Constructive Algebra for Differential Invariants

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Differential invariants arise in equivalence problems and are used in symmetry reduction techniques

We introduce a computationally relevant differential algebraic structure for those.

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Contents

1 Differential Algebra and Related Topics

Differential Polynomial Rings

$$\mathbb{F} = \mathbb{Q}(x, y)$$

$\mathbb{F}$ a field

$$\delta_1 = \frac{\partial}{\partial x},\ \delta_2 = \frac{\partial}{\partial y}$$

$\Delta = \{\delta_1, \ldots, \delta_m\}$ derivations on $\mathbb{F}$

$$\mathcal{Y} = \{\phi, \psi\}$$

$$\mathbb{F}[\![\phi, \psi]\!] = \mathbb{F}[\phi, \phi_x, \phi_y, \ldots, \psi \ldots]$$

$$\mathcal{Y} = \{y_1, \ldots, y_n\}$$

$$\mathbb{F}[\, y_\alpha \mid \alpha \in \mathbb{N}^m,\ y \in \mathcal{Y}\,] = \mathbb{F}[\![\mathcal{Y}]\!]$$

$$\phi_{xxy} \rightsquigarrow \phi_{x^2y} \rightsquigarrow \phi_{(2,1)}$$

$$\frac{\partial}{\partial x}\left(\phi_{xxy}\right) = \phi_{xxxy} \rightsquigarrow \delta_1\left(\phi_{(2,1)}\right) = \phi_{(3,1)}$$

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y} = \frac{\partial}{\partial y}\frac{\partial}{\partial x}$$

$$\delta_i\left(y_\alpha\right) = y_{\alpha+\epsilon_i}$$

$$\epsilon_i = (0, \ldots, \underset{i^{\text{th}}}{1}, \ldots, 0)$$

$$\delta_i\delta_j = \delta_j\delta_i$$

Link with differential geometry

Independent variables $x_1, \ldots, x_n$

$\mathbb{F} = \mathbb{Q}(x_1, \ldots, x_m)$

Dependent variables $y_1, \ldots, y_n$

$\mathcal{Y} = \{y_1, \ldots, y_n\}$ the differential indeterminates

$\mathbb{F}[\![\mathcal{Y}]\!]$ is the coordinate ring for the infinite jet space

Total derivatives:

$$\delta_i = \frac{\partial}{\partial x_i} + \sum_{y \in \mathcal{Y},\, \alpha \in \mathbb{N}^m} y_{\alpha+\epsilon_i} \frac{\partial}{\partial y_\alpha}$$

Derivations with nontrivial commutations

$$\mathcal{Y} = \{y_1, \ldots, y_n\}$$

$$\Delta = \{\delta_1, \ldots, \delta_m\}$$

$$\delta_i\, \delta_j - \delta_j\, \delta_i = \sum_{l=1}^{m} c_{ijl}\, \delta_l$$

$$c_{ijl} \in \mathbb{K}[\![\mathcal{Y}]\!]$$

$$\mathbb{K}[\![\mathcal{Y}]\!]?$$

[H05]

Differential polynomial ring $\mathbb{K}[\![\mathcal{Y}]\!]$ with non commuting derivations

$\mathcal{Y} = \{y_1, \ldots, y_n\}$

$\mathrm{D} = \{\delta_1, \ldots, \delta_m\}$

$\mathbb{K}[y_\alpha \mid \alpha \in \mathbb{N}^m,\ y \in \mathcal{Y}]$

$$\delta_i\,(y_\alpha) = \begin{cases} y_{\alpha+\epsilon_i} \text{ if} \alpha_1 = \ldots = \alpha_{i-1} = 0 \\ \delta_j \delta_i \left(y_{\alpha-\epsilon_j}\right) + \sum_{l=1}^{m} c_{ijl}\, \delta_l \left(y_{\alpha-\epsilon_j}\right) \\ \quad \text{where } j < i \text{ is s.t. } \alpha_j > 0 \\ \quad \text{while } \alpha_1 = \ldots = \alpha_{j-1} = 0 \end{cases}$$

