Differential invariants of a Lie group action: syzygies on a generating set

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Abstract

We exhibit a complete set of syzygies on a generating set of differential invariants. For that we elaborate on the reinterpretation of Cartan's moving frame by Fels and Olver (1999) and provide the constructive tools and software for performing *symmetry reduction* on differential systems.

Key words: Lie group actions, Differential invariants, Syzygies, Differential algebra, Symbolic Computation.
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Introduction

Group actions are ubiquitous in mathematics and arise in diverse fields of science and engineering. Whether algebraic or differential, one can distinguish two families of applications for invariants of group actions: equivalence problems, together with classification and canonical forms, and symmetry reduction. This paper deals more specifically with differential invariants and attends more particularly to symmetry reduction. To that end, invariants are used to take into account the symmetry of a problem, mainly in order to reduce its size or its analysis. The computational requirements include four main components: the explicit computation of a generating set of invariants (1), and the relations among them (2); procedures for rewriting the problem in terms of the invariants (3); and finally procedures for computing in the algebra of invariants (4). In this paper we focus on (2) and (3), in the case of differential invariants, while (1) and (4) were consistently addressed by Hubert and Kogan (2007a,b) and Hubert (2005b) respectively. This paper thus completes an algorithmic suite. While component (4) has been implemented as a generalization

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of the MAPLE library *diffalg* (Boulier and Hubert, 1998; Hubert, 2005a), components (1-3) is implemented in our Maple package AIDA (Hubert, 2007b) that works on top of the MAPLE library *DifferentialGeometry* (Anderson and et al., 2007), as well as *diffalg*. Some non trivial applications of the packages AIDA and the generalized *diffalg* were presented by Hubert and Olver (2007).

With minimal amount of data on the group action, we shall characterize two generating sets of differential invariants. Though not computing them explicitly, we describe inductive processes to rewrite any differential invariants in terms of them and their *invariant derivatives*. For one of those generating set we determine a *complete* set of differential relationships, which we call syzygies. The other generating set is of bounded cardinality and a complete set of syzygies can be computed by differential elimination, that is using component (4) above, addressed in Hubert (2005b). The results in this paper are constructive and our presentation describes very closely their symbolic implementation in AIDA (Hubert, 2007b).

On one hand, the question of the finite generation of differential invariants was addressed by Tresse (1894); Kumpera (1974, 1975a,b); Muñoz et al. (2003), in the more general case of pseudo-groups - see also Ovsiannikov (1982); Olver (1995) for Lie groups. On the other hand, Griffiths's (1974) interpretation of Cartan's (1935; 1937; 1953) moving frame method solved equivalence problem in many geometries (Green, 1978; Jensen, 1977; Gardner, 1989; Ivey and Landsberg, 2003). Recent symbolic implementation (Neut, 2003) lead to explicit computations (Neut and Petitot, 2002; Dridi and Neut, 2006a,b). Besides Fels and Olver (1999) offered another interpretation of Cartan's moving frame method, the application of which goes beyond geometry (Olver, 2005). In particular it includes an explicit approach to the generation properties.

The main original contribution in this paper is to formalize the notion of differential syzygies for a generating set of differential invariants and prove the completeness of a finite set of those. To this end we redevelop the construction of normalized invariants and invariant derivations of Fels and Olver (1999) in a spirit we believe closer to the audience of this journal. We offer alternative proofs, and sometimes more general results. In particular we shall put the emphasis on derivations, rather than differential forms.

One is interested in the action (effective on subsets) of a group $\mathcal{G}$ on a manifold $\mathcal{X} \times \mathcal{U}$ and its prolongation to the higher order jets $\mathrm{J}^k(\mathcal{X},\mathcal{U})$. In other words, $\mathcal{X}$ is the space of independent variables while $\mathcal{U}$ is the set of dependent variables. The jet space is parameterized by the derivatives of the dependent variables with respect to the independent variables. At each order k, a local cross-section to the orbit defines a finite set of normalized invariants. Those latter form a generating set for differential invariants of order k. Rewriting those latter in terms of the normalized invariant is furthermore a trivial substitution. We review this material in Section 2.3, following the presentation of

Hubert and Kogan (2007b).

As the orbit dimension stabilizes at order s the action becomes locally free and, to any local cross-section, we can associate a *moving frame*, i.e. an equivariant map $\rho : \mathrm{J}^s(\mathcal{X},\mathcal{U}) \to \mathcal{G}$ (Fels and Olver, 1999). The moving frame defines in turn a basis of invariant derivations. The great value of this particular set of invariant derivations is the fact that we can write explicitly their action on invariantized functions. This is captured in the so called *recurrence formulae*. They are the key to proving generation, rewriting and syzygies. Fels and Olver (1999) gave the recurrence formulae for the normalized invariants in the case of a coordinate cross-section. We propose generalized recurrence formulae in the case of any cross-section and offer an alternate proof, close in spirit to the one of Mansfield (2008). As an immediate corollary of this new formulation we see that the invariant derivation of a differential invariant is the invariantization of the total derivative of this invariant (Corollary 3.7), a fact we believe has not been noticed before.

We can then show that normalized invariants of order $s+1$ form a generating set with respect to those invariant derivations. Rewriting any differential invariant in terms of those and their derivative is a simple application of the recurrence formulae (Section 4). By exhibiting a canonical rewriting, we can prove the completeness of a set of differential syzygies for those differential invariants in Section 5, after giving this concept a definition.

We formalize the notion of syzygies through the introduction of the algebra of *monotone derivatives*. In the line of Hubert (2005b), this algebra is equipped with derivations that are defined inductively. The syzygies are the elements of the kernel of the differential morphism between the algebra of monotone derivatives and the algebra of differential invariants, equipped with the invariant derivations. The type of differential algebra introduced at this stage, was shown to be a natural generalization of classical differential algebra (Ritt, 1950; Kolchin, 1973). In the polynomial case, it is indeed endowed with an effective differential elimination theory that has been implemented (Hubert, 2005a,b).

For cross-section of minimal order we can also prove that the set of *edge invariants* is generating. This latter set has a cardinality bounded by $m\,r+d_0$, where m, r are the dimension of $\mathcal{X}$ and $\mathcal{G}$ while d_0 is the codimension of the orbits on $\mathcal{X}\times\mathcal{U}$. This is a generalization of the result of Olver (2007b) that bears on coordinate cross-section and where the edge invariants then form a subset of the normalized invariants of order s. Fels and Olver (1999) first conjectured syzygies on this set of generating invariants. We feel that constructing directly a complete and finite set of syzygies for the set of edge invariants is challenging, the problem bearing a high combinatorial difficulty. To obtain those, we suggest to apply generalized differential elimination (Hubert, 2005a,b) on the set of syzygies for the normalized invariants. This is illustrated in the examples of Section 5 and 6.

Similarly, to reduce further the number of generators for the differential invariants we can apply the same generalized differential elimination techniques to the syzygies. This substantially reduces the work of computing explicitly a generating set for a given action. This is an approach that was applied for surfaces in Euclidean, affine, conformal and projective geometry (Olver, 2007a; Hubert and Olver, 2007). Actually, our final aim has been an algorithmic suite for differential elimination in symmetric differential systems (Mansfield, 2001; Hubert, 2005b; Hubert and Kogan, 2007a). Rewriting the symmetric differential system in terms of a generating set of differential invariants and determining the syzygies on those latter is a necessary step - see the motivational example of Hubert (2005b).

Let us stress here the minimal amount of data indeed needed for the determination of a generating set, the rewriting in terms of those and the differential syzygies. All is based on the recurrence formulae that can be written with only the knowledge of the infinitesimal generators of the action and the equations of the cross-section. Furthermore the operations needed consist of derivations, arithmetic operations and test to zero. Provided the coefficients of the infinitesimal generators are rational functions, which provides a general enough class, we are thus in the realm of symbolic computation since we can indeed always choose linear equations for the cross-section. On the other hand, the explicit expression of the invariant derivations, or the differential invariants, requires the knowledge of the moving frame. This latter is obtained by application of the implicit function theorem on the group action. This is therefore not constructive in general, but there are algorithms in the algebraic case (Hubert and Kogan, 2007a,b).

In Section 1 we extract from the books of Olver (1986, 1995) the essential material we need for describing actions and their prolongations. In Section 2 we define invariantization, normalized invariants for the action of a group on a manifold in the line of Hubert and Kogan (2007b). We then extend those notions to differential invariants. In Section 3 we define invariant derivations as the derivations that commute with the infinitesimal generators of the action. We introduce the construction of invariant derivations of Fels and Olver (1999) based on the moving frame together with the *recurrence formulae*. We write those latter in a more general form (Theorem 3.6): the derivations of the invariantization of a function is given explicitly in terms of invariantizations. Section 4 discusses then the generation property of the normalized invariants and effective rewriting. We furthermore show the generalization of Olver (2007a), the generation property of the *edge invariants* in the case of minimal order cross-section. In Section 5 we emphasize the non uniqueness of the rewriting in terms of the normalized invariants. We then introduce the algebra of monotone derivatives, and the inductive derivations acting on it, in order to formalize the concept of syzygies. We can then write a finite set of syzygies and prove its completeness. In the last section we present the examples that many readers are familiar with in order to illustrate our general approach. Some non trivial applications were developed by Hubert and Olver (2007).

1 Group action and their prolongations

This is a preliminary section introducing the definition and notations for Lie group actions and their prolongation to derivatives. We essentially follow the books of Olver (1986, 1995).

1.1 Local action of a Lie group on a manifold

Pullbacks and push-forwards of maps

Consider $\mathcal{M}$ a smooth manifold. $\mathcal{F}(\mathcal{M})$ denotes the ring of smooth functions on $\mathcal{M}$ while $\mathrm{Der}(\mathcal{M})$ denotes the $\mathcal{F}(\mathcal{M})$-module of derivations on $\mathcal{F}(\mathcal{M})$.

If $\mathcal{N}$ is another smooth manifold and $\phi : \mathcal{M} \to \mathcal{N}$ a smooth map, the *pull-back* of ϕ is the map $\phi^* : \mathcal{F}(\mathcal{N}) \to \mathcal{F}(\mathcal{M})$ defined by $\phi^* f = f \circ \phi$ i.e. $(\phi^* f)(z) = f(\phi(z))$ for all $z \in \mathcal{M}$. Through ϕ^*, $\mathcal{F}(\mathcal{N})$ can be viewed as a $\mathcal{F}(\mathcal{M})$ module.

A derivation $\mathrm{V} : \mathcal{F}(\mathcal{M}) \to \mathcal{F}(\mathcal{M})$ on $\mathcal{M}$ induces a derivation$\mathrm{V}|_z : \mathcal{F}(\mathcal{M}) \to \mathbb{R}$ at z defined by $\mathrm{V}|_z(f) = \mathrm{V}(f)(z)$. The set of derivations at a point $z \in \mathcal{M}$ is the tangent space of $\mathcal{M}$ at z.

