CLASSIFICATION OF SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS ADMITTING LIE GROUPS OF FIBRE-PRESERVING POINT SYMMETRIES

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ABSTRACT

We use Elie Cartan's method of equivalence to give a complete classification, in terms of differential invariants, of second-order ordinary differential equations admitting Lie groups of fibre-preserving point symmetries. We then apply our results to the determination of all second-order equations which are equivalent, under fibre-preserving transformations, to the free particle equation. In addition we present those equations of Painlevé type which admit a transitive symmetry group. Finally we determine the symmetry group of some equations of physical interest, such as the Duffing and Holmes–Rand equations, which arise as models of non-linear oscillators.

1. *Introduction*

In the present paper, we give a complete classification (based on Elie Cartan's method of equivalence) of second-order ordinary differential equations,

$$\frac{d^2y}{dx^2} = F\left(x, y, \frac{dy}{dx}\right), \tag{1.1}$$

admitting Lie groups of symmetries of the form

$$\bar{x} \circ \Phi = \phi(x), \quad \bar{y} \circ \Phi = \psi(x, y). \tag{1.2}$$

The role played by the symmetry group of a differential equation in its integration is crucial, as illustrated in the following classical result of Lie [**22, 23**].

Consider a system of n first-order ordinary differential equations

$$\frac{dx^a}{dt} = F^a(x^b, t), \tag{1.3}$$

and suppose that G is an s-dimensional solvable Lie group of symmetries of (1.3) acting regularly on s-dimensional orbits. Then the solutions of (1.3) can be found by *quadratures* from the solutions of a system of $n-s$ first-order ordinary differential equations. In particular, if G is n-dimensional, then the general solution can be found by quadratures alone. (See [**23**] for some illustrations of this result.)

In *Leçons sur les invariants intégraux* [**6**], Cartan points out that Lie groups were in fact *discovered* by considering the problem of integrating certain systems of ordinary differential equations. Indeed, let us associate to (1.3) the Pfaffian system Σ on $\mathbb{R}^{n+1}$ generated by the n one-forms $\eta^a := dx^a - F^a(x^b, t)\, dt$. Any function f which is constant on solution curves of Σ (that is, such that $df \in \Sigma$) is called a *first integral,* and any differential form constructed from first integrals and their exterior derivatives is called an *invariant form.*

Suppose that there exist n non-degenerate infinitesimal symmetries, that is, n

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independent vector fields X_a such that

$$\mathscr{L}_{X_a}\Sigma \subseteq \Sigma, \quad \det(S_a^b := X_a \lrcorner \eta^b) \neq 0. \tag{1.5}$$

The one-forms θ^a defined by

$$\theta^a = (S^{-1})^a_b \eta^b$$

are invariant generators for Σ [6]. As the exterior derivative of an invariant differential form is also invariant, one has

$$d\theta^a = \tfrac{1}{2} C^a_{bc} \theta^b \wedge \theta^c, \tag{1.6}$$

where the C^a_{bc} are first integrals.

One now selects a maximal functionally independent set $\{f^1, \ldots, f^r\}$ from the functions C^a_{bc}. As the f^A are first integrals, the exterior derivatives df^A may be expressed in terms of the θ^a as

$$df^A = f^A_a \theta^a,$$

where the f^A_a are further first integrals. By iterating this procedure, one obtains a functionally independent set $\{f^\alpha \mid \alpha = 1, \ldots, s\}$ of first integrals such that no new functionally independent first integral can be obtained by differentiation. One may now choose as generators for Σ the s one-forms df^α and $n-s$ complementary one-forms among the θ^a:

$$\Sigma = \{df^\alpha, \theta^i \mid \alpha = 1, \ldots, s, i = 1, \ldots, n-s\}. \tag{1.7}$$

On the subset $M := \{f^\alpha = C^\alpha \mid C^\alpha \text{ constant}\}$ which we assume to be an $(n-s)$-dimensional submanifold, (1.6) reads

$$d\theta^i = \tfrac{1}{2} C^i_{jk} \theta^j \wedge \theta^k, \tag{1.8}$$

where the C^i_{jk} are *constants*.

The set of diffeomorphisms Φ of M which satisfy

$$\Phi^* \theta^i = \theta^i, \tag{1.9}$$

forms a group under composition. Furthermore the above equation defines a completely integrable Pfaffian system and thus it follows from the Frobenius Theorem that the coordinate functions of Φ depend on $n-s$ arbitrary constants. Note that equations (1.8) are nothing but the Maurer–Cartan equations for the $(n-s)$-dimensional Lie group whose action preserves the invariant generators θ^i.

It is important to note that in contrast to Lie's method [22, 23] which requires integrating a system of determining equations in order to obtain the differential invariants, Cartan's approach produces these differential invariants by means of differentiations only.

Our paper is organized as follows. In § 2 we state what is meant by a Cartan equivalence problem and its solution and recall necessary and sufficient conditions for the collection of maps which solve an equivalence problem to give the local action of a Lie group on the underlying manifold. In § 3, we recall the main steps of the solution of the equivalence problem for (1.1) under the pseudo-group of fibre-preserving point transformations of the form (1.2) as it appears in references [17] and [18] with some further clarifications which are needed for the symmetry classification. Section 4 contains the statement of our classification theorems for second-order ordinary differential equations admitting fibre-preserving point symmetries. The possible dimensions for a maximal group of fibre-preserving point symmetries, namely six, three, two and one, are considered in turn. Given an equation, our construction gives an adapted co-frame which encodes it as a differential system and contains the Maurer–Cartan forms for a group isomorphic to its symmetry group (the reciprocal group [**11**]). The equivalence class is then characterized by the numerical values of the constant invariants and the functional relations between basic, fundamental, and derived invariants. We then illustrate our results on several examples. We determine and characterize all second-order ordinary differential equations of Painlevé type [**16**] which admit a transitive (that is, a three- or higher-dimensional) Lie group of fibre-preserving point symmetries. The problem of characterizing the Painlevé transcendents, which have no non-trivial symmetries for generic values of the parameters on which they depend, is treated in [**19**]. We then give a complete solution to the linearizability problem under fibre-preserving point transformations, improving the results of Sarlet *et al.* [**26**]. We also determine the symmetry group of some second-order equations of physical importance, such as the Duffing equation [**13**] and the Holmes–Rand non-linear oscillator [**14**], for which we identify symmetries that were not known before (more examples will be studied in a forthcoming work [**15**].) In Section 5 we prove the classification theorems. As the calculations are very extensive (they were performed using the University of Waterloo Maple symbolic system and took about ten hours of CPU time on a VAX 785) we only give the details of the proof in two cases and indicate the method used for the other ones without giving intermediary results.

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2. *Equivalence problems and Lie group actions*

In this section we show that Elie Cartan's method of equivalence provides necessary and sufficient conditions for a system of differential equations to admit a Lie group of symmetries. This result will be used in § 4 to classify all second-order ordinary differential equations admitting Lie groups of fibre-preserving point symmetries.

For our purposes, the Cartan equivalence problem may be stated as follows [**5, 8**]. Given two co-frames ω_U and $\bar{\omega}_V$ on open subsets U and V of a smooth n-manifold M and a Lie subgroup G of $\mathrm{GL}(n, \mathbb{R})$, determine all diffeomorphisms ϕ: $U \to V$ such that

$$\phi^* \bar{\omega}_V = g\omega_U, \tag{2.1}$$

where g is a G-valued function on U.

In the case of the equivalence problem for differential equations, the co-frames ω_U and $\bar{\omega}_V$ are formed from the exterior differential systems Σ and $\bar{\Sigma}$ associated with the original and target equations together with their respective independence conditions, while G is determined from the admissible transformations of the problem (see § 3).

The solution of the Cartan equivalence problem leads either to the structure equations of an *$\{e\}$-structure* or to the structure equations of an *infinite Lie pseudo-group* [**8**, **20**, **21**]. In the former case we obtain a co-frame (ω^a) and structure equations

$$d\omega^a = \tfrac{1}{2} C^a_{bc}\omega^b \wedge \omega^c \tag{2.2}$$

on $U \times G \times G^{(1)} \times \ldots \times G^{(k)}$, where $(G^{(1)}, \ldots, G^{(k)})$ is the sequence of Abelian groups arising in the prolongation procedure and a ranges from 1 to $N := n + \dim G + \sum_{i=1}^{k} \dim G^{(i)}$. The latter case will not be of interest to us as the equivalence problem for second-order ordinary differential equations under fibre-preserving point transformations leads to an *$\{e\}$-structure.*

Since the extension Φ of any equivalence ϕ satisfies $\Phi^*\bar{\omega} = \omega$, it follows that the structure functions C^a_{bc} appearing in (2.2) are invariants, that is,

$$\bar{C}^a_{bc} \circ \Phi = C^a_{bc}, \tag{2.3}$$

where the $\bar{C}^a_{bc}$ are the corresponding structure functions for $\bar{\omega}^a$.

Necessary and sufficient conditions for equivalence are obtained by constructing a maximal set of functionally independent invariants. This can be achieved by the following procedure. One first chooses a maximal functionally independent set $\{I_1, \ldots, I_v\}$ among the basic invariants C^a_{bc}. The covariant derivatives $I_{p|a}$ defined by

$$dI_p = I_{p|a}\omega^a \tag{2.4}$$

are also invariants. If all these derived invariants $I_{p|a}$ are functions of the invariants I_p, there are no new invariants and, by the chain rule, $\{I_1, \ldots, I_v\}$ forms a maximal functionally independent set of fundamental invariants. Otherwise, one selects $v_1 > v$ independent invariants $I_1, \ldots, I_{v_1}$ among the invariants I_p and $I_{p|a}$ with the property that all the invariants I_p and $I_{p|a}$ generated so far are functions of $I_1, \ldots, I_{v_1}$. New invariants are then formed by taking further covariant derivatives $I_{q|a}$, where $q = 1, \ldots, v_1$. After a finite number l of iterations one obtains a set $\{I_1, \ldots, I_r\}$, where $r = v + \sum_{i=1}^{l} v_i$, of functionally independent fundamental invariants with the property that no new invariants can be obtained by taking covariant derivatives $I_{s|a}$, for $s = 1, \ldots, r$. By the chain rule, the set $\{I_1, \ldots, I_r\}$ is thus maximal and so

$$I_{s|a} = F_{sa}(I_t), \quad \text{where } s, t = 1, \ldots, r, \; a = 1, \ldots, N. \tag{2.5}$$

Performing the same operations on $\bar{\omega}_V$, we obtain the following results.

PROPOSITION 2.1. *There exists a diffeomorphism ϕ: $U \to V$ such that*

$$\phi^*\bar{\omega}_V = g\omega_U \tag{2.6}$$

if and only if corresponding constant invariants for ω_U and $\bar{\omega}_V$ have the same numerical values and F_{sa} and $\bar{F}_{sa}$ are the same functions of their arguments.

Proposition 2.1 thus gives a characterization in terms of differential invariants of the equivalence classes under the equivalence relation defined by (2.6).