If the c_{ijl} satisfy

- $c_{ijl} = -c_{jil}$
- $\delta_k(c_{ijl}) + \delta_i(c_{jkl}) + \delta_j(c_{kil}) = \sum_{\mu=1}^{m} c_{ij\mu} c_{\mu kl} + c_{jk\mu} c_{\mu il} + c_{ki\mu} c_{\mu jl}$

& there exists an *admissible ranking* $\prec$

- $|\alpha| < |\beta| \;\;\Rightarrow\;\; y_\alpha \prec y_\beta,$
- $y_\alpha \prec z_\beta \;\;\Rightarrow\;\; y_{\alpha+\gamma} \prec z_{\beta+\gamma},$
- $\sum_{l \in \mathbb{N}_m} c_{ijl}\, \delta_l(y_\alpha) \prec y_{\alpha+\epsilon_i+\epsilon_j}$

then $\delta_i \delta_j(p) - \delta_j \delta_i(p) = \sum_{l=1}^{m} c_{ijl}\, \delta_l(p) \quad \forall p \in \mathbb{K}[\, y_\alpha \,|\, \alpha \in \mathbb{N}^m \,] = \mathbb{K}[\![\mathcal{Y}]\!]$

[H05]

2 Lie Group Actions and their Invariants

Lie (Algebraic) Group $\mathcal{G}$

$\mathcal{G}$ a r-dimensional smooth manifold, locally parameterized by $\mathbb{R}^r$

$$m: \begin{array}{ccc} \mathcal{G} \times \mathcal{G} & \to & \mathcal{G} \\ (\lambda, \mu) & \mapsto & \lambda \cdot \mu \end{array} \qquad \text{and} \qquad i: \begin{array}{ccc} \mathcal{G} & \to & \mathcal{G} \\ \lambda & \mapsto & \lambda^{-1} \end{array} \qquad \text{smooth}$$

$$e \in \mathcal{G} \quad e \cdot \lambda = \lambda \cdot e = \lambda$$

Group action

$\mathcal{G}$ a Lie group $\qquad$ $\mathcal{M}$ an open subset of $\mathbb{R}^n$

$$\text{Action } g: \begin{array}{ccc} \mathcal{G}\times\mathcal{M} & \rightarrow & \mathcal{M} \\ (\lambda, z) & \mapsto & \lambda\star z \end{array} \quad \text{smooth}$$

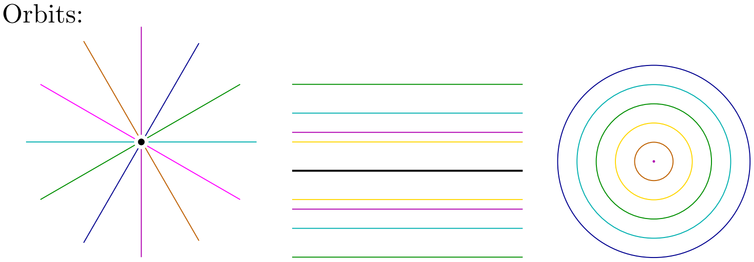
$$e\star z = z \qquad (\lambda\cdot\mu)\star z = \lambda\star(\mu\star z)$$

Orbit of z: $\mathcal{O}_z = \{\lambda\star z \mid \lambda\in\mathcal{G}\}\subset\mathcal{M}$

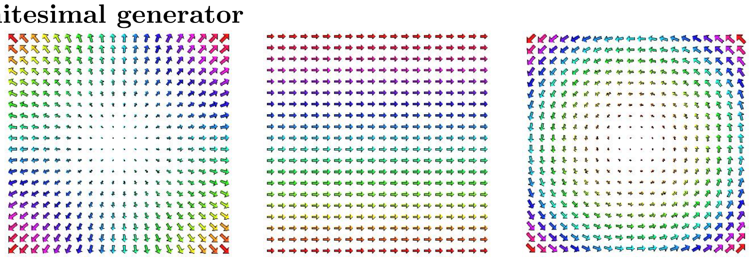
Semi-regular Lie group actions

	scaling	translation+reflection	rotation
$\mathcal{G}$	$\mathbb{R}^*$	$\mathbb{R}\times\{-1,1\}$	$SO(2)$
$\mathcal{M}$	$\mathbb{R}^2\setminus O$	$\mathbb{R}^2$	$\mathbb{R}^2\setminus O$
$\lambda\star z$	$\begin{pmatrix}\lambda z_1\\ \lambda z_2\end{pmatrix}$	$\begin{pmatrix}z_1+\lambda_1\\ \lambda_2 z_2\end{pmatrix}$	$\begin{pmatrix}\cos\lambda & -\sin\lambda\\ \sin\lambda & \cos\lambda\end{pmatrix}\begin{pmatrix}z_1\\ z_2\end{pmatrix}$

