Group classification and exact solutions of a class of variable coefficient nonlinear wave equations

Ding-jiang Huang [†], **Jian-qin Mei** [‡] **and Hong-qing Zhang**[‡]

[†] *Department of Mathematics, East China University of Science and Technology, Shanghai, 200237, P. R. China*
e-mail: hdj8116@163.com, djhuang@ecust.edu.cn

[‡] *Department of Applied Mathematics, Dalian University of Technology, Dalian, 116024, P. R. China*

Complete group classification of a class of variable coefficient (1+1)-dimensional wave equations is performed. The possible additional equivalence transformations between equations from the class under consideration and the conditional equivalence groups are also investigated. These allow simplifying results of classification and further applications of them. The derived Lie symmetries are used to construct exact solutions of special forms of these equations via classical Lie method. Nonclassical symmetries of the wave equations are discussed.
PACS: 02.20.Sv, 02.30.Jr

1 Introduction

Investigation of the flow of one-dimensional gas, longitudinal wave propagation on a moving threadline and dynamics of a finite nonlinear string [1, 2] leads to interesting mathematical models which can be often formulated in terms of partial differential equations. In general case coefficients of model equations explicitly include both dependent and independent model variables that makes difficulties in studying such models.

In this letter a class of (1+1)-dimensional nonlinear wave equations of the general form

$$u_{tt} = (H(u)u_x)_x + k(x)u \tag{1}$$

where $H_u \neq 0$, is investigated within the symmetry framework. Here H and k are arbitrary functions of their argument, t is the time coordinate and x is the one-space coordinate.

The condition $H_u = 0$ corresponds to the linear case of (1) which was completely investigated with the Lie symmetry point of view long time ago [14, 18]. Moreover, the sets of the linear and nonlinear equations of form (1) can be separately investigated under restriction with point symmetries. That is why the linear case is excluded from consideration in the present letter.

The problem of group classification for the degenerate case $k = 0$ (i.e. the class of nonlinear one-dimensional wave equations) was first solved by Ames et.al [2] in 1981. From that well-known paper, the search for symmetries of various kinds of one-dimensional non-linear wave equations has been considered in many papers in the last two decades [3–8, 10, 11, 17, 20–22] . Note also that Lie symmetries of the class of quasilinear Hyperbolic type second-order nonlinear partial differential equations in two independent variables, which has a wide equivalence group and covers all the mentioned classes, were classified in [13]. Recently, by using a compatibility method and additional equivalence transformations, we present a complete group classification of variable coefficient nonlinear telegraph equations [9]. In the present letter, we further extended this method to the wave equation (1).

Study in our letter is concentrated on rigorous and exhaustive group classification of the whole class (1) and construction of exact solutions for some nonlinear wave equations from this class. Additional equivalence transformations and conditional equivalence groups are also found.

These allow to simplify results of classification and further applications of them. To find exact solutions, we apply both classical Lie reduction and nonclassical symmetry approaches.

2 Group classification

Group classification of class (1) is performed in the framework of the classical approach [9, 18]. All necessary objects (the equivalence group, principal group, the kernel and all inequivalent extensions of maximal Lie invariance algebras) are found. Moreover, we extend the classical approach with additional equivalence transformations and conditional equivalence group for simplification of the classification results.

Theorem 1. *The Lie algebra of the kernel of principal groups of* (1) *is an one-dimensional algebra* $A^{\rm ker} = \langle \partial_t \rangle$.

Theorem 2. *The Lie algebra of the equivalence group* $G^{\sim}$ *for class* (1) *is*

$$A^{\sim} = \langle \partial_t,\, \partial_x,\, u\partial_u,\, x\partial_x + 2H\partial_H,\, t\partial_t - 2H\partial_H - 2k\partial_k \rangle.$$

The equivalence group $G^{\sim}$ of class (1) is formed by the transformations

$$\tilde t = t\epsilon_1 + \epsilon_2, \quad \tilde x = x\epsilon_3 + \epsilon_4, \quad \tilde u = \epsilon_5 u, \quad \tilde H = H\epsilon_1^{-2}\epsilon_3^2, \quad \tilde k = k\epsilon_1^{-2},$$

where $\epsilon_1, \dots, \epsilon_5$ are arbitrary constants, $\epsilon_1\epsilon_3\epsilon_5 \neq 0$. The connected component of the unity in $G^{\sim}$ is formed by continuous transformations having $\epsilon_1 > 0, \epsilon_3 > 0$ and $\epsilon_5 > 0$. The complement discrete component of $G^{\sim}$ is generated by three involutive transformations of alternating sign in the sets $\{t, H, k\}, \{x\}$ and $\{u\}$.

