Notes for Math 261A
Lie groups and Lie algebras

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How these notes came to be

Among the Berkeley professors, there was once Allen Knutson, who would teach Math 261. But it happened that professor Knutson was on sabbatical at UCLA, and eventually went there for good. During this turbulent time, Maths 261AB were cancelled two years in a row. The last of these four semesters (Spring 2006), some graduate students gathered together and asked Nicolai Reshetikhin to teach them Lie theory in a giant reading course. When the dust settled, there were two other professors willing to help in the instruction of Math 261A, Vera Serganova and Richard Borcherds. Thus Tag Team 261A was born.

After a few lectures, professor Reshetikhin suggested that the students write up the lecture notes for the benefit of future generations. The first four lectures were produced entirely by the "editors". The remaining lectures were LaTeXed by Anton Geraschenko in class and then edited by the people in the following table. The columns are sorted by lecturer.

Nicolai Reshetikhin		Vera Serganova		Richard Borcherds	
1	Anton Geraschenko	11	Sevak Mkrtchyan	21	Hanh Duc Do
2	Anton Geraschenko	12	Jonah Blasiak	22	An Huang
3	Nathan George	13	Hannes Thiel	23	Santiago Canez
4	Hans Christianson	14	Anton Geraschenko	24	Lilit Martirosyan
5	Emily Peters	15	Lilit Martirosyan	25	Emily Peters
6	Sevak Mkrtchyan	16	Santiago Canez	26	Santiago Canez
7	Lilit Martirosyan	17	Katie Liesinger	27	Martin Vito-Cruz
8	David Cimasoni	18	Aaron McMillan	28	Martin Vito-Cruz
9	Emily Peters	19	Anton Geraschenko	29	Anton Geraschenko
10	Qingtau Chen	20	Hanh Duc Do	30	Lilit Martirosyan
				31	Sevak Mkrtchyan

Richard Borcherds then edited the last third of the notes. The notes were further edited (and often expanded or rearranged) by Crystal Hoyt, Sevak Mkrtchyan, and Anton Geraschenko.

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Dependence of results and other information

Within a lecture, everything uses the same counter, with the exception of exercises. Thus, item $a.b$ is the b-th item in Lecture a, whether it is a theorem, lemma, example, equation, or anything else that deserves a number and isn't an exercise.

Lecture 1

Definition 1.1. A *Lie group* is a smooth manifold G with a group structure such that the multiplication $\mu : G \times G \to G$ and inverse map $\iota : G \to G$ are smooth maps.

▶ **Exercise 1.1.** If we assume only that μ is smooth, does it follow that ι is smooth?

Example 1.2. The group of invertible endomorphisms of $\mathbb{C}^n$, $GL_n(\mathbb{C})$, is a Lie group. The automorphisms of determinant 1, $SL_n(\mathbb{C})$, is also a Lie group.

Example 1.3. If B is a bilinear form on $\mathbb{C}^n$, then we can consider the Lie group

$$\{A \in GL_n(\mathbb{C}) | B(Av, Aw) = B(v, w) \text{ for all } v, w \in \mathbb{C}^n\}.$$

If we take B to be the usual dot product, then we get the group $O_n(\mathbb{C})$. If we let $n = 2m$ be even and set $B(v, w) = v^T \left(\begin{smallmatrix} 0 & 1_m \\ -1_m & 0 \end{smallmatrix}\right) w$, then we get $Sp_{2m}(\mathbb{C})$.

Example 1.4. $SU_n \subseteq SL_n(\mathbb{C})$ is a *real form* (look in lectures 27,28, and 29 for more on real forms).

Example 1.5. We'd like to consider infinite matrices, but the multiplication wouldn't make sense, so we can think of $GL_n \subseteq GL_{n+1}$ via $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right)$, then define GL_∞ as $\bigcup_n GL_n$. That is, invertible infinite matrices which look like the identity almost everywhere.

Lie groups are hard objects to work with because they have global characteristics, but we'd like to know about representations of them. Fortunately, there are things called Lie algebras, which are easier to work with, and representations of Lie algebras tell us about representations of Lie groups.

Definition 1.6. A *Lie algebra* is a vector space V equipped with a *Lie bracket* $[\ ,\ ] : V \times V \to V$, which satisfies

1. Skew symmetry: $[a, a] = 0$ for all $a \in V$, and
2. Jacobi identity: $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$ for all $a, b, c \in V$.

A *Lie subalgebra* of a Lie algebra V is a subspace $W \subseteq V$ which is closed under the bracket: $[W, W] \subseteq W$.

Example 1.7. If A is a finite dimensional associative algebra, you can set $[a, b] = ab - ba$. If you start with $A = \mathbb{M}_n$, the algebra of $n \times n$ matrices, then you get the Lie algebra $\mathfrak{gl}_n$. If you let $A \subseteq \mathbb{M}_n$ be the algebra of matrices preserving a fixed flag $V_0 \subset V_1 \subset \cdots V_k \subseteq \mathbb{C}^n$, then you get *parabolic*indexparabolic subalgebras Lie sub-algebras of $\mathfrak{gl}_n$.

Example 1.8. Consider the set of vector fields on $\mathbb{R}^n$, $\text{Vect}(\mathbb{R}^n) = \{\ell = \Sigma e^i(x)\frac{\partial}{\partial x_i} | [\ell_1, \ell_2] = \ell_1 \circ \ell_2 - \ell_2 \circ \ell_1\}$.

▶ **Exercise 1.2.** Check that $[\ell_1, \ell_2]$ is a first order differential operator.

Example 1.9. If A is an associative algebra, we say that $\partial : A \to A$ is a derivation if $\partial(ab) = (\partial a)b + a\partial b$. Inner derivations are those of the form $[d, \cdot]$ for some $d \in A$; the others are called outer derivations. We denote the set of derivations of A by $\mathcal{D}(A)$, and you can verify that it is a Lie algebra. Note that $\text{Vect}(\mathbb{R}^n)$ above is just $\mathcal{D}(C^\infty(\mathbb{R}^n))$.

The first Hochschild cohomology, denoted $H^1(A, A)$, is the quotient $\mathcal{D}(A)/\{\text{inner derivations}\}$.

Definition 1.10. A *Lie algebra homomorphism* is a linear map $\phi : L \to L'$ that takes the bracket in L to the bracket in L', i.e. $\phi([a, b]_L) = [\phi(a), \phi(b)]_{L'}$. A *Lie algebra isomorphism* is a morphism of Lie algebras that is a linear isomorphism.[1]

A very interesting question is to classify Lie algebras (up to isomorphism) of dimension n for a given n. For $n = 2$, there are only two: the trivial bracket $[\ ,\ ] = 0$, and $[e_1, e_2] = e_2$. For $n = 3$, it can be done without too much trouble. Maybe $n = 4$ has been done, but in general, it is a very hard problem.

If $\{e_i\}$ is a basis for V, with $[e_i, e_j] = c_{ij}^k e_k$ (the c_{ij}^k are called the *structure constants* of V), then the Jacobi identity is some quadratic relation on the c_{ij}^k, so the variety of Lie algebras is some quadratic surface in $\mathbb{C}^{3n}$.

Given a smooth real manifold M^n of dimension n, we can construct $\text{Vect}(M^n)$, the set of smooth vector fields on M^n. For $X \in \text{Vect}(M^n)$, we can define the *Lie derivative* L_X by $(L_X \cdot f)(m) = X_m \cdot f$, so L_X acts on $C^\infty(M^n)$ as a derivation.

▶ **Exercise 1.3.** Verify that $[L_X, L_Y] = L_X \circ L_Y - L_Y \circ L_X$ is of the form L_Z for a unique $Z \in \text{Vect}(M^n)$. Then we put a Lie algebra structure on $\text{Vect}(M^n) = \mathcal{D}(C^\infty(M^n))$ by $[X, Y] = Z$.

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[1] The reader may verify that this implies that the inverse is also a morphism of Lie algebras.

There is a theorem (Ado's Theorem[2]) that any Lie algebra $\mathfrak{g}$ is isomorphic to a Lie subalgebra of $\mathfrak{gl}_n$, so if you understand everything about $\mathfrak{gl}_n$, you're in pretty good shape.

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[2] Notice that if $\mathfrak{g}$ has no center, then the adjoint representation $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is already faithful. See Example 7.4 for more on the adjoint representation. For a proof of Ado's Theorem, see Appendix E of [FH91]

Lecture 2

Last time we talked about Lie groups, Lie algebras, and gave examples. Recall that $M \subseteq L$ is a Lie subalgebra if $[M, M] \subseteq M$. We say that M is a *Lie ideal* if $[M, L] \subseteq M$.

Claim. *If M is an ideal, then L/M has the structure of a Lie algebra such that the canonical projection is a morphism of Lie algebras.*

Proof. Take $l_1, l_2 \in L$, check that $[l_1 + M, l_2 + M] \subseteq [l_1, l_2] + M$. ☐

Claim. *For $\phi : L_1 \to L_2$ a Lie algebra homomorphism,*

1. $\ker \phi \subseteq L_1$ *is an ideal,*
2. $\operatorname{im} \phi \subseteq L_2$ *is a Lie subalgebra,*
3. $L_1/\ker\phi \cong \operatorname{im}\phi$ *as Lie algebras.*

▶ **Exercise 2.1.** Prove this claim.

Tangent Lie algebras to Lie groups

Let's recall some differential geometry. You can look at [Lee03] as a reference. If $f : M \to N$ is a differentiable map, then $df : TM \to TN$ is the derivative. If G is a group, then we have the maps $l_g : x \mapsto gx$ and $r_g : x \mapsto xg$. Recall that a smooth vector field is a smooth section of the tangent bundle $TM \to M$.

Definition 2.1. A vector field X is *left invariant* if $(dl_g) \circ X = X \circ l_g$ for all $g \in G$. The set of left invariant vector fields is called $\mathrm{Vect}_L(G)$.

$$\begin{tikzcd}
TG \arrow[r, "dl_g"] & TG \\
G \arrow[u, "X"] \arrow[r, "l_g"] & G \arrow[u, "X"']
\end{tikzcd}$$

Proposition 2.2. $\mathrm{Vect}_L(G) \subseteq \mathrm{Vect}(G)$ *is a Lie subalgebra.*

Proof. We get an induced map $l_g^* : C^\infty(G) \to C^\infty(G)$, and X is left invariant if and only if L_X commutes with l_G^*. Then
X, Y left invariant $\iff [L_X, L_Y]$ invariant $\iff [X, Y]$ left invariant. ☐

All the same stuff works for right invariant vector fields $\mathrm{Vect}_R(G)$.

Definition 2.3. $\mathfrak{g} = \mathrm{Vect}_L(G)$ is the tangent Lie algebra of G.

Proposition 2.4. *There are vector space isomorphisms* $\mathrm{Vect}_L(G) \simeq T_eG$ *and* $\mathrm{Vect}_R(G) \simeq T_eG$*. Moreover, the Lie algebra structures on* T_eG *induced by these isomorphisms agree.*

Note that it follows that $\dim \mathfrak{g} = \dim G$.

Proof. Recall fibre bundles. $dl_g : T_eG \xrightarrow{\sim} T_gG$, so $TG \simeq T_e \times G$. X is a section of TG, so it can be thought of as $X : G \to T_eG$, in which case the left invariant fields are exactly those which are constant maps, but the set of constants maps to T_eG is isomorphic to T_eG. $\square$

If G is an n dimensional C^ω Lie group, then $\mathfrak{g}$ is an n dimensional Lie algebra. If we take local coordinates near $e \in G$ to be $x^1, \ldots, x^n : U_e \to \mathbb{R}^n$ with $m : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ the multiplication (defined near 0). We have a power series for m near 0,

$$m(x,y) = Ax + By + \alpha_2(x,y) + \alpha_3(x,y) + \cdots$$

where $A, B : \mathbb{R}^n \to \mathbb{R}^n$ are linear, α_i is degree i. Then we can consider the condition that m be associative (only to degree 3): $m(x, m(y,z)) = m(m(x,y),z)$.

$$\begin{aligned} m(x, m(y,z)) &= Ax + Bm(y,z) + \alpha_2(x, m(y,z)) + \alpha_3(x, m(y,z)) + \cdots \\ &= Ax + B(Ay + Bz + \alpha_2(y,z) + \alpha_3(y,z))) \\ &\qquad + \alpha_2(x, Ay + Bz + \alpha_2(y,z)) + \alpha_3(x, Ay + Bz) \\ m(m(x,y),z) &= \end{aligned}$$

Comparing first order terms (remember that A, B must be non-singular), we can get that $A = B = I_n$. From the second order term, we can get that α_2 is bilinear! Changing coordinates ($\phi(x) = x + \phi_2(x) + \phi_3(x) + \cdots$, with $\phi^{-1}(x) = x - \phi_2(x) + \tilde{\phi}_3(x) + \cdots$), we use the fact that $m_\phi(x,y) = \phi^{-1}m(\phi x, \phi y)$ is the new multiplication, we have

$$m_\phi(x,y) = x + y + \underbrace{(\phi_2(x) + \phi_2(y) + \phi_2(x+y))}_{\text{can be any symm form}} + \alpha_2(x,y) + \cdots$$

so we can tweak the coordinates to make α_2 skew-symmetric. Looking at order 3, we have

$$\alpha_2(x, \alpha_2(y,z)) + \alpha_3(x, y+z) = \alpha_2(\alpha_2(x,y),z) + \alpha_3(x+y,z) \tag{2.5}$$

▶ **Exercise 2.2.** Prove that this implies the Jacobi identity for α_2. (hint: skew-symmetrize equation 2.5)

Remarkably, the Jacobi identity is the only obstruction to associativity; all other coefficients can be eliminated by coordinate changes.

Example 2.6. Let G be the set of matrices of the form $\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right)$ for a, b real, $a > 0$. Use coordinates x, y where $e^x = a$, $y = b$, then

$$\begin{aligned} m((x,y),(x',y')) &= (x + x', e^x y' + y e^{-x'}) \\ &= (x + x', y + y' + (\underbrace{xy' - x'y}_{skew}) + \cdots). \end{aligned}$$

The second order term is skew symmetric, so these are good coordinates. There are $H, E \in T_e G$ corresponding to x and y respectively so that $[H, E] = E$[1].

▶ **Exercise 2.3.** Think about this. If a, b commute, then $e^a e^b = e^{a+b}$. If they do not commute, then $e^a e^b = e^{f(a,b)}$. Compute $f(a, b)$ to order 3.

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[1] what does this part mean?

Lecture 3

Last time we saw how to get a Lie algebra Lie(G) from a Lie group G.

> Lie$(G) = \mathrm{Vect}_L(G) \simeq \mathrm{Vect}_R(G)$.
>
> Let $x^1, ..., x^n$ be local coordinates near $e \in G$, and let $m(x, y)^i$ be the i^{th} coordinate of $(x, y) \mapsto m(x, y)$. In this local coordinate system, $m(x, y)^i = x^i + y^i + \frac{1}{2}\sum c^i_{jk} x^j y^k + \cdots$. If $e_1, ..., e_n \in T_eG$ is the basis induced by $x^1, ..., x^n$, $(e_i \sim \partial_i)$, then
>
> $$[e_i, e_j] = \sum_k c^k_{ij} e_k.$$

Example 3.1. Let G be GL_n, and let (g_{ij}) be coordinates. Let $X : GL_n \to TGL_n$ be a vector field.

$$L_X(f)(g) = \sum_{i,j} X_{ij}(g) \frac{\partial f(g)}{\partial g_{ij}}$$

$$\text{, where } L_X(l_h^*(f))(g) = \left\{ \begin{array}{l} l_h : g \mapsto hg \\ l_h^*(f)(g) = f(h^{-1}g) \end{array} \right\}$$

$$= \sum_{i,j} X_{ij}(g) \frac{\partial f(h^{-1}g)}{\partial g_{ij}} = \sum_{i,j} X_{ij}(g) \frac{\partial (h^{-1}g)_{kl}}{\partial g_{ij}} \frac{\partial f(x)}{\partial x_k l}|_{x=h^{-1}g}$$

$$= \langle \frac{\partial (h^{-1}g)_{kl}}{\partial g_{ij}} = \sum_m (h^{-1})_{km} \underbrace{\frac{\partial g_{ml}}{\partial g_{ij}}}_{=\delta_{im}\delta_{lj}} = (h^{-1})_{ki}\delta_{jl}\rangle$$

$$= \sum_{i,j,k} X_{ij}(g)(h^{-1})_{ki} \frac{\partial f}{\partial x_{kj}}|_{x=h^{-1}g}$$

$$= \sum_{j,k} \left(\sum_i (h^{-1})_{ki} X_{ij}(g) \right) \frac{\partial f}{\partial x_{kj}}|_{x=h^{-1}g}$$

If we want X to be left invariant, $\sum_i (h^{-1})_{ki} X_{ij}(g) = X_{kj}(h^{-1}g)$, then $L_X(l_h^*(f)) = l_h^*(L_X(f))$, (left invariance of X).

Example 3.2. All solutions are $X_{ij}(g) = (g \cdot M)_{ij}$, M-constant $n \times n$ matrix. gives that left invariant vector fields on $GL_n \approx n \times n$ matrices $= \mathfrak{gl}_n$. The "Natural Basis" is $e_{ij} = (E_{ij})$, $L_{ij} = \sum_m (g)_{mj} \frac{\partial}{\partial g_{mi}}$.

Example 3.3. Commutation relations between L_{ij} are the same as commutation relations between e_{ij}.

Take $\tau \in T_eG$. Define the vector field: $v_\tau : G \to TG$ by $v_\tau(g) = dl_g(\tau)$, where $l_g : G \to G$ is left multiplication. v_τ is a left invariant vector field by construction.

Consider $\phi : I \to G$, $\frac{d\phi(t)}{dt} = v_\tau(\phi(t))$, $\phi(0) = e$.

Proposition 3.4.

1. *$\phi(t+s) = \phi(t)\phi(s)$*
2. *ϕ extends to a smooth map $\phi : \mathbb{R} \to G$.*

Proof. 1. Fix s and $\alpha(t) = \phi(s)\phi(t), \beta(t) = \phi(s+t)$.

   - $\alpha(0) = \phi(s) = \beta(0)$
   - $\frac{d\beta(a)}{dt} = \frac{d\phi(s+t)}{dt} = v_\tau(\beta(t))$
   - $\frac{d\alpha(t)}{dt} = \frac{d}{dt}(\phi(s)\phi(t)) = dl_{\phi(s)} \cdot v_\tau(\phi(t)) = v_\tau(\phi(s)\phi(t)) = v_\tau(\alpha(t))$, where the second equality is because v_τ is linear.

   $\implies$ α satisfies same equation as β and same initial conditions, so by uniqueness, they coincide for $|t| < \epsilon$.

2. Now we have (1) for $|t+s| < \epsilon$, $|t| < \epsilon$, $|s| < \epsilon$. Then extend ϕ to $|t| < 2\epsilon$. Continue this to cover all of $\mathbb{R}$.

∎

This shows that for all $\tau \in T_eG$, we have a mapping $\mathbb{R} \to G$ and it's image is a 1-parameter (1 dimensional) Lie subgroup in G.

$$\begin{aligned} \exp : \mathfrak{g} = T_eG &\to G \\ \tau &\mapsto \phi_\tau(1) = \exp(\tau) \end{aligned}$$

Notice that $\lambda\tau \mapsto \exp(\lambda\tau) = \phi_{\lambda\tau}(1) = \phi_\tau(\lambda)$

Example 3.5. GL_n, $\tau \in \mathfrak{gl}_n = T_eGL_n$, $\frac{d\phi(t)}{dt} = v_\tau(\phi(t)) \in T_{\phi(t)}GL_n \simeq \mathfrak{gl}_n$.

$$v_\tau(\phi(t)) = \phi(t) \cdot \tau,\ \frac{d\phi(t)}{dt} = \phi(t) \cdot \tau,\ \phi(0) = I,$$

$$\phi(t) = \exp(tI) = \sum_{n=0}^{\infty} \frac{t^n \tau^n}{n!}$$

$\exp : \mathfrak{gl}_n \to GL_n$

$$\left[L_{\gamma(0)=g}(f)(g) = \frac{d}{dt} f(\gamma(t))|_{t=0} \right]$$

Baker-Campbell-Hausdorff formula:

$$e^X \cdot e^Y = e^{H(X,Y)}$$

$$H(X,Y) = \underbrace{X+Y}_{\text{sym}} + \underbrace{\frac{1}{2}[X,Y]}_{\text{skew}} + \underbrace{\frac{1}{12}([X[X,Y]] + [Y[Y,X]])}_{\text{symmetric}} + \cdots$$

Proposition 3.6. *1. Let $f : G \to H$ be a Lie group homomorphism, then the diagram*

$$\begin{tikzcd} G \arrow[r, "f"] & H \\ Lie(G) \arrow[u, "\exp"] \arrow[r, "df_e"] & Lie(H) \arrow[u, "\exp"'] \end{tikzcd}$$

is commutative.

2. If G is connected, *then $(df)_e$ defines the Lie group homomorphism f uniquely.*

Proof. Next time. □

Proposition 3.7. *G, H Lie groups, G* simply connected, *then $\alpha : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ is a Lie algebra homomorphism if and only if there is a Lie group homomorphism $A : G \to H$ lifting α.*

Proof. Next time. □

{Lie algebras } $\xrightarrow{\exp}$ { Lie groups(connected, simply connected)} is an equivalence of categories.

Lecture 4

Theorem 4.1. *Suppose G is a topological group. Let $U \subset G$ be an open neighbourhood of $e \in G$. If G is connected, then*

$$G = \bigcup_{n \geq 1} U^n.$$

Proof. Choose a non-empty open set $V \subset U$ such that $V = V^{-1}$, for example $V = U \cap U^{-1}$. Define $H = \cup_{n \geq 1} V^n$, and observe H is an abstract subgroup, since $V^n V^m \subseteq V^{n+m}$. H is open since it is the union of open sets. If $\sigma \notin H$, then $\sigma H \not\subset H$, since otherwise if $h_1, h_2 \in H$ satisfy $\sigma h_1 = h_2$, then $\sigma = h_2 h_1^{-1} \in H$. Thus H is a complement of the union of all cosets not containing H. Hence H is closed. Since G is connected, $H = G$. □

Theorem 4.2. *Let $f : G \to H$ be a Lie group homomorphism. Then the following diagram commutes:*

$$\begin{tikzcd}
T_e \arrow[r, "(df)_e"] \arrow[d, "\exp"'] & T_e H \arrow[d, "\exp"] \\
G \arrow[r, "f"] & H
\end{tikzcd}$$

Further, if G is connected, $(df)_e$ determines f uniquely.

Proof. 1) Commutative diagram. Fix $\tau \in T_e G$ and set $\eta = df_e \tau \in T_e H$. Recall we defined the vector field $V_\tau(g) = (dl_g)(\tau)$, then if $\phi(t)$ solves

$$\frac{d\phi}{dt} = V_\tau(\phi(t)) \in T_{\phi(t)} G,$$

we have $\exp(\tau) = \phi(1)$. Let ψ solve

$$\frac{d\psi}{dt} = V_\eta(\psi(t)),$$

so that $\exp(\eta) = \psi(1)$. Observe $\tilde{\psi}(t) = f(\phi(t))$ satisfies

$$\frac{d\tilde{\psi}}{dt} = (df)\left(\frac{d\phi}{dt}\right) = V_\eta(\tilde{\psi}),$$

so by uniqueness of solutions to ordinary differential equations, $\psi = \tilde{\psi}$.

2) Uniqueness of f. The exponential map is an isomorphism of a neighborhood of $0 \in \mathfrak{g}$ and a neighborhood of $e \in G$. But if G is connected, $G = \cup_{n \geq 1} (\text{nbd } e)^n$. □

Theorem 4.3. *Suppose G is a topological group, with $G^0 \subset G$ the connected component of e. Then 1) G^0 is normal and 2) G/G^0 is discrete.*

Proof. 2) $G^0 \subset G$ is open implies $\mathrm{pr}^{-1}([e]) = eG^0$ is open in G, which in turn implies $\mathrm{pr}^{-1}([g]) \in G/G^0$ is open for every $g \in G$. Thus each coset is both open and closed, hence G/G^0 is discrete.

1) Fix $g \in G$ and consider the map $G \to G$ defined by $x \mapsto gxg^{-1}$. This map fixes e and is continuous, which implies it maps G^0 into G^0. In other words, $gG^0g^{-1} \subset G^0$, or G^0 is normal. □

We recall some basic notions of algebraic topology. Suppose M is a connected topological space. Let $x, y \in M$, and suppose $\gamma(t) : [0,1] \to M$ is a path from x to y in M. We say $\tilde{\gamma}(t)$ is *homotopic* to γ if there is a continuous map $h(s,t) : [0,1]^2 \to M$ satisfying

$$\bullet h(s,0) = x,\ \ h(s,1) = y$$
$$\bullet h(0,t) = \gamma(t),\ \ h(1,t) = \tilde{\gamma}(t).$$

We call h the *homotopy.* On a smooth manifold, we may replace h with a smooth homotopy. Now fix $x_0 \in M$. We define the first fundamental group of M

$$\pi_1(M, x_0) = \{\text{ homotopy classes of loops based at } x_0\}.$$

It is clear that this is a group with group multiplication composition of paths. It is also a fact that the definition does not depend on the base point x_0:

$$\pi_1(M, x_0) \simeq \pi_1(M, x_0').$$

By $\pi_1(M)$ we denote the isomorphism class of $\pi_1(M, \cdot)$. Lastly, we say M is *simply connected* if $\pi_1(M) = \{e\}$, that is if all closed paths can be deformed to the trivial one.

Theorem 4.4. *Suppose G and H are Lie groups with Lie algebras $\mathfrak{g}, \mathfrak{h}$ respectively. If G is simply connected, then any Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{h}$ lifts to a Lie group homomorphism $R : G \to H$.*

In order to prove this theorem, we will need the following lemma.

Lemma 4.5. *Let $\xi : \mathbb{R} \to \mathfrak{g}$ be a smooth mapping. Then*

$$\frac{dg}{dt} = (dl_g)(\xi(t))$$

has a unique solution on all of $\mathbb{R}$ with $g(t_0) = g_0$.

For convenience, we will write $g\xi := (dl_g)(\xi)$.

Proof. Since $\mathfrak{g}$ is a vector space, we identify it with $\mathbb{R}^n$ and for sufficiently small $r > 0$, we identify $B_r(0) \subset \mathfrak{g}$ with a small neighbourhood of e, $U_e(r) \subset G$, under the exponential map. Here $B_r(0)$ is measured with the usual Euclidean norm $\|\cdot\|$. Note for any $g \in U_e(r)$ and $|t-t_0|$ sufficiently small, we have $\|g\xi(t)\| \le C$. Now according to Exercise 4.1, the solution with $g(t_0) = e$ exists for sufficiently small $|t - t_0|$ and

$$g(t) \in U_e(r) \ \forall |t - t_0| < \frac{r}{C'}.$$

Now define $h(t) = g(t)g_0$ so that $h(t) \in U_{g_0}(r)$ for $|t - t_0| < r/C'$. That is, r and C' do not depend on the choice of initial conditions, and we can cover $\mathbb{R}$ by intervals of length, say r/C'. ☐

► **Exercise 4.1.** Verify that there is a constant C' such that if $|t - t_0|$ is sufficiently small, we have

$$\|g(t)\| \le C'|t - t_0|.$$

Proof of Theorem 4.4. We will construct $R : G \to H$. Beginning with $g(t) : [0, 1] \to G$ satisfying $g(0) = e$, $g(1) = g$, define $\xi(t) \in \mathfrak{g}$ for each t by

$$g(t)\xi(t) = \frac{d}{dt}g(t).$$

Let $\eta(t) = \rho(\xi(t))$, and let $h(t) : [0, 1] \to H$ satisfy

$$\frac{d}{dt}h(t) = h(t)\eta(t), \quad h(0) = e.$$

Define $R(g) = h(1)$.

Claim: $h(1)$ does not depend on the path $g(t)$, only on g.

Proof of Claim. Suppose $g_1(t)$ and $g_2(t)$ are two different paths connecting e to g. Then there is a smooth homotopy $g(t, s)$ satisfying $g(t, 0) = g_1(t)$, $g(t, 1) = g_2(t)$. Define $\xi(t, s)$ and $\eta(t, s)$ by

$$\begin{aligned} \frac{\partial g}{\partial t} &= g(t,s)\xi(t,s); \\ \frac{\partial g}{\partial s} &= g(t,s)\eta(t,s). \end{aligned}$$

Observe

$$\frac{\partial^2 g}{\partial s \partial t} = g\eta \circ \xi + g\frac{\partial \xi}{\partial t} \text{ and} \tag{4.6}$$

$$\frac{\partial^2 g}{\partial t \partial s} = g\xi \circ \eta + g\frac{\partial \eta}{\partial s}, \tag{4.7}$$

and (4.6) is equal to (4.7) since g is smooth. Consequently

$$\frac{\partial \eta}{\partial t} - \frac{\partial \xi}{\partial s} = [\eta, \xi].$$

Now define an s dependent family of solutions $h(\cdot, s)$ to the equations

$$\frac{\partial h}{\partial t}(t,s) = h(t,s)\rho(\xi(t,s)), \quad h(0,s) = e.$$

Define $\theta(t,s)$ by

$$\begin{cases} \dfrac{\partial \theta}{\partial t} - \dfrac{\partial \rho(\xi)}{\partial s} = [\rho(\xi), \theta], \\ \qquad\qquad \theta(0,s) = 0. \end{cases} \tag{4.8}$$

Observe $\tilde{\theta}(t,s) = \rho(\eta(t,s))$ also satisfies equation (4.8), so that $\theta = \tilde{\theta}$ by uniqueness of solutions to ODEs. Finally,

$$g\eta(1,s) = \frac{\partial g}{\partial s}(1,s) = 0 \implies \theta(1,s) = 0 \implies \frac{\partial h}{\partial s}(1,s) = 0,$$

justifying the claim.

We need only show $R : G \to H$ is a homomorphism. Let $g_1, g_2 \in G$ and set $g = g_1 g_2$. Let $\tilde{g}_i(t)$ be a path from e to g_i in G for each $i = 1, 2$. Then the path $\tilde{g}(t)$ defined by

$$\tilde{g}(t) = \begin{cases} \tilde{g}_1(2t), \ 0 \le t \le \frac{1}{2}, \\ g_1 \tilde{g}_2(2t-1), \ \frac{1}{2} \le t \le 1 \end{cases}$$

goes from e to g. Let $\tilde{h}_i$ for $i = 1, 2$ and $\tilde{h}$ be the paths in H corresponding to $\tilde{g}_1$, $\tilde{g}_2$, and $\tilde{g}$ respectively and calculate

$$R(g_1 g_2) = R(g) = \tilde{h}(1) = \tilde{h}_1(1)\tilde{h}_2(1) = R(g_1)R(g_2).$$

□

Lecture 5

Last time we talked about connectedness, and proved the following things:

- Any connected topological group G has the property that $G = \bigcup_n V^n$, where V is any neighborhood of $e \in G$.
- If G is a connected Lie group, with $\alpha : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ a Lie algebra homomorphism, then if there exists $f : G \to H$ with $df_e = \alpha$, it is unique.
- If G is connected, simply connected, with $\alpha : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ a Lie algebra homomorphism, then there is a unique $f : G \to H$ such that $df_e = \alpha$.

Simply Connected Lie Groups

The map p in $Z \to X \xrightarrow{p} Y$ is a *covering map* if it is a locally trivial fiber bundle with discrete fiber Z. Locally trivial means that for any $y \in Y$ there is a neighborhood U such that if $f : U \times Z \to Z$ is the map defined by $f(u, z) = u$, then the following diagram commutes:

$$\begin{tikzcd}
p^{-1}U \arrow[d] \arrow[r, phantom, "\simeq"] & U \times Z \arrow[dl, "f"] \\
Y \supseteq U &
\end{tikzcd}$$

The exact sequence defined below is an important tool. Suppose we have a locally trivial fiber bundle with fiber Z (not necessarily discrete), with X, Y connected. Choose $x_0 \in X, z_0 \in Z, y_0 \in Y$ such that $p(x_0) = y_0$, and $i : Z \to p^{-1}(y_0)$ is an isomorphism such that $i(z_0) = x_0$:

$$\begin{tikzcd}
x_0 \in \pi^{-1}(y_0) \arrow[d] & Z \arrow[l, "i"'] \\
y_0 &
\end{tikzcd}$$

We can define $p_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ in the obvious way (π_1 is a functor). Also define $i_* : \pi_1(Z, z_0) \to \pi_1(X, x_0)$. Then we can define $\partial : \pi_1(Y, y_0) \to \pi_0(Z) = \{\text{connected components of } Z\}$ by taking a loop γ based at y_0 and lifting it to some path $\tilde{\gamma}$. This path is not unique, but up to fiber-preserving homotopy it is. The new path $\tilde{\gamma}$ starts at x_0 and ends at x_0'. Then we define ∂ to be the map associating the connected component of x_0' to the homotopy class of γ.

Claim. *The following sequence is exact:*

$$\pi_1(Z, z_0) \xrightarrow{i_*} \pi_1(X, x_0) \xrightarrow{p_*} \pi_1(Y, y_0) \xrightarrow{\partial} \pi_0(Z) \longrightarrow \{0\}$$

1. $\operatorname{im} i_* = \ker p_*$
2. $\{$*fibers of* $\partial\} \simeq \pi_1(Y, y_0)/\operatorname{im} p_*$
3. ∂ *is surjective.*

Proof.

1. $\ker p_*$ is the set of all loops which map to contractible loops in Y, which are loops which are homotopic to a loop in $\pi^{-1}(y_0)$ based at x_0. These are exactly the loops of $\operatorname{im} i_*$.
2. The fiber of ∂ over the connected component $Z_z \subseteq Z$ is the set of all (homotopy classes of) loops in Y based at y_0 which lift to a path connecting x_0 to a point in the connected component of $\pi^{-1}(y_0)$ containing $i(Z_z)$. If two loops β, γ based at y_0 are in the same fiber, homotope them so that they have the same endpoint. Then $\tilde{\gamma}\tilde{\beta}^{-1}$ is a loop based at x_0. So fibers of ∂ are in one to one correspondence with loops in Y based at y_0, modulo images of loops in X based at x_0, which is just $\pi_1(Y, y_0)/\operatorname{im} p_*$.
3. This is obvious, since X is connected.

□

Now assume we have a covering space with discrete fiber, i.e. maps

$$\begin{tikzcd} X \arrow[d, "p"] & Z \arrow[l] \\ Y & \end{tikzcd}$$

such that $\pi_1(Z, z_0) = \{e\}$ and $\pi_0(Z) = Z$. Then we get the sequence

$$\{e\} \xrightarrow{i_*} \pi_1(X, x_0) \xrightarrow{p_*} \pi_1(Y, y_0) \xrightarrow{\partial} Z \longrightarrow \{0\}$$

and since p_* is injective, $Z = \pi_1(Y)/\pi_1(X)$.

Classifying all covering spaces of Y is therefore the same as describing all subgroups of $\pi_1(Y)$. The *universal cover* of Y is the space $\tilde{Y}$ such that $\pi_1(\tilde{Y}) = \{e\}$, and for any other covering X, we get a factorization of covering maps $\tilde{Y} \xrightarrow{f} X \xrightarrow{p} Y$.

We construct $\tilde{X}$, the universal cover, in the following way: fix $x_0 \in X$, and define $\tilde{X}_{x_0}$ to be the set of basepoint-fixing homotopy classes of paths connecting x_0 to some $x \in X$. We have a natural projection $[\gamma_{x_0,x}] \mapsto x$, and the fiber of this projection (over x_0) can be identified with $\pi_1(X, x_0)$. It is clear for any two basepoints x_0 and x_0', $\tilde{X}_{x_0} \simeq \tilde{X}_{x_0'}$ via any path $\gamma_{x_0,x_0'}$. So we have

$$\begin{tikzcd}
\tilde{X}_{x_0} \arrow[d, "p"] & \pi_1(X) \arrow[l] \\
X &
\end{tikzcd}$$

Claim. *$\tilde{X}_{x_0}$ is simply connected.*

Proof. We need to prove that $\pi_1(\tilde{X}_{x_0})$ is trivial, but we know that the fibers of p can be identified with both $\pi_1(X)$ and $\pi_1(X)/\pi_1(\tilde{X}_{x_0})$, so we're done. ☐

Let G be a connected Lie group. We would like to produce a simply connected Lie group which also has the Lie algebra Lie(G). It turns out that the obvious candidate, $\tilde{G}_e$, is just what we are looking for. It is not hard to see that $\tilde{G}_e$ is a smooth manifold (typist's note: it is not that easy either. See [Hat02], pp. 64-65, for a description of the topology on $\tilde{G}_e$. Once we have a topology and a covering space map, the smooth manifold structure of G lifts to $\tilde{G}_e$. – Emily). We show it is a group as follows.

Write γ_g for $\gamma : [0,1] \to G$ with endpoints e and g. Define multiplication by $[\gamma_g][\gamma_h'] := [\{\gamma_g(t)\gamma_h'(t)\}_{t\in[0,1]}]$. The unit element is the homotopy class of a contractible loop, and the inverse is given by $[\{\gamma(t)^{-1}\}_{t\in[0,1]}]$.

Claim.

1. *$\tilde{G} = \tilde{G}_e$ is a group.*
2. *$p : \tilde{G} \to G$ is a group homomorphism.*
3. *$\pi_1(G) \subseteq \tilde{G}$ is a normal subgroup.*
4. Lie$(\tilde{G}) =$ Lie(G).
5. *$\tilde{G} \to G$ is the universal cover (i.e. $\pi_1(G)$ is discrete).*

Proof.
1. Associativity is inherited from associativity in G, composition with the identity does not change the homotopy class of a path, and the product of an element and its inverse is the identity.
2. This is clear, since $p([\gamma_g][\tilde{\gamma}_h]) = gh$.
3. We know $\pi_1(G) = \ker p$, and kernels of homomorphisms are normal.

4. The topology on $\tilde{G}$ is induced by the topology of G in the following way: If $\mathcal{U}$ is a basis for the topology on G then fix a path $\gamma_{e,g}$ for all $g \in G$. Then $\tilde{\mathcal{U}} = \{\tilde{U}_{\gamma_{e,g}}\}$ is a basis for the topology on $\tilde{G}$ with $\tilde{U}_{\gamma_{e,g}}$ defined to be the set of paths of the form $\gamma_{e,g}^{-1}\beta\gamma_{e,g}$ with β a loop based at g contained entirely in U.

   Now take U a connected, simply connected neighborhood of $e \in G$. Since all paths in U from e to a fixed $g \in G$ are homotopic, we have that U and $\tilde{U}$ are diffeomorphic and isomorphic, hence Lie isomorphic. Thus $Lie(\tilde{G}) = Lie(G)$.

5. As established in (4), G and $\tilde{G}$ are diffeomorphic in a neighborhood of the identity. Thus all points $x \in p^{-1}(e)$ have a neighborhood which does not contain any other inverse images of e, so $p^{-1}(e)$ is discrete; and $p^{-1}(e)$ and $\pi_1(G)$ are isomorphic.

□

We have that for any Lie group G with a given Lie algebra $\text{Lie}(G) = \mathfrak{g}$, there exists a simply connected Lie group $\tilde{G}$ with the same Lie algebra, and $\tilde{G}$ is the universal cover of G.

Lemma 5.1. *A discrete normal subgroup $H \subseteq G$ of a connected topological group G is always central.*

Proof. For any fixed $h \in H$, consider the map $\phi_h : G \to H, g \mapsto ghg^{-1}h^{-1}$, which is continuous. Since G is connected, the image is also connected, but H is discrete, so the image must be a point. In fact, it must be e because $\phi_h(h) = e$. So H is central. □

Corollary 5.2. *$\pi_1(G)$ is central, because it is normal and discrete. In particular, $\pi_1(G)$ is commutative.*

Corollary 5.3. *$G \simeq \tilde{G}/\pi_1(G)$, with $\pi_1(G)$ discrete central.*

The following corollary describes all (connected) Lie groups with a given Lie algebra.

Corollary 5.4. *Given a Lie algebra $\mathfrak{g}$, take $\tilde{G}$ with Lie algebra $\mathfrak{g}$. Then any other connected G with Lie algebra $\mathfrak{g}$ is a quotient of $\tilde{G}$ by a discrete central subgroup of $\tilde{G}$.*

Suppose G is a topological group and G^0 is a connected component of e.

Claim. *$G^0 \subseteq G$ is a normal subgroup, and G/G^0 is a discrete group.*

If we look at {Lie groups} → {Lie algebras}, we have an "inverse" given by exponential: $\exp(\mathfrak{g}) \subseteq G$. Then $G^0 = \bigcup_n (\exp \mathfrak{g})^n$. So for a given Lie algebra, we can construct a well-defined isomorphism class of connected, simply connected Lie groups. When we say "take a Lie group with this Lie algebra", we mean to take the connected, simply connected one.

Coming Attractions: We will talk about $U\mathfrak{g}$, the universal enveloping algebra, $C(G)$, the Hopf algebra, and then we'll do classification of Lie algebras.

Lecture 6 - Hopf Algebras

Last time: We showed that a finite dimensional Lie algebra $\mathfrak{g}$ uniquely determines a connected simply connected Lie group. We also have a "map" in the other direction (taking tangent spaces). So we have a nice correspondence between Lie algebras and connected simply connected Lie groups.

There is another nice kind of structure: Associative algebras. How do these relate to Lie algebras and groups?

Let Γ be a finite group and let $\mathbb{C}[\Gamma] := \{\sum_g c_g g | g \in \Gamma, c_g \in \mathbb{C}\}$ be the $\mathbb{C}$ vector space with basis Γ. We can make $\mathbb{C}[\Gamma]$ into an associative algebra by taking multiplication to be the multiplication in Γ for basis elements and linearly extending this to the rest of $\mathbb{C}[\Gamma]$.[1]

Remark 6.1. Recall that the tensor product V and W is the linear span of elements of the form $v \otimes w$, modulo some linearity relations. If V and W are infinite dimensional, we will look at the *algebraic tensor product* of V and W, i.e. we only allow finite sums of the form $\sum a_i \otimes b_i$.

We have the following maps

> Comultiplication: $\Delta : \mathbb{C}[\Gamma] \to \mathbb{C}[\Gamma] \otimes \mathbb{C}[\Gamma]$, given by $\Delta(\sum x_g g) = \sum x_g g \otimes g$
>
> Counit: $\varepsilon : \mathbb{C}[\Gamma] \to \mathbb{C}$, given by $\varepsilon(\sum x_g g) = \sum x_g$.
>
> Antipode: $S : \mathbb{C}[\Gamma] \to \mathbb{C}[\Gamma]$ given by $S(\sum x_g g) = \sum x_g g^{-1}$.

You can check that

- $\Delta(xy) = \Delta(x)\Delta(y)$ (i.e. Δ is an algebra homomorphism),
- $(\Delta \otimes \mathrm{Id}) \circ \Delta = (\mathrm{Id} \otimes \Delta) \circ \Delta$. (follows from the associativity of $\otimes$),
- $\varepsilon(xy) = \varepsilon(x)\varepsilon(y)$ (i.e. ε is an algebra homomorphism),
- $S(xy) = S(y)S(x)$ (i.e. S is an algebra antihomomorphism).

Consider

$$\mathbb{C}[\Gamma] \xrightarrow{\Delta} \mathbb{C}[\Gamma] \otimes \mathbb{C}[\Gamma] \xrightarrow{S\otimes \mathrm{Id}, \mathrm{Id}\otimes S} \mathbb{C}[\Gamma] \otimes \mathbb{C}[\Gamma] \xrightarrow{m} \mathbb{C}[\Gamma].$$

You get

$$m(S \otimes \mathrm{Id})\Delta(g) = m(g^{-1} \otimes g) = e$$

so the composition sends $\sum x_g g$ to $(\sum_g x_g)e = \varepsilon(x)1_A$.

So we have

---

[1] "If somebody speaks Danish, I would be happy to take lessons."

1. $A = \mathbb{C}[\Gamma]$ an associative algebra with 1_A
2. $\Delta : A \to A \otimes A$ which is coassociative and is a homomorphism of algebras
3. $\varepsilon : A \to \mathbb{C}$ an algebra homomorphism, with $(\varepsilon \otimes \mathrm{Id})\Delta = (\mathrm{Id} \otimes \varepsilon)\Delta = \mathrm{Id}$.

Definition 6.2. Such an A is called a *bialgebra*, with comultiplication Δ and counit ε.

We also have S, the antipode, which is an algebra anti-automorphism, so it is a linear isomorphism with $S(ab) = S(b)S(a)$, such that

$$\begin{tikzcd}
A \otimes A \arrow[rr, "S\otimes\mathrm{Id}", "\mathrm{Id}\otimes S"'] & & A \otimes A \arrow[d, "m"] \\
A \arrow[u, "\Delta"] \arrow[r, "\varepsilon"] & \mathbb{C} \arrow[r, "1_A"] & A
\end{tikzcd}$$

Definition 6.3. A bialgebra with an antipode is a *Hopf algebra*.

If A is finite dimensional, let A^* be the dual vector space. Define the multiplication, Δ_*, S_*, ε_*, 1_{A^*} on A^* in the following way:

- $lm(a) := (l \otimes m)(\Delta a)$ for all $l, m \in A^*$
- $\Delta_*(l)(a \otimes b) := l(ab)$
- $S_*(l)(a) := l(S(a))$
- $\varepsilon_*(l) := l(1_A)$
- $1_{A^*}(a) := \varepsilon(a)$

Theorem 6.4. *A^* is a Hopf algebra with this structure, and we say it is* dual to A. *If A is finite dimensional, then $A^{**} = A$.*

▶ **Exercise 6.1.** Prove it.

We have an example of a Hopf algebra ($\mathbb{C}[\Gamma]$), what is the dual Hopf algebra?[2] Let's compute $A^* = \mathbb{C}[\Gamma]^*$.

Well, $\mathbb{C}[\Gamma]$ has a basis $\{g \in \Gamma\}$. Let $\{\delta_g\}$ be the dual basis, so $\delta_g(h) = 0$ if $g \neq h$ and 1 if $g = h$. Let's look at how we multiply such things

- $\delta_{g_1}\delta_{g_2}(h) = (\delta_{g_1} \otimes \delta_{g_2})(h \otimes h) = \delta_{g_1}(h)\delta_{g_2}(h)$.

---

[2] If you want to read more, look at S. Montgomery's *Hopf algebras*, AMS, early 1990s. [Mon93]

- $\Delta_*(\delta_g)(h_1 \otimes h_2) = \delta_g(h_1 h_2)$
- $S_*(\delta_g)(h) = \delta_g(h^{-1})$
- $\varepsilon_*(\delta_g) = \delta_g(e) = \delta_{g,e}$
- $1_{A^*}(h) = 1.$

It is natural to think of A^* as the set of functions $\Gamma \to \mathbb{C}$, where $(\sum x_g \delta_g)(h) = \sum x_g \delta_g(h)$. Then we can think about functions

- $(f_1 f_2)(h) = f_1(h) f_2(h)$
- $\Delta_*(f)(h_1 \times h_2) = f(h_1 h_2)$
- $S_*(f)(h) = f(h^{-1})$
- $\varepsilon_*(f) = f(e)$
- $1_{A^*} = 1$ constant.

So this is the Hopf algebra $C(\Gamma)$, the space of functions on Γ. If Γ is any affine algebraic group, then $C(\Gamma)$ is the space of polynomial functions on Γ, and all this works. The only concern is that we need $C(\Gamma \times \Gamma) \cong C(\Gamma) \otimes C(\Gamma)$, which we only have in the finite dimensional case; you have to take completions of tensor products otherwise.

So we have the notion of a bialgebra (and duals), and the notion of a Hopf algebra (and duals). We have two examples: $A = \mathbb{C}[\Gamma]$ and $A^* = C(\Gamma)$. A natural question is, "what if Γ is an infinite group or a Lie group?" and "what are some other examples of Hopf algebras?"

Let's look at some infinite dimensional examples. If A is an infinite dimensional Hopf algebra, and $A \otimes A$ is the algebraic tensor product (finite linear combinations of formal $a \otimes b$ s). Then the comultiplication should be $\Delta : A \to A \otimes A$. You can consider cases where you have to take some completion of the tensor product with respect to some topology, but we won't deal with this kind of stuff. In this case, A^* is too big, so instead of the notion of the dual Hopf algebra, we have dual pairs.

Definition 6.5. A *dual pairing* of Hopf algebras A and H is a pair with a bilinear map $\langle\ ,\ \rangle : A \otimes H \to \mathbb{C}$ which is nondegenerate such that

(1) $\langle \Delta a, l \otimes m \rangle = \langle a, lm \rangle$

(2) $\langle ab, l \rangle = \langle a \otimes b, \Delta_* l \rangle$

(3) $\langle Sa, l \rangle = \langle a, S_* l \rangle$

(4) $\varepsilon(a) = \langle a, 1_H \rangle, \varepsilon_H(l) = \langle 1_A, l \rangle$

Exmaple: $A = \mathbb{C}[x]$, then what is A^*? You can evaluate a polynomial at 0, or you can differentiate some number of times before you evaluate at 0. A^* = span of linear functionals on polynomial functions of $\mathbb{C}$ of the form

$$l_n(f) = \left(\frac{d}{dx}\right)^n f(x)\big|_{x=0}.$$

A basis for $\mathbb{C}[x]$ is $1, x^n$ with $n \geq 1$, and we have

$$l_n(x^m) = \begin{cases} m! & ,n = m \\ 0 & ,n \neq m \end{cases}$$

What is the Hopf algebra structure on A? We already have an algebra with identity. Define $\Delta(x) = x \otimes 1 + 1 \otimes x$ and extend it to an algebra homomorphism, then it is clearly coassociative. Define $\varepsilon(1) = 1$ and $\varepsilon(x^n) = 0$ for all $n \geq 1$. Define $S(x) = -x$, and extend to an algebra homomorphism. It is easy to check that this is a Hopf algebra.

Let's compute the Hopf algebra structure on A^*. We have

$$\begin{aligned} l_n l_m(x^N) &= (l_n \otimes l_m)(\Delta(x^N)) \\ &= (l_n \otimes l_m)(\sum \binom{N}{k} x^{N-k} \otimes x^k) \end{aligned}$$

▶ **Exercise 6.2.** Compute this out. The answer is that $A^* = \mathbb{C}[y = \frac{d}{dx}]$, and the Hopf algebra structure is the same as A.

This is an example of a dual pair: $A = \mathbb{C}[x], H = \mathbb{C}[y]$, with $\langle x^n, y^m \rangle = \delta_{n,m} m!$.

Summary: If A is finite dimensional, you get a dual, but in the infinite dimensional case, you have to use dual pairs.

The universal enveloping algebra

The idea is to construct a map from Lie algebras to associative algebras so that the representation theory of the associative algebra is equivalent to the representation theory of the Lie algebra.

1) let V be a vector space, then we can form the free associative algebra (or tensor algebra) of V: $T(V) = \mathbb{C} \oplus (\oplus_{n \geq 1} V^{\otimes n})$. The multiplication is given by concatenation: $(v_1 \otimes \cdots \otimes v_n) \cdot (w_1 \otimes \cdots \otimes w_m) = v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots w_m$. It is graded: $T_n(V) T_m(V) \subseteq T_{n+m}(V)$. It is also a Hopf algebra, with $\Delta(x) = x \otimes 1 + 1 \otimes x$, $S(x) = -x$, $\varepsilon(1) = 1$ and $\varepsilon(x) = 0$. If you choose a basis $e_1, \ldots, e_n$ of V, then $T(V)$ is the free associative algebra $\langle e_1, \ldots, e_n \rangle$. This algebra is $\mathbb{Z}_+$-graded: $T(V) = \oplus_{n \geq 0} T_n(V)$, where the degree of 1 is zero and the degree of each e_i is 1. It is also a $\mathbb{Z}$-graded bialgebra: $\Delta(T_n(V)) \subseteq \oplus(T_i \oplus T_{n-i}), S(T_n(V)) \subset T_n(V), \varepsilon : T(V) \to \mathbb{C}$ is a mapping of graded spaces $((\mathbb{C})_n = \{0\})$.

Definition 6.6. Let A be a Hopf algebra. Then a two-sided ideal $I \subseteq A$ is a *Hopf ideal* if $\Delta(I) \subseteq A \otimes I + I \otimes A$, $S(I) = I$, and $\varepsilon(I) = 0$.

You can check that the quotient of a Hopf algebra by a Hopf ideal is a Hopf algebra (and that the kernel of a map of Hopf algebras is always a Hopf ideal).

► **Exercise 6.3.** Show that $I_0 = \langle v \otimes w - w \otimes v | v, w \in V = T_1(V) \subseteq T(V)\rangle$ is a homogeneous Hopf ideal.

Corollary 6.7. *$S(V) = T(V)/I_0$ is a graded Hopf algebra.*

Choose a basis $e_1, \dots, e_n$ in V, so that $T(V) = \langle e_1, \dots, e_n\rangle$ and $S(V) = \langle e_1, \dots, e_n\rangle / \langle e_i e_j - e_j e_i\rangle$

► **Exercise 6.4.** Prove that the Hopf algebra $S(V)$ is isomorphic to $\mathbb{C}[e_1] \otimes \cdots \otimes \mathbb{C}[e_n]$.

Remark 6.8. From the discussion of $\mathbb{C}[x]$, we know that $S(V)$ and $S(V^*)$ are dual.

► **Exercise 6.5.** Describe the Hopf algebra structure on $T(V^*)$ that is determined by the pairing $\langle v_1 \otimes \cdots \otimes v_n, l_1 \otimes \cdots \otimes l_m\rangle = \delta_{m,n} l_1(v_1) \cdots l_n(v_n)$. (free coalgebra of V^*)

Now assume that $\mathfrak{g}$ is a Lie algebra.

Definition 6.9. The universal enveloping algebra of $\mathfrak{g}$ is $U(\mathfrak{g}) = T(\mathfrak{g})/\langle x \otimes y - y \otimes x - [x, y]\rangle$.

Exercise: prove that $\langle x \otimes y - y \otimes x - [x, y]\rangle$ is a Hopf ideal.

Corollary 6.10. *$U\mathfrak{g}$ is a Hopf algebra.*

If $e_1, \dots, e_n$ is a basis for V. $U\mathfrak{g} = \langle e_1, \dots, e_n | e_i e_j - e_j e_i = \sum_k c_{ij}^k e_k\rangle$, where c_{ij}^k are the structure constants of [,].

Remark 6.11. The ideal $\langle e_i e_j - e_j e_i\rangle$ is homogeneous, but $\langle x \otimes y - y \otimes x - [x, y]\rangle$ is not, so $U\mathfrak{g}$ isn't graded, but it is *filtered.*

Lecture 7

Last time we talked about Hopf algebras. Our basic examples were $\mathbb{C}[\Gamma]$ and $C(\Gamma) = \mathbb{C}[\Gamma]^*$. Also, for a vector space V, $T(V)$ is a Hopf algebra. Then $S(V) = T(V)/\langle x \otimes y - y \otimes x | x, y \in V\rangle$. And we also have $U\mathfrak{g} = T\mathfrak{g}/\langle x \otimes y - y \otimes x - [x, y] | x, y \in \mathfrak{g}\rangle$.

Today we'll talk about the universal enveloping algebra. Later, we'll talk about deformations of associative algebras because that is where recent progress in representation theory has been.

Universality of $U\mathfrak{g}$

We have that $\mathfrak{g} \hookrightarrow T\mathfrak{g} \to U\mathfrak{g}$. And $\sigma : \mathfrak{g} \hookrightarrow U\mathfrak{g}$ canonical embedding (of vector spaces and Lie algebras). Let A be an associative algebra with $\tau : \mathfrak{g} \to L(A) = \{A | [a, b] = ab - ba\}$ a Lie algebra homomorphism such that $\tau([x, y]) = \tau(x)\tau(y) - \tau(y)\tau(x)$.

Proposition 7.1. *For any such τ, there is a unique $\tau' : U\mathfrak{g} \to A$ homomorphism of associative algebras which extends τ:*

$$\begin{tikzcd}
U\mathfrak{g} \arrow[r, "\tau'"] & A \\
\mathfrak{g} \arrow[u, "\sigma"] \arrow[ur, "\tau"'] &
\end{tikzcd}$$

Proof. Because $T(V)$ is generated (freely) by 1 and V, $U\mathfrak{g}$ is generated by 1 and the elements of $\mathfrak{g}$. Choose a basis $e_1, \dots, e_n$ of $\mathfrak{g}$. Then we have that $\tau(e_i)\tau(e_j) - \tau(e_j)\tau(e_i) = \sum_k c_{ij}^k \tau(e_k)$. The elements $e_{i_1} \cdots e_{i_k}$ (this is a product) span $U\mathfrak{g}$ for indices i_j. From the commutativity of the diagram, $\tau'(e_i) = \tau(e_i)$. Since τ' is a homomorphism of associative algebras, we have that $\tau'(e_{i_1} \cdots e_{i_k}) = \tau'(e_{i_1}) \cdots \tau'(e_{i_k})$, so τ' is determined by τ uniquely: $\tau'(e_{i_1} \cdots e_{i_k}) = \tau(e_{i_1}) \cdots \tau(e_{i_k})$. We have to check that the ideal we mod out by is in the kernel. But that ideal is in the kernel because τ is a mapping of Lie algebras. ☐

Definition 7.2. A linear representation of $\mathfrak{g}$ in V is a pair $(V, \phi : \mathfrak{g} \to \mathrm{End}(V))$, where ϕ is a Lie algebra homomorphism. If A is an associative algebra, then $(V, \phi : A \to \mathrm{End}(V))$ a linear representation of A in V.

Corollary 7.3. *There is a bijection between representations of $\mathfrak{g}$ (as a Lie algebra) and representations of $U\mathfrak{g}$ (as an associative algebra).*

Proof. ($\Rightarrow$) By the universality, $A = End(V)$, $\tau = \phi$. ($\Leftarrow$)$\mathfrak{g} \subset L(U\mathfrak{g})$ is a Lie subalgebra. ☐

Example 7.4 (Adjoint representation)**.** $ad : \mathfrak{g} \to \operatorname{End}\mathfrak{g}$ given by $x : y \mapsto [x, y]$. This is also a representation of $U\mathfrak{g}$. Let $e_1, \ldots, e_n$ be a basis in $\mathfrak{g}$. Then we have that $ad_{e_i}(e_j) = [e_i, e_j] = \sum_k c_{ij}^k e_k$, so the matrix representing the adjoint action of the element e_i is the matrix $(ad_{e_i})_{jk} = (c_{ij}^k)$ of structural constants. You can check that $ad_{[e_i,e_j]} = (ad_{e_i})(ad_{e_j}) - (ad_{e_j})(ad_{e_i})$ is same as the Jacobi identity for the c_{ij}^k. We get $ad : U\mathfrak{g} \to \operatorname{End}(\mathfrak{g})$ by defining it on the monomials $e_{i_1} \cdots e_{i_k}$ as $ad_{e_{i_1} \cdots e_{i_k}} = (ad_{e_{i_1}}) \cdots (ad_{e_{i_k}})$ (the product of matrices).

Let's look at some other properties of $U\mathfrak{g}$.

Gradation in $U\mathfrak{g}$

Recall that V is a $\mathbb{Z}_+$*-graded vector space* if $V = \oplus_{n=0}^{\infty} V_n$. A linear map $f : V \to W$ between graded vector spaces is *grading-preserving* if $f(V_n) \subseteq W_n$. If we have a tensor product $V \otimes W$ of graded vector spaces, it has a natural grading given by $(V \otimes W)_n = \oplus_{i=0}^{n} V_i \otimes W_{n-i}$. The "geometric meaning" of this is that there is a linear action of $\mathbb{C}$ on V such that $V_n = \{x | t(x) = t^n \cdot x \text{ for all } t \in \mathbb{C}\}$. A graded morphism is a linear map respecting this action, and the tensor product has the diagonal action of $\mathbb{C}$, given by $t(x \otimes y) = t(x) \otimes t(y)$.

Example 7.5. If $V = \mathbb{C}[x]$, $\frac{d}{dx}$ is not grading preserving, $x\frac{d}{dx}$ is.

We say that $(V, [\ ,\ ])$ is a $\mathbb{Z}_+$*-graded Lie algebra* if $[\ ,\ ] : V \otimes V \to V$ is grading-preserving.

Example 7.6. Let V be the space of polynomial vector fields on $\mathbb{C} = Span(z^n \frac{d}{dz})_{n \geq 0}$. Then $V_n = \mathbb{C} z^n \frac{d}{dz}$.

An associative algebra $(V, m : V \otimes V \to V)$ is $\mathbb{Z}_+$*-graded* if m is grading-preserving.

Example 7.7.

(1) $V = \mathbb{C}[x]$, where the action of $\mathbb{C}$ is given by $x \mapsto tx$.

(2) $V = \mathbb{C}[x_1, \ldots, x_n]$ where the degree of each variable is 1 ... this is the n-th tensor power of the previous example.

(3) Lie algebra: $\operatorname{Vect}(\mathbb{C}) = \{\sum_{n \geq 0} a_n x^{n+1} \frac{d}{dx}\}$ with $\operatorname{Vect}_n(\mathbb{C}) = \mathbb{C} x^{n+1} \frac{d}{dx}$, $deg(x) = 1$. You can embed $\operatorname{Vect}(\mathbb{C})$ into polynomial vector fields on S^1 (Virasoro algebra).

(4) T(V) is a $\mathbb{Z}_+$-graded associative algebra, as is $S(V)$. However, $U\mathfrak{g}$ is not because we have modded out by a non-homogeneous ideal. But the ideal is not so bad. $U\mathfrak{g}$ is a $\mathbb{Z}_+$*-filtered algebra*:

Filtered spaces and algebras

Definition 7.8. V is a *filtered space* if it has an increasing filtration

$$V_0 \subset V_1 \subset V_2 \subset \cdots \subset V$$

such that $V = \bigcup V_i$, and $V_n =$ is a subspace of dimension less than or equal to n. $f : V \to W$ is a *morphism of filtered vector spaces* if $f(V_n) \subseteq W_n$.

We can define filtered Lie algebras and associative algebras as such that the bracket/multiplication are filtered maps.

There is a functor from filtered vector spaces to graded associative algebras $Gr : V \to Gr(V)$, where $Gr(V) = V_0 \oplus V_1/V_0 \oplus V_2/V_1 \cdots$. If $f : V \to W$ is filtration preserving, it induces a map $Gr(f) : Gr(V) \to Gr(W)$ functorially such that this diagram commutes:

$$\begin{tikzcd}
V \arrow[r, "f"] \arrow[d, "Gr"'] & W \arrow[d, "Gr"] \\
Gr(V) \arrow[r, "Gr(f)"] & Gr(W)
\end{tikzcd}$$

Let A be an associative filtered algebra (i.e. $A_iA_j \subseteq A_{i+j}$) such that for all $a \in A_i, b \in A_j$, $ab - ba \in A_{i+j-1}$.

Proposition 7.9. *For such an A,*

(1) $Gr(A)$ has a natural structure of an associative, commutative algebra (that is, the multiplication in A defines an associative, commutative multiplication in $Gr(A)$).

(2) For $a \in A_{i+1}, b \in A_{j+1}$, the operation $\{aA_i, bA_j\} = aA_ibA_j - bA_jaA_i \mod A_{i+j}$ is a lie bracket on $Gr(A)$.

(3) $\{x, yz\} = \{x, y\}z + y\{x, z\}$.

Proof. Exercise$_1$. You need to show that the given bracket is well defined, and then do a little dance, keeping track of which graded component you are in. □

Definition 7.10. A commutative associative algebra B is called a *Poisson algebra* if B is also a Lie algebra with lie bracket $\{\ ,\ \}$ (called a Poisson bracket) such that $\{x, yz\} = \{x, y\}z + y\{x, z\}$ (the bracket is a derivation).

Example 7.11. Let (M, ω) be a symplectic manifold (i.e. ω is a closed non-degenerate 2-form on M), then functions on M form a Poisson algebra. We could have $M = \mathbb{R}^{2n}$ with coordinates $p_1, \dots, p_n, q_1, \dots, q_n$, and $\omega = \sum_i dp_i \wedge dq_i$. Then the multiplication and addition on $C^\infty(M)$ is the usual one, and we can define $\{f, g\} = \sum_{ij} p^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}$, where $\omega = \sum \omega_{ij} dx^i \wedge dx^j$ and (p^{ij}) is the inverse matrix to (ω_{ij}). You can check that this is a Poisson bracket.

Let's look at $U\mathfrak{g} = \langle 1, e_i | e_i e_j - e_j e_i = \sum_k c_{ij}^k e_k \rangle$. Then $U\mathfrak{g}$ is filtered, with $(U\mathfrak{g})_n = Span\{e_{i_1} \cdots e_{i_k} | k \leq n\}$. We have the obvious inclusion $(U\mathfrak{g})_n \subseteq (U\mathfrak{g})_{n+1}$ and $(U\mathfrak{g})_0 = \mathbb{C} \cdot 1$.

Proposition 7.12.

(1) $U\mathfrak{g}$ *is a filtered algebra (i.e.* $(U\mathfrak{g})_r (U\mathfrak{g})_s \subseteq (U\mathfrak{g})_{r+s}$*)*

(2) $[(U\mathfrak{g})_r, (U\mathfrak{g})_s] \subseteq (U\mathfrak{g})_{r+s-1}$.