The *push-forward* or *differential* of ϕ is defined by

$$(\phi_* \mathrm{V})(f)(\phi(x)) = \mathrm{V}(\phi^* f)(x)$$

The coordinate expression for $\phi_* \mathrm{V}$ is given by the chain rule. Yet this *star* formalism allows us to write formulae in a compact way and we shall use it extensively.

Local action on a manifold

We consider a connected Lie group $\mathcal{G}$ of dimension r. The multiplication of two elements $\lambda, \mu \in \mathcal{G}$ is denoted as $\lambda \cdot \mu$.

An action of $\mathcal{G}$ on a manifold $\mathcal{M}$ is defined by a map $g : \mathcal{G} \times \mathcal{M} \to \mathcal{M}$ that satisfies $g(\lambda, g(\mu, z)) = g(\lambda \cdot \mu, z)$. We shall implicitly consider local actions, that is g is defined only on an open subset of $\mathcal{G} \times \mathcal{M}$ that contains $\{e\} \times \mathcal{M}$. We assume that $\mathcal{M}$ is made of a single coordinate chart. If $(z_1, \ldots, z_k)$ are the coordinate functions then $g^* z_i : \mathcal{G} \times \mathcal{M} \to \mathbb{R}$ represents the ith component of the map g.

There is a fine interplay of right and left invariant vector fields in the paper. We thus detail what we mean there now. Given a group action $g : \mathcal{G} \times \mathcal{M} \to \mathcal{M}$ define, for $\lambda \in \mathcal{G}$, $g_\lambda : \mathcal{M} \to \mathcal{M}$ by $g_\lambda(z) = g(\lambda, z)$ for $z \in \mathcal{M}$. A vector field

V on $\mathcal{M}$ is $\mathcal{G}$-*invariant* if $g_\lambda^* \mathrm{V} = \mathrm{V}$ that is for all $\lambda \in \mathcal{G}$

$$\forall f \in \mathcal{F}(\mathcal{M}),\ \forall z \in \mathcal{M},\ \mathrm{V}(f \circ g_\lambda)(z) = \mathrm{V}(f)(g_\lambda(z)).$$

A vector field on $\mathcal{G}$ is right invariant if it is invariant under the action of $\mathcal{G}$ on itself by right multiplication. In other words, if $r_\mu : \mathcal{G} \to \mathcal{G}$ is the right multiplication by μ^{-1}, $r_\mu(\lambda) = \lambda \cdot \mu^{-1}$, a vector field v on $\mathcal{G}$ is right invariant if

$$\mathrm{v}(f \circ r_\mu)(\lambda) = \mathrm{v}(f)(\lambda \cdot \mu^{-1}), \quad \forall f \in \mathcal{F}(\mathcal{G}).$$

A right invariant vector field on $\mathcal{G}$ is completely determined by its value at identity. We can thus find a basis $\mathrm{v} = (\mathrm{v}_1, \ldots, \mathrm{v}_r)$ for the derivations on $\mathcal{F}(\mathcal{G})$ made of right invariant vector fields (Olver, 1995, Chapter 2).

For a right invariant vector field on $\mathcal{G}$, the *exponential* map $e^{\mathrm{v}} : \mathbb{R} \to \mathcal{G}$ is the flow of v such that $e^{\mathrm{v}}(0)$ is the identity. We write $e^{t\mathrm{v}}$ for $e^{v}(t)$. The defining equation for e^{v} is

$$\mathrm{v}(f)(\lambda) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f\left(e^{t\mathrm{v}} \cdot \lambda\right).$$

Similarly the associated *infinitesimal generator* V of the action g of $\mathcal{G}$ on $\mathcal{M}$ is the vector field on $\mathcal{M}$ defined by

$$\mathrm{V}(f)(z) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(g(e^{t\mathrm{v}}, z)), \quad \forall f \in \mathcal{F}(\mathcal{M}). \tag{1.1}$$

Note that v is the infinitesimal generator for the action of $\mathcal{G}$ on $\mathcal{G}$ by left multiplication. The infinitesimal generator associated to v for the action of $\mathcal{G}$ on $\mathcal{G}$ by right multiplication, $r : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, $r(\lambda, \mu) = \mu \cdot \lambda^{-1}$ is

$$\hat{\mathrm{v}}(f)(\lambda) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\lambda \cdot e^{-t\mathrm{v}}).$$

We can observe that $\hat{\mathrm{v}}$ is a left invariant vector field on $\mathcal{G}$. If the right invariant vector fields $\mathrm{v} = (\mathrm{v}_1, \ldots, \mathrm{v}_r)$ is a basis of derivations on $\mathcal{F}(\mathcal{G})$, so is $\hat{\mathrm{v}} = (\hat{\mathrm{v}}_1, \ldots, \hat{\mathrm{v}}_r)$ the associated left invariant vector fields (Olver, 1995, Chapter 2).

The following property is used for the proof of Theorem 3.4 and 3.6. What is used more precisely in Theorem 3.6, is that $\mathrm{v}(g^* f)|_e = \mathrm{V}(f)$, a fact that can also be deduced from Theorem 3.10 by Fels and Olver (1999). In our notations this latter reads as: $\mathrm{v}(g^* z_i) = g^* \xi_i$ where $\xi_i = \mathrm{V}(z_i)$.

Proposition 1.1 *Let* v *be a right invariant vector field on* $\mathcal{G}$, $\hat{\mathrm{v}}$ *the associated infinitesimal generator for the action of* $\mathcal{G}$ *on* $\mathcal{G}$ *by right multiplication and* V *the associated infinitesimal generator of the action* g *of* $\mathcal{G}$ *on* $\mathcal{M}$.

When both $\hat{\mathrm{v}}$ *and* V *are considered as derivations on* $\mathcal{F}(\mathcal{G} \times \mathcal{M})$ *then*

$$\hat{\mathrm{v}}(g^* f) + \mathrm{V}(g^* f) = 0 \quad \text{and} \quad \mathrm{V}(g^* f)(e, z) = \mathrm{V}(f)(z), \quad \forall f \in \mathcal{F}(\mathcal{M}).$$

As a particular case we have $\hat{\mathrm{v}}(f)(e) = -\mathrm{v}(f)(e)$*.*

PROOF: $\hat{\mathrm{v}}$ is a linear combination of derivations with respect to the group parameters, i.e. the coordinate functions on $\mathcal{G}$, while V is a combination of derivations with respect to the coordinate functions on $\mathcal{M}$. By (1.1) we have

$$\mathrm{V}(g^* f)(\lambda, z) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (g^* f)(\lambda, g(e^{t\mathrm{v}}, z))$$

and

$$\hat{\mathrm{v}}(g^* f)(\lambda, z) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (g^* f)(\lambda \cdot e^{-t\mathrm{v}}, z) = -\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (g^* f)(\lambda \cdot e^{t\mathrm{v}}, z).$$

The conclusion follows from the group action property that imposes:

$$(g^* f)(\lambda, g(e^{t\mathrm{v}}, z)) = f(g(\lambda, g(e^{t\mathrm{v}}, z)) = f(g(\lambda \cdot e^{t\mathrm{v}}, z)) = (g^* f)(\lambda \cdot e^{t\mathrm{v}}, z).$$

□

Example 1.2 *We consider the group* $\mathcal{G} = \mathbb{R}^* \ltimes \mathbb{R}$ *with multiplication* $(\lambda_1, \lambda_2) \cdot (\mu_1, \mu_2)^{-1} = (\frac{\lambda_1}{\mu_1}, -\lambda_1 \frac{\mu_2}{\mu_1} + \lambda_2)$*.*

A basis of right invariant vector fields is given by (Olver, 1995, Example 2.46)

$$\mathrm{v}_1 = \lambda_1 \frac{\partial}{\partial \lambda_1} + \lambda_2 \frac{\partial}{\partial \lambda_2}, \quad \mathrm{v}_2 = \frac{\partial}{\partial \lambda_2}.$$

The associated left invariant vector fields, i.e. the infinitesimal generators for the action of $\mathcal{G}$ *on* $\mathcal{G}$ *by right multiplication, are:*

$$\hat{\mathrm{v}}_1 = -\lambda_1 \frac{\partial}{\partial \lambda_1}, \quad \hat{\mathrm{v}}_2 = -\lambda_1 \frac{\partial}{\partial \lambda_2}.$$

If we consider the action g *of* $\mathcal{G}$ *on* $\mathbb{R}$ *given by* $g^* x = \lambda_1 x + \lambda_2$*, the associated infinitesimal generators for this action are*

$$\mathrm{V}_1 = x \frac{\partial}{\partial x}, \quad \mathrm{V}_2 = \frac{\partial}{\partial x}$$

Note that $\hat{\mathrm{v}}_i(g^* x) = -\mathrm{V}_i(g^* x)$ *and* $\hat{\mathrm{v}}_i|_e = -\mathrm{v}_i|e$*.*

1.2 Action prolongations

We shall consider now a manifold $\mathcal{X} \times \mathcal{U}$. We assume that $\mathcal{X}$ and $\mathcal{U}$ are covered by a single coordinate chart with respectively $x = (x_1, \ldots, x_m)$ and $u = (u_1, \ldots, u_n)$ as coordinate functions. The x are considered as the independent variables and the u as dependent variables. We discuss briefly the prolongation of an action of $\mathcal{G}$ on $\mathcal{X} \times \mathcal{U}$ to its jet space following Olver (1986, 1995) and define differential invariants.

Total derivations

The k-th order jet space is noted $\mathrm{J}^k(\mathcal{X},\mathcal{U})$, or J^k for short, while the infinite jet space is J. Besides x and u the coordinate functions of J^k are u_α for u in $\{u_1,\ldots,u_n\}$ and $\alpha \in \mathbb{N}^m$ with $|\alpha| \leq k$.

Pragmatically the set of *total derivations* is the free $\mathcal{F}(\mathrm{J})$-module with basis $\mathrm{D} = \{\mathrm{D}_1,\ldots,\mathrm{D}_m\}$ where a basis is given by

$$\mathrm{D}_i = \frac{\partial}{\partial x_i} + \sum_{u\in\mathcal{U},\,\alpha\in\mathbb{N}^m} u_{\alpha+\epsilon_i}\frac{\partial}{\partial u_\alpha}. \tag{1.2}$$

In other words, D_i is such that for any $u \in \mathcal{U}$ and $\alpha \in \mathbb{N}^m$, $\mathrm{D}_i(u_\alpha) = u_{\alpha+\epsilon_i}$, where ϵ_i is the m-tuple with 1 at the i^{th} position and 0 otherwise.

Geometrically one defines total derivations as the derivations of $\mathcal{F}(\mathrm{J})$ that annihilate the contact forms (Olver, 1995). Alternatively they correspond to the formal derivations in (Kumpera, 1974, 1975a,b; Muñoz et al., 2003).