Theorem 3. *The complete set of* $G^{\sim}$*-inequivalent extensions of* $A^{\max} \neq A^{\rm ker}$ *for equation* (1) *is exhausted by ones given in table 1.*

Table 1. Results of group classification

N	$H(u)$	$h(x)$	Basis of $A^{\max}$
1	$\forall$	$\forall$	∂_t,
2	$\forall$	1	$\partial_t,\, \partial_x$,
3	$\forall$	x^{-2}	$\partial_t,\, t\partial_t + x\partial_x$
4	u^{μ}	ϵx^q	$\partial_t,\, \mu q t\partial_t - 2\mu x\partial_x - 2(q+2)u\partial_u$
5	u^{μ}	ϵe^x	$\partial_t,\, \mu t\partial_t - 2\mu\partial_x - 2u\partial_u$
6	$u^{-\frac{4}{3}}$	$k^1(x)$	$\partial_t,\, -2qt\partial_t + 4(x^2+p)\partial_x - 3(4x+q)u\partial_u$
7	$\forall$	0	$\partial_t,\, \partial_x,\, t\partial_t + x\partial_x$
8	$(u+1)^{-1}$	1	$\partial_t,\, \partial_x,\, e^t\partial_t + 2e^t(u+1)\partial_u$,
9	e^u	0	$\partial_t,\, \partial_x,\, t\partial_t + x\partial_x,\, x\partial_x + 2\partial_u$
10	$u^{\mu},\ \mu \neq -4, -\frac{4}{3}$	ϵ	$\partial_t,\, \partial_x,\, \mu x\partial_x + 2u\partial_u$
11	$(u+\alpha)^{\mu},\ \mu \neq -4, -\frac{4}{3}$	0	$\partial_t,\, \partial_x,\, t\partial_t + x\partial_x,\, \mu x\partial_x + 2(u+\alpha)\partial_u$,
12	$u^{-\frac{4}{3}}$	ϵ	$\partial_t,\, \partial_x,\, 2x\partial_x - 3u\partial_u, x^2\partial_x - 3xu\partial_u$
13	$(u+\alpha)^{-\frac{4}{3}}$	0	$\partial_t,\, \partial_x,\, t\partial_t + x\partial_x,\, 2x\partial_x - 3(u+\alpha)\partial_u, x^2\partial_x - 3x(u+\alpha)\partial_u$
14	$(u+\alpha)^{-4}$	0	$\partial_t,\, \partial_x,\, 2t\partial_t + (u+\alpha)\partial_u,\, 2x\partial_x - (u+\alpha)\partial_u, t^2\partial_t + t(u+\alpha)\partial_u$
15	$(u+\alpha)^{-4}$	1	$\partial_t,\, \partial_x,\, e^{2t}(\partial_t + (u+\alpha)\partial_u),\, 2x\partial_x + (u+\alpha)\partial_u, e^{-2t}(\partial_t - (u+\alpha)\partial_u)$

Here $k^1(x) = \epsilon \exp[\int \frac{q}{x^2+p}]$; $p \in \{-1, 0, 1\}$, $\epsilon = \pm 1$, $\alpha \in \{0, 1\} \bmod G^{\sim}$; $\mu,\ q \neq 0$.