Proof. 1) obvious. 2) Exercise$_2$ (almost obvious). ☐

Now we can consider $Gr(U\mathfrak{g}) = \mathbb{C} \cdot 1 \oplus (\bigoplus_{r \geq 1} (U\mathfrak{g})_r / (U\mathfrak{g})_{r-1})$

Claim. $(U\mathfrak{g})_r / (U\mathfrak{g})_{r-1} \simeq S^r(\mathfrak{g}) =$ *symmetric elements of* $(\mathbb{C}[e_1, \dots, e_n])_r$.

Proof. Exercise$_3$. ☐

So $Gr(U\mathfrak{g}) \simeq S(\mathfrak{g})$ as a commutative algebra.

$S(\mathfrak{g}) \cong$ Polynomial functions on $\mathfrak{g}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathbb{C})$.

Consider $C^\infty(M)$. How can we construct a bracket $\{\ ,\ \}$ which satisfies Liebniz (i.e. $\{f, g_1 g_2\} = \{f, g_1\} g_2 + \{f, g_2\} g_1$). We expect that $\{f, g\}(x) = p^{ij}(x) \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} = \langle p(x), df(x) \wedge dg(x) \rangle$. Such a p is called a bivector field (it is a section of the bundle $TM \wedge TM \to M$). So a Poisson structure on $C^\infty(M)$ is the same as a bivector field p on M satisfying the Jacobi identity. You can check that the Jacobi identity is some bilinear identity on p^{ij} which follows from the Jacobi identity on $\{\ ,\ \}$. This is equivalent to the Schouten identity, which says that the Schouten bracket of some things vanishes [There should be a reference here]. This is more general than the symplectic case because p^{ij} can be degenerate.

Let $\mathfrak{g}$ have the basis $e_1, \dots, e_n$ and corresponding coordinate functions $x^1, \dots, x^n$. On $\mathfrak{g}^*$, we have that dual basis $e^1, \dots, e^n$ (you can identify these with the coordinates $x^1, \dots, x^n$), and coordinates $x_1, \dots, x_n$ (which you can identify with the e_i). The bracket on polynomial functions on $\mathfrak{g}^*$ is given by

$$\{p, q\} = \sum c_{ij}^k x_k \frac{\partial p}{\partial x_i} \frac{\partial q}{\partial x_j}.$$

This is a Lie bracket and clearly acts by derivations.

Next we will study the following. If you have polynomials p, q on $\mathfrak{g}^*$, you can try to construct an associative product $p *_t q = pq + tm_1(p, q) + \cdots$. We will discuss deformations of commutative algebras. The main example will be the universal enveloping algebra as a deformation of polynomial functions on $\mathfrak{g}^*$.

Lecture 8 - The PBW Theorem and Deformations

Last time, we introduced the universal enveloping algebra $U\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$, with its universality property. We discussed graded and filtered spaces and algebras. We showed that under some condition on a filtered algebra A, the graded algebra $Gr(A)$ is a Poisson algebra. We also checked that $U\mathfrak{g}$ satisfies this condition, and that $Gr(U\mathfrak{g}) \simeq S(\mathfrak{g})$ as graded commutative algebras. The latter space can be understood as the space $Pol(\mathfrak{g}^*)$ of polynomial functions on $\mathfrak{g}^*$. It turns out that the Poisson bracket on $Gr(U\mathfrak{g})$, expressed in $Pol(\mathfrak{g}^*)$, is given by

$$\{f, g\}(x) = x([df_x, dg_x])$$

for $f, g \in Pol(\mathfrak{g}^*)$ and $x \in \mathfrak{g}^*$. Note that f is a function on $\mathfrak{g}^*$ and x an element of $\mathfrak{g}^*$, so df_x is a linear form on $T_x\mathfrak{g}^* = \mathfrak{g}^*$, that is, $df_x \in \mathfrak{g}$.

Suppose that V admits a filtration $V_0 \subset V_1 \subset V_2 \subset \cdots$. Then, the associated graded space $Gr(V) = V_0 \oplus \bigoplus_{n\geq 1}(V_n/V_{n+1})$ is also filtered. (Indeed, every graded space $W = \bigoplus_{n\geq 0} W_n$ admits the filtration $W_0 \subset W_0 \oplus W_1 \subset W_0 \oplus W_1 \oplus W_2 \subset \cdots$) A natural question is: When do we have $V \simeq Gr(V)$ as filtered spaces ?

For the filtered space $U\mathfrak{g}$, the answer is a consequence of the following theorem.

Theorem 8.1 (Poincaré-Birkhoff-Witt). *Let $e_1, \dots, e_n$ be any linear basis for $\mathfrak{g}$. Let us also denote by $e_1, \dots, e_n$ the image of this basis in the universal enveloping algebra $U\mathfrak{g}$. Then the monomials $e_1^{m_1} \cdots e_n^{m_n}$ form a basis for $U\mathfrak{g}$.*

Corollary 8.2. *There is an isomorphism of filtered spaces $U\mathfrak{g} \simeq Gr(U\mathfrak{g})$.*

Proof of the corollary. In $S(\mathfrak{g})$, $e_1^{m_1} \cdots e_n^{m_n}$ also forms a basis, so we get an isomorphism $U\mathfrak{g} \simeq S(\mathfrak{g})$ of filtered vector spaces by simple identification of the bases. Since $Gr(U\mathfrak{g}) \simeq S(\mathfrak{g})$ as graded algebras, the corollary is proved. □

Remark 8.3. The point is that these spaces are isomorphic as *filtered* vector spaces. Saying that two infinite dimensional vector spaces are isomorphic is totally useless.

Proof of the theorem. By definition, the unordered monomials $e_{i_1} \cdots e_{i_k}$ for $k \leq p$ span the subspace $T_0 \oplus \cdots \oplus T_p$ of $T(\mathfrak{g})$, where $T_i = \mathfrak{g}^{\otimes i}$. Hence, they also span the quotient $(U\mathfrak{g})_p := T_0 \oplus \cdots \oplus T_p / \langle x \otimes y - y \otimes x - [x, y] \rangle$. The goal is now to show that the *ordered* monomials $e_1^{m_1} \cdots e_n^{m_n}$ for $m_1 + \cdots + m_n \leq p$ still span $(U\mathfrak{g})_p$. Let's prove this by induction on $p \geq 0$.

The case $p = 0$ being trivial, consider $e_{i_1} \cdots e_{i_a} \cdots e_{i_k}$, with $k \le p$, and assume that i_a has the smallest value among the indices $i_1, \dots, i_k$. We can move e_{i_a} to the left as follows

$$e_{i_1} \cdots e_{i_a} \cdots e_{i_k} = e_{i_a} e_{i_1} \cdots \hat{e}_{i_a} \cdots e_{i_k} + \sum_{b=1}^{a-1} e_{i_1} \cdots e_{i_{b-1}} [e_{i_b}, e_{i_a}] \cdots \hat{e}_{i_a} \cdots e_{i_k}.$$

Using the commutation relations $[e_{i_b}, e_{i_a}] = \sum_\ell c^\ell_{i_b i_a} e_\ell$, we see that the term to the right belongs to $(U\mathfrak{g})_{k-1}$. Iterating this procedure leads to an equation of the form

$$e_{i_1} \cdots e_{i_a} \cdots e_{i_k} = e_1^{m_1} \cdots e_n^{m_n} + \text{terms in } (U\mathfrak{g})_{k-1},$$

with $m_1 + \cdots + m_n = k \le p$. We are done by induction. The proof of the theorem is completed by the following homework.[This should really be done here] □

▶ **Exercise 8.1.** Prove that these ordered monomials are linearly independant.

Let's "generalize" the situation. We have $U\mathfrak{g}$ and $S(\mathfrak{g})$, both of which are quotients of $T(\mathfrak{g})$, with kernels $\langle x \otimes y - y \otimes x - [x, y] \rangle$ and $\langle x \otimes y - y \otimes x \rangle$. For any $\varepsilon \in \mathbb{C}$, consider the associative algebra $S_\varepsilon(\mathfrak{g}) = T(\mathfrak{g})/\langle x \otimes y - y \otimes x - \varepsilon[x, y] \rangle$. By construction, $S_0(\mathfrak{g}) = S(g)$ and $S_1(\mathfrak{g}) = U\mathfrak{g}$. Recall that they are isomorphic as filtered vector spaces.

Remark 8.4. If $\varepsilon \neq 0$, the linear map $\phi_\varepsilon : S_\varepsilon(\mathfrak{g}) \to U\mathfrak{g}$ given by $\phi_\varepsilon(x) = \varepsilon x$ for all $x \in \mathfrak{g}$ is an isomorphism of filtered algebras. So, we have nothing new here.

We can think of $S_\varepsilon(\mathfrak{g})$ as a non-commutative deformation of the associative commutative algebra $S(\mathfrak{g})$. (Note that commutative deformations of the algebra of functions on a variety correspond to deformations of the variety.)

Deformations of associative algebras

Let $(A, m : A \otimes A \to A)$ be an associative algebra, that is, the linear map m satisfies the quadratic equation

$$m(m(a, b), c) = m(a, m(b, c)). \tag{8.5}$$

Note that if $\varphi : A \to A$ is a linear automorphism, the multiplication m_φ given by $m_\varphi(a, b) = \varphi^{-1}(m(\varphi(a), \varphi(b)))$ is also associative. We like to think of m and m_φ as equivalent associative algebra structures on A. The "moduli space" of associative algebras on the vector space A is the set of solutions to equation 8.5 modulo this equivalence relation.

One can come up with a notion of deformation for almost any kind of object. In these deformation theories, we are interested in some cohomology theories because they parameterize obstructions to deformations. The knowledge of the cohomology of a given Lie algebra $\mathfrak{g}$, enables us say a lot about the deformations of $\mathfrak{g}$. We'll come back to this question in the next lecture.

Let us turn to our original example: the family of associative algebras $S_\varepsilon(\mathfrak{g})$. Recall that by the PBW theorem, we have an isomorphism of filtered vector spaces $S_\varepsilon(\mathfrak{g}) \xrightarrow{\psi} S(\mathfrak{g}) = Pol(\mathfrak{g}^*)$, but this is not an isomorphisms of associative algebras. Therefore, the multiplication defined by $f * g := \psi(\psi^{-1}(f) \cdot \psi^{-1}(g))$ is not the normal multiplication on $S(\mathfrak{g})$. We claim that the result is of the form

$$f * g = fg + \sum_{n\geq 1} \varepsilon^n m_n(f,g),$$

where m_n is a bidifferential operator of order n, that is, it is of the form

$$m_n(f,g) = \sum_{I,J} p_n^{I,J} \partial^I f \partial^J g,$$

where I and J are multi-indices of length n, and $p_n^{I,J} \in Pol(\mathfrak{g}^*)$. The idea of the proof is to check this for $f = \psi(e_1^{r_1} \cdots e_n^{r_n})$ and $g = \psi(e_1^{l_1} \cdots e_n^{l_n})$ by writing

$$e_1^{r_1} \cdots e_n^{r_n} \cdot e_1^{l_1} \cdots e_n^{l_n} = e_1^{l_1+r_1} \cdots e_n^{l_n+r_n} + \sum_{k\geq 1} \varepsilon^k m_k(e_1^{r_1} \cdots e_n^{r_n}, e_1^{l_1} \cdots e_n^{l_n})$$

in $S_\varepsilon(\mathfrak{g})$ using the commuting relations.

► **Exercise 8.2.** Compute the $p_n^{I,J}$ for the Lie algebra $\mathfrak{g}$ generated by X, Y, and H with bracket $[X,Y] = H, [H,X] = [H,Y] = 0$. This is called the Heisenberg Lie algebra.

So we have a family of products on $Pol(\mathfrak{g}^*)$ which depend on ε in the following way:

$$f * g = fg + \sum_{n\geq 1} \varepsilon^n m_n(f,g)$$

Since f, g are polynomials and m_n is a bidifferential operator of order n, this series terminates, so it is a polynomial in ε. If we try to extend this product to $C^\infty(\mathfrak{g}^*)$, then there are questions about the convergence of the product $*$. There are two ways to deal with this problem. The first one is to take these matters of convergence seriously, consider some topology on $C^\infty(\mathfrak{g}^*)$ and demand that the series converges. The other solution is to forget about convergence and just think in terms of formal power series in ε. This is the so-called "formal deformation" approach. As we shall see, there are interesting things to say with this seemingly rudimentary point of view.

Formal deformations of associative algebras

Let (A, m_0) be an associative algebra over $\mathbb{C}$. Then, a formal deformation of (A, m_0) is a $\mathbb{C}[[h]]$-linear map $m : A[[h]] \otimes_{\mathbb{C}[[h]]} A[[h]] \to A[[h]]$ such that

$$m(a,b) = m_0(a,b) + \sum_{n\geq 1} h^n m_n(a,b)$$

for all $a, b \in A$, and such that $(A[[h]], m)$ is an associative algebra. We say that two formal deformations m and $\tilde{m}$ are equivalent if there is a $\mathbb{C}[[h]]$-automorphism $A[[h]] \xrightarrow{\varphi} A[[h]]$ such that $\tilde{m} = m_\varphi$, with $\varphi(x) = x + \sum_{n\geq 1} h^n \varphi_n(x)$ for all $x \in A$, where φ_n is an endomorphism of A.

<u>Question</u>: Describe the equivalence classes of formal deformations of a given associative algebra.

When (A, m_0) is a commutative algebra, the answer is known. Philosophically and historically, this case is relevant to quantum mechanics. In classical mechanics, observables are smooth functions on a phase space M, i.e they form a commutative associative algebra $C^\infty(M)$. But when you quantize this system (which is needed to describe something on the order of the Planck scale), you cannot think of observables as functions on phase space anymore. You need to deform the commutative algebra $C^\infty(M)$ to a noncommutative algebra. And it works...

From now on, let (A, m_0) be a commutative associative algebra. Let's write $m_0(a,b) = ab$, and $m(a,b) = a * b$. (This is called a star product, and the terminology goes back to the sixties and the work of J. Vey). Then we have

$$a * b = ab + \sum_{n\geq 1} h^n m_n(a,b).$$

Demanding the associativity of $*$ imposes an infinite number of equations for the m_n's, one for each order:

h^0: $a(bc) = (ab)c$

h^1: $am_1(b,c) + m_1(a,bc) = m_1(a,b)c + m_1(ab,c)$

h^2: $\dots$

$\vdots$

▶ **Exercise 8.3.** Show that the bracket $\{a,b\} = m_1(a,b) - m_1(b,a)$ defines a Poisson structure on A. This means that we can think of a Poisson structure on an algebra as the remnants of a deformed product where $a * b - b * a = h\{a,b\} + O(h)$.

One easily checks that if two formal deformations m and $\tilde{m}$ are equivalent via φ (i.e: $\tilde{m} = m_\varphi$), then the associated $m_1, \tilde{m}_1$ are related by $m_1(a,b) = \tilde{m}_1(a,b) + \varphi_1(ab) - \varphi_1(a)b - a\varphi_1(b)$. In particular, two equivalent formal deformations induce the same Poisson structure. Also, it is possible to choose a representative in an equivalence class such that m_1 is skew-symmetric (and then, $m_1(a,b) = \frac{1}{2}\{a,b\}$). This leads to the following program for the classification problem:

1. Classify all Poisson structures on A.
2. Given a Poisson algebra $(A, \{\ ,\ \})$, classify all equivalence classes of star products on A such that $m_1(a,b) = \frac{1}{2}\{a,b\}$.

Under some mild assumption, it can be assumed that a star product is symmetric, i.e. that it satisfies the equation $m_n(a,b) = (-1)^n m_n(b,a)$ for all n. The program given above was completed by Maxim Kontsevitch for the algebra of smooth functions on a manifold M. Recall that Poisson structures on $C^\infty(M)$ are given by bivector fields on M that satisfy the Jacobi identity.

Theorem 8.6 (Kontsevich, 1994)**.** *Let A be the commutative associative algebra $C^\infty(M)$, and let us fix a Poisson bracket $\{\ ,\ \}$ on A. Equivalence classes of symmetric star products on A with $m_1(a,b) = \frac{1}{2}\{a,b\}$ are in bijection with formal deformations of $\{\ ,\ \}$ modulo formal diffeomorphisms of M.*

A *formal deformation* of $\{\ ,\ \}$ is a Poisson bracket $\{\ ,\ \}_h$ on $A[[h]]$ such that

$$\{a,b\}_h = \{a,b\} + \sum_{n\geq 1} h^n \mu_n(a,b)$$

for all a,b in A. A *formal diffeomorphism* of M is an automorphism φ of $A[[h]]$ such that $\varphi(f) = f + \sum_{n\geq 1} h^n \varphi_n(f)$ and $\varphi(fg) = \varphi(f)\varphi(g)$ for all f,g in A.

We won't prove the theorem (it would take about a month) . As Poisson algebras are Lie algebras, it relates deformations of associative algebras to deformations of Lie algebras.

Formal deformations of Lie algebras

Given a Lie algebra $(\mathfrak{g}, [\ ,\ ])$, you want to know how many formal deformations of $\mathfrak{g}$ there are. Sometimes, there are none (like in the case of $\mathfrak{sl}_n$, as we will see later). Sometimes, there are plenty (as for triangular matrices). The goal is now to construct some invariants of Lie algebras which will tell you whether there are deformations, and how many of them there are. In order to do this, we should consider cohomology theories for Lie algebras. We will focus first on the standard complex $C^\bullet(\mathfrak{g},\mathfrak{g}) = \bigoplus_{n\geq 0} C^n(\mathfrak{g},\mathfrak{g})$, where $C^n(\mathfrak{g},\mathfrak{g}) = \mathrm{Hom}(\Lambda^n\mathfrak{g},\mathfrak{g})$.

Lecture 9

Let's summarize what has happened in the last couple of lectures.

1. We talked about $T(\mathfrak{g})$, and then constructed three algebras:
   - $U\mathfrak{g} = T(\mathfrak{g})/\langle x\otimes y - y\otimes x - [x,y]\rangle$, with $U\mathfrak{g} = S_1(\mathfrak{g}) \simeq S_\varepsilon(\mathfrak{g})$ as filtered associative algebras, for all non-zero $\varepsilon \in \mathbb{C}$.
   - $S_\varepsilon(\mathfrak{g}) = T(\mathfrak{g})/\langle x\otimes y - y\otimes x - \varepsilon[x,y]\rangle$ is a family of associative algebras, with $S_\varepsilon(\mathfrak{g}) \simeq S_0(\mathfrak{g})$ as filtered vector spaces.
   - $S_0(\mathfrak{g}) \cong Pol(\mathfrak{g}^*) = T(\mathfrak{g})/\langle x\otimes y - y\otimes x\rangle = S_0(\mathfrak{g})$ is an associative, commutative algebra with a Poisson structure defined by the Lie bracket.
2. We have two "pictures" of deformations of an algebra
   (a) There is a simple "big" algebra B (such as $B = T(\mathfrak{g})$) and a family of ideals I_ε. Then we get a family $B/I_\varepsilon = A_\varepsilon$. This becomes a deformation family of the associative algebra A_0 if we identify $A_\varepsilon \simeq A_0$ as vector spaces (these are called *torsion free* deformations). Fixing this isomorphism gives a family of associative products on A_0.

   We can think of this geometrically as a family of (embedded) varieties.

   (b) Alternatively, we can talk about deformations intrinsically (i.e., without referring to some bigger B). Suppose we have A_0 and a family of associative products $a *_\varepsilon b$ on A_0.

   Example 9.1. Let $Pol(\mathfrak{g}^*) \xrightarrow{\phi} S_\varepsilon(\mathfrak{g})$ be the isomorphism of the PBW theorem. Then define $f * g = \phi^{-1}(\phi(f)\cdot\phi(g)) = fg + \sum_{n\geq 1}\varepsilon^n m_n(f,g)$.

   Understanding deformations makes a connection between representation theory and Poisson geometry. A second course on Lie theory should discuss symplectic leaves of $Pol(\mathfrak{g}^*)$, which happen to be coadjoint orbits and correspond to representations. This is why deformations are relevant to representation theory.

Let A be a Poisson algebra with bracket $\{\ ,\ \}$, so it is a commutative algebra, and a Lie algebra, with the bracket acting by derivations. Typically, $A = C^\infty(M)$. Equivalence classes of formal (i.e., formal power series) symmetric (i.e.,$m_n(f,g) = (-1)^n m_n(g,f)$) star products on $C^\infty(M)$ are in bijection with equivalence classes of formal deformations of $\{\ ,\ \}$ on $C^\infty(M)[[h]]$.

Apply this to the case $A = C^\infty(\mathfrak{g}^*)$. The associative product on $S_\varepsilon(\mathfrak{g})$ comes from the product on $T(\mathfrak{g})$. The question is, "how many equivalence classes of star products are there on A?" Any formal deformation of the Poisson structure on $(A, \{\ ,\ \}_\mathfrak{g})$ is a PBW deformation of some formal deformation of the Lie algebra $C^\infty(\mathfrak{g}^*)$ (with Lie bracket $\{f,g\}(x) = x(df \wedge dg)$). Such a deformation is equivalent to a formal deformation of the Lie algebra structure on $\mathfrak{g}$. This is one of the reasons that deformations of Lie algebras are important — they describe deformations of certain associative algebras. When one asks such questions, some cohomology theory always shows up.

Lie algebra cohomology

Recall that (M,ϕ) is a $\mathfrak{g}$-module if $\phi : \mathfrak{g} \to \operatorname{End}(M)$ is a Lie algebra homomorphism. We will write xm for $\phi(x)m$. Define $C^\bullet(\mathfrak{g}, M) = \bigoplus_{q\geq 0} C^q(\mathfrak{g}, M)$ where $C^q(\mathfrak{g}, M) = \operatorname{Hom}(\Lambda^q\mathfrak{g}, M)$ (linear maps). We define $d : C^q \to C^{q+1}$ by

$$\begin{aligned} dc(x_1 \wedge \cdots \wedge x_{q+1}) &= \\ = \sum_{1\leq s<t\leq q+1} &(-1)^{s+t-1} c([x_s,x_t] \wedge x_1 \wedge \cdots \wedge \hat{x}_s \wedge \cdots \wedge \hat{x}_t \wedge \cdots \wedge x_{q+1}) \\ &+ \sum_{s=1}^{q+1} (-1)^s x_s c(x_1 \wedge \cdots \wedge \hat{x}_s \wedge \cdots \wedge x_{q+1}) \end{aligned}$$

▶ **Exercise 9.1.** Show that $d^2 = 0$.

Motivation: If $\mathfrak{g} = \operatorname{Vect}(\mathcal{M})$, $M = C^\infty(\mathcal{M})$, then $C^q(\mathfrak{g}, M) = \Omega^q(\mathcal{M})$, with the Cartan formula

$$\begin{aligned} (d\omega)(\xi_1 \wedge \cdots \wedge \xi_{q+1}) &= \\ = \sum_{1\leq s<t\leq q+1} &(-1)^{s+t-1} \omega([\xi_s,\xi_t] \wedge \xi_1 \wedge \cdots \wedge \hat{\xi}_s \wedge \cdots \wedge \hat{\xi}_t \wedge \cdots \wedge \xi_{q+1}) \\ &+ \sum_{s=1}^{q+1} (-1)^s \xi_s \omega(\xi_1 \wedge \cdots \wedge \hat{\xi}_s \wedge \cdots \wedge \xi_{q+1}) \end{aligned}$$

for vector fields ξ_i.

Another motivation comes from the following proposition.

Proposition 9.2. *$C^\bullet(\mathfrak{g}, \mathbb{C}) \simeq \Omega^\bullet_R(G) \subseteq \Omega^\bullet(G)$ where $\mathbb{C}$ is the 1 dimensional trivial module over $\mathfrak{g}$ (so $xm = 0$).*

▶ **Exercise 9.2.** Prove it.

Remark 9.3. This was Cartan's original motivation for Lie algebra cohomology. It turns out that the inclusion $\Omega_R^\bullet(G) \hookrightarrow \Omega^\bullet(G)$ is a homotopy equivalence of complexes (i.e. the two complexes have the same homology), and the proposition above tells us that $C^\bullet(\mathfrak{g}, \mathbb{C})$ is homotopy equivalent to $\Omega_R(G)$. Thus, by computing the Lie algebra cohomology of $\mathfrak{g}$ (the homology of the complex $C^\bullet(\mathfrak{g}, \mathbb{C})$), one obtains the De Rham cohomology of G (the homology of the complex $\Omega^\bullet(G)$).

Define $H^q(\mathfrak{g}, M) = \ker(d : C^q \to C^{q+1})/\operatorname{im}(d : C^{q-1} \to C^q)$ as always. Let's focus on the case $M = \mathfrak{g}$, the adjoint representation: $x \cdot m = [x, m]$.

$H^0(\mathfrak{g}, \mathfrak{g})$ We have that $C^0 = \operatorname{Hom}(\mathbb{C}, \mathfrak{g}) \cong \mathfrak{g}$, and

$$dc(y) = y \cdot c = [y, c].$$

so $\ker(d : C^0 \to C^1)$ is the set of $c \in \mathfrak{g}$ such that $[y, c] = 0$ for all $y \in \mathfrak{g}$. That is, the kernel is the center of $\mathfrak{g}$, $Z(\mathfrak{g})$. So $H^0(\mathfrak{g}, \mathfrak{g}) = Z(\mathfrak{g})$.

$H^1(\mathfrak{g}, \mathfrak{g})$ The kernel of $d : C^1(\mathfrak{g}, \mathfrak{g}) \to C^2(\mathfrak{g}, \mathfrak{g})$ is

$$\{\mu : \mathfrak{g} \to \mathfrak{g} | d\mu(x, y) = \mu([x, y]) - [x, \mu(y)] - [\mu(x), y] = 0 \text{ for all } x, y \in \mathfrak{g}\},$$

which is exactly the set of derivations of $\mathfrak{g}$. The image of $d : C^0(\mathfrak{g}, \mathfrak{g}) \to C^1(\mathfrak{g}, \mathfrak{g})$ is the set of *inner derivations*, $\{dc : \mathfrak{g} \to \mathfrak{g} | dc(y) = [y, c]\}$. The Liebniz rule is satisfied because of the Jacobi identity. So

$H^1(\mathfrak{g}, \mathfrak{g}) = \{\text{derivations}\}/\{\text{inner derivations}\} =:$ *outer derivations.*

$H^2(\mathfrak{g}, \mathfrak{g})$ Let's compute $H^2(\mathfrak{g}, \mathfrak{g})$. Suppose $\mu \in C^2$, so $\mu : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g}$ is a linear map. What does $d\mu = 0$ mean?

$$\begin{aligned} d\mu(x_1, x_2, x_3) = \mu([x_1, x_2], x_3) - \mu([x_1, x_3], x_2) + \mu([x_2, x_3], x_1) \\ - [x_1, \mu(x_2, x_3)] + [x_2, \mu(x_1, x_3)] - [x_3, \mu(x_1, x_2)] \\ = -\mu(x_1, [x_2, x_3]) - [x_1, \mu(x_2, x_3)] + \text{cyclic permutations} \end{aligned}$$

Where does this kind of thing show up naturally?

Consider deformations of Lie algebras:

$$[x, y]_h = [x, y] + \sum_{n \geq 1} h^n m_n(x, y)$$

where the $m_n : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ are bilinear. The deformed bracket $[\ ,\ ]_h$ must satisfy the Jacobi identity,

$$[a, [b, c]_h]_h + [b, [c, a]_h]_h + [c, [a, b]_h]_h = 0$$

which gives us relations on the m_n. In degree h^N, we get

$$\begin{aligned}
&[a, m_N(b,c)] + m_N(a,[b,c]) + \sum_{k=1}^{N-1} m_k(a, m_{N-k}(b,c)) + \\
&[b, m_N(c,a)] + m_N(b,[c,a]) + \sum_{k=1}^{N-1} m_k(b, m_{N-k}(c,a)) + \\
&\quad [c, m_N(a,b)] + m_N(c,[a,b]) + \sum_{k=1}^{N-1} m_k(c, m_{N-k}(a,b)) = 0 \qquad (9.4)
\end{aligned}$$

▶ **Exercise 9.3.** Derive equation 9.4.

Define $[m_K, m_{N-K}](a,b,c)$ as

$$m_K\big(a, m_{N-K}(b,c)\big) + m_K\big(b, m_{N-K}(c,a)\big) + m_K\big(c, m_{N-K}(a,b)\big).$$

Then equation 9.4 can be written as

$$dm_N = \sum_{k=1}^{N-1} [m_k, m_{N-k}] \qquad (9.5)$$

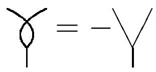
Theorem 9.6. *Assume that for all $n \le N-1$, we have the relation $dm_n = \sum_{k=1}^{n-1}[m_k, m_{n-k}]$. Then $d(\sum_{k=1}^{N-1}[m_k, m_{N-k}]) = 0$.*

▶ **Exercise 9.4.** Prove it.

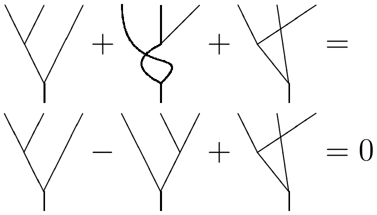
The theorem tells us that if we have a “partial deformation” (i.e. we have found $m_1, \dots, m_{N-1}$), then the expression $\sum_{k=1}^{N-1}[m_k, m_{N-k}]$ is a 3-cocycle. Furthermore, equation 9.5 tells us that if we are to extend our deformation to one higher order, $\sum_{k=1}^{N-1}[m_k, m_{N-k}]$ must represent zero in $H^3(\mathfrak{g}, \mathfrak{g})$.

Taking $N = 1$, we get $dm_1 = 0$, so $\ker(d : C^2 \to C^3) =$ space of first coefficients of formal deformations of $[\ ,\ ]$. It will turn out that H^2 is the space of equivalence classes of m_1.

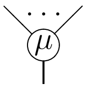
It is worth noting that the following “pictorial calculus” may make some of the above computations easier. In the following pictures, arrows are considered to be oriented downwards, and trivalent vertices with two lines coming in and one going out represent the Lie bracket. So, for example, the antisymmetry of the Lie bracket is expressed as



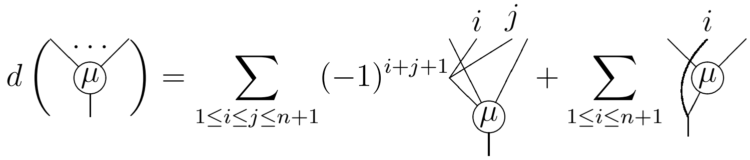
and the Jacobi identity is



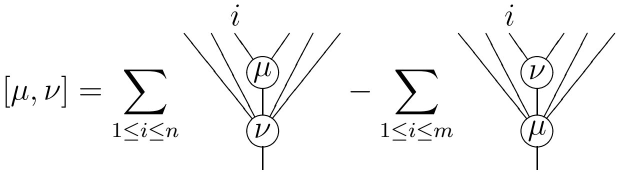
We can also use pictures to represent cocycles. Take $\mu \in H^n(\mathfrak{g}, \mathfrak{g})$. Then we draw μ as



with n lines going in. Then, the Cartan formula for the differential says that



and the bracket of two cocycles $\mu \in H^m$ and $\nu \in H^n$ is



▶ **Exercise 9.5.** Use pictures to show that $d[\mu, \nu] = \pm[d\mu, \nu] \pm [\mu, d\nu]$.

Also, these pictures can be used to do the calculations in Exercises 9.3 and 9.4.

Lecture 10

Here is the take-home exam, it's due on Tuesday:

(1) $B \subset SL_2(\mathbb{C})$ are upper triangular matrices, then

  - Describe $X = SL_2(\mathbb{C})/B$
  - $SL_2(\mathbb{C})$ acts on itself via left multiplication implies that it acts on X. Describe the action.

(2) Find $\exp \begin{pmatrix} 0 & x_1 & & 0 \\ & \ddots & \ddots & \\ & & 0 & x_{n-1} \\ 0 & & & 0 \end{pmatrix}$

(3) Prove that if V, W are filtered vector spaces (with increasing filtration) and $\phi : V \to W$ satisfies $\phi(V_i) \subseteq W_i$, and $Gr(\phi) : Gr(V) \xrightarrow{\sim} Gr(W)$ an isomorphism, then ϕ is a linear isomorphism of filtered spaces.

Lie algebra cohomology

Recall $C^\bullet(\mathfrak{g}, M)$ from the previous lecture, for M a finite dimensional representation of $\mathfrak{g}$ (and $\mathfrak{g}$ finite dimensional). There is a book by D. Fuchs, *Cohomology of ∞ dimensional Lie algebras* [Fuc50].

We computed that $H^0(\mathfrak{g}, \mathfrak{g}) = Z(\mathfrak{g}) \simeq \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ and that $H^1(\mathfrak{g}, \mathfrak{g})$ is the space of exterior derivations of $\mathfrak{g}$. Say $c \in Z^1(\mathfrak{g}, \mathfrak{g})$,[1] so $[c] \in H^1(\mathfrak{g}, \mathfrak{g})$. Define $\tilde{\mathfrak{g}}_c = \mathfrak{g} \oplus \mathbb{C}\partial_c$ with the bracket $[(x, t), (y, s)] = ([x, y] - tc(y) + sc(x), 0)$. So if $e_1, \dots, e_n$ is a basis in $\mathfrak{g}$ with the usual relations $[e_i, e_j] = c_{ij}^k e_k$, then we get one more generator ∂_c such that $[\partial_c, x] = c(x)$. Then $H^1(\mathfrak{g}, \mathfrak{g})$ is the space of equivalence classes of extensions

$$0 \to \mathfrak{g} \to \tilde{\mathfrak{g}} \to \mathbb{C} \to 0$$

up to the equivalences f such that the diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{g} \arrow[r] \arrow[d, "\mathrm{Id}"] & \tilde{\mathfrak{g}} \arrow[r] \arrow[d, "f"] & \mathbb{C} \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\
0 \arrow[r] & \mathfrak{g} \arrow[r] & \tilde{\mathfrak{g}}' \arrow[r] & \mathbb{C} \arrow[r] & 0
\end{tikzcd}$$

This is the same as the space of exterior derivations.

---

[1] $Z^n(\mathfrak{g}, M)$ is the space of n-cocycles, i.e. the kernel of $d : C^n(\mathfrak{g}, M) \to C^{n+1}(\mathfrak{g}, M)$.

$H^2(\mathfrak{g},\mathfrak{g})$ and Deformations of Lie algebras

A *deformation* of $\mathfrak{g}$ is the vector space $\mathfrak{g}[[h]]$ with a bracket $[a,b]_h = [a,b]+\sum_{n\geq 1} h^n m_n(a,b)$ such that $m_n(a,b)=-m_n(b,a)$ and

$$[a,[b,c]_h]_h + [b,[c,a]_h]_h + [c,[a,b]_h]_h = 0.$$

The h^N order term of the Jacobi identity yields equation 9.4, which was

$$[a,m_N(b,c)] + m_N(a,[b,c]) + \sum_{k=1}^{N-1} m_k(a,m_{N-k}(b,c)) + \text{cycle} = 0$$

where "cycle" is the same thing, with a, b, and c permuted cyclically. For $\mu\in C^2(\mathfrak{g},\mathfrak{g})$, we compute

$$d\mu(a,b,c) = -[a,\mu(b,c)] - \mu(a,[b,c]) + \text{cycle}.$$

Define

$$\{m_k,m_{N-k}\}(a,b,c) \overset{def}{=} m_k(a,m_{N-k}(b,c)) + \text{cycle}$$

This is called the Gerstenhaber bracket ... do a Google search for it if you like ... it is a tiny definition from a great big theory.

Then we can rewrite equation 9.4 as equation 9.5, which was

$$dm_N = \sum_{k=1}^{N-1}\{m_k,m_{N-k}\}.$$

In partiular, $dm_1=0$, so m_1 is in $Z^2(\mathfrak{g},\mathfrak{g})$.

Equivalences: $[a,b]'_h \simeq [a,b]_h$ if $[a,b]'_h = \phi^{-1}([\phi(a),\phi(b)]_h)$ for some $\phi(a) = a+\sum_{n\geq 1} h^n\phi_n(a)$. then

$$m_1'(a,b) = m_1(a,b) - \phi_1([a,b]) + [a,\phi_1(b)] + [\phi_1(a),b].$$

which we can write as $m_1' = m_1 + d\phi_1$. From this we can conclude

Claim. *The space of equivalence classes of possible m_1 is exactly $H^2(\mathfrak{g},\mathfrak{g})$.*

Claim (was HW)**.** *If m_1 is a 2-cocycle, and $m_{N-1},\dots,m_2$ satisfy the equations we want, then*

$$d\left(\sum_{k=1}^{N-1}\{m_k,m_{N-k}\}\right) = 0.$$

This is not enough; we know that $\sum_{k=1}^{N+1}\{m_k,m_{N-k}\}$ is in $Z^3(\mathfrak{g},\mathfrak{g})$, but to find m_N, we need it to be trivial in $H^3(\mathfrak{g},\mathfrak{g})$ because of equation 9.5. If the cohomology class of $\sum_{k=1}^{N+1}\{m_k,m_{N-k}\}$ is non-zero, it's class in

$H^3(\mathfrak{g},\mathfrak{g})$ is called an *obstruction* to n-th order deformation. If $H^3(\mathfrak{g},\mathfrak{g})$ is zero, then any first order deformation (element of $H^2(\mathfrak{g},\mathfrak{g})$) extends to a deformation, but if $H^3(\mathfrak{g},\mathfrak{g})$ is non-zero, then we don't know that we can always extend. Thus, $H^3(\mathfrak{g},\mathfrak{g})$ is the space of all possible obstructions to extending a deformation.

Let's keep looking at cohomology spaces. Consider $C^\bullet(\mathfrak{g},\mathbb{C})$, where $\mathbb{C}$ is a one dimensional trivial representation of $\mathfrak{g}$ given by $x \mapsto 0$ for any $x \in \mathfrak{g}$.

First question: take $U\mathfrak{g}$, with the corresponding 1 dimensional representation $\varepsilon : U\mathfrak{g} \to \mathbb{C}$ given by $\varepsilon(x) = 0$ for $x \in \mathfrak{g}$.

▶ **Exercise 10.1.** Show that $(U\mathfrak{g}, \varepsilon, \Delta, S)$ is a Hopf algebra with the ε above, $\Delta(x) = 1\otimes x + x\otimes 1$, and $S(x) = -x$ for $x \in \mathfrak{g}$. Remember that Δ and ε are algebra homomorphisms, and that S is an anti-homomorphism.

Let's compute $H^1(\mathfrak{g},\mathbb{C})$ (H^0 is boring, just a point). This is $\ker(C^1 \xrightarrow{d} C^2)$. Well, $C^1(\mathfrak{g},\mathbb{C}) = \mathrm{Hom}(\mathfrak{g},\mathbb{C})$, $C^2(\mathfrak{g},\mathbb{C}) = \mathrm{Hom}(\Lambda^2\mathfrak{g},\mathbb{C})$, and

$$dc(x,y) = c([x,y]).$$

So the kernel is the set of c such that $c([x,y]) = 0$ for all $x,y \in \mathfrak{g}$. Thus, $\ker(d) \subseteq C^1(\mathfrak{g},\mathbb{C})$ is the space of $\mathfrak{g}$-invariant linear functionals. Recall that $\mathfrak{g}$ acts on $\mathfrak{g}$ by the adjoint action, and on $\mathfrak{g}^* = C^1(\mathfrak{g},\mathfrak{g})$ by the coadjoint action ($x : l \mapsto l_x$ where $l_x(y) = l([x,y])$). Under the coadjoint action, $l \in \mathfrak{g}^*$ is $\mathfrak{g}$-invariant if $l_x = 0$. Note that C^0 is just one point, so its image doesn't have anything in it.

Now let's compute $H^2(\mathfrak{g},\mathbb{C}) = \ker(d : C^2 \to C^3)/\operatorname{im}(d : C^1 \to C^2)$. Let $c \in Z^2$, then

$$dc(x,y,z) = c([x,y],z) - c([x,z],y) + c([y,z],x) = 0$$

for all $x,y,z \in \mathfrak{g}$. Now let's find the image of $d : C^1 \to C^2$: it is the set of functions of the form $dl(x,y) = l([x,y])$ where $l \in \mathfrak{g}^*$. It is clear that $l([x,y])$ are (trivial) 2-cocycles because of the Jacobi identity. Let's see what can we cook with this H^2.

Definition 10.1. A *central extension* of $\mathfrak{g}$ is a short exact sequence

$$0 \to \mathbb{C} \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0.$$

Two such extensions are equivalent if there is a Lie algebra isomorphism $f : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}'$ such that the diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C} \arrow[r] \arrow[d, "\mathrm{Id}"] & \tilde{\mathfrak{g}} \arrow[r] \arrow[d, "f"] & \mathfrak{g} \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\
0 \arrow[r] & \mathbb{C} \arrow[r] & \tilde{\mathfrak{g}}' \arrow[r] & \mathfrak{g} \arrow[r] & 0
\end{tikzcd}$$

Theorem 10.2. *$H^2(\mathfrak{g}, \mathbb{C})$ is isomorphic to the space of equivalence classes of central extensions of $\mathfrak{g}$.*

Proof. Let's describe the map in one direction. If $c \in Z^2$, then consider $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}$ with the bracket $[(x,t),(y,s)] = ([x,y], c(x,y))$. Equivalences of extensions boil down to $c(x,y) \mapsto c(x,y) + l([x,y])$.

▶ **Exercise 10.2.** Finish this proof.

□

Let's do some (infinite dimensional) examples of central extensions.

Example 10.3. [Affine Kac-Moody algebras] If $\mathfrak{g} \subseteq \mathfrak{gl}_n$, then we define the *loop space* or *loop algebra* $\mathcal{L}\mathfrak{g}$ to be the set of maps $S^1 \to \mathfrak{g}$. To make the space more manageable, we only consider Laurent polynomials, $z \mapsto \sum_{m\in\mathbb{Z}} a_m z^m$ for $a_m \in \mathfrak{g}$ with all but finitely many of the a_m equal to zero. The bracket is given by $[f,g]_{\mathcal{L}\mathfrak{g}}(z) = [f(z), g(z)]_{\mathfrak{g}}$.

Since $\mathfrak{g} \subseteq \mathfrak{gl}_n$, there is an induced trace $tr : \mathfrak{g} \to \mathbb{C}$. This gives a non-degenerate inner product on $\mathcal{L}\mathfrak{g}$:

$$(f,g) := \oint_{|z|=1} tr\big(f(z^{-1})g(z)\big)\frac{dz}{z}.$$

There is a natural 2-cocylce on $\mathcal{L}\mathfrak{g}$, given by

$$c(f,g) = \frac{1}{2\pi i}\oint_{|z|=1} tr\big(f(z)g'(z)\big)\frac{dz}{z} = \operatorname*{Res}_{z=0}\Big(tr\big(f(z)g'(z)\big)\Big),$$

and a natural outer derivation $\partial : \mathcal{L}\mathfrak{g} \to \mathcal{L}\mathfrak{g}$ given by $\partial x(z) = \frac{\partial x(z)}{\partial z}$.

The Kac-Moody algebra is $\mathcal{L}\mathfrak{g} \oplus \mathbb{C}\partial \oplus \mathbb{C}c$. A second course on Lie theory should have some discussion of the representation theory of this algebra.

Example 10.4. Let $\mathfrak{gl}_\infty$ be the algebra of matrices with finitely many non-zero entries. It is not very interesting. Let $\mathfrak{gl}^1_\infty$ be the algebra of matrices with finitely many non-zero diagonals. $\mathfrak{gl}^1_\infty$ is "more infinite dimensional" than $\mathfrak{gl}_\infty$, and it is more interesting.

▶ **Exercise 10.3.** Define

$$J = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & -I \end{array}\right).$$

For $x, y \in \mathfrak{gl}_\infty$, show that

$$c(x,y) = tr(x[J,y])$$

is well defined (i.e. is a finite sum).

This c is a non-trivial 1-cocycle, i.e. $[c] \in H^2(\mathfrak{gl}^1_\infty, \mathbb{C})$ is non-zero. By the way, instead of just using linear maps, we require that the maps $\Lambda^2\mathfrak{gl}^1_\infty \to \mathbb{C}$ are graded linear maps. This is H^2_{graded}.

Notice that in $\mathfrak{gl}_n$, $tr(x[J, y]) = tr(J[x, y])$ is a trivial cocycle (it is d of $l(x) = tr(Jx)$. So we have that $H^2(\mathfrak{gl}_n, \mathbb{C}) = \{0\}$.

We can define $a_\infty = \mathfrak{gl}_\infty \oplus \mathbb{C}c$. This is some non-trivial central extension.

To summarize the last lectures:

1. We related Lie algebras and Lie Groups. If you're interested in representations of Lie Groups, looking at Lie algebras is easier.
2. From a Lie algebra $\mathfrak{g}$, we constructed $U\mathfrak{g}$, the universal enveloping algebra. This got us thinking about associative algebras and Hopf algebras.
3. We learned about dual pairings of Hopf algebras. For example, $\mathbb{C}[\Gamma]$ and $C(\Gamma)$ are dual, and $U\mathfrak{g}$ and $C(G)$ are dual (if G is affine algebraic and we are looking at polynomial functions). This pairing is a starting point for many geometric realizations of representations of G. Conceptually, the notion of the universal enveloping algebra is closely related to the notion of the group algebra $\mathbb{C}[\Gamma]$.
4. Finally, we talked about deformations.

Lecture 11 - Engel's Theorem and Lie's Theorem

In the next ten lectures, we will cover

1. Classification of semisimple Lie algebras. This will include root systems and Dynkin diagrams.
2. Representation theory of semisimple Lie algebras and the Weyl character formula.
3. Compact connected Lie Groups.

A reference for this material is Fulton and Harris [FH91].

The first part is purely algebraic: we will study Lie algebras. $\mathfrak{g}$ will be a Lie algebra, usually finite dimensional, over a field k (usually of characteristic 0).

Any Lie algebra $\mathfrak{g}$ contains the ideal $\mathcal{D}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$, the vector subspace generated by elements of the form $[X, Y]$ for $X, Y \in \mathfrak{g}$.

► **Exercise 11.1.** Show that $\mathcal{D}\mathfrak{g}$ is an ideal in $\mathfrak{g}$.

► **Exercise 11.2.** Let G be a simply connected Lie group with Lie algebra $\mathfrak{g}$. Then $[G, G]$ is the subgroup of G generated by elements of the form $ghg^{-1}h^{-1}$ for $g, h \in G$. Show that $[G, G]$ is a connected closed normal Lie subgroup of G, with Lie algebra $\mathcal{D}\mathfrak{g}$.

Warning 11.1. Exercise 11.2 is a tricky problem. Here are some potential pitfalls:

1. For G connected, we do not necessarily know that the exponential map is surjective, because G may not be complete. For example, $\exp : \mathfrak{sl}_2(\mathbb{C}) \to SL_2(\mathbb{C})$ is not surjective.[1]
2. If $H \subseteq G$ is a subgroup with Lie algebra $\mathfrak{h}$, then $\mathfrak{h} \subseteq \mathfrak{g}$ closed is not enough to know that H is closed in G. For example, take G to be a torus, and H to be a line with irrational slope.
3. The statement is false if we relax the condition that G is simply connected. Let

$$H := \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \right\} \times S^1$$

$$K := \left\{ \left(\begin{pmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, c^n \right) \middle| \, n \in \mathbb{Z} \right\} \subseteq H$$

---

[1] Assume $\left(\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$ is in the image, then its pre-image must have eigenvalues $(2n+1)i\pi$ and $-(2n+1)i\pi$ for some integer n. So the pre-image has distinct eigenvalues, so it is diagonalizable. But that implies that $\left(\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$ is diagonalizable, contradiction.

where c is an element of S^1 of infinite order. Then K is normal in H and $G = H/K$ is a counterexample.

Definition 11.2. Define $\mathcal{D}^0\mathfrak{g} = \mathfrak{g}$, and $\mathcal{D}^n\mathfrak{g} = [\mathcal{D}^{n-1}\mathfrak{g}, \mathcal{D}^{n-1}\mathfrak{g}]$. This is called the *derived series* of $\mathfrak{g}$. We say $\mathfrak{g}$ is *solvable* if $\mathcal{D}^n\mathfrak{g} = 0$ for some n sufficiently large.

Definition 11.3. We can also define $\mathcal{D}_0\mathfrak{g} = \mathfrak{g}$, and $\mathcal{D}_n\mathfrak{g} = [\mathfrak{g}, \mathcal{D}_{n-1}\mathfrak{g}]$. This is called the *lower central series* of $\mathfrak{g}$. We say that $\mathfrak{g}$ is *nilpotent* if $\mathcal{D}_n\mathfrak{g} = 0$ for some n sufficiently large.

Note that $\mathcal{D}_1\mathfrak{g} = \mathcal{D}^1\mathfrak{g}$ by $\mathcal{D}\mathfrak{g}$. Solvable and nilpotent Lie algebras are hard to classify. Instead, we will do the classification of semisimple Lie algebras (see Definition 11.15).

The following example is in some sense universal (see corollaries 11.7 and 11.12):

Example 11.4. Let $\mathfrak{gl}(n)$ be the Lie algebra of all $n \times n$ matrices, and let $\mathfrak{b}$ be the subalgebra of upper triangular matrices. I claim that $\mathfrak{b}$ is solvable. To see this, note that $\mathcal{D}\mathfrak{b}$ is the algebra of *strictly* upper triangular matrices, and in general, $\mathcal{D}^k\mathfrak{b}$ has zeros on the main diagonal and the 2^{k-2} diagonals above the main diagonal (for $k \geq 2$). Let $\mathfrak{n} = \mathcal{D}\mathfrak{b}$. You can check that $\mathfrak{n}$ is in fact nilpotent.

Useful facts about solvable/nilpotent Lie algebras:

1. If you have an exact sequence of Lie algebras

$$0 \to \mathfrak{a} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{a} \to 0$$

then $\mathfrak{g}$ is solvable if and only if $\mathfrak{a}$ and $\mathfrak{g}/\mathfrak{a}$ are solvable.

2. If you have an exact sequence of Lie algebras

$$0 \to \mathfrak{a} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{a} \to 0$$

then if $\mathfrak{g}$ is nilpotent, so are $\mathfrak{a}$ and $\mathfrak{g}/\mathfrak{a}$.

Warning 11.5. The converse is not true. Diagonal matrices $\mathfrak{d}$ is nilpotent, and we have

$$0 \to \mathfrak{n} \to \mathfrak{b} \to \mathfrak{d} \to 0.$$

Note that $\mathfrak{b}$ is not nilpotent, because $\mathcal{D}\mathfrak{b} = \mathcal{D}_n\mathfrak{b} = \left(\begin{smallmatrix}0 & * \\ 0 & 0\end{smallmatrix}\right)$.

3. If $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$ are solvable ideals, then the sum $\mathfrak{a} + \mathfrak{b}$ is solvable. To see this, note that we have

$$0 \to \mathfrak{a} \to \mathfrak{a} + \mathfrak{b} \to \underbrace{(\mathfrak{a} + \mathfrak{b})/\mathfrak{a}}_{\simeq \mathfrak{b}/(\mathfrak{a} \cap \mathfrak{b})} \to 0$$

   $\mathfrak{a}$ is solvable by assumption, and $\mathfrak{b}/(\mathfrak{a} \cap \mathfrak{b})$ is a quotient of a solvable algebra, so it is solvable by (1). Applying (1) again, $\mathfrak{a}+\mathfrak{b}$ is solvable.

4. If $k \subseteq F$ is a field extension, with a Lie algebra $\mathfrak{g}$ over k, we can make a Lie algebra $\mathfrak{g} \otimes_k F$ over F. Note that $\mathfrak{g} \otimes_k F$ is solvable (nilpotent) if and only if $\mathfrak{g}$ is.

We will now prove Engel's theorem and Lie's theorem.

For any Lie algebra $\mathfrak{g}$, we have the adjoint representation: $X \mapsto ad_X \in \mathfrak{gl}(\mathfrak{g})$ given by $ad_X(Y) = [X, Y]$. If $\mathfrak{g}$ is nilpotent, then ad_X is a nilpotent operator for any $X \in \mathfrak{g}$. The converse is also true as we will see shortly (Cor. 11.9).

Theorem 11.6 (Engel's Theorem)**.** *Let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$, and assume that X is nilpotent for any $X \in \mathfrak{g}$. Then there is a vector $v \in V$ such that $\mathfrak{g} \cdot v = 0$.*

Note that the theorem holds for any representation ρ of $\mathfrak{g}$ in which every element acts nilpotently; just replace $\mathfrak{g}$ in the statement of the theorem by $\rho(\mathfrak{g})$.

Corollary 11.7. *If V is a representation of $\mathfrak{g}$ in which every element acts nilpotently, then one can find $\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = V$ a complete flag such that $\mathfrak{g}(V_i) \subseteq V_{i-1}$. That is, there is a basis in which all of the elements of $\mathfrak{g}$ are strictly upper triangular.*

Warning 11.8. Note that the theorem isn't true if you say "suppose $\mathfrak{g}$ is nilpotent" instead of the right thing. For example, the set of diagonal matrices $\mathfrak{d} \subset \mathfrak{gl}(V)$ is nilpotent.

Proof. Let's prove the theorem by induction on $\dim \mathfrak{g}$.We first show that $\mathfrak{g}$ has an ideal $\mathfrak{a}$ of codimension 1. To see this, take a maximal proper subalgebra $\mathfrak{a} \subset \mathfrak{g}$. Look at the representation of $\mathfrak{a}$ on the quotient space $\mathfrak{g}/\mathfrak{a}$. This representation, $\mathfrak{a} \to \mathfrak{gl}(\mathfrak{g}/\mathfrak{a})$, satisfies the condition of the theorem,[2] so by induction, there is some $X \in \mathfrak{g}$ such that $ad_{\mathfrak{a}}(X) = 0$ modulo $\mathfrak{a}$. So $[\mathfrak{a}, X] \subseteq \mathfrak{a}$, so $\mathfrak{b} = kX \oplus \mathfrak{a}$ is a new subalgebra of $\mathfrak{g}$ which is larger, so it must be all of $\mathfrak{g}$. Thus, $\mathfrak{a}$ must have had codimension 1. Therefore, $\mathfrak{a} \subseteq \mathfrak{g}$ is actually an ideal (because $[X, \mathfrak{a}] = 0$).

---

[2] For any $X \in \mathfrak{a}$, since X is nilpotent, ad_X is also nilpotent.

Next, we prove the theorem. Let $V_0 = \{v \in V | \mathfrak{a}v = 0\}$, which is non-zero by the inductive hypothesis. We claim that $\mathfrak{g}V_0 \subseteq V_0$. To see this, take $x \in \mathfrak{g}$, $v \in V_0$, and $y \in \mathfrak{a}$. We have to check that $y(xv) = 0$. But

$$yxv = x \underbrace{yv}_{0} + \underbrace{[y,x]}_{\in \mathfrak{a}} v = 0.$$

Now, we have that $\mathfrak{g} = kX \oplus \mathfrak{a}$, and $\mathfrak{a}$ kills V_0, and that $X : V_0 \to V_0$ is nilpotent, so it has a kernel. Thus, there is some $v \in V_0$ which is killed by X, and so v is killed by all of $\mathfrak{g}$. ☐

Corollary 11.9. *If ad_X is nilpotent for every $X \in \mathfrak{g}$, then $\mathfrak{g}$ is nilpotent as a Lie algebra.*

Proof. Let $V = \mathfrak{g}$, so we have $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, which has kernel $Z(\mathfrak{g})$. By Engel's theorem, we know that there is an $x \in \mathfrak{g}$ such that $(ad\, \mathfrak{g})(x) = 0$. This implies that $Z(\mathfrak{g}) \neq 0$. By induction we can assume $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent. But then $\mathfrak{g}$ itself must be nilpotent as well because $\mathcal{D}_n(\mathfrak{g}/Z(\mathfrak{g})) = 0$ implies $\mathcal{D}_{n+1}(\mathfrak{g}) = 0$. ☐

Warning 11.10. If $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ is a nilpotent subalgebra, it does not imply that every $X \in \mathfrak{g}$ is nilpotent (take diagonal matrices for example).

Theorem 11.11 (Lie's Theorem)**.** *Let k be algebraically closed and of characteristic 0. If $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ is a solvable subalgebra, then all elements of $\mathfrak{g}$ have a common eigenvector in V.*

This is a generalization of the statement that two commuting operators have a common eigenvector.

Corollary 11.12. *If $\mathfrak{g}$ is solvable, then there is a complete flag $\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = V$ such that $\mathfrak{g}(V_i) \subseteq V_i$. That is, there is a basis in which all elements of $\mathfrak{g}$ are upper triangular.*

Proof. If $\mathfrak{g}$ is solvable, take any subspace $\mathfrak{a} \subset \mathfrak{g}$ of codimension 1 containing $\mathcal{D}\mathfrak{g}$, then $\mathfrak{a}$ is an ideal. We're going to try to do the same kind of induction as in Engel's theorem.

For a linear functional $\lambda : \mathfrak{a} \to k$, let

$$V_\lambda = \{v \in V | Xv = \lambda(X)v \text{ for all } X \in \mathfrak{a}\}.$$

$V_\lambda \neq 0$ for some λ by induction hypothesis.

Claim. $\mathfrak{g}(V_\lambda) \subseteq V_\lambda$.

Proof of Claim. Choose $v \in V_\lambda$ and $X \in \mathfrak{a}$, $Y \in \mathfrak{g}$. Then

$$X(Yv) = Y(\underbrace{Xv}_{\lambda(X)v}) + \underbrace{[X,Y]v}_{\lambda([X,Y])v}$$

We want to show that $\lambda[X,Y] = 0$. There is a trick. Let r be the largest integer such that $v, Yv, Y^2v, \dots, Y^rv$ is a linearly independent set. We know that $Xv = \lambda(X)v$ for any $X \in \mathfrak{a}$. We claim that $XY^jv \equiv \lambda(X)Y^jv$ mod $(span\{v, Yv, \dots, Y^{j-1}v\})$. This is clear for $j = 0$, and by induction, we have

$$\begin{aligned} XY^jv &= Y\underbrace{XY^{j-1}v}_{\substack{\equiv\lambda(X)Y^{j-1}v \\ \text{mod } span\{v,\dots,Y^{j-2}v\}}} + \underbrace{[X,Y]Y^{j-1}v}_{\substack{\equiv\lambda([X,Y])Y^{j-1}v \\ \text{mod } span\{v,\dots,Y^{j-2}v\}}} \\ &\equiv \lambda(X)Y^jv \quad \text{mod } span\{v,\dots,Y^{j-1}v\} \end{aligned}$$

So the matrix for X can be written as $\lambda(X)$ on the diagonal and stuff above the diagonal (in this basis). So the trace of X is $(r+1)\lambda(X)$. Then we have that $tr([X,Y]) = (r+1)\lambda([X,Y])$, since the above statement was proved for any $X \in \mathfrak{a}$ and $[X,Y] \in \mathfrak{a}$. But the trace of a commutator is always 0. Since the characteristic of k is 0, we can conclude that $\lambda[X,Y] = 0$. $\square_{\text{Claim}}$

To finish the proof, write $\mathfrak{g} = kT \oplus \mathfrak{a}$, with $T : V_\lambda \to V_\lambda$ (we can do this because of the claim). Since k is algebraically closed, T has a nonzero eigenvector w in V_λ. This w is the desired common eigenvector. $\square$

Remark 11.13. If k is not algebraically closed, the theorem doesn't hold. For example, consider the (one dimensional) Lie algebra generated by a rotation of $\mathbb{R}^2$.

The theorem also fails if k is not characteristic 0. Say k is characteristic p, then let x be the permutation matrix of the p-cycle $(p\ p-1\ \cdots\ 2\ 1)$ (i.e. the matrix $\left(\begin{smallmatrix} 0 & I_{p-1} \\ 1 & 0 \end{smallmatrix}\right)$), and let y be the diagonal matrix $diag(0, 1, 2, \dots, p-1)$. Then $[x, y] = x$, so the Lie algebra generated by x and y is solvable. However, y is diagonal, so we know all of its eigenvectors, and none of them is an eigenvector of x.

Corollary 11.14. *Let k be of characteristic 0. Then $\mathfrak{g}$ is solvable if and only if $\mathcal{D}\mathfrak{g}$ is nilpotent.*

If $\mathcal{D}\mathfrak{g}$ is nilpotent, then $\mathfrak{g}$ is solvable from the definitions. If $\mathfrak{g}$ is solvable, then look at everything over the algebraic closure of k, where $\mathfrak{g}$ looks like upper triangular matrices, so $\mathcal{D}\mathfrak{g}$ is nilpotent. All this is independent of coefficients (by useful fact (4)).

The radical

There is a unique maximal solvable ideal in $\mathfrak{g}$ (by useful fact (3): sum of solvable ideals is solvable), which is called the radical of $\mathfrak{g}$.

Definition 11.15. We call $\mathfrak{g}$ *semisimple* if $\operatorname{rad}\mathfrak{g} = 0$.

▶ **Exercise 11.3.** Show that $\mathfrak{g}/\operatorname{rad}\mathfrak{g}$ is always semisimple.

If $\mathfrak{g}$ is one dimensional, generated by X, then we have that $[\mathfrak{g}, \mathfrak{g}] = 0$, so $\mathfrak{g}$ cannot be semisimple.

If $\mathfrak{g}$ is two dimensional, generated by X and Y, then we have that $[\mathfrak{g}, \mathfrak{g}]$ is one dimensional, spanned by $[X, Y]$. Thus, $\mathfrak{g}$ cannot be semisimple because $\mathcal{D}\mathfrak{g}$ is a solvable ideal.

There is a semisimple Lie algebra of dimension 3, namely $\mathfrak{sl}_2$.

Semisimple algebras have really nice properties. Cartan's criterion (Theorem 12.7) says that $\mathfrak{g}$ is semisimple if and only if the Killing form (see next lecture) is non-degenerate. Whitehead's theorem (Theorem 12.10) says that if V is a non-trivial irreducible representation of a semisimple Lie algebra $\mathfrak{g}$, then $H^i(\mathfrak{g}, V) = 0$ for all i. Weyl's theorem (Theorem 12.14) says that every finite dimensional representation of a semisimple Lie algebra is the direct sum of irreducible representations. If G is simply connected and compact, then $\mathfrak{g}$ is semisimple (See Lecture 20).

Lecture 12 - Cartan Criterion, Whitehead and Weyl Theorems

Invariant forms and the Killing form

Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation. To make the notation cleaner, we will write $\bar{X}$ for $\rho(X)$. We can define a bilinear form on $\mathfrak{g}$ by $B_V(X,Y) := tr(\bar{X}\bar{Y})$. This form is symmetric because $tr(AB) = tr(BA)$ for any linear operators A and B.

We also have that

$$\begin{aligned} B_V([X,Y],Z) &= tr(\bar{X}\bar{Y}\bar{Z} - \bar{Y}\bar{X}\bar{Z}) = tr(\bar{X}\bar{Y}\bar{Z}) - tr(\bar{X}\bar{Z}\bar{Y}) \\ &= tr(\bar{X}\bar{Y}\bar{Z} - \bar{X}\bar{Z}\bar{Y}) = B_V(X,[Y,Z]), \end{aligned}$$

so B satisfies

$$B([X,Y],Z) = B(X,[Y,Z]).$$

Such a form is called an *invariant form*. It is called invariant because it is implied by B being Ad-invariant.[1] Assume that for any $g \in G$ and $X, Z \in \mathfrak{g}$, we $B(Ad_g X, Ad_g Z) = B(X,Z)$. Let γ be a path in G with $\gamma'(0) = Y$. We get that

$$B([Y,X],Z) + B(X,[Y,Z]) = \frac{d}{dt}\bigg|_{t=0} B\big(Ad_{\gamma(t)}(X), Ad_{\gamma(t)}(Z)\big) = 0.$$

Definition 12.1. The *Killing form*, denoted by B, is the special case where ρ is the adjoint representation. That is, $B(X,Y) := tr(ad_X \circ ad_Y)$.

▶ **Exercise 12.1.** Let $\mathfrak{g}$ be a simple Lie algebra over an algebraically closed field. Check that two invariant forms on $\mathfrak{g}$ are proportional.

▶ **Exercise 12.2** (In class)**.** If $\mathfrak{g}$ is solvable, then $B(\mathfrak{g}, \mathcal{D}\mathfrak{g}) = 0$.

Solution. First note that if $Z = [X,Y] \in \mathcal{D}\mathfrak{g}$, then $ad_Z = [ad_X, ad_Y] \in \mathcal{D}(ad\, \mathfrak{g})$ since the adjoint representation is a Lie algebra homomorphism. Moreover, $\mathfrak{g}$ solvable implies that the image of the adjoint representation, $ad(\mathfrak{g}) \simeq \mathfrak{g}/Z(\mathfrak{g})$, is solvable. Therefore, in some basis of V of a representation of $ad(\mathfrak{g})$, all matrices of $ad(\mathfrak{g})$ are upper triangular (by Lie's Theorem), and those of $\mathcal{D}(ad\mathfrak{g})$ are all strictly upper triangular. The product of an upper triangular matrix and a strictly upper triangular matrix will be strictly upper triangular and therefore have trace 0. ■

The converse of this exercise is also true. It will follow as a corollary of our next theorem (Corollary 12.6 below).

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[1] If G is connected, the two versions of invariance are equivalent.