A total derivation D is of order l if for all $f \in \mathcal{F}(\mathrm{J}^{l+k})$, $k \geq 0$, $\mathrm{D}(f) \in \mathcal{F}(\mathrm{J}^{l+k+1})$. The total derivations of order l form a $\mathcal{F}(\mathrm{J}^l)$-module.

Prolongation of vector fields

Vector fields on J^k form a free $\mathcal{F}(\mathrm{J}^k)$-module a basis of which is given by $\{\frac{\partial}{\partial x} \mid x \in \mathcal{X}\} \cup \{\frac{\partial}{\partial u_\alpha} \mid u \in \mathcal{U}, |\alpha| \leq k\}$.

Definition 1.3 *Let V^0 be a vector field on J^0. The k-th prolongations V^k, $k \geq 0$, is the unique vector field of $\mathcal{F}(\mathrm{J}^k)$ defined recursively by the conditions*

$$\mathrm{V}^{k+1}|_{\mathcal{F}(\mathrm{J}^k)} = \mathrm{V}^k, \quad \textit{and} \quad \mathrm{V}^{k+1}\circ\mathrm{D}_i - \mathrm{D}_i\circ\mathrm{V}^k \ \textit{is a total derivation for all}\ 1 \leq i \leq m.$$

This definition is to be compared with Proposition 4.33 of Olver (1995) given in terms of contact forms. The explicit form of the prolongations are given in Chapter 4 of Olver (1995).

Proposition 1.4 *The prolongations of a vector field $\mathrm{V}^0 = \sum_{i=1}^n \xi_i \frac{\partial}{\partial x_i} + \sum_{j=1}^n \eta_j \frac{\partial}{\partial u_j}$ on J^0 are the appropriate restrictions of the vector field*

$$\mathrm{V} = \sum_{i=1}^n \xi_i\,\mathrm{D}_i + \sum_{1\leq j\leq n,\,\alpha\in\mathbb{N}^m} \mathrm{D}^\alpha(\zeta_j)\,\frac{\partial}{\partial {u_j}_\alpha} \quad \textit{where} \quad \zeta_j = \eta_j - \sum_{i=1}^m \xi_i\,\mathrm{D}_i(u_j).$$

Furthermore $\mathrm{D}_j \circ \mathrm{V} - \mathrm{V} \circ \mathrm{D}_j = \sum_{i=1}^m \mathrm{D}_j(\xi_i)\mathrm{D}_i, \quad \forall j \in \{1,\ldots,m\}.$

Action prolongations

Consider a connected Lie group $\mathcal{G}$ of dimension r acting on $\mathrm{J}^0 = \mathcal{X} \times \mathcal{U}$.

An action of $\mathcal{G}$ on $\mathrm{J}^0 = \mathcal{X} \times \mathcal{U}$ can be prolonged in a unique way to an action $\mathcal{G} \times \mathrm{J}^\kappa \rightarrow \mathrm{J}^\kappa$ that defines a contact transformation for each $\lambda \in \mathcal{G}$ Olver (1995). We shall write g as well for the action on any J^k. The explicit expressions for $g^* u_\alpha$ is obtained as follows (Olver, 1986, Chapter 4).

In order to obtain compact formulae we introduce vectorial notations. D denotes the vector of total derivations $\mathrm{D} = (\mathrm{D}_1, \dots \mathrm{D}_m)^T$ on $\mathcal{F}(\mathrm{J})$. Define the vector $\tilde{\mathrm{D}} = (\tilde{\mathrm{D}}_1, \dots, \tilde{\mathrm{D}}_m)^T$ of derivations on $\mathcal{F}(\mathcal{G} \times \mathrm{J})$ as

$$\tilde{\mathrm{D}} = A^{-1}\mathrm{D} \quad \text{where} \quad A = \left(\mathrm{D}_i(g^* x_j)\right)_{ij}. \tag{1.3}$$

The total derivations D are here implicitly extended to be derivations on functions of $\mathcal{G} \times \mathrm{J}$. The derivations $\tilde{\mathrm{D}}$ commute and are such that $\tilde{\mathrm{D}}_i(x_j) = \delta_{ij}$ and and $g^* u_\alpha = \tilde{\mathrm{D}}^\alpha(g^* u)$ (Olver, 1995, Chapter 4). The prolongations are then given by:

$$g^*(\mathrm{D} f) = \tilde{\mathrm{D}}(g^* f), \; \forall f \in \mathcal{F}(\mathrm{J}). \tag{1.4}$$

If $\mathrm{V}^0 = (\mathrm{V}_1^0, \dots, \mathrm{V}_r^0)$ are the infinitesimal generators for the action of g on J^0 then their k-th prolongations $\mathrm{V}^k = (\mathrm{V}_1^k, \dots, \mathrm{V}_r^k)$ are the infinitesimal generators for the action of g on J^k.

Example 1.5 *We consider the group of Example 1.2, $\mathcal{G} = \mathbb{R}^* \ltimes \mathbb{R}$ and extend (trivially) its action on $\mathcal{X}^1 \times \mathcal{U}^1$ as follows:*

$$g^* x = \lambda_1 x + \lambda_2, \quad g^* u = u.$$

The derivation $\tilde{\mathrm{D}} = \frac{1}{\lambda_1}\mathrm{D}$ allows to compute the prolongations of the action: $g^* u_k = \frac{u_k}{\lambda_1^k}$. The infinitesimal generators of the action were given in Example 1.2. Their prolongations are:

$$\mathrm{V}_1 = x\,\mathrm{D} - \sum_{k \geq 0} \mathrm{D}^k(x\,u_1)\frac{\partial}{\partial u_k} = x\frac{\partial}{\partial x} - k\,u_k \frac{\partial}{\partial u_k}, \quad \mathrm{V}_2 = \frac{\partial}{\partial x}.$$

2 Local and differential invariants

We first define the normalized invariants in the context of a group action on a manifold $\mathcal{M}$. We then generalize those concepts to differential invariants. The material of this section is essentially borrowed from Fels and Olver (1999) and Hubert and Kogan (2007b), following closely this latter. We refer the readers to those papers for more details and a substantial set of examples.

2.1 Normalized invariants

We consider the action $g : \mathcal{G} \times \mathcal{M} \rightarrow \mathcal{M}$ of the r-dimensional Lie group $\mathcal{G}$ on the smooth manifold $\mathcal{M}$.

Definition 2.1 *A function f of $\mathcal{F}(\mathcal{M})$ is a* local invariant *if* $\mathrm{V}(f) = 0$ *for any infinitesimal generator* V *of the action g of $\mathcal{G}$ on $\mathcal{M}$. The set of local invariants is denoted $\mathcal{F}^{\mathcal{G}}(\mathcal{M})$.*

This is equivalent to say that $g^* f = f$ in an open set of $\mathcal{G} \times \mathcal{M}$ that contains $\{e\} \times \mathcal{M}$.

The orbit of a point $z \in \mathcal{M}$ is the set of points $\mathcal{O}_z = \{g(\lambda, z) | \lambda \in \mathcal{G}\}$. The action is semi-regular if all the orbits have the same dimension, say d. For those the a maximally independent set of local invariants is classically shown to exist by Frobenius theorem (Olver, 1995, Theorem 2.23 and 2.34). Alternatively, a geometric and more constructive approach was described for free action based on a *moving frame* by Fels and Olver (1999) and extended to semi-regular actions with the sole use of a cross-section by Hubert and Kogan (2007b).

Definition 2.2 *An embedded submanifold $\mathcal{P}$ of $\mathcal{M}$ is a* local cross-section *to the orbits if there is an open set $\mathcal{U}$ of $\mathcal{M}$ such that*

- *$\mathcal{P}$ intersects $\mathcal{O}_z^0 \cap \mathcal{U}$ at a unique point $\forall z \in \mathcal{U}$, where $\mathcal{O}_z^0$ is the connected component of $\mathcal{O}_z \cap \mathcal{U}$, containing z.*
- *for all $z \in \mathcal{P} \cap \mathcal{U}$, $\mathcal{O}_z^0$ and $\mathcal{P}$ are transversal and of complementary dimensions.*

Most of the results in this paper restrict to $\mathcal{U}$. We shall thus assume, with no loss, that $\mathcal{U} = \mathcal{M}$.

An embedded submanifold of codimension d can be locally defined as the zero set of a map $P : \mathcal{M} \to \mathbb{R}^d$ where the components $(p_1, \ldots, p_d)$ are independent functions of $\mathcal{F}(\mathcal{M})$. The condition for P to define a local cross section is:

$$\text{the rank of the } r \times d \text{ matrix } \left(\mathrm{V}_i(p_j)\right)_{i=1..r}^{j=1..d} \text{ equals to } d \text{ on } \mathcal{P}. \tag{2.1}$$

When $\mathcal{G}$ acts semi-regularly on $\mathcal{M}$ there is a lot of freedom in choosing a cross-section. In particular we can always choose a coordinate cross-section (Hubert and Kogan, 2007b, Theorem 5.6).

A cross-section on $\mathcal{M}$ defines an invariantization process that is a projection from $\mathcal{F}(\mathcal{M})$ to $\mathcal{F}^{\mathcal{G}}(\mathcal{M})$.

Definition 2.3 *Let $\mathcal{P}$ be a local cross-section to the orbits of the action $g : \mathcal{G} \times \mathcal{M} \to \mathcal{M}$. Let f be a smooth function on $\mathcal{M}$. The* invariantization *$\bar{\iota} f$ of f is the function defined by $\bar{\iota} f(z) = f(z_0)$ for each $z \in \mathcal{M}$, where $z_0 = \mathcal{O}_z^0 \cap \mathcal{P}$.*

The invariantization of the coordinate functions on $\mathcal{M}$ are the *normalized invariants*. Fels and Olver (1999, Definition 4.9) explain how invariantization actually ties in with the normalization procedure in Cartan's work. The following theorem (Hubert and Kogan, 2007b, Theorem 1.8) entails that normalized invariants form a generating set that is equipped with a trivial rewriting process.

Theorem 2.4 *Let a Lie group $\mathcal{G}$ act semi-regularly on a manifold $\mathcal{M}$, and let $\mathcal{P}$ be a cross-section to the orbits. Then the invariantization of f, $\bar{\iota} f$, is the unique local invariant defined on $\mathcal{M}$ whose restriction to $\mathcal{P}$ is equal to the restriction of f to $\mathcal{P}$. In other words $\bar{\iota} f|_{\mathcal{P}} = f|_{\mathcal{P}}$.*

Contained in this theorem as well is the fact that two local invariants are equal iff they have the same restriction on $\mathcal{P}$. In particular if $f \in \mathcal{F}^{\mathcal{G}}(\mathcal{M})$ then $\bar{\iota} f = f$. Now, by comparing the values of the functions involved at the cross-section, it is furthermore easy to check that:

Corollary 2.5 *For $f \in \mathcal{F}(\mathcal{M})$, $\bar{\iota} f(z_1, \ldots, z_n) = f(\bar{\iota} z_1, \ldots, \bar{\iota} z_n)$.*

Thus for $f \in \mathcal{F}^{\mathcal{G}}(\mathcal{M})$ we have $f(z_1, \ldots, z_n) = f(\bar{\iota} z_1, \ldots, \bar{\iota} z_n)$. Therefore the normalized invariants $\{\bar{\iota} z_1, \ldots, \bar{\iota} z_n\}$ form a generating set of local invariants: any local invariant can be written as a function of those. The rewriting is furthermore a simple replacement: we substitute the coordinate functions by their invariantizations.

