory.

Theorem 34.3. *$ker(L) \subset H_{s+l}$ is finite-dimensional.*

Proof. Take the unit ball of $ker(L)$ viewed as $\subset H_s$. Then if $ker(L)$ is ∞-dimensional, there exists a sequence $\eta_1, \eta_2, \dots$ in $ker(L)$, where $(\eta_i, \eta_j)_s = \delta_{ij}$. By the Basic Estimate, $\|\eta_i\|_{s+l}$ is bounded for all i. Now, by the Rellich Lemma, $H_{s+l} \hookrightarrow H_s$ is compact, so $\eta_1, \dots$ has a convergent subsequence in H_s, which is a contradiction. $\square$

Combined with the elliptic regularity, this shows that the kernel of $L : \Gamma(M;E) \to \Gamma(M;F)$ is finite-dimensional.

Theorem 34.4. *$Im(L) \subset H_s$ is closed and $Im(L) = (ker(L^*))^\perp \cap H_s$.*

Here $\perp$ refers to L^2-orthgonality – this means we have fixed a fiberwise inner product. Also, L^* is the *formal adjoint* of L, i.e., it is a linear differential operator which satisfies $(L\phi, \psi)_0 = (\phi, L^*\psi)_0$ for all $\phi, \psi \in C^\infty$. It is left to the reader to verify that if L is of order l, then L^* is of order l, and if L is elliptic, then L^* is elliptic. (Moreover, $\sigma(L^*, \xi)$ is the adjoint of $\sigma(L, \xi)$ as fiberwise linear maps.) Also note that the Laplace-Beltrami operator is self-adjoint, i.e., $\Delta^* = \Delta$.

Proof. If $\psi \in Im(L)$ $(\psi = L\phi)$ and $\omega \in ker(L^*)$, then

$$(\psi, \omega)_0 = (L\phi, \omega)_0 = (\phi, L^*\omega)_0 = 0,$$

so $\psi \in (ker(L^*))^\perp$. Note that elements of $ker(L^*)$ are smooth and $ker(L^*)$ is finite-dimensional since L^* is elliptic.

To prove the converse, we start with the following lemma:

Lemma 34.5. *If $\psi \in (ker(L^*))^\perp \cap H_{s+l}$, then $\|\psi\|_{s+l} \leq C\|L^*\psi\|_s$ for some constant C which does not depend of ψ.*

Proof of lemma. We argue by contradiction. Suppose there exists a sequence $\psi_i \in H_{s+l} \cap (ker(L^*))^\perp$ such that $L^*\psi_i \to 0$ in H_s, yet $\|\psi_i\|_{s+l} = 1$. By the Rellich Lemma, there exists a Cauchy subsequence which we still call ψ_i in H_s. Now, according to the Basic Estimate,

$$\|\psi_i - \psi_j\|_{s+l} \leq C(\|L^*\psi_i - L^*\psi_j\|_s + \|\psi_i - \psi_j\|_s),$$

where the right-hand side approaches zero. Hence $\psi_i \to \psi$ in H_{s+l}. Since $(ker(L^*))^\perp$ is closed, $\psi \in (ker(L^*))^\perp$. However, we also have $L^*\psi_i \to L^*\psi$ in H_s, i.e., $L^*\psi = 0$. Hence $\psi = 0$. This is a contradiction since we cannot have $\psi_i \to \psi = 0$, all the while $\|\psi_i\|_{s+l} = 1$. □

Proof of Theorem 34.1 to be continued next time....

35. Day 35

35.1. **Completion of proof of Thereom 34.1.** Recall that $L : H_{s+l} \to H_s$ is an elliptic operator of order l. We want show that $Im(L) \supset (ker(L^*))^\perp \cap H_s$. Let $\psi \in (ker(L^*))^\perp \cap H_s$.

Strategy: First show that there exists a solution $L(\mathcal{L}_\psi) = \psi$ where $\mathcal{L}_\psi$ is in some H_t for $t << 0$. Then, using elliptic regularity, show that $\mathcal{L}_\psi \in H_{s+l}$.

We proved that:

Lemma 35.1. *If $\eta \in (ker(L^*))^\perp \cap H_{t+l}$, then $\|\eta\|_{t+l} \leq C\|L^*\eta\|_t$.*

The solution $\mathcal{L}_\psi$ we are looking for is a bounded linear functional on some H_t. Since H_t and H_{-t} are duals, this implies that $\mathcal{L}_\psi \in H_{-t}$. We define

$$\mathcal{L}_\psi : L^*\eta \mapsto (\psi, \eta)_0,$$

where $\eta \in C^\infty$ is a test function. $\mathcal{L}_\psi$ is well-defined: suppose η, η' both satisfy $L^*\eta = L^*\eta'$. Then $L^*(\eta - \eta') = 0$, $\eta - \eta' \in ker(L^*)$, and $(\psi, \eta - \eta')_0 = 0$ since $\psi \in (ker(L^*))^\perp$.