Theorem 12.2. *Suppose* $\mathfrak{g} \subseteq \mathfrak{gl}(V)$*, char* $k = 0$*, and* $B_V(\mathfrak{g}, \mathfrak{g}) = 0$*. Then* $\mathfrak{g}$ *is solvable.*

For the proof, we will need the following facts from linear algebra.

Lemma 12.3.[2] *Let* X *be a diagonalizable linear operator in* V*, with* k *algebraically closed. If* $X = A \cdot diag(\lambda_1, \ldots, \lambda_n) \cdot A^{-1}$ *and* $f : k \to k$ *is a function, we define* $f(X)$ *as* $A \cdot diag(f(\lambda_1), \ldots, f(\lambda_n)) \cdot A^{-1}$*. Suppose* $tr(X \cdot f(X)) = 0$ *for any* $\mathbb{Q}$*-linear map* $f : k \to k$ *such that* f *is the identity on* $\mathbb{Q}$*, then* $X = 0$*.*

Proof. Consider only f such that the image of f is $\mathbb{Q}$. Let $\lambda_1, \ldots, \lambda_m$ be the eigenvalues of X with multiplicities $n_1, \ldots, n_m$. We obtain $tr(X \cdot f(X)) = n_1\lambda_1 f(\lambda_1) + \ldots + n_m\lambda_m f(\lambda_n) = 0$. Apply f to this identity to obtain $n_1 f(\lambda_1)^2 + \ldots + n_m f(\lambda_m)^2 = 0$ which implies $f(\lambda_i) = 0$ for all i. If some λ_i is not zero, we can choose f so that $f(\lambda_i) \neq 0$, so $\lambda_i = 0$ for all i. Since X is diagonalizable, $X = 0$. ☐

Lemma 12.4 (Jordan Decomposition)**.** *Given* $X \in \mathfrak{gl}(V)$*, there are unique* $X_s, X_n \in \mathfrak{gl}(V)$ *such that* X_s *is diagonalizable,* X_n *is nilpotent,* $[X_s, X_n] = 0$*, and* $X = X_n + X_s$*. Furthermore,* X_s *and* X_n *are polynomials in* X*.*

Proof. All but the last statement is standard; see, for example, Corollay 2.5 of Chapter XIV of [Lan02]. To see the last statement, let the characteristic polynomial of X be $\prod_i (x - \lambda_i)^{n_i}$. By the chinese remainder theorem, we can find a polynomial f such that $f(x) \equiv \lambda_i \mod (x - \lambda_i)^{n_i}$. Choose a basis so that X is in Jordan form and compute $f(X)$ block by block. On a block with λ_i along the diagonal $(X - \lambda_i I)^{n_i}$ is 0, so $f(X)$ is $\lambda_i I$ on this block. Then $f(X) = X_s$ is diagonalizable and $X_n = X - f(X)$ is nilpotent. ☐

Lemma 12.5. *Let* $\mathfrak{g} \subseteq \mathfrak{gl}(V)$*. The adjoint representation* $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ *preserves Jordan decomposition:* $ad_{X_s} = (ad_X)_s$ *and* $ad_{X_n} = (ad_X)_n$*. In particular,* ad_{X_s} *is a polynomial in* ad_X*.*

Proof. Suppose the eigenvalues of X_s are $\lambda_1, \ldots, \lambda_m$, and we are in a basis where X_s is diagonal. Check that $ad_{X_s}(E_{ij}) = [X_s, E_{ij}] = (\lambda_i - \lambda_j)E_{ij}$. So X_s diagonalizable implies ad_{X_s} is diagonalizable (because it has a basis of eigenvectors). We have that ad_{X_n} is nilpotent because the monomials in the expansion of $(ad_{X_n})^k(Y)$ have X_n to at least the $k/2$ power on one side of Y. So we have that $ad_X = ad_{X_s} + ad_{X_n}$, with ad_{X_s} diagonalizable, ad_{X_n} nilpotent, and the two commute, so by uniqueness of the Jordan decomposition, $ad_{X_s} = (ad_X)_s$ and $ad_{X_n} = (ad_X)_n$. ☐

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[2] This is different from what we did in class. There is an easier way to do this if you are willing to assume $k = \mathbb{C}$ and use complex conjugation. See Fulton and Harris for this method.

Proof of Theorem 12.2. It is enough to show that $\mathcal{D}\mathfrak{g}$ is nilpotent. Let $X \in \mathcal{D}\mathfrak{g}$, so $X = \sum[Y_i, Z_i]$. It suffices to show that $X_s = 0$. To do this, let $f : k \to k$ be any $\mathbb{Q}$-linear map fixing $\mathbb{Q}$.

$$\begin{aligned}
B_V(f(X_s), X_s) &= B_V(f(X_s), X) && (X_n \text{ doesn't contribute})\\
&= B_V\Big(f(X_s), \sum_i [Y_i, Z_i]\Big)\\
&= \sum_i B_V(\underbrace{[f(X_s), Y_i]}_{\in\mathfrak{g}?}, Z_i) && (B_V \text{ invariant})\\
&= 0 && (\text{assuming } [f(X_s), Y_i] \in \mathfrak{g})
\end{aligned}$$

Then by Lemma 12.3, $X_s = 0$.

To see that $[f(X_s), Y_i] \in \mathfrak{g}$, suppose the eigenvalues of X_s are $\lambda_1, \dots, \lambda_m$. Then the eigenvalues of $f(X_s)$ are $f(\lambda_i)$, the eigenvalues of ad_{X_s} are of the form $\mu_{ij} := \lambda_i - \lambda_j$, and eigenvalues of $ad_{f(X_s)}$ are $\nu_{ij} := f(\lambda_i) - f(\lambda_j) = f(\mu_{ij})$. If we define g to be a polynomial such that $g(\mu_{ij}) = \nu_{ij}$, then $ad_{f(X_s)}$ and $g(ad_{X_s})$ are diagonal (in some basis) with the same eigenvalues in the same places, so they are equal. So we have

$$\begin{aligned}
[f(X_s), Y_i] &= g(ad_{X_s})(Y_i)\\
&= h(ad_X)(Y_i) \in \mathfrak{g} && (\text{using Lemma 12.5})
\end{aligned}$$

for some polynomial h.

The above arguments assume k is algebraically closed, so if it's not apply the above to $\mathfrak{g} \otimes_k \bar{k}$. Then $\mathfrak{g} \otimes_k \bar{k}$ solvable implies $\mathfrak{g}$ solvable as mentioned in the previous lecture. □

Corollary 12.6. *$\mathfrak{g}$ is solvable if and only if $B(\mathcal{D}\mathfrak{g}, \mathfrak{g}) = 0$.*

Proof. ($\Leftarrow$) We have that $B(\mathcal{D}\mathfrak{g}, \mathfrak{g}) = 0$ implies $B(\mathcal{D}\mathfrak{g}, \mathcal{D}\mathfrak{g}) = 0$ which implies that $ad(\mathcal{D}\mathfrak{g})$ is solvable. The adjoint representation of $\mathcal{D}\mathfrak{g}$ gives the exact sequence

$$0 \to Z(\mathcal{D}\mathfrak{g}) \to \mathcal{D}\mathfrak{g} \to ad(\mathcal{D}\mathfrak{g}) \to 0.$$

Since $Z(\mathcal{D}\mathfrak{g})$ and $ad(\mathcal{D}\mathfrak{g})$ are solvable, $\mathcal{D}\mathfrak{g}$ is solvable by useful fact (1) of Lecture 11, so $\mathfrak{g}$ is solvable.

($\Rightarrow$) This is exercise 12.2. □

Theorem 12.7 (Cartan's Criterion)**.** *The Killing form is non-degenerate if and only if $\mathfrak{g}$ is semisimple.*

Proof. Say $\mathfrak{g}$ is semisimple. Let $\mathfrak{a} = \ker B$. Because B is invariant, we get that $\mathfrak{a}$ is an ideal, and $B|_{\mathfrak{a}} = 0$. By the previous theorem (12.2), we have that $\mathfrak{a}$ is solvable, so $\mathfrak{a} = 0$ (by definition of semisimple).

Suppose that $\mathfrak{g}$ is not semisimple, so $\mathfrak{g}$ has a non-trivial solvable ideal. Then the last non-zero term in its derived series is some abelian ideal $\mathfrak{a} \subseteq \mathfrak{g}$.[3] For any $X \in \mathfrak{a}$, the matrix of ad_X is of the form $\left(\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right)$ with respect to the (vector space) decomposition $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{g}/\mathfrak{a}$, and for $Y \in \mathfrak{g}$, ad_Y is of the form $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$. Thus, we have that $tr(ad_X \circ ad_Y) = 0$ so $X \in \ker B$, so B is degenerate. □

Theorem 12.8. *Any semisimple Lie algebra is a direct sum of simple algebras.*

Proof. If $\mathfrak{g}$ is simple, then we are done. Otherwise, let $\mathfrak{a} \subseteq \mathfrak{g}$ be an ideal. By invariance of B, $\mathfrak{a}^\perp$ is an ideal. On $\mathfrak{a} \cap \mathfrak{a}^\perp$, B is zero, so the intersection is a solvable ideal, so it is zero by semisimplicity of $\mathfrak{g}$. Thus, we have that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$. The result follows by induction on dimension. □

Remark 12.9. In particular, if $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ is semisimple, with each $\mathfrak{g}_i$ simple, we have that $\mathcal{D}\mathfrak{g} = \bigoplus \mathcal{D}\mathfrak{g}_i$. But $\mathcal{D}\mathfrak{g}_i$ is either 0 or $\mathfrak{g}_i$, and it cannot be 0 (lest $\mathfrak{g}_i$ be a solvable ideal). Thus $\mathcal{D}\mathfrak{g} = \mathfrak{g}$.

Theorem 12.10 (Whitehead)**.** *If $\mathfrak{g}$ is semisimple and V is an irreducible non-trivial representation of $\mathfrak{g}$, then $H^i(\mathfrak{g}, V) = 0$ for all $i \geq 0$.*

Proof. The proof uses the Casimir operator, $C_V \in \mathfrak{gl}(V)$. Assume for the moment that $\mathfrak{g} \subseteq \mathfrak{gl}(V)$. Choose a basis $e_1, \dots, e_n$ in $\mathfrak{g}$, with dual basis $f_1, \dots, f_n$ in $\mathfrak{g}$ (dual with respect to B_V, so $B_V(e_i, f_j) = \delta_{ij}$). It is necessary that B_V be non-degenerate for such a dual basis to exist, and this is where we use that $\mathfrak{g}$ is semisimple. The[4] Casimir operator is defined to be $C_V = \sum e_i \circ f_i \in \mathfrak{gl}(V)$ (where $\circ$ is composition of linear operators on V). The main claim is that $[C_V, X] = 0$ for any $X \in \mathfrak{g}$. This can be checked directly: put $[X, f_i] = \sum a_{ij} f_j$, $[X, e_i] = \sum b_{ij} e_j$, then apply B_V to obtain $a_{ji} = B_V(e_i, [X, f_j]) = B_V([e_i, X], f_j) = -b_{ij}$, where the middle equality is by invariance of B_V.

$$\begin{aligned}
[X, C_V] &= \sum_i X e_i f_i - e_i X f_i + e_i X f_i - e_i f_i X \\
&= \sum_i [X, e_i] f_i + e_i [X, f_i] \\
&= \sum_i \sum_j b_{ij} e_j f_i + a_{ij} e_i f_j \\
&= \sum_i \sum_j (a_{ij} + b_{ji}) e_i f_j = 0.
\end{aligned}$$

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[3] Quick exercise: why is $\mathfrak{a}$ an ideal?

[4] We will soon see that C_V is independent of the basis $e_1, \dots, e_n$, so the article "the" is apropriate.

Suppose V is irreducible, and k is algebraically closed. Then the condition $[C_V, X] = 0$ means precisely that C_V is an intertwiner so by Schur's lemma, $C_V = \lambda \mathrm{Id}$. We can compute

$$\begin{aligned} tr_V C_V &= \sum_{i=1}^{\dim \mathfrak{g}} tr(e_i f_i) \\ &= \sum B_V(e_i, f_i) = \dim \mathfrak{g}. \end{aligned}$$

Thus, we have that $\lambda = \frac{\dim \mathfrak{g}}{\dim V}$, in particular, it is non-zero.

For any representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, we can still talk about C_V, but we define it for the image $\rho(\mathfrak{g})$, so $C_V = \frac{\dim \rho(\mathfrak{g})}{\dim V}\mathrm{Id}$. We get that $[C_V, \rho(X)] = 0$. The point is that if V is non-trivial irreducible, we have that C_V is non-zero.

Now consider the complex calculating the cohomology:

$$\mathrm{Hom}(\Lambda^k \mathfrak{g}, V) \xrightarrow{d} \mathrm{Hom}(\Lambda^{k+1}\mathfrak{g}, V)$$

We will construct a chain homotopy[5] $\gamma : C^{k+1} \to C^k$ between the zero map on the complex and the map $C_V = \frac{\dim \rho(\mathfrak{g})}{\dim V}\mathrm{Id}$:

$$\gamma c(x_1, \dots, x_k) = \sum_i e_i c(f_i, x_1, \dots, x_k)$$

▶ **Exercise 12.3.** Check directly that $(\gamma d + d\gamma)c = C_V c$.

Thus $\gamma d + d\gamma = C_V = \lambda \mathrm{Id}$ (where $\lambda = \frac{\dim \rho(\mathfrak{g})}{\dim V}$). Now suppose $dc = 0$. Then we have that $d\gamma(c) = \lambda c$, so $c = \frac{d(\gamma(c))}{\lambda}$. Thus, $\ker d / \operatorname{im} d = 0$, as desired. □

Remark 12.11. What is $H^1(\mathfrak{g}, k)$, where k is the trivial representation of $\mathfrak{g}$? Recall that the cochain complex is

$$k \to \mathrm{Hom}(\mathfrak{g}, k) \xrightarrow{d} \mathrm{Hom}(\Lambda^2 \mathfrak{g}, k) \to \cdots.$$

If $c \in \mathrm{Hom}(\mathfrak{g}, k)$ and $c \in \ker d$, then $dc(x, y) = c([x, y]) = 0$, so c is 0 on $\mathcal{D}\mathfrak{g} = \mathfrak{g}$. So we get that $H^1(\mathfrak{g}, k) = (\mathfrak{g}/\mathcal{D}\mathfrak{g})^* = 0$.

However, it is not true that $H^i(\mathfrak{g}, k) = 0$ for $i \geq 2$. Recall from Lecture 10 that $H^2(\mathfrak{g}, k)$ parameterizes central extensions of $\mathfrak{g}$ (Theorem 10.2).

▶ **Exercise 12.4.** Compute $H^j(\mathfrak{sl}_2, k)$ for all j.

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[5] Don't worry about the term "chain homotopy" for now. It just means that γ satisfies the equation in Exercise 12.3. See Proposition 2.12 of [Hat02] if you're interested.

Remark 12.12. Note that for $\mathfrak{g}$ semisimple, we have $H^1(\mathfrak{g}, M) = 0$ for *any* finite dimensional representation M (not just irreducibles). We have already seen that this holds when M is trivial and Whitehead's Theorem shows this when M is non-trivial irreducible. If M is not irreducible, use short exact sequences to long exact sequences in cohomology: if

$$0 \to W \to M \to V \to 0$$

is an exact sequence of representations of $\mathfrak{g}$, then

$$\to H^1(\mathfrak{g}, V) \to H^1(\mathfrak{g}, M) \to H^1(\mathfrak{g}, W) \to$$

is exact. The outer guys are 0 by induction on dimension, so the middle guy is zero.

We need a lemma before we do Weyl's Theorem.

Lemma 12.13. *Say we have a short exact sequence*

$$0 \to W \to M \to V \to 0.$$

If $H^1(\mathfrak{g}, \underbrace{\mathrm{Hom}_k(V, W)}_{V^* \otimes W}) = 0$*, then the sequence splits.*

Proof. Let $X \in \mathfrak{g}$. Let X_W represent the induced linear operator on W. Then we can write $X_M = \begin{pmatrix} X_W & c(X) \\ 0 & X_V \end{pmatrix}$. What is $c(X)$? It is an element of $\mathrm{Hom}_k(V, W)$. So c is a linear function from $\mathfrak{g}$ to $\mathrm{Hom}_k(V, W)$. It will be a 1-cocycle: we have $[X_M, Y_M] = [X, Y]_M$ because these are representations, which gives us

$$X_W c(Y) - c(Y) X_V - \big(Y_W c(X) - c(X) Y_V\big) = c([X, Y]).$$

In general, $dc(X, Y) = c([X, Y]) - Xc(Y) + Yc(X)$, where $Xc(Y)$ is given by the action of $X \in \mathfrak{g}$ on $V^* \otimes W$, which is not necessarily composition. In our case this action is by commutation, where $c(Y)$ is extended to an endomorphism of $V \oplus W$ by writing it as $\begin{pmatrix} 0 & c(Y) \\ 0 & 0 \end{pmatrix}$. The line above says exactly that $dc = 0$.

Put $\Gamma = \begin{pmatrix} 1_W & K \\ 0 & 1_V \end{pmatrix}$. Conjugating by Γ gives an equivalent representation. We have

$$\Gamma X_M \Gamma^{-1} = \begin{pmatrix} X_W & c(X) + KX_V - X_W K \\ 0 & X_V \end{pmatrix}$$

We'd like to kill the upper right part (to show that X acts on V and W separately). We have $c \in \mathrm{Hom}(\mathfrak{g}, V^* \otimes W)$, $K \in V^* \otimes W$. Since the first cohomology is zero, $dc = 0$, so we can find a K such that $c = dK$. Since $c(X) = dK(X) = X(K) = X_W K - KX_V$, the upper right part is indeed 0. □

Theorem 12.14 (Weyl)**.** *If $\mathfrak{g}$ is semisimple and V is a finite dimensional representation of $\mathfrak{g}$, then V is semisimple*[6] *(i.e. completely reducible).*

Proof. The theorem follows immediately from Lemma 12.13 and Remark 12.12. ☐

Weyl proved this using the unitary trick, which involves knowing about compact real forms.

Remark 12.15. We know from Lecture 10 that deformations of $\mathfrak{g}$ are enumerated by $H^2(\mathfrak{g},\mathfrak{g})$. This means that semisimple Lie algebras do not have any deformations! This suggests that the variety of semisimple Lie algebras is discrete. Perhaps we can classify them.

Aut $\mathfrak{g}$ is a closed Lie subgroup of $GL(\mathfrak{g})$. Let $X(t)$ be a path in Aut $\mathfrak{g}$ such that $X(0) = 1$, and let $\frac{d}{dt}X(t)\big|_{t=0} = \phi$ be an element of the Lie algebra of Aut $\mathfrak{g}$. We have that

$$\begin{aligned} [X(t)Y, X(t)Z] &= X(t)([Y,Z]) \\ [\phi Y, Z] + [Y, \phi Z] &= \phi[Y,Z] && \text{(differentiating at } t=0) \end{aligned}$$

so Lie(Aut $\mathfrak{g}$) $= \mathcal{D}er(\mathfrak{g})$, the algebra of derivations of $\mathfrak{g}$. (We get equality because any derivation can be exponentiated to an automorphism.)

By the Jacobi identity, ad_X is a derivation on $\mathfrak{g}$. So $ad(\mathfrak{g}) \subseteq \mathcal{D}er(\mathfrak{g})$.

▶ **Exercise 12.5.** Check that $ad(\mathfrak{g})$ is an ideal.

We have seen in lecture 9 (page 42) that $\mathcal{D}er(\mathfrak{g})/ad(\mathfrak{g}) \simeq H^1(\mathfrak{g},\mathfrak{g})$. The conclusion is that $\mathcal{D}er(\mathfrak{g}) = ad(\mathfrak{g}) \cong \mathfrak{g}$—that is, all derivations on a semisimple Lie algebra are inner.

Now we know that G and Aut $\mathfrak{g}$ have the same Lie algebras. If $f \in$ Aut $\mathfrak{g}$ is central (i.e. commutes with all automorphisms), then we have

$$\begin{aligned} \big(\exp(t\cdot ad_x)\big)y &= f \circ \big(\exp(t\cdot ad_x)\big) \circ f^{-1}y && (f \text{ is central}) \\ &= \exp(t \cdot ad_{f(x)})y && (f \text{ an automorphism of } \mathfrak{g}) \end{aligned}$$

Comparing the t^1 coefficients, we see that $ad_{f(x)} = ad_x$ for all x. Since $\mathfrak{g}$ has no center, $f(x) = x$ for all x. Therefore, Aut $\mathfrak{g}$ has trivial center.

It follows that the connected component of the identity of Aut $\mathfrak{g}$ is AdG.

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[6] For any invariant subspace $W \subseteq V$, there is an invariant $W' \subseteq V$ so that $V = W \oplus W'$.

Lecture 13 - The root system of a semisimple Lie algebra

The goal for today is to start with a semisimple Lie algebra over a field k (assumed algebraically closed and characteristic zero), and get a root system.

Recall Jordan decomposition. For $\mathfrak{g} \subseteq \mathfrak{gl}(V)$, any $x \in \mathfrak{g}$ can be written (uniquely) as $x = x_s + x_n$, where x_s is semisimple and x_n is nilpotent, both of which are polynomials in x. In general, x_s and x_n are in $\mathfrak{gl}(V)$, but not necessarily in $\mathfrak{g}$.

Proposition 13.1. *If $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ is semisimple, then $x_s, x_n \in \mathfrak{g}$.*

Proof. Notice that $\mathfrak{g}$ acts on $\mathfrak{gl}(V)$ via commutator, and $\mathfrak{g}$ is an invariant subspace. By complete reducibility (Theorem 12.14), we can write $\mathfrak{gl}(V) = \mathfrak{g} \oplus \mathfrak{m}$ where $\mathfrak{m}$ is $\mathfrak{g}$-invariant, so

$$[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g} \qquad \text{and} \qquad [\mathfrak{g}, \mathfrak{m}] \subseteq \mathfrak{m}.$$

We have that ad_{x_s} and ad_{x_n} are polynomials in ad_x (by Lemma 12.5), so

$$[x_n, \mathfrak{g}] \subseteq \mathfrak{g}\ ,\ [x_s, \mathfrak{g}] \subseteq \mathfrak{g} \qquad \text{and} \qquad [x_n, \mathfrak{m}] \subseteq \mathfrak{m}\ ,\ [x_s, \mathfrak{m}] \subseteq \mathfrak{m}.$$

Take $x_n = a + b \in \mathfrak{g} \oplus \mathfrak{m}$, where $a \in \mathfrak{g}$ and $b \in \mathfrak{m}$. We would like to show that $b = 0$, for then we would have that $x_n \in \mathfrak{g}$, from which it would follow that $x_s \in \mathfrak{g}$.

Decompose $V = V_1 \oplus \cdots \oplus V_n$ with the V_i irreducible. Since x_n is a polynomial in x, we have that $x_n(V_i) \subseteq V_i$, and $a(V_i) \subseteq V_i$ since $a \in \mathfrak{g}$, so $b(V_i) \subseteq V_i$. Moreover, we have that

$$[x_n, \mathfrak{g}] = \underbrace{[a, \mathfrak{g}]}_{\in \mathfrak{g}} + \underbrace{[b, \mathfrak{g}]}_{\in \mathfrak{m}} \subseteq \mathfrak{g},$$

so $[b, \mathfrak{g}] = 0$ (i.e. b is an intertwiner). By Schur's lemma, b must be a scalar operator on V_i (i.e. $b|_{V_i} = \lambda_i \mathrm{Id}$). We have $tr_{V_i}(x_n) = 0$ because x_n is nilpotent. Also $tr_{V_i}(a) = 0$ because $\mathfrak{g}$ is semisimple implies $\mathcal{D}\mathfrak{g} = \mathfrak{g}$, so $a = \sum [x_k, y_k]$, and the traces of commutators are 0. Thus, $tr_{V_i}(b) = 0$, so $\lambda_i = 0$ and $b = 0$. Now $x_n = a \in \mathfrak{g}$, and so $x_s \in \mathfrak{g}$. □

Since the image of a semisimple Lie algebra is semisimple, the proposition tells us that for any representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, the semisimple and nilpotent parts of $\rho(x)$ are in the image of $\mathfrak{g}$. In fact, the following corollary shows that there is an *absolute* Jordan decomposition $x = x_s + x_n$ within $\mathfrak{g}$.

Corollary 13.2. *If* $\mathfrak{g}$ *is semisimple, and* $x \in \mathfrak{g}$*, then there are* $x_s, x_n \in \mathfrak{g}$ *such that for any representation* $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$*, we have* $\rho(x_s) = \rho(x)_s$ *and* $\rho(x_n) = \rho(x)_n$*.*

Proof. Consider the (faithful) representation $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. By the proposition, there are some $x_s, x_n \in \mathfrak{g}$ such that $(ad_x)_s = ad_{x_s}$ and $(ad_x)_n = ad_{x_n}$. Since ad is faithful, $ad_x = ad_{x_n} + ad_{x_s}$ and $ad_{[x_n,x_s]} = [ad_{x_n}, ad_{x_s}] = 0$ tell us that $x = x_n + x_s$ and $[x_s, x_n] = 0$. These are our candidates for the absolute Jordan decomposition.

Given any surjective Lie algebra homomorphism $\sigma : \mathfrak{g} \to \mathfrak{g}'$, we have that $ad_{\sigma(y)}(\sigma(z)) = \sigma(ad_y(z))$, from which it follows that $ad_{\sigma(x_s)}$ is diagonalizable and $ad_{\sigma(x_n)}$ is nilpotent (note that we've used surjectivity of σ). Thus, $\sigma(x)_n = \sigma(x_n)$ and $\sigma(x)_s = \sigma(x_s)$. That is, our candidates are preserved by surjective homomorphisms.

Now given any representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, the previous paragraph allows us to replace $\mathfrak{g}$ by its image, so we may assume ρ is faithful. By the proposition, there are some $y, z \in \mathfrak{g}$ such that $\rho(x)_s = \rho(y), \rho(x)_n = \rho(z)$. Then $[\rho(y), -]_{\mathfrak{gl}(\rho(\mathfrak{g}))}$ is a diagonalizable operator on $\mathfrak{gl}\big(\rho(\mathfrak{g})\big) \cong \mathfrak{gl}(\mathfrak{g})$, and $[\rho(z), -]_{\mathfrak{gl}(\rho(\mathfrak{g}))}$ is nilpotent. Uniqueness of the Jordan decomposition implies that $\rho(y) = \rho(x_s)$ and $\rho(z) = \rho(x_n)$. Since ρ is faithful, it follows that $y = x_s$ and $z = x_n$. □

Definition 13.3. We say $x \in \mathfrak{g}$ is *semisimple* if ad_x is diagonalizable. We say x is *nilpotent* if ad_x is nilpotent.

Given any representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ with $\mathfrak{g}$ semisimple, the corollary tells us that if x is semisimple, then $\rho(x)$ is diagonalizable, and if x is nilpotent, then $\rho(x)$ is nilpotent. If ρ is faithful, then x is semisimple (resp. nilpotent) if and only if $\rho(x)$ is semisimple (resp. nilpotent).

Definition 13.4. We denote the set of all semisimple elements in $\mathfrak{g}$ by $\mathfrak{g}_{ss}$. We call an $x \in \mathfrak{g}_{ss}$ *regular* if $\dim(\ker ad_x)$ is minimal (i.e. the dimension of the centralizer is minimal).

Example 13.5. Let $\mathfrak{g} = \mathfrak{sl}_n$. Semisimple elements of $\mathfrak{sl}_n$ are exactly the diagonalizable matrices, and nilpotent elements are exactly the nilpotent matrices. If $x \in \mathfrak{g}$ is diagonalizable, then the centralizer is minimal exactly when all the eigenvalues are distinct. So the regular elements are the diagonalizable matrices with distinct eigenvalues.

Let $h \in \mathfrak{g}_{ss}$ be regular. We have that ad_h is diagonalizable, so we can write $\mathfrak{g} = \bigoplus_{\mu \in k} \mathfrak{g}_\mu$, where $\mathfrak{g}_\mu = \{x \in \mathfrak{g} | [h, x] = \mu x\}$ are eigenspaces of ad_h. We know that $\mathfrak{g}_0 \neq 0$ because it contains h. There are some other properties:

1. $[\mathfrak{g}_\mu, \mathfrak{g}_\nu] \subseteq \mathfrak{g}_{\mu+\nu}$.

2. $\mathfrak{g}_0 \subseteq \mathfrak{g}$ is a subalgebra.
3. $B(\mathfrak{g}_\mu, \mathfrak{g}_\nu) = 0$ if $\mu \neq -\nu$ (here, B is the Killing form, as usual).
4. $B|_{\mathfrak{g}_\mu \oplus \mathfrak{g}_{-\mu}}$ is non-degenerate, and $B|_{\mathfrak{g}_0}$ is non-degenerate.

Proof. Property 1 follows from the Jacobi identity: if $x \in \mathfrak{g}_\mu$ and $y \in \mathfrak{g}_\nu$, then

$$[h, [x, y]] = [[h, x], y] + [x, [h, y]] = \mu[x, y] + \nu[x, y],$$

so $[x, y] \in \mathfrak{g}_{\mu+\nu}$. Property 2 follows immediately from 1. Property 3 follows from 1 because $ad_x \circ ad_y : \mathfrak{g}_\gamma \to \mathfrak{g}_{\gamma+\mu+\nu}$, so $B(x, y) = tr(ad_x \circ ad_y) = 0$ whenever $\mu + \nu \neq 0$. Finally, Cartan's criterion says that B must be non-degenerate, so property 4 follows from 3. ☐

Proposition 13.6. *In the situation above ($\mathfrak{g}$ is semisimple and $h \in \mathfrak{g}_{ss}$ is regular), $\mathfrak{g}_0$ is abelian.*

Proof. Take $x \in \mathfrak{g}_0$, and write $x = x_s + x_n$. Since ad_{x_n} is a polynomial of ad_x, we have $[x_n, h] = 0$, so $x_n \in \mathfrak{g}_0$, from which we get $x_s \in \mathfrak{g}_0$. Since $[x_s, h] = 0$, we know that ad_{x_s} and ad_h are simultaneously diagonalizable (recall that ad_{x_s} is diagonalizable). Thus, for generic $t \in k$, we have that $\ker ad_{h+tx_s} \subseteq \ker ad_h$. Since h is regular, $\ker ad_{x_s} = \ker ad_h = \mathfrak{g}_0$. So $[x_s, \mathfrak{g}_0] = 0$, which implies that $\mathfrak{g}_0$ is nilpotent by Corollary 11.9 to Engel's Theorem. Now we have that $ad_x : \mathfrak{g}_0 \to \mathfrak{g}_0$ is nilpotent, and we want ad_x to be the zero map. Notice that $B(\mathfrak{g}_0, \mathcal{D}\mathfrak{g}_0) = 0$ since $\mathfrak{g}_0$ is nilpotent, but $B|_{\mathfrak{g}_0}$ is non-degenerate by property 4 above, so $\mathcal{D}\mathfrak{g}_0 = 0$, so $\mathfrak{g}_0$ is abelian. ☐

Definition 13.7. We call $\mathfrak{h} := \mathfrak{g}_0$ the *Cartan subalgebra* of $\mathfrak{g}$ (associated to h).

In Theorem 14.1, we will show that any two Cartan subalgebras of a semisimple Lie algebra $\mathfrak{g}$ are related by an automorphism of $\mathfrak{g}$, but for now we just fix one. See [Hum78, §15] for a more general definition of Cartan subalgebras.

► **Exercise 13.1.** Show that if $\mathfrak{g}$ is semisimple, $\mathfrak{h}$ consists of semisimple elements.

All elements of $\mathfrak{h}$ are simultaneously diagonalizable because they are all diagonalizable (by the above exercise) and they all commute (by the above proposition). For $\alpha \in \mathfrak{h}^* \smallsetminus \{0\}$ consider

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} | [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$$

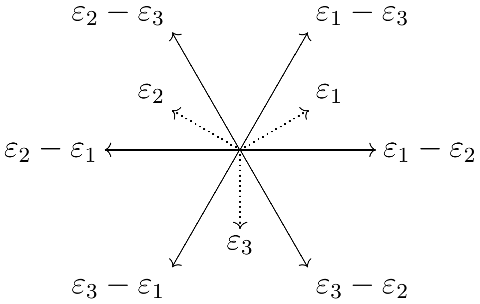
If this $\mathfrak{g}_\alpha$ is non-trivial, it is called a *root space* and the α is called a *root*. The *root decomposition* (or *Cartan decomposition*) of $\mathfrak{g}$ is $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^* \smallsetminus \{0\}} \mathfrak{g}_\alpha$.

Example 13.8. $\mathfrak{g} = \mathfrak{sl}(2)$. Take $H = \left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$, a regular element. The Cartan subalgebra is $\mathfrak{h} = k \cdot H$, a one dimensional subspace. We have $\mathfrak{g}_2 = \left\{\left(\begin{smallmatrix}0&t\\0&0\end{smallmatrix}\right)\right\}$ and $\mathfrak{g}_{-2} = \left\{\left(\begin{smallmatrix}0&0\\t&0\end{smallmatrix}\right)\right\}$, and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_{-2}$.

Example 13.9. $\mathfrak{g} = \mathfrak{sl}(3)$. Take

$$\mathfrak{h} = \left\{ \left(\begin{matrix} x_1 & & \text{\huge 0} \\ & x_2 & \\ \text{\huge 0} & & x_3 \end{matrix} \right) \middle| x_1 + x_2 + x_3 = 0 \right\}.$$

Let E_{ij} be the elementary matrices. We have that $[diag(x_1, x_2, x_3), E_{ij}] = (x_i - x_j)E_{ij}$. If we take the basis $\varepsilon_i(x_1, x_2, x_3) = x_i$ for $\mathfrak{h}^*$, then we have roots $\varepsilon_i - \varepsilon_j$. They can be arranged in a diagram:



This generalizes to $\mathfrak{sl}(n)$.

The *rank* of $\mathfrak{g}$ is defined to be $\dim \mathfrak{h}$. In particular, the rank of $\mathfrak{sl}(n)$ is going to be $n - 1$.

Basic properties of the root decomposition are:

1. $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.
2. $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ if $\alpha + \beta \neq 0$.
3. $B|_{\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}}$ is non-degenerate.
4. $B|_{\mathfrak{h}}$ is non-degenerate

Note that 3 implies that α is a root if and only if $-\alpha$ is a root.

▶ **Exercise 13.2.** Check these properties.

Now let's try to say as much as we can about this root decomposition. Define $\mathfrak{h}_\alpha \subseteq \mathfrak{h}$ as $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$. Take $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$ and $h \in \mathfrak{h}$. Then compute

$$\begin{aligned} B(\overbrace{[x,y]}^{\in \mathfrak{h}_\alpha}, h) &= B(x, \overbrace{[y,h]}^{\in \mathfrak{g}_\alpha}) && (B \text{ is invariant}) \\ &= \alpha(h) B(x, y) && (\text{since } y \in \mathfrak{g}_\alpha) \end{aligned}$$

It follows that $\mathfrak{h}_\alpha^\perp = \ker(\alpha)$, which is of codimension one. Thus, $\mathfrak{h}_\alpha$ is one dimensional.

Proposition 13.10. *If $\mathfrak{g}$ is semisimple and α is a root, then $\alpha(\mathfrak{h}_\alpha) \neq 0$.*

Proof. Assume that $\alpha(\mathfrak{h}_\alpha) = 0$. Then pick $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$ such that $[x, y] = h \neq 0$. If $\alpha(h) = 0$, then we have that $[h, x] = \alpha(h)x = 0$, $[h, y] = 0$. Thus $\langle x, y, h\rangle$ is a copy of the Heisenberg algebra, which is solvable (in fact, nilpotent). By Lie's Theorem, $ad_{\mathfrak{g}}(x)$ and $ad_{\mathfrak{g}}(y)$ are simultaneously upper triangularizable, so $ad_{\mathfrak{g}}(h) = [ad_{\mathfrak{g}}(x), ad_{\mathfrak{g}}(y)]$ is nilpotent. This is a contradiction because h is an element of the Cartan subalgebra, so it is semisimple. □

For each root α, we will take $H_\alpha \in \mathfrak{h}_\alpha$ such that $\alpha(H_\alpha) = 2$ (we can always scale H_α to get this). We can choose $X_\alpha \in \mathfrak{g}_\alpha$ and $Y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $[X_\alpha, Y_\alpha] = H_\alpha$. We have that $[H_\alpha, X_\alpha] = \alpha(H_\alpha)X_\alpha = 2X_\alpha$ and $[H_\alpha, Y_\alpha] = -2Y_\alpha$. That means we have a little copy of $\mathfrak{sl}(2) \cong \langle H_\alpha, X_\alpha, Y_\alpha\rangle$. Note that this makes $\mathfrak{g}$ a representation of $\mathfrak{sl}_2$ via $\mathfrak{sl}_2 \hookrightarrow \mathfrak{g} \overset{ad}{\hookrightarrow} \mathfrak{gl}(\mathfrak{g})$.

We normalize $\alpha(h_\alpha)$ to 2 so that we get the standard basis of $\mathfrak{sl}_2$. This way, the representations behave well (namely, that various coefficients are integers). Next we study these representations.

Irreducible finite dimensional representations of $\mathfrak{sl}(2)$

Let H, X, Y be the standard basis of $\mathfrak{sl}(2)$, and let V be an irreducible representation. By Corollary 13.2, the action of H on V is diagonalizable and the actions of X and Y on V are nilpotent. By Lie's Theorem (applied to the solvable subalgebra generated by H and X), X and H have a common eigenvector v: $Hv = \lambda v$ and $Xv = 0$ (since X is nilpotent, its only eigenvalues are zero). Verify by induction that

$$\begin{aligned} HY^r v &= YHY^{r-1}v + [H,Y]Y^{r-1}v = \big(\lambda - 2(r-1)\big)Y^r v + 2Y^r v \\ &= (\lambda - 2r)Y^r v && (13.11) \\ XY^r v &= YXY^{r-1}v + [X,Y]Y^{r-1}v \\ &= (r-1)\big(\lambda - (r-1) + 1\big)Y^{r-1}v + \big(\lambda - 2(r-1)\big)Y^{r-1}v \\ &= r(\lambda - r + 1)Y^{r-1}v && (13.12) \end{aligned}$$

Thus, the span of $v, Yv, Y^2v, \ldots$ is a subrepresentation, so it must be all of V (since V is irreducible). Since Y is nilpotent, there is a minimal n such that $Y^n v = 0$. From (13.12), we get that $\lambda = n - 1$ is a non-negative integer. Since $v, Yv, \ldots, Y^{n-1}v$ have distinct eigenvalues (under H), they are linearly independent.

Conclusion: For every non-negative integer n, there is exactly one irreducible representation of $\mathfrak{sl}_2$ of dimension $n+1$, and the H-eigenvalues on that representation are $n, n-2, n-4, \ldots, 2-n, -n$.

Remark 13.13. As a consequence, we have that in a general root decomposition, $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} g_\alpha$, each root space is one dimensional. Assume that $\dim \mathfrak{g}_{-\alpha} > 1$. Consider an $\mathfrak{sl}(2)$ in $\mathfrak{g}$, generated by $\langle X_\alpha, Y_\alpha, H_\alpha = [X_\alpha, Y_\alpha]\rangle$ where $Y_\alpha \in \mathfrak{g}_{-\alpha}$ and $X_\alpha \in \mathfrak{g}_\alpha$. Then there is some $Z \in \mathfrak{g}_{-\alpha}$ such that $[X_\alpha, Z] = 0$ (since $\mathfrak{h}_\alpha$ is one dimensional). Hence, Z is a highest vector with respect to the adjoint action of this $\mathfrak{sl}(2)$. But we have that $ad_{H_\alpha}(Z) = -2Z$, and the eigenvalue of a highest vector must be positive! This shows that the choice of X_α and Y_α is really unique.

Definition 13.14. Thinking of $\mathfrak{g}$ as a representation of $\mathfrak{sl}_2 = \langle X_\alpha, Y_\alpha, H_\alpha\rangle$, the irreducible subrepresentation containing $\mathfrak{g}_\beta$ is called the *α-string through β*.

Let Δ denote the set of roots. Then Δ is a finite subset of $\mathfrak{h}^*$ with the following properties:

1. Δ spans $\mathfrak{h}^*$.
2. If $\alpha, \beta \in \Delta$, then $\beta(H_\alpha) \in \mathbb{Z}$, and $\beta - \big(\beta(H_\alpha)\big)\alpha \in \Delta$.
3. If $\alpha, c\alpha \in \Delta$, then $c = \pm 1$.

▶ **Exercise 13.3.** Prove these properties.

Lecture 14 - More on Root Systems

Assume $\mathfrak{g}$ is semisimple. Last time, we started with a regular element $h \in \mathfrak{g}_{ss}$ and constructed the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$, where $\Delta \subseteq \mathfrak{h}^*$ is the set of roots. We proved that each $\mathfrak{g}_\alpha$ is one dimensional (we do not call 0 a root). For each root, we associated an $\mathfrak{sl}(2)$ subalgebra. Given $X_\alpha \in \mathfrak{g}_\alpha, Y_\alpha \in \mathfrak{g}_{-\alpha}$, we set $H_\alpha = [X_\alpha, Y_\alpha]$, and normalized so that $\alpha(H_\alpha) = 2$.

Furthermore, we showed that

1. Δ spans $\mathfrak{h}^*$,
2. $\alpha(H_\beta) \in \mathbb{Z}$, with $\alpha - \alpha(H_\beta)\beta \in \Delta$ for all $\alpha, \beta \in \Delta$, and
3. if $\alpha, k\alpha \in \Delta$, then $k = \pm 1$.

How unique is this decomposition? We started with some choice of a regular semisimple element. Maybe a different one would have produced a different Cartan subalgebra.

Theorem 14.1. *Let $\mathfrak{h}$ and $\mathfrak{h}'$ be two Cartan subalgebras of a semisimple Lie algebra $\mathfrak{g}$ (over an algebraically closed field of characteristic zero). Then there is some $\phi \in Ad\,G$ such that $\phi(\mathfrak{h}) = \mathfrak{h}'$. Here, G is the Lie group associated to $\mathfrak{g}$.*

Proof. Consider the map

$$\begin{aligned} \Phi : \mathfrak{h}^{\text{reg}} \times \mathfrak{g}_{\alpha_1} \times \cdots \times \mathfrak{g}_{\alpha_N} &\to \mathfrak{g} \\ (h, x_1, \dots, x_N) &\mapsto \exp(ad_{x_1}) \cdots \exp(ad_{x_N})h. \end{aligned}$$

Note that $ad_{x_i}h$ is linear in both x_i and h, and each $ad_{\mathfrak{g}_{\alpha_i}}$ is nilpotent, so the power series for exp is finite. It follows that Φ is a polynomial function. Since $\frac{d}{dt}\exp(ad_{tx_i})h\big|_{t=0} = \alpha_i(h)x_i \in \mathfrak{g}_{\alpha_i}$, the differential if Φ at $(h, 0, \dots, 0)$ is

$$D\Phi|_{(h,0,\dots,0)} = \left(\begin{array}{c|ccc} \mathrm{Id}_{\mathfrak{h}} & & 0 & \\ \hline & \alpha_1(h) & & \text{\Large 0} \\ 0 & & \ddots & \\ & \text{\Large 0} & & \alpha_N(h) \end{array}\right)$$

with respect to the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}_{\alpha_1} \oplus \cdots \oplus \mathfrak{g}_{\alpha_N}$. $D\Phi|_{(h,0,\dots,0)}$ is non-degenerate because $h \in \mathfrak{h}^{\text{reg}}$ implies that $\alpha_i(h) \neq 0$. So $\operatorname{im}\Phi$ contains a Zariski open set.[1] Let Φ' be the analogous map for $\mathfrak{h}'$. Since Zariski open sets are dense, we have that $\operatorname{im}\Phi \cap \operatorname{im}\Phi' \neq \varnothing$. So there are $\psi, \psi' \in Ad\,G$, and $h \in \mathfrak{h}, h' \in \mathfrak{h}'$ such that $\psi(h) = \psi'(h')$. Thus, we have that $\mathfrak{h} = \psi^{-1}\psi'(\mathfrak{h})$. □

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[1] This is a theorem from algebraic geometry. [FH91] claims in §D.3 that this result is in [Har77], but I cannot find it.

Abstract Root systems

We'd like to forget that any of this came from a Lie algebra. Let's just study an abstract set of vectors in $\mathfrak{h}^*$ satisfying some properties. We know that B is non-degenerate on $\mathfrak{h}$, so there is an induced isomorphism $s : \mathfrak{h} \to \mathfrak{h}^*$. By definition, $\langle s(h), h' \rangle = B(h, h')$.

Let's calculate

$$\begin{aligned}
\langle sH_\beta, H_\alpha \rangle &= B(H_\beta, H_\alpha) = B(H_\alpha, H_\beta) && (B \text{ symmetric})\\
&= B(H_\alpha, [X_\beta, Y_\beta]) = B([H_\alpha, X_\beta], Y_\beta) && (B \text{ invariant})\\
&= B(X_\beta, Y_\beta)\beta(H_\alpha)\\
&= \frac{1}{2}B([H_\beta, X_\beta], Y_\beta)\beta(H_\alpha) && (2X_\beta = [H_\beta, X_\beta])\\
&= \frac{1}{2}B(H_\beta, H_\beta)\beta(H_\alpha) && (B \text{ invariant})
\end{aligned}$$

Thus, we have that $s(H_\beta) = \frac{B(H_\beta, H_\beta)}{2}\beta$. Also, compute

$$\begin{aligned}
(\alpha, \beta) &:= \langle \alpha, s^{-1}\beta \rangle\\
&= \alpha\left(\frac{2H_\beta}{B(H_\beta, H_\beta)}\right)\\
&= \frac{2\alpha(H_\beta)}{B(H_\beta, H_\beta)}.
\end{aligned} \tag{14.2}$$

In particular, letting $\alpha = \beta$, we get $s(H_\beta) = \frac{2\beta}{(\beta,\beta)}$. This is sometimes called the *coroot* of β, and denoted $\check{\beta}$. We may use (14.2) to rewrite fact 2 from last time as:

$$\text{For } \alpha, \beta \in \Delta,\ \frac{2(\alpha, \beta)}{(\beta, \beta)} \in \mathbb{Z}, \text{ and } \alpha - \frac{2(\alpha, \beta)}{(\beta, \beta)}\beta \in \Delta. \tag{2'}$$

Now you can define $r_\beta : \mathfrak{h}^* \to \mathfrak{h}^*$ by $r_\beta(x) = x - \frac{2(x,\beta)}{(\beta,\beta)}\beta$. This is the reflection through the hyperplane orthogonal to β in $\mathfrak{h}^*$. The group generated by the r_β for $\beta \in \Delta$ is a Coxeter group. If we want to study Coxeter groups, we'd better classify root systems.[2]

We want to be working in Euclidean space, but we are now in $\mathfrak{h}^*$. Let $\mathfrak{h}_r$ be the real span[3] of the H_α's. We claim that B is positive definite on $\mathfrak{h}_r$. To see this, note that $X_\alpha, Y_\alpha, H_\alpha$ make a little $\mathfrak{sl}(2)$ in $\mathfrak{g}$, and that $\mathfrak{g}$ is therefore a representation of $\mathfrak{sl}(2)$ via the adjoint actions $ad_{X_\alpha}, ad_{Y_\alpha}, ad_{H_\alpha}$. But we know that in any representation of $\mathfrak{sl}(2)$, the eigenvalues of H_α must be integers. so $ad_{H_\alpha} \circ ad_{H_\alpha}$ has only positive eigenvalues, so $B(H_\alpha, H_\alpha) = tr(ad_{H_\alpha} \circ ad_{H_\alpha}) > 0$.

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[2] We will not talk about Coxeter groups in depth in this class.

[3] Assuming we are working over $\mathbb{C}$. Otherwise, we can use the $\mathbb{Q}$ span.

Thus, we may think of our root systems in Euclidean space, where the inner product on $\mathfrak{h}^*$ is given by $(\mu, \nu) \overset{def}{=} B(s^{-1}(\mu), s^{-1}(\nu)) = \langle \mu, s^{-1}\nu \rangle$.

Definition 14.3. An *abstract reduced root system* is a finite set $\Delta \subseteq \mathbb{R}^n \smallsetminus \{0\}$ which satisfies

(RS1) Δ spans $\mathbb{R}^n$,

(RS2) if $\alpha, \beta \in \Delta$, then $\frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$, and $r_\beta(\Delta) = \Delta$

(i.e. $\alpha, \beta \in \Delta \Rightarrow r_\beta(\alpha) \in \Delta$, with $\alpha - r_\beta(\alpha) \in \mathbb{Z}\beta$), and

(RS3) if $\alpha, k\alpha \in \Delta$, then $k = \pm 1$ (this is the "reduced" part).

The number n is called the *rank* of Δ.

Notice that given root systems $\Delta_1 \subset \mathbb{R}^n$, and $\Delta_2 \subset \mathbb{R}^m$, we get that $\Delta_1 \coprod \Delta_2 \subset \mathbb{R}^n \oplus \mathbb{R}^m$ is a root system.

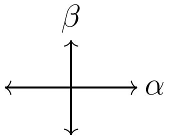
Definition 14.4. A root system is *irreducible* if it cannot be decomposed into the union of two root systems of smaller rank.

► **Exercise 14.1.** Let $\mathfrak{g}$ be a semisimple Lie algebra and let Δ be its root system. Show that Δ is irreducible if and only if $\mathfrak{g}$ is simple.

Now we will classify all systems of rank 2. Observe that $\frac{2(\alpha,\beta)}{(\alpha,\alpha)}\frac{2(\alpha,\beta)}{(\beta,\beta)} = 4\cos^2\theta$, where θ is the angle between α and β. This thing must be an integer. Thus, there are not many choices for θ:

$\cos\theta$	0	$\pm\frac{1}{2}$	$\pm\frac{1}{\sqrt{2}}$	$\pm\frac{\sqrt{3}}{2}$
θ	$\frac{\pi}{2}$	$\frac{\pi}{3}, \frac{2\pi}{3}$	$\frac{\pi}{4}, \frac{3\pi}{4}$	$\frac{\pi}{6}, \frac{5\pi}{6}$

Choose two vectors with minimal angle between them. If the minimal angle is $\pi/2$, then the system is reducible.



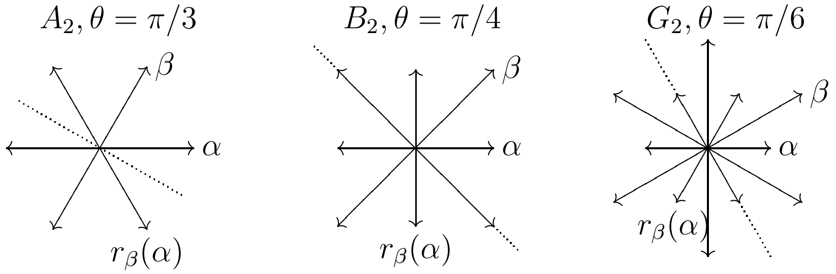
Notice that α and β can be scaled independently.

If the minimal angle is smaller than $\pi/2$, then $r_\beta(\alpha) \neq \alpha$, so the difference $\alpha - r_\beta(\alpha)$ is a non-zero integer multiple of β (in fact, a positive multiple of β since $\theta < \pi/2$). If we assume $\|\alpha\| \leq \|\beta\|$ (we can always switch them), we get that $\|\alpha - r_\beta(\alpha)\| < 2\|\alpha\| \leq 2\|\beta\|$. It follows that $\alpha - r_\beta(\alpha) = \beta$.

Remark 14.5. Observe that we have shown that for any roots α and β, if $\theta_{\alpha,\beta} < \pi/2$, then $\alpha - \beta$ is a root.

Remark 14.6. We have also shown that once we set the direction of the longer root, β (thus determining r_β), its length is determined relative to the length of α.

Now we can obtain the remaining elements of the root system from the condition that Δ is invariant under r_α and r_β, observing that no additional vectors can be added without violating RS2, RS3, or the prescribed minimal angle. Thus, all the irreducible rank two root systems are



$A_2, \theta = \pi/3$ $B_2, \theta = \pi/4$ $G_2, \theta = \pi/6$

The Weyl group

Given a root system $\Delta = \{\alpha_1, \dots, \alpha_N\}$, we call the group generated by the r_{α_i}s the *Weyl group*, denoted $\mathfrak{W}$.

Remark 14.7. If G is a Lie group with Lie algebra $\mathfrak{g}$, then for each $r_\alpha \in \mathfrak{W}$, there is a group element $S_\alpha \in G$, such that Ad_{S_α} takes $\mathfrak{h}$ to itself, and induces r_α. Consider the $\mathfrak{sl}_2 \subseteq \mathfrak{g}$ generated by X_α, Y_α, and H_α. The embedding $\mathfrak{sl}_2 \to \mathfrak{g}$ induces a homomorphism $SL(2) \to G$, and S_α is the image of $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ under this homomorphism.

▶ **Exercise 14.2.** Let $\mathfrak{g}$ be a semisimple Lie algebra, and let $\mathfrak{h}$ be a Cartan subalgebra. For each root α define

$$S_\alpha = \exp(X_\alpha)\exp(-Y_\alpha)\exp(X_\alpha).$$

Prove that $Ad_{S_\alpha}(\mathfrak{h}) = \mathfrak{h}$ and that

$$\langle \lambda, Ad_{S_\alpha}(h)\rangle = \langle r_\alpha(\lambda), h\rangle$$

for any $h \in \mathfrak{h}$ and $\lambda \in \mathfrak{h}^*$, where r_α is the reflection in $\alpha^\perp$.

If G is a connected group with Lie algebra $\mathfrak{g}$, then define the *Cartan subgroup* $H \subseteq G$ to be the subgroup generated by the image of $\mathfrak{h}$ under the exponential map $\exp : \mathfrak{g} \to G$. Let

$$N(H) = \{g \in G | gHg^{-1} = H\}$$

be the normalizer of H. Then we get a sequence of homomorphisms

$$\begin{tikzcd} N(H) \arrow[r] & \mathrm{Aut}\,H \arrow[r] & \mathrm{Aut}\,\mathfrak{h} \arrow[r] & \mathrm{Aut}\,\mathfrak{h}^* \\ g \arrow[rr, mapsto] & & Ad_g \arrow[r, mapsto] & Ad_g^*. \end{tikzcd}$$

The first map is given by conjugation, the second by differentiation at the identity, and the third by the identification of $\mathfrak{h}$ with $\mathfrak{h}^*$ via the Killing form. The final map is given by $g \mapsto Ad_g^*$, where $\big(Ad_g^*(l)\big)(h) = l(Ad_{g^{-1}}h)$.

Proposition 14.8. *The kernel of the composition above is exactly H, and the image is the Weyl group. In particular, $\mathfrak{W} \cong N(H)/H$.*

Before we prove this proposition, we need a lemma.

Lemma 14.9. *The centralizer of H is H.*

Proof. If g centralizes H, then Ad_g is the identity on $\mathfrak{h}$. Furthermore, for any $h \in \mathfrak{h}$ and $x \in \mathfrak{g}_\alpha$,

$$[h, Ad_g x] = Ad_g([h, x]) = Ad_g(\alpha(h)x) = \alpha(\mathfrak{h}) \cdot Ad_g x$$

so $Ad_g(\mathfrak{g}_\alpha) = \mathfrak{g}_\alpha$. Say $Ad_g(X_i) = c_i X_i$, where X_i spans the simple root space $\mathfrak{g}_{\alpha_i}$. Then $Ad_g(Y_i) = \frac{1}{c}Y_i$. Since the simple roots are linearly independent, we can find an $h \in \mathfrak{h}$ such that $Ad_{\exp h} X_i = c_i X_i$. Now we have that $Ad_{g\cdot(\exp h)^{-1}}$ is the identity on $\mathfrak{g}$, so $g \cdot (\exp h)^{-1}$ is in the center of G, which is in H, so $g \in H$, as desired. $\square$

Proof of Proposition 14.8. It is clear that H is in the kernel of the composition. To see that H is exactly the kernel, observe that Ad_g^* can only be the identity map if Ad_g is the identity map, which can only happen if conjugation by g is the identity map on H, i.e. if g is in the centralizer of H. By Lemma 14.9, $g \in H$.

Since the S_α in the previous exercise preserves $\mathfrak{h}$ under the Ad action, it is in the normalizer of H. It is easy to see (given Exercise 14.2) that the image of S_α in $\mathrm{Aut}\,\mathfrak{h}^*$ is exactly r_α. Thus, every element of the Weyl group is in the image.

We can show that the map preserves the set of roots. If α is a root, with a root vector x, then we have $ad_h(x) = \alpha(h)x$ for all $h \in \mathfrak{h}$. We would like to show that $Ad_g^*\alpha$ is also a root. It is enough to observe that $Ad_g x$ is a root vector:

$$\begin{aligned} ad_h(Ad_g x) &= [h, Ad_g x] = Ad_g\big([Ad_{g^{-1}}h, x]\big) \\ &= Ad_g\big(\alpha(Ad_{g^{-1}}h)x\big) = \alpha(Ad_{g^{-1}}h)Ad_g(x) \\ &= \big(Ad_g^*(\alpha)\big)(h)\,(Ad_g x) \end{aligned}$$

Therefore, we can find some element w in the Weyl group so that $w \circ Ad_g^*$ preserves the set Π of simple roots. Since w is in the image of Ad^*, it is enough to show that whenever Ad_g^* preserves Π, it is the identity map on $\mathfrak{h}^*$.

□

Example 14.10 (Also see Example 13.9)**.** The root system of $\mathfrak{sl}_{n+1}$ is called A_n. We pick an orthonormal basis $\varepsilon_1, \dots, \varepsilon_{n+1}$ of $\mathbb{R}^{n+1}$, the the root system is the set of all the differences: $\Delta = \{\varepsilon_i - \varepsilon_j | i \neq j\}$. We have that

$$r_{\varepsilon_i - \varepsilon_j}(\varepsilon_k) = \begin{cases} \varepsilon_k & k \neq i, j \\ \varepsilon_j & k = i \\ \varepsilon_i & k = j \end{cases}$$

is a transposition, so we have that $\mathfrak{W} \simeq S_{n+1}$.

Now back to classification of abstract root systems.

Draw a hyperplane in general position (so that it doesn't contain any roots). This divides Δ into two parts, $\Delta = \Delta^+ \coprod \Delta^-$. The roots in Δ^+ are called *positive roots*, and the elements of Δ^- are called negative roots. We say that $\alpha \in \Delta^+$ is *simple* if it cannot be written as the sum of other positive roots. Let Π be the set of simple roots, sometimes called a base. It has the properties

1. Any $\alpha \in \Delta^+$ is a sum of simple roots (perhaps with repitition): $\alpha = \sum_{\beta \in \Pi} m_\beta \beta$ where $m_\beta \in \mathbb{Z}_{\geq 0}$.

2. If $\alpha, \beta \in \Pi$, then $(\alpha, \beta) \leq 0$.

   This follows from the fact that if $(\alpha, \beta) > 0$, then $\alpha - \beta$ and $\beta - \alpha$ are again roots (as we showed when we classified rank 2 root systems), and one of them is positive, say $\alpha - \beta$. Then $\alpha = \beta + (\alpha - \beta)$, contradicting simplicity of α.

3. Π is a linearly independent set.

   If they were linearly dependent, the relation $\sum_{\alpha_i \in \Pi} a_i \alpha_i = 0$ must have some negative coefficients (because all of Π is in one half space), so we can always write

   $$0 \neq a_1\alpha_1 + \cdots + a_r\alpha_r = a_{r+1}\alpha_{r+1} + \cdots + a_n\alpha_n$$

   with all the $a_i \geq 0$. Taking the inner product with the left hand side, we get

   $$\begin{aligned} &\|a_1\alpha_1 + \cdots + a_r\alpha_r\|^2 \\ &\qquad = (a_1\alpha_1 + \cdots + a_r\alpha_r, a_{r+1}\alpha_{r+1} + \cdots + a_n\alpha_n) \leq 0 \end{aligned}$$

   by 2, which is absurd.

Remark 14.11. Notice that the hyperplane is $t^\perp$ for some t, and the positive roots are the $\alpha \in \Delta$ for which $(t, \alpha) > 0$. This gives an order on the roots. You can inductively prove 1 using this order.

Remark 14.12. Notice that when you talk about two roots, they always generate one of the rank 2 systems, and we know what all the rank 2 systems are.

Lemma 14.13 (Key Lemma)**.** *Suppose we have chosen a set of positive roots Δ^+, with simple roots Π. Then for $\alpha \in \Pi$, we have that $r_\alpha(\Delta^+) = \Delta^+ \cup \{-\alpha\} \smallsetminus \{\alpha\}$.*

Proof. For a simple root $\beta \neq \alpha$, we have $r_\alpha(\beta) = \beta + k\alpha$ for some non-negative k; this must be on the positive side of the hyperplane, so it is a positive root. Now assume you have a positive root of the form $\gamma = m\alpha + \sum_{\alpha_i \neq \alpha} m_i \alpha_i$, with $m, m_i \geq 0$. Then we have that $r_\alpha(\gamma) = -m\alpha + \sum_{\alpha_i \neq \alpha} m_i(\alpha_i - k_i\alpha) \in \Delta$. If any of the m_i are strictly positive, then the coefficient of α_i in $r_i(\gamma)$ is positive, so $r_\alpha(\gamma)$ must be positive because every root can be (uniquely) written as either a non-negative or a non-positive combination of the simple roots. □

Proposition 14.14. *The group generated by simple reflections (with respect to some fixed $\Pi = \{\alpha_1, \dots, \alpha_n\}$) acts transitively on the set of sets of positive roots.*

Proof. It is enough to show that we can get from Δ^+ to any other set of simple roots $\bar{\Delta}^+$.

If $\bar{\Delta}^+$ contains Π, then $\bar{\Delta}^+ = \Delta^+$ and we are done. Otherwise, there is some $\alpha_i \notin \bar{\Delta}^+$ (equivalently, $-\alpha_i \in \bar{\Delta}^+$). Applying r_i, Lemma 14.13 tells us that

$$\left| r_i(\Delta^+) \smallsetminus \bar{\Delta}^+ \right| < \left| \Delta^+ \smallsetminus \bar{\Delta}^+ \right|.$$

If we can show for any root α which is simple *with respect to* $r_i(\Delta)$, that r_α is a product of simple reflections, then we are done by induction. But we have that $\alpha = r_i(\alpha_j)$ for some j, from which we get that $r_\alpha = r_i r_j r_i$. □

Corollary 14.15. *$\mathfrak{W}$ is generated by simple reflections.*

Proof. Any root α is a simple root for some choice of positive roots. To see this, draw the hyperplane really close to the given root. Then we know that α is obtained from our initial set Π by simple reflections. We get that if $\alpha = r_{i_1} \cdots r_{i_k}(\alpha_j)$, then $r_\alpha = (r_{i_1} \cdots r_{i_k}) r_j (r_{i_1} \cdots r_{i_k})^{-1}$. □

We define the *length* of an element $w \in \mathfrak{W}$ to be the smallest number k so that $w = r_{i_1} \cdots r_{i_k}$, for some simple reflections r_{i_j}.

Next, we'd like to prove that $\mathfrak{W}$ acts *simply transitively* on the set of sets of simple roots. To do this, we need the following lemma, which essentially says that if you have a string of simple reflections so that some positive root becomes negative and then positive again, then you can get the same element of $\mathfrak{W}$ with fewer simple reflections.

Lemma 14.16. *Let* $\beta_1, \beta_2, \ldots, \beta_t$ *be a sequence in* Π *(possibly with repetition) with* $t \geq 2$. *Let* $r_i = r_{\beta_i}$. *If* $r_1 r_2 \cdots r_t(\beta_t) \in \Delta^+$, *then there is some* $s < t$ *such that*

$$r_1 \cdots r_t = r_1 \cdots r_{s-1} r_{s+1} \cdots r_{t-1}.$$

(Note that the right hand side omits r_s *and* r_t*.)*

Proof. Note that β_t is positive and $r_1 \cdots r_{t-1}(\beta_t)$ is negative, so there is a smallest number s for which $r_{s+1} \cdots r_{t-1}(\beta_t) = \gamma$ is positive. Then $r_s(\gamma)$ is negative, so by Lemma 14.13, we get $\gamma = \beta_s$. This gives us

$$\begin{aligned} r_s &= (r_{s+1} \cdots r_{t-1}) r_t (r_{s+1} \cdots r_{t-1})^{-1} \\ r_s r_{s+1} \cdots r_{t-1} &= r_{s+1} \cdots r_{t-1} r_t. \end{aligned}$$

Multiplying both sides of the second equation on the left by $r_1 \cdots r_{s-1}$ to get the result. ☐

Proposition 14.17. $\mathfrak{W}$ *acts* simply transitively *on the set of sets of positive roots.*

Proof. Proposition 14.14 shows that the action is transitive, so we need only show that any $w \in \mathfrak{W}$ which fixes Δ^+ must be the identity element. If w is a simple reflection, then it does not preserve Δ^+. So we may assume that the shortest way to express w as a product simple reflections uses at least two simple reflections, say $w = r_{i_1} \cdots r_{i_t}$. Then by Lemma 14.16, we can reduce the length of w by two, contradicting the minimality of t. ☐

Corollary 14.18. *The length of an element* $w \in \mathfrak{W}$ *is exactly* $\left| w(\Delta^+) \smallsetminus \Delta^+ \right|$.