The normalized invariants are nonetheless not functionally independent. Characterizing the functions that vanish on $(\bar{\iota} z_1, \ldots, \bar{\iota} z_n)$ amounts to characterize the functions the invariantization of which is zero. The functions that cut out the cross-section are an example of those.

Proposition 2.6 *Assume the cross-section $\mathcal{P}$ is the zero set of the map $P = (p_1, \ldots, p_d) : \mathcal{M} \to \mathbb{R}^d$ which is of maximal rank d. The invariantization of $f \in \mathcal{F}(\mathcal{M})$ is zero iff there exists $a_1, \ldots, a_d \in \mathcal{F}(\mathcal{M})$ such that $f = \sum_{i=1}^{d} a_i \, p_i$ on an open set that contains $\mathcal{P}$.*

PROOF: Taylor's formula with integral remainder shows the following (Bourbaki, 1967, Paragraph 2.5). For a smooth function f on an open set $I_1 \times \ldots \times I_d \times U \subset \mathbb{R}^k \times \mathbb{R}^l$, where the I_i are intervals of $\mathbb{R}$ that contain zero, there are smooth functions f_0 on U, and f_i on $I_1 \times \ldots \times I_i \times U$, $1 \leq i \leq d$ such that $f(t_1, \ldots, t_d, x) = f_0(x) + \sum_{j=1}^{l} t_j \, f_j(t_1, \ldots, t_j, x)$ where $f_0(x) = f(0, \ldots, 0, x)$.

Since $(p_1, \ldots, p_d)$ is of rank d along $\mathcal{P}$ we can find $x_{d+1}, \ldots, x_n \in \mathcal{F}(\mathcal{M})$ such that $(p_1, \ldots, p_d, x_{d+1}, \ldots, x_n)$ is a coordinate system on an open set that contains $\mathcal{P}$. In this coordinate system we have $f(0, \ldots, 0, x_{d+1}, \ldots, x_n) = 0$ since $\bar{\iota} f = 0 \Leftrightarrow f|_{\mathcal{P}} = 0$. The result therefore follows from the above Taylor formula. □

When $\mathcal{G}$ is an algebraic group and g a rational action, the normalized invariants $(\bar{\iota} z_1, \ldots, \bar{\iota} z_n)$ can be computed effectively (Hubert and Kogan, 2007b, Theorem 3.6). The method of Fels and Olver (1999) proceed through the moving frame.

Invariantization was first defined by Fels and Olver (1999) in terms of an equivariant map $\rho : \mathcal{M} \to \mathcal{G}$. They called such a map a *moving frame* in reference to the *repère mobile* of Cartan (1935, 1937) of which they offer a new interpretation. As noted already by Griffiths (1974); Green (1978); Jensen (1977); Ivey and Landsberg (2003), the geometric idea of classical moving frames, like the Frenet frame for space curves in Euclidean geometry, can indeed be understood as maps to the group. This geometric vision of moving frames as frames is arguably misleading.

An action of a Lie group $\mathcal{G}$ on a manifold $\mathcal{M}$ is *locally free* if for every point $z \in \mathcal{M}$ its isotropy group $\mathcal{G}_z = \{\lambda \in \mathcal{G} \,|\, \lambda \cdot z = z\}$ is discrete. Local freeness implies semi-regularity with the dimension of each orbit being equal to the dimension of the group. The following result (Fels and Olver, 1999, Theorem 4.4) establishes the existence of moving frames for actions with this property.

Theorem 2.7 *A Lie group $\mathcal{G}$ acts locally freely on $\mathcal{M}$ if and only if every point of $\mathcal{M}$ has an open neighborhood $\mathcal{U}$ such that there exists a map $\rho\colon \mathcal{U} \to \mathcal{G}$ that makes the following diagram commute. Here right multiplication is chosen for the action of $\mathcal{G}$ on itself, and λ is taken in a suitable neighborhood of the identity in $\mathcal{G}$.*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\lambda} & \mathcal{M} \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\rho} \\ \mathcal{G} & \xrightarrow[\lambda]{} & \mathcal{G} \end{array}$$

The map ρ in the theorem is called a *moving frame*. In other words, a *moving frame* is a locally $\mathcal{G}$-equivariant map, that is $\rho(\lambda \cdot z) = \rho(z) \cdot \lambda^{-1}$ for λ sufficiently close to the identity. As before, we shall restrict our attention to $\mathcal{U}$ and therefore we assume it is equal to $\mathcal{M}$.

A local cross-section to the orbit of a locally free action defines a moving frame. Indeed, if $\mathcal{P}$ is a local cross-section, then the equation

$$g(\rho(z), z) \in \mathcal{P} \text{ for } z \in \mathcal{M} \text{ and } \rho(z) = e, \forall z \in \mathcal{P} \tag{2.2}$$

uniquely defines a smooth map $\rho : \mathcal{M} \to \mathcal{G}$ in a sufficiently small neighborhood of any point of the cross-section. In particular, $\rho(z) = e$ for all $z \in \mathcal{P}$. This map is seen to be equivariant. If $\mathcal{P}$ is the zero set of the map $P = (p_1, \ldots, p_r)$ then $p_1(g(\rho, z)) = 0, \ldots, p_r(g(\rho, z)) = 0$ are implicit equations for the moving frame . If we can solve those, ρ provides an explicit construction for the invariantization process. To make that explicit let us introduce the following maps.

$$\begin{array}{rcl} \sigma : \mathcal{M} & \rightarrow & \mathcal{G} \times \mathcal{M} \\ z & \mapsto & (\rho(z),\, z) \end{array} \quad \text{and} \quad \begin{array}{rcl} \pi = g \circ \sigma : \mathcal{M} & \rightarrow & \mathcal{M} \\ z & \mapsto & g(\rho(z),\, z) \end{array} \tag{2.3}$$

Proposition 1.16 of Hubert and Kogan (2007b) can be restated as:

Proposition 2.8 *$\bar{\iota} f = \pi^* f$, that is $\bar{\iota} f(z) = f(g(\rho(z), z))$ for all $z \in \mathcal{M}$.*

2.3 Differential invariants

We consider an action g of $\mathcal{G}$ on $\mathrm{J}^0 = \mathcal{X} \times \mathcal{U}$ and its prolongations to the jet spaces J^k. The prolongation of the infinitesimal generators on J^k are denoted $\mathrm{V}^k = (\mathrm{V}_1^k, \ldots, \mathrm{V}_d^k)$ while their prolongations to J is denoted $\mathrm{V} = (\mathrm{V}_1, \ldots, \mathrm{V}_r)$.

Definition 2.9 *A differential invariant of order k is a function f of $\mathcal{F}(\mathrm{J}^k)$ such that $\mathrm{V}_1^k(f) = 0, \ldots, \mathrm{V}_r^k(f) = 0$.*

A differential invariant of order k is thus a local invariant of the action prolonged to J^k.

The maximal dimension of the orbits can only increase as the action is prolonged to higher order jets. It can not go beyond the dimension of the group though. The stabilization order is the order at which the maximal dimension of the orbits becomes stationary. If the action on J^0 is locally effective on subsets, i.e. the global isotropy group is discrete, then, for s bigger than the stabilization order, the action on J^s is locally free on an open subset of J^s (Olver, 1995, Theorem 5.11). We shall make this assumption of a locally effective action. The dimension of the orbits in J^s is then r, the dimension of the group.

For any k, a cross-section to the orbits of g in J^k defines an invariantization and a set of normalized invariants on an open set of J^k. As previously we tacitly restrict to this open set though we keep the global notation J^k. Let s be equal or bigger than the stabilization order and $\mathcal{P}^s$ a cross-section to the orbits in J^s. Its pre-image $\mathcal{P}^{s+k}$ in J^{s+k} by the projection map $\pi_s^{s+k} : \mathrm{J}^{s+k} \rightarrow \mathrm{J}^s$ is a cross-section to the orbits in J^{s+k}. It defines an invariantization $\bar{\iota} : \mathcal{F}(\mathrm{J}^{s+k}) \rightarrow \mathcal{F}^{\mathcal{G}}(\mathrm{J}^{s+k})$. The *normalized invariants of order $s+k$* are the invariantizations of the coordinate functions on J^{s+k}. We note the set of those:

$$\mathcal{I}^{s+k} = \{\bar{\iota} x_1, \ldots, \bar{\iota} x_m\} \cup \{\bar{\iota} u_\alpha \,|\, u \in \mathcal{U},\ |\alpha| \leq s+k\}.$$

We can immediately extend Theorem 2.4 and its Corollary 2.5 to show that $\mathcal{I}^{s+k}$ is a generating set of differential invariants of order $s+k$ endowed with a trivial rewriting.

Theorem 2.10 *Let s be greater than the stabilization order and let $\mathcal{P}^s$ be a cross-section in J^s. For $f \in \mathcal{F}(\mathrm{J}^{s+k})$, $k \in \mathbb{N}$, $\bar{\iota} f$ is the unique differential invariant (of order $s+k$) whose restriction to $\mathcal{P}^{s+k}$ is equal to the restriction of f to $\mathcal{P}^{s+k}$.*

Corollary 2.11 *For $f \in \mathcal{F}(\mathrm{J}^{s+k})$, $\bar{\iota} f(x, u_\alpha) = f(\bar{\iota} x, \bar{\iota} u_\alpha)$.*

In particular, if $f \in \mathcal{F}^{\mathcal{G}}(\mathrm{J}^{s+k})$ then $\bar{\iota} f = f$ and $f(x, u_\alpha) = f(\bar{\iota} x, \bar{\iota} u_\alpha)$.

We furthermore know the functional relationships among the elements in $\mathcal{I}^{s+k}$. They are given by the functions the invariantization of which is zero. Those are essentially characterized by Proposition 2.6.