Using Lemma 35.1, we obtain

$$|\mathcal{L}_\psi(L^*\eta)| = |(\psi, \eta)_0| \leq \|\psi\|_{-l}\|\eta\|_l \leq C\|L^*\eta\|_0.$$

Therefore, $\mathcal{L}_\psi$ is a bounded linear functional on $H_0 \cap Im(L^*)$. Now, using the Hahn-Banach theorem, we can extend $\mathcal{L}_\psi$ to a bounded linear functional on all of H_0; by duality, $\mathcal{L}_\psi \in H_0$. For test functions $\eta \in C^\infty$,

$$(L(\mathcal{L}_\psi), \eta)_0 = (\mathcal{L}_\psi, L^*\eta)_0 = (\psi, \eta)_0.$$

This shows that $L(\mathcal{L}_\psi) = \psi$ as elements in H_{-l}. Since $\psi \in H_s$, by elliptic regularity, $\mathcal{L}_\psi \in H_{s+l}$. Hence $\psi \in Im(L) \subset H_s$. □

We have proved:

Theorem 35.2. *$L : H_{s+l} \to H_s$ is* Fredholm.

Let us specialize to the case where $L = \Delta$, the Laplace-Beltrami operator. Noting that $\Delta^* = \Delta$, we have $\Delta(H_{s+l}(\Omega^k(M))) = (ker(\Delta))^\perp \cap H_s(\Omega^k(M))$. Passing to the C^∞ limit, we have the Hodge Decomposition Theorem:

$$\Omega^k(M) = \Delta(\Omega^k(M)) \oplus \mathcal{H}^k.$$

35.2. **Green's operator.** Let $\Pi : \Omega^k(M) \to \mathcal{H}^k$ be the orthogonal projection onto $\mathcal{H}^k$. We define the Green's operator $G : H_s(\Omega^k(M)) \to H_{s+2}(\Omega^k(M))$ as the map which sends $\alpha \mapsto \omega$, where $\Delta\omega = \alpha - \Pi(\alpha)$ and ω is the unique element in $(\mathcal{H}^k)^\perp$. The Hodge Decomposition Theorem implies that there exists an ω, and the uniqueness is due to the fact we are restricting to $(\mathcal{H}^k)^\perp$. $G|_{(\mathcal{H}^k)^\perp}$ is a bounded inverse to $\Delta|_{(\mathcal{H}^k)^\perp}$, since

$$\|\omega\|_{s+2} \leq C\|\Delta\omega\|_s,$$

for $\omega \in (\mathcal{H}^k)^\perp \cap H_{s+2}(\Omega^k(M))$, by Lemma 35.1.

36. Yang-Mills Equations

36.1. **Electromagnetism.** Let us write down Maxwell's equations in a vacuum on $\mathbf{R}^4$:

1. $\nabla \times E + \frac{\partial}{\partial t} B = 0$,
2. $\nabla \times B - \frac{\partial}{\partial t} E = 0$,
3. $\nabla \cdot B = 0$,
4. $\nabla \cdot E = 0$.

Here $B = (B_x, B_y, B_z)$ and $E = (E_x, E_y, E_z)$.

If we write

$$\begin{aligned} F &= B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy \\ &\quad + E_x dx \wedge dt + E_y dy \wedge dt + E_z dz \wedge dt, \end{aligned}$$

then

$$\begin{aligned} *F &= B_x dx \wedge dt + B_y dy \wedge dt + B_z dz \wedge dt \\ &\quad - E_x dy \wedge dz - E_y dz \wedge dx - E_z dx \wedge dy, \end{aligned}$$

The sign discrepancy arises from using the Hodge star with respect to the Minkowski metric $g = dt^2 - dx^2 - dy^2 - dz^2$. (Here, on an n-manifold, $*e_1 \wedge \cdots \wedge e_k = \eta_1 \ldots \eta_k e_{k+1} \wedge \cdots \wedge e_n$ if $(e_1, \ldots, e_n)$ is an oriented orthonormal basis and $g(e_i, e_i) = \eta_i = \pm 1$.) Then Maxwell's equations become

$$dF = 0, d^* F = 0.$$

Therefore, electromagnetism on a manifold is the same as Hodge theory on M.

Also note that there exists a potential function A such that $F = dA$, which combines the electromagnetic potential for E and the vector potential for B.

36.2. **Yang-Mills Functional.** In view of the above and the fact that in Hodge theory we are looking for critical points of the functional

$$\int_M \omega \wedge *\omega,$$

we can generalize to the setting that we now describe. Let $P \to M$ be a principal G-bundle and $\mathcal{A}(P)$ be the space of connections on P. Here we will conveniently take G to be a matrix Lie group. Then we define

$$YM : \mathcal{A}(P) \to \mathbf{R},$$
$$A \mapsto \int_M tr(F_A \wedge *F_A).$$

We now compute the derivative map. Although $\mathcal{A}(P)$ is an infinite-dimensional space, undaunted, we compute:

$$\frac{d}{dt} F_{A+t\eta}|_{t=0} = \frac{d}{dt}(d(A + t\eta) + (A + t\eta) \wedge (A + t\eta))|_{t=0} = d\eta + \eta \wedge A + A \wedge \eta = \nabla_A \eta.$$

Here we used the definition $\nabla_A \eta = d\eta + [A, \eta]$.