PROPOSITION 2.2. *If the necessary and sufficient conditions of Proposition* 2.1 *are satisfied and if $N - r = p > 0$, then the set of diffeomorphisms*

$$\Phi\colon\; U \times G \times G^{(1)} \times \ldots \times G^{(k)} \to V \times G \times G^{(1)} \times \ldots \times G^{(k)}$$

which satisfy

$$\Phi^* \bar{\omega} = \omega, \tag{2.7}$$

defines a p-dimensional Lie group of (local) transformations. Conversely, if the set of diffeomorphisms satisfying (2.7) *defines a p-dimensional Lie group of* (*local*) *transformations then* $N - r = p > 0$, *corresponding constant invariants for* ω_U *and* $\bar{\omega}_V$ *have the same numerical values, and* F_{sa} *and* $\bar{F}_{sa}$ *are the same functions of their arguments.*

Propositions 2.1 and 2.2 are proved by using the technique of the graph and the Frobenius theorem [**8, 28**]. The main lines of the proof can be found in [**8**].

In the following sections, we apply Elie Cartan's method of equivalence to obtain generators for the Pfaffian system $\Sigma = \{dy - p\,dx, dp - F(x, y, p)\,dx\}$ associated with the second-order ordinary differential equation $d^2y/dx^2 = F(x, y, dy/dx)$ which will be invariant under its Lie group of symmetries.

3. *The equivalence problem*

In this section we recall the main steps of the solution given in [**17**] to the local equivalence problem for the equation

$$\frac{d^2y}{dx^2} = F\left(x, y, \frac{dy}{dx}\right)$$

under the first prolongation

$$\begin{aligned} p^1\Phi\colon\ & J^1(\mathbb{R}, \mathbb{R}) \to J^1(\mathbb{R}, \mathbb{R}), \\ & (x, y, p) \to \left(\phi(x), \psi(x, y), \frac{\psi_x + p\psi_y}{\phi_x}\right) \end{aligned}$$

of fibre-preserving point transformations of the form

$$\begin{aligned} \Phi\colon\ & J^0(\mathbb{R}, \mathbb{R}) \to J^0(\mathbb{R}, \mathbb{R}), \\ & (x, y) \to (\phi(x), \psi(x, y)). \end{aligned}$$

The solutions of (1.1) are curves c in $J^1(\mathbb{R}, \mathbb{R})$ which satisfy

$$\begin{aligned} & c^* dx \neq 0, \\ & c^*(dy - p\,dx) = 0, \\ & c^*(dp - F(x, y, p)\,dx) = 0, \end{aligned} \tag{3.1}$$

where (x, y, p) are the standard jet coordinates on $J^1(\mathbb{R}, \mathbb{R})$.

Introducing the differential system

$$\Sigma := \{dy - p\,dx, dp - F(x, y, p)\,dx\} \tag{3.2a}$$

associated with (1.1), and the differential system

$$\bar{\Sigma} := \{d\bar{y} - \bar{p}\,d\bar{x}, d\bar{p} - \bar{F}(\bar{x}, \bar{y}, \bar{p})\,d\bar{x}\} \tag{3.2b}$$

associated with the target equation

$$\frac{d^2\bar{y}}{d\bar{x}^2} = \bar{F}\left(\bar{x}, \bar{y}, \frac{d\bar{y}}{d\bar{x}}\right),$$

we call the two differential equations *equivalent* if there exists a diffeomorphism Φ of $J^0(\mathbb{R}, \mathbb{R})$ such that

$$(p^1\Phi)^*\bar{\Sigma} = \Sigma. \tag{3.3}$$

It immediately follows from (1.2) and (3.1) that Φ will be an equivalence if and only if

$$(p^1\Phi)^*\bar{\omega}_V = g\omega_U, \tag{3.4a}$$

where ω_U and $\bar{\omega}_V$ are co-frames on open neighbourhoods, U and V respectively, of $J^1(\mathbb{R}, \mathbb{R})$ given by

$$\omega_U := \begin{bmatrix} dx \\ dy - p\,dx \\ dp - F(x, y, p)\,dx \end{bmatrix}, \quad \bar{\omega}_V := \begin{bmatrix} d\bar{x} \\ d\bar{y} - \bar{p}\,d\bar{x} \\ d\bar{p} - \bar{F}(\bar{x}, \bar{y}, \bar{p})\,d\bar{x} \end{bmatrix}, \tag{3.4b}$$

and g is a function taking values in the subgroup G of $\mathrm{GL}(3, \mathbb{R})$ of matrices of the form

$$\begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & BC & B/A \end{bmatrix}. \tag{3.4c}$$

This is exactly the form of the Cartan equivalence problem treated by Gardner in **[12]** and we now proceed by applying the algorithm he describes.

Let ω be the collection of one-forms on $J^1(\mathbb{R}, \mathbb{R}) \times G$ obtained from the left action of G on the co-frame ω_U:

$$\begin{bmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{bmatrix} = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & BC & B/A \end{bmatrix} \begin{bmatrix} \omega_U^1 \\ \omega_U^2 \\ \omega_U^3 \end{bmatrix}. \tag{3.5}$$

After absorption of torsion, the structure equations are given by

$$\begin{aligned} d\omega^1 &= \alpha \wedge \omega^1, \\ d\omega^2 &= \beta \wedge \omega^2 + \omega^1 \wedge \omega^3, \\ d\omega^3 &= \gamma \wedge \omega^2 + (\beta - \alpha) \wedge \omega^3, \end{aligned} \tag{3.6}$$

where α, β and γ are congruent mod ω^i to right-invariant one-forms on G.

Since all the torsion in (3.6) is constant, the question arises of whether (3.6) gives the structure equations of an *infinite Lie pseudo-group*. The answer is given by applying Cartan's involutivity test **[4]**. The Cartan characters are $s_1 = 3$ and $s_2 = 0$. On the other hand, writing (3.6) as

$$d\omega^i = a^i_{j\rho}\pi^\rho \wedge \omega^j + \tfrac{1}{2}C^i_{jk}\omega^j \wedge \omega^k, \tag{3.7}$$

where $(\pi^\rho) \equiv {}^t(\alpha, \beta, \gamma) \bmod \omega^i$, we see that the space of solutions of the system

$$a^i_{j\rho}v^\rho_k\omega^j \wedge \omega^k = 0 \tag{3.8}$$

has dimension $2 < 3$. Hence the system (3.6) is not involutive and we have to prolong.

The structure equations for the prolonged system read

$$\begin{aligned} d\omega^1 &= \alpha \wedge \omega^1, \\ d\omega^2 &= \beta \wedge \omega^2 + \omega^1 \wedge \omega^3, \\ d\omega^3 &= \gamma \wedge \omega^2 + (\beta - \alpha) \wedge \omega^3, \\ d\alpha &= -2\gamma \wedge \omega^1 + a\omega^1 \wedge \omega^2 + b\omega^1 \wedge \omega^3, \\ d\beta &= \rho \wedge \omega^2 - \gamma \wedge \omega^1, \\ d\gamma &= \delta \wedge \omega^2 + \rho \wedge \omega^3 - a\omega^1 \wedge \omega^3 - \alpha \wedge \gamma, \end{aligned} \tag{3.9}$$

where ρ and δ are congruent mod (ω^i, π^ρ) to right-invariant one-forms of the Abelian Lie subgroup $G^{(1)}$ of GL(6, $\mathbb{R}$) associated to the prolonged problem. It is parametrized by the solutions of (3.8) and acts as follows:

$$\begin{bmatrix} \bar{\omega} \\ \bar{\pi} \end{bmatrix} = \begin{bmatrix} \mathbb{1}_3 & 0 \\ v & \mathbb{1}_3 \end{bmatrix} \begin{bmatrix} \omega \\ \pi \end{bmatrix},$$

where

$$v := \begin{bmatrix} 0 & 0 & 0 \\ 0 & r & 0 \\ 0 & s & r \end{bmatrix}.$$

The $G^{(1)}$-action on a and b is given in infinitesimal form by computing the fibre variation of a and b. We have

$$\left.\begin{aligned} da + 2\delta \equiv 0 \\ db + 2\rho \equiv 0 \end{aligned}\right\} \bmod \omega^i, \pi^\rho, \tag{3.10}$$

from which it follows that the $G^{(1)}$-action can always be used to translate a and b to zero. Having done so, we see that the structure equations become

$$\begin{aligned} d\omega^1 &= \alpha \wedge \omega^1, \\ d\omega^2 &= \beta \wedge \omega^2 + \omega^1 \wedge \omega^3, \\ d\omega^3 &= \gamma \wedge \omega^2 + (\beta - \alpha) \wedge \omega^3, \\ d\alpha &= -2\gamma \wedge \omega^1, \\ d\beta &\equiv \omega^1 \wedge \gamma + I_2\omega^1 \wedge \omega^2 - I_1\omega^3 \wedge \omega^2, \\ d\gamma &= \gamma \wedge \alpha + I_3\omega^1 \wedge \omega^2 + I_2\omega^1 \wedge \omega^3. \end{aligned} \tag{3.11a}$$

We have now attained an $\{e\}$-structure on $J(\mathbb{R}, \mathbb{R}) \times G$.

Parametrically, the basic invariants I_1, I_2 and I_3 are given by

$$\begin{aligned} I_1 &= -\frac{A}{2B^2} F_{ppp}, \\ I_2 &= \frac{1}{2AB}\left(\frac{d}{dx} F_{pp} - F_{py}\right), \\ I_3 &= -CI_2 + \frac{1}{2A^2B}\left(\frac{d}{dx} F_{py} + F_{pp}F_y - F_{py}F_p - 2F_{yy}\right), \end{aligned} \tag{3.11b}$$

while the invariant one-forms α, β and γ take the form

$$
\begin{aligned}
\alpha &= \frac{dA}{A} - \left(2C + \frac{F_p}{A}\right)\omega^1, \\
\beta &= \frac{dB}{B} - C\omega^1 + \frac{F_{pp}}{2B}\omega^2, \\
\gamma &= dC + C\frac{dA}{A} + \left(\frac{F_y}{A^2} - \frac{CF_p}{A} - C^2\right)\omega^1 + \left(\frac{F_{py}}{2AB} - \frac{CF_{pp}}{2B}\right)\omega^2 + \frac{F_{pp}}{2B}\omega^3,
\end{aligned}
\tag{3.11c}
$$

with the total derivative operator d/dx defined by

$$\frac{d}{dx} := \frac{\partial}{\partial x} + p\frac{\partial}{\partial y} + F\frac{\partial}{\partial p}.$$

The invariants I_1, I_2 and I_3 are functions on $J^1(\mathbb{R}, \mathbb{R}) \times G$ which can be used to produce necessary and sufficient conditions for the equivalence of two equations of the form (1.1) under fibre-preserving point transformations (1.2). However, it has been shown in **[18]** that unless all three invariants vanish, the natural G-action can be used to cast I_1, I_2, I_3 and further torsion coefficients arising in the reduction process into normal forms, thereby reducing the $\{e\}$-structure on $J^1(\mathbb{R}, \mathbb{R}) \times G$ given by (3.11) to an $\{e\}$-structure on $J^1(\mathbb{R}, \mathbb{R})$. This procedure allows us to identify the equivalence classes of equations admitting symmetry groups as well as the structure of these groups.