Proposition 2.12 *Let s be greater than the stabilization order. Consider the cross-section $\mathcal{P}^s$ in J^s that we assume given as the zero set of $P = (p_1, \ldots, p_r) : \mathrm{J}^s \to \mathbb{R}^r$, a map of maximal rank r along $\mathcal{P}^s$. The invariantization of $f \in \mathcal{F}(\mathrm{J}^{s+k})$, for $k \in \mathbb{N}$, is zero iff there exists $a_1, \ldots, a_r \in \mathcal{F}(\mathrm{J}^{s+k})$ such that $f = \sum_{i=1}^r a_i\, p_i$ on an open set of J^{s+k} that contains $\mathcal{P}^{s+k}$.*

Example 2.13 *We carry on with Example 1.5.*

The map $\rho : \mathrm{J}^1 \to \mathcal{G}$ defined by $\rho^*\lambda_1 = u_1, \rho^*\lambda_2 = -x\, u_1$ is equivariant for the affine action: $\rho(g(\lambda, z)) = \rho(z) \cdot \lambda^{-1}$. The section defined by this equivariant map is $P = (x, u_1 - 1)$ so that $\bar{\iota} x = 0, \bar{\iota} u_0 = u_0, \bar{\iota} u_1 = 1$. As $g^* u_i = \frac{u_i}{\lambda_1^i}$, $\bar{\iota} u_i = \bar{\iota} u_i = \frac{u_i}{u_1^i}$.

Example 2.14 *We consider the action of $\mathcal{G} = \mathbb{R}^* \times \mathbb{R}^2$ on $\mathrm{J}^0 = \mathcal{X}^2 \times \mathcal{U}^1$, with coordinate (x, y, u), given by:*

$$g^* x = \lambda_1\, x + \lambda_2, \quad g^* y = \lambda_1\, y + \lambda_3, \quad g^* u = u.$$

The derivations $\tilde{\mathrm{D}}_1 = \frac{1}{\lambda_1}\mathrm{D}_1$ and $\tilde{\mathrm{D}}_2 = \frac{1}{\lambda_1}\mathrm{D}_2$ allow to compute its prolongations:

$$g^* u_{ij} = \frac{u_{ij}}{\lambda_1^{i+j}}.$$

The action is locally free on $\mathrm{J}^1 \setminus \mathcal{S}$ where $\mathcal{S}$ are the points where both u_{10} and u_{01} are zero. The moving frame associated with the cross-section defined by $P = (x, y, u_{10} - 1)$ is $\rho^*\lambda_1 = u_{10}$, $\rho^*\lambda_2 = -x\, u_{10}$, $\rho^*\lambda_3 = -y\, u_{10}$. It is defined only on a proper subset of $\mathrm{J}^1 \setminus \mathcal{S}$, as are the normalized invariants: $\bar{\iota} u_{ij} = \frac{u_{ij}}{u_{10}^{i+j}}$

On the other hand, if we choose the cross-section defined by

$$P = \left(x,\ y,\ \frac{1}{2} - \frac{1}{2}(u_{10}^2 + u_{01}^2) \right)$$

the associated moving frame is well defined on the whole of $\mathrm{J}^1 \setminus \mathcal{S}$:

$$\rho^*\lambda_1 = \sqrt{u_{10}^2 + u_{01}^2}, \quad \rho^*\lambda_2 = -x\,\sqrt{u_{10}^2 + u_{01}^2}, \quad \rho^*\lambda_3 = -y\,\sqrt{u_{10}^2 + u_{01}^2}.$$

The invariantizations associated to this second cross-section are given by:

$$\bar{\iota}x = 0, \quad \bar{\iota}y = 0, \text{ and } \bar{\iota}u_{ij} = \frac{u_{ij}}{(u_{10}^2 + u_{01}^2)^{\frac{i+j}{2}}}.$$

3 Invariant derivations

An invariant derivation is a total derivation that commutes with the infinitesimal generators. It maps differential invariants of order k to differential invariant of order $k+1$, for k large enough. Classically a basis of commuting invariant derivations is constructed with the use of sufficiently many differential invariants (Olver, 1995; Ovsiannikov, 1982; Kumpera, 1974, 1975a,b; Muñoz et al., 2003). The novel construction proposed by Fels and Olver (1999) is based on a moving frame. The constructed invariant derivations do not commute in general. Their formidable benefit is that it brings an explicit formula for the derivation of a normalized invariants. This has been known as the *recurrence formulae*. They are the key to most results about generation and syzygies in this paper. All the algebraic and algorithmic treatments of differential invariants and their applications (Mansfield, 2001; Olver, 2007a; Hubert and Olver, 2007; Hubert, 2008) come as an exploitation of those formulae.

In Theorem 3.6 we present the derivation formulae for any invariantized functions. For the proof we take the dual approach of the one of Fels and Olver (1999) which is therefore close in essence to the one presented by Mansfield (2008), based on the application of the chain rule.

We always consider the action g of a connected r-dimensional Lie group $\mathcal{G}$ on $\mathrm{J}^0 = \mathcal{X} \times \mathcal{U}$ and its prolongations. We make use of a basis of right invariant vector fields on $\mathcal{G}$, $\mathrm{v} = (\mathrm{v}_1, \ldots, \mathrm{v}_r)$, and the associated infinitesimal generators:

- $\mathrm{V} = (\mathrm{V}_1, \ldots, \mathrm{V}_r)^T$ is the vector of infinitesimal generators for the action g of $\mathcal{G}$ on J
- $\hat{\mathrm{v}} = (\hat{\mathrm{v}}_1, \ldots, \hat{\mathrm{v}}_r)^T$ is the vector of infinitesimal generators for the action of $\mathcal{G}$ on itself by right multiplication.

3.1 Infinitesimal criterion

Definition 3.1 *An invariant derivation $\mathcal{D}$ is a total derivation such that $[\mathcal{D}, \mathrm{V}] = 0$ for any infinitesimal generator* V *of the action.*

As an immediate consequence of this definition we see: If f is a differential invariant and $\mathcal{D}$ an invariant derivation then $\mathcal{D}(f)$ is an differential invariant.

Proposition 3.2 *Let $A = (a_{ij})$ be an invertible $m \times m$ matrix with entries in $\mathcal{F}(\mathrm{J}^k)$. A vector of total derivations $\mathcal{D} = (\mathcal{D}_1, \ldots, \mathcal{D}_m)^T$ defined by $\mathcal{D} = A^{-1}\,\mathrm{D}$*

is a vector of invariant derivations if and only if, for all infinitesimal generator V *of the action,*

$$\mathrm{V}(a_{ij}) + \sum_{k=1}^{m} \mathrm{D}_i(\xi_k)\, a_{kj} = 0, \qquad \text{where } \xi_k = \mathrm{V}(x_k), \quad \forall\, 1 \le i, j \le m.$$

PROOF: For all i we have $\mathrm{D}_i = \sum_{j=1}^{m} a_{ij}\, \mathcal{D}_j$. By expanding the equality $[\mathrm{D}_i, \mathrm{V}] = \sum_{k=1}^{m} \mathrm{D}_i(\xi_k)\, \mathrm{D}_k$ (Proposition 1.4) we obtain, for all i,

$$\sum_{j=1}^{m} a_{ij}\, [\mathcal{D}_j, V] = \sum_{j=1}^{m} \left(V(a_{ij}) + \sum_{k=1}^{m} \mathrm{D}_i(\xi_k) a_{kj} \right) \mathcal{D}_j$$

Since A is of non-zero determinant $[\mathcal{D}_j, V] = 0$ for all j iff $V(a_{ij}) + \sum_{k=1}^{m} \mathrm{D}_i(\xi_k) a_{kj} = 0, \forall i, j$. □

As illustration, a classical construction of invariant derivations is given by the following proposition (Kumpera, 1974, 1975a,b; Olver, 1995; Ovsiannikov, 1982; Muñoz et al., 2003):

Proposition 3.3 *If $f_1, \ldots, f_m$ are differential invariants such that the matrix $A = (\mathrm{D}_i(f_j))_{i,j}$ is invertible then the derivations $\mathcal{D} = A^{-1}\mathrm{D}$ are invariant derivations.*

PROOF: If $a_{ij} = \mathrm{D}_i(f_j)$ then, by Proposition 1.4,

$$V(a_{ij}) = V(\mathrm{D}_j(f_i)) = \mathrm{D}_j(V(f_i)) - \sum_k \mathrm{D}_j(\xi_k)\, \mathrm{D}_k(f_i) = \mathrm{D}_j(V(f_i)) - \sum_k \mathrm{D}_j(\xi_k)\, a_{ik}.$$

By hypothesis $V(f_i) = 0$ so that the result follows from Proposition 3.2. □

The above derivations commute. They can be understood as derivations with respect to the new *independent variables* $f_1, \ldots, f_m$.

As a side remark, note that Definition 3.1 is dual to the infinitesimal condition for a 1-form to be contact invariant (Olver, 1995, Theorem 2.91). The invariant derivations of Proposition 3.3 are dual to the contact invariant 1-forms $\mathrm{d}_H f_1, \ldots, \mathrm{d}_H f_m$.

3.2 Moving frame construction of invariant derivations

Assume that there exists on J^s a moving frame $\rho : \mathrm{J}^s \to \mathcal{G}$. As in Section 2 we construct the additional maps

$$\begin{array}{rcl} \sigma : \mathrm{J}^{s+k} & \to & \mathcal{G} \times \mathrm{J}^{s+k} \\ z & \mapsto & (\rho(z),\, z) \end{array} \quad \text{and} \quad \begin{array}{rcl} \pi = g \circ \sigma : \mathrm{J}^{s+k} & \to & \mathrm{J}^{s+k} \\ z & \mapsto & g(\rho(z),\, z) \end{array} \tag{3.1}$$

Theorem 3.4 *The vector of derivations* $\mathcal{D} = (\sigma^* A)^{-1}\,\mathrm{D}$*, where* A *is the* $m \times m$ *matrix* $(\mathrm{D}_i(g^* x_j))_{ij}$*, is a vector of invariant derivations.*

The matrix A has entries in $\mathcal{F}(\mathcal{G}\times \mathrm{J}^1)$. Its pull back $\sigma^* A$ has entries in $\mathcal{F}(\mathrm{J}^s)$. The above result is proved by checking that the formula of Proposition 1.4 holds.