Now, we compute the critical points of the Yang-Mills functional:

$$\begin{aligned} \delta YM(A)(\eta) &= \int_M tr(\nabla_A \eta \wedge *F_A) + tr(F_A \wedge *\nabla_A \eta) \\ &= 2\int_M tr(\nabla_A \eta \wedge *F_A), \end{aligned}$$

and since

$$dtr(\eta \wedge *F_A) = tr(\nabla_A \eta \wedge *F_A) - tr(\eta \wedge \nabla_A * F_A),$$

we integrate by parts to get that $\nabla_A * F_A = 0$. We also have $\nabla_A F_A = 0$ (this is the Bianchi identity). Therefore, Hodge theory in this context is

$$\nabla_A F_A = 0,$$

$$\nabla_A^* F_A = 0.$$

36.3. **Yang-Mills moduli spaces.** We define the Yang-Mills moduli space as follows:

$$\mathcal{M}(P) = \{A \in \mathcal{A}(P) | \nabla_A F_A = 0, \nabla_A^* F_A = 0\}/\mathcal{G}(P),$$

where $\mathcal{G}(P)$ is the gauge group (the group of bundle automorphisms). This turns out to be a finite-dimensional space, but not necessarily a manifold, since you have quotient singularities.

Example: Let $G = S^1$ and identify $\mathfrak{g} = i\mathbf{R}$. Then we view $\mathcal{A}(P) \simeq \Omega^1(M; i\mathbf{R})$ (non-canonical identification – depends on a choice of fixed connection). We have $F_A = dA$, since $A \wedge A = 0$. Also, we may identify $\mathcal{G}(P) = Map(M, S^1)$. [Recall that $\mathcal{G}(P)$ is the set of maps $\phi : P \to G$ for which $\phi(pg) = g^{-1}\phi(p)g$. If $G = S^1$, then ϕ is constant on each fiber, and can be pushed down to M.]

We have:

$$\mathcal{M}(P) = \{A \in \Omega^1(M; i\mathbf{R}) | dF_A = 0, d * F_A = 0\}/Map(M, S^1).$$

This implies that F_A is harmonic. However, $F_A = dA$, and since $d\Omega^1 \perp \mathcal{H}^2$, we have $F_A = 0$. Thus,

$$\mathcal{M}(P) = \mathcal{Z}^1(M; i\mathbf{R})/Map(M, S^1),$$

where $\mathcal{Z}^1(M; i\mathbf{R})$ is the space of closed 1-forms (with values in $i\mathbf{R}$).

Next we analyze $Map(M, S^1)$. First, suppose $g \in Map(M, S^1)$ can be lifted to $\phi : M \to \mathbf{R}$, namely, $g = e^{i\phi}$. Since the action of $\mathcal{G}(P)$ is given by:

$$A \mapsto g^{-1}Ag + g^{-1}dg = A + g^{-1}dg,$$

and $g^{-1}dg = id\phi$, $\mathcal{M}(P)$ is a quotient of $\mathcal{Z}^1(M; i\mathbf{R})/d\Omega^0(M; i\mathbf{R}) = H^1(M; i\mathbf{R})$.

Now, from algebraic topology we have:

Claim. *If $[M, S^1]$ is the space of homotopy classes of maps from M to S^1, then $[M, S^1] \simeq H^1(M; \mathbf{Z})$.*

It is also easy to see that the maps which are liftable to $\phi : M \to \mathbf{R}$ are precisely the ones in the class $0 \in H^1(M; \mathbf{Z})$. Therefore,

$$\mathcal{M}(P) \simeq H^1(M; \mathbf{R})/H^1(M; \mathbf{Z}).$$

In other words, the moduli space is a torus of dimension $\dim H^1(M; \mathbf{R})$.

37. Day 37

37.1. **Anti-self-dual moduli spaces.** Let M be a 4-manifold, G be a matrix Lie group (such as $SU(2)$), $P \to M$ be a principal G-bundle, and $\mathcal{A}(P)$ be the space of connections. Recall the Yang-Mills equations:

$$\nabla_A F_A = 0, \nabla_A^* F_A = 0.$$

In dimension 4, the star operator $* : \bigwedge^2 T^*M \to \bigwedge^2 T^*M$ has the property that $*^2 = 1$. Hence we can decompose $\bigwedge^2 T^*M$ into ± 1 eigenspaces, i.e., $\bigwedge^* = \bigwedge^+ \oplus \bigwedge^-$, where $\bigwedge^\pm$ has rank three. We say ω is *self-dual* (SD) if $*\omega = \omega$ and *anti-self-dual* (ASD) if $*\omega = -\omega$.

HW: Write down a basis for $\bigwedge_x^+$ at the point $x \in M$. Do the same for $\bigwedge_x^-$.

We define the ASD moduli space:

$$\mathcal{M}_{ASD}(P) = \{A \in \mathcal{A}(P) \text{ irreducible and ASD}\}/\mathcal{G}(P).$$

Here ASD means that $*F_A = -F_A$, and a connection is *irreducible* if the holonomy group is all of G. Observe that since $\nabla_A F_A = 0$ is the Bianchi identity, $*F_A = -F_A$ implies that $\nabla_A * F_A = 0$. It is also not hard to see that the elements of the ASD moduli space minimize the Yang-Mills functional.