Orbits:



Infinitesimal generator



$x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$ $\qquad$ $\frac{\partial}{\partial x}$ $\qquad$ $y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}$

Infinitesimal generators

$$\xi_1\frac{\partial}{\partial z_1}+\ldots+\xi_d\frac{\partial}{\partial z_d}$$

a vector field the flow of which is the action of a one-dimensional (connected) subgroup of $\mathcal{G}$.
$\mathrm{V}_1,\ldots,\mathrm{V}_r$ a basis of infinitesimal generators for the action on $\mathcal{M}$ of the r-dimensional group $\mathcal{G}$.

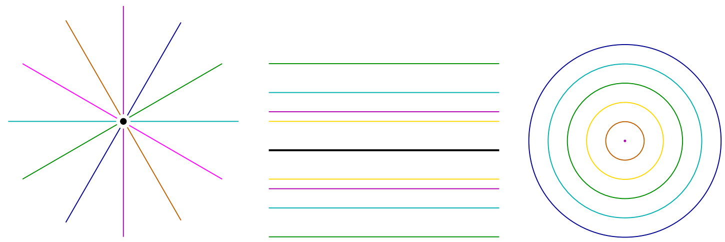
Local Invariants

$f:\mathcal{U}\subset\mathcal{M}\rightarrow\mathbb{RK}$ smooth

$$f(\lambda\star z)=f(z) \text{ for } \lambda\in\mathcal{G} \text{ close to } e$$
$$\Leftrightarrow$$
$$f \text{ is constant on orbits within } \mathcal{U}$$
$$\Leftrightarrow$$
$$\mathrm{V}_1(f)=0,\ldots,\mathrm{V}_r(f)=0$$

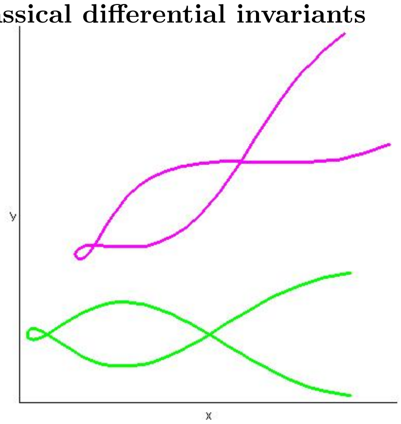
Examples

$\mathcal{G}$	$\mathbb{K}^*$	$\mathbb{K} \times \{-1, 1\}$	$SO(2)$



rational	$\frac{x}{y}$	y^2	$x^2 + y^2$
local	$\frac{x}{y}$	y	$\sqrt{x^2 + y^2}$

Classical differential invariants



$E(2)$ $\qquad\qquad \alpha^2 + \beta^2 = 1$

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

$$Y_X = \frac{\beta + \alpha y_x}{\alpha - \beta\, y} \quad Y_{XX} = \frac{y_{xx}}{(\alpha - \beta\, y)^3}$$

Curvature: $\sigma = \sqrt{\frac{y_{xx}^2}{(1+y_x^2)^3}}$ a differential invariant

Arc length: $ds = \sqrt{1 + y_x^2}\, dx$

Invariant derivation: $\frac{d}{ds} = \frac{1}{\sqrt{1 + y_x^2}} \frac{d}{dx}$

Jets / Differential algebraProlongation

$\mathrm{J}^0 = \mathcal{X} \times \mathcal{U}$ $\qquad g^{(0)} : \mathcal{G} \times \mathrm{J}^0 \to \mathrm{J}^0$ $\qquad \mathrm{V}_1^0, \ldots, \mathrm{V}_r^0$

$(x_1, \ldots, x_m)$ coordinates on $\mathcal{X} \rightsquigarrow$ independent variables