Proof. By Proposition 14.17, w is the unique element taking Δ^+ to $w(\Delta^+)$. Say we are building a word, as in Proposition 14.14, to get from Δ^+ to $w(\Delta^+)$. Assume we've already applied $r_{i_1} \cdots r_{i_k}$, and next we are going to reflect through $r_{i_1} \cdots r_{i_k}(\alpha_j)$. Then we will have applied the element

$$(r_{i_1} \cdots r_{i_k}) r_j (r_{i_1} \cdots r_{i_k})^{-1} (r_{i_1} \cdots r_{i_k}) = r_{i_1} \cdots r_{i_k} r_j.$$

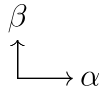
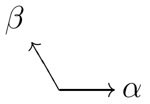
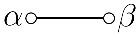
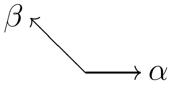
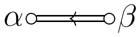
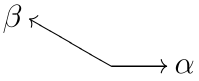
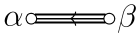
Thus, each time we reduce $\left| r_{i_1} \cdots r_{i_k}(\Delta^+) \smallsetminus w(\Delta^+) \right|$ by one, we add one simple reflection. This shows that we can express w in the desired number of simple reflections.

On the other hand, Lemma 14.13 tells us that for any sequence of simple reflections $r_{i_1}, \ldots, r_{i_k}$, $\left| r_{i_1} \cdots r_{i_k}(\Delta^+) \smallsetminus \Delta^+ \right| \leq k$, so w cannot be written as a product of fewer than $\left| w(\Delta^+) \smallsetminus \Delta^+ \right|$ simple reflections. ☐

Lecture 15 - Dynkin diagrams, Classification of root systems

Last time, we talked about root systems $\Delta \subset \mathbb{R}^n$. We constructed the Weyl group $\mathfrak{W}$, the finite group generated by reflections. We considered $\Pi \subset \Delta$, the simple roots. We showed that Π forms a basis for $\mathbb{R}^n$, and that every root is a non-negative (or non-positive) linear combination of simple roots.

If $\alpha, \beta \in \Pi$, then define $n_{\alpha\beta} = \frac{2(\alpha,\beta)}{(\beta,\beta)}$. We showed that $n_{\alpha\beta}$ is a non-positive integer. Since $n_{\alpha\beta}n_{\beta\alpha} = 4\cos^2\theta_{\alpha\beta}$, $n_{\alpha,\beta}$ can only be $0, -1, -2$, or -3. If $n_{\alpha\beta} = 0$, the two are orthogonal. If $n_{\alpha\beta} = -1$, then the angle must be $2\pi/3$ and the two are the same length. If $n_{\alpha\beta} = -2$, the angle must be $3\pi/4$ and $||\beta|| = \sqrt{2}\,||\alpha||$. If $n_{\alpha\beta} = -3$, the angle is $5\pi/6$ and $||\beta|| = \sqrt{3}\,||\alpha||$. Thus we get:

$n_{\beta\alpha}$	$n_{\alpha\beta}$	relationship	Dynkin picture
0	0		
-1	-1		
-2	-1		
-3	-1		

Definition 15.1. Given a root system, the *Dynkin diagram* of the root system is obtained in the following way. For each simple root, draw a node. We join two nodes by $n_{\alpha\beta}n_{\beta\alpha}$ lines. If there are two or three lines (i.e. if the roots are not the same length), then we draw an arrow from the longer root to the shorter root. (As always, the alligator eats the big one.)

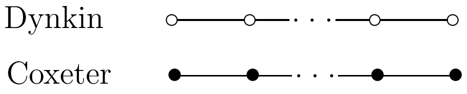
The Dynkin diagram is independent of the choice of simple roots. For any other choice of simple roots, there is an element of the Weyl group that translates between the two, and the Weyl group preserves inner products.

We would really like to classify Dynkin diagrams. To aid the classification, we define an undirected version of the Dynkin diagram. Define $e_i = \frac{\alpha_i}{(\alpha_i,\alpha_i)^{1/2}}$, for $\alpha_i \in \Pi$. Then the number of lines between two vertices is $n_{\alpha_i\alpha_j}n_{\alpha_j\alpha_i} = 4 \cdot \frac{(\alpha_i,\alpha_j)^2}{(\alpha_i,\alpha_i)(\alpha_j,\alpha_j)} = 4(e_i, e_j)^2$.

Definition 15.2. Given a set $\{e_1, \ldots, e_n\}$ of linearly independent unit vectors in some Euclidean space with the property that $(e_i, e_j) \le 0$ and $4(e_i, e_j)^2 \in \mathbb{Z}$ for all i and j, the *Coxeter diagram* associated to the set is obtained in the following way. For each unit vector, draw a node. Join the nodes of e_i and e_j by $4(e_i, e_j)^2$ lines.

Since every Dynkin diagram gives a Coxeter diagram, understanding Coxeter diagrams is a good start in classifying Dynkin diagrams.

Example 15.3. A_n has n simple roots, given by $\varepsilon_i - \varepsilon_{i+1}$. So the graphs are



Let's prove some properties of Coxeter diagrams.

(CX1) A subgraph of a Coxeter diagram is a Coxeter diagram. This is obvious from the definition.

(CX2) A Coxeter diagram is acyclic.

Proof. Let $e_1, \ldots, e_k$ be a cycle in the Coxeter diagram. Then

$$\left(\sum e_i, \sum e_i\right) = k + \sum_{\substack{i<j \\ i,j \text{ adjacent}}} \underbrace{2(e_i, e_j)}_{\le -1} \le 0$$

which contradicts that the inner product is positive definite. □

(CX3) The degree of each vertex in a Coxeter diagram is less than or equal to 3, where double and triple edges count as two and three edges, respectively.

Proof. Let e_0 have $e_1, \ldots, e_k$ adjacent to it. Since there are no cycles, $e_1, \ldots, e_k$ are orthogonal to each other. So we can compute

$$\left(e_0 - \sum (e_0, e_i) e_i, e_0 - \sum (e_0, e_i) e_i\right) > 0$$
$$1 - \sum (e_0, e_i)^2 > 0$$

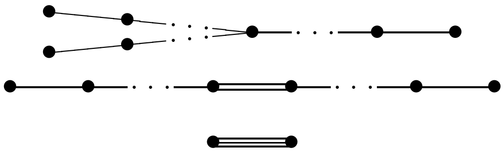
but $(e_0, e_i)^2$ is one fourth of the number of edges connecting e_0 and e_i. So k cannot be bigger than 3. □

(CX4) Suppose a Coxeter diagram has a subgraph of type A_n, and only the endpoints of this subgraph have additional edges (say Γ_1 at one end and Γ_2 at the other end). Then we can "contract" the stuff in the middle and just fuse Γ_1 with Γ_2, and the result is a Coxeter diagram.

Proof. Let $e_1, \ldots, e_k$ be the vertices in the A_k. Let $e_0 = e_1 + \cdots + e_k$. Then we can compute that $(e_0, e_0) = 1$. If $e_s \in \Gamma_1$ and $e_t \in \Gamma_2$, we get that $(e_0, e_s) = (e_1, e_s)$ and $(e_0, e_t) = (e_k, e_t)$. ☐

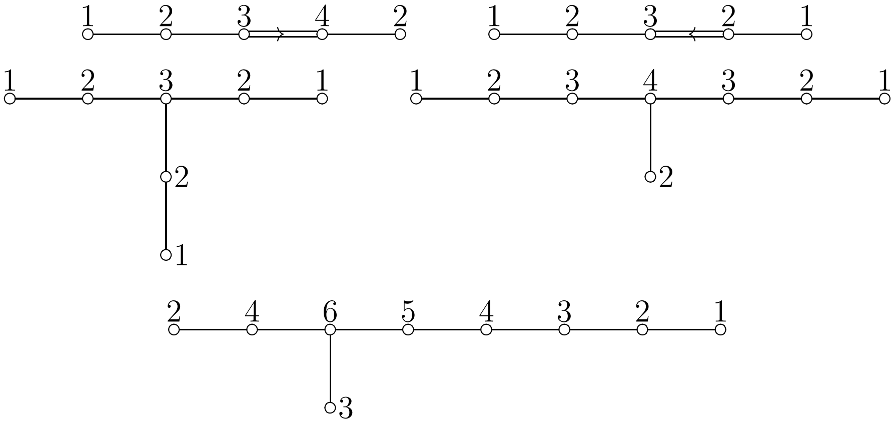
Thus, a connected Coxeter diagram can have at most one fork (two could be glued to give valence 4), at most one double edge, and if there is a triple edge, nothing else can be connected to it.

So the only possible connected Coxeter diagrams (an therefore Dynkin diagrams) so far are of the form



Now we switch gears back to Dynkin diagrams. Note that a subgraph of a Dynkin diagram is a Dynkin diagram. We will calculate that some diagrams are forbidden. We label the vertex corresponding to α_i with a number m_i, and check that

$$\left(\sum m_i \alpha_i, \sum m_i \alpha_i\right) = 0.$$



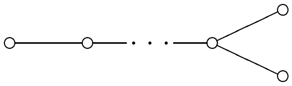
Thus we have narrowed our list of possible Dynkin diagrams to a short list.

The “classical” connected Dynkin diagrams are shown below (n is the total number of vertices).

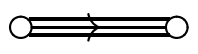

(A_n)

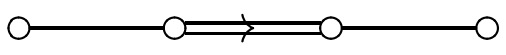

(B_n)

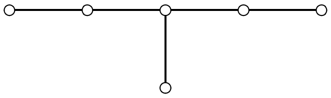

(C_n)

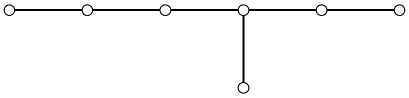

(D_n)

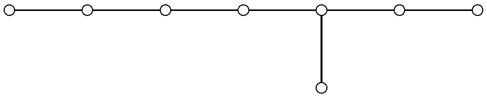
The “exceptional” Dynkin diagrams are


(G_2)


(F_4)


(E_6)


(E_7)


(E_8)

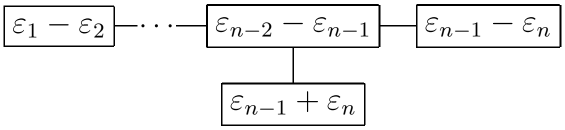
It remains to show that each of these is indeed the Dynkin diagram of some root system.

We have already constructed the root system A_n in Example 14.10.

Next we construct D_n. Let $\varepsilon_1, \dots, \varepsilon_n$ be an orthonormal basis for $\mathbb{R}^n$. Then let the roots be

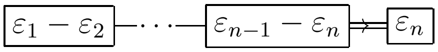
$$\Delta = \{\pm(\varepsilon_i \pm \varepsilon_j) | i < j \leq n\}.$$

We choose the simple roots to be



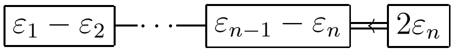
(D_n)

To get the root system for B_n, take D_n and add $\{\pm\varepsilon_i | i \leq n\}$, in which case the simple roots are



(B_n)

To get C_n, take D_n and add $\{\pm 2\varepsilon_i | i \leq n\}$, then the simple roots are



(C_n)

Remark 15.4. Recall that we can define coroots $\check{\alpha} = \frac{2\alpha}{(\alpha,\alpha)}$. Replacing all the roots with their coroots will reverse the arrows in the Dynkin diagram. The dual root system is usually the same as the original, but is sometimes different. For example, C_n and B_n are dual.

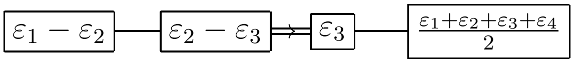
Now let's construct the exceptional root systems.

We constructed G_2 when we classified rank two root systems on page 72.

F_4 comes from some special properties of a cube in 4-space. Let ε_1, ε_2, ε_3, ε_4 be an orthonormal basis for $\mathbb{R}^4$. Then let the roots be

$$\left\{ \pm(\varepsilon_i \pm \varepsilon_j), \pm\varepsilon_i, \frac{\pm(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)}{2} \right\}$$

The simple roots are



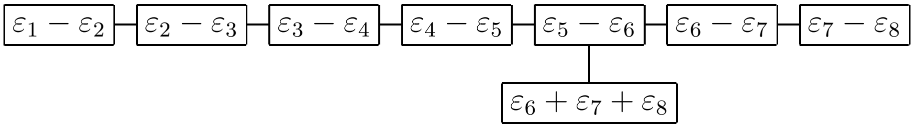
(F_4)

There are 48 roots total. Remember that the dimension of the Lie algebra (which we have yet to construct) is the number of roots plus the dimension of the Cartan subalgebra (the rank of $\mathfrak{g}$, which is 4 here), so the dimension is 52 in this case.

To construct E_8, look at $\mathbb{R}^9$ with our usual orthonormal basis. The trick is that we are going to project on to the plane orthogonal to $\varepsilon_1 + \cdots + \varepsilon_9$. The roots are

$$\{\varepsilon_i - \varepsilon_j | i \neq j\} \cup \{\pm(\varepsilon_i + \varepsilon_j + \varepsilon_k) | i \neq j \neq k\}$$

The total number of roots is $|\Delta| = 9 \cdot 8 + 2\binom{9}{3} = 240$. So the dimension of the algebra is 248! The simple roots are



(E_8)

The root systems E_6 and E_7 are contained in the obvious way in the root system E_8.

Warning 15.5. Remember to project onto the orthogonal complement of $\varepsilon_1 + \cdots + \varepsilon_9$. Thus, $\varepsilon_6 + \varepsilon_7 + \varepsilon_8$ is really $\frac{2}{3}(\varepsilon_6 + \varepsilon_7 + \varepsilon_8) - \frac{1}{3}(\varepsilon_1 + \cdots + \varepsilon_5 + \varepsilon_9)$. There is another way to construct this root system, which is discussed in Lecture 25.

▶ **Exercise 15.1.** Verify that F_4 and E_8 are root systems, and that the given sets are simple roots.

We have now classified all indecomposable root systems. The diagram of the root system $\Delta_1 \coprod \Delta_2$ is the disjoint union of the diagrams of Δ_1 and Δ_2.

Construction of the Lie algebras A_n, B_n, C_n, and D_n

Next lecture, we will prove Serre's Theorem (Theorem 16.1), which states that for each irreducible root system and for each algebraically closed field of characteristic zero, there is a unique simple Lie algebra with the given root system (it actually gives explicit generators and relations for this Lie algebra). Meanwhile, we will explicitly construct Lie algebras with the classical root systems.

$\underline{A_n}$: Example 14.10 shows that $\mathfrak{sl}(n+1)$ has root system A_n.

$\underline{D_n}$: Consider $\mathfrak{so}(2n)$, the Lie algebra of linear maps of k^{2n} preserving some non-degenerate symmetric form. We can choose a basis for k^{2n} so that the matrix of the form is $I = \left(\begin{smallmatrix} 0 & 1_n \\ 1_n & 0 \end{smallmatrix}\right)$. Let $X \in \mathfrak{so}(2n)$, then we have that $X^t I + IX = 0$. It follows that X is of the form

$$X = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \text{ with } B^t = -B, C^t = -C.$$

We guess[1] that and element H of the Cartan subalgebra should have the form $A = diag(t_1, \dots, t_n)$ and $B = C = 0$ (to check this guess, it is enough to demonstrate that we get a root decomposition). To compute the root spaces, we try bracketing H with various elements of $\mathfrak{so}(2n)$. We have that $\left(\begin{smallmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{smallmatrix}\right)$ has eigenvalue $t_i - t_j$, that $\left(\begin{smallmatrix} 0 & E_{ij}-E_{ji} \\ 0 & 0 \end{smallmatrix}\right)$ has eigenvalue $t_i + t_j$, and that $\left(\begin{smallmatrix} 0 & 0 \\ E_{ij}-E_{ji} & 0 \end{smallmatrix}\right)$ eigenvalue $-t_i - t_j$. Since these matrices span $\mathfrak{so}(2n)$, we know that we are done. Thus, D_n is the root system of $\mathfrak{so}(2n)$.

$\underline{B_n}$: Consider $\mathfrak{sp}(2n)$, the linear operators on k^{2n} which preserve a non-degenerate skew-symmetric form. In some basis, the form is $J = \left(\begin{smallmatrix} 0 & 1_n \\ -1_n & 0 \end{smallmatrix}\right)$, so an element $X \in \mathfrak{sp}(2n)$ satisfies $X^t J + JX = 0$. It follows that X is of the form

$$X = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \text{ with } B^t = B, C^t = C.$$

Let the Cartan subalgebra be the diagonal matrices. We get all the same roots as for $\mathfrak{so}(2n)$, and a few more. $\left(\begin{smallmatrix} 0 & E_{ii} \\ 0 & 0 \end{smallmatrix}\right)$ has eigenvalue $2t_i$, and $\left(\begin{smallmatrix} 0 & 0 \\ E_{ii} & 0 \end{smallmatrix}\right)$ has eigenvalue $-2t_i$. Thus, B_n is the root system of $\mathfrak{sp}(2n)$.

$\underline{C_n}$: Consider $\mathfrak{so}(2n+1)$. Choose a basis so that the non-degenerate symmetric form is

$$I = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 1_n \\ 0 & 1_n & 0 \end{array}\right).$$

---

[1] This will always be the right guess. The elements of the Cartan are simultaneously diagonalizable, so in some basis, the Cartan is exactly the set of diagonal matrices in the Lie algebra. The guess would be wrong if the Lie algebra did not have enough diagonal elements, but this would just mean that we had chosen the wrong basis.

Then an element $X \in \mathfrak{so}(2n+1)$, satisfying $X^tI + IX = 0$, has the form

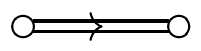
$$X = \left(\begin{array}{c|cc} 0 & u & v \\ \hline -v^t & A & B \\ -u^t & C & -A^t \end{array}\right), \text{ with } B^t = -B, C^t = -C,$$

where u and v are row vectors of length n. Again, we take the Cartan subalgebra to be the diagonal matrices. We get all the same roots as we for $\mathfrak{so}(2n)$, and a few more. If e_i is the row vector with a one in the i-th spot and zeros elsewhere, then $\left(\begin{array}{c|cc} 0 & e_i & 0 \\ \hline 0 & 0 & 0 \\ -e_i^t & 0 & 0 \end{array}\right)$ has eigenvalue t_i, and $\left(\begin{array}{c|cc} 0 & 0 & e_i \\ \hline -e_i^t & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$ has eigenvalue $-t_i$. Thus, C_n is the root system of $\mathfrak{so}(2n+1)$.

Isomorphisms of small dimension

Let's say that we believe Serre's Theorem. Then you can see that for small n some of the Dynkin diagrams coincide, so the corresponding Lie algebras are isomorphic.

$B_2 = C_2$

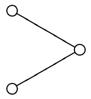


$\mathfrak{so}(5) \simeq \mathfrak{sp}(4)$

$D_2 = A_1 \coprod A_1$

$\mathfrak{so}(4) \simeq \mathfrak{sl}(2) \times \mathfrak{sl}(2)$

$D_3 = A_3$



$\mathfrak{so}(6) \simeq \mathfrak{sl}(4)$

We can see some of these isomorphisms directly on the level of groups! Let's construct a map of groups $SL(2) \times SL(2) \to SO(4)$, whose kernel is discrete. Let W be the space of 2×2 matrices, then the $SL(2) \times SL(2)$ acts on W by $(X, Y)w = XwY^{-1}$. This action preserves the determinant of $w = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. That is, the quadratic form $ad - bc$ is preserved, so the corresponding non-degenerate bilinear form is preserved.[2] The Lie group preserving such a form is $SO(4)$, so we have a map $SL_(2) \times SL(2) \to SO(4)$. It is easy to check that the kernel is the set $\{(I, I), (-I, -I)\}$, and since the domain and range each have dimension 6, we get $SL(2) \times SL(2)/(\pm I, \pm I) \cong SO(4)$ (we are also using that $SO(4)$ is connected). This yields an isomorphism on the level of Lie algebras.

Now let's see that $\mathfrak{so}(6) \simeq \mathfrak{sl}(4)$. The approach is the same. Let V be the standard 4 dimensional representation of $SL(4)$. Let $W = \Lambda^2 V$,

---

[2] Given a quadratic form Q, one gets a symmetric bilinear form $(w, w') := Q(w + w') - Q(w) - Q(w')$. In the case $Q(w) = \det w$, the form is non-degenerate. Indeed, assume $\det(w + w') = \det w + \det w$ for all w. Then by choosing a basis so that w' is in Jordan form and letting w vary over diagonal matrices, we see that $w' = 0$.

which is a 6 dimensional representation of $SL(4)$. Note that you have a pairing

$$W \times W = \Lambda^2 V \times \Lambda^2 V \to \Lambda^4 V \overset{\det}{\simeq} k$$

where the last map is an isomorphism of representations of $SL(4)$ (because the determinant of any element of $SL(4)$ is 1). Thus, $W = \Lambda^2 V$ has some $SL(4)$-invariant non-degenerate symmetric bilinear form, so we have a map $SL(4) \to SO(W) \simeq SO(6)$. It is not hard to check that the kernel is $\pm I$, and the dimensions match, so we get an isomorphism of Lie algebras.

Lecture 16 - Serre's Theorem

Start with a semisimple Lie algebra $\mathfrak{g}$ over an algebraically closed field k of characteristic zero, with Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Then we have the root system $\Delta \subseteq \mathfrak{h}^*$, with a fixed set of simple roots $\Pi = \{\alpha_1, \dots, \alpha_n\}$. We have a copy of $\mathfrak{sl}_2$—generated by X_i, Y_i, and H_i—associated to each simple root.

The *Cartan matrix* (a_{ij}) of $\mathfrak{g}$ is given by $a_{ij} = \langle \check{\alpha}_i, \alpha_j \rangle = \alpha_j(H_i) = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$. From the definition of coroots and from properties of simple roots, we know that $a_{ij} \in \mathbb{Z}_{\le 0}$ for $i \neq j$, that $a_{ii} = 2$, and that $a_{ij} = 0$ implies $a_{ji} = 0$.

Claim. *The following relations (called* Serre relations[1]*) are satisfied in* $\mathfrak{g}$.

$$\begin{array}{llll} [H_i, X_j] = a_{ij} X_j & \text{(a)} & [H_i, H_j] = 0 & \text{(c)} \\ [H_i, Y_j] = -a_{ij} Y_j & \text{(b)} & [X_i, Y_j] = \delta_{ij} H_i & \text{(d)} \end{array} \tag{Ser1}$$

$$\begin{array}{l} \theta^+_{ij} := (ad_{X_i})^{1-a_{ij}} X_j = 0 \\ \theta^-_{ij} := (ad_{Y_i})^{1-a_{ij}} Y_j = 0 \end{array} \text{, for } i \neq j. \tag{Ser2}$$

Proof. (Ser1a), (Ser1b), and (Ser1c) are immediate because $X_i \in \mathfrak{g}_{\alpha_i}$, $Y_i \in \mathfrak{g}_{-\alpha_i}$, and $H_i = [X_i, Y_i] \in \mathfrak{h}$. To show (Ser1d), we need to show that $[X_i, Y_i] = 0$ for $i \neq j$. This is because $[X_i, Y_j] \in \mathfrak{g}_{\alpha_i - \alpha_j}$, which is not in Δ because every element of Δ is a non-negative or non-positive combination of the α_i.

Since $ad_{X_i}(Y_j) = 0$, we get that Y_j is a highest vector for the $\mathfrak{sl}(2)$ generated by X_i, Y_i, and H_i. We also have that $ad_{H_i}(Y_j) = -a_{ij} Y_j$. Thus, the α_i-string through Y_j is spanned by Y_j, $ad_{Y_i} Y_j$, $\dots$, $ad_{Y_i}^{-a_{ij}} Y_j$. In particular, $\theta^-_{ij} = ad_{Y_i}^{1-a_{ij}} Y_j = 0$. Similarly, $\theta^+_{ij} = 0$, so the relations (Ser2) hold. □

So far, all we know is that any Lie algebra with root system Δ satisfies these relations. We have yet to show that such an algebra exists, that it is unique, and that these relations define it.

Theorem 16.1 (Serre's Theorem)**.** *Let Δ be a root system, with a fixed set of simple roots $\Pi = \{\alpha_1, \dots, \alpha_n\}$, yielding the Cartan matrix $a_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$. Let $\mathfrak{g}$ be the Lie algebra generated by H_i, X_i, Y_i for $1 \le i \le n$, with relations (Ser1) and (Ser2). Then $\mathfrak{g}$ is a finite dimensional semisimple Lie algebra with a Cartan subalgebra spanned by $H_1, \dots, H_n$, and with root system Δ.*

---

[1] Serre called them Weyl relations.

Remark 16.2. In order to talk about a Lie algebra given by certain generators and relations, it is necessary to understand the notion of a *free Lie algebra* $L(X)$ on a set of generators X, which is non-trivial (because of the Jacobi identity). We define $L(X)$ as the Lie subalgebra of the tensor algebra $T(X)$ generated by the set X. This algebra has the universal property that for any Lie algebra L' and for any function $f : X \to L'$, there is a *unique* extension of f to a Lie algebra homomorphism $\tilde{f} : L(X) \to L'$ (to prove this, one needs the PBW theorem).

To impose a set of relations R, quotient $L(X)$ by the smallest ideal containing R. The resulting Lie algebra $L(X, R)$ has the universal property that for any Lie algebra L' and for any function $f : X \to L'$ such that the image satisfies the relations R, there is a *unique* extension of f to a Lie algebra homomorphism $\tilde{f} : L(X, R) \to L'$.

Remark 16.3. Serre's Theorem proves that for any root system Δ there is a finite dimensional semisimple Lie algebra $\mathfrak{g}$ with root system Δ. But since any other Lie algebra $\mathfrak{g}'$ with root system Δ satisfies (Ser1) and (Ser2), and since $\mathfrak{g}$ is the universal Lie algebra satisfying these relations, we get an induced Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{g}'$. This homomorphism is surjective because $\mathfrak{g}'$ is spanned by $\phi(X_i)$, $\phi(Y_i)$, and $\phi(H_i)$. Moreover, both $\dim \mathfrak{g}$ and $\dim \mathfrak{g}'$ must be equal to $|\Delta|+\mathrm{rank}(\Delta)$, so ϕ must be an isomorphism. Therefore, we get uniqueness of $\mathfrak{g}$.

Proof of Serre's Theorem.

Step 1. Decompose $\tilde{\mathfrak{g}}$: Consider the free Lie algebra with generators X_i, Y_i, H_i for $1 \le i \le n$ and impose the relations (Ser1). Call the result $\tilde{\mathfrak{g}}$. Let $\mathfrak{h}$ be the abelian Lie subalgebra generated by $H_1, \dots, H_n$, and let $\tilde{\mathfrak{n}}^+$ (resp. $\tilde{\mathfrak{n}}^-$) be the Lie subalgebra generated by the X_i (resp. Y_i). The goal is to show that $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}^- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}^+$ as a vector space.

There is a standard trick for doing such things. It is easy to see from (Ser1) that $U\tilde{\mathfrak{g}} = U\tilde{\mathfrak{n}}^- \cdot U\mathfrak{h} \cdot U\tilde{\mathfrak{n}}^+$.[2] Let $T(X)$ be the tensor algebra on the X_i, let $T(Y)$ be the tensor algebra on the Y_i, and let $S\mathfrak{h}$ be the symmetric algebra on the H_i. We define a representation $U\tilde{\mathfrak{g}} \to \mathrm{End}\big(T(Y) \otimes S\mathfrak{h} \otimes T(X)\big)$. For $a \in T(Y)$, $b \in S\mathfrak{h}$, and $c \in T(X)$, define

$$\begin{aligned} X_i(1 \otimes 1 \otimes c) &= 1 \otimes 1 \otimes X_i c, \\ H_i(1 \otimes b \otimes c) &= 1 \otimes H_i b \otimes c, \text{ and} \\ Y_i(a \otimes b \otimes c) &= (Y_i a) \otimes b \otimes c. \end{aligned}$$

---

[2] By $U\tilde{\mathfrak{n}}^- \cdot U\mathfrak{h} \cdot U\tilde{\mathfrak{n}}^+$, we mean the set of linear combinations of terms of the form $y \cdot h \cdot x$, where $y \in U\tilde{\mathfrak{n}}^-$, $h \in U\mathfrak{h}$, and $x \in U\tilde{\mathfrak{n}}^+$.

Then extend inductively by

$$\begin{aligned} H_i(Y_j a \otimes b \otimes c) &= Y_j H_i(a \otimes b \otimes c) - a_{ij} Y_j (a \otimes b \otimes c) \\ X_i(1 \otimes H_j b \otimes c) &= H_j X_i(1 \otimes b \otimes c) - a_{ji} X_i(1 \otimes b \otimes c) \\ X_i(Y_j a \otimes b \otimes c) &= Y_j X_i(a \otimes b \otimes c) + \delta_{ij} H_i(a \otimes b \otimes c). \end{aligned}$$

▶ **Exercise 16.1.** Check that this is a representation.

Observe that the canonical (graded vector space) homomorphism $T(Y) \otimes S\mathfrak{h} \otimes T(X) \to U\tilde{\mathfrak{n}}^- \cdot U\mathfrak{h} \cdot U\tilde{\mathfrak{n}}^+ = U\tilde{\mathfrak{g}}$ is the inverse of the map $w \mapsto w(1 \otimes 1 \otimes 1)$, so $U\tilde{\mathfrak{g}} \simeq T(Y) \otimes S\mathfrak{h} \otimes T(X)$ as graded vector spaces.[3] Looking at the degree 1 parts, we get the vector space isomorphism $\tilde{\mathfrak{g}} \simeq \tilde{\mathfrak{n}}^- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}^+$.

Step 2. Construct $\mathfrak{g}$: We have that $\theta_{ij}^{\pm} \in \tilde{\mathfrak{n}}^{\pm}$. Let $\mathfrak{j}^+$ (resp. $\mathfrak{j}^-$) be the ideal in $\tilde{\mathfrak{n}}^+$ (resp. $\tilde{\mathfrak{n}}^-$) generated by the set $\{\theta_{ij}^+\}$ (resp. $\{\theta_{ij}^-\}$).

▶ **Exercise 16.2.** Check that

$$[Y_k, \theta_{ij}^+] = 0 \qquad \text{and} \qquad [H_k, \theta_{ij}^+] = c_{kij}\theta_{ij}^+$$

for some constants c_{kij}. Therefore, $\mathfrak{j}^{\pm}$ are ideals in $\tilde{\mathfrak{g}}$.

Now define $\mathfrak{n}^+ = \tilde{\mathfrak{n}}^+/\mathfrak{j}^+$, $\mathfrak{n}^- = \tilde{\mathfrak{n}}^-/\mathfrak{j}^-$, and $\mathfrak{g} = \tilde{\mathfrak{g}}/(\mathfrak{j}^+ + \mathfrak{j}^-) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. From relations (Ser1), we know that $\mathfrak{h}$ acts diagonalizably on $\mathfrak{n}^+$, $\mathfrak{n}^-$, and $\mathfrak{h}$, so we get the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\mathfrak{h}^*} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} | [h, x] = \alpha(h)x\}$. Note that each $\mathfrak{g}_\alpha$ is either in $\mathfrak{n}^+$ or in $\mathfrak{n}^-$.

Define R as the set of non-zero $\alpha \in \mathfrak{h}^*$ such that $\mathfrak{g}_\alpha \neq 0$. We know that $\pm\alpha_1, \ldots, \pm\alpha_n \in R$ because $X_i \in \mathfrak{g}_{\alpha_i}$ and $Y_i \in \mathfrak{g}_{-\alpha_i}$. Since each $\mathfrak{g}_\alpha$ is either in $\mathfrak{n}^+$ or in $\mathfrak{n}^-$, α must be a non-negative or a non-positive combination of the α_i (recalling that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$). This gives us the decomposition $R = R^+ \amalg R^-$.

Since $\mathfrak{g}$ is generated by the X_i and Y_i, the relation $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ tells us that R is contained in the lattice $\sum_{i=1}^n \mathbb{Z}\alpha_i$. Since $\mathfrak{n}^+ = \bigoplus_{\alpha\in R^+} \mathfrak{g}_\alpha$ is a quotient of $\tilde{\mathfrak{n}}^+$, it is generated as Lie algebra by the X_i. Together with the relation $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ and the linear independence of the α_i, this tells us that $\mathfrak{g}_{\alpha_i}$ is one dimensional, spanned by X_i, and that $\mathfrak{g}_{n\alpha_i} = [\mathfrak{g}_{\alpha_i}, \mathfrak{g}_{(n-1)\alpha_i}] = 0$ for $n > 1$.

Step 3. R is $\mathfrak{W}$-invariant: Let $\mathfrak{W}$ be the Weyl group of the root system Δ, generated by the simple reflections $r_i : \lambda \mapsto \lambda - \lambda(H_i)\alpha_i$. We would like to show that R is invariant under the action of $\mathfrak{W}$. To do this, we need to make sense of the element $s_i = \exp(ad_{X_i}) \exp(-ad_{Y_i}) \exp(ad_{X_i}) \in \operatorname{Aut}\mathfrak{g}$.

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[3] $U\tilde{\mathfrak{g}}$ is graded as a vector space, but only *filtered* as an algebra.

The main idea is that (Ser1) and (Ser2) imply that ad_{X_i}, ad_{Y_i} are locally nilpotent operators on $\mathfrak{g}$.[4] The Serre relations say that ad_{X_i} and ad_{Y_i} are nilpotent on generators, and then the Jacobi identity implies that they are locally nilpotent. Thus, $s_i = \exp(ad_{X_i})\exp(-ad_{Y_i})\exp(ad_{X_i})$ is a well-defined automorphism of $\mathfrak{g}$ because each power series is (locally) finite.

As in Exercise 14.2, we get $s_i(\mathfrak{h}) \subseteq \mathfrak{h}$ and

$$\lambda\big(s_i(h)\big) = \langle \lambda, s_i(h)\rangle = \langle r_i(\lambda), h\rangle = (r_i\lambda)(h) \tag{16.4}$$

for any $h \in \mathfrak{h}$ and any $\lambda \in \mathfrak{h}^*$.

Now we are ready to show that R is $\mathfrak{W}$-invariant. If $\alpha \in R$, with $X \in \mathfrak{g}_\alpha$, then we will show that $s_i^{-1}X$ is a root vector for $r_i\alpha$. For $h \in \mathfrak{h}$, we have

$$\begin{aligned} [h, s_i^{-1}X] &= s_i^{-1}([s_ih, X]) = s_i^{-1}\big(\alpha(s_ih)X\big) \\ &= \alpha(s_ih)\, s_i^{-1}X = (r_i\alpha)(h)\, s_i^{-1}X, \qquad \text{(by 16.4)} \end{aligned}$$

so $r_i\alpha \in R$. So $\mathfrak{W}$ preserves R.

On the other hand, we know that $\pm\alpha_i \subseteq R$ from the end of Step 2, so we get $\Delta \subseteq R$. Moreover, for any $\alpha \in \Delta$, we have that $\dim \mathfrak{g}_\alpha = 1$ because we can choose $w = r_{i_1}\cdots r_{i_k}$ and $s = s_{i_1}\cdots s_{i_k}$ so that $\alpha = w(\alpha_i)$ for some i; then $\mathfrak{g}_\alpha = s(\mathfrak{g}_{\alpha_i})$ has dimension one by the last sentence of Step 2.

Step 4. Prove that $\Delta = R$: Let $\lambda \in R \smallsetminus \Delta$. Then λ is not proportional to any $\alpha \in \Delta$. One can find some h in the real span of the H_i such that $\langle\lambda, h\rangle = 0$ and $\langle\alpha, h\rangle \neq 0$ for all $\alpha \in \Delta$. This decomposes Δ as $\Delta^{+'} \coprod \Delta^{-'}$, and gives a new basis of simple roots $\{\beta_1, \dots, \beta_n\} = \Pi' \subseteq \Delta^{+'}$. By Proposition 14.14, $\mathfrak{W}$ acts transitively on the sets of simple roots, so we can find some $w \in \mathfrak{W}$ such that $w(\alpha_i) = \beta_i$ (after permutation of the β_i, if necessary). Then look at $w^{-1}(\lambda) \in R$.

By construction λ is neither in the non-negative span nor the non-positive span of the β_i, so $w^{-1}(\lambda)$ is neither in the non-negative nor the non-positive span of the α_i. But we had the decomposition $R = R^+ \coprod R^-$ from Step 2, so this is a contradiction. Hence $\Delta = R$.

Step 5. Check that $\mathfrak{g}$ is semisimple: It is enough to show that $\mathfrak{h}$ has no nontrivial abelian ideals. We already know that $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta}\mathfrak{g}_\alpha$ and that each $\mathfrak{g}_\alpha$ is 1 dimensional. In particular, $\mathfrak{g}$ is *finite dimensional*. We also know that the Serre relations hold. Notice that for any ideal $\mathfrak{a}$, $ad_{\mathfrak{h}}$-invariance implies that $\mathfrak{a} = \mathfrak{h}' \oplus_{\alpha\in S}\mathfrak{g}_\alpha$ for some subspace $\mathfrak{h}' \subseteq \mathfrak{h}$ and some subset $S \subseteq \Delta$. If $\mathfrak{g}_\alpha \subseteq \mathfrak{a}$, then $X_\alpha \in \mathfrak{a}$, so $[X_\alpha, Y_\alpha] = H_\alpha \in \mathfrak{a}$ ($\mathfrak{a}$ is an ideal), and $[Y_\alpha, H_\alpha] = 2Y_\alpha \in \mathfrak{a}$. Thus, we have the whole $\mathfrak{sl}(2)$

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[4] An operator A on V is *locally nilpotent* if for any vector $v \in V$, there is some $n(v)$ such that $A^{n(v)}v = 0$.

in $\mathfrak{a}$, so it cannot be abelian. So $\mathfrak{a} = \mathfrak{h}' \subseteq \mathfrak{h}$. Take a nonzero element $h \in \mathfrak{h}'$. Since $\{\alpha_1, \dots, \alpha_n\}$ spans $\mathfrak{h}^*$, there is some α_i with $\alpha_i(h) \neq 0$, then $[h, X_i] = \alpha_i(h) X_i \in \mathfrak{a}$, contradicting $\mathfrak{a} \subseteq \mathfrak{h}$. □

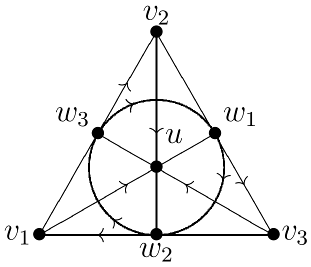
In the non-exceptional cases, we have nice geometric descriptions of these Lie algebras. Next time, we will explicitly construct the exceptional Lie algebras.

Lecture 17 - Constructions of Exceptional simple Lie Algebras

We'll begin with the construction of G_2.

We saw here that G_2 is isomorphic to the Lie algebra of automorphisms of a generic 3-form in 7 dimensional space. The picture of the projective plane is related to Cayley numbers, an important nonassociative division algebra, of which G_2 is the algebra of automorphisms.

Consider the picture



This is the projective plane over $\mathbb{F}_2$.

Consider the standard 7 dimensional representation of $\mathfrak{gl}(7)$, call it E. Take a basis given by points in the projective plane above.

► **Exercise 17.1.** Consider the following element of $\Lambda^3 E$.

$$\omega = v_1 \wedge v_2 \wedge v_3 + w_1 \wedge w_2 \wedge w_3 + u \wedge v_1 \wedge w_1 + u \wedge v_2 \wedge w_2 + u \wedge v_3 \wedge w_3$$

Prove that $\mathfrak{gl}(7)\omega = \Lambda^3 E$.

Warning 17.1. Don't forget that $\mathfrak{gl}(7)$ acts on $\Lambda^3 E$ as a *Lie algebra*, not as an associative algebra. That is, $x(a \wedge b \wedge c) = (xa) \wedge b \wedge c + a \wedge xb \wedge c + a \wedge b \wedge xc$. In particular, the action of x followed by the action of y is *not* the same as the action of yx.

Claim. $\mathfrak{g} = \{x \in \mathfrak{gl}(7) | x\omega = 0\}$ *is a simple Lie algebra with root system* G_2.

Proof. It is immediate that $\mathfrak{g}$ is a Lie algebra. Let's pick a candidate for the Cartan subalgebra. Consider linear operators which are diagonal in the given basis $u, v_1, v_2, v_3, w_1, w_2, w_3$, take $h = diag(c, a_1, a_2, a_3, b_1, b_2, b_3)$. If we want $h \in \mathfrak{g}$, we must have

$$\begin{aligned} h\omega = (a_1 + a_2 + a_3)v_1v_2v_3 + (b_1 + b_2 + b_3)w_1w_2w_3 + \\ + (c + a_1 + b_1)uv_1w_1 + (c + a_2 + b_2)uv_2w_2 + (c + a_3 + b_3)uv_3w_3 = 0 \end{aligned}$$

which is equivalent to $c = 0$, $a_1 + a_2 + a_3 = 0$, and $b_i = -a_i$. So our Cartan subalgebra is two dimensional.

If you consider the root diagram for G_2 and look at only the long roots, you get a copy of A_2. This should mean that you have an embedding $A_2 \subseteq G_2$,[1] so we should look for a copy of $\mathfrak{sl}(3)$ in our $\mathfrak{g}$. We can write $E = ku \oplus V \oplus W$, where $V = \langle v_1, v_2, v_3\rangle$ and $W = \langle w_1, w_2, w_3\rangle$. Let's consider the subalgebra which not only kills ω, but also kills u. Let $\mathfrak{g}_0 = \{x \in \mathfrak{g} | xu = 0\}$.

Say $x \in \mathfrak{g}_0$ is of the form

$$x = \left(\begin{array}{c|cc} 0 & a & b \\ \hline 0 & A & B \\ 0 & C & D \end{array}\right),$$

where a, b are row vectors, then

$$0 = x \cdot \omega = \underset{\substack{vvv \\ \boxed{uvv}_3 \\ \boxed{wvv}_1}}{x(v_1v_2v_3)} + \underset{\substack{www \\ \boxed{uww}_3 \\ \boxed{vww}_1}}{x(w_1w_2w_3)} + u \wedge \underset{\substack{\boxed{uvw}_4 \\ \cancel{uuw} \\ \cancel{uuv} \\ \boxed{\substack{uvv \\ uww}}_2}}{x(v_1w_1 + v_2w_2 + v_3w_3)}$$

where each term lies in the span of the basis vectors below it. Since the terms in boxes labelled 1 appear in only one way, we must have $B = C = 0$. From that, it follows that the terms boxed an labelled 2 cannot appear. Thus, the terms in boxes labelled 3 only appear in one way, so we must have $a = b = 0$. Since the terms in boxes labelled 2 appear in only one place (though in two ways), we must have $D = -A^t$. Finally, since vvv only appears in one place (in three different ways), we must have $tr\, A = 0$.

For $x \in \mathfrak{g}_0$ we have $x(v_1 \wedge v_2 \wedge v_3) = 0$ and $x(w_1 \wedge w_2 \wedge w_3) = 0$, so x preserves V and W. It also must kill the 2-form $\alpha = v_1 \wedge w_1 + v_2 \wedge w_2 + v_3 \wedge w_3$, since $0 = x(u \wedge \alpha) = xu \wedge \alpha + u \wedge x\alpha = u \wedge x\alpha$ forces $x\alpha = 0$. This 2-form gives a pairing, so that $V^* \simeq W$. We can compute exactly what the pairing is, $\langle v_i, w_j\rangle = \delta_{ij}$. Therefore the operator x must be of the form

$$x = \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & A & 0 \\ 0 & 0 & -A^t \end{array}\right), \text{ where } tr(A) = 0.$$

The total dimension of G_2 is 14, which we also know by the exercise is the dimension of $\mathfrak{g}$. We saw the Cartan subalgebra has dimension 2, and this $\mathfrak{g}_0$ piece has dimension 8 (two of which are the Cartan). So we still need another 6 dimensional piece.

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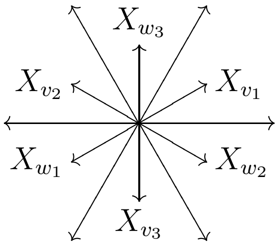
[1] Note that this is not true of the short roots because the bracket of elements in "adjacent" short root spaces produces an element in a long root space, so the short root spaces will generate all of G_2.

For each $v \in V$, define a linear operator X_v which acts by $X_v(u) = v$. $\Lambda^2 V \simeq_\gamma W$ is dual to V (since $\Lambda^3 V = \mathbb{C}$). Then X_v acts by $X_v(v') = \gamma(v \wedge v')$ and $X_v(w) = 2\langle v, w\rangle u$. Check that this kills ω, and hence is in $\mathfrak{g}$.

Similarly, you can define X_w for each $w \in W$ by $X_w(u) = w, X_w(w') = \gamma(w \wedge w'), X_w(v) = 2\langle w, v\rangle u$.

If you think about a linear operator which takes $u \mapsto v_i$, it must be in some root space, this tells you about how it should act on V and W. This is how we constructed X_v and X_w, so we know that X_{v_i} and X_{w_i} are in some root spaces. We can check that their roots are the short roots in the diagram,



and so they span the remaining 6 dimensions of G_2. To properly complete this construction, we should check that this is semisimple, but we're not going to. $\square$

Let's analyze what we did with G_2, so that we can do a similar thing to construct E_8. We discovered certain phenomena, we can write $\mathfrak{g} = \underbrace{\mathfrak{g}_0}_{\mathfrak{sl}(3)} \oplus \underbrace{\mathfrak{g}_1}_{V} \oplus \underbrace{\mathfrak{g}_2}_{W}$. This gives us a $\mathbb{Z}/3$-grading: $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j(mod\ 3)}$. As an $\mathfrak{sl}(3)$ representation, it has three components: *ad*, standard, and the dual to the standard. We get that $W \cong V^* \simeq \Lambda^2 V$. Similarly, $V \simeq \Lambda^2 W$. This is called *Triality*.

More generally, say we have $\mathfrak{g}_0$ a semisimple Lie algebra, and V, W representations of $\mathfrak{g}_0$, with intertwining maps

$$\begin{aligned} \alpha &: \Lambda^2 V \to W \\ \beta &: \Lambda^2 W \to V \\ & V \simeq W^*. \end{aligned}$$

We also have $\gamma : V \otimes W \simeq V \otimes V^* \to \mathfrak{g}_0$ (representations are semisimple, so the map $\mathfrak{g}_0 \to \mathfrak{gl}(V) \simeq V \otimes V^*$ splits). We normalize γ in the following way. Let B be the Killing form, and normalize γ so that $B(\gamma(v \otimes w), X) = \langle w, Xv\rangle$. Make a Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus V \oplus W$ by defining $[X, v] = Xv$ and $[X, w] = Xw$ for $X \in \mathfrak{g}_0, v \in V, w \in W$. We also need to define

[,] on V,W and between V and W. These are actually forced, up to coefficients:

$$[v_1, v_2] = a\alpha(v_1 \wedge v_2)$$
$$[w_1, w_2] = b\beta(w_1 \wedge w_2)$$
$$[v, w] = c\gamma(v \otimes w).$$

There are some conditions on the coefficients a, b, c imposed by the Jacobi identity; $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$. Suppose $x \in \mathfrak{g}_0$, with $y, z \in \mathfrak{g}_i$ for $i = 0, 1, 2$, then there is nothing to check, these identities come for free because α, β, γ are $\mathfrak{g}_0$-invariant maps. There are only a few more cases to check, and only one of them gives you a condition. Look at

$$[v_0, [v_1, v_2]] = ca\gamma(v_0 \otimes \alpha(v_1 \wedge v_2)) \tag{RHS}$$

and it must be equal to

$$\begin{aligned}[[v_0, v_1], v_2] + [v_1, [v_0, v_2]] = \\ - ac\gamma(v_2 \otimes \alpha(v_0 \wedge v_1)) + ac\gamma(v_1 \otimes \alpha(v_0 \wedge v_1))\end{aligned} \tag{LHS}$$

This doesn't give a condition on ac, but we need to check that it is satisfied. It suffices to check that $B(RHS, X) = B(LHS, X)$ for any $X \in \mathfrak{g}_0$. This gives us the following condition:

$$\langle \alpha(v_1 \wedge v_2), Xv_0 \rangle = \langle \alpha(v_0 \wedge v_2), Xv_1 \rangle - \langle \alpha(v_0 \wedge v_1), Xv_2 \rangle$$

The fact that α is an intertwining map for $\mathfrak{g}_0$ gives us the identity:

$$\langle \alpha(v_1 \wedge v_2), Xv_0 \rangle = \langle \alpha(Xv_1 \wedge v_2), v_0 \rangle - \langle \alpha(v_1 \wedge Xv_2), v_0 \rangle$$

and we also have that

$$\langle \alpha(v_1 \wedge v_2), v_0 \rangle = \langle \alpha(v_0 \wedge v_2), v_1 \rangle = \langle \alpha(v_0 \wedge v_1), v_2 \rangle$$

With these two identities it is easy to show that the equation (and hence this Jacobi identity) is satisfied.

We also get the Jacobi identity on $[w, [v_1, v_2]]$, which is equivalent to:

$$ab\beta(w \wedge \alpha(v_1 \wedge v_2)) = c(\gamma(v_1 \otimes w)v_2 - \gamma(v_2 \otimes w)v_1)$$

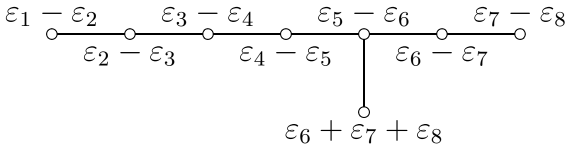
It suffices to show for any $w' \in W$ that the pairings of each side with w' are equal,

$$\begin{aligned} ab\langle w', \beta(w \wedge \alpha(v_1 \wedge v_2)) \rangle = cB(\gamma(v_1 \otimes w), \gamma(v_2 \otimes w')) \\ - cB(\gamma(v_2 \otimes w), \gamma(v_1 \otimes w')) \end{aligned}$$

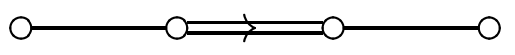
This time we will get a condition on a, b, and c. You can check that any of the other cases of the Jacobi identity give you the same conditions.

Now we will use this to construct E_8. Write $\mathfrak{g} = \mathfrak{g}_0 \oplus V \oplus W$, where we take $\mathfrak{g}_0 = \mathfrak{sl}(9)$. Let E be the 9 dimensional representation of $\mathfrak{g}_0$. Then take $V = \Lambda^3 E$ and $W = \Lambda^3 E^* \simeq \Lambda^6 E$. We have a pairing $\Lambda^3 E \otimes \Lambda^6 E \to k$, so we have $V \simeq W^*$. We would like to construct $\alpha : \Lambda^2 \to W$, but this is just given by $v_1 \wedge v_2$ including into $\Lambda^6 E \simeq W$. Similarly, we get $\beta : \Lambda^2 W \to V$. You get that the rank of $\mathfrak{g}$ is 8 (= rank of $\mathfrak{g}_0$). Notice that $\dim V = \binom{9}{3} = 84$, which is the same as $\dim W$, and $\dim \mathfrak{g}_0 = \dim \mathfrak{sl}(9) = 80$. Thus, we have that $\dim \mathfrak{g} = 84+84+80 = 248$, which is the dimension of E_8, and this is indeed E_8.

Remember that we previously got E_7 and E_6 from E_8. Look at the diagram for E_8:



The extra guy, $\varepsilon_6 + \varepsilon_7 + \varepsilon_8$, corresponds to the 3-form. When you cut out $\varepsilon_1 - \varepsilon_2$, you can figure out what is left and you get E_7. Then you can additionally cut out $\varepsilon_2 - \varepsilon_3$ and get E_6.

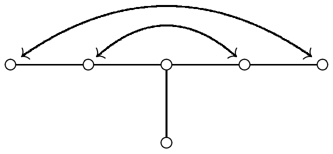
Fianally, we construct F_4: 

We know that any simple algebra can be determined by generators and relations, with a X_i, Y_i, H_i for each node i. But sometimes our diagram has a symmetry, like switching the horns on a D_n, which induces an automorphism of the Lie algebra given by $\gamma(X_i) = X_i$ for $i < n-1$ and switches X_{n-1} and X_n. Because the arrows are preserved, you can check that the Serre relations still hold. Thus, in general, an automorphism of the diagram induces an automorphism of the Lie algebra (in a very concrete way).

Theorem 17.2. $(\operatorname{Aut} \mathfrak{g})/(\operatorname{Aut}_0 \mathfrak{g}) = \operatorname{Aut} \Gamma$. *So the connected component of the identity gives some automorphisms, and the connected components are parameterized by automorphisms of the diagram.*

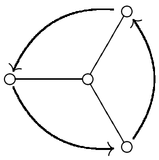
D_n is the diagram for $SO(2n)$. We have that $SO(2n) \subset O(2n)$, and the group of automorphisms of $SO(2n)$ is $O(2n)$. This isn't true of $SO(2n+1)$, because the automorphisms given by $O(2n+1)$ are the same as those from $SO(2n+1)$. This corresponds the the fact that D_n has a nontrivial automorphism, but B_n doesn't.

Notice that E_6 has a symmetry; the involution:

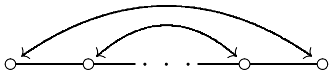


Define $X_1' = X_1 + X_5, X_2' = X_2 + X_4, X_3' = X_3, X_6' = X_6$, and the same with Y's, the fixed elements of this automorphism. We have that $H_1' = H_1 + H_5$ (you have to check that this works), and similarly for the other H's. As the set of fixed elements, you get an algebra of rank 4 (which must be our F_4). You can check that $\alpha_1'(H_2') = -1, \alpha_2'(H_1') = -1, \alpha_3'(H_1') = 0, \alpha_3'(H_2') = -2, \alpha_2'(H_3') = -1$, so this is indeed F_4 as desired. In fact, any diagram with multiple edges can be obtained as the fixed algebra of some automorphism:

▶ **Exercise 17.2.** Check that G_2 is the fixed algebra of the automorphism of D_4:



Check that B_n, C_n can be obtained from A_{2n}, A_{2n+1}



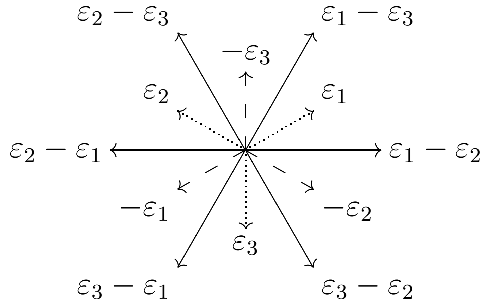
Lecture 18 - Representations of Lie algebras

Let $\mathfrak{g}$ be a semisimple Lie algebra over an algebraically closed field k of characteristic 0. Then we have the root decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_\alpha \mathfrak{g}_\alpha$. Let V be a finite dimensional representation of $\mathfrak{g}$. Because all elements of $\mathfrak{h}$ are semisimple, and because Jordan decomposition is preserved, the elements of $\mathfrak{h}$ can be simultaneously diagonalized. That is, we have a *weight decomposition* $V = \bigoplus_{\mu\in\mathfrak{h}^*} V_\mu$, where $V_\mu = \{v \in V | hv = \mu(h)v \text{ for all } h \in \mathfrak{h}\}$. We call V_μ a *weight space*, and μ a *weight*. Define $P(V) = \{\mu \in \mathfrak{h}^* | V_\mu \neq 0\}$. The multiplicity of a weight $\mu \in P(V)$ is $\dim V_\mu$, and is denoted m_μ.

Example 18.1. You can take $V = k$ (the trivial representation). Then $P(V) = \{0\}$ and $m_0 = 1$.

Example 18.2. If $V = \mathfrak{g}$ and we take the adjoint representation, then we have that $P(V) = \Delta \cup \{0\}$, with $m_\alpha = 1$ for $\alpha \in \Delta$, and m_0 is equal to the rank of $\mathfrak{g}$.

Example 18.3. Let $\mathfrak{g} = \mathfrak{sl}(3)$. The weights of the adjoint representation are shown by the solid arrows (together with zero, which has multiplicity two).



The weights of the standard 3-dimensional representation are $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$, shown in dotted lines.

In general, the weights of the dual of a representation are the negatives of the original representation because $\langle h\phi, v\rangle$ is defined as $-\langle\phi, hv\rangle$. Thus, the dashed lines show the weights of the dual of the standard representation.

If V is a finite dimensional representation, then its weight decomposition has the following are properties.

1. For any root α and $\mu \in P(V)$, $\mu(H_\alpha) \in \mathbb{Z}$.

   To see this, consider V as a representation of the $\mathfrak{sl}(2)$ spanned by X_α, H_α, and Y_α. Our characterization of finite dimensional representations of $\mathfrak{sl}(2)$ implies the result.

2. For $\alpha \in \Delta$ and $\mu \in P(V)$, $\mathfrak{g}_\alpha V_\mu \subseteq V_{\mu+\alpha}$

   This follows from the standard calculation:

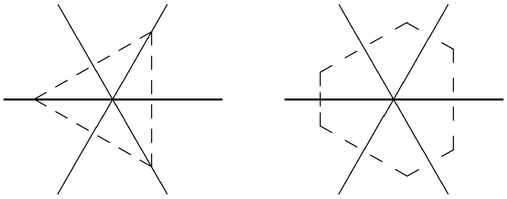
$$\begin{aligned} h(x_\alpha v) &= x_\alpha h v + [h, x_\alpha] v \\ &= x_\alpha \mu(h) v + \alpha(h) x_\alpha v \\ &= (\mu + \alpha)(h) x_\alpha v. \end{aligned}$$

3. If $\mu \in P(V)$ and $w \in \mathfrak{W}$, then $w(\mu) \in P(V)$ and $m_\mu = m_{w(\mu)}$.

   It is sufficient to check this when w is a simple reflection r_i. Consider V as a representation of the copy of $\mathfrak{sl}(2)$ spanned by X_i, Y_i, and H_i. If $v \in V_\mu$, then we have that $h \cdot v = \mu(h)v$ for all $h \in \mathfrak{h}$. By property 1, we know that $\mu(H_i) = l$ is a non-negative integer. From the characterization of finite dimensional representations of $\mathfrak{sl}(2)$, we know that there is a corresponding vector with H_i-eigenvalue $-l$, namely $u = Y_i^l v$. By property 2, $u \in V_{\mu - l\alpha_i}$. But $\mu - l\alpha_i = \mu - \mu(H_i)\alpha_i = \mu - \frac{2(\mu,\alpha_i)}{(\alpha_i,\alpha_i)}\alpha_i = r_i\mu$. Putting it all together, if we consider the $\mathfrak{sl}(2)$ subrepresentation of V generated by V_μ and $V_{r_i\mu}$, the symmetry of finite dimensional representations of $\mathfrak{sl}(2)$ tells us that $\dim V_{r_i\mu} = \dim V_\mu$, as desired.

Remark 18.4. Note that the proof of property 2 did not require that V be finite dimensional. Properties 1 and 3 used finite dimensionality, but in a weak way. Consider the $\mathfrak{sl}(2)$ spanned by X_i, Y_i, and H_i. It is enough for each vector v in a weight space of V to be contained in a finite dimensional $\mathfrak{sl}(2)$ subrepresentation. In particular, if each X_i and Y_i act locally nilpotently,[1] then all three properties hold.

Example 18.5. If $\mathfrak{g} = \mathfrak{sl}(3)$, then we get $\mathfrak{W} = D_{2\cdot 3} = S_3$. The orbit of a point can have a couple of different forms. If the point is on a hyperplane orthogonal to a root, then you get a triangle. For a generic point, you get a hexagon (which is not regular, but still symmetric).



It is pretty clear that knowing the weights and multiplicities gives us a lot of information about the representation, so we'd better find a good way to exploit this information.

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[1] We say that a linear operator A is *locally nilpotent* if for each vector v there is an integer $n(v)$ such that $A^{n(v)}v = 0$.

Let V be a representation of $\mathfrak{g}$. Then V is also a representation of the associated simply connected group G, and we get the commutative square

$$\begin{tikzcd} G \arrow[r] & GL(V) \\ \mathfrak{g} \arrow[r] \arrow[u, "\exp"] & \mathfrak{gl}(V) \arrow[u, "\exp"'] \end{tikzcd}$$

If $h \in \mathfrak{h}$, then $\exp h \in G$, and we can evaluate the group character of the representation V on $\exp h$ as

$$\chi_V(\exp h) = tr(\exp h) = \sum_{\mu \in P(V)} m_\mu e^{\mu(h)}$$

where the second equality is because every eigenvalue $\mu(h)$ of h yields an eigenvalue $e^{\mu(h)}$ of $\exp h$. Since characters tell us a lot about finite dimensional representations, it makes sense to consider the following definition.

Definition 18.6. The *character* of the representation V is the formal sum

$$ch\, V = \sum_{\mu \in P(V)} m_\mu e^\mu.$$

You can and multiply these (formal) expressions; *ch* is additive with respect to direct sum and multiplicative with respect to tensor products:

$$\begin{aligned} ch(V \oplus W) &= ch\, V + ch\, W \\ ch(V \otimes W) &= (ch\, V)(ch\, W) \end{aligned}$$

This is because $V_\mu \otimes W_\nu \subseteq (V \otimes W)_{\mu+\nu}$ (or you can use the relationship with group characters). You can also check that the $ch\, V^*$ is $\sum m_\mu e^{-\mu}$.

Remark 18.7. We only evaluated χ_V on the image of the Cartan subalgebra. Is it possible that we've lost some information about the behavior of χ_V on the rest of G? The answer is no. Since χ_V is constant on conjugacy classes, and any Cartan subalgebra is conjugate to any other Cartan subalgebra (Theorem 14.1), we know how χ_V behaves on the union of all Cartan subalgebras. Since the union of all Cartan subalgebras is dense in $\mathfrak{g}$, $\exp \mathfrak{g}$ is dense in G, and χ_V is continuous, the behavior of χ_V on the image of a single Cartan determines it completely.

► **Exercise 18.1.** Show that the union of all Cartan subalgebras is dense in $\mathfrak{g}$.

Highest weights

Fix a set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$. A *highest weight* of a representation V is a $\lambda \in P(V)$ such that $\alpha_i + \lambda \notin P(V)$ for all $\alpha_i \in \Pi$. A *highest weight vector* is a vector in V_λ.

Let V be irreducible, let λ be a highest weight, and let $v \in V_\lambda$ be a highest weight vector. Since V is irreducible, v generates: $V = (U\mathfrak{g})v$. We know that $\mathfrak{n}^+ v = 0$ and that $\mathfrak{h}$ acts on v by scalars. By PBW, $U\mathfrak{g} = U\mathfrak{n}^- \otimes U\mathfrak{h} \otimes U\mathfrak{n}^+$, so $V = U\mathfrak{n}^- v$. Thus, V is generated from v by applying various Y_α, where $\alpha \in \Delta^+$. In particular, the multiplicity m_λ is one. This also tells us that any other weight μ is "less than" λ in the sense that $\lambda - \mu = \sum_{\alpha \in \Delta^+} l_\alpha \alpha$, where the l_α are non-negative.

It follows that in an irreducible representation, the highest weight is unique. If μ is another highest weight, then we get $\lambda \le \mu$ and $\mu \le \lambda$, which implies $\mu = \lambda$.

Remark 18.8. If V is an irreducible *finite dimensional* representation with highest weight λ, then for any $w \in \mathfrak{W}$, property 3 tells us that $w(\lambda)$ is a highest weight with respect to the set of simple roots $\{w\alpha_1, \ldots, w\alpha_n\}$. So $P(V)$ is contained in the convex hull of the set $\{w\lambda\}_{w \in \mathfrak{W}}$.

We also know that λ is a highest weight for each $\mathfrak{sl}(2)$ spanned by X_α, Y_α, and H_α, with $\alpha \in \Delta^+$ (from the definition of highest weight). So $\lambda(H_i) = (\lambda, \check{\alpha}_i) \in \mathbb{Z}_{\ge 0}$ for each i.

Definition 18.9. The lattice generated by the roots, $Q = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_n$, is called the *root lattice*.

Definition 18.10. The lattice $P = \{\mu \in \mathfrak{h}^* | (\mu, \check{\alpha}_i) \in \mathbb{Z} \text{ for } 1 \le i \le n\}$ is called the *weight lattice*.

Definition 18.11. The set $\{\mu \in \mathfrak{h}^* | (\mu, \check{\alpha}_i) \ge 0 \text{ for } 1 \le i \le n\}$ is called the *Weyl chamber*, and the intersection of the Weyl chamber with the weight lattice is called the set of *dominant integral weights*, and is denoted P^+.

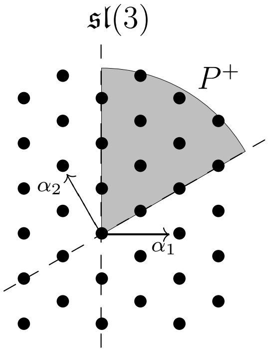
P and Q have the same rank. It is clear that Q is contained in P, and in general this containment is strict.

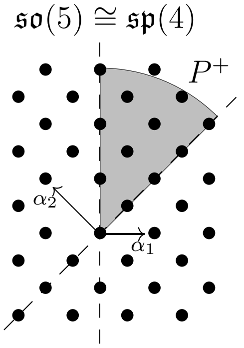
P/Q is isomorphic to the center of the simply connected group corresponding to $\mathfrak{g}$.