Additional equivalence transformations:

1. $6|_{p=-1} \to 4|_{\tilde\mu=-4/3, \tilde q = q/2}$: $\tilde t = t,\, \tilde x = \frac{x-1}{x+1},\, \tilde u = 2^{-3/2}(x+1)^3 u$;
2. $6|_{p=0} \to 5|_{\tilde\mu=-4/3}$: $\tilde t = t,\, \tilde x = x^{-1},\, \tilde u = x^3 u$;
3. $11|_{\alpha\neq 0} \to 11|_{\alpha=0}$ $(\mu = -\frac{4}{3})$, $13|_{\alpha\neq 0} \to 13|_{\alpha=0}$, $14|_{\alpha\neq 0} \to 14|_{\alpha=0}$, $15|_{\alpha\neq 0} \to 15|_{\alpha=0}$: $\tilde t = t,\, \tilde x = x,\, \tilde u = u+\alpha$;

Remark 1. The parameter function $k^1(x)$ equals to the following functions depending on values of p:

$$p=-1: \ k^1(x)=\epsilon\left|\frac{x-1}{x+1}\right|^{\frac{q}{2}}; \quad p=0: \ k^1(x)=\epsilon e^{-\frac{q}{x}}; \quad p=1: \ k^1(x)=\epsilon e^{q\arctan x}.$$

Additionally we can assume $q=-1 \bmod G^{\sim}$ if $p=0$.

Some cases from Table 1 are equivalent with respect to point transformations which obviously do not belong to $G^{\sim}$. These transformations are called *additional equivalence transformations* and lead to simplification of further application of group classification results. The simplest way to find such additional equivalences between previously classified equations is based on the fact that equivalent equations have equivalent maximal Lie invariance algebras. Explicit formulas for pairs of point-equivalent extension cases and the corresponding additional equivalence transformations are adduced after the tables. One can check that there exist no other point transformations between the equations from tables 1. Using this we can formulate the following theorem.

Theorem 4. *Up to point transformations, a complete list of extensions of the maximal Lie invariance algebra of equations from class* (1) *is exhausted by the cases 1–5,* $6|_{p=1}$*, 7–10,* $11|_{\alpha=0}$*, 12,* $13|_{\alpha=0}$*,* $14|_{\alpha=0}$ *and* $15|_{\alpha=0}$*.*

The singularity of the wave coefficient such as $H=u^{-4/3}$ with a number of different values of $k(x)$ admitting extensions of Lie invariance algebra can be explained in the framework of conditional equivalence groups. The equivalence group is extended under the condition $H=u^{-4/3}$. More precisely, the equivalence group $G_1^{\sim}$ of the subclass of equations (1) with $H=u^{-\frac{4}{3}}$ is formed by the transformations

$$\tilde t=t\epsilon_1+\epsilon_2, \quad \tilde x=\frac{x\epsilon_3+\epsilon_4}{x\epsilon_5+\epsilon_6}, \quad \tilde u=\pm\epsilon_1^2(\epsilon_5x+\epsilon_6)^3u, \quad \tilde k=k\epsilon_1^{-2}.$$

where ϵ_i, $i=1,...,6$, are arbitrary constants, $\epsilon_1>0$ and $\epsilon_3\epsilon_6-\epsilon_4\epsilon_5=1$. $G_1^{\sim}$ is a non-trivial conditional equivalence group of class (1). Two first additional equivalence transformations belong to $G_1^{\sim}$.

Another example of a conditional equivalence group in class (1) arises under the condition $k=0$. The equivalence group $G^{\sim}$ of the whole class is then extended with translations with respect to u, i.e. the complete equivalence group $G_2^{\sim}$ of nonlinear wave equations ($k=0$) is formed by the transformations

$$\tilde t=t\epsilon_1+\epsilon_2, \quad \tilde x=x\epsilon_3+\epsilon_4, \quad \tilde u=u\epsilon_5+\epsilon_6, \quad \tilde k=k\epsilon_1^{-2}\epsilon_3^2.$$

where $\epsilon_i, i=1,...,6$, are arbitrary constants, $\epsilon_1\epsilon_3\epsilon_5\neq 0$. The third additional equivalence transformation belongs to $G_2^{\sim}$.

The subclass of equations (1) with k being a constant admits an extension of generalized equivalence group. The prefix generalized means that transformations of the variables t,x and u can depend on arbitrary elements [15]. The associated generalized equivalence group $G_4^{\sim}$ is generated by transformations from $G^{\sim}$, where ϵ is replaced by k.