$(u_1, \ldots, u_n)$ coordinates on $\mathcal{U} \rightsquigarrow$ dependent variables

$\mathrm{J}^k = \mathcal{X} \times \mathcal{U}^{(k)}$ $\qquad g^{(k)} : \mathcal{G} \times \mathrm{J}^k \to \mathrm{J}^k$ $\qquad \mathrm{V}_1^k, \ldots, \mathrm{V}_r^k$

[DifferentialGeometry]

additional coordinates $u_\alpha = \dfrac{\partial^{|\alpha|} u}{\partial x^\alpha}$, $|\alpha| \leq k$

$\rightsquigarrow$ the derivatives of u w.r.t x up to order k

Differential polynomial ring: $\mathbb{K}(x) [\![u]\!] = \mathbb{K}(x) [u_\alpha \mid \alpha \in \mathbb{N}^m]$

$$\mathrm{D}_i u_\alpha = u_{\alpha+\epsilon_i}.$$

[diffalg]

$f : \mathrm{J}^k \to \mathbb{R}$ differential invariant of order k if $\mathrm{V}^k(f) = 0$.

Invariant derivation

$$\mathcal{D} : \mathcal{F}(\mathrm{J}^k) \to \mathcal{F}(\mathrm{J}^{k+1}) \text{ s.t } \mathcal{D} \circ \mathrm{V} = \mathrm{V} \circ \mathcal{D}$$

$f : \mathrm{J}^k \to \mathbb{R}$ a differential invariant

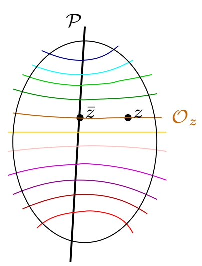
$\Rightarrow$ $\mathcal{D}(\mathrm{f})$ a differential invariant of order $k+1$.

What is a computationnally relevant algebraic structure for differential invariants?

$$\mathbb{K} [\![y_1, \ldots, y_n]\!] \;/\; [\![S]\!]$$

3 Normalized Invariants: Geometric and Algebraic Construction

Local cross-section $\mathcal{P}$



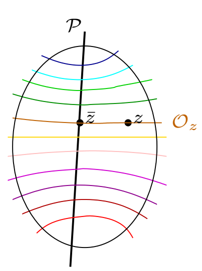
- $\mathcal{P}$ an embedded manifold of dimension $n - d$
  $\mathcal{P} = \{z \in \mathcal{U} \mid p_1(z) = \ldots = p_d(z) = 0\}$
- $\mathcal{P}$ is transverse to $\mathcal{O}_z$ at $z \in \mathcal{P}$.
- $\mathcal{P}$ intersect $\mathcal{O}_z^0$ at a unique point, $\forall z \in \mathcal{U}$.

$\Leftrightarrow$ the matrix $(V_i(p_j))_{1 \leq i \leq r, 1 \leq j \leq d}$ has rank d on $\mathcal{P}$.

A local invariant is uniquely determined by a function on $\mathcal{P}$.

[Fels Olver 99, H. Kogan 07b]

Invariantization $\bar{\iota} f$ of a function f



$f : \mathcal{U} \to \mathbb{R}$ smooth

$\bar{\iota} f$ is the unique *local* invariant with $\bar{\iota} f|_{\mathcal{P}} = f|_{\mathcal{P}}$

$\bar{\iota} f(z) = f(\bar{z})$

Normalized invariants: $\bar{\iota} z_1, \ldots, \bar{\iota} z_n$.

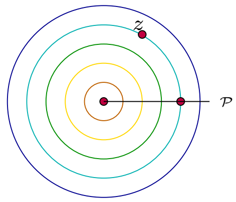
$\bar{\iota} f(z) = f(\bar{\iota} z)$

Generation and rewriting:
f local invariant $\Rightarrow$ $f(z_1, \ldots, z_n) = f(\bar{\iota} z_1, \ldots, \bar{\iota} z_n)$

Relations: $p_1(\bar{\iota} z_1, \ldots, \bar{\iota} z_n) = 0, \ldots, p_d(\bar{\iota} z_1, \ldots, \bar{\iota} z_n) = 0$

[Fels Olver 99, H. Kogan 07b]

Normalized invariants. Example.