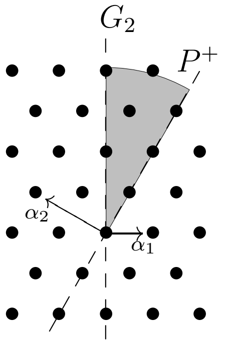
Example 18.12. For $\mathfrak{g} = \mathfrak{sl}(2)$, the root lattice is $2\mathbb{Z}$ (because $[H, X] = 2X$), and the weight lattice is $\mathbb{Z}$.

Example 18.13. In the three rank two cases, the weight lattices and

Weyl chambers are







▶ **Exercise 18.2.** Show that P^+ is the fundamental domain of the action of $\mathfrak{W}$ on P. That is, show that for every $\mu \in P$, the $\mathfrak{W}$-orbit of μ intersects P^+ in exactly one point. (Hint: use Proposition 14.17)

We have already shown that the highest weight of an irreducible finite dimensional representation is an element of P^+ (this is exactly the second part of Remark 18.8). The rest of the lecture will be devoted to proving the following remarkable theorem.

Theorem 18.14. *There is a bijection between P^+ and the set of (isomorphism classes of) finite dimensional irreducible representations of $\mathfrak{g}$, in which an irreducible representation corresponds to its highest weight.*

It remains to show that two non-isomorphic finite dimensional irreducible representations cannot have the same highest weight, and that any element of P^+ appears as the highest weight of some finite dimensional representation. To prove these things, we will use Verma modules.

Let V be an irreducible representation with highest weight λ. Then V_λ is a 1-dimensional representation of the subalgebra $\mathfrak{b}^+ := \mathfrak{h} \oplus \mathfrak{n}^+ \subseteq \mathfrak{g}$. There is an induced representation $U\mathfrak{g} \otimes_{U\mathfrak{b}^+} V_\lambda$ of $\mathfrak{g}$, and an induced homomorphism $U\mathfrak{g} \otimes_{U\mathfrak{b}^+} V_\lambda \to V$ given by $x \otimes v \mapsto x \cdot v$.

Definition 18.15. A *Verma module* is $M(\lambda) = U\mathfrak{g} \otimes_{U\mathfrak{b}^+} V_\lambda$.

The Verma module is universal in the sense that for any representation V with highest weight vector v of weight λ, there is a unique homomorphism of representations $M(\lambda) \to V$ sending the highest vector of $M(\lambda)$ to v. However, there is a problem: $M(\lambda)$ is infinite dimensional.

To understand $M(\lambda)$ as a vector space, we use PBW to get that $U\mathfrak{g} = U\mathfrak{n}^- \otimes_k U\mathfrak{h} \otimes_k U\mathfrak{n}^+ = U(\mathfrak{n}^-) \otimes_k U\mathfrak{b}^+$. Since $U\mathfrak{b}^+$ acts on V_λ by scalars, we get

$$M(\lambda) = U\mathfrak{n}^- \otimes_k U\mathfrak{b}^+ \otimes_{U\mathfrak{b}^+} V_\lambda = U\mathfrak{n}^- \otimes_k V_\lambda.$$

If $\Delta^+ = \{\alpha_1, \dots, \alpha_N\}$, with $\Pi = \{\alpha_1, \dots, \alpha_n\}$, then by PBW, $\{Y_{\alpha_1}^{k_1} \cdots Y_{\alpha_N}^{k_N}\}$ is a basis for $U\mathfrak{n}^-$, so $\{Y_{\alpha_1}^{k_1} \cdots Y_{\alpha_N}^{k_N} v\}$ is a basis for $M(\lambda) = U(\mathfrak{n}^-) \otimes_k V_\lambda$. Thus, even though the Verma module is infinite dimensional, it still has a weight decomposition with finite dimensional weight spaces:

$$h(Y_{\alpha_1}^{k_1} \cdots Y_{\alpha_N}^{k_N} v) = (\lambda - k_1\alpha_1 - \cdots - k_N\alpha_N)(h)(Y_{\alpha_1}^{k_1} \cdots Y_{\alpha_N}^{k_N} v).$$

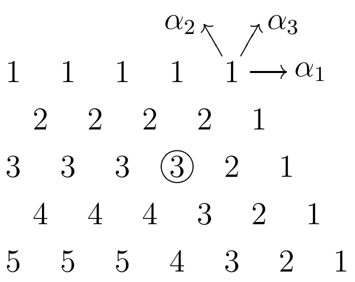
In particular, we get a nice formula for the multiplicity of a weight. The multiplicity of μ is given by the number of different ways $\lambda - \mu$ can be written as a non-negative sum of positive roots, corresponding to the number of basis vectors $Y_{\alpha_1}^{k_1} \cdots Y_{\alpha_N}^{k_N} v$ lying in V_μ.

$$m_\mu = \#\Big\{\lambda - \mu = \sum_{\alpha_i \in \Delta^+} k_i\alpha_i \;\Big|\; k_i \in \mathbb{Z}_{\geq 0}\Big\}.$$

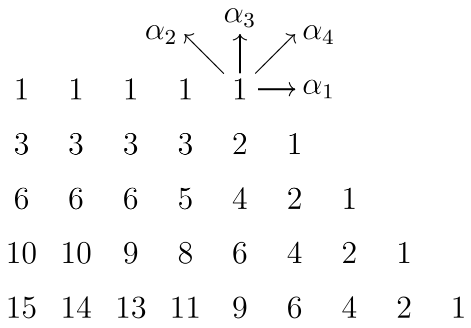
This is called the Kostant partition function.

Example 18.16. We are now in a position to calculate the characters of Verma modules. In the rank two cases, we get the characters below. For example, since $2\alpha_3 = \alpha_3 + \alpha_2 + \alpha_1 = 2\alpha_1 + 2\alpha_2$ can be written in these three ways as a sum of positive roots, the circled multiplicity (in the $\mathfrak{sl}(3)$ case) is 3.

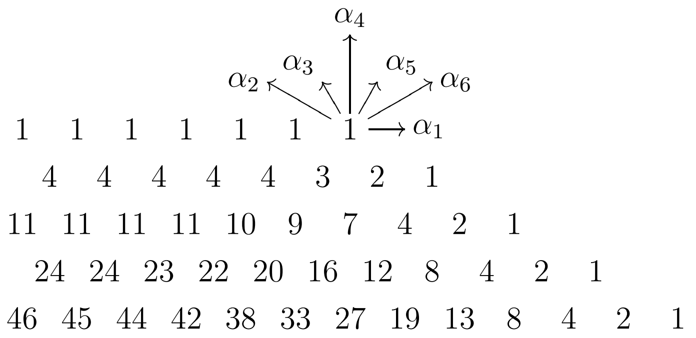
$\mathfrak{sl}(3)$



$\mathfrak{so}(5) \cong \mathfrak{sp}(4)$



G_2



▶ **Exercise 18.3.** Check that the characters in Example 18.16 are correct. (Hint: For $\mathfrak{so}(5)$, at each lattice point, keep track of four numbers: the number of ways to write $\lambda - \mu$ as a non-negative sum in the sets $\{\alpha_1, \dots, \alpha_4\}$, $\{\alpha_2, \alpha_3, \alpha_4\}$, $\{\alpha_2, \alpha_3\}$, and $\{\alpha_2\}$)

Lemma 18.17. *A Verma module $M(\lambda)$ has a unique proper maximal submodule $N(\lambda)$.*

Proof. N being proper is equivalent to $N \cap V_\lambda = 0$. This property is clearly preserved under taking sums, so you get a unique maximal submodule. □

Remark 18.18. If V and W are irreducible representations with the same highest weight, then they are both isomorphic to the unique irreducible quotient $M(\lambda)/N(\lambda)$, so they are isomorphic.

Lemma 18.19. *If $\lambda \in P^+$, then the quotient $V(\lambda) = M(\lambda)/N(\lambda)$ is finite dimensional.*

Proof. If w is a weight vector (but not the highest weight vector) in $M(\lambda)$ such that $X_i w = 0$ for $i = 1, \dots, n$, then we claim that $w \in N(\lambda)$. To see this, you note that

$$(U\mathfrak{g})w = (U\mathfrak{n}^- \otimes U\mathfrak{h} \otimes U\mathfrak{n}^+)w = (U\mathfrak{n}^-)w$$

so the submodule generated by w contains only lower weight spaces. In particular, the highest weight space V_λ cannot be obtained from w.

Fix an $i \leq n$. By assumption, $\lambda(H_i) = \langle \lambda, \breve{\alpha}_i \rangle = l_i \in \mathbb{Z}_{\geq 0}$. Letting $w = Y_i^{l_i+1} v$, we get that

$$X_i w = (l_i + 1)\big(l_i - (l_i + 1) + 1\big)Y_i^{l_i+1} w = 0 \qquad \text{(by Equation 13.12)}$$
$$X_j w = Y_i^{l_i+1} X_j w = 0 \qquad \text{(since } [X_j, Y_i] = 0\text{)}$$

so $w \in N(\lambda)$. It follows from the Serre relations that in the quotient $V(\lambda)$, the Y_i act locally nilpotently. The X_i act locally nilpotently on $M(\lambda)$, so they act locally nilpotently on $V(\lambda)$. By Remark 18.4, $P\big(V(\lambda)\big)$ is invariant under $\mathfrak{W}$, so it is contained in the convex hull of the orbit of λ. Since each weight space is finite dimensional, it follows that $V(\lambda)$ is finite dimensional. □

Putting it all together, we can prove the Theorem.

Proof of Theorem 18.14. By Remark 18.8, the highest weight of an irreducible finite dimensional representation is an element of P^+. By Remark 18.18, non-isomorphic representations have distinct highest weights. Finally, by Lemmas 18.17 and 18.19, every element of P^+ appears as the highest weight of some finite dimensional irreducible representation. □

Corollary 18.20. *If V and W are finite dimensional representations, and if $ch\, V = ch\, W$, then $V \simeq W$.*

Proof. Since their characters are equal, V and W have a common highest weight λ, so they both contain a copy of $V(\lambda)$. By complete reducibility (Theorem 12.14), $V(\lambda)$ is a direct summand in both V and W. It is enough to show that the direct complements are isomorphic, but this follows from the fact that they have equal characters and fewer irreducible components. □

So it is desirable to be able to compute the character of $V(\lambda)$. This is what we will do next lecture.

Lecture 19 - The Weyl character formula

If $\lambda \in P^+$ (i.e. $(\lambda, \check{\alpha}_i) \in \mathbb{Z}_{\geq 0}$ for all i), then we can construct an irreducible representation with highest weight λ, which we called $V(\lambda)$. We define the *fundamental weights* $\omega_1, \dots, \omega_n$ of a Lie algebra to be those weights for which $(\omega_i, \check{\alpha}_j) = \delta_{ij}$. It is clear that any dominant integral weight can be written as $\lambda = \lambda_1\omega_1 + \cdots + \lambda_n\omega_n$ for $\lambda_i \geq 0$, so people often talk about $V(\lambda)$ by drawing the Dynkin diagram with the the i-th vertex labelled by λ_i.

With this notation, the first fundamental representation $V(\omega_1)$ for $\mathfrak{sl}(n)$ is written $\overset{1}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{0}{\circ}\!\!-\cdots-\!\!\overset{0}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{0}{\circ}$, which happens to be the standard representation (see Example 19.2 below). Similarly, the adjoint representation is $\overset{1}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{0}{\circ}\!\!-\cdots-\!\!\overset{0}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{1}{\circ}$.

Warning 19.1. Another common notation (incompatible with this one) is to write $\lambda = \sum k_i\alpha_i$ and label the i-th vertex by k_i. In this notation, the standard representation is $1 \text{ --- } 0 \text{ -}\cdots\text{- } 0 \text{ --- } 0$ and the adjoint representation is $1 \text{ --- } 1 \text{ -}\cdots\text{- } 1 \text{ --- } 1$. In these notes, we will draw the diagram differently to distinguish between the two notations.

Observe that if $v \in V$ a highest vector of weight λ, and $w \in W$ another highest weight vector of weight μ in another representation, then $v \otimes w \in V \otimes W$ is a highest weight vector of weight $\lambda + \mu$. It follows that every finite dimensional irreducible representation can be realized as a subrepresentation of a tensor product of fundamental representations.

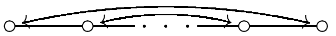
Example 19.2. Let's calculate the fundamental weights for $\mathfrak{sl}(n+1)$. Recall that we have simple roots $\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_n - \varepsilon_{n+1}$, and they are equal to their coroots (since they have length $\sqrt{2}$). It follows that $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$ for $i = 1, \dots, n$.

Let E be the standard $(n+1)$-dimensional representation of $\mathfrak{sl}(n+1)$. Let $e_1, \dots, e_{n+1}$ be a basis for E. Note that e_i has weight ε_i, and $\varepsilon_i - \varepsilon_j$ can be written as a non-negative sum of positive roots exactly when $i \leq j$. Thus, the weights of E, in decreasing order, are ε_1, ε_2, $\dots$, ε_{n+1}.

Consider the representation $\Lambda^k E$. We'd like to write down its weights. Note that $\Lambda^k E$ is spanned by the vectors $e_{i_1} \wedge \cdots \wedge e_{i_k}$, which have weights $\varepsilon_{i_1} + \cdots + \varepsilon_{i_k}$. Thus, the highest weight is $\varepsilon_1 + \cdots + \varepsilon_k = \omega_k$, so we know that $V(\omega_k) \subseteq \Lambda^k E$.

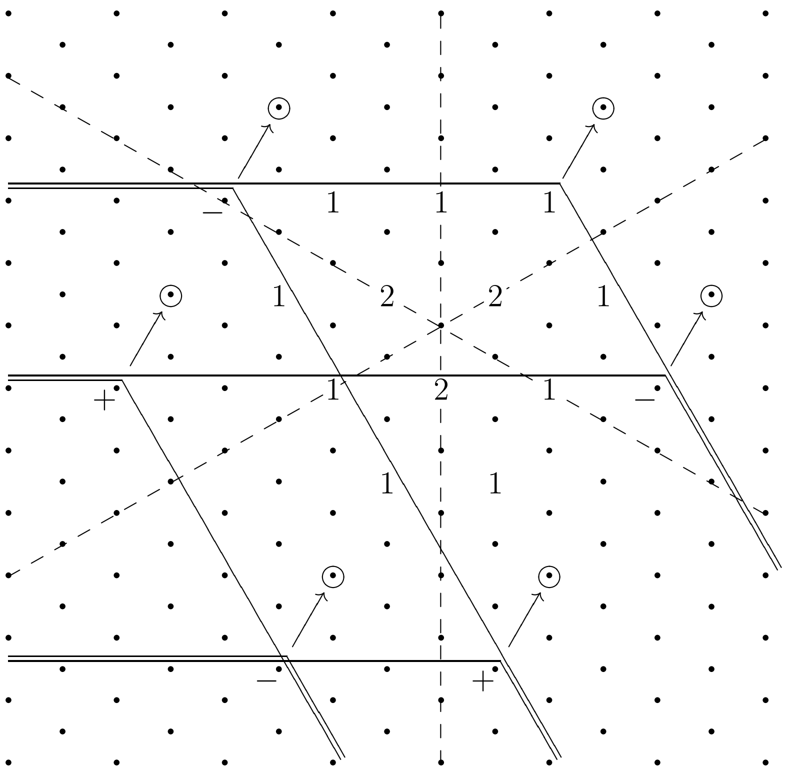
Note also that $\mathfrak{W} \cong S_{n+1}$ acts by permutation of the e_i, so it can take any weight space to any other weight space. Such a representation (where all the weight spaces form a single orbit of the Weyl group) is called *minuscule*. Since the character of any subrepresentation must be $\mathfrak{W}$-invariant, minuscule representations are always irreducible. So $\Lambda^k E = V(\omega_k)$ is a fundamental representation.

Remark 19.3 (Highest weights of duals). One of the weights of $V(\lambda)^*$ is $-\lambda$, but to compute the highest weight, we need to get back into P^+, so we apply the longest word w in the Weyl group. Thus, $-w(\lambda)$ is the highest weight of $V(\lambda)^*$. This means that there is a fixed involution of the Weyl chamber (namely, $-w$) which takes the highest weight of a representation to the highest weight of its dual. It is clear that $-w$ preserves the set of simple roots and preserves inner products, so it corresponds to an involution of the Dynkin diagram.

In the case of $\mathfrak{sl}(n+1)$, the involution is . In particular, the dual of the standard representation $V(\omega_1)$ is $V(\omega_n)$.

The key to computing the character of $V(\lambda)$ is to write it as a linear combination of characters of Verma modules, as in the following example.

Example 19.4. Let $\mathfrak{g} = \mathfrak{sl}(3)$ and let $\lambda = 2\omega_1 + \omega_2$. We try to write $ch\,V(\lambda)$ as a linear combination of characters of Verma modules in the naïve way. We know that $M(\lambda)$ must appear once and that $ch\,V(\lambda)$ must end up symmetric with respect to the Weyl group. We must subtract off two Verma modules to keep the symmetry. Then we find that we must add back two more and subtract one in order to get zeros outside of the hexagon. In the picture below, each dot can be read as a zero.



For now, just observe that if we shift the weights that appear (by something we will call the Weyl vector), we get an orbit of the Weyl group, with signs alternating according to the length of the element of the Weyl group.

Some notation: if $w \in \mathfrak{W}$, we define $(-1)^w := \det(w)$. Since each simple reflection has determinant -1, this is the same as $(-1)^{\mathrm{length}(w)}$. Note that $(-1)^{w'w} = (-1)^{w'}(-1)^w$.

The *Weyl vector* is $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$. Note that $r_i(\rho) = \rho - \alpha_i$ by Lemma 14.13. On the other hand, $r_i(\rho) = \rho - (\rho, \check{\alpha}_i)\alpha_i$, so we know that $(\rho, \check{\alpha}_i) = 1$ for all i. Thus, ρ is the sum of all the fundamental weights.

Theorem 19.5 (Weyl Character Formula)**.** *For $\lambda \in P^+$, the character of the irreducible finite dimensional representation with highest weight λ is*[1]

$$ch\, V(\lambda) = \frac{\sum_{w\in\mathfrak{W}}(-1)^w e^{w(\lambda+\rho)}}{\sum_{w\in\mathfrak{W}}(-1)^w e^{w(\rho)}}.$$

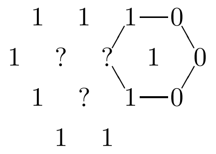
The denominator is called the *Weyl denominator*. It is not yet obvious that the Weyl denominator divides the numerator (as formal sums), so one may prefer to rewrite the equation as $ch\, V(\lambda)\cdot\sum_{w\in\mathfrak{W}}(-1)^w e^{w(\rho)} = \sum_{w\in\mathfrak{W}}(-1)^w e^{w(\lambda+\rho)}$.

Proof. Step 1. Compute $ch\, M(\gamma)$: Recall from the previous lecture that the multiplicity of μ in $M(\gamma)$ is the number of ways $\gamma - \mu$ can be written as a sum of positive roots. Thus, it is easy to see that $ch\, M(\gamma)$ is given by the following generating function.

$$\begin{aligned} ch\, M(\gamma) &= e^\gamma \prod_{\alpha\in\Delta^+}(1 + e^{-\alpha} + e^{-2\alpha} + \cdots) \\ &= \frac{e^\gamma}{\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})} \\ &= \frac{e^{\gamma+\rho}}{\prod_{\alpha\in\Delta^+}(e^{\alpha/2}-e^{-\alpha/2})} \qquad \left(\textstyle\prod_{\alpha\in\Delta^+} e^{\alpha/2} = e^\rho\right) \end{aligned}$$

---

[1] This formula may look ugly, but it is *sweet*. It says that you can compute the character of $V(\lambda)$ in the following way. Translate the Weyl vector ρ around by the Weyl group; this will form some polytope. Make a piece of cardboard shaped like this polytope (ok, so maybe this is only practical for rank 2), and put $(-1)^w$ at the vertex $w(\rho)$. This is your *cardboard denominator*. Now the formula tells you that when you center your cardboard denominator around any weight, and then add the multiplicities of the weights of $V(\lambda)$ at the vertices with the appropriate sign, you'll get zero (unless you centered your cardboard at $w(\lambda+\rho)$, in which case only one non-zero multiplicity shows up in the sum, so you'll get ± 1). Since we know that the highest weight has multiplicity 1, we can use this to compute the rest of the character. For $\mathfrak{sl}(3)$, your cardboard denominator will be a hexagon, and one step of computing the character of $V_{2\omega_1+\omega_2}$ might look like: $0 = 0 - 1 + ? - 1 + 0 - 0$, so $? = 2$. Since $ch\, V$ is symmetric with respect to $\mathfrak{W}$, all three of the ?s must be 2.



Step 2. The action of the Casimir operator: Recall the Casimir operator from the proof of Whitehead's Theorem (Theorem 12.10). If $\{e_i\}$ is a basis for $\mathfrak{g}$, and $\{f_i\}$ is the dual basis (with respect to the Killing form), then $\Omega = \sum e_i f_i \in U\mathfrak{g}$. We showed that Ω is in the center of $U\mathfrak{g}$ (i.e. that $\Omega x = x\Omega$ for all $x \in \mathfrak{g}$).

Claim. *Ω acts on $M(\gamma)$ as $(\gamma, \gamma + 2\rho)\mathrm{Id}$.*

Proof of Claim. Since Ω is in the center of $U\mathfrak{g}$, it is enough to show that $\Omega v = (\gamma + 2\rho, \gamma)v$ for a highest weight vector $v \in V_\gamma$.

Let $\{u_i\}$ be an orthonormal basis for $\mathfrak{h}$, and let $\{X_\alpha\}_{\alpha\in\Delta}$ be a basis for the rest of $\mathfrak{g}$. The dual basis is $\left\{\frac{Y_\alpha}{(X_\alpha, Y_\alpha)}\right\}$. Then we get

$$\begin{aligned}\Omega &= \sum_{i=1}^n u_i^2 + \sum_{\alpha\in\Delta} \frac{X_\alpha Y\alpha}{(X_\alpha, Y_\alpha)} \\ &= \sum_{i=1}^n u_i^2 + \sum_{\alpha\in\Delta^+} \frac{X_\alpha Y\alpha}{(X_\alpha, Y_\alpha)} + \frac{Y_\alpha X_\alpha}{(X_\alpha, Y_\alpha)} \qquad (X_{-\alpha}Y_{-\alpha} = Y_\alpha X_\alpha).\end{aligned}$$

Using the equalities

$$\begin{aligned} u_i v &= \gamma(u_i)v, & (\gamma,\gamma) &= \textstyle\sum_{i=1}^n \gamma(u_i)^2, \text{ and} \\ X_\alpha v &= 0, & (X_\alpha, Y_\alpha) &= \tfrac{1}{2}([H_\alpha, X_\alpha], Y_\alpha) \\ X_\alpha Y_\alpha v &= H_\alpha v - Y_\alpha X_\alpha v & &= \tfrac{1}{2}(H_\alpha, [X_\alpha, Y_\alpha]) \\ &= \gamma(H_\alpha)v, & &= \tfrac{1}{2}(H_\alpha, H_\alpha) = \tfrac{1}{2}\cdot\tfrac{2\alpha(H_\alpha)}{(\alpha,\alpha)} \\ \gamma(H_\alpha) &= \tfrac{2(\gamma,\alpha)}{(\alpha,\alpha)}, & &= \tfrac{2}{(\alpha,\alpha)} \end{aligned}$$

we get

$$\begin{aligned}\Omega v &= \Big(\sum_{i=1}^n \gamma(u_i)^2\Big)v + \sum_{\alpha\in\Delta^+} \frac{\gamma(H_\alpha)}{(X_\alpha, Y_\alpha)}v \\ &= (\gamma,\gamma)v + \sum_{\alpha\in\Delta^+}(\gamma,\alpha)v = (\gamma, \gamma+2\rho)v \qquad \square_{\text{Claim}}\end{aligned}$$

Note that the universal property of Verma modules implies that the action of Ω on any representation generated by a highest vector of weight γ is given by $(\gamma, \gamma + 2\rho)\mathrm{Id}$.

Finally, consider the set

$$\Omega^\gamma = \{\mu \in P | (\mu + \rho, \mu + \rho) = (\gamma + \rho, \gamma + \rho)\}.$$

This is the intersection of the weight lattice P with the sphere of radius $\|\gamma + \rho\|$ centered at $-\rho$. In particular, it is a *finite set*. On the other hand, since $(\gamma, \gamma+2\rho) = (\gamma+\rho, \gamma+\rho) - (\rho, \rho)$, it is also the set of weights μ such that Ω acts on $M(\mu)$ in the same way it acts on $M(\gamma)$.

Step 3. Filter $M(\gamma)$ for another formula: We say that a weight vector v is a *singular vector* if $\mathfrak{n}^+ v = 0$. If a representation is generated by some highest vector v, and if all singular vectors are proportional to v, then the representation is irreducible. To see this, note that a highest weight vector of any proper subrepresentation must be singular, and it cannot be proportional to v, lest it generate the whole representation.

Now let w be a singular vector of weight μ in $M(\gamma)$. Then w generates a subrepresentation which is a quotient of $M(\mu)$. By the claim in Step 2, Ω acts on this subrepresentation by $(\mu, \mu + 2\rho)$. On the other hand, since we are in $M(\gamma)$, Ω must act by $(\gamma, \gamma + 2\rho)$. It follows that $\mu \in \Omega^\gamma$.

In particular, since Ω^γ is finite, there is a minimal singular vector w, which generates some irreducible subrepresentation; we will call that representation $F_1 M(\gamma)$. Mod out my $F_1 M(\gamma)$ and repeat the process. Any singular vector in $M(\gamma)/F_i M(\gamma)$ must be in Ω^γ, so there is a minimal one, w, which generates an irreducible subrepresentation. Define $F_{i+1} M(\gamma) \subseteq M(\gamma)$ to be the pre-image of that representation. Since Ω^γ is finite and each V_μ is finite dimensional, the process terminates. The result is a filtration

$$0 = F_0 M(\gamma) \subseteq F_1 M(\gamma) \subseteq \cdots \subseteq F_k M(\gamma) = M(\gamma)$$

such that $F_i M(\gamma)/F_{i+1} M(\gamma)$ is isomorphic to the irreducible representation $V(\mu)$ for some $\mu \in \Omega^\gamma$.[2] We also know that each μ that appears is less than or equal to λ.

This gives us the nice formula

$$ch\, M(\gamma) = \sum_{\mu \le \gamma,\, \mu \in \Omega^\gamma} b_{\gamma\mu} ch\, V(\mu)$$

for some non-negative integers $b_{\gamma\mu}$.[3] Moreover, $V(\gamma)$ appears as a quotient exactly once, so $b_{\gamma\gamma} = 1$.

Step 4. Invert and simplify the equation: We've shown that the matrix $(b_{\gamma\mu})_{\gamma,\mu\in\Omega^\lambda}$ is lower triangular with ones on the diagonal, so it has a lower triangular inverse $(c_{\gamma\mu})_{\gamma,\mu\in\Omega^\lambda}$ with ones on the diagonal.[4] This gives us the formula

$$ch\, V(\lambda) = \sum_{\mu \le \lambda,\, \mu \in \Omega^\lambda} c_{\lambda\mu} ch\, M(\mu).$$

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[2] We showed in Lecture 18 that for every $\mu \in \mathfrak{h}^*$, there is a unique irreducible representation $V(\mu)$ with highest weight μ. However, we only showed that $V(\mu)$ is finite dimensional when $\mu \in P^+$. In general, it is infinite dimensional. In fact, sometimes it happens that $V(\mu) = M(\mu)$.

[3] These $b_{\gamma\mu}$ are called *Kazhdan-Luztig multiplicities*, and they are hard to compute for general γ and μ.

[4] We will prove that each non-zero $c_{\gamma\mu}$ is ± 1. It was once conjectured that even if $\lambda \notin P^+$, each non-zero $c_{\gamma\mu}$ is ± 1, but this is false.

Using Step 1, we can rewrite this as

$$ch\, V(\lambda) \cdot \prod_{\alpha\in\Delta^+}(e^{\alpha/2} - e^{-\alpha/2}) = \sum_{\mu\le\lambda,\, \mu\in\Omega^\lambda} c_{\lambda\mu} e^{\mu+\rho}.$$

For any element w of the Weyl group, we know that $w(\text{LHS}) = (-1)^w\text{LHS}$, so the same must be true of the RHS, i.e.

$$\sum c_{\lambda\mu} e^{w(\mu+\rho)} = \sum (-1)^w c_{\lambda\mu} e^{\mu+\rho}.$$

This is equivalent to the condition $c_{\lambda, w(\mu+\rho)-\rho} = c_{\lambda\mu}$. Since P^+ is the fundamental domain of $\mathfrak{W}$, and since $c_{\lambda\lambda} = 1$, we get

$$ch\, V(\lambda) \cdot \prod_{\alpha\in\Delta^+}(e^{\alpha/2} - e^{-\alpha/2}) = \sum_{w\in\mathfrak{W}} (-1)^w e^{w(\lambda+\rho)} + \sum_{\substack{\mu<\lambda,\, \mu\in\Omega^\lambda \\ \mu+\rho\in P^+}} (-1)^w c_{\lambda\mu} e^{w(\mu+\rho)}.$$

We would like to eliminate the second sum on the right hand side. The following claim does that nicely by showing that the sum is empty.

Claim. *If* $\mu \le \lambda$, $\mu \in \Omega^\lambda$, *and* $\mu + \rho \ge 0$, *then* $\mu = \lambda$.

Proof. We assume that $(\mu+\rho, \mu+\rho) = (\lambda+\rho, \lambda+\rho)$ and $\lambda-\mu = \sum_{i=1}^n k_i\alpha_i$ for some non-negative k_i. Then we get

$$\begin{aligned} 0 &= \big((\lambda+\rho) - (\mu+\rho), (\lambda+\rho) + (\mu+\rho)\big) \\ &= (\lambda-\mu, \lambda+\mu+2\rho) \\ &= \sum_{i=1}^n k_i(\alpha_i, \lambda+\mu+2\rho) \end{aligned}$$

But $\lambda \ge 0$ and $\mu+\rho \ge 0$, so $(\alpha_i, \lambda+\mu+\rho) \ge 0$. Also, $(\alpha_i, \rho) > 0$ for each i, so $(\alpha_i, \lambda+\mu+2\rho) > 0$. It follows that each k_i is zero. $\square_{\text{Claim}}$

Now we have

$$ch\, V(\lambda) \cdot \prod_{\alpha\in\Delta^+}(e^{\alpha/2} - e^{-\alpha/2}) = \sum_{w\in\mathfrak{W}} (-1)^w e^{w(\lambda+\rho)}.$$

Specializing to the case $\lambda = 0$, we know that $V(0)$ is the trivial representation, so $ch\, V(0) = 1$. This tells us that

$$\prod_{\alpha\in\Delta^+}(e^{\alpha/2} - e^{-\alpha/2}) = \sum_{w\in\mathfrak{W}} (-1)^w e^{w(\rho)}, \tag{19.6}$$

so we get the desired

$$ch\, V(\lambda) = \frac{\sum_{w\in\mathfrak{W}}(-1)^w e^{w(\lambda+\rho)}}{\sum_{w\in\mathfrak{W}}(-1)^w e^{w(\rho)}}. \qquad \square$$

Corollary 19.7 (Weyl dimension formula)**.** $\dim V(\lambda) = \prod_{\alpha\in\Delta^+} \frac{(\lambda+\rho,\alpha)}{(\rho,\alpha)}$.

Proof. The point is that e^μ is a formal expression. The only property that we use is $e^\mu e^\gamma = e^{\mu+\gamma}$, so everything we've ever done with characters works if we replace e^μ by any other expression satisfying that relation. In particular, if replace e^μ with $6^{t(\gamma+\rho,\mu)}$, where t is a real number,[5] then Equation 19.6 says

$$\prod_{\alpha\in\Delta^+}\Big(6^{t(\gamma+\rho,\alpha/2)} - 6^{-t(\gamma+\rho,\alpha/2)}\Big) = \sum_{w\in\mathfrak{W}}(-1)^w 6^{t(\gamma+\rho,w(\rho))}$$
$$= \sum_{w\in\mathfrak{W}}(-1)^w 6^{t(w(\gamma+\rho),\rho)} \qquad (19.8)$$

where the second equality is obtained by replacing w by w^{-1} and observing that $(x, w^{-1}y) = (w\,x, y)$ and that $(-1)^{w^{-1}} = (-1)^w$.

Now we switch things up and replace e^μ by $6^{t(\mu,\rho)}$, so the character formula becomes

$$ch\, V(\lambda) = \frac{\sum_{w\in\mathfrak{W}}(-1)^w 6^{t(w(\lambda+\rho),\rho)}}{\sum_{w\in\mathfrak{W}}(-1)^w 6^{t(w(\rho),\rho)}}.$$

Applying Equation 19.8 to the numerator (with $\gamma = \lambda$) and to the denominator (with $\gamma = 0$), we get

$$ch\, V(\lambda) = \prod_{\alpha\in\Delta^+} \frac{\big(6^{t(\lambda+\rho,\alpha/2)} - 6^{-t(\lambda+\rho,\alpha/2)}\big)}{\big(6^{t(\rho,\alpha/2)} - 6^{-t(\rho,\alpha/2)}\big)}.$$

The dimension of $V(\lambda)$ is equal to the expression $ch\, V(\lambda)$ with e^μ replaced by 1. We can obtain this by letting t tend to zero in $6^{t(\mu,\rho)}$. This gives

$$\dim V(\lambda) = \lim_{t\to 0}\prod_{\alpha\in\Delta^+} \frac{\big(6^{t(\lambda+\rho,\alpha/2)} - 6^{-t(\lambda+\rho,\alpha/2)}\big)}{\big(6^{t(\rho,\alpha/2)} - 6^{-t(\rho,\alpha/2)}\big)}$$
$$= \prod_{\alpha\in\Delta^+}\frac{(\lambda+\rho,\alpha)}{(\rho,\alpha)}. \qquad \text{(By l'Hôpital's rule)} \quad \square$$

Example 19.9. Let $\mathfrak{g} = \mathfrak{sl}(n+1)$. We choose the standard set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ so that $\Delta^+ = \{\alpha_i + \alpha_{i+1} + \cdots + \alpha_j\}_{1\le i\le j\le n}$. Recall

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[5] Obviously, there is nothing special about the base 6; just about any number would work. It is important to understand that for any μ, $t \mapsto 6^{t(\gamma+\rho,\mu)}$ is an honest real-valued function in t. Equation 19.8 is an equality of *real-valued functions* in t! Similarly, $ch\, V(\lambda)$ becomes a real-valued function in t.

that $(\rho, \alpha_i) = 1$ for $1 \le i \le n$ and that $(\omega_i, \alpha_j) = \delta_{ij}$. If $\lambda+\rho = \sum_{i=1}^n a_i\omega_i$, the dimension formula tells us that

$$\begin{aligned}\dim V(\lambda) &= \prod_{\alpha\in\Delta^+} \frac{(\lambda+\rho,\alpha)}{(\rho,\alpha)}\\ &= \prod_{1\le i\le j\le n} \frac{a_i + a_{i+1} + \cdots + a_{j-1} + a_j}{j-i+1}\\ &= \frac{1}{n!!} \prod_{1\le i\le j\le n} \sum_{k=i}^{j} a_k\end{aligned}$$

where $n!! := n!\,(n-1)!\cdots 3!\,2!\,1!$.

If $\mathfrak{g} = \mathfrak{sl}(3)$, and if $\lambda+\rho = 3\omega_1 + 2\omega_2$, we get $\dim V(\lambda) = \frac{1}{2!!}\cdot 2\cdot 3\cdot (2+3) = 15$, computing the dimension of the representation in Example 19.4. This formula is nice because the calculation does not get big as λ gets big. If $\lambda+\rho = 20\omega_1 + 91\omega_2$, it would be really annoying to compute $ch\, V(\lambda)$ completely, but we can get $\dim V(\lambda) = \frac{1}{2}20\cdot 91\cdot 111 = 101010$ easily.

Even for larger n, this formula is pretty good. Say we want the dimension of , then $\lambda+\rho = 2\omega_1 + 3\omega_2 + 1\omega_3 + 7\omega_4$, so we get

$$\frac{1}{4!!} 2\cdot 3\cdot 1\cdot 7\cdot (2+3)(3+1)(1+7)(2+3+1)(3+1+7)(2+3+1+7) = 20020.$$

Remark 19.10. Given complete reducibility, knowing the characters of all irreducible representations allows you to decompose tensor products, just like in representation theory of finite groups. That is, we can now compute the coefficients in $V(\lambda)\otimes V(\mu) = \bigoplus b^{\nu}_{\lambda\mu} V(\nu)$. In the finite group case, we make this easier by choosing an inner product on class functions so that characters of irreducible representations form an orthonormal basis. Now we would like to come up with an inner product on formal expressions $\sum m_\mu e^\mu$ so that characters of irreducible representations are orthonormal.

The obvious inner product is $\langle e^\lambda, e^\mu\rangle = \delta_{\lambda,\mu}$, under which the e^μ are an orthonormal basis. There is no hope for the $ch\, V(\lambda)$ to be orthogonal, but we can tweak it. Another inner product is

$$(e^\lambda, e^\mu) = \frac{1}{|\mathfrak{W}|}\langle D\cdot e^\lambda, D\cdot e^\mu\rangle$$

where D is the Weyl denominator. The character formula tells us that under this inner product, the $ch\, V(\lambda)$ are orthonormal, and form a basis for $\mathfrak{W}$-*symmetric* expressions where $m_\mu = 0$ for $\mu \notin P$.

As with the character formula, this may not look so impressive, but it makes decomposing tensor products very fast. We want to compute

$$\big(ch\,V(\lambda)\cdot ch\,V(\mu), ch\,V(\gamma)\big) = \frac{1}{|\mathfrak{W}|}\big\langle \underbrace{D\cdot ch\,V(\lambda)}_{\sum(-1)^w e^{w(\lambda+\rho)}} \cdot ch\,V(\mu), \underbrace{D\cdot ch\,V(\gamma)}_{\sum(-1)^w e^{w(\gamma+\rho)}}\big\rangle$$

for all $\gamma\in P^+$. Since we know that the result must be $\mathfrak{W}$-symmetric, we can remove the $\frac{1}{|\mathfrak{W}|}$ and restrict our attention to the Weyl chamber. That is, we can just compute $\big\langle\sum(-1)^w e^{w(\lambda+\rho)}\cdot ch\,V(\mu), e^{\gamma+\rho}\big\rangle$, which is the multiplicity of γ in $\sum(-1)^w e^{w(\lambda+\rho)-\rho} ch\,V(\mu)$. In practice, we choose $|\mu|\le|\lambda|$, so most of the summands lie outside of the Weyl chamber, so we can ignore them.

Example 19.11 (For those who know about $\mathfrak{gl}(n)$)**.** We know that $\mathfrak{gl}(n+1)$ is the direct sum (as a Lie algebra) of its center, $k\cdot\mathrm{Id}$, and $\mathfrak{sl}(n+1)$.[6] Let $\{\varepsilon_1,\dots,\varepsilon_{n+1}\}$ be the image of an orthonormal basis of k^{n+1} in k^n (under the usual projection, so that $\sum\varepsilon_i = 0$). Let $z_i = e^{\varepsilon_i}$, so $z_1\cdots z_{n+1} = 1$. The Weyl group $W\simeq S_{n+1}$ acts on the z_i by permutation. We have that

$$\begin{aligned}\rho &= \frac{1}{2}\sum_{i<j}\varepsilon_i-\varepsilon_j\\ &= \frac{n}{2}\varepsilon_1 + \frac{n-2}{2}\varepsilon_2+\cdots+\frac{-n}{2}\varepsilon_{n+1}\\ &= n\varepsilon_1+(n-1)\varepsilon_2+\cdots+2\varepsilon_{n-1}+\varepsilon_n+0\varepsilon_{n+1} \qquad \big(\textstyle\sum\varepsilon_i=0\big)\end{aligned}$$

so

$$e^\rho = z_1^n z_2^{n-1}\cdots z_n^1 z_{n+1}^0.$$

If $\lambda=\sum_{i=1}^{n+1}a_i\varepsilon_i$ (with $\sum a_i = 0$) is a dominant integral weight, we have $(\lambda,\check\alpha_i) = a_i - a_{i+1}\ge 0$. The character formula says that

$$ch\,V(\lambda) = \frac{\sum_{\sigma\in S_{n+1}}(-1)^\sigma z_{\sigma(1)}^{a_1+n}\cdots z_{\sigma(n)}^{a_n+1} z_{\sigma(n+1)}^{a_{n+1}}}{\sum_{\sigma\in S_{n+1}}(-1)^\sigma z_{\sigma(1)}^{n}\cdots z_{\sigma(n)}^{1} z_{\sigma(n+1)}^{0}}$$

The denominator (call it D) is the famous Vandermonde determinant,

$$\det\begin{pmatrix} z_1^n & z_2^n & \cdots & z_{n+1}^n\\ z_1^{n-1} & z_2^{n-1} & \cdots & z_{n+1}^{n-1}\\ \vdots & \vdots & \ddots & \vdots\\ z_1 & z_2 & \cdots & z_{n+1}\\ 1 & 1 & \cdots & 1\end{pmatrix} \begin{aligned}&= \sum_{\sigma\in S_{n+1}}(-1)^\sigma z_{\sigma(1)}^n\cdots z_{\sigma(n)}^1 z_{\sigma(n+1)}^0\\ &= \prod_{1\le i<j\le n+1}(z_j - z_i)\end{aligned}$$

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[6] In general, a Lie algebra which is the the direct sum of its center and its semisimple part is called *reductive*.

The numerator is

$$D_\lambda = \det \begin{pmatrix} z_1^{a_1+n} & z_2^{a_1+n} & \cdots & z_{n+1}^{a_1+n} \\ z_1^{a_2+n-1} & z_2^{a_2+n-1} & \cdots & z_{n-1}^{a_2+n+1} \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{a_n+1} & z_2^{a_n+1} & \cdots & z_{n+1}^{a_n+1} \\ z_1^{a_{n+1}} & z_1^{a_{n+1}} & \cdots & z_1^{a_{n+1}} \end{pmatrix}$$

So the character is the Schur polynomial.

Usually, the representations are encoded as Young diagrams. The marks on the dynkin diagram are the differences in consecutive rows in the young diagram.

Lecture 20 - Compact Lie groups

So far we classified semisimple Lie algebras over an algebraically closed field characteristic 0. Now we will discuss the connection to compact groups. Representations of Lie groups are always taken to be smooth.

Example 20.1. $SU(n) = \{X \in GL(n, \mathbb{C}) | \bar{X}^t X = \mathrm{Id} \text{ and } \det X = 1\}$ is a compact connected Lie group over $\mathbb{R}$. It is the group of linear transformations of $\mathbb{C}^n$ preserving some hermitian form.

You may already know that $SU(2)$ is topologically a 3-sphere.

▶ **Exercise 20.1.** If G is an abelian compact connected Lie group, then it is a product of circles, so it is $\mathbb{T}^n$.

There exists the G-invariant volume form[1] ω satisfying

1. The volume of G is one: $\int_G \omega = 1$, and
2. ω is left invariant: $\int_G f\omega = \int_G L_h^* f\,\omega$ for all $h \in G$. Recall that $L_h^* f$ is defined by $(L_h^* f)(g) = f(hg)$.

To construct ω pick $\omega_e \in \Lambda^{\mathrm{top}}(T_e G)^*$ and define $\omega_g = L_{g^{-1}}^* \omega_e$.

▶ **Exercise 20.2.** If G is connected, show that this ω is also right invariant. Even if G is not connected, show that the measure obtained from a right invariant form agrees with the measure obtained from a left invariant form.

Theorem 20.2. *If G is a compact group and V is a real representation of G, then there exists a positive definite G-invariant inner product on V. That is, $(gv, gw) = (v, w)$.*

Proof. Pick any positive definite inner product[2] $\langle v, w\rangle$, and define

$$(v, w) = \int_G \langle gv, gw\rangle \omega$$

which is positive definite and invariant. □

It follows that any finite dimensional representation of a compact group G is completely reducible (i.e. splits into a direct sum of irreducibles) because the orthogonal complement to a subrepresentation is a subrepresentation.

In particular, the representation $Ad : G \to GL(\mathfrak{g})$ is completely reducible, and the irreducible subrepresentations are exactly the irreducible

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[1] A *volume form* is a non-vanishing top degree form.

[2] Pick any basis, and declare it to be orthonormal.

subrepresentations of the derivative, $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. Thus, we get the decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \mathfrak{g}_k \oplus \mathfrak{a}$, with each $\mathfrak{g}_i$ is a one dimensional or simple ideal. We dump all the one dimensional $\mathfrak{g}_i$ into $\mathfrak{a}$, which is then the center of $\mathfrak{g}$. Thus, the Lie algebra of a compact group is the direct sum of its center and a semisimple Lie algebra. Such a Lie algebra is called *reductive*.

If G is simply connected, then I claim that $\mathfrak{a}$ is trivial. This is because the simply connected group connected to $\mathfrak{a}$ must be a torus, so a center gives you some fundamental group. Thus, if G is simply connected, then $\mathfrak{g}$ is semisimple.

Theorem 20.3. *If the Lie group G of $\mathfrak{g}$ is compact, then the Killing form B on $\mathfrak{g}$ is negative semi-definite. If the Killing form on $\mathfrak{g}$ is negative definite, then there is some compact group G with Lie algebra $\mathfrak{g}$.*

Proof. If you have $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, and you know that $\mathfrak{g}$ has an *ad*-invariant positive definite product, so it lies in $\mathfrak{so}(\mathfrak{g})$. Here you have $A^t = -A$, so you have to check that $tr(A^2) < 0$. It is not hard to check that the eigenvalues of A are imaginary (as soon as $A^t = -A$), so we have that the trace of the square is negative (or zero).

If B is negative definite, then it is non-degenerate, so $\mathfrak{g}$ is semisimple by Theorem 12.7, and $-B$ is an inner product. Moreover, we have that

$$-B(ad_X Y, Z) = B(Y, ad_X Z)$$

so $ad_X = -ad_X^t$ with respect to this inner product. That is, the image of ad lies in $\mathfrak{so}(\mathfrak{g})$. It follows that the image under Ad of the simply connected group $\widetilde{G}$ with Lie algebra $\mathfrak{g}$ lies in $SO(\mathfrak{g})$. Thus, the image is a closed subgroup of a compact group, so it is compact. Since Ad has a discrete kernel, the image has the same Lie algebra. □

How to classify compact Lie algebras? We know the classification over $\mathbb{C}$, so we can always take $\mathfrak{g} \rightsquigarrow \mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$, which remains semisimple. However, this process might not be injective. For example, take $\mathfrak{su}(2) = \{ \left(\begin{smallmatrix} a & b \\ -\bar{b} & a \end{smallmatrix}\right) | a \in \mathbb{R}i, b \in \mathbb{C}\}$ and $\mathfrak{sl}(2, \mathbb{R})$, then they complexify to the same thing.

$\mathfrak{g}$ in this case is called a *real form* of $\mathfrak{g}_\mathbb{C}$. So you can start with $\mathfrak{g}_\mathbb{C}$ and classify all real forms.

Theorem 20.4 (Cartan)**.** *Every semisimple Lie algebra has exactly one (up to isomorphism) compact real form.*

For example, for $\mathfrak{sl}(2)$ it is $\mathfrak{su}(2)$.

Classical Lie groups: $SL(n, \mathbb{C}), SO(n, \mathbb{C})$ (SO has lots of real forms of this, because in the real case, you get a signiture of a form; in the complex case, all forms are isomorphic), $Sp(2n,\mathbb{C})$. What are the corresponding compact simple Lie groups?

Compact real forms: $SU(n)$ = the group of linear operators on $\mathbb{C}^n$ preserving a positive definite Hermitian form. $SL(n)$ = the group of linear operators on $\mathbb{R}^n$ preserving a positive definite symmetric bilinear form. $Sp(2n)$ =the group of linear operators on $\mathbb{H}^n$ preserving a positive definite Hermitian form

We're not going to prove this theorem because we don't have time, but let's show existence.

Proof of existence. Suppose $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes_\mathbb{R} \mathbb{C} = \mathfrak{g} \oplus i\mathfrak{g}$. Then you can construct $\sigma : \mathfrak{g}_\mathbb{C} \to \mathfrak{g}_\mathbb{C}$ "complex conjugation". Then σ preserves the commutator, but it is an anti-linear involution. Classifying real forms amounts to classifying all anti-linear involutions. There should be one that corresponds to the compact algebra. Take $X_1, \dots, X_n, H_1, \dots, H_n, Y_1, \dots, Y_n$ generators for the algebra. Then we just need to define σ on the generators: $\sigma(X_i) = -Y_i, \sigma(Y_i) = -X_i, \sigma(H_i) = -H_i$, and extend anti-linearly. This particular σ is called the *Cartan involution.*

Now we claim that $\mathfrak{g} = (\mathfrak{g}_\mathbb{C})^\sigma = \{X|\sigma(X) = X\}$ is a compact simple Lie algebra. We just have to check that the Killing form is negative definite. If you take $h \in \mathfrak{h}$, written as $h = \sum a_i H_i$, then $\sigma(h) = h$ implies that all the a_i are purely imaginary. This implies that the eigenvalues of h are imaginary, which implies that $B(h,h) < 0$. You also have to check it on X_i, Y_i. The fixed things will be of the form $(aX_i - \bar{a}Y_i) \in \mathfrak{g}$. The Weyl group action shows that B is negative on all of the root space. ☐

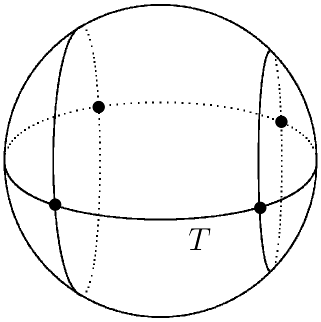
Look at $\exp \mathfrak{h}^\sigma \subset G$ (simply connected), which is called the maximal torus T. I'm going to tell you several facts now. You can always think of T as $\mathbb{R}^n/L$. The point is that $\mathbb{R}^n$ can be identified with $\mathfrak{h}_{re}$, and $\mathfrak{h}^*_{re}$ has two natural lattices: Q (the root lattice) and P (the weight lattice). So one can identify $T = \mathbb{R}^n/L = \mathfrak{h}_{re}/\check{P}$, where $\check{P}$ is the natural dual lattice to P, the set of $h \in \mathfrak{h}$ such that $\langle \omega, h\rangle \in \mathbb{Z}$ for all $\omega \in P$. G is simply connected, and when you quotient by the center, you get $Ad\, G$, and all other groups with the same algebra are in between. $Ad\, T = \mathfrak{h}_{re}/\check{Q}$. We have the sequence $\{1\} \to Z(G) \to T \to Ad\, T \to \{1\}$. You can check that any element is semisimple in a compact group, so the center of G is the quotient $P/Q \simeq \check{Q}/\check{P}$. Observe that $|P/Q|$ = the determinant of the Cartan matrix. For example, if $\mathfrak{g} = \mathfrak{sl}(3)$, then we have $\det \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right) = 3$, and the center of $SU(3)$ is the set of all elements of the form $diag(\omega, \omega, \omega)$ where $\omega^3 = 1$.

G_2 has only one real form because the det is 1?

Orthogonality relations for compact groups:

$$\int_G \chi(g)\bar{\psi}(g^{-1})\omega = \delta_{\chi,\psi}$$

where χ and ψ are characters of irreducible representations. You know that the character is constant on conjugacy classes, so you can integrate over the conjugacy classes. There is a nice picture for $SU(2)$.



The integral can be written as

$$\frac{1}{|\mathfrak{W}|}\int_T \chi(t)\bar{\psi}(t)Vol(C(t))dt$$

And $Vol(C(t)) = \mathcal{D}(t)\bar{\mathcal{D}}(t)$. You divide by $|\mathfrak{W}|$ because that is how many times each class hits T.

Lecture 21 - An overview of Lie groups

The (unofficial) goal of the last third of the course is to prove no theorems. We'll talk about

1. Lie groups in general,
2. Clifford algebras and Spin groups,
3. Construction of all Lie groups and all representations. You might say this is impossible, so let's just try to do all simple ones, and in particular E_8, E_7, E_6.
4. Representations of $SL_2(\mathbb{R})$.

Lie groups in general

In general, a Lie group G can be broken up into a number of pieces.

The connected component of the identity, $G_{\text{conn}} \subseteq G$, is a normal subgroup, and G/G_{conn} is a discrete group.

$$1 \longrightarrow G_{\text{conn}} \longrightarrow G \longrightarrow G_{\text{discrete}} \longrightarrow 1$$

The maximal connected normal solvable subgroup of G_{conn} is called G_{sol}. Recall that a group is *solvable* if there is a chain of subgroups $G_{\text{sol}} \supseteq \cdots \supseteq 1$, where consecutive quotients are abelian. The Lie algebra of a solvable group is solvable (by Exercise 11.2), so Lie's theorem (Theorem 11.11) tells us that G_{sol} is isomorphic to a subgroup of the group of upper triangular matrices.

Every normal solvable subgroup of $G_{\text{conn}}/G_{\text{sol}}$ is discrete, and therefore in the center (which is itself discrete). We call the pre-image of the center G_*. Then G/G_* is a product of simple groups (groups with no normal subgroups).

$$G_{\text{sol}} \subseteq \left\{ \begin{pmatrix} * & & & \ast \\ & \ddots & & \\ & & * & \\ 0 & & & * \end{pmatrix} \right\} \qquad G_{\text{nil}} \subseteq \left\{ \begin{pmatrix} 1 & & & \ast \\ & \ddots & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} \right\}$$

Since G_{sol} is solvable, $G_{\text{nil}} := [G_{\text{sol}}, G_{\text{sol}}]$ is nilpotent, i.e. there is a chain of subgroups $G_{\text{nil}} \supseteq G_1 \supseteq \cdots \supseteq G_k = 1$ such that G_i/G_{i+1} is in the center of G_{nil}/G_{i+1}. In fact, G_{nil} must be isomorphic to a subgroup of the group of upper triangular matrices with ones on the diagonal. Such a group is called *unipotent*.

We have the picture

$$\begin{array}{rcl}
 & G & \\
 & | & \Big\}\text{discrete; classification hopeless}\\
\text{connected}\left\{\begin{array}{c} G_{\text{conn}} \\ | \\ G_* \\ | \\ G_{\text{sol}} \\ | \\ G_{\text{nil}} \\ | \\ 1 \end{array}\right. & & \begin{array}{l} \Big\}\prod\text{connected simples; classified}\\ \left.\begin{array}{l}\Big\}\text{abelian discrete}\\ \Big\}\text{abelian}\end{array}\right\}\text{classification trivial}\\ \Big\}\text{nilpotent; classification a mess}\end{array}
\end{array}$$

The classification of connected simple Lie groups is quite long. There are many infinite series and a lot of exceptional cases. Some infinite series are $PSU(n)$, $PSL_n(\mathbb{R})$, and $PSL_n(\mathbb{C})$.[1]

One way to get many connected simple Lie groups is not observe that there is a unique connected simple Lie group for each simple Lie algebra. We've already classified complex Lie algebras, and it turns out that there a finite number of real Lie algebras which complexify to any given complex Lie algebra. We will classify all such real forms in Lecture 29.

For example, $\mathfrak{sl}_2(\mathbb{R}) \not\simeq \mathfrak{su}_2(\mathbb{R})$, but $\mathfrak{sl}_2(\mathbb{R})\otimes\mathbb{C} \simeq \mathfrak{su}_2(\mathbb{R})\otimes\mathbb{C} \simeq \mathfrak{sl}_2(\mathbb{C})$. By the way, $\mathfrak{sl}_2(\mathbb{C})$ is simple as a *real* Lie algebra, but its complexification is $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$, which is not simple. Thus, we cannot obtain all connected simple groups this way.

Example 21.1. Let G be the group of all shape-preserving transformations of $\mathbb{R}^4$ (i.e. translations, reflections, rotations, and scaling). It is sometimes called $\mathbb{R}^4 \cdot GO_4(\mathbb{R})$. The $\mathbb{R}^4$ stands for translations, the G means that you can multiply by scalars, and the O means that you can reflect and rotate. The $\mathbb{R}^4$ is a normal subgroup. In this case, we have

$$\begin{array}{rrl}
 & \mathbb{R}^4 \cdot GO_4(\mathbb{R}) = G & \\
 & & G/G_{\text{conn}} = \mathbb{Z}/2\mathbb{Z}\\
\begin{array}{l} G_{\text{conn}}/G_{\text{sol}} \\ = SO_4(\mathbb{R})\end{array} & \left\{\begin{array}{r} \mathbb{R}^4\cdot GO_4^+(\mathbb{R}) = G_{\text{conn}} \\ \\ \mathbb{R}^4\cdot\mathbb{R}^\times = G_* \\ \\ \mathbb{R}^4\cdot\mathbb{R}^+ = G_{\text{sol}}\end{array}\right. & \begin{array}{l} G_{\text{conn}}/G_* = PSO_4(\mathbb{R}) \\ \qquad\qquad\big(\simeq SO_3(\mathbb{R})\times SO_3(\mathbb{R})\big) \\ G_*/G_{\text{sol}} = \mathbb{Z}/2\mathbb{Z}\end{array}\\
 & & G_{\text{sol}}/G_{\text{nil}} = \mathbb{R}^+\\
 & \mathbb{R}^4 = G_{\text{nil}} &
\end{array}$$

---

[1] The P means "mod out by the center".

where $GO_4^+(\mathbb{R})$ is the connected component of the identity (those transformations that preserve orientation), $\mathbb{R}^\times$ is scaling by something other than zero, and $\mathbb{R}^+$ is scaling by something positive. Note that $SO_3(\mathbb{R}) = PSO_3(\mathbb{R})$ is simple.

$SO_4(\mathbb{R})$ is "almost" the product $SO_3(\mathbb{R}) \times SO_3(\mathbb{R})$. To see this, consider the associative (but not commutative) algebra of quaternions, $\mathbb{H}$. Since $q\bar{q} = a^2 + b^2 + c^2 + d^2 > 0$ whenever $q \neq 0$, any non-zero quaternion has an inverse (namely, $\bar{q}/q\bar{q}$). Thus, $\mathbb{H}$ is a division algebra. Think of $\mathbb{H}$ as $\mathbb{R}^4$ and let S^3 be the unit sphere, consisting of the quaternions such that $\|q\| = q\bar{q} = 1$. It is easy to check that $\|pq\| = \|p\| \cdot \|q\|$, from which we get that left (right) multiplication by an element of S^3 is a norm-preserving transformation of $\mathbb{R}^4$. So we have a map $S^3 \times S^3 \to O_4(\mathbb{R})$. Since $S^3 \times S^3$ is connected, the image must lie in $SO_4(\mathbb{R})$. It is not hard to check that $SO_4(\mathbb{R})$ is the image. The kernel is $\{(1,1),(-1,-1)\}$. So we have $S^3 \times S^3/\{(1,1),(-1,-1)\} \simeq SO_4(\mathbb{R})$.

Conjugating a purely imaginary quaternion by some $q \in S^3$ yields a purely imaginary quaternion of the same norm as the original, so we have a homomorphism $S^3 \to O_3(\mathbb{R})$. Again, it is easy to check that the image is $SO_3(\mathbb{R})$ and that the kernel is ± 1, so $S^3/\{\pm 1\} \simeq SO_3(\mathbb{R})$.

So the universal cover of $SO_4(\mathbb{R})$ (a double cover) is the cartesian square of the universal cover of $SO_3(\mathbb{R})$ (also a double cover). Orthogonal groups in dimension 4 have a strong tendency to split up like this. Orthogonal groups in general tend to have these double covers, as we shall see in Lectures 23 and 24. These double covers are important if you want to study fermions.

Lie groups and Lie algebras

Let $\mathfrak{g}$ be a Lie algebra. We can set $\mathfrak{g}_{\text{sol}} = \operatorname{rad}\mathfrak{g}$ to be the maximal solvable ideal (normal subalgebra), and $\mathfrak{g}_{\text{nil}} = [\mathfrak{g}_{\text{sol}}, \mathfrak{g}_{\text{sol}}]$. Then we get the chain

$$\begin{array}{l}
\mathfrak{g} \\
\Big| \ \Big) \prod \text{simples; classification known} \\
\mathfrak{g}_{\text{sol}} \\
\Big| \ \Big) \text{abelian; easy to classify} \\
\mathfrak{g}_{\text{nil}} \\
\Big| \ \Big) \text{nilpotent; classification a mess} \\
0
\end{array}$$

We have an equivalence of categories between simply connected Lie groups and Lie algebras. The correspondence cannot detect

- Non-trivial components of G. For example, SO_n and O_n have the same Lie algebra.

  - Discrete normal (therefore central, Lemma 5.1) subgroups of G. If $Z \subseteq G$ is any discrete normal subgroup, then G and G/Z have the same Lie algebra. For example, $SU(2)$ has the same Lie algebra as $PSU(2) \simeq SO_3(\mathbb{R})$.

If $\tilde{G}$ is a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$, then any other connected group G with Lie algebra $\mathfrak{g}$ must be isomorphic to $\tilde{G}/Z$, where Z is some discrete subgroup of the center. Thus, if you know all the discrete subgroups of the center of $\tilde{G}$, you can read off all the connected Lie groups with the given Lie algebra.

Let's find all the groups with the algebra $\mathfrak{so}_4(\mathbb{R})$. First let's find a simply connected group with this Lie algebra. You might guess $SO_4(\mathbb{R})$, but that isn't simply connected. The simply connected one is $S^3 \times S^3$ as we saw earlier (it is a product of two simply connected groups, so it is simply connected). The center of S^3 is generated by -1, so the center of $S^3 \times S^3$ is $(\mathbb{Z}/2\mathbb{Z})^2$, the Klein four group. There are three subgroups of order 2

$$\begin{tikzcd}
 & (\mathbb{Z}/2\mathbb{Z})^2 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
(-1,1) \arrow[dr, dash] & (-1,-1) \arrow[d, dash] & (1,-1) \arrow[dl, dash] \\
 & 1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & PSO_4(\mathbb{R}) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
SO_3(\mathbb{R}) \times S^3 \arrow[dr, dash] & SO_4(\mathbb{R}) \arrow[d, dash] & S^3 \times SO_3(\mathbb{R}) \arrow[dl, dash] \\
 & S^3 \times S^3 &
\end{tikzcd}$$

Therefore, there are 5 groups with Lie algebra $\mathfrak{so}_4$.

Lie groups and finite groups

1. The classification of finite simple groups resembles the classification of connected simple Lie groups when $n \geq 2$.

   For example, $PSL_n(\mathbb{R})$ is a simple Lie group, and $PSL_n(\mathbb{F}_q)$ is a finite simple group except when $n = q = 2$ or $n = 2, q = 3$. Simple finite groups form about 18 series similar to Lie groups, and 26 or 27 exceptions, called sporadic groups, which don't seem to have any analogues for Lie groups.

2. Finite groups and Lie groups are both built up from simple and abelian groups. However, the way that finite groups are built is much more complicated than the way Lie groups are built. Finite groups can contain simple subgroups in very complicated ways; not just as direct factors.

   For example, there are *wreath products*. Let G and H be finite simple groups with an action of H on a set of n points. Then H

acts on G^n by permuting the factors. We can form the semi-direct product $G^n \ltimes H$, sometimes denoted $G \wr H$. There is no analogue for (finite dimensional) Lie groups. There *is* an analogue for infinite dimensional Lie groups, which is why the theory becomes hard in infinite dimensions.

3. The commutator subgroup of a solvable finite group need not be a nilpotent group. For example, the symmetric group S_4 has commutator subgroup A_4, which is not nilpotent.

Lie groups and Algebraic groups (over $\mathbb{R}$)

By algebraic group, we mean an algebraic variety which is also a group, such as $GL_n(\mathbb{R})$. Any algebraic group is a Lie group. Probably all the Lie groups you've come across have been algebraic groups. Since they are so similar, we'll list some differences.

1. Unipotent and semisimple abelian algebraic groups are totally different, but for Lie groups they are nearly the same. For example $\mathbb{R} \simeq \{\left(\begin{smallmatrix}1 & *\\ 0 & 1\end{smallmatrix}\right)\}$ is unipotent and $\mathbb{R}^\times \simeq \left\{\left(\begin{smallmatrix}a & 0\\ 0 & a^{-1}\end{smallmatrix}\right)\right\}$ is semisimple. As Lie groups, they are closely related (nearly the same), but the Lie group homomorphism $\exp : \mathbb{R} \to \mathbb{R}^\times$ is not algebraic (polynomial), so they look quite different as algebraic groups.

2. Abelian varieties are different from affine algebraic groups. For example, consider the (projective) elliptic curve $y^2 = x^3 + x$ with its usual group operation and the group of matrices of the form $\left(\begin{smallmatrix}a & b\\ -b & a\end{smallmatrix}\right)$ with $a^2 + b^2 = 1$. Both are isomorphic to S^1 as Lie groups, but they are completely different as algebraic groups; one is projective and the other is affine.