The G-action on I_1, I_2 and I_3 is obtained by computing the integrability conditions of the structure equations (3.11):

$$
\begin{aligned}
&[dI_2 + I_2(\alpha + \beta)]\omega^1 \wedge \omega^2 + [dI_1 + I_1(2\beta - \alpha)]\omega^2 \wedge \omega^3 = 0, \\
&[dI_3 + I_3(2\alpha + \beta) + \gamma I_2]\omega^1 \wedge \omega^2 + [dI_2 + I_2(\alpha + \beta)]\omega^1 \wedge \omega^3 = 0.
\end{aligned}
\tag{3.12}
$$

Equations (3.12) lead to an exhaustive set of cases in the reduction procedure. These cases are defined by conditions which are invariant under fibre-preserving point transformations and are expressed in terms of the function $F(x, y, p)$ appearing in (1.1) and its derivatives.

In the following, we summarize the reduction process, listing only the final $\{e\}$-structure on $J^1(\mathbb{R}, \mathbb{R})$ and the relevant parametric expressions. Details of these reductions have been given in **[18]**. We should mention that the results listed below are obtained directly from the structure equations rather than from the integrability conditions as was done in the previous reference.

There are five main cases, labelled by A, B, C, D and E.

Case A: $I_1 I_2 \neq 0$, that is,

$$F_{ppp}\left(\frac{d}{dx}F_{pp} - F_{py}\right) \neq 0. \tag{3.13}$$

It follows from equations (3.12) that one can use the G-action to scale I_1 and I_2 to one and translate I_3 to zero. Parametrically, this amounts to setting

$$A = -(2F_{ppp})^{-1/3}\left(\frac{d}{dx}F_{pp} - F_{py}\right)^{2/3}, \tag{3.14a}$$

$$B = -\left[\frac{F_{ppp}}{4}\left(\frac{d}{dx}F_{pp} - F_{py}\right)\right]^{1/3}, \tag{3.14b}$$

$$C = -(2F_{ppp})^{1/3}\left(\frac{d}{dx}F_{pp} - F_{py}\right)^{-5/3}\left(\frac{d}{dx}F_{py} + F_{pp}F_y - F_{py}F_p - 2F_{yy}\right). \tag{3.14c}$$

The final $\{e\}$-structure on $J^1(\mathbb{R}, \mathbb{R})$ is given by

$$d\omega^1 \doteq l\omega^2 \wedge \omega^1 + m\omega^3 \wedge \omega^1, \tag{3.15a}$$

$$d\omega^2 = r\omega^1 \wedge \omega^2 + (k - 2r - m)\omega^3 \wedge \omega^2 + \omega^1 \wedge \omega^3, \tag{3.15b}$$

$$d\omega^3 = u\omega^1 \wedge \omega^2 + (r - k)\omega^1 \wedge \omega^3 + 2l\omega^3 \wedge \omega^2. \tag{3.15c}$$

The $d\pi^\rho$ equations take the form

$$\begin{aligned} dk \wedge \omega^1 + dl \wedge \omega^2 + dm \wedge \omega^3 &+ [l(k - r) - mu + 2v]\omega^2 \wedge \omega^1 \\ &+ [m(2k - r) + 2s + l]\omega^3 \wedge \omega^1 \\ &+ l(k - 2r + m)\omega^3 \wedge \omega^2 = 0, \end{aligned} \tag{3.15d}$$

$$\begin{aligned} dr \wedge \omega^1 + ds \wedge \omega^2 &+ d(k - 2r - m) \wedge \omega^3 \\ &+ [u(2r - k + m) + (l - s)r + v + 1]\omega^2 \wedge \omega^1 \\ &+ [m(2r - k) + (2r - 3k)r + k^2 + l]\omega^3 \wedge \omega^1 \\ &+ [(s + 2l)(k - 2r - m) + 1]\omega^3 \wedge \omega^2 = 0, \end{aligned} \tag{3.15e}$$

$$\begin{aligned} du \wedge \omega^1 + dv \wedge \omega^2 + d(l + s) \wedge \omega^3 &+ [u(l - s) - v(k + r)]\omega^2 \wedge \omega^1 \\ &+ [2mu - r(l + s) - v + 1]\omega^3 \wedge \omega^1 \\ &+ [v(k - 2r) + l(l + s)]\omega^3 \wedge \omega^2 = 0. \end{aligned} \tag{3.15f}$$

The parametric expressions of the invariants (k, l, m, r, s, u and v) appearing in equations (3.15) are given by:

$$k = \frac{1}{A}\left(\frac{d}{dx}\ln A - 2AC - F_p\right), \tag{3.16a}$$

$$l = \frac{1}{AB}(A_y - ACA_p), \tag{3.16b}$$

$$m = \frac{A_p}{B}, \tag{3.16c}$$

$$r = \frac{1}{AB}\frac{d}{dx}B - C, \tag{3.16d}$$

$$s = \frac{1}{B^2}(B_y - ACB_p + \tfrac{1}{2}BF_{pp}), \tag{3.16e}$$

$$u = \frac{1}{A^2}\left[\frac{d}{dx}(AC) + F_y - ACF_p - (AC)^2\right], \tag{3.16f}$$

$$v = \frac{1}{AB}\left[(AC)_y - AC(AC)_p + \tfrac{1}{2}F_{py} - \tfrac{1}{2}ACF_{pp}\right]. \tag{3.16g}$$

Case B: $I_1 = 0$ and $I_2 \neq 0$, that is,

$$F(x, y, p) = p^2M(x, y) + pN(x, y) + Q(x, y), \tag{3.17a}$$

$$G(x, y) := 2M_x - N_y \neq 0. \tag{3.17b}$$

It follows from equations (3.12) that one can use the G-action to translate I_3 to zero and scale I_2 to one. Parametrically, this amounts to setting

$$2AB = G, \tag{3.18a}$$

$$AC = \frac{pG_y + H}{G}, \tag{3.18b}$$

where

$$H(x, y) := -2Q_{yy} + N_{yx} - N_yN + 2(QM)_y. \tag{3.18c}$$

There are two cases to consider, Case B(i) and Case B(ii).

Case B(i): $\frac{d}{dx}(\ln G) - 3AC - F_p \neq 0$. Further normalizations of invariants give

$$A = \frac{d}{dx}(\ln G) - 3AC - F_p. \tag{3.19}$$

The final $\{e\}$-structure on $J^1(\mathbb{R}, \mathbb{R})$ is given by

$$d\omega^1 = l\omega^2 \wedge \omega^1 - 2f\omega^3 \wedge \omega^1, \tag{3.20a}$$

$$d\omega^2 = (1 - k)\omega^1 \wedge \omega^2 + 2f\omega^3 \wedge \omega^2 + \omega^1 \wedge \omega^3, \tag{3.20b}$$

$$d\omega^3 = c\omega^1 \wedge \omega^2 + (1 - 2k)\omega^1 \wedge \omega^3 - 2l\omega^2 \wedge \omega^3. \tag{3.20c}$$

The $d\pi^\rho$ equations now read

$$df \wedge \omega^2 + 2f^2\omega^3 \wedge \omega^2 + [f(1 - k) - (3e + l + 1)]\omega^1 \wedge \omega^2 = 0, \tag{3,20d}$$

$$dc \wedge \omega^1 + de \wedge \omega^2 + df \wedge \omega^3 + [e + c(f - 2l)]\omega^1 \wedge \omega^2$$
$$- fl\omega^2 \wedge \omega^3 + [f(1 + 4c - k) + e - 1]\omega^1 \wedge \omega^3 = 0, \tag{3.20e}$$

$$dk \wedge \omega^1 + dl \wedge \omega^2 - 2df \wedge \omega^3 + [(2k - 1)l + 2fc + 2e]\omega^2 \wedge \omega^1$$
$$- 2fl\omega^3 \wedge \omega^2 + [2f(2 - 3k) - l]\omega^3 \wedge \omega^1 = 0. \tag{3.20f}$$

The parametric expressions of the invariants (c, e, f, k and l) appearing in equations (3.20) are given by

$$c=\frac{1}{A^2}\left[\frac{d}{dx}(AC)+F_y-ACF_p-(AC)^2\right], \tag{3.21a}$$

$$e=\frac{1}{AB}[(AC)_y-AC(AC)_p+\tfrac{1}{2}F_{py}-\tfrac{1}{2}ACF_{pp}], \tag{3.21b}$$

$$f=\frac{2A}{G}[(\ln G)_y+M], \tag{3.21c}$$

$$k=\frac{1}{A}\left(\frac{d}{dx}\ln A-2AC-F_p\right), \tag{3.21d}$$

$$l=\frac{1}{AB}(A_y-ACA_p). \tag{3.21e}$$

Case B(ii): $\frac{d}{dx}(\ln G)-3AC-F_p=0$, which implies that

$$(\ln G)_y+M=0, \tag{3.22a}$$

$$AB=-3[(AC)_y-AC(AC)_p+\tfrac{1}{2}F_{py}-\tfrac{1}{2}ACF_{pp}], \tag{3.22b}$$

with

$$\frac{d}{dx}(AC)+F_y-ACF_p-(AC)^2\neq 0. \tag{3.22c}$$

Further normalizations of invariants give

$$\varepsilon A^2=\frac{d}{dx}(AC)+F_y-ACF_p-(AC)^2. \tag{3.22d}$$

The final $\{e\}$-structure on $J^1(\mathbb{R},\mathbb{R})$ is given by

$$d\omega^1=-\tfrac{2}{3}\varepsilon\omega^3\wedge\omega^1, \tag{3.23a}$$

$$d\omega^2=-r\omega^1\wedge\omega^2+\tfrac{2}{3}\varepsilon\omega^3\wedge\omega^2+\omega^1\wedge\omega^3, \tag{3.23b}$$

$$d\omega^3=-2r\omega^1\wedge\omega^3+\varepsilon\omega^1\wedge\omega^2. \tag{3.23c}$$

The $d\pi^\rho$ equations become

$$dr\wedge\omega^1+2\varepsilon r\omega^1\wedge\omega^3=0. \tag{3.23d}$$

The parametric expression of the invariant r appearing in equations (3.23) is given by

$$r=\frac{1}{A}\left(\frac{d}{dx}\ln A-2AC-F_p\right). \tag{3.24}$$

Case C: $I_2=0$ and $I_1I_3\neq 0$, that is,

$$\frac{d}{dx}F_{pp}-F_{py}=0, \tag{3.25a}$$

$$F_{ppp}\left(\frac{d}{dx}F_{py}+F_{pp}F_y-F_{py}F_p-2F_{yy}\right)\neq 0. \tag{3.25b}$$

It follows from equations (3.12) that one can use the G-action to scale I_1 and I_3 to one. Parametrically this amounts to setting

$$A = -(2F_{ppp})^{-1/5}\left(\frac{d}{dx}F_{py} + F_{pp}F_y - F_{py}F_p - 2F_{yy}\right)^{2/5}, \tag{3.26a}$$

$$B = 2^{-3/5}F_{ppp}^{2/5}\left(\frac{d}{dx}F_{py} + F_{pp}F_y - F_{py}F_p - 2F_{yy}\right)^{1/5}. \tag{3.26b}$$