$\mathcal{G} = SO(2)$, $\mathcal{M} = \mathbb{R}^2 \setminus O$

$\mathcal{P}: z_2 = 0,\ z_1 > 0$ $\mathcal{U} = \mathcal{M}$

$$(\bar\iota z_1, \bar\iota z_2) = \left(\sqrt{z_1^2 + z_2^2}, 0\right)$$

Replacement property:

$$f(z_1, z_2) \text{ invariant } \Rightarrow f(z_1, z_2) = f(\bar\iota z_1, 0).$$

Normalized invariants in practice

We mostly do not need $(\bar\iota z_1, \ldots, \bar\iota z_n)$ explicitly.

We can work formally with $(\bar\iota z_1, \ldots, \bar\iota z_n)$, subject to the relationships $p_1(\bar\iota z) = 0, \ldots, p_d(\bar\iota z) = 0$.

Computing normalized invariants

In the algebraic case, the normalized invariants $(\bar\iota z_1, \ldots, \bar\iota z_n)$ for a $\overline{\mathbb{K}(z)}^G$-zero of the *graph-section* ideal

$$(G + (Z - \lambda \star z) + P) \cap \mathbb{K}(z)[Z]$$

The coefficients of the reduced Gröbner basis of the graph-section ideal form a generating set for $\mathbb{K}(z)^G$ endowed with a simple rewriting algorithm.

[H. Kogan 07a 07b]

4 Differential Invariant Derivation, Syzygies

Differential invariants

$\mathrm{J}^0 = \mathcal{X} \times \mathcal{U}$ $g^{(0)}: \mathcal{G} \times \mathrm{J}^0 \to \mathrm{J}^0$ $\mathrm{V}_1^0, \ldots, \mathrm{V}_r^0$

$(x_1, \ldots, x_m)$ coordinates on $\mathcal{X}$

$(u_1, \ldots, u_n)$ coordinates on $\mathcal{U}$

$\mathrm{J}^k = \mathcal{X} \times \mathcal{U}^{(k)}$ $g^{(k)}: \mathcal{G} \times \mathrm{J}^k \to \mathrm{J}^k$ $\mathrm{V}_1^k, \ldots, \mathrm{V}_r^k$

additional coordinates $u_\alpha = \dfrac{\partial^{|\alpha|} u}{\partial x^\alpha}$, $|\alpha| \le k$

Normalized invariants of order k

$$\mathcal{I}^k = \{\bar\iota x_1, \ldots, \bar\iota x_m\} \cup \{\bar\iota u_\alpha \,|\, |\alpha| \le k\}$$

Generation in finite terms

r_k, the dimension of orbits on J^k, stabilizes at order s

$$r_0 \leq r_1 \leq \ldots \leq r_s = r_{s+1} = \ldots = r.$$

$\mathcal{P}^s : p_1 = 0, \ldots, p_r = 0$ defines a cross-section on J^{s+k}

$$\mathcal{I}^{s+k} = \{\bar{\iota}x_1, \ldots, \bar{\iota}x_m\} \cup \{\bar{\iota}u_\alpha \,|\, |\alpha| \leq s+k\}$$

Construct: $\mathcal{D}_1, \ldots, \mathcal{D}_m : \mathcal{F}(\mathrm{J}^{s+k}) \to \mathcal{F}(\mathrm{J}^{s+k+1})$ s.t. $\mathcal{D}_i \mathrm{V}_a = \mathrm{V}_a \mathcal{D}_i$

Key Prop: $\bar{\iota}u_{\alpha+\epsilon_i} = \mathcal{D}_i(\bar{\iota}u_\alpha) + K_{ia}\,\bar{\iota}\left(\mathrm{V}_a(u_\alpha)\right)$ $\qquad K = \bar{\iota}\left(\mathrm{D}(P)\,\mathrm{V}(P)^{-1}\right)$

Col: Any differential invariants can be contructively written in terms of $\mathcal{I}^{s+1}$ and their derivatives.