3. Some Lie groups do not correspond to ANY algebraic group. We give two examples here.

   The *Heisenberg group* is the subgroup of symmetries of $L^2(\mathbb{R})$ generated by translations $(f(t) \mapsto f(t + x))$, multiplication by $e^{2\pi i t y}$ $(f(t) \mapsto e^{2\pi i t y} f(t))$, and multiplication by $e^{2\pi i z}$ $(f(t) \mapsto e^{2\pi i z} f(t))$. The general element is of the form $f(t) \mapsto e^{2\pi i (yt+z)} f(t + x)$. This can also be modelled as

   $$\left\{\begin{pmatrix} 1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}\right\} \Bigg/ \left\{\begin{pmatrix} 1 & 0 & n\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \,\middle|\, n \in \mathbb{Z}\right\}$$

   It has the property that in any finite dimensional representation, the center (elements with $x = y = 0$) acts trivially, so it cannot be isomorphic to any algebraic group.

The *metaplectic group.* Let's try to find all connected groups with Lie algebra $\mathfrak{sl}_2(\mathbb{R}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) | a + d = 0\}$. There are two obvious ones: $SL_2(\mathbb{R})$ and $PSL_2(\mathbb{R})$. There aren't any other ones that can be represented as groups of finite dimensional matrices. However, if you look at $SL_2(\mathbb{R})$, you'll find that it is not simply connected. To see this, we will use Iwasawa decomposition (without proof).

Theorem 21.2 (Iwasawa decomposition)**.** *If G is a (connected) semisimple Lie group, then there are closed subgroups K, A, and N, with K compact, A abelian, and N unipotent, such that the multiplication map $K \times A \times N \to G$ is a surjective diffeomorphism. Moreover, A and N are simply connected.*

In the case of SL_n, this is the statement that any basis can be obtained uniquely by taking an orthonormal basis ($K = SO_n$), scaling by positive reals (A is the group of diagonal matrices with positive real entries), and shearing (N is the group $\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$). This is exactly the result of the Gram-Schmidt process.

The upshot is that $G \simeq K \times A \times N$ (topologically), and A and N do not contribute to the fundamental group, so the fundamental group of G is the same as that of K. In our case, $K = SO_2(\mathbb{R})$ is isomorphic to a circle, so the fundamental group of $SL_2(\mathbb{R})$ is $\mathbb{Z}$.

So the universal cover $\widetilde{SL_2(\mathbb{R})}$ has center $\mathbb{Z}$. Any finite dimensional representation of $\widetilde{SL_2}(\mathbb{R})$ factors through $SL_2(\mathbb{R})$, so none of the covers of $SL_2(\mathbb{R})$ can be written as a group of finite dimensional matrices. Representing such groups is a pain.

The most important case is the metaplectic group $Mp_2(\mathbb{R})$, which is the connected double cover of $SL_2(\mathbb{R})$. It turns up in the theory of modular forms of half-integral weight and has a representation called the metaplectic representation.

Important Lie groups

Dimension 1: There are just $\mathbb{R}$ and $S^1 = \mathbb{R}/\mathbb{Z}$.

Dimension 2: The abelian groups are quotients of $\mathbb{R}^2$ by some discrete subgroup; there are three cases: $\mathbb{R}^2$, $\mathbb{R}^2/\mathbb{Z} = \mathbb{R} \times S^1$, and $\mathbb{R}^2/\mathbb{Z}^2 = S^1 \times S^1$.

There is also a non-abelian group, the group of all matrices of the form $\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right)$, where $a > 0$. The Lie algebra is the subalgebra of 2×2 matrices of the form $\left(\begin{smallmatrix} h & x \\ 0 & -h \end{smallmatrix}\right)$, which is generated by two elements H and X, with $[H, X] = 2X$.

Dimension 3: There are some boring abelian and solvable groups, such as $\mathbb{R}^2 \ltimes \mathbb{R}^1$, or the direct sum of $\mathbb{R}^1$ with one of the two dimensional groups. As the dimension increases, the number of boring solvable groups gets huge, and nobody can do anything about them, so we ignore them from here on.

You get the group $SL_2(\mathbb{R})$, which is the most important Lie group of all. We saw earlier that $SL_2(\mathbb{R})$ has fundamental group $\mathbb{Z}$. The double cover $Mp_2(\mathbb{R})$ is important. The quotient $PSL_2(\mathbb{R})$ is simple, and acts on the open upper half plane by linear fractional transformations

Closely related to $SL_2(\mathbb{R})$ is the compact group SU_2. We know that $SU_2 \simeq S^3$, and it covers $SO_3(\mathbb{R})$, with kernel ± 1. After we learn about Spin groups, we will see that $SU_2 \cong \mathrm{Spin}_3(\mathbb{R})$. The Lie algebra $\mathfrak{su}_2$ is generated by three elements X, Y, and Z with relations $[X,Y]=2Z$, $[Y,Z]=2X$, and $[Z,X]=2Y$.[2]

The Lie algebras $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{su}_2$ are non-isomorphic, but when you complexify, they both become isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.

There is another interesting 3 dimensional algebra. The Heisenberg algebra is the Lie algebra of the Heisenberg group. It is generated by X, Y, Z, with $[X,Y]=Z$ and Z central. You can think of this as strictly upper triangular matrices.

Dimension 6: (nothing interesting happens in dimensions 4,5) We get the group $SL_2(\mathbb{C})$. Later, we will see that it is also called $\mathrm{Spin}_{1,3}(\mathbb{R})$.

Dimension 8: We have $SU_3(\mathbb{R})$ and $SL_3(\mathbb{R})$. This is the first time we get a non-trivial root system.

Dimension 14: G_2, which we will discuss a little.

Dimension 248: E_8, which we will discuss in detail.

This class is mostly about finite dimensional algebras, but let's mention some infinite dimensional Lie groups or Lie algebras.

1. Automorphisms of a Hilbert space form a Lie group.

2. Diffeomorphisms of a manifold form a Lie group. There is some physics stuff related to this.

3. *Gauge groups* are (continuous, smooth, analytic, or whatever) maps from a manifold M to a group G.

4. The *Virasoro algebra* is generated by L_n for $n \in \mathbb{Z}$ and c, with relations $[L_n, L_m] = (n-m)L_{n+m} + \delta_{n+m,0}\frac{n^3-n}{12}c$, where c is central (called the *central charge*). If you set $c=0$, you get (complexified) vector fields on S^1, where we think of L_n as $ie^{in\theta}\frac{\partial}{\partial\theta}$. Thus, the

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[2] An explicit representation is given by $X = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $Y = \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$, and $Z = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$. The cross product on $\mathbb{R}^3$ gives it the structure of this Lie algebra.

Virasoro algebra is a central extension

$$0 \to c\mathbb{C} \to \text{Virasoro} \to \text{Vect}(S^1) \to 0.$$

5. Affine Kac-Moody algebras, which are more or less central extensions of certain gauge groups over the circle.

Lecture 22 - Clifford algebras

With Lie algebras of small dimensions, there are accidental isomorphisms. Almost all of these can be explained with Clifford algebras and Spin groups.

Motivational examples that we'd like to explain:

1. $SO_2(\mathbb{R}) = S^1$: S^1 can double cover S^1 itself.
2. $SO_3(\mathbb{R})$: has a simply connected double cover S^3.
3. $SO_4(\mathbb{R})$: has a simply connected double cover $S^3 \times S^3$.
4. $SO_5(\mathbb{C})$: Look at $Sp_4(\mathbb{C})$, which acts on $\mathbb{C}^4$ and on $\Lambda^2(\mathbb{C}^4)$, which is 6 dimensional, and decomposes as $5 \oplus 1$. $\Lambda^2(\mathbb{C}^4)$ has a symmetric bilinear form given by $\Lambda^2(\mathbb{C}^4) \otimes \Lambda^2(\mathbb{C}^4) \to \Lambda^4(\mathbb{C}^4) \simeq \mathbb{C}$, and $Sp_4(\mathbb{C})$ preserves this form. You get that $Sp_4(\mathbb{C})$ acts on $\mathbb{C}^5$, preserving a symmetric bilinear form, so it maps to $SO_5(\mathbb{C})$. You can check that the kernel is ± 1. So $Sp_4(\mathbb{C})$ is a double cover of $SO_5(\mathbb{C})$.
5. $SO_5(\mathbb{C})$: $SL_4(\mathbb{C})$ acts on $\mathbb{C}^4$, and we still have our 6 dimensional $\Lambda^2(\mathbb{C}^4)$, with a symmetric bilinear form. So you get a homomorphism $SL_4(\mathbb{C}) \to SO_6(\mathbb{C})$, which you can check is surjective, with kernel ± 1.

So we have double covers S^1, S^3, $S^3 \times S^3$, $Sp_4(\mathbb{C})$, $SL_4(\mathbb{C})$ of the orthogonal groups in dimensions 2,3,4,5, and 6, respectively. All of these look completely unrelated. By the end of the next lecture, we will have an understanding of these groups, which will be called $\mathrm{Spin}_2(\mathbb{R})$, $\mathrm{Spin}_3(\mathbb{R})$, $\mathrm{Spin}_4(\mathbb{R})$, $\mathrm{Spin}_5(\mathbb{C})$, and $\mathrm{Spin}_6(\mathbb{C})$, respectively.

Example 22.1. We have not yet defined Clifford algebras, but here are some examples of Clifford algebras over $\mathbb{R}$.

- $\mathbb{C}$ is generated by $\mathbb{R}$, together with i, with $i^2 = -1$
- $\mathbb{H}$ is generated by $\mathbb{R}$, together with i, j, each squaring to -1, with $ij + ji = 0$.
- Dirac wanted a square root for the operator $\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2}$ (the wave operator in 4 dimensions). He supposed that the square root is of the form $A = \gamma_1 \frac{\partial}{\partial x} + \gamma_2 \frac{\partial}{\partial y} + \gamma_3 \frac{\partial}{\partial z} + \gamma_4 \frac{\partial}{\partial t}$ and compared coefficients in the equation $A^2 = \nabla$. Doing this yields $\gamma_1^2 = \gamma_2^2 = \gamma_3^2 = 1$, $\gamma_4^2 = -1$, and $\gamma_i \gamma_j + \gamma_j \gamma_i = 0$ for $i \neq j$.

  Dirac solved this by taking the γ_i to be 4×4 complex matrices. A operates on vector-valued functions on space-time.

Definition 22.2. A general *Clifford algebra* over $\mathbb{R}$ should be generated by elements $\gamma_1, \dots, \gamma_n$ such that γ_i^2 is some given real, and $\gamma_i\gamma_j + \gamma_j\gamma_i = 0$ for $i \neq j$.

Definition 22.3 (better definition)**.** Suppose V is a vector space over a field K, with some quadratic form[1] $N : V \to K$. Then the *Clifford algebra* $C_V(K)$ is generated by the vector space V, with relations $v^2 = N(v)$.

We know that $N(\lambda v) = \lambda^2 N(v)$ and that the expression $(a, b) := N(a+b) - N(a) - N(b)$ is bilinear. If the characteristic of K is not 2, we have $N(a) = \frac{(a,a)}{2}$. Thus, you can work with symmetric bilinear forms instead of quadratic forms so long as the characteristic of K is not 2. We'll use quadratic forms so that everything works in characteristic 2.

Warning 22.4. A few authors (mainly in index theory) use the relations $v^2 = -N(v)$. Some people add a factor of 2, which usually doesn't matter, but is wrong in characteristic 2.

Example 22.5. Take $V = \mathbb{R}^2$ with basis i, j, and with $N(xi + yj) = -x^2 - y^2$. Then the relations are $(xi + yj)^2 = -x^2 - y^2$ are exactly the relations for the quaternions: $i^2 = j^2 = -1$ and $(i+j)^2 = i^2 + ij + ji + j^2 = -2$, so $ij + ji = 0$.

Remark 22.6. If the characteristic of K is not 2, a "completing the square" argument shows that any quadratic form is isomorphic to $c_1x_1^2 + \cdots + c_nx_n^2$, and if one can be obtained from another other by permuting the c_i and multiplying each c_i by a non-zero square, the two forms are isomorphic.

It follows that every quadratic form on a vector space over $\mathbb{C}$ is isomorphic to $x_1^2 + \cdots + x_n^2$, and that every quadratic form on a vector space over $\mathbb{R}$ is isomorphic to $x_1^2 + \cdots + x_m^2 - x_{m+1}^2 - \cdots - x_{m+n}^2$ (m pluses and n minuses) for some m and n. One can check that these forms over $\mathbb{R}$ are non-isomorphic.

We will always assume that N is non-degenerate (i.e. that the associated bilinear form is non-degenerate), but one could study Clifford algebras arising from degenerate forms.

Warning 22.7. The criterion in the remark is not sufficient for classifying quadratic forms. For example, over the field $\mathbb{F}_3$, the forms x^2+y^2 and $-x^2-y^2$ are isomorphic via the isomorphism $\left(\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right) : \mathbb{F}_3^2 \to \mathbb{F}_3^2$, but -1 is not a square in $\mathbb{F}_3$. Also, completing the square doesn't work in characteristic 2.

---

[1] N is a *quadratic form* if it is a homogeneous polynomial of degree 2 in the coefficients with respect to some basis.

Remark 22.8. The tensor algebra TV has a natural $\mathbb{Z}$-grading, and to form the Clifford algebra $C_V(K)$, we quotient by the ideal generated by the even elements $v^2 - N(v)$. Thus, the algebra $C_V(K) = C_V^0(K) \oplus C_V^1(K)$ is $\mathbb{Z}/2\mathbb{Z}$-graded. A $\mathbb{Z}/2\mathbb{Z}$-graded algebra is called a *superalgebra*.

Problem: Find the structure of $C_{m,n}(\mathbb{R})$, the Clifford algebra over $\mathbb{R}^{n+m}$ with the form $x_1^2 + \cdots + x_m^2 - x_{m+1}^2 - \cdots - x_{m+n}^2$.

Example 22.9.

- $C_{0,0}(\mathbb{R})$ is $\mathbb{R}$.
- $C_{1,0}(\mathbb{R})$ is $\mathbb{R}[\varepsilon]/(\varepsilon^2 - 1) = \mathbb{R}(1+\varepsilon) \oplus \mathbb{R}(1-\varepsilon) = \mathbb{R} \oplus \mathbb{R}$. Note that the given basis, this is a direct sum of *algebras* over $\mathbb{R}$.
- $C_{0,1}(\mathbb{R})$ is $\mathbb{R}[i]/(i^2+1) = \mathbb{C}$, with i odd.
- $C_{2,0}(\mathbb{R})$ is $\mathbb{R}[\alpha, \beta]/(\alpha^2 - 1, \beta^2 - 1, \alpha\beta + \beta\alpha)$. We get a homomorphism $C_{2,0}(\mathbb{R}) \to \mathbb{M}_2(\mathbb{R})$, given by $\alpha \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\beta \mapsto \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The homomorphism is onto because the two given matrices generate $\mathbb{M}_2(\mathbb{R})$ as an algebra. The dimension of $\mathbb{M}_2(\mathbb{R})$ is 4, and the dimension of $C_{2,0}(\mathbb{R})$ is at most 4 because it is spanned by 1, α, β, and $\alpha\beta$. So we have that $C_{2,0}(\mathbb{R}) \simeq \mathbb{M}_2(\mathbb{R})$.
- $C_{1,1}(\mathbb{R})$ is $\mathbb{R}[\alpha, \beta]/(\alpha^2 - 1, \beta^2 + 1, \alpha\beta + \beta\alpha)$. Again, we get an isomorphism with $\mathbb{M}_2(\mathbb{R})$, given by $\alpha \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\beta \mapsto \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$

Thus, we've computed the Clifford algebras

$m \backslash n$	0	1	2
0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
1	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{M}_2(\mathbb{R})$	
2	$\mathbb{M}_2(\mathbb{R})$		

Remark 22.10. If $\{v_1, \ldots, v_n\}$ is a basis for V, then $\{v_{i_1} \cdots v_{i_k} | i_1 < \cdots < i_k,\ k \leq n\}$ spans $C_V(K)$, so the dimension of $C_V(K)$ is less than or equal to $2^{\dim V}$. The tough part of Clifford algebras is showing that it cannot be smaller.

Now let's try to analyze larger Clifford algebras more systematically. What is $C_{U \oplus V}$ in terms of C_U and C_V? One might guess $C_{U \oplus V} \cong C_U \otimes C_V$. For the usual definition of tensor product, this is false (e.g. $C_{1,1}(\mathbb{R}) \neq C_{1,0}(\mathbb{R}) \otimes C_{0,1}(\mathbb{R})$). However, for the *superalgebra* definition of tensor product, this is correct. The superalgebra tensor product is the regular tensor product of vector spaces, with the product given by $(a \otimes b)(c \otimes d) = (-1)^{\deg b \cdot \deg c} ac \otimes bd$ for homogeneous elements a, b, c, and d.

Let's specialize to the case $K = \mathbb{R}$ and try to compute $C_{U\oplus V}(K)$. Assume for the moment that $\dim U = m$ is even. Take $\alpha_1, \dots, \alpha_m$ to be an orthogonal basis for U and let $\beta_1, \dots, \beta_n$ to be an orthogonal basis for V. Then set $\gamma_i = \alpha_1\alpha_2 \cdots \alpha_m \beta_i$. What are the relations between the α_i and the γ_j? We have

$$\alpha_i\gamma_j = \alpha_i\alpha_1\alpha_2\cdots\alpha_m\beta_j = \alpha_1\alpha_2\cdots\alpha_m\beta_i\alpha_i = \gamma_j\alpha_i$$

since $\dim U$ is even, and α_i anti-commutes with everything except itself.

$$\begin{aligned}\gamma_i\gamma_j &= \gamma_i\alpha_1\cdots\alpha_m\beta_j = \alpha_1\cdots\alpha_m\gamma_i\beta_j\\ &= \alpha_1\cdots\alpha_m\alpha_1\cdots\alpha_m \underbrace{\beta_i\beta_j}_{-\beta_j\beta_i} = -\gamma_j\gamma_i\\ \gamma_i^2 &= \alpha_1\cdots\alpha_m\alpha_1\cdots\alpha_m\beta_i\beta_i = (-1)^{\frac{m(m-1)}{2}}\alpha_1^2\cdots\alpha_m^2\beta_i^2\\ &= (-1)^{m/2}\alpha_1^2\cdots\alpha_m^2\beta_i^2 \qquad\qquad (m \text{ even})\end{aligned}$$

So the γ_i's commute with the α_i and satisfy the relations of some Clifford algebra. Thus, we've shown that $C_{U\oplus V}(K) \cong C_U(K) \otimes C_W(K)$, where W is V with the quadratic form multiplied by $(-1)^{\frac{1}{2}\dim U}\alpha_1^2\cdots\alpha_m^2 = (-1)^{\frac{1}{2}\dim U}\cdot$ discriminant(U), and this is the usual tensor product of algebras over $\mathbb{R}$.

Taking $\dim U = 2$, we find that

$$\begin{aligned} C_{m+2,n}(\mathbb{R}) &\cong \mathbb{M}_2(\mathbb{R}) \otimes C_{n,m}(\mathbb{R})\\ C_{m+1,n+1}(\mathbb{R}) &\cong \mathbb{M}_2(\mathbb{R}) \otimes C_{m,n}(\mathbb{R})\\ C_{m,n+2}(\mathbb{R}) &\cong \mathbb{H} \otimes C_{n,m}(\mathbb{R})\end{aligned}$$

where the indices switch whenever the discriminant is positive. Using these formulas, we can reduce any Clifford algebra to tensor products of things like $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{M}_2(\mathbb{R})$.

Recall the rules for taking tensor products of matrix algebras (all tensor products are over $\mathbb{R}$).

- $\mathbb{R} \otimes X \cong X$.
- $\mathbb{C} \otimes \mathbb{H} \cong \mathbb{M}_2(\mathbb{C})$.

  This follows from the isomorphism $\mathbb{C} \otimes C_{m,n}(\mathbb{R}) \cong C_{m+n}(\mathbb{C})$.
- $\mathbb{C} \otimes \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$.
- $\mathbb{H} \otimes \mathbb{H} \cong \mathbb{M}_4(\mathbb{R})$.

  You can see by thinking of the action on $\mathbb{H} \cong \mathbb{R}^4$ given by $(x \otimes y) \cdot z = xzy^{-1}$.

- $\mathbb{M}_m\big(\mathbb{M}_n(X)\big) \cong \mathbb{M}_{mn}(X)$.
- $\mathbb{M}_m(X) \otimes \mathbb{M}_n(Y) \cong \mathbb{M}_{mn}(X \otimes Y)$.

Filling in the middle of the table is easy because you can move diagonally by tensoring with $\mathbb{M}_2(\mathbb{R})$. It is easy to see that $C_{8+m,n}(\mathbb{R}) \cong C_{m,n+8}(\mathbb{R}) \cong C_{m,n} \otimes \mathbb{M}_{16}(\mathbb{R})$, which gives the table a kind of mod 8 periodicity. There is a more precise way to state this: $C_{m,n}(\mathbb{R})$ and $C_{m',n'}(\mathbb{R})$ are *super Morita equivalent* if and only if $m - n \equiv m' - n' \mod 8$.

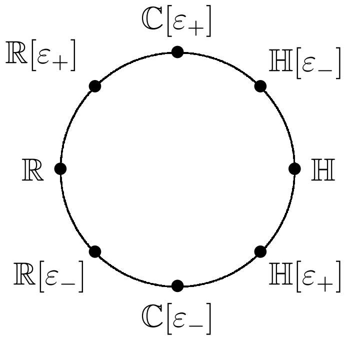
$$\begin{tikzcd}
 & n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
m \\
0 & & \mathbb{R} \arrow[rrrrrrrr, bend left, "\otimes \mathbb{M}_{16}"] \arrow[dddddddd, bend right, "\otimes \mathbb{M}_{16}"'] & \mathbb{C} \arrow[dddl, "\otimes \mathbb{M}_2"] & \mathbb{H} \arrow[ddddll] & \mathbb{H} \oplus \mathbb{H} & \mathbb{M}_2(\mathbb{H}) & \mathbb{M}_4(\mathbb{C}) & \mathbb{M}_8(\mathbb{R}) & \mathbb{M}_8(\mathbb{R}) \oplus \mathbb{M}_8(\mathbb{R}) & \mathbb{M}_{16}(\mathbb{R}) \\
1 & & \mathbb{R} \oplus \mathbb{R} \arrow[urrr, "\otimes \mathbb{H}"'] \\
2 & & \mathbb{M}_2(\mathbb{R}) \arrow[uurrrr] \\
3 & & \mathbb{M}_2(\mathbb{C}) \\
4 & & \mathbb{M}_2(\mathbb{H}) \\
5 & & \mathbb{M}_2(\mathbb{H}) \oplus \mathbb{M}_2(\mathbb{H}) \arrow[dr, "\otimes \mathbb{M}_2"] \\
6 & & \mathbb{M}_4(\mathbb{H}) & \mathbb{M}_4(\mathbb{H}) \oplus \mathbb{M}_4(\mathbb{H}) \arrow[dr, "\otimes \mathbb{M}_2"] \\
7 & & \mathbb{M}_8(\mathbb{C}) & & \text{etc.} \\
8 & & \mathbb{M}_{16}(\mathbb{R}) \\
9 & & \mathbb{M}_{16}(\mathbb{R}) \oplus \mathbb{M}_{16}(\mathbb{R})
\end{tikzcd}$$

Lecture 23

Last time we defined the Clifford algebra $C_V(K)$, where V is a vector space over K with a quadratic form N. $C_V(K)$ is generated by V with $x^2 = N(x)$. $C_{m,n}(\mathbb{R})$ uses the form $x_1^2 + \cdots + x_m^2 - x_{m+1}^2 - \cdots - x_{m+n}^2$. We found that the structure depends heavily on $m - n \mod 8$.

Remark 23.1. This mod 8 periodicity turns up in several other places:

1. Real Clifford algebras $C_{m,n}(\mathbb{R})$ and $C_{m',n'}(\mathbb{R})$ are super Morita equivalent if and only if $m - n \equiv m' - n' \mod 8$.
2. *Bott periodicity*, which says that stable homotopy groups of orthogonal groups are periodic mod 8.
3. Real K-theory is periodic with a period of 8.
4. Even unimodular lattices (such as the E_8 lattice) exist in $\mathbb{R}^{m,n}$ if and only if $m - n \equiv 0 \mod 8$.
5. The Super Brauer group of $\mathbb{R}$ is $\mathbb{Z}/8\mathbb{Z}$. The Super Brauer group consists of super division algebras over $\mathbb{R}$ (algebras in which every non-zero homogeneous element is invertible) with the operation of tensor product modulo super Morita equivalence.[1]



where $\varepsilon_\pm$ are odd with $\varepsilon_\pm^2 = \pm 1$, and $i \in \mathbb{C}$ is odd,[2] but $i, j, k \in \mathbb{H}$ are even.

Recall that $C_V(\mathbb{R}) = C_V^0(\mathbb{R}) \oplus C_V^1(\mathbb{R})$, where $C_V^1(\mathbb{R})$ is the odd part and $C_V^0(\mathbb{R})$ is the even part. It turns out that we will need to know the structure of $C_{m,n}^0(\mathbb{R})$. Fortunately, this is easy to compute in terms of smaller Clifford algebras. Let $\dim U = 1$, with γ a basis for U and let

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[1] See `http://math.ucr.edu/home/baez/trimble/superdivision.html`

[2] One could make i even since $\mathbb{R}[i, \varepsilon_\pm] = \mathbb{R}[\mp\varepsilon_\pm i, \varepsilon_\pm]$, and $\mathbb{R}[\mp\varepsilon_\pm i] \cong \mathbb{C}$ is entirely even.

$\gamma_1, \dots, \gamma_n$ an orthogonal basis for V. Then $C^0_{U\oplus V}(K)$ is generated by $\gamma\gamma_1, \dots, \gamma\gamma_n$. We compute the relations

$$\gamma\gamma_i \cdot \gamma\gamma_j = -\gamma\gamma_j \cdot \gamma\gamma_i$$

for $i \neq j$, and

$$(\gamma\gamma_i)^2 = (-\gamma^2)\gamma_i^2$$

So $C^0_{U\oplus V}(K)$ is itself the Clifford algebra $C_W(K)$, where W is V with the quadratic form multiplied by $-\gamma^2 = -\mathrm{disc}(U)$. Over $\mathbb{R}$, this tells us that

$$\begin{aligned} &C^0_{m+1,n}(\mathbb{R}) \cong C_{n,m}(\mathbb{R}) && \text{(mind the indices)} \\ &C^0_{m,n+1}(\mathbb{R}) \cong C_{m,n}(\mathbb{R}). \end{aligned}$$

Remark 23.2. For complex Clifford algebras, the situation is similar, but easier. One finds that $C_{2m}(\mathbb{C}) \cong \mathbb{M}_{2^m}(\mathbb{C})$ and $C_{2m+1}(\mathbb{C}) \cong \mathbb{M}_{2^m}(\mathbb{C}) \oplus \mathbb{M}_{2^m}(\mathbb{C})$, with $C^0_n(\mathbb{C}) \cong C_{n-1}(\mathbb{C})$. You could figure these out by tensoring the real algebras with $\mathbb{C}$ if you wanted. We see a mod 2 periodicity now. Bott periodicity for the unitary group is mod 2.

Clifford groups, Spin groups, and Pin groups

In this section, we define Clifford groups, denoted $\Gamma_V(K)$, and find an exact sequence

$$1 \to K^\times \xrightarrow{\text{central}} \Gamma_V(K) \to O_V(K) \to 1.$$

Definition 23.3. $\Gamma_V(K) = \{x \in C_V(K)$ homogeneous[3]$|xV\alpha(x)^{-1} \subseteq V\}$ (recall that $V \subseteq C_V(K)$), where α is the automorphism of $C_V(K)$ induced by -1 on V (i.e. the automorphism which acts by -1 on odd elements and 1 on even elements).

Note that $\Gamma_V(K)$ acts on V by $x \cdot v = xv\alpha(x)^{-1}$.

Many books leave out the α, which is a mistake, though not a serious one. They use xVx^{-1} instead of $xV\alpha(x)^{-1}$. Our definition is better for the following reasons:

1. It is the correct superalgebra sign. The superalgebra convention says that whenever you exchange two elements of odd degree, you pick up a minus sign, and V is odd.

---

[3] We assume that $\Gamma_V(K)$ consists of homogeneous elements, but this can actually be proven.

2. Putting α in makes the theory much cleaner in odd dimensions. For example, we will see that the described action gives a map $\Gamma_V(K) \to O_V(K)$ which is onto if we use α, but not if we do not. (You get $SO_V(K)$ without the α, which isn't too bad, but is still annoying.)

Lemma 23.4.[4] *The elements of $\Gamma_V(K)$ which act trivially on V are the elements of $K^\times \subseteq \Gamma_V(K) \subseteq C_V(K)$.*

Proof. Suppose $a_0 + a_1 \in \Gamma_V(K)$ acts trivially on V, with a_0 even and a_1 odd. Then $(a_0 + a_1)v = v\alpha(a_0 + a_1) = v(a_0 - a_1)$. Matching up even and odd parts, we get $a_0 v = v a_0$ and $a_1 v = -v a_1$. Choose an orthogonal basis $\gamma_1, \dots, \gamma_n$ for V.[5] We may write

$$a_0 = x + \gamma_1 y$$

where $x \in C_V^0(K)$ and $y \in C_V^1(K)$ and neither x nor y contain a factor of γ_1, so $\gamma_1 x = x\gamma_1$ and $\gamma_1 y = y\gamma_1$. Applying the relation $a_0 v = v a_0$ with $v = \gamma_1$, we see that $y = 0$, so a_0 contains no monomials with a factor γ_1.

Repeat this procedure with v equal to the other basis elements to show that $a_0 \in K^\times$ (since it cannot have any γ's in it). Similarly, write $a_1 = y + \gamma_1 x$, with x and y not containing a factor of γ_1. Then the relation $a_1\gamma_1 = -\gamma_1 a_1$ implies that $x = 0$. Repeating with the other basis vectors, we conclude that $a_1 = 0$.

So $a_0 + a_1 = a_0 \in K \cap \Gamma_V(K) = K^\times$. □

Now we define $-^T$ to be the identity on V, and extend it to an anti-automorphism of $C_V(K)$ ("anti" means that $(ab)^T = b^T a^T$). Do not confuse $a \mapsto \alpha(a)$ (automorphism), $a \mapsto a^T$ (anti-automorphism), and $a \mapsto \alpha(a^T)$ (anti-automorphism).

Notice that on V, N coincides with the quadratic form N. Many authors seem not to have noticed this, and use different letters. Sometimes they use a sign convention which makes them different.

Now we define the *spinor norm* of $a \in C_V(K)$ by $N(a) = aa^T$. We also define a twisted version: $N^\alpha(a) = a\alpha(a)^T$.

Proposition 23.5.

1. *The restriction of N to $\Gamma_V(K)$ is a homomorphism whose image lies in $K^\times$. N is a mess on the rest of $C_V(K)$.*

2. *The action of $\Gamma_V(K)$ on V is orthogonal. That is, we have a homomorphism $\Gamma_V(K) \to O_V(K)$.*

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[4] I promised no Lemmas or Theorems, but I was lying to you.

[5] All these results are true in characteristic 2, but you have to work harder ... you can't go around choosing orthogonal bases because they may not exist.

Proof. First we show that if $a \in \Gamma_V(K)$, then $N^\alpha(a)$ acts trivially on V.

$$\begin{aligned} N^\alpha(a)\, v\, \alpha\big(N^\alpha(a)\big)^{-1} &= a\alpha(a)^T v \Big(\alpha(a)\, \underbrace{\alpha\big(\alpha(a)^T\big)}_{=a^T}\Big)^{-1} \\ &= a\, \underbrace{\alpha(a)^T v (a^{-1})^T}_{=(a^{-1}v^T\alpha(a))^T}\, \alpha(a)^{-1} \\ &= aa^{-1}v\alpha(a)\alpha(a)^{-1} \qquad (T|_V = \mathrm{Id}_V \text{ and } a^{-1}v\alpha(a) \in V) \\ &= v \end{aligned}$$

So by Lemma 23.4, $N^\alpha(a) \in K^\times$. This implies that N^α is a homomorphism on $\Gamma_V(K)$ because

$$\begin{aligned} N^\alpha(a)N^\alpha(b) &= a\alpha(a)^T N^\alpha(b) \\ &= aN^\alpha(b)\alpha(a)^T \qquad\qquad (N^\alpha(b) \text{ is central}) \\ &= ab\alpha(b)^T\alpha(a)^T \\ &= (ab)\alpha(ab)^T = N^\alpha(ab). \end{aligned}$$

After all this work with N^α, what we're really interested is N. On the even elements of $\Gamma_V(K)$, N agrees with N^α, and on the odd elements, $N = -N^\alpha$. Since $\Gamma_V(K)$ consists of homogeneous elements, N is also a homomorphism from $\Gamma_V(K)$ to $K^\times$. This proves the first statement of the Proposition.

Finally, since N is a homomorphism on $\Gamma_V(K)$, the action on V preserves the quadratic form $N|_V$. Thus, we have a homomorphism $\Gamma_V(K) \to O_V(K)$. □

Now let's analyze the homomorphism $\Gamma_V(K) \to O_V(K)$. Lemma 23.4 says exactly that the kernel is $K^\times$. Next we will show that the image is all of $O_V(K)$. Say $r \in V$ and $N(r) \neq 0$.

$$\begin{aligned} rv\alpha(r)^{-1} &= -rv\frac{r}{N(r)} = v - \frac{vr^2 + rvr}{N(r)} \\ &= v - \frac{(v,r)}{N(r)}r \qquad (23.6) \\ &= \begin{cases} -r & \text{if } v = r \\ v & \text{if } (v,r) = 0 \end{cases} \qquad (23.7) \end{aligned}$$

Thus, r is in $\Gamma_V(K)$, and it acts on V by reflection through the hyperplane $r^\perp$. One might deduce that the homomorphism $\Gamma_V(K) \to O_V(K)$ is surjective because $O_V(K)$ is generated by reflections. This is wrong; $O_V(K)$ is *not* always generated by reflections!

▶ **Exercise 23.1.** Let $H = \mathbb{F}_2^2$, with the quadratic form $x^2 + y^2 + xy$, and let $V = H \oplus H$. Prove that $O_V(\mathbb{F}_2)$ is not generated by reflections.

Remark 23.8. It turns out that this is the *only* counterexample. For any other vector space and/or any other non-degenerate quadratic form, $O_V(K)$ is generated by reflections. The map $\Gamma_V(K) \to O_V(K)$ is surjective even in the example above. Also, in every case except the example above, $\Gamma_V(K)$ is generated as a group by non-zero elements of V (i.e. every element of $\Gamma_V(K)$ is a monomial).

Remark 23.9. Equation 23.6 is the definition of the reflection of v through r. It is only possible to reflect through vectors of non-zero norm. Reflections in characteristic 2 are strange; strange enough that people don't call them reflections, they call them *transvections*.

Thus, we have the diagram

$$\begin{tikzcd}
 & 1 \arrow[r] & K^\times \arrow[r] \arrow[d, phantom, "\|"] & \Gamma_V(K) \arrow[r] \arrow[d, "N"] & O_V(K) \arrow[r] \arrow[d, dotted, "N"] & 1 \\
1 \arrow[r] & \pm 1 \arrow[r] & K^\times \arrow[r, "x\mapsto x^2"] & K^\times \arrow[r] & K^\times/(K^\times)^2 \arrow[r] & 1
\end{tikzcd} \tag{23.10}$$

where the rows are exact, $K^\times$ is in the center of $\Gamma_V(K)$ (this is obvious, since $K^\times$ is in the center of $C_V(K)$), and $N : O_V(K) \to K^\times/(K^\times)^2$ is the unique homomorphism sending reflection through $r^\perp$ to $N(r)$ modulo $(K^\times)^2$.

Definition 23.11. $\mathrm{Pin}_V(K) = \{x \in \Gamma_V(K) | N(x) = 1\}$, and $\mathrm{Spin}_V(K) = \mathrm{Pin}_V^0(K)$, the even elements of $\mathrm{Pin}_V(K)$.

On $K^\times$, the spinor norm is given by $x \mapsto x^2$, so the elements of spinor norm 1 are $= \pm 1$. By restricting the top row of (23.10) to elements of norm 1 and even elements of norm 1, respectively, we get exact sequences

$$\begin{tikzcd}
1 \arrow[r] & \pm 1 \arrow[r] & \mathrm{Pin}_V(K) \arrow[r] & O_V(K) \arrow[r, dotted, "N"] & K^\times/(K^\times)^2
\end{tikzcd}$$

$$\begin{tikzcd}
1 \arrow[r] & \pm 1 \arrow[r] & \mathrm{Spin}_V(K) \arrow[r] & SO_V(K) \arrow[r, dotted, "N"] & K^\times/(K^\times)^2
\end{tikzcd}$$

To see exactness of the top sequence, note that the kernel of ϕ is $K^\times \cap \mathrm{Pin}_V(K) = \pm 1$, and that the image of $\mathrm{Pin}_V(K)$ in $O_V(K)$ is exactly the elements of norm 1. The bottom sequence is similar, except that the image of $\mathrm{Spin}_V(K)$ is not all of $O_V(K)$, it is only $SO_V(K)$; by Remark 23.8, every element of $\Gamma_V(K)$ is a product of elements of V, so every element of $\mathrm{Spin}_V(K)$ is a product of an even number of elements of V. Thus, its image is a product of an even number of reflections, so it is in $SO_V(K)$.

??

These maps are NOT always onto, but there are many important cases when they are, like when V has a positive definite quadratic form. The image is the set of elements of $O_V(K)$ or $SO_V(K)$ which have spinor norm 1 in $K^\times/(K^\times)^2$.

What is $N : O_V(K) \to K^\times/(K^\times)^2$? It is the UNIQUE homomorphism such that $N(a) = N(r)$ if a is reflection in $r^\perp$, and r is a vector of norm $N(r)$.

Example 23.12. Take V to be a positive definite vector space over $\mathbb{R}$. Then N maps to 1 in $\mathbb{R}^\times/(\mathbb{R}^\times)^2 = \pm 1$ (because N is positive definite). So the spinor norm on $O_V(\mathbb{R})$ is TRIVIAL.

So if V is positive definite, we get double covers

$$1 \to \pm 1 \to \mathrm{Pin}_V(\mathbb{R}) \to O_V(\mathbb{R}) \to 1$$

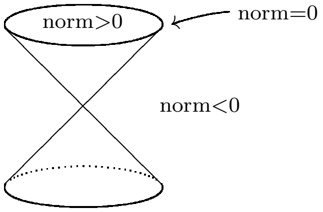
$$1 \to \pm 1 \to \mathrm{Spin}_V(\mathbb{R}) \to SO_V(\mathbb{R}) \to 1$$

This will account for the weird double covers we saw before.

What if V is negative definite. Every reflection now has image -1 in $\mathbb{R}^\times/(\mathbb{R}^\times)^2$, so the spinor norm N is the same as the determinant map $O_V(\mathbb{R}) \to \pm 1$.

So in order to find interesting examples of the spinor norm, you have to look at cases that are neither positive definite nor negative definite.

Let's look at Losrentz space: $\mathbb{R}^{1,3}$.



Reflection through a vector of norm < 0 (spacelike vector, P: parity reversal) has spinor norm -1, det -1 and reflection through a vector of norm > 0 (timelike vector, T: time reversal) has spinor norm $+1$, det -1. So $O_{1,3}(\mathbb{R})$ has 4 components (it is not hard to check that these are all the components), usually called 1, P, T, and PT.

Remark 23.13. For those who know Galois cohomology. We get an exact sequence of algebraic groups

$$1 \to GL_1 \to \Gamma_V \to O_V \to 1$$

(algebraic group means you don't put a field). You do not necessarily get an exact sequence when you put in a field.

If

$$1 \to A \to B \to C \to 1$$

is exact,

$$1 \to A(K) \to B(K) \to C(K)$$

is exact. What you really get is

$$\begin{aligned} 1 &\to H^0(\mathrm{Gal}(\bar{K}/K), A) \to H^0(\mathrm{Gal}(\bar{K}/K), B) \to H^0(\mathrm{Gal}(\bar{K}/K), C) \to \\ &\to H^1(\mathrm{Gal}(\bar{K}/K), A) \to \cdots \end{aligned}$$

It turns out that $H^1(Gal(\bar{K}/K), GL_1) = 1$. However, $H^1(Gal(\bar{K}/K), \pm 1) = K^\times/(K^\times)^2$.

So from

$$1 \to GL_1 \to \Gamma_V \to O_V \to 1$$

you get

$$1 \to K^\times \to \Gamma_V(K) \to O_V(K) \to 1 = H^1(Gal(\bar{K}/K), GL_1)$$

However, taking

$$1 \to \mu_2 \to \mathrm{Spin}_V \to SO_V \to 1$$

you get

$$1 \to \pm 1 \to \mathrm{Spin}_V(K) \to SO_V(K) \xrightarrow{N} K^\times/(K^\times)^2 = H^1(\bar{K}/K, \mu_2)$$

so the non-surjectivity of N is some kind of higher Galois cohomology.

Warning 23.14. $\mathrm{Spin}_V \to SO_V$ is onto as a map of ALGEBRAIC GROUPS, but $\mathrm{Spin}_V(K) \to SO_V(K)$ need NOT be onto.

Example 23.15. Take $O_3(\mathbb{R}) \cong SO_3(\mathbb{R}) \times \{\pm 1\}$ as 3 is odd (in general $O_{2n+1}(\mathbb{R}) \cong SO_{2n+1}(\mathbb{R}) \times \{\pm 1\}$). So we have a sequence

$$1 \to \pm 1 \to \mathrm{Spin}_3(\mathbb{R}) \to SO_3(\mathbb{R}) \to 1.$$

Notice that $\mathrm{Spin}_3(\mathbb{R}) \subseteq C_3^0(\mathbb{R}) \cong \mathbb{H}$, so $\mathrm{Spin}_3(\mathbb{R}) \subseteq \mathbb{H}^\times$, and in fact we saw that it is S^3.

Lecture 24

Last time we constructed the sequences

$$1 \to K^\times \to \Gamma_V(K) \to O_V(K) \to 1$$

$$1 \to \pm 1 \to \mathrm{Pin}_V(K) \to O_V(K) \xrightarrow{N} K^\times/(K^\times)^2$$

$$1 \to \pm 1 \to \mathrm{Spin}_V(K) \to SO_V(K) \xrightarrow{N} K^\times/(K^\times)^2$$

Spin representations of Spin and Pin groups

Notice that $\mathrm{Pin}_V(K) \subseteq C_V(K)^\times$, so any module over $C_V(K)$ gives a representation of $\mathrm{Pin}_V(K)$. We already figured out that $C_V(K)$ are direct sums of matrix algebras over $\mathbb{R}, \mathbb{C}$, and $\mathbb{H}$.

What are the representations (modules) of complex Clifford algebras? Recall that $C_{2n}(\mathbb{C}) \cong \mathbb{M}_{2^n}(\mathbb{C})$, which has a representations of dimension 2^n, which is called the spin representation of $\mathrm{Pin}_V(K)$ and $C_{2n+1}(\mathbb{C}) \cong \mathbb{M}_{2^n}(\mathbb{C}) \times \mathbb{M}_{2^n}(\mathbb{C})$, which has 2 representations, called the spin representations of $\mathrm{Pin}_{2n+1}(K)$.

What happens if we restrict these to $\mathrm{Spin}_V(\mathbb{C}) \subseteq \mathrm{Pin}_V(\mathbb{C})$? To do that, we have to recall that $C^0_{2n}(\mathbb{C}) \cong \mathbb{M}_{2^{n-1}}(\mathbb{C}) \times \mathbb{M}_{2^{n-1}}(\mathbb{C})$ and $C^0_{2n+1}(\mathbb{C}) \cong \mathbb{M}_{2^n}(\mathbb{C})$. So in EVEN dimensions $\mathrm{Pin}_{2n}(\mathbb{C})$ has 1 spin representation of dimension 2^n splitting into 2 HALF SPIN representations of dimension 2^{n-1} and in ODD dimensions, $\mathrm{Pin}_{2n+1}(\mathbb{C})$ has 2 spin representations of dimension 2^n which become the same on restriction to $\mathrm{Spin}_V(\mathbb{C})$.

Now we'll give a second description of spin representations. We'll just do the even dimensional case (odd is similar). Say $\dim V = 2n$, and say we're over $\mathbb{C}$. Choose an orthonormal basis $\gamma_1, \dots, \gamma_{2n}$ for V, so that $\gamma_i^2 = 1$ and $\gamma_i\gamma_j = -\gamma_j\gamma_i$. Now look at the group G generated by $\gamma_1, \dots, \gamma_{2n}$, which is finite, with order 2^{1+2n} (you can write all its elements explicitly). You can see that representations of $C_V(\mathbb{C})$ correspond to representations of G, with -1 acting as -1 (as opposed to acting as 1). So another way to look at representations of the Clifford algebra, you can look at representations of G.

Let's look at the structure of G:

(1) The center is ± 1. This uses the fact that we are in even dimensions, lest $\gamma_1 \cdots \gamma_{2n}$ also be central.

(2) The conjugacy classes: 2 of size 1 (1 and -1), $2^{2n} - 1$ of size 2 ($\pm\gamma_{i_1} \cdots \gamma_{i_n}$), so we have a total of $2^{2n} + 1$ conjugacy classes, so we should have that many representations. G/center is abelian,

isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{2n}$, which gives us 2^{2n} representations of dimension 1, so there is only one more left to find! We can figure out its dimension by recalling that the sum of the squares of the dimensions of irreducible representations gives us the order of G, which is 2^{2n+1}. So $2^{2n} \times 1^1 + 1 \times d^2 = 2^{2n+1}$, where d is the dimension of the mystery representation. Thus, $d = \pm 2^n$, so $d = 2^n$. Thus, G, and therefore $C_V(\mathbb{C})$, has an irreducible representation of dimension 2^n (as we found earlier in another way).

Example 24.1. Consider $O_{2,1}(\mathbb{R})$. As before, $O_{2,1}(\mathbb{R}) \cong SO_{2,1}(\mathbb{R}) \times (\pm 1)$, and $SO_{2,1}(\mathbb{R})$ is not connected: it has two components, separated by the spinor norm N. We have maps

$$1 \to \pm 1 \to \mathrm{Spin}_{2,1}(\mathbb{R}) \to SO_{2,1}(\mathbb{R}) \xrightarrow{N} \pm 1.$$

$\mathrm{Spin}_{2,1}(\mathbb{R}) \subseteq C^*_{2,1}(\mathbb{R}) \cong \mathbb{M}_2(\mathbb{R})$, so $\mathrm{Spin}_{2,1}(\mathbb{R})$ has one 2 dimensional spin representation. So there is a map $\mathrm{Spin}_{2,1}(\mathbb{R}) \to SL_2(\mathbb{R})$; by counting dimensions and such, you can show it is an isomorphism. So $\mathrm{Spin}_{2,1}(\mathbb{R}) \cong SL_2(\mathbb{R})$.

Now let's look at some 4 dimensional orthogonal groups

Example 24.2. Look at $SO_4(\mathbb{R})$, which is compact. It has a complex spin representation of dimension $2^{4/2} = 4$, which splits into two half spin representations of dimension 2. We have the sequence

$$1 \to \pm 1 \to \mathrm{Spin}_4(\mathbb{R}) \to SO_4(\mathbb{R}) \to 1 \qquad (N = 1)$$

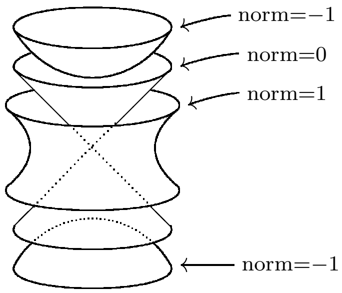
$\mathrm{Spin}_4(\mathbb{R})$ is also compact, so the image in any complex representation is contained in some unitary group. So we get two maps $\mathrm{Spin}_4(\mathbb{R}) \to SU(2) \times SU(2)$, and both sides have dimension 6 and centers of order 4. Thus, we find that $\mathrm{Spin}_4(\mathbb{R}) \cong SU(2) \times SU(2) \cong S^3 \times S^3$, which give you the two half spin representations.

So now we've done the positive definite case.

Example 24.3. Look at $SO_{3,1}(\mathbb{R})$. Notice that $O_{3,1}(\mathbb{R})$ has four components distinguished by the maps $\det, N \to \pm 1$. So we get

$$1 \to \pm 1 \to \mathrm{Spin}_{3,1}(\mathbb{R}) \to SO_{3,1}(\mathbb{R}) \xrightarrow{N} \pm 1 \to 1$$

We expect 2 half spin representations, which give us two homomorphisms $\mathrm{Spin}_{3,1}(\mathbb{R}) \to SL_2(\mathbb{C})$. This time, each of these homomorphisms is an isomorphism (I can't think of why right now). The $SL_2(\mathbb{C})$s are double covers of simple groups. Here, we don't get the splitting into a product as in the positive definite case. This isomorphism is heavily used in quantum field theory because $\mathrm{Spin}_{3,1}(\mathbb{R})$ is a double cover of the connected component of the Lorentz group (and $SL_2(\mathbb{C})$ is easy to work with). Note also that the center of $\mathrm{Spin}_{3,1}(\mathbb{R})$ has order 2, not 4, as for $\mathrm{Spin}_{4,0}(\mathbb{R})$. Also note that the group $PSL_2(\mathbb{C})$ acts on the compactified $\mathbb{C} \cup \{\infty\}$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(\tau) = \frac{a\tau+b}{c\tau+d}$. Subgroups of this group are called KLEINIAN groups. On the other hand, the group $SO_{3,1}(\mathbb{R})^+$ (identity component) acts on $\mathbb{H}^3$ (three dimensional hyperbolic space). To see this, look at



One sheet of norm -1 hyperboloid is isomorphic to $\mathbb{H}^3$ under the induced metric. In fact, we'll define hyperbolic space that way. If you're a topologist, you're very interested in hyperbolic 3-manifolds, which are $\mathbb{H}^3/(\text{discrete subgroup of } SO_{3,1}(\mathbb{R}))$. If you use the fact that $SO_{3,1}(\mathbb{R}) \cong PSL_2(\mathbb{R})$, then you see that these discrete subgroups are in fact Klienian groups.

There are lots of exceptional isomorphisms in small dimension, all of which are very interesting, and almost all of them can be explained by spin groups.

Example 24.4. $O_{2,2}(\mathbb{R})$ has 4 components (given by det, N); $C^0_{2,2}(\mathbb{R}) \cong \mathbb{M}_2(\mathbb{R}) \times \mathbb{M}_2(\mathbb{R})$, which induces an isomorphism $\mathrm{Spin}_{2,2}(\mathbb{R}) \to SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$, which give you the two half spin representations. Both sides have dimension 6 with centers of order 4. So this time we get two non-compact groups. Let's look at the fundamental group of $SL_2(\mathbb{R})$, which is $\mathbb{Z}$, so the fundamental group of $\mathrm{Spin}_{2,2}(\mathbb{R})$ is $\mathbb{Z} \oplus \mathbb{Z}$. As we recall, $\mathrm{Spin}_{4,0}(\mathbb{R})$ and $\mathrm{Spin}_{3,1}(\mathbb{R})$ were both simply connected. This shows that SPIN GROUPS NEED NOT BE SIMPLY CONNECTED. So we can take covers of it. What do the corresponding covers (e.g. the universal cover) of $\mathrm{Spin}_{2,2}(\mathbb{R})$ look like? This is hard to describe because for FINITE dimensional complex representations, you get finite dimensional representations of the Lie algebra L, which correspond to the finite dimensional representations of $L \otimes \mathbb{C}$, which correspond to the finite dimensional representations of $L' =$ Lie algebra of $\mathrm{Spin}_{4,0}(\mathbb{R})$, which correspond to the finite dimensional representations of $\mathrm{Spin}_{4,0}(\mathbb{R})$, which has no covers because it is simply connected. This means that any finite dimensional representation of a cover of $\mathrm{Spin}_{2,2}(\mathbb{R})$ actually factors through $\mathrm{Spin}_{2,2}(\mathbb{R})$. So there is no way you can talk about these things with finite matrices, and infinite dimensional representations are hard.

To summarize, the ALGEBRAIC GROUP $\mathrm{Spin}_{2,2}$ is simply connected (as an algebraic group) (think of an algebraic group as a functor from rings to groups), which means that it has no algebraic central extensions. However, the LIE GROUP $\mathrm{Spin}_{2,2}(\mathbb{R})$ is NOT simply connected; it has fundamental group $\mathbb{Z} \oplus \mathbb{Z}$. This problem does not happen for COMPACT Lie groups (where every finite cover is algebraic).

We've done $O_{4,0}, O_{3,1}$, and $O_{2,2}$, from which we can obviously get $O_{1,3}$ and $O_{0,4}$. Note that $O_{4,0}(\mathbb{R}) \cong O_{0,4}(\mathbb{R})$, $SO_{4,0}(\mathbb{R}) \cong SO_{0,4}(\mathbb{R})$, $\mathrm{Spin}_{4,0}(\mathbb{R}) \cong \mathrm{Spin}_{0,4}(\mathbb{R})$. However, $\mathrm{Pin}_{4,0}(\mathbb{R}) \not\cong \mathrm{Pin}_{0,4}(\mathbb{R})$. These two are hard to distinguish. We have

$$\begin{tikzcd}
\mathrm{Pin}_{4,0}(\mathbb{R}) \arrow[d] & \mathrm{Pin}_{0,4}(\mathbb{R}) \arrow[d] \\
O_{4,0}(\mathbb{R}) \arrow[r, phantom, "="] & O_{0,4}(\mathbb{R})
\end{tikzcd}$$

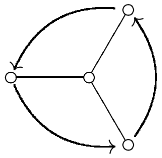
Take a reflection (of order 2) in $O_{4,0}(\mathbb{R})$, and lift it to the Pin groups. What is the order of the lift? The reflection vector v, with $v^2 = \pm 1$ lifts to the element $v \in \Gamma_V(\mathbb{R}) \subseteq C_V^*(\mathbb{R})$. Notice that $v^2 = 1$ in the case of $\mathbb{R}^{4,0}$ and $v^2 = -1$ in the case of $\mathbb{R}^{0,4}$, so in $\mathrm{Pin}_{4,0}(\mathbb{R})$, the reflection lifts to something of order 2, but in $\mathrm{Pin}_{0,4}(\mathbb{R})$, you get an element of order 4!. So these two groups are different.

Two groups are *isoclinic* if they are confusingly similar. A similar phenomenon is common for groups of the form $2 \cdot G \cdot 2$, which means it has a center of order 2, then some group G, and the abelianization has order 2. Watch out.

▶ **Exercise 24.1.** $\mathrm{Spin}_{3,3}(\mathbb{R}) \cong SL_4(\mathbb{R})$.

Triality

This is a special property of 8 dimensional orthogonal groups. Recall that $O_8(\mathbb{C})$ has the Dynkin diagram D_4, which has a symmetry of order three:



But $O_8(\mathbb{C})$ and $SO_8(\mathbb{C})$ do NOT have corresponding symmetries of order three. The thing that does have the symmetry of order three is the spin group! The group $\mathrm{Spin}_8(\mathbb{R})$ DOES have "extra" order three symmetry. You can see it as follows. Look at the half spin representations of $\mathrm{Spin}_8(\mathbb{R})$. Since this is a spin group in even dimension, there are two. $C_{8,0}(\mathbb{R}) \cong \mathbb{M}_{2^{8/2-1}}(\mathbb{R}) \times \mathbb{M}_{2^{8/2-1}}(\mathbb{R}) \cong \mathbb{M}_8(\mathbb{R}) \times \mathbb{M}_8(\mathbb{R})$. So $\mathrm{Spin}_8(\mathbb{R})$ has two 8 dimensional real half spin representations. But the spin group is compact, so it preserves some quadratic form, so you get 2 homomorphisms $\mathrm{Spin}_8(\mathbb{R}) \to SO_8(\mathbb{R})$. So $\mathrm{Spin}_8(\mathbb{R})$ has THREE 8 dimensional representations: the half spins, and the one from the map to $SO_8(\mathbb{R})$. These maps $\mathrm{Spin}_8(\mathbb{R}) \to SO_8(\mathbb{R})$ lift to Triality automorphisms $\mathrm{Spin}_8(\mathbb{R}) \to \mathrm{Spin}_8(\mathbb{R})$. The center of $\mathrm{Spin}_8(\mathbb{R})$ is $(\mathbb{Z}/2) + (\mathbb{Z}/2)$ because the center of the Clifford group is $\pm 1, \pm\gamma_1 \cdots \gamma_8$. There are 3 non-trivial elements of the center, and quotienting by any of these gives you something isomorphic to $SO_8(\mathbb{R})$. This is special to 8 dimensions.

More about Orthogonal groups

Is $O_V(K)$ a simple group? NO, for the following reasons:

(1) There is a determinant map $O_V(K) \to \pm 1$, which is usually onto, so it can't be simple.

(2) There is a spinor norm map $O_V(K) \to K^\times/(K^\times)^2$

(3) $-1 \in$ center of $O_V(K)$.

(4) $SO_V(K)$ tends to split if $\dim V = 4$, abelian if $\dim V = 2$, and trivial if $\dim V = 1$.

It turns out that they are usually simple apart from these four reasons why they're not. Let's mod out by the determinant, to get to SO, then look at $\mathrm{Spin}_V(K)$, then quotient by the center, and assume that $\dim V \geq 5$. Then this is usually simple. The center tends to have order 1,2, or 4. If K is a FINITE field, then this gives many finite simple groups.

Note that $SO_V(K)$ is NOT a subgroup of $O_V(K)$, elements of determinant 1 in general, it is the image of $\Gamma_V^0(K) \subseteq \Gamma_V(K) \to O_V(K)$, which is the correct definition. Let's look at why this is right and the definition you know is wrong. There is a homomorphism $\Gamma_V(K) \to \mathbb{Z}/2\mathbb{Z}$, which takes $\Gamma_V^0(K)$ to 0 and $\Gamma_V^1(K)$ to 1 (called the DICKSON INVARIANT). It is easy to check that $\det(v) = (-1)^{\text{dickson invariant}(v)}$. So if the characteristic of K is not 2, det $= 1$ is equivalent to dickson $= 0$, but in characteristic 2, determinant is the wrong invariant (because determinant is always 1).

Special properties of $O_{1,n}(\mathbb{R})$ and $O_{2,n}(\mathbb{R})$. $O_{1,n}(\mathbb{R})$ acts on hyperbolic space $\mathbb{H}^n$, which is a component of norm -1 vectors in $\mathbb{R}^{n,1}$. $O_{2,n}(\mathbb{R})$ acts on the "Hermitian symmetric space" (Hermitian means it has a complex structure, and symmetric means really nice). There are three ways to construct this space:

(1) It is the set of positive definite 2 dimensional subspaces of $\mathbb{R}^{2,n}$

(2) It is the norm 0 vectors ω of $\mathbb{PC}^{2,n}$ with $(\omega, \bar{\omega}) = 0$.

(3) It is the vectors $x + iy \in \mathbb{R}^{1,n-1}$ with $y \in C$, where the cone C is the interior of the norm 0 cone.

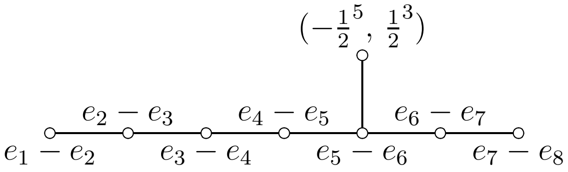
► **Exercise 24.2.** Show that these are the same.

Next week, we'll mess around with E_8.

Lecture 25 - E_8

In this lecture we use a vector notation in which powers represent repetitions: so $(1^8) = (1,1,1,1,1,1,1,1)$ and $(\pm\frac{1}{2}^2, 0^6) = (\pm\frac{1}{2}, \pm\frac{1}{2}, 0,0,0,0,0,0)$.

Recall that E_8 has the Dynkin diagram



where each vertex is a root r with $(r,r) = 2$; $(r,s) = 0$ when r and s are not joined, and $(r,s) = -1$ when r and s are joined. We choose an orthonormal basis $e_1, \ldots, e_8$, in which the roots are as given.

We want to figure out what the root lattice L of E_8 is (this is the lattice generated by the roots). If you take $\{e_i - e_{i+1}\} \cup (-1^5, 1^3)$ (all the A_7 vectors plus twice the strange vector), they generate the D_8 lattice $= \{(x_1, \ldots, x_8) | x_i \in \mathbb{Z}, \quad \sum x_i \text{ even}\}$. So the E_8 lattice consists of two cosets of this lattice, where the other coset is $\{(x_1, \ldots, x_8) | x_i \in \mathbb{Z} + \frac{1}{2}, \quad \sum x_i \text{ odd}\}$.

Alternative version: If you reflect this lattice through the hyperplane $e_1^\perp$, then you get the same thing except that $\sum x_i$ is always even. We will freely use both characterizations, depending on which is more convenient for the calculation at hand.

We should also work out the weight lattice, which is the vectors s such that $(r,r)/2$ divides (r,s) for all roots r. Notice that the weight lattice of E_8 is contained in the weight lattice of D_8, which is the union of four cosets of D_8: D_8, $D_8 + (1, 0^7)$, $D_8 + (\frac{1}{2}^8)$ and $D_8 + (-\frac{1}{2}, \frac{1}{2}^7)$. Which of these have integral inner product with the vector $(-\frac{1}{2}^5, \frac{1}{2}^3)$? They are the first and the last, so the weight lattice of E_8 is $D_8 \cup D_8 + (-\frac{1}{2}, \frac{1}{2}^7)$, which is equal to the root lattice of E_8.

In other words, the E_8 lattice L is UNIMODULAR (equal to its dual L'), where the dual is the lattice of vectors having integral inner product with all lattice vectors. This is also true of G_2 and F_4, but is not in general true of Lie algebra lattices.

The E_8 lattice is EVEN, which means that the inner product of any vector with itself is always even.

Even unimodular lattices in $\mathbb{R}^n$ only exist if $8|n$ (this 8 is the same 8 that shows up in the periodicity of Clifford groups). The E_8 lattice is the only example in dimension equal to 8 (up to isomorphism, of course). There are two in dimension 16 (one of which is $L \oplus L$, the other is $D_{16} \cup$ some coset). There are 24 in dimension 24, which are the Niemeier lattices. In 32 dimensions, there are more than a billion!

The Weyl group of E_8 is generated by the reflections through $s^\perp$ where $s \in L$ and $(s,s) = 2$ (these are called roots). First, let's find all the roots: $(x_1, \ldots, x_8)$ such that $\sum x_i^2 = 2$ with $x_i \in \mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$ and $\sum x_i$ even. If $x_i \in \mathbb{Z}$, obviously the only solutions are permutations of $(\pm 1, \pm 1, 0^6)$, of which there are $\binom{8}{2} \times 2^2 = 112$ choices. In the $\mathbb{Z} + \frac{1}{2}$ case, you can choose the first 7 places to be $\pm\frac{1}{2}$, and the last coordinate is forced, so there are 2^7 choices. Thus, you get 240 roots.

Let's find the orbits of the roots under the action of the Weyl group. We don't yet know what the Weyl group looks like, but we can find a large subgroup that is easy to work with. Let's use the Weyl group of D_8, which consists of the following: we can apply all permutations of the coordinates, or we can change the sign of an even number of coordinates (e.g., reflection in $(1, -1, 0^6)$ swaps the first two coordinates, and reflection in $(1,\ -1,\ 0^6)$ followed by reflection in $(1, 1, 0^6)$ changes the sign of the first two coordinates.)

Notice that under the Weyl group of D_8, the roots form two orbits: the set which is all permutations of $(\pm 1^2, 0^6)$, and the set $(\pm\frac{1}{2}^8)$. Do these become the same orbit under the Weyl group of E_8? Yes; to show this, we just need one element of the Weyl group of E_8 taking some element of the first orbit to the second orbit. Take reflection in $(\frac{1}{2}^8)^\perp$ and apply it to $(1^2, 0^6)$: you get $(\frac{1}{2}^2, -\frac{1}{2}^6)$, which is in the second orbit. So there is just one orbit of roots under the Weyl group.

What do orbits of $W(E_8)$ on other vectors look like? We're interested in this because we might want to do representation theory. The character of a representation is a map from weights to integers, which is $W(E_8)$-invariant. Let's look at vectors of norm 4 for example. So $\sum x_i^2 = 4$, $\sum x_i$ even, and $x_i \in \mathbb{Z}$ or $x_i \in \mathbb{Z} + \frac{1}{2}$. There are 8×2 possibilities which are permutations of $(\pm 2, 0^7)$. There are $\binom{8}{4} \times 2^4$ permutations of $(\pm 1^4, 0^4)$, and there are 8×2^7 permutations of $(\pm\frac{3}{2}, \pm\frac{1}{2}^7)$. So there are a total of 240×9 of these vectors. There are 3 orbits under $W(D_8)$, and as before, they are all one orbit under the action of $W(E_8)$. Just reflect $(2,\ 0^7)$ and $(1^3, -1, 0^4)$ through $(\frac{1}{2}^8)$.

► **Exercise 25.1.** Show that the number of norm 6 vectors is 240×28, and they form one orbit

(If you've seen a course on modular forms, you'll know that the number of vectors of norm $2n$ is given by $240 \times \sum_{d|n} d^3$. If you let call these c_n, then $\sum c_n q^n$ is a modular form of level 1 (E_8 even, unimodular), weight 4 ($\dim E_8/2$).)

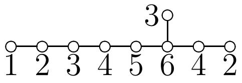
For norm 8 there are two orbits, because you have vectors that are twice a norm 2 vector, and vectors that aren't. As the norm gets bigger, you'll get a large number of orbits.