Further normalizations of invariants give

$$\frac{d}{dx}A = (2AC + F_p)A. \tag{3.26c}$$

The final $\{e\}$-structure on $J^1(\mathbb{R}, \mathbb{R})$ is given by

$$d\omega^1 = s\omega^2 \wedge \omega^1 - l\omega^3 \wedge \omega^1, \tag{3.27a}$$

$$d\omega^2 = 2l\omega^3 \wedge \omega^2 + \omega^1 \wedge \omega^3, \tag{3.27b}$$

$$d\omega^3 = c\omega^1 \wedge \omega^2 + (k - s - f)\omega^2 \wedge \omega^3, \tag{3.27c}$$

where $5f = 2s + k$, while the $d\pi^\rho$ equations take the form

$$ds \wedge \omega^2 - dl \wedge \omega^3 - (2e + lc)\omega^1 \wedge \omega^2 + (s - 2f)\omega^1 \wedge \omega^3 + l(f - s - k)\omega^2 \wedge \omega^3 = 0, \tag{3.27d}$$

$$dk \wedge \omega^2 + 2dl \wedge \omega^3 + (2lc - e)\omega^1 \wedge \omega^2 + (k - f)\omega^1 \wedge \omega^3 - [1 + 2l(f + s)]\omega^2 \wedge \omega^3 = 0, \tag{3.27e}$$

$$dc \wedge \omega^1 + de \wedge \omega^2 + df \wedge \omega^3 + [(2s - f)c + 1]\omega^2 \wedge \omega^1 - (2lc + e)\omega^3 \wedge \omega^1 + [el + f(f - k)]\omega^3 \wedge \omega^2 = 0. \tag{3.27f}$$

The parametric expressions of the invariants (c, e, k, l and s) appearing in equations (3.27) are given by

$$c = \frac{1}{A^2}\left[\frac{d}{dx}(AC) + F_y - ACF_p - (AC)^2\right], \tag{3.28a}$$

$$e = \frac{1}{AB}[(AC)_y - AC(AC)_p + \tfrac{1}{2}F_{py} - \tfrac{1}{2}ACF_{pp}], \tag{3.28b}$$

$$k = \frac{1}{B^2}(B_y - ACB_p + \tfrac{1}{2}BF_{pp}), \tag{3.28c}$$

$$l = -\frac{A_p}{B}, \tag{3.28d}$$

$$s = \frac{1}{AB}(A_y - ACA_p). \tag{3.28e}$$

Case D: $I_1 \neq 0$ and $I_2 = I_3 = 0$, that is,

$$F_{ppp} \neq 0, \tag{3.29a}$$

$$\frac{d}{dx} F_{pp} - F_{py} = 0, \tag{3.29b}$$

$$\frac{d}{dx} F_{py} + F_{pp}F_y - F_{py}F_p - 2F_{yy} = 0. \tag{3.29c}$$

From equations (3.12), it follows that the G-action can be used to scale I_3 to one. Parametrically, this amounts to having

$$\frac{B^2}{A} = -\tfrac{1}{2}F_{ppp}. \tag{3.30a}$$

Further normalizations of invariants give

$$B = -\frac{\varepsilon F_{ppp}^2}{F_{pppp}} \quad \text{with} \quad F_{pppp} \neq 0, \tag{3.30b}$$

$$C = -\frac{F_{pppp}}{2F_{ppp}^3}(F_{pppy} + F_{pp}F_{ppp}). \tag{3.30c}$$

The final $\{e\}$-structure on $J^1(\mathbb{R}, \mathbb{R})$ is given by

$$d\omega^1 = b\omega^2 \wedge \omega^1 + c\omega^3 \wedge \omega^1, \tag{3.31a}$$

$$d\omega^2 = \tfrac{1}{2}(c + 2\varepsilon)\omega^3 \wedge \omega^2 + \omega^1 \wedge \omega^3, \tag{3.31b}$$

$$d\omega^3 = -\varepsilon\omega^3 \wedge \omega^2, \tag{3.31c}$$

while the $d\pi^\rho$ equations now become

$$db \wedge \omega^2 + dc \wedge \omega^3 + 2k\omega^2 \wedge \omega^1 + [\tfrac{1}{2}bc + (b - c)\varepsilon]\omega^3 \wedge \omega^2 + 2(b + \varepsilon)\omega^1 \wedge \omega^3 = 0, \tag{3.31d}$$

$$2dk \wedge \omega^2 - db \wedge \omega^3 + [(3c + 2\varepsilon)k + (3\varepsilon + b)b + 2]\omega^3 \wedge \omega^2 + 2k\omega^1 \wedge \omega^3 = 0. \tag{3.31e}$$

The parametric expressions of the invariants (b, c and k) appearing in equations (3.31) are given by

$$b = \frac{1}{AB}(A_y - ACA_p), \tag{3.32a}$$

$$c = \frac{A_p}{B}, \tag{3.32b}$$

$$k = \frac{1}{AB}[(AC)_y - AC(AC)_p + \tfrac{1}{2}F_{py} - \tfrac{1}{2}ACF_{pp}]. \tag{3.32c}$$

Case E: $I_1 = I_2 = 0$ and $I_3 \neq 0$, that is,

$$F(x, y, p) = \tfrac{1}{2}p^2M_y + pM_x + N \tag{3.33}$$

for some functions $M(x, y)$ and $N(x, y)$. From equations (3.12) we see that the

G-action can be used to scale I_3 to one. Parametrically, this amounts to having

$$2A^2B = G, \tag{3.34a}$$

where

$$G(x, y) := (M_{xx} + NM_y - 2N_y - \tfrac{1}{2}M_x^2)_y. \tag{3.34b}$$

These are two cases to consider, Case E(i) and Case E(ii).

Case E(i): $(2\ln G + M)_y = 0$. Further normalizations of invariants give

$$\varepsilon A^2 = \frac{1}{5}\frac{d}{dx}H - \tfrac{1}{25}H^2 - \tfrac{1}{5}H\frac{d}{dx}M + \tfrac{1}{2}p^2M_{yy} + pM_{xy} + N_y, \tag{3.35a}$$

$$5AC = \frac{d}{dx}(\ln G - 2M), \tag{3.35b}$$

where

$$H(x, y, p) := \frac{d}{dx}(\ln G - 2M). \tag{3.35c}$$

The final $\{e\}$-structure on $J^1(\mathbb{R}, \mathbb{R})$ is given by

$$d\omega^1 = \tfrac{1}{2}\varepsilon\omega^1 \wedge \omega^2, \tag{3.36a}$$

$$d\omega^2 = 2s\omega^2 \wedge \omega^1 + \omega^1 \wedge \omega^3, \tag{3.36b}$$

$$d\omega^3 = \varepsilon\omega^1 \wedge \omega^2 + 3s\omega^3 \wedge \omega^1 + \tfrac{3}{2}\varepsilon\omega^2 \wedge \omega^3, \tag{3.36c}$$

while the $d\pi^p$ equations now read

$$ds \wedge \omega^1 - \tfrac{3}{2}\varepsilon s\omega^2 \wedge \omega^1 + \tfrac{1}{2}\varepsilon\omega^3 \wedge \omega^1 = 0. \tag{3.36d}$$

The parametric expression of the invariant appearing in equations (3.36) is given by

$$s = \frac{1}{5A}\frac{d}{dx}(5\ln A - 2\ln G - M). \tag{3.37}$$

Case E(ii): $(2\ln G + M)_y \neq 0$. Further normalizations of invariants give

$$2B = \varepsilon(2\ln G + M)_y, \tag{3.38a}$$

$$5AC = \frac{d}{dx}(\ln G - 2M). \tag{3.38b}$$

The final $\{e\}$-structure on $J^1(\mathbb{R}, \mathbb{R})$ is given by

$$d\omega^1 = s\omega^2 \wedge \omega^1, \tag{3.39a}$$

$$d\omega^2 = 2r\omega^2 \wedge \omega^1 + \omega^1 \wedge \omega^3, \tag{3.39b}$$

$$d\omega^3 = u\omega^1 \wedge \omega^2 - 3r\omega^1 \wedge \omega^3 + (\tfrac{4}{5}\varepsilon - 3s)\omega^2 \wedge \omega^3, \tag{3.39c}$$

while the $d\pi^\rho$ equations now read

$$dr \wedge \omega^1 + ds \wedge \omega^2 + (3s - \tfrac{4}{5}\varepsilon)r\omega^2 \wedge \omega^1 + (s - \tfrac{2}{5}\varepsilon)\omega^1 \wedge \omega^3 = 0, \tag{3.39d}$$

$$du \wedge \omega^1 - \tfrac{2}{5}\varepsilon dr \wedge \omega^2 + (2us - \tfrac{2}{5}\varepsilon r^2 - \tfrac{1}{5}\varepsilon u + 1)\omega^2 \wedge \omega_1 - \tfrac{4}{5}\varepsilon r\omega^1 \wedge \omega^3 + (\tfrac{4}{25} - \tfrac{2}{5}\varepsilon s)\omega^2 \wedge \omega^3 = 0. \tag{3.39e}$$

The parametric expressions of the invariants (r, s and u) appearing in equations (3.39) are given by

$$r = \frac{1}{5A}\frac{d}{dx}(5 \ln A - 2 \ln G - M), \tag{3.40a}$$

$$s = \frac{1}{B}(\ln A)_y, \tag{3.40b}$$

$$u = \frac{1}{A^2}\left(\frac{1}{5}\frac{d}{dx}H - \tfrac{1}{25}H^2 - \tfrac{1}{5}H\frac{d}{dx}M + \tfrac{1}{2}p^2 M_{yy} + pM_{xy} + N_y\right). \tag{3.40c}$$

In the above, it is important to note that the $d\pi^\rho$ equations are integrability conditions for the $\{e\}$-structure on $J^1(\mathbb{R}, \mathbb{R})$ obtained by reducing the $\{e\}$-structure (3.11) on the bundle of G-frames over $J^1(\mathbb{R}, \mathbb{R})$ to the base. In what follows, this $\{e\}$-structure, together with the associated integrability conditions (the $d\pi^\rho$-equations), will be referred to as the *reduced* $\{e\}$*-structure.*

We conclude the present section by recalling a theorem of Cartan [7] which states that necessary and sufficient conditions for the existence of a collection of one-forms $\{\omega^i, \pi^\rho\}$ satisfying structure equations of the form

$$d\omega^i = a^i_{j\rho}\pi^\rho \wedge \omega^j + \tfrac{1}{2}C^i_{jk}\omega^j \wedge \omega^k, \tag{3.41}$$

are that the $a^i_{j\rho}$ form an involutive tableau and that the system of differential equations obtained by substituting

$$d\pi^\rho = \tfrac{1}{2}\gamma^\rho_{\mu\nu}\pi^\mu \wedge \pi^\nu + \delta^\rho_{i\alpha}\omega^i \wedge \pi^\alpha + \tfrac{1}{2}\varepsilon^\rho_{kh}\omega^k \wedge \omega^h, \tag{3.42}$$

in the integrability conditions, $d^2\omega^i = 0$, admit a solution.

This result generalizes Lie's Third Fundamental Theorem and is proved using the Cartan–Kähler existence theorem for involutive exterior differential systems [3, 4]. We shall use it in § 5 to investigate the existence of second-order ordinary differential equations admitting a Lie group of fibre-preserving point symmetries.

4. *Existence of symmetries*

The *symmetry group* of a differential equation is the *maximal* pseudo-group of transformations mapping solutions to solutions. A vector field X will thus generate a one-dimensional group of symmetries if and only if

$$\mathcal{L}_X\Sigma \subseteq \Sigma, \tag{4.1}$$

where Σ is an exterior differential system associated with the differential equation.