$\mathcal{M}_{ASD}(P)$ is usually not compact. Therefore we must compactify to get $\overline{\mathcal{M}}_{ASD}(P)$. Cohomology classes in $H^*(\overline{\mathcal{M}}_{ASD}(P))$ are the so-called *Donaldson invariants*. They distinguish some 4-manifolds which are homeomorphic but *not* diffeomorphic!

37.2. **Morse theory introduction.** For Morse theory, please refer to Milnor's *Morse Theory* for a wonderful account. We make a few preliminary definitions. Let M be a manifold and $f : M \to \mathbf{R}$ be a smooth map. Then $p \in M$ is a *critical point* of f if $df(p) = 0$. A critical point p is *nondegenerate* if the Hessian $(\frac{\partial^2 f}{\partial x_i \partial x_j}(p))$ is nonsingular. Here we have chosen coordinates $x_1, \dots, x_n$ about p. The *index* of a nondegenerate critical point is the number of negative eigenvalues when the Hessian is diagonalized. It is not hard to see that the Hessian remains nondegenerate under a change of coordinates, and the index does not depend on the choice of coordinates.

We look at the following motivating example:

Motivating Example: Consider the situation as in Figure 1. Here the function $f : M \to \mathbf{R}$ is a height function (say height in the z-direction). Then the critical points of f are p, q, r, s. We study $M^t = f^{-1}((-\infty, t])$ for regular values a, b, c given in Figure 1. Figure 2 gives the sequence of modifications as we go from M^a to M^b to M^c to the whole M.

Let e^k be a k-cell, i.e., a closed ball of dimension k. Then, homotopically, to get from M^a to M^b we attach a 1-cell e^1 onto M^a along ∂e^1, and similarly for M_b to M_c we attach e^1 along ∂e^1. Also, from M^c to M, we attach e^2 along ∂e^2. (We can also think of the process of getting from $\emptyset$ to M^a as attaching e^0.)

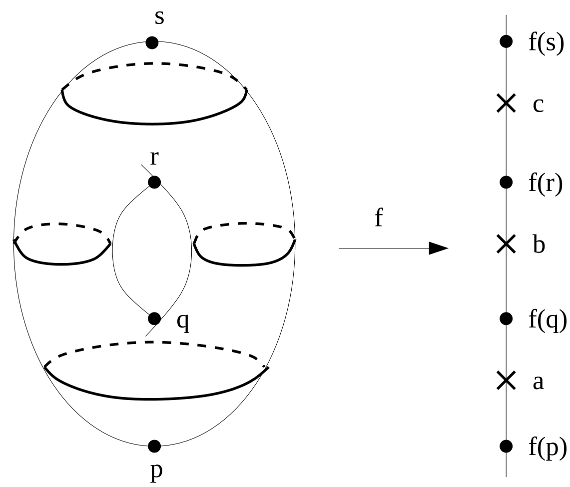

FIGURE 1

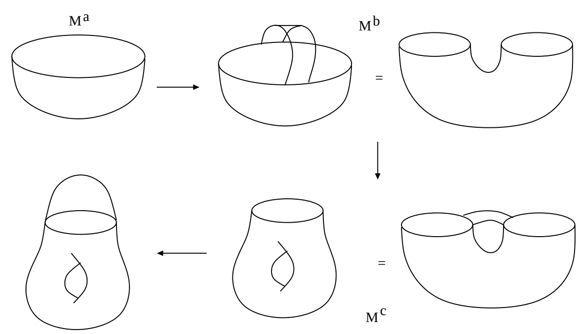

FIGURE 2

Now observe that near p, $f(x, y)$ can be written as $f(p) + x^2 + y^2$ (with $p = (0, 0)$). Similarly, near q and r, $f(x, y) = f(p) + x^2 - y^2$, and, near s, $f(x, y) = f(p) - x^2 - y^2$. Therefore to get past each critical point we are attaching a k-cell, where k is the index of the Hessian at the critical point.

Next time, we make some of these ideas more precise.

38. Morse Functions

Many assertions today will not be proven carefully. You are referred to Milnor's *Morse Theory.*

Lemma 38.1. *Let p be a nondegenerate critical point of f. Then there exist local coordinates $y_1, \dots, y_n$ near $p = 0$ such that*

$$f = f(p) - y_1^2 - \cdots - y_k^2 + y_{k+1}^2 + \cdots + y_n^2.$$

Proof. Proof omitted. □

Definition 38.2. *A function $f : M \to \mathbf{R}$ is a* Morse function *if*

1. *f is proper, i.e., $f^{-1}(K)$ is compact if K is compact, and*
2. *the critical points of f are nondegenerate.*

We will also assume for convenience that Morse functions also have the following property: (3) for each critical value there is a unique critical point. (Note that this is usually not included in the definition of a Morse function.) Recall that $t \in \mathbf{R}$ is a *critical value* if $f^{-1}(t)$ contains a critical point.