Knowledge on conditional equivalence groups allows us to describe the set of admissible (from-preserving) transformations in class (1) completely. See e.g. [12] and references therein.

3 Similarity Solutions

The Lie symmetry operators found as a result of solving the group classification problem can be applied to construction of exact solutions of the corresponding equations. The method of reductions with respect to subalgebras of Lie invariance algebras is well-known and quite algorithmic to use in most cases; we refer to the standard textbooks on the subject [16, 18]. In what follows, we select case 10 as an example to implement the relevant Lie reduction algorithm. The Lie reductions and similarity solutions of some truly nonlinear 'variable coefficient' wave equations (cases 4-6 of table 1) will be a subject in a subsequent publication.

As shown in the previous section, the equation (Case 10 of Table 1)

$$u_{tt} = (u^{\mu}u_x)_x + \epsilon u, \quad \epsilon = \pm 1 \tag{2}$$

admits the three-dimensional Lie invariance algebra $\mathfrak{g}$ generated by the operators

$$Q_1 = \partial_x, \quad Q_2 = x\partial_x + \frac{2u}{\mu}\partial_u, \quad Q_3 = \partial_t.$$

These operator satisfy the commutation relations

$$[Q_1,\ Q_2] = Q_1, \quad [Q_1,\ Q_3] = 0, \quad [Q_2,\ Q_3] = 0.$$

It means that the algebras $\mathfrak{g}$ are isomorphic to the algebra $\mathfrak{g}_2 \oplus \mathfrak{g}_1$ being the direct sum of the two-dimensional non-Abelian Lie algebra $\mathfrak{g}_2$ and the one-dimensional Lie algebra $\mathfrak{g}_1$. An optimal set of subalgebras of $\mathfrak{g}_2 \oplus \mathfrak{g}_1$ can be easily constructed with application of a standard technique [16, 18]. Another way is to take the set from [19]. (In this paper optimal sets of subalgebras are listed for all three- and four-dimensional algebras.) The used optimal set consists of

$$\begin{array}{ll} \text{one-dimensional subalgebras :} & \langle Q_2 - \alpha Q_3\rangle, \ \ \langle Q_3\rangle, \ \ \langle Q_1 \pm Q_3\rangle, \ \ \langle Q_1\rangle; \\ \text{two-dimensional subalgebras :} & \langle Q_1,\ Q_3 - \beta Q_2\rangle, \ \ \langle Q_1,\ Q_2\rangle, \end{array}$$

where α and β are arbitrary constants.

Lie reduction to algebraic equations can be made only with the first two-dimensional subalgebra; the second one does not satisfy the transversality condition [16]. The corresponding ansatzes and reduced algebraic equations have the form:

$$\begin{array}{l} u = Ce^{\sigma t}, \ \ \text{where} \ \ \sigma = -\frac{2\beta}{\mu}; \ \ C(\sigma^2 - \epsilon) = 0, \ \ \text{if} \ \ \epsilon = 1; \\ u = C\cos(\sigma t), \ \ \text{where} \ \ \sigma = -\frac{2\beta}{\mu}; \ \ C(\sigma^2 + \epsilon) = 0, \ \ \text{if} \ \ \epsilon = -1; \end{array}$$

Here C is an unknown constant to be found. The reduced equations are compatible and have non-trivial (non-zero) solutions only for some values of σ and, moreover, become identities for these values of σ. As a result, the following x-free solutions are constructed:

$$\begin{array}{l} u = Ce^{\sigma t}, \ \ \text{where} \ \ \sigma^2 - \epsilon = 0, \ \ \text{if} \ \ \epsilon = 1; \\ u = C\cos(\sigma t), \ \ \text{where} \ \ \sigma^2 + \epsilon = 0, \ \ \text{if} \ \ \epsilon = -1; \end{array}$$

Here C is an arbitrary constant. These solutions can be also obtained with step-by-step reductions with respect to one-dimensional subalgebras.

The ansatzes and reduced equations corresponding to the one-dimensional subalgebras from the optimal system are collected in table 2.

Table 2. Reduced ODEs for equation (2).