A differential equation is said to admit an n-dimensional Lie group of symmetries if there exist n linearly independent vector fields which satisfy (4.1) and span a Lie algebra. From Proposition 2.2 and the solution to the equivalence problem given in § 3, it follows that the existence of a p-dimensional Lie group of symmetries is equivalent to the existence of a co-frame satisfying (2.7) with $N-r=p$. In this section we use these results and Proposition 2.1 to classify invariantly in terms of differential invariants *all* second-order ordinary differential equations admitting a non-trivial group of *fibre-preserving* point symmetries.

We begin by recalling [**17**] that a second-order ordinary differential equation can have at most a six-dimensional group of fibre-preserving symmetries (isomorphic to the affine group in the plane Aff $(\mathbb{R}^2)$, which is the semi-direct product of GL(2, $\mathbb{R}$) with $\mathbb{R}^2$). All second-order equations with this property are characterized in the following proposition.

PROPOSITION 4.1. *A second-order ordinary differential equation admits a six-dimensional Lie group of fibre-preserving point symmetries if and only if it is of the form*

$$y'' = \tfrac{1}{2}y'^2 M_y + y' M_x + N \tag{4.2a}$$

with

$$M_{xxy} + (NM_y)_y - M_{xy}M_x - 2N_{yy} = 0 \tag{4.2b}$$

for some functions M and N depending on x and y. Furthermore, every equation with this property is equivalent to the free particle equation

$$y'' = 0. \tag{4.2c}$$

In particular, every linear equation is equivalent to (4.2c).

REMARKS. 1. The above result answers completely the question of linearizability of an arbitrary second-order ordinary differential equation under fibre-preserving point transformations. In this regard we should mention the work of Sarlet *et al.* [**26**] who considered a related question: linearizability of second-order equations under point transformations (see also Arnold [**1**]). There they were able to show that every equation of the form

$$y'' = F(y'), \tag{4.3}$$

where F is at most cubic in y', is linearizable via point transformations. By a trivial application of the above proposition, we see that (4.3) is linearizable via fibre-preserving transformations if and only if it is at most quadratic in y'.

2. As a further application, we consider the problem of determining which of the fifty Painlevé equations [**16**] (discovered in the classification of second-order ordinary differential equations whose solutions have no essential singularities or branch points which are movable in the sense of depending on initial conditions) admit a six-dimensional Lie group of symmetries and so are locally equivalent under a fibre-preserving point transformation to the free particle equation $y''=0$. Proposition 4.1 completely solves the problem; the result follows.

The Painlevé equations which are equivalent to $y''=0$ are as follows:

$$p_{\mathrm{I}}:\ y'' = 0,$$

$$p_{\mathrm{II}}:\ y'' = \frac{1}{y}y'^2,$$

$$p_{17}:\ y''=\frac{(m-1)}{my}y'^2,$$

$$p_{37}:\ y''=\left(\frac{1}{2y}+\frac{1}{y-1}\right)y'^2,$$

$$p_{41}:\ y''=\frac{2}{3}\left(\frac{1}{y}+\frac{1}{y-1}\right)y'^2,$$

$$p_{43}:\ y''=\frac{3}{4}\left(\frac{1}{y}+\frac{1}{y-1}\right)y'^2.$$

It was shown by Kamran and Shadwick [**18**] that no second-order ordinary differential equation admits a maximal four- or five-dimensional Lie group of fibre-preserving point symmetries. We now consider the case of three-dimensional symmetry groups.

PROPOSITION 4.2. *A second-order ordinary differential equation admits a maximal three-dimensional Lie group of fibre-preserving point symmetries if and only if all the basic invariants of its reduced $\{e\}$-structure are constants. All possible classes of equations satisfying these conditions are invariantly characterized in Table* 1.

TABLE 1. *Equations admitting a maximal three-dimensional symmetry group*

Case	Invariant characterization	Representative equation	Description
Case A:	k, l, m, r and u constant		
3-dA$_{\text{IV}}$	$k=l=u=v=0$ $rs=1,\ m=-r,\ m^3=2$	$y''=e^{y'}$	The symmetry group in this case is *isolated*. Thus every equation in this branch is equivalent to the representative equation.
3-dA$_{\text{VI}}$	$l=u=v=0,\ k\neq 0$ $rs=1,\ k-m=r$ $m(m^2-k^2)=2$	$y''=y'^{3/2}$	These cases provide *one-parameter* families of equivalence classes admitting three-dimensional symmetry groups.
3-dA$_{\text{VIII}}$	$l=0,\ k=2r,\ 2v=mu$ $sm=1,\ 3m^2r+2=0$ $3m^2u+2(m-r)=0$		
Case B:	r constant	$y''=(3y'-y^2)y$	These cases provide *isolated* equivalence classes admitting three-dimensional symmetry groups. Thus every equation in these branches is equivalent to their respective representative equation.
3-dB(ii$_{\text{VIII}}$)	$r=0$		
Case D:	b and c constant		
3-dD$_{\text{VIII}}$	$b=-\varepsilon,\ c=-\frac{2}{3}\varepsilon,\ k=0$		
Case E:	r, s and u constant	$y''=y^{-3}$	
3-dE(ii$_{\text{VIII}}$)	$r=0,\ s=\frac{2}{5}\varepsilon,\ u=-\frac{5}{3}\varepsilon$		

REMARKS. 1. The roman numerals that appear in the classification refer to the Bianchi types [2] of three-dimensional real Lie algebras. It should be noted that not all Bianchi types occur in Table 1. In particular, Bianchi type I cannot occur and thus no second-order equation admits an Abelian three-dimensional symmetry group.

2. The representative equation of 3-dB(ii_{VIII}) is especially interesting in the sense that it highlights the difference between point and fibre-preserving symmetries. Mahomed and Leach [26] have shown that the equation

$$y'' = (3y' - y^2)y \tag{4.4}$$

admits an eight-dimensional Lie group of point symmetries (isomorphic to $SL(3, \mathbb{R})$) and is equivalent (under point transformations) to the free particle equation. By the above proposition we see that (4.4) admits only a three-dimensional fibre-preserving symmetry group and thus cannot be equivalent (under fibre-preserving transformations) to $y'' = 0$.

3. As a further application we again consider the Painlevé equations. Using Proposition 4.2, it is not hard to show that the only Painlevé equation which admits a maximal three-dimensional Lie group of fibre-preserving symmetries is

$$p_{32}\colon\ y'' = \frac{1}{2y}(y'^2 - 1).$$

This equation is a member of 3-dE(ii_{VIII}) and so, since the symmetry group of this class is unique, is equivalent to the representative equation $y'' = y^{-3}$.

It is important to note that the above does not exclude the possibility of other Painlevé equations degenerating into one which admits the required symmetries. An interesting example is

$$p_{34}\colon\ y'' = \frac{1}{2y}(y'^2 - 1) + 4\alpha y^2 - xy.$$

Generically this equation admits no symmetries, but if $\alpha = 0$ we obtain

$$y'' = \frac{1}{2y}(y'^2 - 1) - xy.$$

It can easily be shown that this degenerate Painlevé equation again admits a three-dimensional symmetry group 3-dE(ii_{VIII}).

The above propositions exhaust all second-order equations admitting a transitive symmetry group. We now consider the non-transitive cases.

Let us start by considering the case of two-dimensional symmetry groups.

PROPOSITION 4.3. *A second-order differential equation admits a maximal two-dimensional Lie group of fibre-preserving point symmetries if and only if the basic invariants of its reduced $\{e\}$-structure produce only one fundamental invariant. All possible classes of equations satisfying this symmetry condition are invariantly characterized in Table* 2.

REMARKS. 1. In the above, the derived invariants of an invariant I is denoted by I_a. These correspond exactly to the covariant derivatives of I with respect to the adapted co-frame ω^a (see § 2).

2. All the representative equations given in Table 2 arise in a physical context. Here we will briefly mention the context in which some of these equations occur. For these equations new symmetries, in addition to the classically known ones, are exhibited.

TABLE 2. *Equations admitting a maximal two-dimensional symmetry group*

Case	Invariant characterization	Representative equation	Description
Case A:			
2-dA_1	$l=0$, k and m constant $dr\neq 0$, $dr \wedge dr_1 = 0$		Some of these cases may be vacuous (see p. 407).
2-dA_2	k and l constant, $dm\neq 0$ $dm \wedge d(r, s, u, m_3) = 0$		
2-dA_3	k constant, $dl\neq 0$ $dl \wedge d(m, r, s, u, v, l_2) = 0$		
2-dA_4	$dk \wedge d(l, m, r, s, u, k_1) = 0$	$y''=(1-x)y'^3$	
Case B:			Existence of an *isolated* two-dimensional symmetry group follows from Cartan's theorem (p. 401)
2-dB(i_1)	c, e and f constant $dk \wedge dk_1 = 0$		
2-dB(i_2)	e and f constant $dc \wedge dc_1 = 0$	$y''=(\alpha y'-y^2)y$ $\alpha^2\neq 9$	This case provides a *one-parameter* family of equivalence classes admitting two-dimensional symmetry groups.
2-dB(i_3)	$df\neq 0$, $df \wedge d(e, k) = 0$	$y''=-(4\alpha^{-1}+\alpha y^2)y'$ $-3\alpha^{-2}y-y^3$	
Case C:			
2-dC_1	s constant $dl\neq 0$, $dk \wedge dl = 0$		Some of these cases may be vacuous (see p. 407).
2-dC_2	$ds\neq 0$, $ds \wedge d(c, e, l, s_2) = 0$		
Case D:			Existence of an *isolated* and a *one-parameter* family of equivalence classes admitting two-dimensional symmetry group for 2-dD_1 and 2-dD_2 follows from Cartan's theorem (p. 401).
2-dD_1	$b=-2\varepsilon$, k constant $dc \wedge dc_3 = 0$		
2-dD_2	k constant $db \wedge dc = 0$		
Case E:			These cases, 2-dE(i) and 2-dE(ii) respectively, provide an *isolated* and a *one-parameter* family of equivalence classes admitting two-dimensional symmetry groups.
2-dE(i)	$ds \wedge ds_1 = 0$	$y''=6y^2$	
2-dE(ii)	$dr\neq 0$, $dr \wedge du = 0$	$y''=-\delta y'-\frac{2}{9}\delta^2 y-y^3$	

The representative equation of 2-dB(i_3)

$$y''=-(4\alpha^{-1}+\alpha y^2)y'-3\alpha^{-2}y-y^3, \quad \alpha \text{ constant},$$

occurs as a simple model in certain flow-induced structural vibration problems in which the structural non-linearities act to maintain overall stability [**14**], while the representative equation of 2-dE(ii) (which is a special case of Duffing's equation)

$$y''=-\delta y'-\tfrac{2}{9}\delta^2 y-y^3, \quad \delta \text{ constant},$$

describes the dynamics of a buckled beam or plate when only one mode of vibration is considered [**13**]. In addition to the obvious translational symmetry of these equations, Proposition 4.3 allows us to establish the existence of a second independent symmetry.