Theorem 38.3. *Every compact manifold without boundary admits a Morse function. (In fact, Morse functions are "dense" in the space of smooth functions.)*

Proof. The proof relies on transversality theory and Sard's Theorem. For an elementary proof, see Guillemin-Pollack, *Differential Topology.* □

Let $M^t = f^{-1}((-\infty, t])$. Then we have:

Proposition 38.4. *Suppose $a < b$. If there are no critical points of f on $f^{-1}([a, b])$, then M^a is diffeomorphic to M^b.*

Proof. We will show that M_a is a deformation retract of M_b, and a slight modification of the vector field involved will give the desired diffeomorphism. Consider the vector field ∇f defined by

$$\langle \nabla f, Y \rangle = df(Y),$$

pointwise, where $\langle , \rangle$ is the Riemannian metric. (Basically, ∇f is the dual of df given by $TM \stackrel{\sim}{\to} T^*M$ induced from the Riemannian metric.) ∇f is called the *gradient of f*.

Notice that $\nabla f \neq 0$ on all of $f^{-1}([a, b])$ since $df(\nabla f) = \langle \nabla f, \nabla f \rangle = \|\nabla f\|^2$, which is nonzero unless $df = 0$. Then set $X = -\phi(x) \frac{\nabla X}{\|\nabla f\|^2}$, where $\phi(x) = 1$ on $f^{-1}([a, b])$ and has support on $f^{-1}([a - \varepsilon, b])$. Then $df(X) = -1$ on $f^{-1}([a, b])$. X is complete vector field whose time $(b - a)$-flow maps $f^{-1}(b)$ diffeomorphically onto $f^{-1}(a)$ and M^b diffeomorphically onto M^a. □

Therefore it remains to see what happens as you go from $M^{c-\varepsilon}$ to $M^{c+\varepsilon}$, if c is a critical value of f.

Proposition 38.5. *Let $f: M \to \mathbf{R}$ be a Morse function and let $c \in \mathbf{R}$ be a critical value of f with corresponding critical point $p \in f^{-1}(c)$ of index k. Then, for sufficiently small $\varepsilon > 0$, $M^{c+\varepsilon}$ has the homotopy type of $M^{c-\varepsilon}$ with a k-cell attached.*

Sketch of proof. By the Morse Lemma, we can write $f = f(p) - y_1^2 - \cdots - y_k^2 + y_{k+1}^2 + \cdots + y_n^2$ near the critical p. Now look at Figure 3, which is a diagram of M near p. It is not difficult to see that $M^{c+\varepsilon}$ is obtained by attaching a $D^k \times D^{n-k}$ onto $M^{c-\varepsilon}$ along $(\partial D^k) \times D^{n-k}$, as can be seen by Figure 3. Homotopically, this is the same as attaching D^k along ∂D^k. The shaded region on the right-hand side diagram is the $D^k \times D^{n-k}$. To get that $M^{c+\varepsilon}$ is diffeomorphic to $M^{c-\varepsilon} \cup (D^k \times D^{n-k})$ (after rounding corners), we use the flow X which has been slightly modified near p as in the figure. □

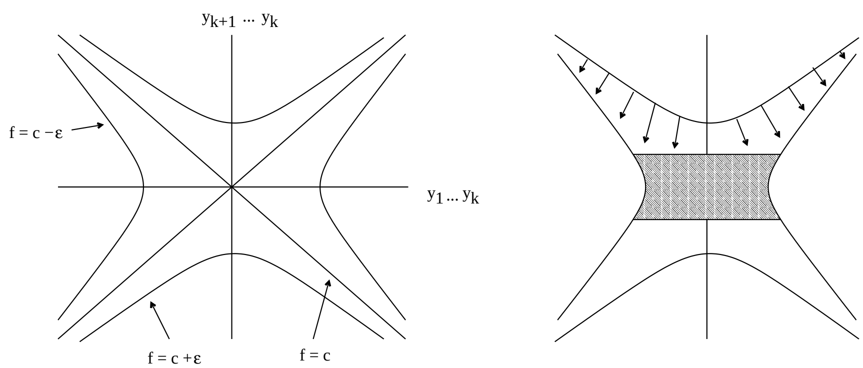

FIGURE 3

39. The Morse Inequalities

Let M be a compact manifold without boundary (usually such a manifold is called "closed") and $f : M \to \mathbf{R}$ be a Morse function. Define $M^t = f^{-1}((-\infty, t])$. Suppose $c_0 < c_1 < \cdots < c_n \in \mathbf{R}$ are the critical values with corresponding critical points $p_i \in M$. Last time we showed the following:

- If there are no critical values on $[t, t']$, then M^t is diffeomorphic to $M^{t'}$.
- If p_i is a critical point of index k_i, then $M^{c_i+\varepsilon}$ has the homotopy type of $M^{c_i-\varepsilon}$ with a k_i-cell attached.