PROOF: The equivariance of ρ implies $\rho(g(e^{t\mathrm{v}}, z)) = \rho(z)\cdot e^{-t\mathrm{v}}$ so that $\rho_* \mathrm{V} = \hat{\mathrm{v}}$. Thus $\sigma_* V = \hat{\mathrm{v}} + \mathrm{V}$ that is $\sigma_* V(a_{ij}) = \hat{\mathrm{v}}(\mathrm{D}_i(g^* x_j)) + V(\mathrm{D}_i(g^* x_j))$. As derivations on $\mathcal{F}(\mathcal{G}\times \mathrm{J}^s)$, D_i and $\hat{\mathrm{v}}$ commute while the commutator of D_i and V is given by Proposition 1.4. It follows that $\sigma_* V(a_{ij}) = \mathrm{D}_i(\hat{\mathrm{v}}(g^* x_j)) + \mathrm{D}_i(\mathrm{V}(g^* x_j)) - \sum_{k=1}^{m} \mathrm{D}_i(\xi_k)\,\mathrm{D}_k(g^* x_j)$. By Proposition 1.1 the two first terms cancel and since $V(\sigma^* a_{ij}) = \sigma^*(\sigma_* V)(a_{ij})$ we have $V(\sigma^* a_{ij}) = -\sum_{k=1}^{m} \mathrm{D}_i(\xi_k)\,\sigma^* a_{kj}$. We can conclude with Proposition 3.2. □

Example 3.5 *We carry on with Example 1.5 and 2.13.*

We found that the equivariant map associated to $P = (x, u_1 - 1)$ *is given by* $\rho^*\lambda_1 = u_1, \rho^*\lambda_2 = -x\,u_1$. *In addition* $\tilde{\mathrm{D}} = \frac{1}{\lambda_1}\mathrm{D}$ *while* $\mathrm{V}_1 = x\frac{\partial}{\partial u} - \sum_{k\geq 0} k\, u_k \frac{\partial}{\partial u_k}$, $\quad \mathrm{V}_2 = \frac{\partial}{\partial x}$.

Accordingly define $\mathcal{D} = \frac{1}{u_1}\mathrm{D}$ *and we can verify that:* $[\mathrm{V}_1, \mathcal{D}] = 0, \quad [\mathrm{V}_2, \mathcal{D}] = 0$. *The application of* $\mathcal{D}$ *to a differential invariant thus produces a differential invariant. For instance*

$$\mathcal{D}\left(\frac{u_i}{u_1^i}\right) = \frac{u_{i+1}}{u_1^{i+1}} - \frac{u_i}{u_1^{i+2}}u_2 = \frac{u_{i+1}}{u_1^{i+1}} - \frac{u_i}{u_1^i}\frac{u_2}{u_1^2}.$$

Remembering that $\bar{\iota} u_i = \frac{u_i}{u_1^i}$ *we have to observe that* $\mathcal{D}(\bar{\iota} u_i) \neq \bar{\iota} u_{i+1}$. *The relationship between these two quantities is the subject of Theorem 3.6. Note nonetheless that* $\mathcal{D}(\frac{u_i}{u_1^i}) = \bar{\iota}(\mathrm{D}(\frac{u_i}{u_1^i}))$ *(Corollary 3.7).*

3.3 Derivation of invariantized functions.

An essential property of the invariant derivations of Theorem 3.4 is that we can write explicitly their action on the invariantized functions. Theorem 3.6 below is a general form for the recurrence formulae of Fels and Olver (1999, Equation 13.7).

Assume that the action of g on J^s is locally free and that $P = (p_1, \ldots, p_r)$ defines the cross-section $\mathcal{P}$. Let $\rho : \mathrm{J}^s \to \mathcal{G}$ be the associated moving frame. We construct the vector of invariant derivations $\mathcal{D} = (\mathcal{D}_1, \ldots, \mathcal{D}_m)$ as in Theorem 3.4.

Denote by $\mathrm{D}(P)$ the $m \times r$ matrix $(\mathrm{D}_i(p_j))_{i,j}$ with entries in $\mathcal{F}(\mathrm{J}^{s+1})$ while $\mathrm{V}(P)$ is the $r \times r$ matrix $(\mathrm{V}_i(p_j))_{i,j}$ with entries in $\mathcal{F}(\mathrm{J}^s)$. As $\mathcal{P}$ is transverse to the orbits of $\mathcal{P}$, the matrix $\mathrm{V}(P)$ has non zero determinant along $\mathcal{P}$ and therefore in a neighborhood of each of its points.

Theorem 3.6 *Let $P = (p_1, \ldots, p_d)$ define a cross-section $\mathcal{P}$ to the orbits in* J^s*, where s is greater than the stabilization order. Consider $\rho : \mathrm{J}^s \to \mathcal{G}$ the associated moving frame and $\bar\iota : \mathcal{F}(\mathrm{J}) \to \mathcal{F}^{\mathcal{G}}(\mathrm{J})$ the associated invariantization. Consider $\mathcal{D} = (\mathcal{D}_1, \ldots, \mathcal{D}_m)^T$ the vector of invariant derivations constructed in Theorem 3.4. Let K be the $m \times r$ matrix obtained by invariantizing the entries of* $\mathrm{D}(P)\,\mathrm{V}(P)^{-1}$*. Then*

$$\mathcal{D}(\bar\iota f) = \bar\iota(\mathrm{D} f) - K\,\bar\iota(\mathrm{V}(f)).$$

PROOF: From the definition of $\sigma : z \mapsto (\,\rho(z), z\,)$ and the chain rule we have

$$\mathcal{D}(\bar\iota f)(z) = \mathcal{D}(\sigma^* g^* f)(z) = \mathcal{D}(g^* f)(\rho(z), z) + (\rho_* \mathcal{D})(g^* f)(\rho(z), z). \quad (3.2)$$

Recall the definition of $\tilde{\mathrm{D}}$ in Section 1.2 that satisfies $\tilde{\mathrm{D}}_j(g^* f) = g^*(\mathrm{D}_j f)$ for all $f \in \mathcal{F}(\mathrm{J})$. We have $\mathcal{D}(g^* f)(\rho(z), z) = (\sigma^* \tilde{\mathrm{D}}(g^* f))(z) = \sigma^* g^*(\mathrm{D} f)(z) = \bar\iota(\mathrm{D} f)(z)$ and (3.2) becomes

$$\mathcal{D}(\bar\iota f)(z) = \bar\iota(\mathrm{D} f)(z) + \sigma^*(\rho_* \mathcal{D})(g^* f)(z). \quad (3.3)$$

Since $\hat{\mathrm{v}} = (\hat{\mathrm{v}}_1, \ldots, \hat{\mathrm{v}}_d)$ form a basis for the derivations on $\mathcal{G}$ there is a matrix [1] $\tilde{K}$ with entries in $\mathcal{F}(\mathcal{G} \times \mathrm{J}^s)$ such that $\rho_* \mathcal{D} = \tilde{K}\,\hat{\mathrm{v}}$.

We can write (3.3) as $\mathcal{D}(\bar\iota f)(z) = \bar\iota(\mathrm{D} f)(z) + \sigma^* \left(\tilde{K} \hat{\mathrm{v}}(g^* f)\right)(z)$ so that, by Proposition 1.1,

$$\mathcal{D}(\bar\iota f)(z) = \bar\iota(\mathrm{D} f)(z) - \sigma^* \left(\tilde{K} \mathrm{V}(g^* f)\right)(z). \quad (3.4)$$

This latter equation shows that $\sigma^* \left(\tilde{K} \mathrm{V}(g^* f)\right) = \bar\iota(\mathrm{D} f) - \mathcal{D}(\bar\iota f)$ is a differential invariant. As such it is equal to its invariantization and thus

$$\sigma^* \left(\tilde{K} \mathrm{V}(g^* f)\right) = \bar\iota(\sigma^* \tilde{K})\,\bar\iota(\sigma^* \mathrm{V}(g^* f)).$$

For all $z \in \mathcal{P}$, $\rho(z) = e$ and therefore $\sigma^* \mathrm{V}(g^* f)$ and $\mathrm{V}(f)$ agree on $\mathcal{P}$: for all $z \in \mathcal{P}$, $\sigma^* \mathrm{V}(g^* f)\,(z) = \mathrm{V}(g^* f)\,(e, z) = \mathrm{V}(f)(z)$ by Proposition 1.1. It follows that $\bar\iota(\sigma^* \mathrm{V}(g^* f)) = \bar\iota(\mathrm{V}(f))$ so that (3.4) becomes

$$\mathcal{D}(\bar\iota f)(z) = \bar\iota(\mathrm{D} f)(z) - \bar\iota(\sigma^* \tilde{K})\,\bar\iota(\mathrm{V}(f)). \quad (3.5)$$

To find the matrix $K = \bar\iota(\sigma^* \tilde{K})$ we use the fact that $\bar\iota p_i = 0$ for all $1 \leq i \leq r$. Applying $\mathcal{D}$ and (3.5) to this equality we obtain: $\bar\iota(\mathrm{D} p_i) = K\,\bar\iota(\mathrm{V}(p_i))$ so that

[1] With $\mathcal{D}$ known explicitly, we can write $\tilde{K}$ explicitly in terms of coordinates $\lambda = (\lambda_1, \ldots, \lambda_d)$. $\tilde{K}$ is the matrix obtained by multiplying the matrix $\mathcal{D}(\rho) = \left(\mathcal{D}_j(\rho^* \lambda_i)\right)$ with the inverse of $\hat{\mathrm{v}}(\lambda) = (\hat{\mathrm{v}}_i(\lambda_j))$. Yet $\sigma^* \tilde{K}$ needs not have differential invariants as entries and we shall seek $\bar\iota(\sigma^* \tilde{K})$ in a more direct way. See Example 3.9.

$\bar{\iota}(\mathrm{D}(P)) = K\,\bar{\iota}(\mathrm{V}(P))$. The transversality of $\mathcal{P}$ imposes that $\mathrm{V}(P)$ is invertible along $\mathcal{P}$, and thus so is $\bar{\iota}(\mathrm{V}(P))$.

We thus have proved that $\mathcal{D}(\bar{\iota}f) = \bar{\iota}(\mathrm{D}f) - K\,\bar{\iota}(\mathrm{V}(f))$ where $K = \bar{\iota}(\sigma^*\tilde{K}) = \bar{\iota}(\mathrm{D}(P)V(P)^{-1})$.

□

If f is a differential invariant, $\mathcal{D}(f)$ is also a differential invariant, while $\mathrm{D}(f)$ need not be. Yet the relationship between the two follows immediately from this new way of writing the *recurrence formulae*. We have not seen the following corollary in previous papers on the subject.

Corollary 3.7 *If f is a differential invariant then $\mathcal{D}(f) = \bar{\iota}(\mathrm{D}(f))$.*

PROOF: If f is a differential invariant then $\bar{\iota}f = f$ and $\mathrm{V}(f) = 0$. The result thus follows from the above theorem. □

By deriving a recurrence formula for forms, (Fels and Olver, 1999, Section 13) derived explicitly the commutators of the invariant derivations . It can actually be derived directly from Theorem 3.6 through the use of *formal invariant derivations* (Hubert, 2008).

Proposition 3.8 *For all $1 \leq i, j \leq m$, $\left[\mathcal{D}_i, \mathcal{D}_j\right] = \sum_{k=1}^{m} \Lambda_{ijk}\,\mathcal{D}_k$ where*

$$\Lambda_{ijk} = \sum_{c=1}^{d} K_{ic}\,\bar{\iota}(\mathrm{D}_j(\xi_{ck})) - K_{jc}\,\bar{\iota}(\mathrm{D}_i(\xi_{ck})) \;\in\; \mathcal{F}^{\mathcal{G}}(\mathrm{J}^{s+1}),$$

$K = \bar{\iota}\,(D(P)\,V(P)^{-1})$, and $\xi_{ck} = \mathrm{V}_c(x_k)$.