TABLE 3. *Equations admitting a maximal one-dimensional symmetry group*

Case	Invariant characterization	Case	Invariant characterization
Case A:		1-dA_9	$dk \wedge dt \neq 0,\ t := k_1$
1-dA_1	k, l, m and r constant, $t := u_1$		$dk \wedge d(l, m, r, s, u, v) = 0$
	$du \wedge dt \neq 0,\ du \wedge dt \wedge dt_1 = 0$		$dk \wedge dt \wedge dt_1 = 0$
1-dA_2	k, l, m constant, $t := r_1$	1-dA_{10}	$dk \wedge dv \neq 0,$
	$dr \wedge dt \neq 0,\ dr \wedge dt \wedge dt_1 = 0$		$dk \wedge d(l, m, r, s, u) = 0$
1-dA_3	k and l constant, $dm \wedge ds \neq 0$		$dk \wedge dv \wedge dk_1 = 0$
	$dm \wedge dr = 0,\ dm \wedge ds \wedge du = 0$	1-dA_{11}	$dk \wedge du \neq 0,$
1-dA_4	k and l constant, $dm \wedge dr \neq 0$		$dk \wedge d(l, m, r, s) = 0$
	$dm \wedge dr \wedge d(s, m_3, r_1) = 0$		$dk \wedge du \wedge d(v, k_1, u_1) = 0$
1-dA_5	k constant	1-dA_{12}	$dk \wedge ds \neq 0,\ dk \wedge d(l, m, r) = 0$
	$dl \wedge du \neq 0,\ dl \wedge d(m, r, s) = 0$		$dk \wedge ds \wedge d(u, v, k_1) = 0$
	$dl \wedge du \wedge d(v, l_2, u_1) = 0$	1-dA_{13}	$dk \wedge dr \neq 0,\ dk \wedge d(l, m) = 0$
1-dA_6	k constant		$dk \wedge dr \wedge d(s, u, v, k_1, r_1) = 0$
	$dl \wedge ds \neq 0,\ dl \wedge d(m, r) = 0$	1-dA_{14}	$dk \wedge dm \neq 0,\ dk \wedge dl = 0$
	$dl \wedge ds \wedge d(u, l_2) = 0$		$dk \wedge dm \wedge d(r, s, u, v, k_1, k_2, m_3) = 0$
1-dA_7	k constant	1-dA_{15}	$dk \wedge dl \neq 0$
	$dl \wedge dr \neq 0,\ dl \wedge dm = 0$		$dk \wedge dl \wedge d(m, r, s, u, v, k_\alpha, l_2, l_3) = 0$
	$dl \wedge dr \wedge d(u, v, l_2, r_1) = 0$		
1-dA_8	k constant, $dl \wedge dm \neq 0$		
	$dl \wedge dm \wedge d(r, s, u, v, l_2, l_3, m_3 = 0)$		
Case B:			
1-dB(i_1)	c, e and f constant, $t := k_1$	1-dB(i_5)	$df \wedge dk \neq 0,\ df \wedge d(c, e)$
	$dk \wedge dt \neq 0,\ dk \wedge dt \wedge dt_1 = 0$		$df \wedge dk \wedge dk_1 = 0$
1-dB(i_2)	e and f constant, $t := c_1$	1-dB(i_6)	$df \wedge de \neq 0,\ df \wedge dc = 0$
	$dc \wedge dt \neq 0,\ dc \wedge dk = 0,\ dc \wedge dt_1 = 0$		$df \wedge de \wedge d(k, f_2) = 0$
1-dB(i_3)	e and f constant, $dc \wedge dk \neq 0$	1-dB(i_7)	$df \wedge dc \neq 0$
	$dc \wedge dk \wedge d(c_1, k_1) = 0$		$df \wedge dc \wedge d(e, k, l, c_1) = 0$
1-dB(i_4)	f constant, $dc \wedge de \neq 0$		
	$dc \wedge de \wedge d(k, c_1) = 0$		
1-dB(ii)	$dr \wedge dt \neq 0,\ t := r_1$		
	$dr \wedge dt \wedge dt_1 = 0$		
Case C:			
1-dC_1	s constant, $dc \wedge dl \neq 0$	1-dC_3	$ds \wedge de \neq 0,\ ds \wedge dc = 0$
	$dc \wedge dl \wedge d(k, c_1) = 0$		$ds \wedge de \wedge d(l, k, e_2, s_2) = 0$
1-dC_2	$ds \wedge dl \neq 0,\ ds \wedge d(c, e) = 0$	1-dC_4	$ds \wedge dc \neq 0$
	$ds \wedge dl \wedge d(k, l_3, s_2, s_3) = 0$		$ds \wedge dc \wedge d(e, l, c_1, s_2) = 0$
Case D:			
1-dD_1	$b = -2\varepsilon$, k constant, $t := c_1$	1-dD_3	k constant
	$dc \wedge dt \neq 0,\ dc \wedge dt \wedge dt_3 = 0$		$db \wedge dc \neq 0,\ db \wedge dc \wedge dc_3 = 0$
1-dD_2	$b = -\varepsilon$, k constant, $t := c_3$	1-dD_4	$db \wedge dk \neq 0$
	$dc \wedge dt \neq 0,\ dc \wedge dt \wedge dt_3 = 0$		$db \wedge dk \wedge d(c, b_3) = 0$
Case E:			
1-dE(i)	$ds \wedge dt \neq 0,\ t := s_1$		
	$ds \wedge dt \wedge dt_1 = 0$		
1-dE(ii_1)	r and s constant, $t := u_1$	1-dE(ii_3)	$dr \wedge ds \neq 0$
	$du \wedge dt \neq 0,\ du \wedge dt \wedge dt_1 = 0$		$dr \wedge ds \wedge dr_1 = 0$
1-dE(ii_2)	s constant		
	$dr \wedge du \neq 0,\ dr \wedge du \wedge du_1 = 0$		

Finally, to conclude this section, we consider equations admitting a one-dimensional symmetry group.

PROPOSITION 4.4. *A second-order ordinary differential equation admits a maximal one-dimensional Lie group of fibre-preserving point symmetries if and only if the basic and derived invariants of its reduced $\{e\}$-structure produce two independent fundamental invariants. All possible classes of equations satisfying this symmetry condition are invariantly characterized in Table* 3.

Since we have very few examples to present, we will format Table 3 differently.

REMARKS. 1. Here we present the few representative equations we have been able to obtain for the above classification. Some of these equations again arise in physically interesting situations.

1-dA$_{15}$: $y'' = (\alpha - \beta y'^2)y' - \gamma y$. This equation arises in the study of non-linear effects in acoustic phenomena. It was first studied by Lord Rayleigh [**25**].

1-dB(i$_3$): $y'' = -(12 + y')y + y^3$. This is a special case of p_{10}, the tenth Painlevé equation.

1-dB(i$_7$): $y'' = -(\delta + \alpha y^2)y' + \beta y - y^3$. This is the Holmes–Rand non-linear oscillator. It was shown earlier that when the coefficients α, β and γ are related by $\beta = -3\alpha^{-2}$, $\delta = 4\alpha^{-1}$ the equation admits a second independent symmetry.

1-dB(ii): $y'' = (1 - 3y)y' + y^2 - y^3$. This is a special case of p_6, the sixth Painlevé equation.

1-dE(i): $y'' = y' + y^2$. This equation occurs in the study of shear-free plane-symmetric relativistic cosmological models [**9**].

1-dE(ii)$_2$: $y'' = -\delta y' + \beta y - y^3$. This is Duffing's equation described before. As was shown earlier, when the coefficients β and δ are related by $\beta = -2\delta^2/9$ the equation admits a second independent symmetry.

1-dE(ii$_3$): $y'' = y'^2/y + y^3 + y^2 + 1 + y^{-1}$. This is a special case of p_{33}, the thirty-third Painlevé equation.

2. In addition to the above cases where an explicit representative equation exists, there are other cases where we have been able to establish existence of second-order equations satisfying the respective symmetry conditions (see following remarks). These cases are listed below:

1-dA$_1$, 1-dA$_2$, 1-dA$_{15}$, 1-dB(i$_1$), 1-dB(i$_2$), 1-dB(i$_3$), 1-dB(i$_5$), 1-dB(i$_7$),
1-dB(ii), 1-dD$_1$, 1-dD$_2$, 1-dE(i), 1-dE(ii$_1$), 1-dE(ii$_2$), 1-dE(ii$_3$).

In connection with these propositions one should note the following important point. Although all second-order ordinary differential equations admitting a Lie group of fibre-preserving point symmetries belong to one of the mutually exclusive cases listed in Propositions 4.1–4, we cannot *a priori* claim that all these cases are non-vacuous. For the transitive cases, since the integrability conditions reduce to algebraic constraints (see § 5), the existence of second-order ordinary differential equations with the prescribed symmetries is guaranteed by the theorem of Cartan stated at the end of § 3. However, for the non-transitive cases the integrability conditions involve differential relations between basic, fundamental and derived invariants. These integrability conditions manifest themselves as over-determined systems of ordinary and partial differential equations (see § 5) for cases admitting two and one-dimensional symmetry groups respectively, i.e. Propositions 4.3 and 4.4. To establish existence for these cases we need to show that the associated over-determined systems of equations admit solutions. In principle these can be handled using geometric theory of over-determined systems [**4**, **24**, **27**]. We have been able to establish existence for the cases listed in Remark 2, but this question has not been investigated fully. However, in any application, we are first provided with a differential equation. Thus if the equation admits symmetries, it must belong to one of the cases listed in the preceding propositions. A computer programme based on these symmetry conditions is being developed, on the University of Waterloo Maple symbolic system, to test for existence of fibre-preserving symmetries of an arbitrary second-order ordinary differential equation [**10**].

5. *Proof of Propositions* 4.1–4.4

In the following we give complete proofs for the transitive cases, namely Propositions 4.1 and 4.2. The proofs of Propositions 4.3 and 4.4 are similar. However, in view of the large number of possible subcases in the latter cases, a complete proof will not be given. Instead a detailed analysis of Case E and an outline for Case B will be presented for Proposition 4.3 while for Proposition 4.4 only Case E will be analysed.

Let us start with the transitive case:

Proof of Proposition 4.1. An equation admits a six-dimensional symmetry group if and only if the basic invariants I_i of its $\{e\}$-structure (3.11a) are constant (since the structure equations then become the Maurer–Cartan equations for a Lie group). The integrability conditions of these Maurer–Cartan equations then imply that $I_i \equiv 0$. From the parametric expressions of these invariants (3.11b) we obtain

$$F(x, y, y') = \tfrac{1}{2}y'^2 M_y + y' M_x + N \tag{5.1a}$$

and

$$M_{xxy} + (NM_y)_y - M_{xy}M_x - 2N_{yy} = 0 \tag{5.1b}$$

for some functions M and N depending on x and y. Since $F \equiv 0$ satisfies (5.1), and (3.11a) with $I_i \equiv 0$ are the structure equations of a transitive pseudo-group, every equation in this branch is equivalent to $y'' = 0$.

Proof of Proposition 4.2. It follows from Proposition 2.2 that in order to have a three-dimensional group of symmetries, the basic invariants appearing in the reduced $\{e\}$-structure must be constants. We now proceed to show that the cases listed in Table 1 are exhaustive. Let us consider Case E. There are two subcases:

Case E(i). From equation (3.36d) we have

$$ds \wedge \omega^1 = \tfrac{3}{2}\varepsilon s \omega^2 \wedge \omega^1 - \tfrac{1}{2}\varepsilon \omega^3 \wedge \omega^1. \tag{5.2}$$

Clearly s cannot be constant and thus no equation in this class admits a three-dimensional symmetry group.