Now, in order to reconstruct the homology of M, we use the following two results from algebraic topology (we will use $\mathbf{R}$ coefficients):

1. (Relative homology sequence)

$$\cdots \to H_j(M^{c_i-\varepsilon}) \to H_j(M^{c_i+\varepsilon}) \to H_j(M^{c_i+\varepsilon}, M^{c_i-\varepsilon}) \xrightarrow{\partial} H_{j-1}(M^{c_i-\varepsilon}) \to \dots$$

2. (Excision)

$$\begin{aligned} H_j(M^{c_i+\varepsilon}, M^{c_i-\varepsilon}) &= H_j(M^{c_i-\varepsilon} \cup e^{k_i}, M^{c_i-\varepsilon}) \\ &= H_j(e^{k_i}, \partial e^{k_i}) \\ &= \begin{cases} \mathbf{R} & \text{if } j = k_i, \\ 0 & \text{if } j \neq k_i. \end{cases} \end{aligned}$$

Hence, $H_j(M^{c_i-\varepsilon}) = H_j(M^{c_i+\varepsilon})$ unless $j = k_i$ or $j = k_i - 1$, and

$$0 \to H_{k_i}(M^{c_i-\varepsilon}) \to H_{k_i}(M^{c_i+\varepsilon}) \to H_{k_i}(M^{c_i+\varepsilon}, M^{c_i-\varepsilon}) = \mathbf{R} \to$$
$$\xrightarrow{\partial} H_{k_i-1}(M^{c_i-\varepsilon}) \to H_{k_i-1}(M^{c_i+\varepsilon}) \to 0.$$

Now, there are two cases:

- If $\partial \neq 0$, then
$$\dim H_{k_i}(M^{c_i-\varepsilon}) = \dim H_{k_i}(M^{c_i+\varepsilon}),$$
$$\dim H_{k_i-1}(M^{c_i-\varepsilon}) - 1 = \dim H_{k_i-1}(M^{c_i+\varepsilon}).$$
- If $\partial = 0$, then
$$\dim H_{k_i}(M^{c_i-\varepsilon}) + 1 = \dim H_{k_i}(M^{c_i+\varepsilon}),$$
$$\dim H_{k_i-1}(M^{c_i-\varepsilon}) = \dim H_{k_i-1}(M^{c_i+\varepsilon}).$$

Let $b_j = \dim H_j(M; \mathbf{R})$ and $c_j = \#$ critical points of index j. We then have the following:

Theorem 39.1 (Morse Inequalities, Weak Form)**.** $b_j \leq c_j$.

The above algebraic topology computation actually yields a better result. Set $b(t) = \sum_j b_j t^j$ (this is called the Poincaré polynomial) and $c(t) = \sum_j c_j t^j$. Then

Theorem 39.2 (Morse Inequalities, Strong Form)**.** $c(t) - b(t) = P(t)(1+t)$, *where* $P(t)$ *is a polynomial with nonnegative integer coefficients.*

Corollary 39.3. *The Euler characteristic* $\sum_j (-1)^j b_j$ *equals* $\sum_j (-1)^j c_j$.

40. Morse Theory from a Modern Viewpoint

Today we will reinterpret Morse theory from a more modern perspective. References for this material are Milnor, *Lectures on the h-cobordism Theorem*, and Schwarz, *Morse Homology.*

Let M be a closed manifold and $f: M \to \mathbf{R}$ be a Morse function. Let $C(f)$ be the set of critical points of f, and ∇f be the gradient of f (with respect to some Riemannian metric).

The goal is to define a chain complex (called the *Morse complex*) (C_*, ∂) out of the critical point data whose homology is $H_*(M;\mathbf{R})$. We define C_i to be the $\mathbf{R}$-vector space spanned by the index i critical points. In order to define the boundary map ∂, we consider *flow lines* of a generic $-\nabla f$ which begin and end at critical points. More precisely, we are looking for maps $u: \mathbf{R} \to M$ which satisfy:

$$\begin{cases} \frac{du}{dt} = -\nabla f(u(t)), \\ \lim_{t\to-\infty} u(t) = p, \\ \lim_{t\to\infty} u(t) = q, \end{cases}$$

where $p, q \in C(f)$. Given $p \in C(f)$ of index i, we define

$$\partial p = \sum_{q \text{ of index } i-1} (\#\text{ flow lines from } p \text{ to } q)\ q,$$

where the number of flow lines from p to q is counted *with sign.* We will not say more about how the sign is computed (see Schwarz's book), but simply note that if we wanted to do homology with $\mathbf{Z}/2\mathbf{Z}$ coefficients, then we don't have to worry about signs.

We will give examples of computations of $H_*(M;\mathbf{R})$.

1. Consider the left-hand side diagram of Figure 4. There are 4 critical points, p, q, r, s. We "compute" ∂s. s has index 2 and there are 2 flow lines to r and 2 flow lines to q. Since the flow lines emanate from s in opposite directions, the flow lines from s to r are assigned opposite signs; similarly for q. Therefore, $\partial s = 0$. Similarly, $\partial r = \partial q = 0$. Hence,

$$0 \to C_2 \to C_1 \to C_0 \to 0,$$

is given by:

$$0 \to \mathbf{R} = \mathbf{R}\{s\} \stackrel{\partial_2}{\to} \mathbf{R}^2 = \mathbf{R}\{r, q\} \stackrel{\partial_1}{\to} \mathbf{R} = \mathbf{R}\{p\} \to 0,$$

where $\partial_2 = \partial_1 = 0$. Therefore, the homology of the Morse complex is:

$$H_2(M) = \mathbf{R}, H_1(M) = \mathbf{R}^2, H_0(M) = \mathbf{R}.$$

2. Consider the right-hand side diagram of Figure 4. There are now 6 critical points, p, q, r, s, t, u. The Morse complex is:

$$0 \to \mathbf{R}^2 = \mathbf{R}\{u, s\} \stackrel{\partial_2}{\to} \mathbf{R}^3 = \mathbf{R}\{t, r, q\} \stackrel{\partial_1}{\to} \mathbf{R} = \mathbf{R}\{p\} \to 0,$$