Example 3.9 *We carry on with Example 1.5, 2.13, and 3.5.*

We chose $P = (x, u_1 - 1)$ and showed that $\mathcal{D} = \frac{1}{u_1}\mathrm{D}$ while $\bar{\iota}u_i = \frac{u_i}{u_1^i}$. We computed

$$\mathcal{D}(\bar{\iota}u_i) = \frac{u_{i+1}}{u_1^{i+1}} - i\,\frac{u_i}{u_1^i}\frac{u_2}{u_1^2} = \bar{\iota}u_{i+1} - i\,\bar{\iota}u_2\,\bar{\iota}u_i.$$

We have $\mathrm{D}(P) = (\,1\;u_2\,)$ and $\mathrm{V}(P) = \begin{pmatrix} x & -u_1 \\ 1 & 0 \end{pmatrix}$. The matrix K of Theorem 3.6 is thus $K = \bar{\iota}\,(D(P)\,V(P)^{-1}) = (\,-\bar{\iota}u_2\;1\,)$ and the formula is verified:

$$\mathcal{D}(\bar{\iota}u_i) = \bar{\iota}u_{i+1} - (\,-\bar{\iota}u_2\;1\,)\begin{pmatrix} \bar{\iota}\mathrm{V}_1(u_i) \\ \bar{\iota}\mathrm{V}_2(u_i) \end{pmatrix} \quad \text{since} \quad \bar{\iota}\mathrm{V}(u_i) = \begin{pmatrix} -i\,u_i & 0 \end{pmatrix}^T.$$

What we shall do next is illustrate the proof by exhibiting the matrix $\tilde{K}$ that arises there. It is defined by $\rho_*\mathcal{D} = \tilde{K}\,\hat{\mathrm{v}}$ and the fact that $\sigma^*\tilde{K}\mathrm{V}(g^*f)$ is an invariant for any $f \in \mathcal{F}(\mathrm{J})$.

We have $\hat{\mathrm{v}}_1 = -\lambda_1 \frac{\partial}{\partial \lambda_1}, \quad \hat{\mathrm{v}}_2 = -\lambda_1 \frac{\partial}{\partial \lambda_2}$ *and saw that* $\rho^*\lambda_1 = u_1$ *and* $\rho^*\lambda_2 = -x u_1$. *Thus*

$$\rho_*\mathcal{D} = \begin{pmatrix} \mathcal{D}(\rho^*\lambda_1) & \mathcal{D}(\rho^*\lambda_2) \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial \lambda_1} \\ \frac{\partial}{\partial \lambda_2} \end{pmatrix} = \begin{pmatrix} -\frac{u_2}{u_1}\frac{1}{\lambda_1} & \frac{u_1 + x\, u_2}{u_1}\frac{1}{\lambda_1} \end{pmatrix} \begin{pmatrix} \hat{\mathrm{v}}_1 \\ \hat{\mathrm{v}}_2 \end{pmatrix}.$$

So here $\sigma^*\tilde{K} = \left(-\frac{u_2}{u_1^2}, \frac{u_1 + x u_2}{u_1^2}\right)$. *We indeed have that* $\bar{\iota}\sigma^*\tilde{K} = K$ *as used in the proof. We verify here that* $\sigma^*\left(\tilde{K}\mathrm{V}(g^*f)\right)$ *is a vector of differential invariants. We have*

$$\mathrm{V}(g^*x) = \begin{pmatrix} \lambda_1\, x \\ \lambda_1 \end{pmatrix}, \; \mathrm{V}(g^*u_i) = \begin{pmatrix} -i\,\frac{u_i}{\lambda_1^i} \\ 0 \end{pmatrix}$$

so that $\sigma^*\tilde{K}\mathrm{V}(g^*x) = 1$ *and* $\sigma^*\tilde{K}\mathrm{V}(g^*u_i) = i\,\frac{u_2}{u_1^2}\,\frac{u_1}{u_1^i} = i\,\bar{\iota}u_2\,\bar{\iota}u_i$.

Example 3.10 *We carry on with Example 2.14.*
We chose

$$P = \left(x, y, \frac{1}{2} - \frac{1}{2}(u_{10}^2 + u_{01}^2)\right).$$

On one hand the prolongations of the infinitesimal generators to J *are*

$$\mathrm{V}_1 = \frac{\partial}{\partial x}, \quad \mathrm{V}_2 = \frac{\partial}{\partial y}, \quad \mathrm{V}_3 = x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y} - \sum_{i,j\geq 0} (i+j)\,u_{ij}\,\frac{\partial}{\partial u_{ij}}$$

so that

$$\mathrm{V}(P) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & y & u_{10}^2 + u_{01}^2 \end{pmatrix} \text{ while } \mathrm{D}(P) = \begin{pmatrix} 1 & 0 & v \\ 0 & 1 & w \end{pmatrix}$$

where

$$v = -(u_{10}u_{20} + u_{01}u_{11}) \text{ and } w = -(u_{10}u_{11} + u_{01}u_{02}).$$

Since $\bar{\iota}x = 0$, $\bar{\iota}y = 0$ *and* $\bar{\iota}(u_{10}^2 + u_{01}^2) = 1$, $\bar{\iota}\mathrm{V}(P)$ *is the identity matrix so that*

$$K = \bar{\iota}(\mathrm{D}(P)\mathrm{V}(P)^{-1}) = \begin{pmatrix} 1 & 0 & \bar{\iota}v \\ 0 & 1 & \bar{\iota}w \end{pmatrix}.$$

On the other hand the normalized invariants and invariant derivations are

$$\bar{\iota}u_{ij} = \frac{u_{ij}}{(u_{10}^2 + u_{01}^2)^{\frac{i+j}{2}}}, \; \forall i,j; \qquad \mathcal{D}_i = \frac{1}{\sqrt{u_{10}^2 + u_{01}^2}}\mathrm{D}_i, \; i = 1, 2.$$

We can thus check that

$$\begin{pmatrix} \mathcal{D}_1(\bar\iota u_{ij}) \\ \mathcal{D}_2(\bar\iota u_{ij}) \end{pmatrix} = \begin{pmatrix} \bar\iota(u_{i+1,j}) \\ \bar\iota(u_{i,j+1}) \end{pmatrix} - K \begin{pmatrix} 0 \\ 0 \\ -(i+j)\,\bar\iota u_{ij} \end{pmatrix},$$

as predicted by Theorem 3.6, and that $[\mathcal{D}_2, \mathcal{D}_1] = \bar\iota w\, \mathcal{D}_1 - \bar\iota v\, \mathcal{D}_2$*, as predicted by Theorem 3.8.*

4 Finite generation and rewriting

The recurrence formulae, Theorem 3.6, together with the replacement theorem, Theorem 2.11, show that any differential invariant can be written in terms of the normalized invariants of order $s+1$, where s is the order of the moving frame, and their invariant derivatives. The rewriting is effective.

In the case of a cross-section of minimal order, we exhibit another generating set of differential invariants with bounded cardinality. This bound is mr in the case of action transitive on J^0. When in addition we choose a coordinate cross-section, this set consists of normalized invariants and we retrieve the result of Olver (2007b). This was incorrectly stated for any cross-section by Fels and Olver (1999, Theorem 13.3).

4.1 Rewriting in terms of normalized invariants of order $s+1$

Let s be greater or equal to the stabilization order and let $\mathcal{P}$ be a cross-section to the orbits in J^s defined by $P = (p_1, \ldots, p_r)$ with $p_i \in \mathcal{F}(\mathrm{J}^s)$. Recall from Section 2.3 that

$$\mathcal{I}^{s+k} = \{\bar\iota x_1, \ldots, \bar\iota x_m\} \cup \{\bar\iota u_\alpha \,|\, u \in \mathcal{U},\ |\alpha| \leq s+k\},$$

where $\bar\iota : \mathcal{F}(\mathrm{J}^{s+k}) \to \mathcal{F}^{\mathcal{G}}(\mathrm{J}^{s+k})$ is the invariantization associated to $\mathcal{P}$, forms a generating set of local invariants for the action of g on J^{s+k}. Those invariants have additional very desirable properties: we can trivially rewrite any differential invariants of order $s+k$ in terms of them. Yet it is even more desirable to describe the differential invariants of all order in finite terms.

Theorem 3.6 implies in particular that

$$\bar\iota(\mathrm{D}_i u_\alpha) = \mathcal{D}_i(\bar\iota u_\alpha) + \sum_{a=1}^{r} K_{ia}\, \bar\iota(\mathrm{V}_a(u_\alpha))$$

where $K = \bar\iota(\mathrm{D}(P)\mathrm{V}(P)^{-1})$ has entries that are function of $\mathcal{I}^{s+1}$. It is then an easy inductive argument to show that any $\bar\iota u_\alpha$ can be written as a function of

$\mathcal{I}^{s+1}$ and their derivatives of order $\max(0, |\alpha| - s - 1)$. Combining with the replacement property, Theorem 2.11, we have a constructive way of rewriting any differential invariants in terms of the elements of $\mathcal{I}^{s+1}$ and their derivatives: A differential invariant of order k is first trivially rewritten in terms of $\mathcal{I}^k$ by Theorem 2.11. If $k \leq s+1$ we are done. Otherwise, any element $\bar{\iota}u_\alpha$ of $\mathcal{I}^k$ with $|\alpha| = k$ is a $\bar{\iota}(\mathrm{D}_i u_\beta)$, for some $1 \leq i \leq m$ and $|\beta| = k-1$. We can thus write it as:

$$\bar{\iota}u_\alpha = \bar{\iota}(\mathrm{D}_i u_\beta) = \mathcal{D}_i(\bar{\iota}u_\beta) + \sum_a K_{ia}\,\bar{\iota}\,(\mathrm{V}(u_\beta))\,.$$

This involves only elements of $\mathcal{I}^{k-1}$ and their derivatives. Carrying on recursively we can rewrite everything in terms of the elements of $\mathcal{I}^{s+1}$ and their derivatives.

This leads to the following result that will be refined in Section 5. Indeed the rewriting is not unique: at each step there might be several choices of pairs (i, β) such that $u_\alpha = \mathrm{D}_i u_\beta$.

Theorem 4.1 *Any differential invariant of order $s+k$ can be written in terms of the elements of $\mathcal{I}^{s+1}$ and their derivatives of order k or less.*

4.2 Case of minimal order cross-section

A natural question is to determine a smaller set of differential invariants that is generating. Olver (2007b) proved that when choosing a coordinate cross-section of *minimal order* the normalized invariants corresponding to the derivatives of the coordinates functions which are set to constant form a generating set of differential invariants. Here we generalize the result to non coordinate cross-sections. The proof is based on the same idea.