Case E(ii). From the constancy of the basic invariants and (3.39) we obtain the following algebraic system:

$$\begin{aligned}&(15s-4\varepsilon)r=0,\\&5s-2\varepsilon=0,\\&10su-2\varepsilon r^2-\varepsilon u+5=0,\\&r=0,\end{aligned} \tag{5.3}$$

whose solution set is given by

$$r=0, \quad s=\tfrac{2}{5}\varepsilon, \quad u=-\tfrac{5}{3}\varepsilon. \tag{5.4}$$

Thus we have obtained an isolated equivalence class admitting a three-dimensional symmetry group: 3-dE($\mathrm{ii}_{\mathrm{VIII}}$).

The proofs of the remaining cases proceed in exactly the same fashion. It can easily be shown that for Cases B(i) and C, the algebraic systems obtained from the integrability conditions admit no solution. For Cases A, B(ii) and D the solution sets are listed in Table 1.

In what follows, we give a proof of Proposition 4.3. Since the method of proof is similar for all cases we will give details only for Case E and an outline for Case B.

Proof of Proposition 4.3. It follows from Proposition 2.2 that in order to have a two-dimensional group of symmetries the basic invariants appearing in the reduced $\{e\}$-structure must produce only one fundamental invariant. Thus to complete the proof, we need to show that the list given in Table 2 is complete. Let us start with a detailed analysis of Case E. There are two subcases:

Case E(i). From equation (3.36d) we have

$$ds=s_1\omega^1+\tfrac{3}{2}\varepsilon s\omega^2-\tfrac{1}{2}\varepsilon\omega^3, \tag{5.5}$$

where s_1 denotes the derived invariant of s. Without loss of generality, s can be chosen as the fundamental invariant. Furthermore, to obtain a two-dimensional group of symmetries, the differentiation of s must fail to produce new fundamental invariants. Thus we must have $s_1=\phi(s)$. From the integrability condition, $d^2s=0$, we obtain

$$d\phi\wedge\omega^1+(2\varepsilon\phi-3\varepsilon s^2-\tfrac{1}{2})\omega^1\wedge\omega^2+3\varepsilon s\omega^1\wedge\omega^3=0.$$

and so we are led to an over-determined system of two first-order ordinary differential equations for $s_1=\phi(s)$ given by

$$\phi'+6s=0, \tag{5.6a}$$

$$3s\phi'-4\phi+6s^2+\varepsilon=0. \tag{5.6b}$$

The unique solution of this system is

$$\phi(s)=\tfrac{1}{4}\varepsilon-3s^2. \tag{5.7}$$

Thus we have obtained an isolated equivalence class admitting a two-dimensional symmetry group: 2-dE(i).

Case E(ii). From (3.39d) we have

$$ds=s_1\omega^1+s_2\omega^2. \tag{5.8}$$

There are two cases to consider: that when s is constant and that where s can be chosen as the fundamental invariant.

(a) *s constant.* Equation (3.39d) gives

$$dr = r_1\omega^1 + (\tfrac{4}{5}\varepsilon - 3s)r\omega^2 + (s - \tfrac{2}{5}\varepsilon)\omega^3. \tag{5.9}$$

There are two further cases to consider: that when r is constant or that where r can be chosen as the fundamental invariant.

(a_i) *r and s constant.* Equations (5.9) and (3.39e) imply that

$$r = 0 \quad \text{and} \quad s = \tfrac{2}{5}\varepsilon, \tag{5.10}$$

$$du = u_1\omega^1 - (1 + \tfrac{3}{5}\varepsilon u)\omega^2. \tag{5.11}$$

In order to obtain a two-dimensional group of symmetries, we necessarily have u as a fundamental invariant. Furthermore, the differentiation of u must fail to produce new fundamental invariants. Thus we must have $u_1 = \phi(u)$. From the integrability condition, $d^2u = 0$, we obtain

$$u = \tfrac{5}{3}\varepsilon. \tag{5.12}$$

Hence any equation in this branch with a two-dimensional group of symmetries actually has a three-dimensional symmetry group.

(a_{ii}) *s constant and r as the fundamental invariant.* Recall that from (5.9) we have

$$dr = r_1\omega^1 + (\tfrac{4}{5}\varepsilon - 3s)r\omega^2 + (s - \tfrac{2}{5}\varepsilon)\omega^3.$$

In order to obtain a two-dimensional group of symmetries, we necessarily have $u = u(r)$. From (3.39e) we obtain the system

$$(5s - 2\varepsilon)u' + 4\varepsilon r = 0, \tag{5.13a}$$

$$(4\varepsilon - 15s)ru' + 2\varepsilon r_1 + 10su - 2\varepsilon r^2 - \varepsilon u + 5 = 0. \tag{5.13b}$$

Solving for u and r_1 we obtain

$$(5s - 2\varepsilon)u = \alpha - 2\varepsilon r^2, \quad \alpha \text{ constant}, \tag{5.14a}$$

$$2(5s - 2\varepsilon)r_1 + 10(3s - \varepsilon)r^2 + 10\varepsilon\alpha s - \alpha + 25\varepsilon s - 10 = 0. \tag{5.14b}$$

Notice that in this branch $u = u(r)$ implies $r_1 = \phi(r)$. Furthermore, from the integrability condition, $d^2r = 0$, we have the constraint

$$(20\alpha + 50)\varepsilon s^2 - (30 + 5\alpha)s + 4\varepsilon = 0. \tag{5.14c}$$

Thus we obtain a one-parameter family of equivalence classes admitting two-dimensional symmetry groups: 2-dE(ii).

(b) *s as the fundamental invariant.* Recall that from (5.8) we have

$$ds = s_1\omega^1 + s_2\omega^2.$$

To obtain a two-dimensional group of symmetries, we necessarily have $r = r(s)$ and $u = u(s)$. Then (3.39d) implies that

$$s = \tfrac{2}{5}\varepsilon. \tag{5.15}$$

Hence any equation in this branch with a two-dimensional group of symmetries actually has a three-dimensional symmetry group.

Let us now outline the proof of Case B.

Case B. There are again two subcases: Case B(i) and Case B(ii).

Case B(i). From equation (3.20d) we have

$$df = [f(k-1) + 3e + l + 1]\omega^1 + f_2\omega^2 - 2f^2\omega^3, \tag{5.16}$$

and so there are two cases to consider: f constant or that where f can be chosen as the fundamental invariant.

(a) *f constant.* Equations (5.16) and (3.20e) imply that

$$f = 0 \quad \text{and} \quad l = -(1+3e), \tag{5.17}$$

$$de = e_1\omega^1 + e_2\omega^2, \tag{5.18}$$

and so there are two further cases to consider: e constant, or e chosen as the fundamental invariant.

(a_i) *$f = 0$ and e constant.* Equations (5.17) and (3.20e) imply that

$$\begin{aligned} &l \text{ is constant,} \\ &dc = c_1\omega^1 + (e - 2cl)\omega^2 + (e-1)\omega^3, \end{aligned} \tag{5.19}$$

and so there are two more cases to consider: c constant or c chosen as the fundamental invariant.

(a_{i_1}) *$f = 0$, $l = -(1+3e)$, c and e constant.* From equations (5.19), (5.17) and (3.20f) we have

$$c = -\tfrac{1}{8}, \quad e = 1 \quad \text{and} \quad l = -4, \tag{5.20}$$

$$dk = k_1\omega^1 + (8k-6)\omega^2 - 4\omega^3. \tag{5.21}$$

Thus, with no loss of generality, k can be chosen as the fundamental invariant. The symmetry condition (i.e. existence of exactly one fundamental invariant) then implies that we must have $k_1 = \phi(k)$. It can readily be shown that there exists an isolated equivalence class admitting a two-dimensional symmetry group 2-dB(i_1).

(a_{i_2}) *$f = 0$, $l = -(1+3e)$, e constant and c as the fundamental invariant.* Recall that from (5.19) we have

$$dc = c_1\omega^1 + (e - 2cl)\omega^2 + (e-1)\omega^3.$$

To obtain a two-dimensional group of symmetries, we necessarily have $c_1 = \phi(c)$ which implies $k = k(c)$. It can readily be shown that there exists a one-parameter family of equivalence classes admitting two-dimensional symmetry groups: 2-dB(i_2).

(a_{ii}) *$f = 0$, $l = -(1+3e)$ and e as the fundamental invariant.* From equation (5.18) we obtain

$$de = e_1\omega^1 + e_2\omega^2.$$

A detailed analysis shows that this case admits no maximal two-dimensional symmetry group.

(b) *f as the fundamental invariant.* From equation (5.16) we obtain

$$df = [f(k-1) + 3e + l + 1]\omega^1 + f_2\omega^2 - 2f^2\omega^3.$$

To obtain a two-dimensional group of symmetries, we necessarily have $c = c(f)$ and $l = l(f)$ which implies that $e = e(f)$, $k = k(f)$, and $f_2 = \phi(f)$ where f_2 is the derived invariant of f. It is not hard to show that there exist equations in this class possessing maximal two-dimensional symmetry groups: 2-dB(i_3).

Case B(ii). A detailed analysis shows that this branch admits no maximal two-dimensional symmetry group.

Finally we conclude this section by proving Proposition 4.4. Again since the proof is similar for all cases, we will only give a detailed analysis for Case E.

Proof of Proposition 4.4. It follows from Proposition 2.2 that in order to have a one-dimensional group of symmetries, the invariants appearing in the reduced $\{e\}$-structure must produce two fundamental invariants. Thus to complete the proof, we need to show that Table 3 is exhaustive. Let us again consider a detailed analysis of Case E. As before, there are two subcases:

Case E(i). Recall that from (5.5) we have

$$ds = s_1\omega^1 + \tfrac{3}{2}\varepsilon s\omega^2 - \tfrac{1}{2}\varepsilon\omega^3.$$

Thus, without loss of generality, s can be chosen as a fundamental invariant. To obtain a one-dimensional group of symmetries, the differentiation of s must produce a new fundamental invariant. Thus s_1, relabelled by t, must be independent of s. From the integrability, $d^2s = 0$, we have

$$dt = t_1\omega^1 + (2\varepsilon t - 3\varepsilon s^2 - \tfrac{1}{2})\omega^2 + 3\varepsilon s\omega^3. \tag{5.22}$$

Since s and t are two independent invariants, the differentiation of t must fail to produce new fundamental invariants. Thus we must have $t_1 = \phi(s, t)$. From the integrability condition, $d^2t = 0$, we obtain

$$d\phi \wedge \omega^1 + (\tfrac{5}{2}\varepsilon\phi - 10\varepsilon st + 6\varepsilon s^3 + 4s)\omega^1 \wedge \omega^2 + (5\varepsilon t - 12\varepsilon s^2 - \tfrac{1}{2})\omega^1 \wedge \omega^3 = 0,$$

and so we are led to an over-determined system of two first-order partial differential equations for $t_1 = \phi(s, t)$ given by

$$\tfrac{3}{2}\varepsilon s\phi_s + (2\varepsilon t - 3\varepsilon s^2 - \tfrac{1}{2})\phi_t - \tfrac{5}{2}\varepsilon\phi + 10\varepsilon st - 6\varepsilon s^3 - 4s = 0, \tag{5.23a}$$

$$-\tfrac{1}{2}\varepsilon\phi_s + 3\varepsilon s\phi_t - 5\varepsilon t + 12\varepsilon s^2 + \tfrac{1}{2} = 0. \tag{5.23b}$$

It can readily be shown that the general solution of (5.23) is given by

$$\phi(r, s) = \alpha(4\varepsilon r + 12\varepsilon s^2 - 1)^{5/4} - 12s^3 + \varepsilon s - 10rs, \quad \alpha \text{ constant}. \tag{5.23c}$$

Hence we obtain a one-parameter family of equivalence classes admitting one-dimensional symmetry groups: 1-dE(i).