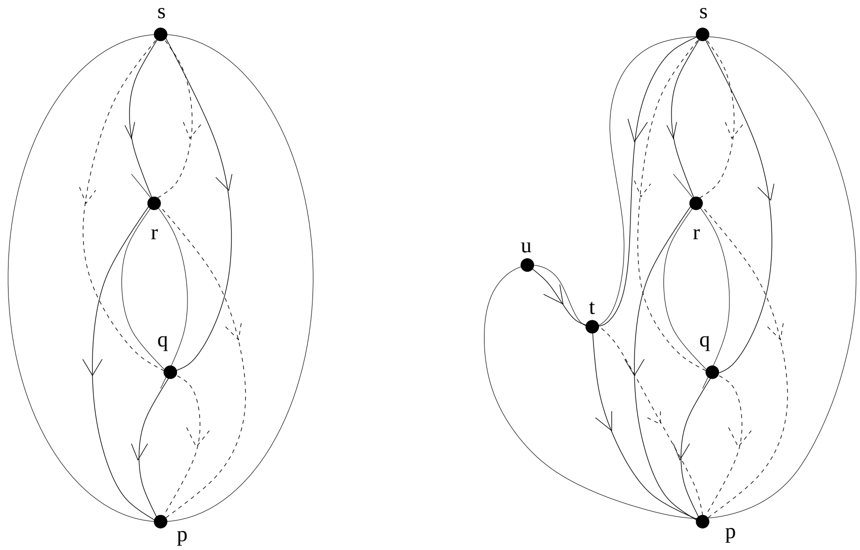

FIGURE 4

where $\partial_2 s = t$, $\partial_s u = t$, and the other boundary maps are zero. Therefore,

$$\begin{aligned} H_2(M) &= \ker \partial_2 = \mathbf{R}\{s - u\} = \mathbf{R}, \\ H_1(M) &= \ker \partial_1 / \mathrm{Im}\ \partial_2 = \mathbf{R}\{t, r, q\}/\mathbf{R}\{t\} = \mathbf{R}^2, \\ H_0(M) &= \mathbf{R}. \end{aligned}$$

41. Witten's approach to Morse theory

Today we give a rather informal account of Witten's paper *Supersymmetry and Morse theory*, J. Differential Geom. **17** (1982), 661–692.

41.1. **"Twisted" de Rham complex.** Let M be a closed, oriented n-dimensional manifold and $f : M \to \mathbf{R}$ a Morse function. We define the *twisted* de Rham operator $d_t = e^{-tf} \circ d \circ e^{tf}$, where $t \in \mathbf{R}$. Note that d_0 is the usual exterior derivative d. Our goal is to try to understand the behavior of d_t as $t \to +\infty$.

It is easy to verify that $(\Omega^*(M), d_t)$

$$0 \to \Omega^0(M) \stackrel{d_t}{\to} \Omega^1(M) \stackrel{d_t}{\to} \Omega^2(M) \stackrel{d_t}{\to} \dots,$$

is a chain complex:

$$d_t^2 = (e^{-tf} d e^{tf})(e^{-tf} d e^{tf}) = e^{-tf} d^2 e^{tf} = 0.$$

Lemma 41.1. $H^*(\Omega^i(M), d_t) \simeq H^*(M; \mathbf{R})$.

Proof. The isomorphism $\Omega^i(M) \to \Omega^i(M)$ given by $\omega \mapsto e^{-tf}\omega$ maps the d-closed forms isomorphically onto the d_t-closed forms and the d-exact forms onto the d_t-exact forms. □

Now let g be a Riemannian metric on M and $*$ be the corresponding star operator.

Lemma 41.2. *The adjoint d_t^* of d^* is $e^{tf} \circ d^* \circ e^{-tf}$.*

Note: d_t^* is NOT $e^{-tf} \circ d^* \circ e^{tf}$.

Proof.

$$\begin{aligned} (d_t\alpha, \beta)_{L^2} &= (e^{-tf} d(e^{tf}\alpha), \beta) \\ &= (d(e^{tf}\alpha), e^{-tf}\beta) \\ &= (e^{tf}\alpha, d^*(e^{-tf}\beta)) \\ &= (\alpha, e^{tf} d^*(e^{-tf}\beta)), \end{aligned}$$

so we have $d_t^* = e^{tf} \circ d^* \circ e^{-tf}$. □

We define $\Delta_t = d_t \circ d_t^* + d_t^* d_t$, and denote by $\mathcal{H}_t^i$ the kernel of $\Delta_t : \Omega^i(M) \to \Omega^i(M)$.