What is the order of the Weyl group of E_8? We'll do this by 4 different methods, which illustrate the different techniques for this kind of thing:

(1) This is a good one as a mnemonic. The order of E_8 is given by

$$|W(E_8)| = 8! \times \prod \begin{pmatrix} \text{numbers on the} \\ \text{affine } E_8 \text{ diagram[1]} \end{pmatrix} \times \frac{\text{Weight lattice of } E_8}{\text{Root lattice of } E_8}$$

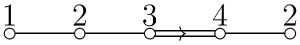
$$= 8! \times \Bigg(\qquad \Bigg) \times 1$$



$$= 2^{14} \times 3^5 \times 5^2 \times 7$$

We can do the same thing for any other Lie algebra, for example,

$$|W(F_4)| = 4! \times (\qquad) \times 1$$



$$= 2^7 \times 3^2$$

(2) The order of a reflection group is equal to the products of degrees of the fundamental invariants. For E_8, the fundamental invariants are of degrees 2,8,12,14,18,20,24,30 (primes +1).

(3) This one is actually an honest method (without quoting weird facts). The only fact we will use is the following: suppose G acts transitively on a set X with $H =$ the group fixing some point; then $|G| = |H| \cdot |X|$.

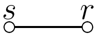
This is a general purpose method for working out the orders of groups. First, we need a set acted on by the Weyl group of E_8. Let's take the root vectors (vectors of norm 2). This set has 240 elements, and the Weyl group of E_8 acts transitively on it. So $|W(E_8)| = 240 \times |\text{subgroup fixing } (1, -1, 0^6)|$. But what is the order of this subgroup (call it G_1)? Let's find a set acted on by this group. It acts on the set of norm 2 vectors, but the action is NOT transitive. What are the orbits? G_1 fixes $s = (1, -1, 0^6)$. For other roots r, G_1 obviously fixes (r, s). So how many roots are there with a given inner product with s?

(s, r)	number	choices
2	1	s
1	56	$(1, 0, \pm 1^6), (0, -1, \pm 1^6), (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}^6)$
0	126	
−1	56	
−2	1	$-s$

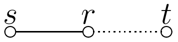
---

[1] These are the numbers giving highest root.

So there are at least 5 orbits under G_1. In fact, each of these sets is a single orbit under G_1. How can we see this? Find a large subgroup of G_1. Take $W(D_6)$, which is all permutations of the last 6 coordinates and all even sign changes of the last 6 coordinates. It is generated by reflections associated to the roots orthogonal to e_1 and e_2 (those that start with two 0s). The three cases with inner product 1 are three orbits under $W(D_6)$. To see that there is a single orbit under G_1, we just need some reflections that mess up these orbits. If you take a vector $(\frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2}^6)$ and reflect norm 2 vectors through it, you will get exactly 5 orbits. So G_1 acts transitively on these orbits.

We'll use the orbit of vectors r with $(r, s) = -1$. Let G_2 be the vectors fixing s and r:  We have that $|G_1| = |G_2| \cdot 56$.

Keep going ... it gets tedious, but here are the answers up to the last step:

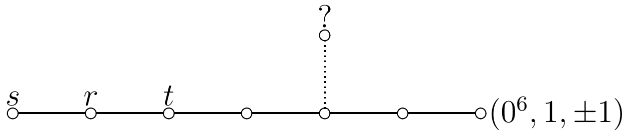
Our plan is to chose vectors acted on by G_i, fixed by G_{i+1} which give us the Dynkin diagram of E_8. So the next step is to try to find vectors t that give us the picture , i.e, they have inner product -1 with r and 0 with s. The possibilities for t are $(-1, -1, 0, 0^5)$ (one of these), $(0, 0, 1, \pm1, 0^4)$ and permutations of its last five coordinates (10 of these), and $(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2}^5)$ (there are 16 of these), so we get 27 total. Then we could check that they form one orbit, which is boring.

Next find vectors which go next to t in our picture:
, i.e., whose inner product is -1 with t and zero with r, s. The possibilities are permutations of the last four coords of $(0, 0, 0, 1, \pm1, 0^3)$ (8 of these) and $(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2}^4)$ (8 of these), so there are 16 total. Again check transitivity.

Find a fifth vector; the possibilities are $(0^4, 1, \pm1, 0^2)$ and perms of the last three coords (6 of these), and $(-\frac{1}{2}^4, \frac{1}{2}, \pm\frac{1}{2}^3)$ (4 of these) for a total of 10.

For the sixth vector, we can have $(0^5, 1, \pm1, 0)$ or $(0^5, 1, 0, \pm1)$ (4 possibilites) or $(-\frac{1}{2}^5, \frac{1}{2}, \pm\frac{1}{2}^2)$ (2 possibilities), so we get 6 total.

NEXT CASE IS TRICKY: finding the seventh one, the possibilities are $(0^6, 1, \pm1)$ (2 of these) and $((-\frac{1}{2})^6, \frac{1}{2}, \frac{1}{2})$ (just 1). The proof of transitivity fails at this point. The group we're using by now doesn't even act transitively on the pair (you can't get between them by changing an even number of signs). What elements of $W(E_8)$ fix all of these first 6 points ? We want to find roots perpendicular to all of these vectors, and the only possibility is $((\frac{1}{2})^8)$. How does reflection in this root act on the three vectors above? $(0^6, 1^2) \mapsto ((-\frac{1}{2})^6, \frac{1}{2}^2)$ and $(0^6, 1, -1)$ maps to itself. Is this last vector in the same orbit? In fact they are in different orbits. To see this, look for vectors



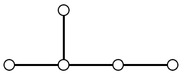
completing the E_8 diagram. In the $(0^6, 1, 1)$ case, you can take the vector $((-\frac{1}{2})^5, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$. But in the other case, you can show that there are no possibilities. So these really are different orbits.

Use the orbit with 2 elements, and you get

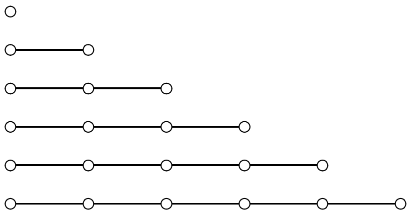
$$|W(E_8)| = 240 \times \underbrace{56 \times \overbrace{27 \times 16 \times 10 \times 6 \times 2 \times 1}^{\text{order of } W(E_6)}}_{\text{order of } W(E_7)}$$

because the group fixing all 8 vectors must be trivial. You also get that

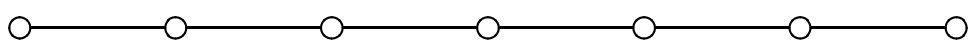
$$|W(\text{“}E_5\text{”})| = 16 \times \underbrace{10 \times \overbrace{6 \times 2 \times 1}^{|W(A_2 \times A_1)|}}_{|W(A_4)|}$$

where “E_5” is the algebra with diagram  (that is, D_5). Similarly, E_4 is A_4 and E_3 is $A_2 \times A_1$.

We got some other information. We found that the Weyl group of E_8 acts transitively on all the configurations



but not on



(4) We'll slip this in to next lecture

Also, next time we'll construct the Lie algebra E_8.

Lecture 26

Today we'll finish looking at $W(E_8)$, then we'll construct E_8.

Remember that we still have a fourth method of finding the order of $W(E_8)$. Let L be the E_8 lattice. Look at $L/2L$, which has 256 elements. Look at this as a set acted on by $W(E_8)$. There is an orbit of size 1 (represented by 0). There is an orbit of size $240/2 = 120$, which are the roots (a root is congruent mod $2L$ to it's negative). Left over are 135 elements. Let's look at norm 4 vectors. Each norm 4 vector, r, satisfies $r \equiv -r \mod 2$, and there are $240 \cdot 9$ of them, which is a lot, so norm 4 vectors must be congruent to a bunch of stuff. Let's look at $r = (2, 0, 0, 0, 0, 0, 0, 0)$. Notice that it is congruent to vectors of the form $(0 \cdots \pm 2 \dots 0)$, of which there are 16. It is easy to check that these are the only norm 4 vectors congruent to r mod 2. So we can partition the norm 4 vectors into $240 \cdot 9/16 = 135$ subsets of 16 elements. So $L/2L$ has 1+120+135 elements, where 1 is the zero, 120 is represented by 2 elements of norm 2, and 135 is represented by 16 elements of norm 4. A set of 16 elements of norm 4 which are all congruent is called a FRAME. It consists of elements $\pm e_1, \dots, \pm e_8$, where $e_i^2 = 4$ and $(e_i, e_j) = 1$ for $i \neq j$, so up to sign it is an orthogonal basis.

Then we have

$$|W(E_8)| = (\#\text{ frames}) \times |\text{subgroup fixing a frame}|$$

because we know that $W(E_8)$ acts transitively on frames. So we need to know what the automorphisms of an orthogonal base are. A frame is 8 subsets of the form $(r, -r)$, and isometries of a frame form the group $(\mathbb{Z}/2\mathbb{Z})^8 \cdot S_8$, but these are not all in the Weyl group. In the Weyl group, we found a $(\mathbb{Z}/2\mathbb{Z})^7 \cdot S_8$, where the first part is the group of sign changes of an EVEN number of coordinates. So the subgroup fixing a frame must be in between these two groups, and since these groups differ by a factor of 2, it must be one of them. Observe that changing an odd number of signs doesn't preserve the E_8 lattice, so it must be the group $(\mathbb{Z}/2\mathbb{Z})^7 \cdot S_8$, which has order $2^7 \cdot 8!$. So the order of the Weyl group is

$$135 \cdot 2^7 \cdot 8! = |2^7 \cdot S_8| \times \frac{\#\text{ norm 4 elements}}{2 \times \dim L}$$

Remark 26.1. Similarly, if Λ is the Leech lattice, you actually get the order of Conway's group to be

$$|2^{12} \cdot M_{24}| \cdot \frac{\#\text{ norm 8 elements}}{2 \times \dim \Lambda}$$

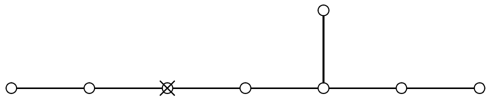
where M_{24} is the Mathieu group (one of the sporadic simple groups). The Leech lattice seems very much to be trying to be the root lattice of the monster group, or something like that. There are a lot of analogies, but nobody can make sense of it.

$W(E_8)$ acts on $(\mathbb{Z}/2\mathbb{Z})^8$, which is a vector space over $\mathbb{F}_2$, with quadratic form $N(a) = \frac{(a,a)}{2} \mod 2$, so you get a map

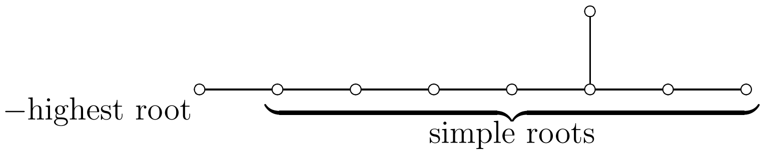
$$\pm 1 \to W(E_8) \to O_8^+(\mathbb{F}_2)$$

which has kernel ±1 and is surjective. O_8^+ is one of the 8 dimensional orthogonal groups over $\mathbb{F}_2$. So the Weyl group is very close to being an orthogonal group of a vector space over $\mathbb{F}_2$.

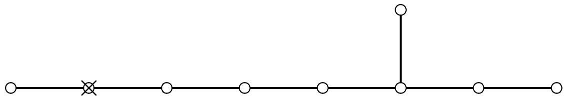
What is inside the root lattice/Lie algebra/Lie group E_8? One obvious way to find things inside is to cover nodes of the E_8 diagram:



If we remove the shown node, you see that E_8 contains $A_2 \times D_5$. We can do better by showing that we can embed the affine $\tilde{E}_8$ in the E_8 lattice.

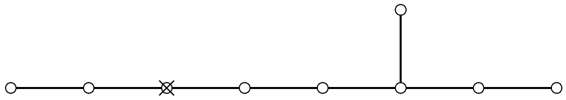


Now you can remove nodes here and get some bigger sub-diagrams. For example, if we cover



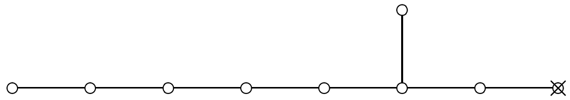
you get that an $A_1 \times E_7$ in E_8. The E_7 consisted of 126 roots orthogonal to a given root. This gives an easy construction of E_7 root system, as all the elements of the E_8 lattice perpendicular to $(1, -1, 0 \dots)$

We can cover



Then we get an $A_2 \times E_6$, where the E_6 are all the vectors with the first 3 coordinates equal. So we get the E_6 lattice for free too.

If you cover



you see that there is a D_8 in E_8, which is all vectors of the E_8 lattice with integer coordinates. We sort of constructed the E_8 lattice this way in the first place.

We can ask questions like: What is the E_8 Lie algebra as a representation of D_8? To answer this, we look at the weights of the E_8 algebra, considered as a module over D_8, which are the 112 roots of the form $(\cdots \pm 1 \cdots \pm 1 \dots)$ and the 128 roots of the form $(\pm 1/2, \dots)$ and 1 vector 0, with multiplicity 8. These give you the Lie algebra of D_8. Recall that D_8 is the Lie algebra of SO_{16}. The double cover has a half spin representation of dimension $2^{16/2-1} = 128$. So E_8 decomposes as a representation of D_8 as the adjoint representation (of dimension 120) plus a half spin representation of dimension 128. This is often used to construct the Lie algebra E_8. We'll do a better construction in a little while.

We've found that the Lie algebra of D_8, which is the Lie algebra of SO_{16}, is contained in the Lie algebra of E_8. Which *group* is contained in the the compact form of the E_8? We found that there were groups

$$\begin{tikzcd}
& \mathrm{Spin}_{16}(\mathbb{R}) \arrow[dl, dash] \arrow[d, dash] \arrow[drr, dash] & & \\
SO_{16}(\mathbb{R}) \arrow[dr, dash] & \mathrm{Spin}_{16}(\mathbb{R})/(\mathbb{Z}/2\mathbb{Z}) \arrow[d, dash] & \cong & \boxed{\mathrm{Spin}_{16}(\mathbb{R})/(\mathbb{Z}/2\mathbb{Z})} \arrow[dll, dash] \\
& PSO_{16}(\mathbb{R}) & &
\end{tikzcd}$$

corresponding to subgroups of the center $(\mathbb{Z}/2\mathbb{Z})^2$:

$$\begin{tikzcd}
& 1 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathbb{Z}/2\mathbb{Z} \arrow[dr, dash] & \mathbb{Z}/2\mathbb{Z} \arrow[d, dash] & \mathbb{Z}/2\mathbb{Z} \arrow[dl, dash] \\
& (\mathbb{Z}/2\mathbb{Z})^2 &
\end{tikzcd}$$

We have a homomorphism $\mathrm{Spin}_{16}(\mathbb{R}) \to E_8(\text{compact})$. What is the kernel? The kernel are elements which act trivially on the Lie algebra of E_8, which is equal to the Lie algebra D_8 plus the half spin representation. On the Lie algebra of D_8, everything in the center is trivial, and on the half spin representation, one of the elements of order 2 is trivial. So the subgroup that you get is the circled one.

▶ **Exercise 26.1.** Show $SU(2) \times E_7(\text{compact})/(-1,-1)$ is a subgroup of E_8 (compact). Similarly, show that $SU(9)/(\mathbb{Z}/3\mathbb{Z})$ is also. These are similar to the example above.

Construction of E_8

Earlier in the course, we had some constructions:

1. using the Serre relations, but you don't really have an idea of what it looks like
2. Take D_8 plus a half spin representation

Today, we'll try to find a natural map from root lattices to Lie algebras. The idea is as follows: Take a basis element e^α (as a formal symbol) for each root α; then take the Lie algebra to be the direct sum of 1 dimensional spaces generated by each e^α and L (L root lattice $\cong$ Cartan subalgebra) . Then we have to define the Lie bracket by setting $[e^\alpha, e^\beta] = e^{\alpha+\beta}$, but then we have a sign problem because $[e^\alpha, e^\beta] \neq -[e^\beta, e^\alpha]$. Is there some way to resolve the sign problem? The answer is that there is no good way to solve this problem (not true, but whatever). Suppose we had a nice functor from root lattices to Lie algebras. Then we would get that the automorphism group of the lattice has to be contained in the automorphism group of the Lie algebra (which is contained in the Lie group), and the automorphism group of the Lattice contains the Weyl group of the lattice. But the Weyl group is NOT usually a subgroup of the Lie group.

We can see this going wrong even in the case of $\mathfrak{sl}_2(\mathbb{R})$. Remember that the Weyl group is $N(T)/T$ where $T = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1}\end{smallmatrix}\right)$ and $N(T) = T \cup \left(\begin{smallmatrix} 0 & b \\ -b^{-1} & 0\end{smallmatrix}\right)$, and this second part is stuff having order 4, so you cannot possibly write this as a semi-direct product of T and the Weyl group.

So the Weyl group is not usually a subgroup of $N(T)$. The best we can do is to find a group of the form $2^n \cdot W \subseteq N(T)$ where n is the rank. For example, let's do it for $SL(n+1, \mathbb{R})$ Then $T = diag(a_1, \dots, a_n)$ with $a_1 \cdots a_n = 1$. Then we take the normalizer of the torus to be $N(T)$ =all permutation matrices with ± 1's with determinant 1, so this is $2^n \cdot S_n$, and it does not split. The problem we had with signs can be traced back to the fact that this group doesn't split.

We can construct the Lie algebra from something acted on by $2^n \cdot W$ (but not from something acted on by W). We take a CENTRAL EXTENSION of the lattice by a group of order 2. Notation is a pain because the lattice is written additively and the extension is nonabelian, so you want it to be written multiplicatively. Write elements of the lattice in the form e^α formally, so we have converted the lattice operation to multiplication. We will use the central extension

$$1 \to \pm 1 \to \hat{e}^L \to \underbrace{e^L}_{\cong L} \to 1$$

We want $\hat{e}^L$ to have the property that $\hat{e}^\alpha \hat{e}^\beta = (-1)^{(\alpha,\beta)} \hat{e}^\beta \hat{e}^\alpha$, where $\hat{e}^\alpha$ is something mapping to e^α. What do the automorphisms of $\hat{e}^L$ look like?

We get

$$1 \to \underbrace{(L/2L)}_{(\mathbb{Z}/2)^{\mathrm{rank}(L)}} \to \mathrm{Aut}(\hat{e}^L) \to \mathrm{Aut}(e^L)$$

for $\alpha \in L/2L$, we get the map $\hat{e}^\beta \to (-1)^{(\alpha,\beta)}\hat{e}^\beta$. The map turns out to be onto, and the group $\mathrm{Aut}(e^L)$ contains the reflection group of the lattice. This extension is usually non-split.

Now the Lie algebra is $L \oplus \{$1 dimensional spaces spanned by $(\hat{e}^\alpha, -\hat{e}^\alpha)\}$ for $\alpha^2 = 2$ with the convention that $-\hat{e}^\alpha$ (-1 in the vector space) is $-\hat{e}^\alpha$ (-1 in the group $\hat{e}^L$). Now define a Lie bracket by the "obvious rules" $[\alpha, \beta] = 0$ for $\alpha, \beta \in L$ (the Cartan subalgebra is abelian), $[\alpha, \hat{e}^\beta] = (\alpha, \beta)\hat{e}^\beta$ ($\hat{e}^\beta$ is in the root space of β), and $[\hat{e}^\alpha, \hat{e}^\beta] = 0$ if $(\alpha, \beta) \geq 0$ (since $(\alpha+\beta)^2 > 2$), $[\hat{e}^\alpha, \hat{e}^\beta] = \hat{e}^\alpha\hat{e}^\beta$ if $(\alpha, \beta) < 0$ (product in the group $\hat{e}^L$), and $[\hat{e}^\alpha, (\hat{e}^\alpha)^{-1}] = \alpha$.

Theorem 26.2. *Assume L is positive definite. Then this Lie bracket forms a Lie algebra (so it is skew and satisfies Jacobi).*

Proof. Easy but tiresome, because there are a lot of cases; let's do them (or most of them).

We check the Jacobi identity: We want $[[a,b],c]+[[b,c],a]+[[c,a],b] = 0$

1. all of a, b, c in L. Trivial because all brackets are zero.
2. two of a, b, c in L. Say α, β, e^γ

   $$\underbrace{[[\alpha,\beta],e^\gamma]}_{0} + \underbrace{[[\beta,e^\gamma],\alpha]}_{(\beta,\alpha)(-\alpha,\beta)e^\gamma} + [[e^\gamma,\alpha],\beta]$$

   and similar for the third term, giving a sum of 0.
3. one of a, b, c in L. $\alpha, e^\beta, e^\gamma$. e^β has weight β and e^γ has weight γ and $e^\beta e^\gamma$ has weight $\beta+\gamma$. So check the cases, and you get Jacobi:

   $$\begin{aligned}
   [[\alpha, e^\beta], e^\gamma] &= (\alpha,\beta)[e^\beta, e^\gamma] \\
   [[e^\beta, e^\gamma], \alpha] &= -[\alpha, [e^\beta, e^\gamma]] = -(\alpha, \beta+\gamma)[e^\beta, e^\gamma] \\
   [[e^\gamma, \alpha], e^\beta] &= -[[\alpha, e^\gamma], e^\beta] = (\alpha,\gamma)[e^\beta, e^\gamma],
   \end{aligned}$$

   so the sum is zero.
4. none of a, b, c in L. This is the really tiresome one, $e^\alpha, e^\beta, e^\gamma$. The main point of going through this is to show that it isn't as tiresome as you might think. You can reduce it to two or three cases. Let's make our cases depending on (α,β), (α,γ), (β,γ).

(a) if 2 of these are 0, then all the $[[*,*],*]$ are zero.

(b) $\alpha = -\beta$. By case a, γ cannot be orthogonal to them, so say $(\alpha,\gamma) = 1$ $(\gamma,\beta) = -1$; adjust so that $e^\alpha e^\beta = 1$, then calculate

$$\begin{aligned}[[e^\gamma, e^\beta], e^\alpha] - [[e^\alpha, e^\beta], e^\gamma] + [[e^\alpha, e^\gamma], e^\beta] &= e^\alpha e^\beta e^\gamma - (\alpha,\gamma)e^\gamma + 0\\ &= e^\gamma - e^\gamma = 0.\end{aligned}$$

(c) $\alpha = -\beta = \gamma$, easy because $[e^\alpha, e^\gamma] = 0$ and $[[e^\alpha, e^\beta], e^\gamma] = -[[e^\gamma, e^\beta], e^\alpha]$

(d) We have that each of the inner products is 1, 0 or -1. If some $(\alpha,\beta) = 1$, all brackets are 0.

This leaves two cases, which we'll do next time

$\square$

Lecture 27

Last week we talked about $\hat{e}^L$, which was a double cover of e^L. L is the root lattice of E_8. We had the sequence

$$1 \to \pm 1 \to \hat{e}^L \to e^L \to 1.$$

The Lie algebra structure on $\hat{e}^L$ was given by

$$\begin{aligned}
[\alpha, \beta] &= 0 \\
[\alpha, e^\beta] &= (\alpha, \beta) e^\beta \\
[e^\alpha, e^\beta] &= \begin{cases} 0 & \text{if } (\alpha,\beta) \geq 0 \\ e^\alpha e^\beta & \text{if } (\alpha,\beta) = -1 \\ \alpha & \text{if } (\alpha,\beta) = -2 \end{cases}
\end{aligned}$$

The Lie algebra is $L \oplus \bigoplus_{\alpha^2=2} \hat{e}^\alpha$.

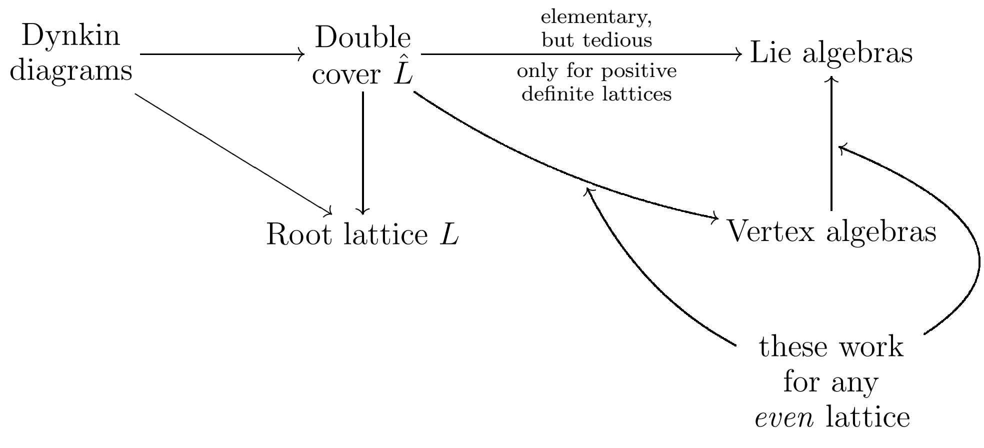
Let's finish checking the Jacobi identity. We had two cases left:

$$[[e^\alpha, e^\beta], e^\gamma] + [[e^\beta, e^\gamma], e^\alpha] + [[e^\gamma, e^\alpha], e^\beta] = 0$$

- $(\alpha, \beta) = (\beta, \gamma) = (\gamma, \alpha) = -1$, in which case $\alpha + \beta + \gamma = 0$. then $[[e^\alpha, e^\beta], e^\gamma] = [e^\alpha e^\beta, e^\gamma] = \alpha + \beta$. By symmetry, the other two terms are $\beta + \gamma$ and $\gamma + \alpha$;the sum of all three terms is $2(\alpha + \beta + \gamma) = 0$.
- $(\alpha, \beta) = (\beta, \gamma) = -1$, $(\alpha, \gamma) = 0$, in which case $[e^\alpha, e^\gamma] = 0$. We check that $[[e^\alpha, e^\beta], e^\alpha] = [e^\alpha e^\beta, e^\gamma] = e^\alpha e^\beta e^\gamma$ (since $(\alpha + \beta, \gamma) = -1$). Similarly, we have $[[e^\beta, e^\gamma], e^\alpha] = [e^\beta e^\gamma, e^\alpha] = e^\beta e^\gamma e^\alpha$. We notice that $e^\alpha e^\beta = -e^\beta e^\alpha$ and $e^\gamma e^\alpha = e^\alpha e^\gamma$ so $e^\alpha e^\beta e^\gamma = -e^\beta e^\gamma e^\alpha$; again, the sum of all three terms in the Jacobi identity is 0.

This concludes the verification of the Jacobi identity, so we have a Lie algebra.

Is there a proof avoiding case-by-case check? Good news: yes! Bad news: it's actually more work. We really have functors as follows:



where $\hat{L}$ is generated by $\hat{e}^{\alpha_i}$ (the i's are the dots in your Dynkin diagram), with $\hat{e}^{\alpha_i}\hat{e}^{\alpha_j} = (-1)^{(\alpha_i,\alpha_j)}\hat{e}^{\alpha_j}\hat{e}^{\alpha_i}$, and -1 is central of order 2.

Unfortunately, you have to spend several weeks learning vertex algebras. In fact, the construction we did was the vertex algebra approach, with all the vertex algebras removed. So there is a more general construction which gives a much larger class of infinite dimensional Lie algebras.

Now we should study the double cover $\hat{L}$, and in particular prove its existence. Given a Dynkin diagram, we can construct $\hat{L}$ as generated by the elements e^{α_i} for α_i simple roots with the given relations. It is easy to check that we get a surjective homomorphism $\hat{L} \to L$ with kernel generated by z with $z^2 = 1$. What's a little harder to show is that $z \neq 1$ (i.e., show that $\hat{L} \neq L$). The easiest way to do it is to use cohomology of groups, but since we have such an explicit case, we'll do it bare hands: Problem: Given Z, H groups with Z abelian, construct central extensions

$$1 \to Z \to G \to H \to 1$$

(where Z lands in the center of G). Let G be the set of pairs (z, h), and set the product $(z_1, h_1)(z_2, h_2) = (z_1 z_2 c(h_1, h_2), h_1 h_2)$, where $c(h_1, h_2) \in Z$ ($c(h_1, h_2)$ will be a cocycle in group cohomology). We obviously get a homomorphism by mapping $(z, h) \mapsto h$. If $c(1, h) = c(h, 1) = 1$ (normalization), then $z \mapsto (z, 1)$ is a homomorphism mapping Z to the center of G. In particular, $(1, 1)$ is the identity. We'll leave it as an exercise to figure out what the inverses are. When is this thing *associative*? Let's just write everything out:

$$\big((z_1, h_1)(z_2, h_2)\big)(z_3, h_3) = (z_1 z_2 z_3 c(h_1, h_2)c(h_1 h_2, h_3), h_1 h_2 h_3)$$
$$(z_1, h_1)\big((z_2, h_2)(z_3, h_3)\big) = (z_1 z_2 z_3 c(h_1, h_2 h_3)c(h_2, h_3), h_1 h_2 h_3)$$

so we must have

$$c(h_1, h_2)c(h_1 h_2, h_3) = c(h_1 h_2, h_3)c(h_2, h_3).$$

This identity is actually very easy to satisfy in one particular case: when c is bimultiplicative: $c(h_1, h_2 h_3) = c(h_1, h_2)c(h_1, h_3)$ and $c(h_1 h_2, h_3) = c(h_1, h_3)c(h_2, h_3)$. That is, we have a map $H \times H \to Z$. Not all cocycles come from such maps, but this is the case we care about.

To construct the double cover, let $Z = \pm 1$ and $H = L$ (free abelian). If we write H additively, we want c to be a bilinear map $L \times L \to \pm 1$. It is really easy to construct bilinear maps on free abelian groups. Just take any basis $\alpha_1, \ldots, \alpha_n$ of L, choose $c(\alpha_1, \alpha_j)$ arbitrarily for each i, j and extend c via bilinearity to $L \times L$. In our case, we want to find a double cover $\hat{L}$ satisfying $\hat{e}^{\alpha}\hat{e}^{\beta} = (-1)^{(\alpha,\beta)}\hat{e}^{\beta}\hat{e}^{\alpha}$ where $\hat{e}^{\alpha}$ is a lift of e^{α}.

This just means that $c(\alpha,\beta) = (-1)^{(\alpha,\beta)}c(\beta,\alpha)$. To satisfy this, just choose $c(\alpha_i,\alpha_j)$ on the basis $\{\alpha_i\}$ so that $c(\alpha_i,\alpha_j) = (-1)^{(\alpha_i,\alpha_j)}c(\alpha_j,\alpha_i)$. This is trivial to do as $(-1)^{(\alpha_i,\alpha_i)} = 1$. Notice that this uses the fact that the lattice is even. There is no canonical way to choose this 2-cocycle (otherwise, the central extension would split as a product), but all the different double covers are isomorphic because we can specify $\hat{L}$ by generators and relations. Thus, we have constructed $\hat{L}$ (or rather, verified that the kernel of $\hat{L} \to L$ has order 2, not 1).

Let's now look at lifts of automorphisms of L to $\hat{L}$.

▶ **Exercise 27.1.** Any automorphism of L preserving $(\ ,\ )$ lifts to an automorphism of $\hat{L}$

There are two special cases:

1. -1 is an automorphism of L, and we want to lift it to $\hat{L}$ explicitly. First attempt: try sending $\hat{e}^\alpha$ to $\hat{e}^{-\alpha} := (\hat{e}^\alpha)^{-1}$, which doesn't work because $a \mapsto a^{-1}$ is not an automorphism on non-abelian groups.

   Better: $\omega : \hat{e}^\alpha \mapsto (-1)^{\alpha^2/2}(\hat{e}^\alpha)^{-1}$ is an automorphism of $\hat{L}$. To see this, check

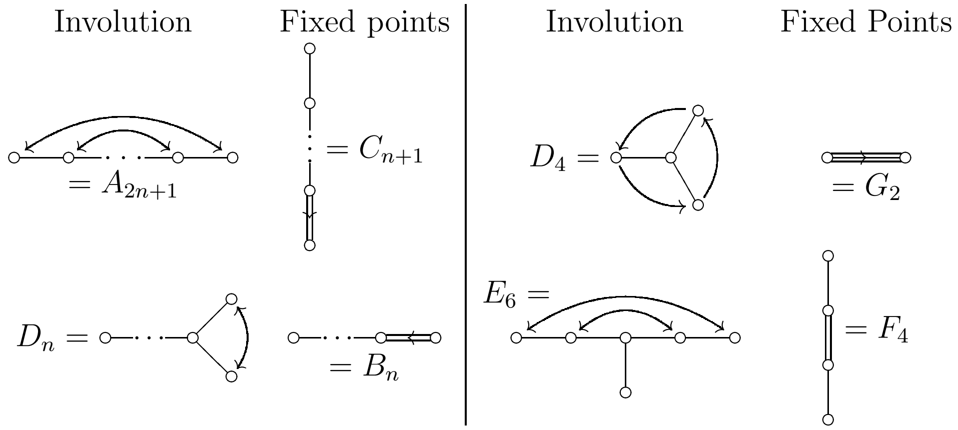
   $$\begin{aligned} \omega(\hat{e}^\alpha)\omega(\hat{e}^\beta) &= (-1)^{(\alpha^2+\beta^2)/2}(\hat{e}^\alpha)^{-1}(\hat{e}^\beta)^{-1} \\ \omega(\hat{e}^\alpha\hat{e}^\beta) &= (-1)^{(\alpha+\beta)^2/2}(\hat{e}^\beta)^{-1}(\hat{e}^\alpha)^{-1} \end{aligned}$$

   which work out just right

2. If $r^2 = 2$, then $\alpha \mapsto \alpha - (\alpha,r)r$ is an automorphism of L (reflection through $r^\perp$). You can lift this by $\hat{e}^\alpha \mapsto \hat{e}^\alpha(\hat{e}^r)^{-(\alpha,r)} \times (-1)^{\binom{(\alpha,r)}{2}}$. This is a homomorphism (check it!) of order (usually) 4!

   Remark 27.1. Although automorphisms of L lift to automorphisms of $\hat{L}$, the lift might have larger order.

This construction works for the root lattices of A_n, D_n, E_6, E_7, and E_8; these are the lattices which are even, positive definite, and generated by vectors of norm 2 (in fact, all such lattices are sums of the given ones). What about B_n, C_n, F_4 and G_2? The reason the construction doesn't work for these cases is because there are roots of different lengths. These all occur as fixed points of diagram automorphisms of A_n, D_n and E_6. In fact, we have a *functor* from Dynkin diagrams to Lie algebras, so and automorphism of the diagram gives an automorphism of the algebra



A_{2n} doesn't really give you a new algebra: it corresponds to some superalgebra stuff.

Construction of the Lie group of E_8

It is the group of automorphisms of the Lie algebra generated by the elements $\exp(\lambda Ad(\hat{e}^\alpha))$, where λ is some real number, $\hat{e}^\alpha$ is one of the basis elements of the Lie algebra corresponding to the root α, and $Ad(\hat{e}^\alpha)(a) = [\hat{e}^\alpha, a]$. In other words,

$$\exp(\lambda Ad(\hat{e}^\alpha))(a) = 1 + \lambda[\hat{e}^\alpha, a] + \frac{\lambda^2}{2}[\hat{e}^\alpha, [\hat{e}^\alpha, a]].$$

and all the higher terms are zero. To see that $Ad(\hat{e}^\alpha)^3 = 0$, note that if β is a root, then $\beta + 3\alpha$ is not a root (or 0).

Warning 27.2. In general, the group generated by these automorphisms is NOT the whole automorphism group of the Lie algebra. There might be extra diagram automorphisms, for example.

We get some other things from this construction. We can get simple groups over finite fields: note that the construction of a Lie algebra above works over any commutative ring (e.g. over $\mathbb{Z}$). The only place we used division is in $\exp(\lambda Ad(\hat{e}^\alpha))$ (where we divided by 2). The only time this term is non-zero is when we apply $\exp(\lambda Ad(\hat{e}^\alpha))$ to $\hat{e}^{-\alpha}$, in which case we find that $[\hat{e}^\alpha, [\hat{e}^\alpha, \hat{e}^{-\alpha}]] = [\hat{e}^\alpha, \alpha] = -(\alpha, \alpha)\hat{e}^\alpha$, and the fact that $(\alpha, \alpha) = 2$ cancels the division by 2. So we can in fact construct the E_8 group over *any* commutative ring. You can mumble something about group schemes over $\mathbb{Z}$ at this point. In particular, we have groups of type E_8 over *finite fields*, which are actually finite simple groups (these are called Chevalley groups; it takes work to show that they are simple, there is a book by Carter called *Finite Simple Groups* which you can look at).

Real forms

So we've constructed the Lie group and Lie algebra of type E_8. There are in fact several *different* groups of type E_8. There is one *complex* Lie algebra of type E_8, which corresponds to several different real Lie algebras of type E_8.

Let's look at some smaller groups:

Example 27.3. $\mathfrak{sl}_2(\mathbb{R}) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with a, b, c, d real $a + d = 0$; this is not compact. On the other hand, $\mathfrak{su}_2(\mathbb{R}) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $d = -a$ imaginary $b = -\bar{c}$, is compact. These have the same Lie algebra over $\mathbb{C}$.

Let's look at what happens for E_8. In general, suppose L is a Lie algebra with complexification $L \otimes \mathbb{C}$. How can we find another Lie algebra M with the same complexification? $L \otimes \mathbb{C}$ has an anti-linear involution $\omega_L : l \otimes z \mapsto l \otimes \bar{z}$. Similarly, it has an anti-linear involution ω_M. Notice that $\omega_L \omega_M$ is a linear involution of $L \otimes \mathbb{C}$. Conversely, if we know this involution, we can reconstruct M from it. Given an involution ω of $L \otimes \mathbb{C}$, we can get M as the fixed points of the map $a \mapsto \omega_L \omega(a)$"$=$" $\overline{\omega(a)}$. Another way is to put $L = L^+ \oplus L^-$, which are the $+1$ and -1 eigenspaces, then $M = L^+ \oplus iL^-$.

Thus, to find other real forms, we have to study the involutions of the complexification of L. The exact relation is kind of subtle, but this is a good way to go.

Example 27.4. Let $L = \mathfrak{sl}_2(\mathbb{R})$. It has an involution $\omega(m) = -m^T$. $\mathfrak{su}_2(\mathbb{R})$ is the set of fixed points of the involution ω times complex conjugation on $\mathfrak{sl}_2(\mathbb{C})$, by definition.

So to construct real forms of E_8, we want some involutions of the Lie algebra E_8 which we constructed. What involutions do we know about? There are two obvious ways to construct involutions:

1. Lift -1 on L to $\hat{e}^\alpha \mapsto (-1)^{\alpha^2/2}(\hat{e}^\alpha)^{-1}$, which induces an involution on the Lie algebra.
2. Take $\beta \in L/2L$, and look at the involution $\hat{e}^\alpha \mapsto (-1)^{(\alpha,\beta)}\hat{e}^\alpha$.

(2) gives nothing new ... you get the Lie algebra you started with. (1) only gives you one real form. To get all real forms, you multiply these two kinds of involutions together.

Recall that $L/2L$ has 3 orbits under the action of the Weyl group, of size 1, 120, and 135. These will correspond to the three real forms of E_8. How do we distinguish different real forms? The answer was found by Cartan: look at the signature of an invariant quadratic form on the Lie algebra!

A bilinear form (,) on a Lie algebra is called *invariant* if $([a, b], c) + (b[a, c]) = 0$ for all a, b, c. This is called invariant because it corresponds to the form being invariant under the corresponding group action. Now we can construct an invariant bilinear form on E_8 as follows:

1. $(\alpha, \beta)_{\text{in the Lie algebra}} = (\alpha, \beta)_{\text{in the lattice}}$
2. $(\hat{e}^\alpha, (\hat{e}^\alpha)^{-1}) = 1$
3. $(a, b) = 0$ if a and b are in root spaces α and β with $\alpha + \beta \neq 0$.

This gives an invariant inner product on E_8, which you prove by case-by-case check

▶ **Exercise 27.2.** do these checks

Next time, we'll use this to produce bilinear forms on all the real forms and then we'll calculate the signatures.

Lecture 28

Last time, we constructed a Lie algebra of type E_8, which was $L \oplus \bigoplus \hat{e}^\alpha$, where L is the root lattice and $\alpha^2 = 2$. This gives a double cover of the root lattice:

$$1 \to \pm 1 \to \hat{e}^L \to e^L \to 1.$$

We had a lift for $\omega(\alpha) = -\alpha$, given by $\omega(\hat{e}^\alpha) = (-1)^{(\alpha^2/2)}(\hat{e}^\alpha)^{-1}$. So ω becomes an automorphism of order 2 on the Lie algebra. $e^\alpha \mapsto (-1)^{(\alpha,\beta)}e^\alpha$ is also an automorphism of the Lie algebra.

Suppose σ is an automorphism of order 2 of the real Lie algebra $L = L^+ + L^-$ (eigenspaces of σ). We saw that you can construct another real form given by $L^+ + iL^-$. Thus, we have a map from conjugacy classes of automorphisms with $\sigma^2 = 1$ to real forms of L. This is not in general in isomorphism.

Today we'll construct some more real forms of E_8. E_8 has an invariant symmetric bilinear form $(e^\alpha, (e^\alpha)^{-1}) = 1$, $(\alpha, \beta) = (\beta, \alpha)$. The form is unique up to multiplication by a constant since E_8 is an irreducible representation of E_8. So the *absolute value of the signature* is an invariant of the Lie algebra.

For the split form of E_8, what is the signature of the invariant bilinear form (the split form is the one we just constructed)? On the Cartan subalgebra L, $(\ ,\ )$ is positive definite, so we get $+8$ contribution to the signature. On $\{e^\alpha, (e^\alpha)^{-1}\}$, the form is $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, so it has signature $0 \cdot 120$. Thus, the signature is 8. So if we find any real form with a different signature, we'll have found a new Lie algebra.

Let's first try involutions $e^\alpha \mapsto (-1)^{(\alpha,\beta)}e^\alpha$. But this doesn't change the signature. L is still positive definite, and you still have $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right)$ on the other parts. These Lie algebras actually turn out to be isomorphic to what we started with (though we haven't shown that they are isomorphic).

Now try $\omega : e^\alpha \mapsto (-1)^{\alpha^2/2}(e^\alpha)^{-1}$, $\alpha \mapsto -\alpha$. What is the signature of the form? Let's write down the $+$ and $-$ eigenspaces of ω. The $+$ eigenspace will be spanned by $e^\alpha - e^{-\alpha}$, and these vectors have norm -2 and are orthogonal. The $-$ eigenspace will be spanned by $e^\alpha + e^{-\alpha}$ and L, which have norm 2 and are orthogonal, and L is positive definite. What is the Lie algebra corresponding to the involution ω? It will be spanned by $e^\alpha - e^{-\alpha}$ where $\alpha^2 = 2$ (norm -2), and $i(e^\alpha + e^{-\alpha})$ (norm -2), and iL (which is now negative definite). So the bilinear form is *negative definite*, with signature $-248 (\neq \pm 8)$.

With some more work, you can actually show that this is the Lie algebra of the *compact* form of E_8. This is because the automorphism group of E_8 preserves the invariant bilinear form, so it is contained in $O_{0,248}(\mathbb{R})$, which is compact.

Now let's look at involutions of the form $e^\alpha \mapsto (-1)^{(\alpha,\beta)}\omega(e^\alpha)$. Notice that ω commutes with $e^\alpha \mapsto (-1)^{(\alpha,\beta)}e^\alpha$. The β's in (α, β) correspond to $L/2L$ modulo the action of the Weyl group $W(E_8)$. Remember this has three orbits, with 1 norm 0 vector, 120 norm 2 vectors, and 135 norm 4 vectors. The norm 0 vector gives us the compact form. Let's look at the other cases and see what we get.

Suppose V has a negative definite symmetric inner product $(\ ,\ )$, and suppose σ is an involution of $V = V_+ \oplus V_-$ (eigenspaces of σ). What is the signature of the invariant inner product on $V_+ \oplus iV_-$? On V_+, it is negative definite, and on iV_- it is positive definite. Thus, the signature is $\dim V_- - \dim V_+ = -\mathrm{tr}(\sigma)$. So we want to work out the traces of these involutions.

Given some $\beta \in L/2L$, what is $\mathrm{tr}(e^\alpha \mapsto (-1)^{(\alpha,\beta)}e^\alpha)$? If $\beta = 0$, the traces is obviously 248 because we just have the identity map. If $\beta^2 = 2$, we need to figure how many roots have a given inner product with β. Recall that this was determined before:

(α,β)	# of roots α with given inner product	eigenvalue
2	1	1
1	56	-1
0	126	1
-1	56	-1
-2	1	1

Thus, the trace is $1 - 56 + 126 - 56 + 1 + 8 = 24$ (the 8 is from the Cartan subalgebra). So the signature of the corresponding form on the Lie algebra is -24. We've found a third Lie algebra.

If we also look at the case when $\beta^2 = 4$, what happens? How many α with $\alpha^2 = 2$ and with given (α, β) are there? In this case, we have:

(α,β)	# of roots α with given inner product	eigenvalue
2	14	1
1	64	-1
0	84	1
-1	64	-1
-2	14	1

The trace will be $14 - 64 + 84 - 64 + 14 + 8 = -8$. This is just the split form again.

Summary: We've found 3 forms of E_8, corresponding to 3 classes in $L/2L$, with signatures 8, -24, -248, corresponding to involutions $e^\alpha \mapsto (-1)^{(\alpha,\beta)}e^{-\alpha}$ of the *compact* form. If L is the *compact* form of a simple Lie algebra, then Cartan showed that the other forms correspond exactly to the conjugacy classes of involutions in the automorphism group of L (this doesn't work if you don't start with the compact form — so always start with the compact form).

In fact, these three are the *only* forms of E_8, but we won't prove that.

Working with simple Lie groups

As an example of how to work with simple Lie groups, we will look at the general question: Given a simple Lie group, what is its homotopy type? Answer: G has a unique conjugacy class of maximal compact subgroups K, and G is homotopy equivalent to K.

Proof for $GL_n(\mathbb{R})$. First pretend $GL_n(\mathbb{R})$ is simple, even though it isn't; whatever. There is an obvious compact subgroup: $O_n(\mathbb{R})$. Suppose K is *any* compact subgroup of $GL_n(\mathbb{R})$. Choose any positive definite form $(\ , \ )$ on $\mathbb{R}^n$. This will probably not be invariant under K, but since K is compact, we can average it over K get one that is: define a new form $(a, b)_{\text{new}} = \int_K (ka, kb)\, dk$. This gives an invariant positive definite bilinear form (since integral of something positive definite is positive definite). Thus, any compact subgroup preserves some positive definite form. But the subgroup fixing some positive definite bilinear form is conjugate to a subgroup of $O_n(\mathbb{R})$ (to see this, diagonalize the form). So K is contained in a conjugate of $O_n(\mathbb{R})$.

Next we want to show that $G = GL_n(\mathbb{R})$ is homotopy equivalent to $O_n(\mathbb{R}) = K$. We will show that $G = KAN$, where K is O_n, A is all diagonal matrices with positive coefficients, and N is matrices which are upper triangular with 1s on the diagonal. This is the *Iwasawa decomposition.* In general, we get K compact, A semisimple abelian, and N is unipotent. The proof of this you saw before was called the Grahm-Schmidt process for orthonormalizing a basis. Suppose $v_1, \dots, v_n$ is any basis for $\mathbb{R}^n$.

1. Make it orthogonal by subtracting some stuff, you'll get v_1, $v_2 - *v_1$, $v_3 - *v_2 - *v_1$, $\dots$.
2. Normalize by multiplying each basis vector so that it has norm 1. Now we have an orthonormal basis.

This is just another way to say that GL_n can be written as KAN. Making things orthogonal is just multiplying by something in N, and normalizing is just multiplication by some diagonal matrix with positive entries. An orthonormal basis is an element of O_n. Tada! This decomposition is just a topological one, not a decomposition as groups. Uniqueness is easy to check.

Now we can get at the homotopy type of GL_n. $N \cong \mathbb{R}^{n(n-1)/2}$, and $A \cong (\mathbb{R}^+)^n$, which are contractible. Thus, $GL_n(\mathbb{R})$ has the same homotopy type as $O_n(\mathbb{R})$, its maximal compact subgroup. $\square$

If you wanted to know $\pi_1(GL_3(\mathbb{R}))$, you could calculate $\pi_1(O_3(\mathbb{R})) \cong \mathbb{Z}/2\mathbb{Z}$, so $GL_3(\mathbb{R})$ has a double cover. Nobody has shown you this double cover because it is *not algebraic*.

Example 28.1. Let's go back to various forms of E_8 and figure out (guess) the fundamental groups. We need to know the maximal compact subgroups.

1. One of them is easy: the compact form is its own maximal compact subgroup. What is the fundamental group? Remember or quote the fact that for compact simple groups, $\pi_1 \cong \frac{\text{weight lattice}}{\text{root lattice}}$, which is 1. So this form is simply connected.

2. $\beta^2 = 2$ case (signature -24). Recall that there were 1, 56, 126, 56, and 1 roots α with $(\alpha, \beta) = 2, 1, 0, -1$, and -2 respectively, and there are another 8 dimensions for the Cartan subalgebra. On the $1, 126, 1, 8$ parts, the form is negative definite. The sum of these root spaces gives a Lie algebra of type E_7A_1 with a negative definite bilinear form (the 126 gives you the roots of an E_7, and the 1s are the roots of an A_1). So it a reasonable guess that the maximal compact subgroup has something to do with E_7A_1. E_7 and A_1 are not simply connected: the compact form of E_7 has $\pi_1 = \mathbb{Z}/2$ and the compact form of A_1 also has $\pi_1 = \mathbb{Z}/2$. So the universal cover of E_7A_1 has center $(\mathbb{Z}/2)^2$. Which part of this acts trivially on E_8? We look at the E_8 Lie algebra as a representation of $E_7 \times A_1$. You can read off how it splits form the picture above: $E_8 \cong E_7 \oplus A_1 \oplus 56 \otimes 2$, where 56 and 2 are irreducible, and the centers of E_7 and A_1 both act as -1 on them. So the maximal compact subgroup of this form of E_8 is the simply connected compact form of $E_7 \times A_1/(-1, -1)$. This means that $\pi_1(E_8)$ is the same as π_1 of the compact subgroup, which is $(\mathbb{Z}/2)^2/(-1, -1) \cong \mathbb{Z}/2$. So this simple group has a nontrivial double cover (which is non-algebraic).

3. For the other (split) form of E_8 with signature 8, the maximal compact subgroup is $\mathrm{Spin}_{16}(\mathbb{R})/(\mathbb{Z}/2)$, and $\pi_1(E_8)$ is $\mathbb{Z}/2$.

You can compute any other homotopy invariants with this method.

Let's look at the 56 dimensional representation of E_7 in more detail.

We had the picture

(α, β)	# of α's
2	1
1	56
0	126
-1	56
-2	1

The Lie algebra E_7 fixes these 5 spaces of E_8 of dimensions $1, 56, 126 + 8, 56, 1$. From this we can get some representations of E_7. The $126 + 8$ splits as $1 + (126 + 7)$. But we also get a 56 dimensional representation of E_7. Let's show that this is actually an irreducible representation. Recall that in calculating $W(E_8)$, we showed that $W(E_7)$ acts transitively on this set of 56 roots of E_8, which can be considered as weights of E_7.

An irreducible representation is called *minuscule* if the Weyl group acts transitively on the weights. This kind of representation is particularly easy to work with. It is really easy to work out the character for example: just translate the 1 at the highest weight around, so every weight has multiplicity 1.

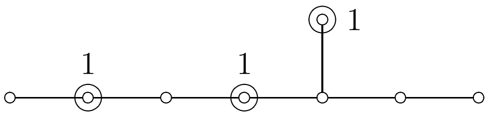
So the 56 dimensional representation of E_7 must actually be the irreducible representation with whatever highest weight corresponds to one of the vectors.

Every possible simple Lie group

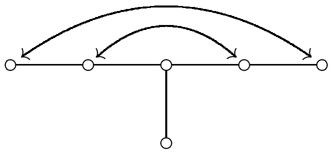
We will construct them as follows: Take an involution σ of the compact form $L = L^+ + L^-$ of the Lie algebra, and form $L^+ + iL^-$. The way we constructed these was to first construct A_n, D_n, E_6, and E_7 as for E_8. Then construct the involution $\omega : e^\alpha \mapsto -e^{-\alpha}$. We get B_n, C_n, F_4, and G_2 as fixed points of the involution ω.

Kac classified all automorphisms of finite order of any compact simple Lie group. The method we'll use to classify involutions is extracted from his method. We can construct lots of involutions as follows:

1. Take any Dynkin diagram, say E_8, and select some of its vertices, corresponding to simple roots. Get an involution by taking $e^\alpha \mapsto \pm e^\alpha$ where the sign depends on whether α is one of the simple roots we've selected. However, this is not a great method. For one thing, you get a lot of repeats (recall that there are only 3, and we've found 2^8 this way).



2. Take any diagram automorphism of order 2, such as



   This gives you more involutions.

Next time, we'll see how to cut down this set of involutions.

Lecture 29

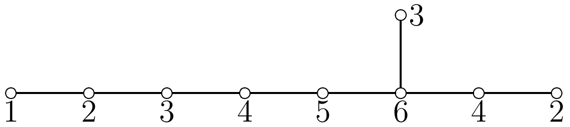
Split form of Lie algebra (we did this for A_n, D_n, E_6, E_7, E_8): $A = \bigoplus \hat{e}^\alpha \oplus L$. Compact form $A^+ + iA^-$, where $A^\pm$ eigenspaces of $\omega : \hat{e}^\alpha \mapsto (-1)^{\alpha^2/2}\hat{e}^{-\alpha}$.

We talked about other involutions of the compact form. You get all the other forms this way.

The idea now is to find ALL real simple Lie algebras by listing all involutions of the compact form. We will construct all of them, but we won't prove that we have all of them.

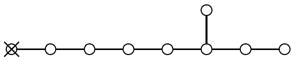
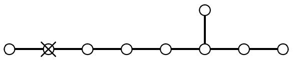
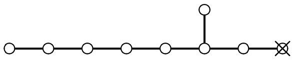
We'll use Kac's method for classifying all automorphisms of order N of a compact Lie algebra (and we'll only use the case $N = 2$). First let's look at inner automorphisms. Write down the AFFINE Dynkin diagram

−highest weight =



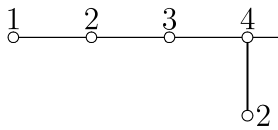
Choose n_i with $\sum n_i m_i = N$ where the m_i are the numbers on the diagram. We have an automorphism $e^{\alpha_j} \mapsto e^{2\pi i n_j/N} e^{\alpha_j}$ induces an automorphism of order dividing N. This is obvious. The point of Kac's theorem is that all inner automorphisms of order dividing N are obtained this way and are conjugate if and only if they are conjugate by an automorphism of the Dynkin diagram. We won't actually prove Kac's theorem because we just want to get a bunch of examples. See [Kac90] or [Hel01].

Example 29.1. Real forms of E_8. We've already found three, and it took us a long time. We can now do it fast. We need to solve $\sum n_i m_i = 2$ where $n_i \geq 0$; there are only a few possibilities:

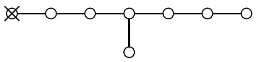
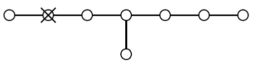
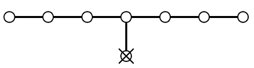
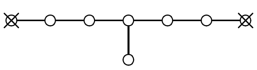
$\sum n_i m_i = 2$	# of ways	how to do it	maximal compact subgroup K
2×1	one way		E_8 (compact form)
1×2	two ways		A_1E_7
			D_8 (split form)
$1 \times 1 + 1 \times 1$	no ways		

The points NOT crossed off form the Dynkin diagram of the maximal compact subgroup. Thus, by just looking at the diagram, we can see what all the real forms are!

Example 29.2. Let's do E_7. Write down the affine diagram:

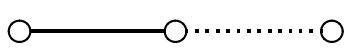


We get the possibilities

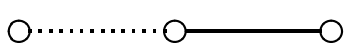
$\sum n_i m_i = 2$	# of ways	how to do it	maximal compact subgroup K
2×1	one way*		E_7 (compact form)
1×2	two ways*		$A_1 D_6$
			A_7 (split form)**
$1 \times 1 + 1 \times 1$	one way		$E_6 \oplus \mathbb{R}$ ***

(*) The number of ways is counted up to automorphisms of the diagram.
(**) In the split real form, the maximal compact subgroup has dimension equal to half the number of roots. The roots of A_7 look like $\varepsilon_i - \varepsilon_j$ for $i, j \leq 8$ and $i \neq j$, so the dimension is $8 \cdot 7 + 7 = 56 = \frac{112}{2}$.
(***) The maximal compact subgroup is $E_6 \oplus \mathbb{R}$ because the fixed subalgebra contains the whole Cartan subalgebra, and the E_6 only accounts for 6 of the 7 dimensions. You can use this to construct some interesting representations of E_6 (the minuscule ones). How does the algebra E_7 decompose as a representation of the algebra $E_6 \oplus \mathbb{R}$?

We can decompose it according to the eigenvalues of $\mathbb{R}$. The $E_6 \oplus \mathbb{R}$ is the zero eigenvalue of $\mathbb{R}$ [why?], and the rest is 54 dimensional. The easy way to see the decomposition is to look at the roots. Remember when we computed the Weyl group we looked for vectors like



or

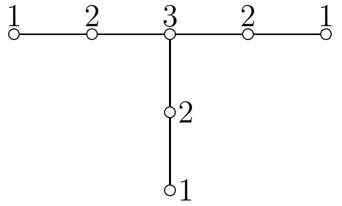


The 27 possibilities (for each) form the weights of a 27 dimensional representation of E_6. The orthogonal complement of the two nodes is an E_6 root system whose Weyl group acts transitively on these 27 vectors (we showed that these form a single orbit, remember?). Vectors of the E_7 root system are the vectors of the E_6 root system plus these 27 vectors plus the other 27 vectors. This splits up the E_7 explicitly. The two 27s form single orbits, so they are irreducible. Thus, $E_7 \cong E_6 \oplus \mathbb{R} \oplus 27 \oplus 27$, and the 27s are minuscule.

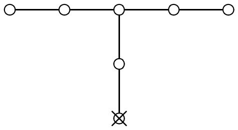
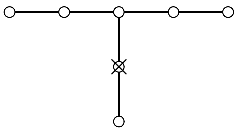
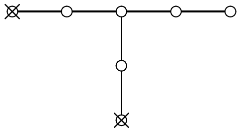
Let K be a maximal compact subgroup, with Lie algebra $\mathbb{R} + E_6$. The factor of $\mathbb{R}$ means that K has an S^1 in its center. Now look at the space G/K, where G is the Lie group of type E_7, and K is the maximal compact subgroup. It is a *Hermitian symmetric space*. Symmetric space means that it is a (simply connected) Riemannian manifold M such that for each point $p \in M$, there is an automorphism fixing p and acting as -1 on the tangent space. This looks weird, but it turns out that all kinds of nice objects you know about are symmetric spaces. Typical examples you may have seen: spheres S^n, hyperbolic space $\mathbb{H}^n$, and Euclidean space $\mathbb{R}^n$. Roughly speaking, symmetric spaces have nice properties of these spaces. Cartan classified all symmetric spaces: they are non-compact simple Lie groups modulo the maximal compact subgroup (more or less ... depending on simply connectedness hypotheses 'n such). Historically, Cartan classified simple Lie groups, and then later classified symmetric spaces, and was surprised to find the same result. Hermitian symmetric spaces are just symmetric spaces with a complex structure. A standard example of this is the upper half plane $\{x + iy | y > 0\}$. It is acted on by $SL_2(\mathbb{R})$, which acts by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \tau = \frac{a\tau+b}{c\tau+d}$.

Let's go back to this G/K and try to explain why we get a Hermitian symmetric space from it. We'll be rather sketchy here. First of all, to make it a symmetric space, we have to find a nice invariant Riemannian metric on it. It is sufficient to find a positive definite bilinear form on the tangent space at p which is invariant under K ... then you can translate it around. We can do this as K is compact (so you have the averaging trick). Why is it Hermitian? We'll show that there is an almost complex structure. We have S^1 acting on the tangent space of each point because we have an S^1 in the center of the stabilizer of any given point. Identify this S^1 with complex numbers of absolute value 1. This gives an invariant almost complex structure on G/K. That is, each tangent space is a complex vector space. Almost complex structures don't always come from complex structures, but this one does (it is integrable). Notice that it is a little unexpected that G/K has a complex structure (G and K are odd dimensional in the case of $G = E_7$, $K = E_6 \oplus \mathbb{R}$, so they have no hope of having a complex structure).

Example 29.3. Let's look at E_6, with affine Dynkin diagram



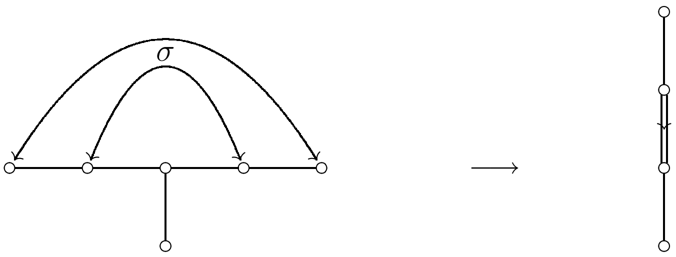
We get the possibilities

$\sum n_i m_i = 2$	# of ways	how to do it	maximal compact subgroup K
2×1	one way		E_6 (compact form)
1×2	one way		$A_1 A_5$
$1 \times 1 + 1 \times 1$	one way		$D_5 \oplus \mathbb{R}$

In the last one, the maximal compact subalgebra is $D_5 \oplus \mathbb{R}$. Just as before, we get a Hermitian symmetric space. Let's compute its dimension (over $\mathbb{C}$). The dimension will be the dimension of E_6 minus the dimension of $D_5 \oplus \mathbb{R}$, all divided by 2 (because we want complex dimension), which is $(78 - 46)/2 = 16$.

So we have found two non-compact simply connected Hermitian symmetric spaces of dimensions 16 and 27. These are the only "exceptional" cases; all the others fall into infinite families!

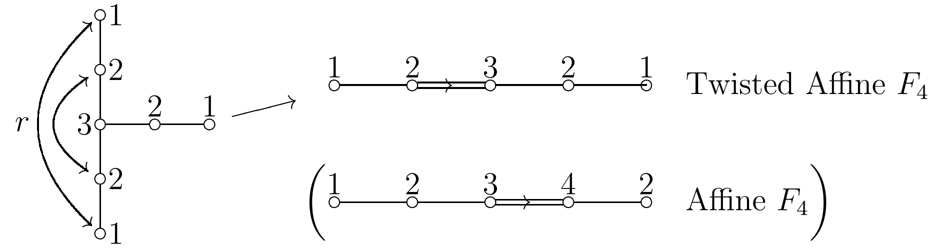
There are also some OUTER automorphisms of E_6 coming from the diagram automorphism



The fixed point subalgebra has Dynkin diagram obtained by folding the E_6 on itself. This is the F_4 Dynkin diagram. The fixed points of E_6 under the diagram automorphism is an F_4 Lie algebra. So we get a real form of E_6 with maximal compact subgroup F_4. This is probably the easiest way to construct F_4, by the way. Moreover, we can decompose E_6 as a representation of F_4. $\dim E_6 = 78$ and $\dim F_4 = 52$, so $E_6 = F_4 \oplus 26$, where 26 turns out to be irreducible (the smallest non-trivial representation of F_4 ... the only one anybody actually works with). The roots of F_4 look like $(\dots, \pm 1, \pm 1 \dots)$ (24 of these) and $(\pm\frac{1}{2} \cdots \pm \frac{1}{2})$ (16 of these), and $(\dots, \pm 1 \dots)$ (8 of them) ... the last two types are in the same orbit of the Weyl group.

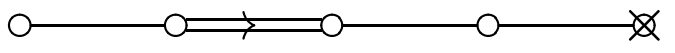
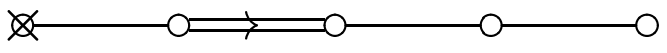
The 26 dimensional representation has the following character: it has all norm 1 roots with multiplicity 1 and 0 with multiplicity 2 (note that this is not minuscule).

There is one other real form of E_6. To get at it, we have to talk about Kac's description of non-inner automorphisms of order N. The non-inner automorphisms all turn out to be related to diagram automorphisms. Choose a diagram automorphism of order r, which divides N. Let's take the standard thing on E_6. Fold the diagram (take the fixed points), and form a TWISTED affine Dynkin diagram (note that the arrow goes the wrong way from the affine F_4)

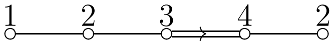


There are also numbers on the twisted diagram, but nevermind them. Find n_i so that $r\sum n_i m_i = N$. This is Kac's general rule. We'll only use the case $N = 2$.

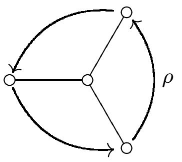
If $r > 1$, the only possibility is $r = 2$ and one n_1 is 1 and the corresponding m_i is 1. So we just have to find points of weight 1 in the twisted affine Dynkin diagram. There are just two ways of doing this in the case of E_6

 and 

one of these gives us F_4, and the other has maximal compact subalgebra C_4, which is the split form since $\dim C_4 = \#\text{roots of } F_4/2 = 24$.