[Fels Olver 99]

Algebra of Differential Invariants

$$\mathbb{K}\,[\![\mathfrak{x}_i, \mathfrak{u}_\alpha \,|\, |\alpha| \leq s+1]\!] \;/\; [\![S]\!]$$

$$\left[\mathcal{D}_i, \mathcal{D}_j\right] = \sum_{k=1}^{m} \Lambda_{ijk}\,\mathcal{D}_k$$

where $\Lambda_{ijk} = \sum_{c=1}^{r} K_{ic}\,\bar{\iota}(\mathrm{D}_j(\mathrm{V}_c(x_k))) - K_{jc}\,\bar{\iota}(\mathrm{D}_i(\mathrm{V}_c(x_k)))$.

The monotone derivatives of $\mathcal{I}^{s+1}$,
$\left\{\mathcal{D}_1^{\beta_1} \ldots \mathcal{D}_m^{\beta_m}(\bar{\iota}x_i)\right\} \cup \left\{\mathcal{D}_1^{\beta_1} \ldots \mathcal{D}_m^{\beta_m}(\bar{\iota}u_\alpha) \,|\, |\alpha| \leq s+1\right\}$,
generate all differential invariants.

[H 05, 08]

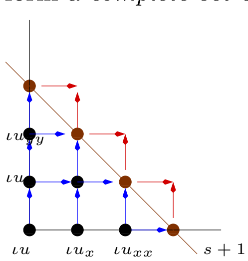
Syzygies=Differential relationships

A subset S of the following relationships

$p_1(\bar{\iota}x, \bar{\iota}u_\alpha) = 0, \ldots, p_r(\bar{\iota}x, \bar{\iota}u_\alpha) = 0$
$\mathcal{D}_i(\bar{\iota}x_j) = \delta_{ij} - K_{ia}\bar{\iota}\left(\mathrm{V}(x_j)\right),$
$\mathcal{D}_i(\bar{\iota}u_\alpha) = \bar{\iota}u_{\alpha+\epsilon_i} - K_{ia}\bar{\iota}\left(\mathrm{V}(u_\alpha)\right),\ |\alpha| \leq s$
$\mathcal{D}_i(\bar{\iota}u_\alpha) - \mathcal{D}_j(\bar{\iota}u_\beta) = K_{ja}\bar{\iota}\left(\mathrm{V}(u_\beta)\right) - K_{ia}\bar{\iota}\left(\mathrm{V}(u_\alpha)\right),$

$$\alpha + \epsilon_i = \beta + \epsilon_j,\ |\alpha| = |\beta| = s+1.$$

form a *complete set of differential syzygies.*



[H08]

$$\bar{\iota}u_{\alpha+\epsilon_i} = \mathcal{D}_i(\bar{\iota}u_\alpha) + K_{ia}\,\bar{\iota}\left(\mathrm{V}_a(u_\alpha)\right)$$

$$K = \bar{\iota}\left(\mathrm{D}(P)\,\mathrm{V}(P)^{-1}\right)$$

Smaller set of generators

- Differential elimination on a complete set of syzygies allows to reduce the number of differential invariants.
- We can reduce to two or even 1 the number of generating invariants by differential elimination on the syzygies.
- Euclidean and affine surfaces [Olver 07]
- Conformal and projective surfaces [H. Olver 07]

We can always restrict to $m\,r + d_0$ generating invariants

$\bar{\iota} u_{\alpha+\epsilon_i} = \mathcal{D}_i(\bar{\iota} u_\alpha) + K_{ia}\,\bar{\iota}\,(\mathrm{V}_a(u_\alpha))$ $\quad\quad K = \bar{\iota}\left(\mathrm{D}(P)\,\mathrm{V}(P)^{-1}\right)$

The *edge invariants* $\mathcal{E} = \{\bar{\iota}(\mathcal{D}_i(p_a))\} \cup \mathcal{I}^0$ form a generating set when the the cross-section is of minimal order.

We can obtain their syzygies by elimination.

The *Maurer-Cartan invariants* $\{K_{ia}\} \cup \mathcal{I}^0$ form a generating set of differential invariants.

We can obtain their syzygies from the structure equations.

[H07, H08]

Merci. Thanks.

5 Bibliography

Origins of the project: Symmetry reduction

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Computing (algebraic) invariants

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Differential Algebraic Structure of Invariants

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Computing in the Algebra of Differential Invariants

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Software

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