N	Subalgebra	Ansatz $u =$	ω	Reduced ODE
1	$\langle Q_1\rangle$	$\varphi(\omega)$	t	$\varphi_{\omega\omega} - \epsilon\varphi = 0$
2	$\langle Q_3\rangle$	$\varphi(\omega)$	x	$(\varphi^{\mu}\varphi_{\omega})_{\omega} + \epsilon\varphi = 0$
3	$\langle Q_1 \pm Q_3\rangle$	$\varphi(\omega)$	$x \pm t$	$\varphi_{\omega\omega} - (\varphi^{\mu}\varphi_{\omega})_{\omega} - \epsilon\varphi = 0$
4	$\langle Q_2 - \alpha Q_3\rangle$	$\exp(-\frac{2}{\alpha\mu}t)\varphi(\omega)$	$\exp(\frac{1}{\alpha}t)x$	$\mu^2\omega^2\varphi_{\omega\omega} - (\mu^2 - 4\mu)\omega\varphi_{\omega} + (4 - \alpha^2\mu^2\epsilon)\varphi$ $-\alpha^2\mu^2(\varphi^{\mu}\varphi_{\omega})_{\omega} = 0$

Some reduced equations are integrated completely. Thus we have the following solutions of equation (2):

$$\varphi = C_1 e^t + C_2 e^{-t}, \quad \text{if} \quad \epsilon = 1; \quad u = C_1 \cos t + C_2 \sin t, \quad \text{if} \quad \epsilon = -1;$$

$$\pm \frac{(\mu+2)\text{hypergeom}([\frac{\mu+1}{\mu+2}, \frac{1}{2}], [1+\frac{\mu+1}{\mu+2}], \frac{2u^{\mu+2}}{C_1}\epsilon)u^{\mu+1}}{(\mu+1)\sqrt{(\mu+2)C_1}} = x + C_2;$$

$$\int^u \pm \frac{(\mu+2)(z^n-1)}{\sqrt{(\mu+2)(-2\epsilon z^{\mu+2} + \epsilon(\mu+2)z^2 + C_1(\mu+2)}} dz = x \pm t + C_2.$$

4 On conditional symmetries

We also study conditional (nonclassical) symmetries of equations from class (1). As well-known, the operators with the vanishing coefficient of ∂_t gives so-called 'no-go' case in study of conditional symmetries of an arbitrary $(1+1)$-dimensional evolution equation since the problem on their finding is reduced to a single equation which is equivalent to the initial one (see e.g. [23]). Since the determining equation has more independent variables and, therefore, more freedom degrees, it is more convenient often to guess a simple solution or a simple ansatz for the determining equation, which can give a parametric set of complicated solutions of the initial equation. For example, the wave equation

$$u_{tt} = (uu_x)_x + \epsilon u, \quad \epsilon = \pm 1 \tag{3}$$

is conditionally invariant with respect to the operator $\partial_x + (\frac{u}{2x} - \frac{\epsilon}{4}x)\partial_u$. The associated ansatz $u = -\frac{\epsilon}{6}x^2 + \varphi(\omega)x^{\frac{1}{2}}, \omega = t$, reduces equation (3) to the equation $\varphi_{\omega\omega} = \frac{3}{8}\varphi$, i.e. $u = -\frac{\epsilon}{6}x^2 + (c_1 e^{\frac{\sqrt{6}}{4}t} + c_2 e^{-\frac{\sqrt{6}}{4}t})x^{\frac{1}{2}}$ is its non-Lie exact solution which can be additionally extended with symmetry transformations.

It is known that there also exist conditional symmetry operators of equations (1), which have non-vanishing coefficients of ∂_t, are inequivalent to Lie-invariance operators and even lead to truly non-Lie exact solutions. We omit these discussions because of the limited space. Exhaustive description of nonclassical symmetry operators of equations (1) will be a subject of a forthcoming paper.

Acknowledgements

This work was partially supported by the National Key Basic Research Project of China under the Grant NO.2004CB318000.

References

[1] Ames W.F., Nonlinear partial differential equations in engineering, V.1, New York, Academic, 1965, V.2, New York, Academic, 1972.