Example 29.4. F_4. The affine Dynkin is  We can cross out one node of weight 1, giving the compact form (split form), or a node of weight 2 (in two ways), giving maximal compacts A_1C_3 or B_4. This gives us three real forms.

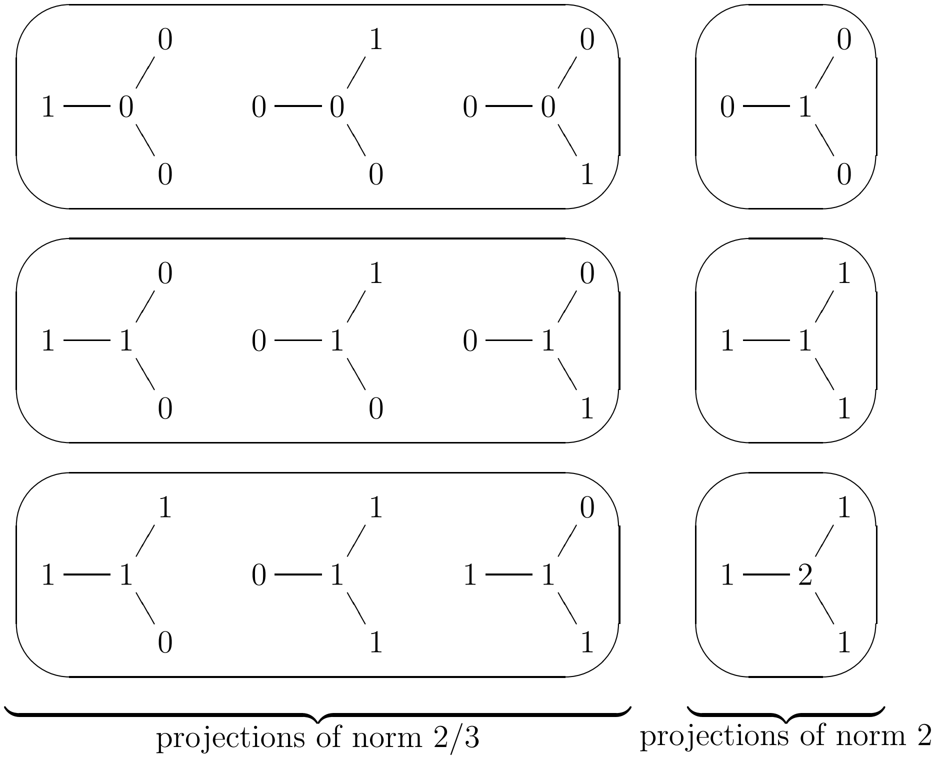
Example 29.5. G_2. We can actually draw this root system ... UCB won't supply me with a four dimensional board. The construction is to take the D_4 algebra and look at the fixed points of:



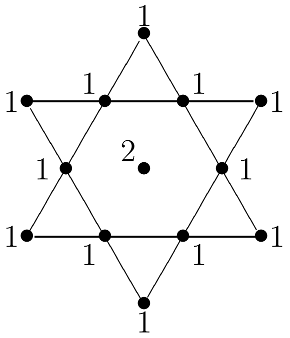
We want to find the fixed point subalgebra.

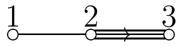
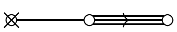
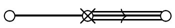
Fixed points on Cartan subalgebra: ρ fixes a two dimensional space, and has 1 dimensional eigenspaces corresponding to ω and $\bar{\omega}$, where $\omega^3 = 1$. The 2 dimensional space will be the Cartan subalgebra of G_2.

Positive roots of D_4 as linear combinations of simple roots (not fundamental weights):

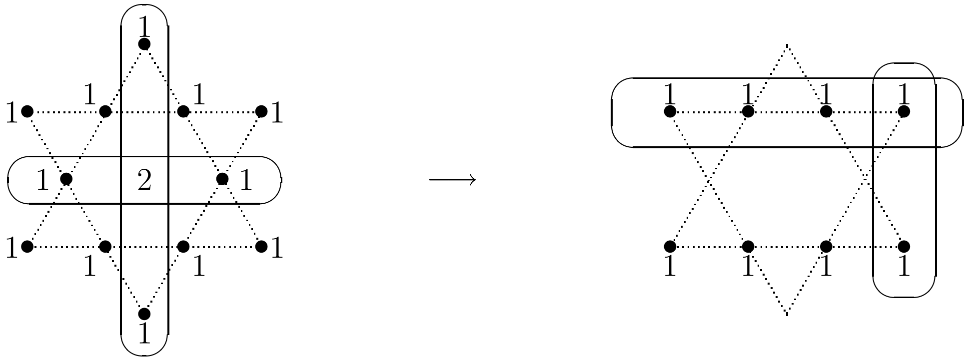


There are six orbits under ρ, grouped above. It obviously acts on the negative roots in exactly the same way. What we have is a root system with six roots of norm 2 and six roots of norm 2/3. Thus, the root system is G_2:



One of the only root systems to appear on a country's national flag. Now let's work out the real forms. Look at the affine: . we can delete the node of weight 1, giving the compact form: . We can delete the node of weight 2, giving A_1A_1 as the compact subalgebra:  ... this must be the split form because there is nothing else the split form can be.

Let's say some more about the split form. What does the Lie algebra of G_2 look like as a representation of the maximal compact subalgebra $A_1 \times A_1$? In this case, it is small enough that we can just draw a picture:



We have two orthogonal A_1s, and we have leftover the stuff on the right. This thing on the right is a tensor product of the 4 dimensional irreducible representation of the horizontal and the 2 dimensional of the vertical. Thus, $G_2 = 3 \times 1 + 1 \otimes 3 + 4 \otimes 2$ as irreducible representations of $A_1^{\text{(horizontal)}} \otimes A_1^{\text{(vertical)}}$.

Let's use this to determine exactly what the maximal compact subgroup is. It is a quotient of the simply connected compact group $SU(2) \times SU(2)$, with Lie algebra $A_1 \times A_1$. Just as for E_8, we need to identify which elements of the center act trivially on G_2. The center is $\mathbb{Z}/2 \times \mathbb{Z}/2$. Since we've decomposed G_2, we can compute this easily. A non-trivial element of the center of $SU(2)$ acts as 1 (on odd dimensional representations) or -1 (on even dimensional representations). So the element $z \times z \in SU(2) \times SU(2)$ acts trivially on $3 \otimes 1 + 1 \otimes 3 + 4 \times 2$. Thus the maximal compact subgroup of the non-compact simple G_2 is $SU(2) \times SU(2)/(z \times z) \cong SO_4(\mathbb{R})$, where z is the non-trivial element of $\mathbb{Z}/2$.

So we have constructed $3 + 4 + 5 + 3 + 2$ (from E_8, E_7, E_6, F_4, G_2) real forms of exceptional simple Lie groups.

There are another 5 exceptional real Lie groups: Take COMPLEX groups $E_8(\mathbb{C})$, $E_7(\mathbb{C})$, $E_6(\mathbb{C})$, $F_4(\mathbb{C})$, and $G_2(\mathbb{C})$, and consider them as REAL. These give simple real Lie groups of dimensions 248×2, 133×2, 78×2, 52×2, and 14×2.

Lecture 30 - Irreducible unitary representations of $SL_2(\mathbb{R})$

$SL_2(\mathbb{R})$ is non-compact. For compact Lie groups, all unitary representations are finite dimensional, and are all known well. For non-compact groups, the theory is much more complicated. Before doing the infinite dimensional representations, we'll review finite dimensional (usually not unitary) representations of $SL_2(\mathbb{R})$.

Finite dimensional representations

Finite dimensional complex representations of the following are much the same: $SL_2(\mathbb{R})$, $\mathfrak{sl}_2\mathbb{R}$, $\mathfrak{sl}_2\mathbb{C}$ [branch $SL_2(\mathbb{C})$ as a complex Lie group] (as a complex Lie algebra), $\mathfrak{su}_2\mathbb{R}$ (as a real Lie algebra), and SU_2 (as a real Lie group). This is because finite dimensional representations of a simply connected Lie group are in bijection with representations of the Lie algebra. Complex representations of a REAL Lie algebra L correspond to complex representations of its complexification $L \otimes \mathbb{C}$ considered as a COMPLEX Lie algebra.

Note: Representations of a COMPLEX Lie algebra $L \otimes \mathbb{C}$ are not the same as representations of the REAL Lie algebra $L \otimes \mathbb{C} \cong L + L$. The representations of the real Lie algebra correspond roughly to (reps of L)$\otimes$(reps of L).

Strictly speaking, $SL_2(\mathbb{R})$ is not simply connected, which is not important for finite dimensional representations.

Recall the main results for representations of SU_2:

1. For each positive integer n, there is one irreducible representation of dimension n.

2. The representations are completely reducible (every representation is a sum of irreducible ones). This is perhaps the most important fact.

   The finite dimensional representation theory of SU_2 is EASIER than the representation theory of the ABELIAN Lie group $\mathbb{R}^2$, and that is because representations of SU_2 are completely reducible.

   For example, it is very difficult to classify pairs of commuting nilpotent matrices.

Completely reducible representations:

1. Complex representations of finite groups.

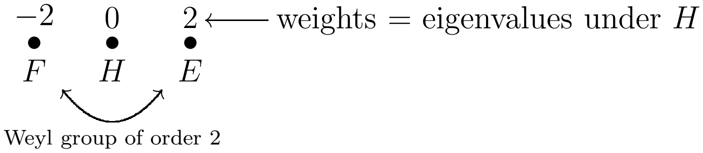
2. Representations of compact groups (Weyl character formula)

3. More generally, unitary representations of anything (you can take orthogonal complements of subrepresentations)
4. Finite dimensional representations of semisimple Lie groups.

Representations which are not completely reducible:

1. Representations of a finite group G over fields of characteristic $p|\,|G|$.
2. Infinite dimensional representations of non-compact Lie groups (even if they are semisimple).

We'll work with the Lie algebra $\mathfrak{sl}_2\mathbb{R}$, which has basis $H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $E = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, and $F = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. H is a basis for the Cartan subalgebra $\left(\begin{smallmatrix} a & 0 \\ 0 & -a \end{smallmatrix}\right)$. E spans the root space of the simple root. F spans the root space of the negative of the simple root. We find that $[H,E] = 2E$, $[H,F] = -2F$ (so E and F are eigenvectors of H), and you can check that $[E,F] = H$.



The Weyl group is generated by $\omega = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\omega^2 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

Let V be a finite dimensional irreducible complex representation of $\mathfrak{sl}_2\mathbb{R}$. First decompose V into eigenspaces of the Cartan subalgebra (weight spaces) (i.e. eigenspaces of the element H). Note that eigenspaces of H exist because V is FINITE-DIMENSIONAL (remember this is a complex representation). Look at the LARGEST eigenvalue of H (exists since V is finite dimensional), with eigenvector v. We have that $Hv = nv$ for some n. Compute

$$\begin{aligned} H(Ev) &= [H,E]v + E(Hv) \\ &= 2Ev + Env = (n+2)Ev \end{aligned}$$

So $Ev = 0$ (lest it be an eigenvector of H with higher eigenvalue). $[E,-]$ increases weights by 2 and $[F,-]$ decreases weights by 2, and $[H,-]$ fixes weights.

We have that E kills v, and H multiplies it by n. What does F do to v?

$$\begin{tikzcd}
 & nv & (n-2)Fv & (n-4)F^2v & (n-6)F^3v & \cdots \\
0 & v \arrow[u, "H"] \arrow[l, bend left, "E"] \arrow[r, bend left, "F"] & Fv \arrow[u, "H"] \arrow[l, bend left, "\substack{E \\ \times n}"] \arrow[r, bend left, "F"] & F^2v \arrow[u, "H"] \arrow[l, bend left, "\substack{E \\ \times (2n-2)}"] \arrow[r, bend left, "F"] & F^3v \arrow[u, "H"] \arrow[l, bend left, "\substack{E \\ \times (3n-6)}"] & \cdots
\end{tikzcd}$$

What is $E(Fv)$? Well,

$$\begin{aligned} EFv &= FEv + [E,F]v \\ &= 0 + Hv = nv \end{aligned}$$

In general, we have

$$\begin{aligned} H(F^i v) &= (n-2i)F^i v \\ E(F^i v) &= (ni - i(i-1))F^{i-1}v \\ F(F^i v) &= F^{i+1}v \end{aligned}$$

So the vectors $F^i v$ span V because they span an invariant subspace. This gives us an infinite number of vectors in distinct eigenspaces of H, and V is finite dimensional. Thus, $F^k v = 0$ for some k. Suppose k is the SMALLEST integer such that $F^k v = 0$. Then

$$0 = E(F^k v) = (nk - k(k-1))\underbrace{EF^{k-1}v}_{\neq 0}$$

So $nk - k(k-1) = 0$, and $k \neq 0$, so $n - (k-1) = 0$, so $\boxed{k = n+1}$. So V has a basis consisting of $v, Fv, \ldots, F^n v$. The formulas become a little better if we use the basis $w_n = v, w_{n-2} = Fv, w_{n-4} = \frac{F^2v}{2!}, \frac{F^3v}{3!}, \ldots, \frac{F^nv}{n!}$.

$$\begin{tikzcd}
 & & & & & & & E \\
w_{-6} \arrow[r, bend left, "1"] & w_{-4} \arrow[l, bend left, "6"] \arrow[r, bend left, "2"] & w_{-2} \arrow[l, bend left, "5"] \arrow[r, bend left, "3"] & w_0 \arrow[l, bend left, "4"] \arrow[r, bend left, "4"] & w_2 \arrow[l, bend left, "3"] \arrow[r, bend left, "5"] & w_4 \arrow[l, bend left, "2"] \arrow[r, bend left, "6"] & w_6 \arrow[l, bend left, "1"] & \\
 & & & & & & & F
\end{tikzcd}$$

This says that $E(w_2) = 5w_4$ for example. So we've found a complete description of all finite dimensional irreducible complex representations of $\mathfrak{sl}_2\mathbb{R}$. This is as explicit as you could possibly want.

These representations all lift to the group $SL_2(\mathbb{R})$: $SL_2(\mathbb{R})$ acts on homogeneous polynomials of degree n by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) f(x,y) = f(ax+by, cx+dy)$. This is an $n+1$ dimensional space, and you can check that the eigenspaces are $x^i y^{n-i}$.

We have implicitly constructed VERMA MODULES. We have a basis $w_n, w_{n-2}, \ldots, w_{n-2i}, \ldots$ with relations $H(w_{n-2i}) = (n-2i)w_{n-2i}$, $Ew_{n-2i} = (n-i+1)w_{n-2i+2}$, and $Fw_{n-2i} = (i+1)w_{n-2i-2}$. These are obtained by copying the formulas from the finite dimensional case, but allow it to be infinite dimensional. This is the universal representation generated by the highest weight vector w_n with eigenvalue n under H (highest weight just means $E(w_n) = 0$).

Let's look at some things that go wrong in infinite dimensions.

Warning 30.1. Representations corresponding to the Verma modules do NOT lift to representations of $SL_2(\mathbb{R})$, or even to its universal cover. The reason: look at the Weyl group (generated by $(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right))$ of $SL_2(\mathbb{R})$ acting on $\langle H\rangle$; it changes H to $-H$. It maps eigenspaces with eigenvalue m to eigenvalue $-m$. But if you look at the Verma module, it has eigenspaces $n, n-2, n-4, \dots$, and this set is obviously not invariant under changing sign. The usual proof that representations of the Lie algebra lifts uses the exponential map of matrices, which doesn't converge in infinite dimensions.

Remark 30.2. The universal cover $\widetilde{SL_2(\mathbb{R})}$ of $SL_2(\mathbb{R})$, or even the double cover $Mp_2(\mathbb{R})$, has NO faithful finite dimensional representations.

Proof. Any finite dimensional representation comes from a finite dimensional representation of the Lie algebra $\mathfrak{sl}_2\mathbb{R}$. All such finite dimensional representations factor through $SL_2(\mathbb{R})$. □

All finite dimensional representations of $SL_2(\mathbb{R})$ are completely reducible. Weyl did this by Weyl's unitarian trick:

Notice that finite dimensional representations of $SL_2(\mathbb{R})$ are isomorphic (sort of) to finite dimensional representations of the COMPACT group SU_2 (because they have the same complexified Lie algebras. Thus, we just have to show it for SU_2. But representations of ANY compact group are completely reducible. Reason:

1. All unitary representations are completely reducible (if $U \subseteq V$, then $V = U \oplus U^\perp$).
2. Any representation V of a COMPACT group G can be made unitary: take any unitary form on V (not necessarily invariant under G), and average it over G to get an invariant unitary form. We can average because G is compact, so we can integrate any continuous function over G. This form is positive definite since it is the average of positive definite forms (if you try this with non-(positive definite) forms, you might get zero as a result).

The Casimir operator

Set $\Omega = 2EF + 2FE + H^2 \in U(\mathfrak{sl}_2\mathbb{R})$. The main point is that Ω commutes with $\mathfrak{sl}_2\mathbb{R}$. You can check this by brute force:

$$\begin{aligned}
[H,\Omega] &= 2\underbrace{([H,E]F + E[H,F])}_{0} + \cdots \\
[E,\Omega] &= 2[E,E]F + 2E[F,E] + 2[E,F]E \\
&\qquad + 2F[E,E] + [E,H]H + H[E,H] = 0 \\
[F,\Omega] &= \text{Similar}
\end{aligned}$$

Thus, Ω is in the center of $U(\mathfrak{sl}_2\mathbb{R})$. In fact, it generates the center. This doesn't really explain where Ω comes from.

Remark 30.3. Why does Ω exist? The answer is that it comes from a symmetric invariant bilinear form on the Lie algebra $\mathfrak{sl}_2\mathbb{R}$ given by $(E,F)=1$, $(E,E)=(F,F)=(F,H)=(E,H)=0$, $(H,H)=2$. This bilinear form is an invariant map $L\otimes L\to\mathbb{C}$, where $L=\mathfrak{sl}_2\mathbb{R}$, which by duality gives an invariant element in $L\otimes L$, which turns out to be $2E\otimes F+2F\otimes E+H\otimes H$. The invariance of this element corresponds to Ω being in the center of $U(\mathfrak{sl}_2\mathbb{R})$.

Since Ω is in the center of $U(\mathfrak{sl}_2\mathbb{R})$, it acts on each irreducible representation as multiplication by a constant. We can work out what this constant is for the finite dimensional representations. Apply Ω to the highest vector w_n:

$$\begin{aligned}(2EF+2FE+HH)w_n &= (2n+0+n^2)w_n\\ &= (2n+n^2)w_n\end{aligned}$$

So Ω has eigenvalue $2n+n^2$ on the irreducible representation of dimension $n+1$. Thus, Ω has DISTINCT eigenvalues on different irreducible representations, so it can be used to separate different irreducible representations. The main use of Ω will be in the next lecture, where we'll use it to deal with infinite dimensional representation.

To finish today's lecture, let's look at an application of Ω. We'll sketch an algebraic argument that the representations of $\mathfrak{sl}_2\mathbb{R}$ are completely reducible. Given an exact sequence of representations

$$0\to U\to V\to W\to 0$$

we want to find a splitting $W\to V$, so that $V=U\oplus W$.

Step 1: Reduce to the case where $W=\mathbb{C}$. The idea is to look at

$$0\to \mathrm{Hom}_{\mathbb{C}}(W,U)\to \mathrm{Hom}_{\mathbb{C}}(W,V)\to \mathrm{Hom}_{\mathbb{C}}(W,W)\to 0$$

and $\mathrm{Hom}_{\mathbb{C}}(W,W)$ has an obvious one dimensional subspace, so we can get a smaller exact sequence

$$0\to \mathrm{Hom}_{\mathbb{C}}(W,U)\to \text{subspace of } \mathrm{Hom}_{\mathbb{C}}(W,V)\to \mathbb{C}\to 0$$

and if we can split this, the original sequence splits.

Step 2: Reduce to the case where U is irreducible. This is an easy induction on the number of irreducible components of U.

▶ **Exercise 30.1.** Do this.

Step 3: This is the key step. We have

$$0 \to U \to V \to \mathbb{C} \to 0$$

with U irreducible. Now apply the Casimir operator Ω. V splits as eigenvalues of Ω, so is $U \oplus \mathbb{C}$ UNLESS U has the same eigenvalue as $\mathbb{C}$ (i.e. unless $U = \mathbb{C}$).

Step 4: We have reduced to

$$0 \to \mathbb{C} \to V \to \mathbb{C} \to 0$$

which splits because $\mathfrak{sl}_2(\mathbb{R})$ is perfect[1] (no homomorphisms to the abelian algebra $\left(\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right)$).

Next time, in the final lecture, we'll talk about infinite dimensional unitary representations.

---

[1] L is *perfect* if $[L, L] = L$

Lecture 31 - Unitary representations of $SL_2(\mathbb{R})$

Last lecture, we found the finite dimensional (non-unitary) representations of $SL_2(\mathbb{R})$.

Background about infinite dimensional representations

(of a Lie group G) What is an finite dimensional representation?

1st guess Banach space acted on by G?

This is no good for some reasons: Look at the action of G on the functions on G (by left translation). We could use L^2 functions, or L^1 or L^p. These are completely different Banach spaces, but they are essentially the same representation.

2nd guess Hilbert space acted on by G? This is sort of okay.

The problem is that finite dimensional representations of $SL_2(\mathbb{R})$ are NOT Hilbert space representations, so we are throwing away some interesting representations.

Solution (Harish-Chandra) Take $\mathfrak{g}$ to be the Lie algebra of G, and let K be the maximal compact subgroup. If V is an infinite dimensional representation of G, there is no reason why $\mathfrak{g}$ should act on V.

The simplest example fails. Let $\mathbb{R}$ act on $L^2(\mathbb{R})$ by left translation. Then the Lie algebra is generated by $\frac{d}{dx}$ (or $i\frac{d}{dx}$) acting on $L^2(\mathbb{R})$, but $\frac{d}{dx}$ of an L^2 function is not in L^2 in general.

Let V be a Hilbert space. Set V_ω to be the K-finite vectors of V, which are the vectors contained in a finite dimensional representation of K. The point is that K is compact, so V splits into a Hilbert space direct sum finite dimensional representations of K, at least if V is a Hilbert space. Then V_ω is a representation of the Lie algebra $\mathfrak{g}$, not a representation of G. V_ω is a representation of the group K. It is a $(\mathfrak{g}, K)$-module, which means that it is acted on by $\mathfrak{g}$ and K in a "compatible" way, where compatible means that

1. they give the same representations of the Lie algebra of K.
2. $k(u)v = k(u(k^{-1}v))$ for $k \in K$, $u \in \mathfrak{g}$, and $v \in V$.

The K-finite vectors of an irreducible unitary representation of G is ADMISSIBLE, which means that every representation of K only occurs a *finite* number of times. The GOOD category of representations is the representations of admissible $(\mathfrak{g}, K)$-modules. It turns out that this is a really well behaved category.

We want to find the unitary irreducible representations of G. We will do this in several steps:

1. Classify all irreducible admissible representations of G. This was solved by Langlands, Harish-Chandra et. al.
2. Find which have hermitian inner products $(\, , \,)$. This is easy.
3. Find which ones are positive definite. This is VERY HARD. We'll only do this for the simplest case: $SL_2(\mathbb{R})$.

The group $SL_2(\mathbb{R})$

We found some generators (in $Lie(SL_2(\mathbb{R})) \otimes \mathbb{C}$ last time: E, F, H, with $[H,E] = 2E$, $[H,F] = -2F$, and $[E,F] = H$. We have that $H = -i \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $E = \frac{1}{2}\left(\begin{smallmatrix} 1 & i \\ i & -1 \end{smallmatrix}\right)$, and $F = \frac{1}{2}\left(\begin{smallmatrix} 1 & -i \\ -i & -1 \end{smallmatrix}\right)$. Why not use the old $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$?

Because $SL_2(\mathbb{R})$ has two different classes of Cartan subgroup: $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$, spanned by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$, spanned by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, and the second one is COMPACT. The point is that non-compact (abelian) groups need not have eigenvectors on infinite dimensional spaces. An eigenvector is the same as a weight space. The first thing you do is split it into weight spaces, and if your Cartan subgroup is not compact, you can't get started. We work with the compact subalgebra so that the weight spaces exist.

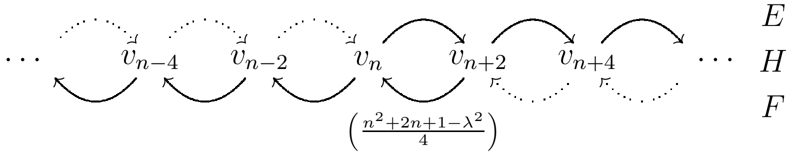
Given the representation V, we can write it as some direct sum of eigenspaces of H, as the Lie group H generates is compact (isomorphic to S^1). In the finite dimensional case, we found a HIGHEST weight, which gave us complete control over the representation. The trouble is that in infinite dimensions, there is no reason for the highest weight to exist, and in general they don't. The highest weight requires a finite number of eigenvalues.

A good substituted for the highest weight vector: Look at the Casimir operator $\Omega = 2EF + 2FE + H^2 + 1$. The key point is that Ω is in the center of the universal enveloping algebra. As V is assumed admissible, we can conclude that Ω has eigenvectors (because we can find a finite dimensional space acted on by Ω). As V is irreducible and Ω commutes with G, all of V is an eigenspace of Ω. We'll see that this gives us about as much information as a highest weight vector.

Let the eigenvalue of Ω on V be λ^2 (the square will make the interesting representations have integral λ; the $+1$ in Ω is for the same reason).

Suppose $v \in V_n$, where V_n is the space of vectors where H has eigenvalue n. In the finite dimensional case, we looked at Ev, and saw that $HEv = (n+2)Ev$. What is FEv? If v was a highest weight vector, we could control this. Notice that $\Omega = 4FE + H^2 + 2H + 1$ (using $[E, F] = H$), and $\Omega v = \lambda^2 v$. This says that $4FEv + n^2v + 2nv + v = \lambda^2 v$. This shows that FEv is a multiple of v.

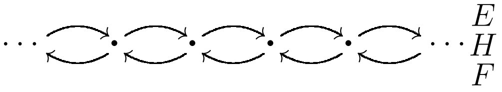
Now we can draw a picture of what the representation looks like:



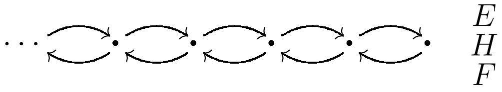
Thus, V_ω is spanned by V_{n+2k}, where k is an integer. The non-zero elements among the V_{n+2k} are linearly independent as they have different eigenvalues. The only question remaining is whether any of the V_{n+2k} vanish.

There are four possible shapes for an irreducible representation

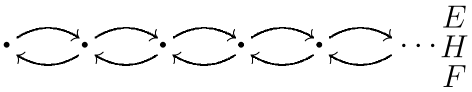
– infinite in both directions:



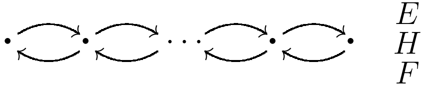
– a lowest weight, and infinite in the other direction:



– a highest weight, and infinite in the other direction:

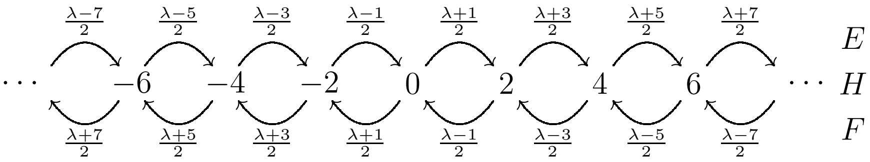


– we have a highest weight and a lowest weight, in which case it is finite dimensional



We'll see that all these show up. We also see that an irreducible representation is completely determined once we know λ and some n for which $V_n \neq 0$. The remaining question is to construct representations with all possible values of $\lambda \in \mathbb{C}$ and $n \in \mathbb{Z}$. n is an integer because it must be a representations of the circle.

If n is even, we have



It is easy to check that these maps satisfy $[E,F]=H$, $[H,E]=2E$, and $[H,F]=-2F$

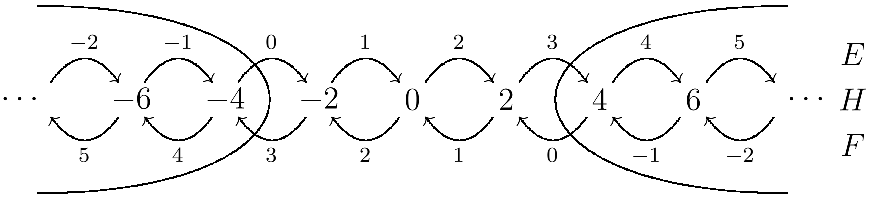
▶ **Exercise 31.1.** Do the case of n odd.

Problem: These may not be irreducible, and we want to decompose them into irreducible representations. The only way they can fail to be irreducible if if $Ev_n = 0$ of $Fv_n = 0$ for some n (otherwise, from any vector, you can generate the whole space). The only ways that can happen is if

n even: λ an odd integer
n odd: λ an even integer.

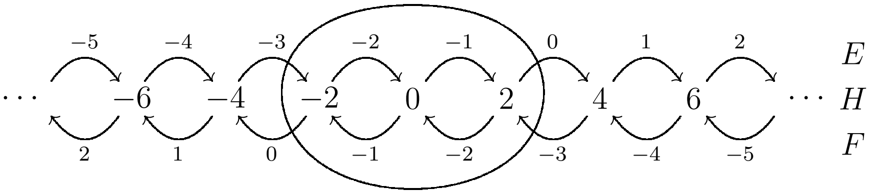
What happens in these cases? The easiest thing is probably just to write out an example.

Example 31.1. Take n even, and $\lambda = 3$, so we have



You can just see what the irreducible subrepresentations are ... they are shown in the picture. So V has two irreducible subrepresentations V_- and V_+, and $V/(V_- \oplus V_+)$ is an irreducible 3 dimensional representation.

Example 31.2. If n is even, but λ is negative, say $\lambda = -3$, we get



Here we have an irreducible finite dimensional representation. If you quotient out by that subrepresentation, you get $V_+ \oplus V_-$.

▶ **Exercise 31.2.** Show that for n odd, and $\lambda = 0$, $V = V_+ \oplus V_-$.

So we have a complete list of all irreducible admissible representations:

1. if $\lambda \notin \mathbb{Z}$, you get one representation (remember $\lambda \equiv -\lambda$). This is the bi-infinite case.

2. Finite dimensional representation for each $n \geq 1$ ($\lambda = \pm n$)

3. Discrete series for each $\lambda \in \mathbb{Z} \smallsetminus \{0\}$, which is the half infinite case: you get a lowest weight when $\lambda < 0$ and a highest weight when $\lambda > 0$.
4. two "limits of discrete series" where n is odd and $\lambda = 0$.

Which of these can be made into *unitary* representations? $H^\dagger = -H$, $E^\dagger = F$, and $F^\dagger = E$. If we have a hermitian inner product $(\ , \ )$, we see that

$$\begin{aligned}(v_{j+2}, v_{j+2}) &= \frac{2}{\lambda + j + 1}(Ev_j, v_{j+2}) \\ &= \frac{2}{\lambda + j + 1}(v_j, -Fv_{j+2}) \\ &= -\frac{2}{\lambda + j + 1}\frac{\overline{\lambda - j - 1}}{2}(v_j, v_j) > 0\end{aligned}$$

where we fix the sign errors. So we want $-\frac{\overline{\lambda-1-j}}{\lambda+j+1}$ to be real and positive whenever $j, j+2$ are non-zero eigenvectors. So

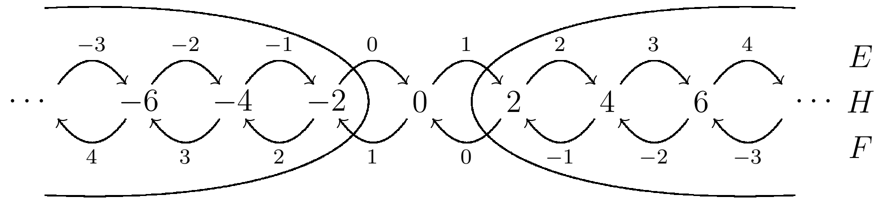
$$-(\lambda - 1 - j)(\lambda + 1 + j) = -\lambda^2 + (j+1)^2$$

should be positive for all j. Conversely, when you have this, blah.

This condition is satisfied in the following cases:

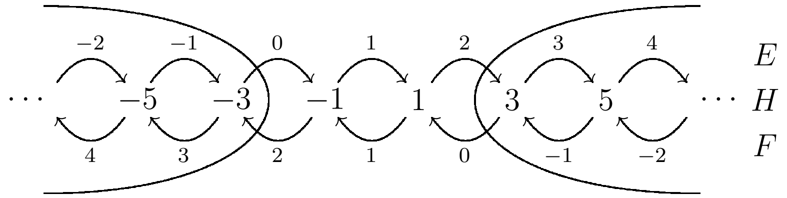
1. $\lambda^2 \leq 0$. These representations are called PRINCIPAL SERIES representations. These are all irreducible *except* when $\lambda = 0$ and n is odd, in which case it is the sum of two limits of discrete series representations
2. $0 < \lambda < 1$ and j even. These are called COMPLEMENTARY SERIES. They are annoying, and you spend a lot of time trying to show that they don't occur.
3. $\lambda^2 = n^2$ for $n \geq 1$ (for some of the irreducible pieces).

   If $\lambda = 1$, we get



   We see that we get two discrete series and a 1 dimensional representation, all of which are unitary

For $\lambda = 2$ (this is the more generic one), we have



The middle representation (where $(j+1)^2 < \lambda^2 = 4$ is NOT unitary, which we already knew. So the DISCRETE SERIES representations ARE unitary, and the FINITE dimensional representations of dimension greater than or equal to 2 are NOT.

Summary: the irreducible unitary representations of $SL_2(\mathbb{R})$ are given by

1. the 1 dimensional representation
2. Discrete series representations for any $\lambda \in \mathbb{Z} \smallsetminus \{0\}$
3. Two limit of discrete series representations for $\lambda = 0$
4. Two series of principal series representations:

$$\begin{array}{l} j \text{ even: } \lambda \in i\mathbb{R},\ \lambda \geq 0 \\ j \text{ odd: } \lambda \in i\mathbb{R},\ \lambda > 0 \end{array}$$

5. Complementary series: parameterized by λ, with $0 < \lambda < 1$.

The nice stuff that happened for $SL_2(\mathbb{R})$ breaks down for more complicated Lie groups.

Representations of finite covers of $SL_2(\mathbb{R})$ are similar, except j need not be integral. For example, for the double cover $\widehat{SL_2(\mathbb{R})} = Mp_2(\mathbb{R})$, $2j \in \mathbb{Z}$.

▶ **Exercise 31.3.** Find the irreducible unitary representations of $Mp_2(\mathbb{R})$.

Solutions to (some) Exercises

Solution 1.1. Yes. Consider $\mu^{-1}(e) \subseteq G \times G$. We would like to use the implicit function theorem to show that there is a function f (which is as smooth as μ) such that $(h, g) \in \mu^{-1}(e)$ if and only if $g = f(h)$. This function will be ι. You need to check that for every g, the derivative of left multiplication by g at g^{-1} is non-singular (i.e. that $dl_g(g^{-1})$ is a non-singular matrix). This is obvious because we have an inverse, namely $dl_{g^{-1}}(e)$.

Solution 1.2. Just do it.

Solution 4.1. We calculate:

$$\begin{aligned} \frac{d}{dt}\|g(t)\|^2 &= 2\langle \frac{d}{dt}g, g\rangle \\ &\leq 2\left\|\frac{d}{dt}g\right\| \|g\| \\ &\leq 2\|\xi\|\|g\|^2. \end{aligned}$$

That is, $\eta(t) := \|g(t)\|^2$ satisfies the differential inequality:

$$\frac{d}{dt}\eta(t) \leq \|\xi\|\eta(t),$$

which in turn implies (Gronwall's inequality) that

$$\eta(t) \leq e^{2\int_{t_0}^{t} \|\xi(s)\|ds}$$

so that

$$\begin{aligned} \|g\| &\leq e^{\int_{t_0}^{t} \|\xi(s)\|ds} \\ &\leq C'|t - t_0| \end{aligned}$$

since for $|t - t_0|$ sufficiently small, exponentiation is Lipschitz.

Solution 8.2. We would like to compute the coefficients of the product $(X^a H^b Y^r)(X^s H^c Y^d)$ once it is rewritten in the PBW basis by repeatedly applying the relations $XY - YX = \varepsilon H$, $HX = XH$, and $HY = YH$. Check by induction that

$$Y^r X^s = \sum_{n=0}^{\infty} \varepsilon^n (-1)^n n! \binom{r}{n}\binom{s}{n} X^{r-n} H^l Y^{s-n}.$$

It follows that $p_n^{I,J}$ is zero unless $I = (Y, \ldots, Y)$ and $J = (X, \ldots, X)$, in which case $p_n^{I,J} = \frac{(-1)^n}{n!} H^n$.

Solution 9.3. We have $[a,b]_h := \sum_{n=0}^{\infty} h^n m_n(a,b)$, where $m_0(a,b) = [a,b]$. Now we compute

$$
\begin{aligned}
[a,[b,c]_h]_h &= [a, \sum_{l\geq 0} h^l m_l(b,c)]_h \\
&= \sum_{l\geq 0} h^l \sum_{k\geq 0} h^k m_k(a, m_l(b,c)) \\
&= \sum_{N\geq 0} h^N m_k(a, m_{N-k}(b,c)) \qquad (N = k+l)
\end{aligned}
$$

Adding the cyclic permutations and looking at the coefficient of h^N, we get the desired result.

Solution 11.1. $[\mathfrak{g}, \mathcal{D}\mathfrak{g}] \subseteq [\mathfrak{g},\mathfrak{g}] = \mathcal{D}\mathfrak{g}$, so $\mathcal{D}\mathfrak{g}$ is an ideal.

Solution 11.2. $[G,G]$ is normal because $r[g,h]r^{-1} = [rgr^{-1}, rhr^{-1}]$. To see that $[G,G]$ is connected, let $\gamma_{gh} : [0,1] \to G$ be a path from g to h. Then $t \mapsto g\gamma(t)g^{-1}\gamma(t)^{-1}$ is a path in $[G,G]$ from the identity to $[g,h]$. Since all the generators of $[G,G]$ are connected to $e \in G$ by paths, all of $[G,G]$ is connected to e.

Now we show that the Lie algebra of $[G,G]$ is $\mathcal{D}\mathfrak{g}$. Consider the Lie algebra homomorphism $\pi : \mathfrak{g} \to \mathfrak{g}/\mathcal{D}\mathfrak{g}$. Since G is simply connected, Theorem 4.4 says there is a Lie group homomorphism $p : G \to H$ lifting π.

$$\begin{tikzcd}
\mathcal{D}\mathfrak{g} \arrow[r] & \mathfrak{g} \arrow[r, "\pi"] \arrow[d, "\exp"'] & \mathfrak{g}/\mathcal{D}\mathfrak{g} \arrow[d, "\exp"] \\
{[G,G]} \arrow[r] & G \arrow[r, dotted, "p"] & H \cong \mathbb{R}^n
\end{tikzcd}$$

where H is the simply connected Lie group with Lie algebra $\mathfrak{g}/\mathcal{D}\mathfrak{g}$. Note that the Lie algebra of the kernel of p must be contained in $\ker \pi = \mathcal{D}\mathfrak{g}$. Also, $\mathfrak{g}/\mathcal{D}\mathfrak{g}$ is abelian, so H is abelian, so $[G,G]$ is in the kernel of p. This shows that $\mathrm{Lie}([G,G]) \subseteq \mathcal{D}\mathfrak{g}$.

To see that $\mathcal{D}\mathfrak{g} \subseteq \mathrm{Lie}([G,G])$, assume that $\mathfrak{g} \subseteq \mathfrak{gl}(V)$. Then for $X, Y \in \mathfrak{g}$ consider the path $\gamma(t) = \exp(X\sqrt{t})\exp(Y\sqrt{t})\exp(-X\sqrt{t})\exp(-Y\sqrt{t})$ in $[G,G]$:

$$
\begin{aligned}
\gamma(t) &= \left(1 + X\sqrt{t} + \frac{1}{2}X^2 t + \cdots\right)\left(1 + Y\sqrt{t} + \frac{1}{2}Y^2 t + \cdots\right) \times \\
&\qquad \left(1 - X\sqrt{t} + \frac{1}{2}X^2 t + \cdots\right)\left(1 - Y\sqrt{t} + \frac{1}{2}Y^2 t + \cdots\right) \\
&= 1 + \sqrt{t}(X + Y - X - Y) + \\
&\qquad t(XY - X^2 - XY - YX - Y^2 + X^2 + Y^2) + \cdots \\
&= 1 + t[X,Y] + O(t^{3/2})
\end{aligned}
$$

So $\gamma'(0) = [X, Y]$. This shows that $[G, G]$ is a connected component of the kernel of p.

Since $[G, G]$ is a connected component of $p^{-1}(0)$, it is closed in G.

Solution 11.3. Let $\pi : \mathfrak{g} \to \mathfrak{g}/\operatorname{rad}\mathfrak{g}$ be the canonical projection, and assume $\mathfrak{a} \in \mathfrak{g}/\operatorname{rad}\mathfrak{g}$ is solvable. Then $\mathcal{D}^k\mathfrak{a} = 0$ for some k, so $\mathcal{D}^k\pi^{-1}(\mathfrak{a}) \subseteq \operatorname{rad}\mathfrak{g}$. Since $\operatorname{rad}\mathfrak{g}$ is solvable, we have that $\mathcal{D}^N\pi^{-1}(\mathfrak{a}) = 0$ for some N. By definition of $\operatorname{rad}\mathfrak{g}$, we get that $\pi^{-1}(\mathfrak{a}) \subseteq \operatorname{rad}\mathfrak{g}$, so $\mathfrak{a} = 0 \subseteq \mathfrak{g}/\operatorname{rad}\mathfrak{g}$. Thus, $\mathfrak{g}/\operatorname{rad}\mathfrak{g}$ is semisimple.

Solution 12.1. An invariant form B induces a homomorphism $\mathfrak{g} \to \mathfrak{g}^*$. Invariance says that this homomorphism is an intertwiner of representations of $\mathfrak{g}$ (with the adjoint action on $\mathfrak{g}$ and the coadjoint action on $\mathfrak{g}^*$). Since $\mathfrak{g}$ is simple, these are both irreducible representations. By Schur's Lemma, any two such homomorphisms must be proportional, so any two invariant forms must be proportional.

Solution 12.2. Done in class.

Solution 12.3. yuck.

Solution 12.4. The complex for computing cohomology is

$$\begin{array}{l} 0 \longrightarrow k \xrightarrow{d_0} \operatorname{Hom}(\mathfrak{sl}_2, k) \xrightarrow{d_1} \operatorname{Hom}(\Lambda^2\mathfrak{sl}_2, k) \xrightarrow{d_2} \operatorname{Hom}(\Lambda^3\mathfrak{sl}_2, k) \longrightarrow 0 \\ \qquad c \longmapsto dc(x) = -x \cdot c = 0 \\ \qquad\qquad\qquad f \longmapsto df(x, y) = f([x, y]) \\ \qquad\qquad\qquad\qquad\qquad\quad \alpha \mapsto d\alpha(x, y, z) = \alpha([x, y], z) - \alpha([x, z], y) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad + \alpha([y, z], x) \end{array}$$

We have that $\ker d_1 = k$, so $H^0(\mathfrak{sl}_2, k) = k$. The kernel of d_1 is zero, as we computed in Remark 12.11. Since $\operatorname{Hom}(\mathfrak{sl}_2, k)$ and $\operatorname{Hom}(\Lambda^2\mathfrak{sl}_2, k)$ are both three dimensional, it follows that d_1 is surjective, and since the kernel of d_2 must contain the image of d_1, we know that d_2 is the zero map. This tells us that $H^1(\mathfrak{sl}_2, k) = 0$, $H^2(\mathfrak{sl}_2, k) = 0$, and $H^3(\mathfrak{sl}_2, k) = \operatorname{Hom}(\Lambda^3\mathfrak{sl}_2, k) \cong k$.

Solution 12.5. Let $D \in \mathcal{D}er(\mathfrak{g})$ and let $X, Y \in \mathfrak{g}$. Then

$$\begin{aligned} [D, ad_X]_{\mathcal{D}er(\mathfrak{g})}(Y) &= D([X, Y]) - [X, D(Y)] \\ &= [D(X), Y] + [X, D(Y)] - [X, D(Y)] \\ &= ad_{D(X)}(Y). \end{aligned}$$

Solution 13.1. Let $x \in \mathfrak{h}$, so $[x, h] = 0$. Since ad_{x_n} is a polynomial in ad_x, we get that $[x_n, h] = 0$, so $x_n \in \mathfrak{h}$. Thus, it is enough to show that any nilpotent element in $\mathfrak{h}$ is zero (then $x = x_s + x_n = x_s$ is semisimple). We do this using property 4, that the Killing form is non-degenerate on $\mathfrak{h}$. If $y \in \mathfrak{h}$, then $B(x_n, y) = tr(ad_{x_n} \circ ad_y)$. By Proposition 13.6, $\mathfrak{h}$ is abelian, so $[x_n, y] = 0$, so ad_{x_n} commutes with ad_y. Thus, we can simultaneously upper triangularize ad_{x_n} and ad_y by Engel's theorem. Since ad_{x_n} is nilpotent, it is *strictly* upper triangular so $tr(ad_{x_n} \circ ad_y) = 0$. So $x_n = 0$ by non-degeneracy of B.

Solution 13.2. Since Δ is a finite set in $\mathfrak{h}^*$, we can find some $h \in \mathfrak{h}$ so that $\alpha(h) \neq \beta(h)$ for distinct roots α and β. Then this h is a regular element which gives the right Cartan subalgebra, and the desired properties follow from the properties on page 64.

Solution 13.3. If Δ does not span $\mathfrak{h}^*$, then there is some non-zero $h \in \mathfrak{h}$ such that $\alpha(h) = 0$ for all $\alpha \in \Delta$. This means that all of the eigenvalues of ad_h are zero. Since h is semisimple, $ad_h = 0$. And since ad is faithful, we get $h = 0$, proving property 1.

To prove 2, consider the α-string through β. It must be of the form $\mathfrak{g}_{\beta+n\alpha} \oplus \mathfrak{g}_{\beta+(n-1)\alpha} \oplus \cdots \oplus \mathfrak{g}_{\beta+m\alpha}$ for some integers $n \geq 0 \geq m$. From the characterization of irreducible finite dimensional representations of $\mathfrak{sl}_2$, we know that each eigenvalue of H_α is an integer, so $\beta(H_\alpha) = r \in \mathbb{Z}$ (since $[H_\alpha, X_\beta] = \beta(H_\alpha)X_\beta$). We also know that the eigenvalues of H_α are symmetric around zero, so we must have $-r = (\beta + s\alpha)(H_\alpha)$ for some s for which $\mathfrak{g}_{\beta+s\alpha}$ is in the α-string through β. Then we get $\beta(H_\alpha) + s\alpha(H_\alpha) = \beta(\alpha) + 2s = -r = -\beta(\alpha)$, from which we know that $s = -\beta(H_\alpha)$. Thus, $\mathfrak{g}_{\beta-\beta(H_\alpha)\alpha} \neq 0$, so $\beta - \big(\beta(H_\alpha)\big)\alpha$ is a root.

Finally, we prove 3. If α and $\beta = c\alpha$ are roots, then by property 2, we know that $\alpha(H_\beta) = 2/c$ and $\beta(H_\alpha) = 2c$ are integers (note that $H_\beta = H_\alpha/c$). It follows that $c = \pm\frac{1}{2}, \pm 1$, or ± 2. Therefore, it is enough to show that α and 2α cannot both be roots. To see this, consider the α-string through 2α. We have that $[H_\alpha, X_{2\alpha}] = 2\alpha(H_\alpha)X_{2\alpha} = 4X_{2\alpha}$, so the α-string must have a non-zero element $[Y_\alpha, X_{2\alpha}] \in \mathfrak{g}_\alpha$, which is spanned by X_α. But then we would have that $X_{2\alpha}$ is a multiple of $[X_\alpha, X_\alpha] = 0$, which is a contradiction.

Solution 14.1. If Δ is reducible, with $\Delta = \Delta_1 \cup \Delta_2$, then set $\mathfrak{h}_i^*$ to be the span of Δ_i, and set $\mathfrak{g}_i = \mathfrak{h}_i \oplus \bigoplus_{\alpha\in\Delta_i} \mathfrak{g}_\alpha$ (for $i = 1, 2$). Then we have that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ as a vector space. We must check that $\mathfrak{g}_1$ is an ideal (the by symmetry, $\mathfrak{g}_2$ will also be an ideal). From the relation $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$, we know that it is enough to check that for $\alpha \in \Delta_1$,

$$[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{h}_1, \text{ and} \tag{1}$$

$$[\mathfrak{g}_\alpha, \mathfrak{h}_2] = 0. \tag{2}$$

Letting $\beta \in \Delta_2$, we have that $\beta([X_\alpha, Y_\alpha]) = \beta(H_\alpha) = \frac{2(\alpha,\beta)}{(\beta,\beta)} = 0$ because Δ_1 and Δ_2 are orthogonal; 1 follows because Δ_2 spans the orthogonal complement of $\mathfrak{h}_1$ in $\mathfrak{h}$. Similarly, we have $[X_\alpha, H_\beta] = \alpha(H_\beta)X_\alpha = 0$; 2 follows because the H_β span $\mathfrak{h}_2$.

Conversely, if $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ as a Lie algebra, then take root decompositions $\mathfrak{g}_1 = \mathfrak{h}_1 \oplus \bigoplus_{\alpha\in\Delta_1} \mathfrak{g}_\alpha$ and $\mathfrak{g}_2 = \mathfrak{h}_2 \oplus \bigoplus_{\beta\in\Delta_2} \mathfrak{g}_\beta$, with respect to regular elements $h_1 \in \mathfrak{h}_1$ and $h_2 \in \mathfrak{h}_2$. Then for $x_1 \in \mathfrak{g}_1$ and $x_2 \in \mathfrak{g}_2$, we have that $[h_1 + h_2, x_1 + x_2] = [h_1, x_1] + [h_2, x_2]$; it follows that $h_1 + h_2$ is a regular element in $\mathfrak{g}$. The Cartan given by this element is clearly $\mathfrak{h}_1 \oplus \mathfrak{h}_2$. If $x \in \mathfrak{g}_\alpha \subseteq \mathfrak{g}_1$, then we have $[h_1 + h_2, x] = \alpha(h_1)x + 0$, so α is a root. Similarly, each $\beta \in \Delta_2$ is a root. Since we have accounted for all the root spaces of $\mathfrak{g}_1$ and of $\mathfrak{g}_2$, we have a root decomposition $\mathfrak{g} = (\mathfrak{h}_1 \oplus \mathfrak{h}_2) \oplus \bigoplus_{\alpha\in\Delta_1} \mathfrak{g}_\alpha \oplus \bigoplus_{\beta\in\Delta_2} \mathfrak{g}_\beta$. This shows that $\Delta = \Delta_1 \cup \Delta_2$.

Solution 14.2. Note that $Ad_{S_\alpha} = \exp(ad_{X_\alpha})\exp(-ad_{Y_\alpha})\exp(ad_{X_\alpha})$. If $h \in \mathfrak{h}$, then $ad_{X_\alpha}h = -\alpha(h)X_\alpha$ and $ad_{X_\alpha}ad_{X_\alpha}(h) = \alpha(h)ad_{X_\alpha}(X_\alpha) = 0$. Using the power series expansion for exp, we get that

$$\exp(ad_{X_\alpha})(h) = h - \alpha(h)X_\alpha.$$

Similarly, we apply $\exp(-ad_{Y_\alpha})$ to the result

$$\begin{aligned}\exp(-ad_{Y_\alpha})\big(h-\alpha(h)X_\alpha\big) \\ &= h - \alpha(h)Y_\alpha - \alpha(h)\big(X_\alpha - \underbrace{[Y_\alpha, X_\alpha]}_{-H_\alpha} + \underbrace{\frac{1}{2}[Y_\alpha,[Y_\alpha,X_\alpha]]}_{-\frac{1}{2}\alpha(H_\alpha)Y_\alpha=-Y_\alpha} + 0\big) \\ &= h - \alpha(h)(X_\alpha + H_\alpha)\end{aligned}$$

and then apply $\exp(ad_{X_\alpha})$

$$\begin{aligned}\exp(ad_{X_\alpha})\big(h-\alpha(h)(X_\alpha + H_\alpha)\big) \\ &= h - \alpha(h)X_\alpha - \alpha(h)\Big(\big(X_\alpha + 0\big) + \big(H_\alpha - \alpha(H_\alpha)X_\alpha + 0\big)\Big) \\ &= h - \alpha(h)H_\alpha.\end{aligned}$$

This shows that $Ad_{S_\alpha}(\mathfrak{h}) = \mathfrak{h}$. For $\lambda \in \mathfrak{h}^*$, we get

$$\begin{aligned}\langle r_\alpha(\lambda), h\rangle &= \lambda(h) - \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}\alpha(h) \\ &= \lambda(h) - \frac{2\lambda(H_\alpha)}{\alpha(H_\alpha)}\alpha(h) &&\text{(using Equation 14.2)} \\ &= \lambda\big(h - \alpha(h)H_\alpha\big) &&\big(\alpha(H_\alpha) = 2\big) \\ &= \langle \lambda, Ad_{S_\alpha}(h)\rangle.\end{aligned}$$

Solution 15.1. It is immediate to verify RS1 and RS3. One may check that the proposed sets of simple roots are correct by checking that every root can be written as a non-positive or non-negative integer combination of the proposed simple roots. It is not hard to verify that the given root systems satisfy $r_\alpha(\Delta) = \Delta$ for each $\alpha \in \Delta$.

Finally, it is enough to verify RS2 in the case where β is a simple root. Since every root is an integer sum of simple roots, it is enough to consider the case where α is also a simple root. This amounts to checking that the given number of lines between α and β is correct, which is relatively straightforward (keeping in mind Warning 15.5).

Solution 16.1. It is enough to check that the proposed endomorphisms of $T(Y) \otimes S\mathfrak{h} \otimes T(X)$ satisfy (Ser1). Then the universal property $\tilde{\mathfrak{g}}$ (from Remark 16.2) and the universal property of $U\tilde{\mathfrak{g}}$ (from Proposition 7.1) tell us exactly that there is a unique algebra homomorphism $U\tilde{\mathfrak{g}} \to \operatorname{End}\big(T(Y) \otimes S\mathfrak{h} \otimes T(X)\big)$ such that X_i, Y_i, and H_i act as described. We get (Ser1a), (Ser1b), and (Ser1d) by construction. We need only check that H_iH_j acts in the same way as H_jH_i. It is clear that $H_iH_j(1 \otimes b \otimes c) = H_jH_i(1 \otimes b \otimes c)$. Now we induct on the degree of a.

$$\begin{aligned} H_iH_j(Y_ka \otimes b \otimes c) &= (H_iY_kH_j - a_{jk}H_iY_k)(a \otimes b \otimes c) && \text{(Ser1b)} \\ &= (Y_kH_iH_j - a_{ik}Y_kH_j \\ &\qquad - a_{jk}Y_kH_i + a_{jk}a_{ik}Y_k)(a \otimes b \otimes c) && \text{(Ser1b)} \\ &= H_jH_i(Y_ka \otimes b \otimes c) && (i,\, j \text{ symmetric}) \end{aligned}$$

This shows that the representation is well defined.

Solution 16.2. It is easy to check by induction that in $U\tilde{\mathfrak{g}}$,

$$\begin{aligned} H_kX_i^r &= X_i^rH_k + ra_{ki}X_i^r, \text{ and} \\ Y_kX_i^r &= X_i^rY_k - r\delta_{ik}\big(X_i^{r-1}H_i + (r-1)X_i^{r-1}\big). \end{aligned}$$

Since ad is a representation, it follows that

$$\begin{aligned} [H_k, \theta_{ij}^+] &= ad_{H_k} ad_{X_i}^{1-a_{ij}} X_j \\ &= ad_{X_i}^{1-a_{ij}} ad_{H_k} X_j + (1-a_{ij})a_{ki} ad_{X_i}^{1-a_{ij}} X_j \\ &= (a_{kj} + a_{ki} - a_{ij}a_{ki})\theta_{ij}^+ \\ [Y_k, \theta_{ij}^+] &= ad_{X_i}^{1-a_{ij}}[Y_k, X_j] = 0 && (\text{if } k \neq j) \\ [Y_j, \theta_{ij}^+] &= ad_{X_i}^{1-a_{ij}} \overbrace{[Y_j, X_j]}^{-H_j} - (1-a_{ij}) ad_{X_i}^{-a_{ij}} \overbrace{[H_i, X_j]}^{a_{ij}X_j} \\ &\qquad\qquad + (1-a_{ij})a_{ij} ad_{X_i}^{-a_{ij}} X_j \\ &= a_{ji} ad_{X_i}^{-a_{ij}} X_i \end{aligned}$$

which is zero if $a_{ij} = a_{ji} = 0$, and is zero if $a_{ij} < 0$.

Solution 17.1. It is enough to show that each basis vector of $\Lambda^3 E$ is in the orbit of ω. Let p_u, p_v, and p_1 be the projections onto $span\{u\}$, $span\{v_1, v_2, v_3\}$, and $span\{v_1, w_1\}$ respectively. For $x, y \in S := \{u, v_1, v_2, v_3, w_1, w_2, w_3\}$, let $\phi_{x\to y}$ be the element of $\mathfrak{gl}(7)$ sending x to y, and sending the rest of S to zero. Then a little messing around produces

x	$x \cdot \omega$	x	$x \cdot \omega$
$\frac{1}{3}(p_v - p_u)$	$v_1 \wedge v_2 \wedge v_3$	$\phi_{v_1\to u}$	$u \wedge v_2 \wedge v_3$
$\frac{1}{2}p_1 + \frac{1}{2}p_u - \frac{1}{6}\mathrm{Id}$	$u \wedge v_1 \wedge w_1$	$\phi_{w_1\to w_2}$	$u \wedge v_1 \wedge w_2$
$\phi_{v_3\to w_1} + \phi_{w_2\to u}$	$v_1 \wedge v_2 \wedge w_1$	$\phi_{v_3\to w_3}$	$v_1 \wedge v_2 \wedge w_3$

Any other basis vector can be obtained from one of these (up to a sign) by permuting indices and/or swapping v's and w's, so we can get all of them.[1]

Solution 18.1. Every regular semisimple element is in some Cartan subalgebra; namely, the Cartan subalgebra of elements that commute with it. We will show that regular semisimple elements are dense in $\mathfrak{g}$.

Choose a basis for $\mathfrak{g}$, which gives you a corresponding basis for $\mathfrak{gl}(\mathfrak{g})$. Say $\mathfrak{g}$ has rank r. Let I be an indexing set so that for a matrix $A \in \mathfrak{gl}(\mathfrak{g})$, the set $\{M_\gamma(A)\}_{\gamma\in I}$ is the set of all $(n-r)\times(n-r)$ minors of A. Define $f_\gamma : \mathfrak{g} \to k$ by $f_\gamma(x) = \det\big(M_\gamma(ad_x)\big)$. Since ad is linear, f_γ is a polynomial map for each γ. Now consider union of all of the zero sets of all of the f_γ. This is a Zariski closed set, so its complement in $\mathfrak{g}$ is a Zariski open set. Since $\mathfrak{g}$ has a regular element (a semisimple element h, where ad_h is rank $n-r$), that open set is non-empty, and since $\mathfrak{g} \cong \mathbb{A}^{\dim \mathfrak{g}}$ is irreducible, this set is dense.

Solution 18.2.

Solution 18.3. It is not hard to set up a recursive calculation with the numbers in the hint. Alternatively, note that the Kostant partition function tells us exactly that

$$\begin{aligned} ch\, M(\lambda) &= e^\lambda \prod_{\alpha\in\Delta^+} (1 + e^{-\alpha} + e^{-2\alpha} + \cdots) \\ &= e^\lambda \prod_{\alpha\in\Delta^+} (1 - e^{-\alpha})^{-1}. \end{aligned}$$

You can easily (have your computer) compute the coefficients of this power series. For example, to compute the character of a Verma

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[1] Since ω is not quite invariant under permutations of the indices or swapping of v's and w's, you sometimes have to tweak a sign (e.g. to get $w_1 \wedge w_2 \wedge v_1$, take $x = \phi_{w_3\to v_1} - \phi_{v_2\to u}$).

module of G_2, I think of e^{α_1} as `x` and of e^{α_2} as `y`. Then the following Mathematica code returns the first 144 multiplicities.

```
Nmax = 12;
mySeries=Series[
  ((1-x)(1-y)(1- x y)(1- x^2 y)(1- x^3 y)(1- x^3 y^2))^(-1),
   {x,0,Nmax},{y,0,Nmax}];
TableForm[Table[SeriesCoefficient[
                    mySeries,{i,j}],{i,0,Nmax},{j,0,Nmax}]]
```

Solution 20.1. Since G is abelian, $\mathfrak{g}$ is the abelian Lie algebra $\mathbb{R}^n$, whose simply connected Lie group is $\mathbb{R}^n$. Thus, G is a quotient of $\mathbb{R}^n$ by a discrete subgroup (i.e. a lattice). Since G is compact, this lattice must be full rank, so $G \cong \mathbb{T}^n$.

Solution 20.2. Consider the representation $G \to \mathrm{End}(\Lambda^{\mathrm{top}} T_eG) \simeq \mathrm{End}(\mathbb{R}) = \mathbb{R}^\times$ given by $h \mapsto \Lambda^{\mathrm{top}} Ad_h$. Since G is compact, its image must also be compact, but the only compact subgroups of $\mathbb{R}^\times$ are $\{1\}$ and $\{\pm 1\}$.

If G is connected, the image must be $\{1\}$, so the adjoint action on $\Lambda^{\mathrm{top}} T_eG$ is trivial. It follows that $R_h^*\omega_e = L_h^*\omega_e = \omega_h$, i.e. that ω is right invariant.

If G is not connected, then we may have $R_h^*\omega_e = -\omega_h$. That is, the left invariant form agrees with the right invariant form up to sign. Since the volume form determines the orientation, changing it by a sign does not change the measure.

Solution 23.1. In H, the norm of any non-zero vector is 1. It is immediate to check that the reflection of a non-zero vector v through another non-zero vector u is

$$r_u(v) = \begin{cases} u & \text{if } u = v \\ v + u & \text{if } u \neq v \end{cases}$$

so reflection through a non-zero vector fixes that vector and swaps the two other non-zero vectors. Thus, the reflection in H generate the symmetric group on three elements S_3, acting on the three non-zero vectors.

If u and v are non-zero vectors, then $(u, v) \in H \oplus H$ has norm $1 + 1 = 0$, so one cannot reflect through it. Thus, every reflection in V is “in one of the H’s,” so the group generated by reflections is $S_3 \times S_3$. However, swapping the two H’s is clearly an orthogonal transformation, so reflections do not generate $O_V(\mathbb{F}_2)$.

References

[Ada69] J. Frank Adams. *Lectures on Lie groups.* W. A. Benjamin, Inc., New York-Amsterdam, 1969.

[Ada96] J. Frank Adams. *Lectures on exceptional Lie groups.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1996.

[Bor91] Armand Borel. *Linear algebraic groups*, volume 126 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second enlarged edition, 1991.

[CSM95] Roger Carter, Graeme Segal, and Ian Macdonald. *Lectures on Lie groups and Lie algebras*, volume 32 of *London Mathematical Society Student Texts.* Cambridge University Press, Cambridge, 1995.

[FH91] William Fulton and Joe and Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1991. 8, 50, 69

[Fuc50] Dmitry Fuchs. *Cohomology of infinite dimensional Lie algebras?* I don't know if this book exists, 2050. 45

[Har77] Robin Hartshorne. *Algebraic geometry.* Springer-Verlag, New York, 1977. 69

[Hat02] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002. 21, 60

[Hel01] Sigurdur Helgason. *Differential geometry, Lie groups, and symmetric spaces*, volume 34 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2001. 168

[HS97] P. J. Hilton and U. Stammbach. *A course in homological algebra*, volume 4 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1997.

[Hum78] James E. Humphreys. *Introduction to Lie algebras and representation theory*, volume 9 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1978. 65

[Kac90] Victor G. Kac. *Infinite dimensional Lie algebras.* Cambridge University Press, Cambridge, 1990. 168

[Kna02] Anthony W. Knapp. *Lie groups beyond an introduction*, volume 140 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 2002.

[Knu01] Allen Knutson. *Math 261A - Lie groups and Lie algebras*. `http://math.berkeley.edu/~allenk/courses/fall01/261/`, Fall 2001.

[Knu02] Allen Knutson. *Math 261B - Lie groups and Lie algebras*. `http://math.berkeley.edu/~allenk/courses/spr02/261b/`, Spring 2002.

[Lan02] Serge Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2002. 57

[Lee03] John M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2003. 9

[Mon93] Susan Montgomery. *Hopf algebras and their actions on rings*, volume 82 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1993. 25

[OV90] A. L. Onishchik and È. B. Vinberg. *Lie groups and algebraic groups*. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1990.

[Ser87] Jean-Pierre Serre. *Complex semisimple Lie algebras*. Springer-Verlag, New York, 1987.

[War83] Frank W. Warner. *Foundations of differentiable manifolds and Lie groups*, volume 94 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1983.

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