Lemma 41.3. $d_t\omega = d\omega + tdt \wedge \omega$ *and* $d_t^*\omega = d^*\omega + ti_{\nabla f}\omega$. *Here, ∇f is the gradient of f, i.e., the dual to df under the metric g, and i_X is the interior product with X.*

Proof. We compute

$$d_t\omega = e^{-tf} d(e^{tf}\omega) = e^{-tf}(e^{tf} tdf \wedge \omega + e^{tf} d\omega) = d\omega + tdf \wedge \omega.$$

For d_t^*, it suffices to note that wedging with df is the adjoint of contracting with ∇f. □

Lemma 41.3 indicates that the symbols for d and d_t are the same, and hence d_t is an *elliptic complex.*

Proposition 41.4. $\mathcal{H}_t^i \simeq H^*(\Omega^i(M), d_t) \simeq H^*(M; \mathbf{R})$.

Proof. This follows from the discussion on the elliptic complex, namely that for the complex $(\Omega^i(M), d_t)$, the corresponding cohomology $H^i(\Omega^*(M), d_t)$ is equal to $\mathcal{H}_t^i$, the kernel of $\square = d_t d_t^* + d_t^* d_t$. The second isomorphism comes from Lemma 41.1. □

The following can be computed (computation omitted):

Proposition 41.5. *For the flat metric,*

$$\Delta_t = \Delta + t^2 \sum_i \left(\frac{\partial f}{\partial x_i} \right)^2 + t \sum_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} [e_i^*, e_j]. \tag{14}$$

Here, $\Delta = -\sum_i \frac{\partial^2}{\partial x_i^2}$, $\{e_1, \dots, e_n\}$ *is an orthonormal basis for* $T_x^* M$, *and, in the commutator* $[\cdot, \cdot]$, e_j *is wedging with* e_j *and* e_i^* *is interior product with the dual of* e_i.

The term $t^2 \sum_i (\frac{\partial f}{\partial x_i})^2$ in Equation 14 represents the *potential energy* $V(x)$, and, for $t >> 0$, $V(x) >> 0$ except when $\frac{\partial f}{\partial x_i} = 0$ for all i, i.e., near a critical point of f. Near a critical point p of f, we write $f = f(p) + \frac{1}{2} \sum_i a_i x_i^2$, and then Δ_t has the form:

$$\Delta_t = -\sum_i \frac{\partial^2}{\partial x_i^2} + t^2 \sum_i a_i^2 x_i^2 + t \sum_i a_i [e_i^*, e_i]. \tag{15}$$

The first two terms of Equation 15 constitutes the *harmonic oscillator*, which we now briefly review.

41.2. **Harmonic oscillators.** Consider the harmonic oscillator on $\mathbf{R}$ given by $H = -\frac{d^2}{dx^2} + k^2 x^2$, where $k \in \mathbf{R}$. Using the creation and annihilation operators

$$a = \frac{d}{dx} + kx \text{ (annihilation)},$$

$$a^* = -\frac{d}{dx} + kx \text{ (creation)},$$

we compute the eigenfunctions ϕ and corresponding eigenvalues λ.

Theorem 41.6. $L^2(\mathbf{R}) = \oplus_{N=0}^{\infty} V_{k(1+2N)}$. *Here, the eigenspace* $V_{k(1+2N)}$ *corresponding to the eigenvalue* $k(1+2N)$ *is 1-dimensional and is spanned by*

$$(a^*)^N e^{-kx^2/2}.$$

For a discussion of the harmonic oscillator, refer to any quantum mechanics textbook, for example Liboff, *Introductory Quantum Mechanics* or Sakurai, *Modern Quantum Mechanics.* For the L^2-completeness of the eigenfunctions, see Reed & Simon, *Methods of Modern Mathematical Physics.*

Since the n harmonic oscillators in Equation 15 are uncoupled, the eigenvalues of Δ_t are (close to):

$$t\sum_{i=1}^{n}\left[|a_i|(1+2N_i)+(\pm 1)a_i\right].$$

Here ± 1 are the eigenvalues of $[e_i^*, e_i]$. We are now looking to find eigenvalues which are close to 0. To realize this, we choose the ground state near each critical point, i.e., $N_i = 0$ for all i, and choose the sign of the eigenvalue of $[e_i^*, e_i]$ so that the eigenvalue of Δ_t is (close to) zero. Hence we have:

Proposition 41.7. *If we let* $\widetilde{\mathcal{H}}_t^i = \{\Delta_t\omega = \delta\omega, \delta << 1\}$*, then*

$$\widetilde{\mathcal{H}}_t^i = \oplus_p \mathbf{R}\{p\},$$

where the sum is over critical points p *of* f *of index* i*.*

Now, since $\widetilde{\mathcal{H}}_t^i$ may contain “fake” eigenfunctions whose eigenvalues are close to zero but not exactly zero, we have:

$$\dim \widetilde{\mathcal{H}}_t^i \geq \dim \mathcal{H}_t^i.$$

This implies the weak Morse inequalities

$$\#\text{ critical points of index } i \geq \dim H^i(M;\mathbf{R}).$$

Witten goes on to compute that d_t is precisely the boundary operator from the last lecture (modulo some normalizing factor) — at this point I do not understand why. Fortunately, we are out of time!