Case E(ii). Recall that from (5.8) we have

$$ds = s_1\omega^1 + s_2\omega^2.$$

There are two cases to consider: s constant or s chosen as a fundamental invariant.

(a) *s constant.* From (5.9) we obtain

$$dr = r_1\omega^1 + (\tfrac{4}{5}\varepsilon - 3s)r\omega^2 + (s - \tfrac{2}{5}\varepsilon)\omega^3.$$

There are two further cases to consider: r constant or r chosen as a fundamental invariant.

(a_i) *r and s constant.* Recall that from (5.10) and (5.11) we have

$$r = 0 \quad \text{and} \quad s = \tfrac{2}{5}\varepsilon,$$
$$du = u_1\omega^1 - (1 + \tfrac{3}{5}\varepsilon u)\omega^2.$$

In order to obtain a one-dimensional group of symmetries, we necessarily have u as a fundamental invariant. Furthermore, the differentiation of u must produce a new fundamental invariant. Thus u_1, relabelled by t, must be independent of u. From the integrability condition, $d^2u = 0$, we have

$$dt = t_1\omega^1 - \varepsilon t\omega^2 - (1 + \tfrac{3}{5}\varepsilon u)\omega^3. \tag{5.24}$$

Since u and t form two independent fundamental invariants, we necessarily have $t_1 = \phi(u, t)$. The integrability condition, $d^2t = 0$, then implies that

$$d\phi \wedge \omega^1 + (\tfrac{7}{5}\varepsilon\phi + u + \tfrac{3}{5}\varepsilon u^2)\omega^2 \wedge \omega^1 - \tfrac{8}{5}\varepsilon t\omega^1 \wedge \omega^3 = 0,$$

and so we are led to an over-determined system of two first-order partial differential equations for $t_1 = \phi(u, t)$ given by

$$(5 + 3\varepsilon u)\phi_t = 8\varepsilon t, \tag{5.25a}$$
$$5\varepsilon t\phi_t + (5 + 3\varepsilon u)\phi_u = 7\varepsilon\phi + (5 + 3\varepsilon u)u. \tag{5.25b}$$

It can readily be shown that the general solution of (5.25) is given by

$$\phi(t, u) = \frac{4\varepsilon t^2 + a(u)}{5 + 3\varepsilon u}, \tag{5.26a}$$

where $a(u)$ is defined by

$$a(u) = \tfrac{1}{12}[5(5 + 3\varepsilon u)^2 - 4(5 + 3\varepsilon u)^3 + \alpha(5 + 3\varepsilon u)^{10/3}], \quad \alpha \text{ constant}. \tag{5.26b}$$

Hence we obtain a one-parameter family of equivalence classes admitting one-dimensional symmetry groups: 1-dE(ii_1).

(a_{ii}) *s constant and r as a fundamental invariant.* Recall that from (5.9) we have

$$dr = r_1\omega^1 + (\tfrac{4}{5}\varepsilon - 3s)r\omega^2 + (s - \tfrac{2}{5}\varepsilon)\omega^3,$$

and so there are two further cases to consider: $u = u(r)$ or u chosen as the final fundamental invariant.

(a_{ii_1}) *s constant, $u = u(r)$ and r as a fundamental invariant.* In order to obtain a one-dimensional group of symmetries the differentiation of r must produce a new fundamental invariant. However, as noticed before, in this branch $u = u(r)$ implies $r_1 = \phi(r)$ and so there exist no additional independent fundamental invariants. Hence any equation in this branch with a one-dimensional group of symmetries actually has a two-dimensional symmetry group.

(a_{ii_2}) *s constant, r and u as the fundamental invariants.* From equation (3.39e) we obtain

$$du = u_1\omega^1 - \tfrac{1}{5}(2\varepsilon r_1 + 10su - 2\varepsilon r^2 - \varepsilon u + 5)\omega^2 - \tfrac{4}{5}\varepsilon r\omega^3. \tag{5.27}$$

In order to obtain a one-dimensional group of symmetries the differentiation of r and u must fail to produce new fundamental invariants. Thus we must have $u_1 = \psi(r, u)$ which in turn implies $r_1 = \phi(r, u)$. From the integrability conditions,

$d^2r = d^2u = 0$, we obtain

$$d\phi \wedge \omega^1 + \tfrac{1}{5}(20s\phi - 4\varepsilon\phi + 8\varepsilon r^2 - 30r^2s - 5su + 2\varepsilon u)\omega^2 \wedge \omega^1 + 2(\varepsilon - 3s)r\omega^1 \wedge \omega^3 = 0,$$

$$d\psi \wedge \omega^1 - \tfrac{2}{5}\varepsilon d\phi \wedge \omega^2 + \tfrac{1}{5}(15s\psi - \varepsilon\psi - 8\varepsilon r\phi - 20rsu + 4\varepsilon r^3 + 6\varepsilon ru - 10r)\omega^2 \wedge \omega^1 - \tfrac{1}{5}(6\varepsilon\phi + 10su - 14\varepsilon r^2 - \varepsilon u + 5)\omega^1 \wedge \omega^3 + \tfrac{4}{5}(1 - 3\varepsilon s)r\omega^3 \wedge \omega^2 = 0,$$

and so we are led to an over-determined system of four first-order partial differential equations for $r_1 = \phi(r, u)$ and $u_1 = \psi(r, u)$ given by

$$(r^2 - \varepsilon u + 5)\phi_u + 4(5s - \varepsilon)\phi + 8\varepsilon r^2 - 30r^2s - 5su + 2\varepsilon u = 0, \tag{5.28a}$$

$$(5s - 2\varepsilon)\phi_r - 4\varepsilon r\phi_u + 10(3s - \varepsilon)r = 0, \tag{5.28b}$$

$$(4\varepsilon - 15s)r\psi_r - (2\varepsilon\phi + 10su - 2\varepsilon r^2 - \varepsilon u + 5)\phi_u + (15s - \varepsilon)\psi + 2\varepsilon\phi\phi_r + 2\varepsilon\psi\phi_u - 8\varepsilon r\phi - 20rsu + 4\varepsilon r^3 + 6\varepsilon ru - 10r = 0, \tag{5.28c}$$

$$(5s - 2\varepsilon)\psi_r - 4\varepsilon r\psi_u + 6\varepsilon\phi + 10su - 14\varepsilon r^2 - \varepsilon u + 5 = 0. \tag{5.28d}$$

It can readily be shown that equations (5.28) admit non-trivial solutions and thus there exist equations in this class admitting maximal one-dimensional symmetry groups: 1-dE(ii_2).

(b) *s as a fundamental invariant.* Recall that from (5.8) we have

$$ds = s_1\omega^1 + s_2\omega^2,$$

and so we are led to consider two final cases: $r = r(s)$ or r chosen as the final fundamental invariant.

(b_i) *$r = r(s)$ and s as a fundamental invariant.* From equation (3.39d) we obtain

$$dr \wedge \omega^1 = [s_1 + (\tfrac{4}{5}\varepsilon - 3s)r]\omega^2 \wedge \omega^1 + (s - \tfrac{2}{5}\varepsilon)\omega^3 \wedge \omega^1.$$

The relation $r = r(s)$ coupled with (5.8) imply that $s = \tfrac{2}{5}\varepsilon$. Thus no equation in this branch admits a maximal one-dimensional symmetry group.

(b_ii) *r and s as fundamental invariants.* From equation (3.39d) we obtain

$$dr = r_1\omega^1 + [s_1 + (\tfrac{4}{5}\varepsilon - 3s)r]\omega^2 + (s - \tfrac{2}{5}\varepsilon)\omega^3. \tag{5.29}$$

To obtain a one-dimensional group of symmetries we necessarily have $u = u(r, s)$. Furthermore, the differentiation of r and s must fail to produce new invariants. Thus we must have $r_1 = \phi(r, s)$, $s_1 = \psi(r, s)$, and $s_2 = \phi(r, s)$. From the integrability conditions, $d^2r = d^2s = 0$, and (3.39e), we obtain

$$d\phi \wedge \omega^1 + d\psi \wedge \omega^2 + \tfrac{1}{5}(20s\phi - 4\varepsilon\phi + 25r\psi + 8\varepsilon r^2 - 30sr^2 - 5su + 2\varepsilon u)\omega^2 \wedge \omega^1 + (2\psi + 2\varepsilon r - 6rs)\omega^1 \wedge \omega^3 - \pi\omega^3 \wedge \omega^2 = 0,$$

$$d\psi \wedge \omega^1 + d\pi \wedge \omega^2 + \pi\omega^1 \wedge \omega^3 = 0,$$

$$du \wedge \omega^1 - \tfrac{1}{5}(2\varepsilon\phi + 10su - 2\varepsilon r^2 - \varepsilon u + 5)\omega^1 \wedge \omega^2 - \tfrac{4}{5}\varepsilon r\omega^1 \wedge \omega^3 = 0,$$

and so we are led to an over-determined system of seven first-order partial differential equations for $u = u(r, s)$, $r_1 = \phi(r, s)$, $s_1 = \psi(r, s)$, and $s_2 = \pi(r, s)$

given by

$$(5\psi + 4\varepsilon r - 15rs)\phi_r + 5\pi\phi_2 - 5\phi\psi_r - 5\psi\psi_s + 5(4s - \varepsilon)\phi + 25r\psi + 8\varepsilon r^2 - 30r^2 s - 5su + 2\varepsilon u = 0, \tag{5.30a}$$

$$(5s - 2\varepsilon)\phi_r - 10\psi + 10(3s - \varepsilon)r = 0, \tag{5.30b}$$

$$(5s - 2\varepsilon)\psi_r - 5\pi = 0, \tag{5.30c}$$

$$(5\psi + 4\varepsilon r - 15rs)\psi_r + 5\pi\psi_r - 5\phi\pi_r - 5\psi\pi_s + 5s\psi + 10r\pi = 0, \tag{5.30d}$$

$$(5s - 2\varepsilon)\pi_r = 0, \tag{5.30e}$$

$$(5\psi + 4\varepsilon r - 15rs)u_r + 5\pi u_s + 2\varepsilon\phi + 2\varepsilon r^2 - \varepsilon u + 5 = 0, \tag{5.30f}$$

$$(5s - 2\varepsilon)u_r + 4\varepsilon r = 0. \tag{5.30g}$$

Notice that the above, coupled with the relation $r_1 = \phi(r, s)$, implies that $u = u(r, s)$, $s_1 = \psi(r, s)$, and $s_2 = \pi(r, s)$. It can readily be shown that equations (5.30) admit non-trivial solutions and thus there exist equations in this class admitting maximal one-dimensional symmetry groups: 1-dE(ii$_3$).

The proofs of the remaining cases of Propositions 4.3 and 4.4 proceed in exactly the same fashion. As in the above, the integrability conditions in these cases produce over-determined systems of differential equations. Existence of solutions to these systems is necessary and sufficient for existence of second-order equations with the prescribed symmetries.

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