[2] Ames W.F., Adams E. and Lohner R.J., Group properties of $u_{tt} = [f(u)u_x]_x$, *Int. J. Non-Linear Mech.*, 1981, V.16, 439–447.

[3] Bluman G and Cheviakov A F 2007 Nonlocally related systems, linearization and nonlocal symmetries for the nonlinear wave equation , *J. Math. Anal. Appl.* **333** 93-111.

[4] Bluman G.W. and Kumei S., Symmetries and Differential Equations, Springer, New York, 1989.

[5] Bluman G., Temuerchaolu and Sahadevan R., Local and nonlocal symmetries for nonlinear telegraph equation, *J. Math. Phys.*, 2005, V.46, 023505.

[6] Chikwendu S.C., Non-linear wave propagation solutions by Fourier transform perturbation, *Int. J. Non-Linear Mech.*, 1981, V.16, 117–128.

[7] Donato A., Similarity analysis and nonlinear wave propagation, *Int. J. Non-Linear Mech.*, 1987, V.22, 307–314.

[8] Gandarias M.L., Torrisi M., and Valenti A., Symmetry classification and optimal systems of a non-linear wave equation, *Int. J. Non-Linear Mech.*, 2004, V.39, 389–398.

[9] Huang Dingjiang, Ivanova N M. Group analysis and exact solutions of a class of variable coefficient nonlinear telegraph equations. J. Math. Phys, 2007, 48: 073507 (23 pages)

[10] Ibragimov N.H. (Editor), Lie group analysis of differential equations — symmetries, exact solutions and conservation laws, V.1, Boca Raton, FL, CRC Press, 1994.

[11] Ibragimov N.H., Torrisi M. and Valenti A., Preliminary group classification of equations $v_{tt} = f(x, v_x)v_{xx} + g(x, v_x)$, *J. Math. Phys.*, 1991, V.32, 2988–2995.

[12] Kingston J.G. and Sophocleous C., On form-preserving point transformations of partial differential equations, *J. Phys. A: Math. Gen.*, 1998, V.31, 1597–1619.

[13] Lahno V., Zhdanov R. and Magda O., Group classification and exact solutions of nonlinear wave equations *Acta Appl. Math.*, 2006, V.91, 253–313.

[14] Lie S., On integration of a Class of Linear Partial Differential Equations by Means of Definite Integrals, *CRC Handbook of Lie Group Analysis of Differential Equations*, V.2, 473–508. (Translation by N.H. Ibragimov of Arch. for Math., Bd. VI, Heft 3, 328–368, Kristiania 1881).

[15] Meleshko S.V., Group classification of equations of two-dimensional gas motions, *Prikl. Mat. Mekh.*, V.58, 1994, 56–62 (in Russian); translation in *J. Appl. Math. Mech.*, V.58, 1994, 629–635.

[16] Olver P.J., Application of Lie Groups to Differential Equations, New York, Springer-Verlag, 1986.

[17] Oron A. and Rosenau P., Some symmetries of the nonlinear heat and wave equations, *Phys. Lett. A*, 1986, V.118, 172–176.

[18] Ovsiannikov L.V., Group analysis of differential equations, 1982, New York: Academic Press.

[19] Patera J. and Winternitz P., Subalgebras of real three- and four-dimensional Lie algebras *J. Math. Phys.*, 1977, V.18, 1449–1455.

[20] Pucci E., Group analysis of the equation $u_{tt} + \lambda u_{xx} = g(u, u_x)$, *Riv. Mat. Univ. Parma*, 1987, V.12 N 4, 71–87.

[21] Pucci E. and Salvatori M.C., Group properties of a class of semilinear hyperbolic equations, *Int. J. Non-Linear Mech.*, 1986, V.21, 147–155.

[22] Torrisi M. and Valenti A., Group properties and invariant solutions for infinitesimal transformations of a nonlinear wave equation, *Int. J. Non-Linear Mech.*, 1985, V.20, 135–144.

[23] Zhdanov R.Z. and Lahno V.I., Conditional symmetry of a porous medium equation, Physica D: Nonlinear Phenomena, 1998, 122, 178-186.