Let s be equal or greater than the stabilization order. A local cross-section $\mathcal{P}$ in J^s is of *minimal order* if its projection on J^k, for all $k \leq s$, is a local cross-section to the orbits of the action of g on J^k (Olver, 2007b). Assume $P = (p_1, \ldots, p_r)$ defines a cross-section $\mathcal{P}$ of minimal order. Without loss of generality we can assume that $P_k = (p_1, \ldots, p_{r_k})$ where r_k is the dimension of the orbits of the action of g on J^k, defines the projection of $\mathcal{P}$ on J^k.

Theorem 4.2 *If $P = (p_1, \ldots, p_r)$ defines a cross-section for the action of g on J such that $P_k = (p_1, \ldots, p_{r_k})$ defines a cross-section for the action of g on J^k, for all k, then $\mathcal{E} = \{\bar{\iota}(\mathrm{D}_i(p_j)) \,|\, 1 \leq i \leq m,\ 1 \leq j \leq r\}$ together with $\mathcal{I}^0$ form a generating set of differential invariants.*

PROOF: The minimal order condition imposes that the $r \times r_k$ matrix $\mathrm{V}(P_k)$ has maximal rank r_k on $\mathcal{P}$, and therefore on an open neighborhood. As V^k has rank r_k, for any f in $\mathcal{F}(\mathrm{J}^k)$, $\mathrm{V}(f)$ is linearly dependent on $\mathrm{V}(p_1), \ldots, \mathrm{V}(p_{r_k})$.

On a neighborhood of $\mathcal{P}^k$ there is thus a relation

$$\mathrm{V}(f) = \sum_{i=1}^{r_k} a_i(f)\,\mathrm{V}(p_i), \text{ where } a_i \in \mathcal{F}(\mathrm{J}^k).$$

On one hand, by Theorem 3.6, we have $\bar\iota(\mathrm{D}f) = \mathcal{D}(\bar\iota f) + K\,\bar\iota(V(f))$ so that $\bar\iota(\mathrm{D}f) = \mathcal{D}(\bar\iota f) + \sum_{i=1}^{r_k} \bar\iota(a_i)\,K\,\bar\iota(\mathrm{V}(p_i))$. On the other hand $\bar\iota(p_i) = 0$ so that $\bar\iota(\mathrm{D}p_i) = K\bar\iota(\mathrm{V}(p_i))$ It follows that

$$\bar\iota(\mathrm{D}f) = \mathcal{D}(\bar\iota f) + \sum_{i=1}^{r_k} \bar\iota(a_i)\,\bar\iota(\mathrm{D}p_i).$$

Note that $\bar\iota(a_i)$ can be written in terms of the $\bar\iota(u_\beta)$ with $|\beta| \leq k$. So the formula implies that any $\bar\iota u_\alpha$, with $|\alpha| = k+1$, can be written in terms of $\{\,\bar\iota(\mathrm{D}p_i) \mid 1 \leq i \leq r_k\,\}$ and $\{\,\bar\iota(u_\beta) \mid |\beta| \leq k\,\}$ together with their derivatives with respect to the invariant derivations $\mathcal{D}$. By induction, it follows that any $\bar\iota u_\alpha$ can be written in terms of the zero-th order normalized invariants together with the elements of $\mathcal{E}$ and their derivatives. □

In the case of coordinate cross-section $\mathcal{E}$ is a subset of normalized invariants $\mathcal{I}^{s+1}$ that Olver (2007b) named the *edge invariants* for the representation of the derivatives of a dependent function on a lattice. We shall extend this name in the case of non coordinate cross-section though the pictorial representation is no longer valid.

Minimality is necessary for the edge invariants to be generating in general. Olver (2007a) exhibits a choice of non minimal (coordinate) cross-section for which the edge invariants are not generating. We review this example in Section 6.1.

A consequence of Theorem 4.2 is that we can bound the number of differential invariants necessary to form a generating set. The bound is $m\,r + d_0$, where $d_0 = m + n - r_0$ is the codimension of the orbits of the action of g on J^0. Transitive actions on J^0 are of particular interest. There $d_0 = 0$ and the bound is simply $m\,r$. Hubert (2007a) exhibits a generating set of such cardinality even in the case of non minimal cross-section.

Example 4.3 *Consider Example 3.10 again. The chosen cross-section is of minimal order. Specializing Theorem 3.6 we obtained*

$$\begin{pmatrix} \bar\iota(u_{i+1,j}) \\ \bar\iota(u_{i,j+1}) \end{pmatrix} = \begin{pmatrix} \mathcal{D}_1(\bar\iota u_{ij}) \\ \mathcal{D}_2(\bar\iota u_{ij}) \end{pmatrix} - (i+j)\,\bar\iota u_{ij} \begin{pmatrix} \bar\iota v \\ \bar\iota w \end{pmatrix}$$

from which it is clear that all the normalized invariants can be inductively written in terms of $\bar\iota u_{00}, \bar\iota v$ and $\bar\iota w$, i.e the non constant elements of $\mathcal{I}^0 \cup \mathcal{E}$, and their derivatives.

5 Syzygies

Loosely speaking, a *differential syzygy* is a relationship among a (generating) set of differential invariants and their derivatives. A set of differential syzygies is complete if any other syzygyies is inferred by those and their derivatives. In this section we formalize a definition of syzygies by introducing the appropriate differential algebra. We then show the completeness of a finite set of differential syzygies on the normalized invariants of order $s+1$.

Fels and Olver (1999, Theorem 13.2) claimed a complete set of syzygies for edge invariants, in the case of coordinate cross-section. It has so far remained unproven [2] . As we are finishing this paper Olver and Pohjanpelto (2007) announce a syzygy theorem for pseudo-groups. The *symbol module* of the infinitesimal determining system takes there a prominent place: on one hand it dictates the coordinate cross-section to be used and, on the other hand, its (algebraic) syzygies prescribe the syzygies on the differential invariants. Let us note here two immediate advantages of our result for Lie group actions: we do not need to have any side algebraic computations (over a ring of functions) nor are we restricted in our choice of cross-section In particular we are neither restricted to minimal order nor coordinate cross-section.

The commutation rules, Theorem 3.8, imply infinitely many relationships on derivatives of normalized invariants. Fels and Olver (1999), as well as Olver and Pohjanpelto (2007), considered those as syzygies. Our approach is in the line of Hubert (2005b). We encapsulate those relationships in a recursive definition of the derivations and work exclusively with *monotone derivatives.* [3] The differential algebra of monotone derivatives that arises there is a generalization of the classical differential algebras considered by Ritt (1950) and Kolchin (1973). Of great importance is the fact that it is endowed with a proper differential elimination theory (Hubert, 2005b). This generalization is effective and has been implemented (Hubert, 2005a).

Refining the discussion of Section 4, we first observe that any differential invariant can be written in terms of the monotone derivatives of the normalized invariants of order $s+1$. The rewriting is nonetheless not unique in general. The syzygies can be understood as the relationships among the monotone derivatives that govern this indeterminacy.

For the normalized invariants of order $s+1$ we introduce the concept of *normal derivatives*. They provide a canonical rewriting of any differential invariant. The set of relationships that allows one to rewrite any monotone derivative in terms of normal derivatives is then a complete set of differential syzygies for

[2] An necessary amendment of the statement is that K might be taken as the empty set in (iii).

[3] The use of non monotone derivations leads to surprising results regarding the generation of differential invariants (Olver, 2007a; Hubert and Olver, 2007).

the normalized invariants of order $s+1$ (Theorem 5.13).

To prove these results we formalize the notion of syzygies by introducing the algebra of monotone derivatives. We endow this algebra with derivations so as to have a differential morphism on the algebra of differential invariants. The syzygies are the elements of the kernel of this morphism. It is a differential ideal and Theorem 5.13 actually exhibits a set of generators.

5.1 Monotone and normal derivatives

In Section 4 we showed that any differential invariant can be written in terms of $\mathcal{I}^{s+1}$ and its derivatives. However, this rewriting is not unique. We can actually restrict the derivatives to be used in this rewriting, first to *monotone derivatives*, then to *normal derivatives*. Normal derivatives provide a canonical rewriting.

Definition 5.1 *A derivation operator $\mathcal{D}_{j_1} \ldots \mathcal{D}_{j_k}$ is monotone if $j_1 \leq \ldots \leq j_k$. The monotone derivation operators are noted $\mathcal{D}^\alpha$ where $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$ and α_i is the cardinality of $\{j_l \mid j_l = i\}$.*

There is an inductive process to rewrite any normalized invariants, and therefore any differential invariants, in terms of the monotone derivatives of $\mathcal{I}^{s+1}$. For the inductive rewriting of $\bar{\iota} u_\beta$, for $|\beta| > s+1$, in terms of the monotone derivatives of $\mathcal{I}^{s+1}$ we can proceed as follows: split β in $\beta = \hat{\beta} + \bar{\beta}$ where $|\bar{\beta}| = s+1$ and then rewrite $\bar{\iota} u_\beta - \mathcal{D}^{\hat{\beta}}(\bar{\iota} u_{\bar{\beta}})$ which is of lower order. There might be several inequivalent ways to split β, each leading to a different rewriting. The following definition imposes a single choice of splitting.

Notation 5.2 *For $\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{N}^m$, we denote*

$$\bar{\beta} = \begin{cases} \beta \text{ if } |\beta| \leq s+1 \\ (0, \ldots, 0, \beta'_i, \beta_{i+1}, \ldots, \beta_m) \text{ otherwise} \\ \quad \text{with } i = \max\{j \mid \beta_j + \ldots + \beta_m \geq s+1\} \\ \quad \text{and } \beta'_i = (s+1) - \beta_{i+1} - \ldots - \beta_m \end{cases}$$

and $\hat{\beta} = \beta - \bar{\beta}$.

For $\beta \in \mathbb{N}^m$, $|\beta| > 0$, we define $f(\beta)$ and $l(\beta)$ respectively as the first and last non zero component of β, i.e.

$$f(\beta) = \min\{j \,|\, \beta_j \neq 0\} \quad \text{and} \quad l(\beta) = \max\{j \,|\, \beta_j \neq 0\}.$$

With those notations, $|\bar{\beta}|$ is always less or equal to $s+1$ and $\hat{\beta} = 0$ when $|\beta| \leq s+1$. Furthermore $l(\hat{\beta}) \leq f(\bar{\beta})$ for all $\beta \neq 0$.

Definition 5.3 *The normal derivatives of $\mathcal{I}^{s+1}$ are the elements of the set*

$$\mathcal{N} = \mathcal{I}^{s+1} \cup \left\{ \mathcal{D}^{\hat{\beta}}(\bar{\iota} u_{\bar{\beta}}) \mid \beta \in \mathbb{N}^m,\ |\beta| > s+1 \right\}.$$

The set $\mathcal{N}^k$ of the normal derivatives of order k is the subset thereof with $|\hat{\beta}| \leq k$.

Proposition 5.4 *Any differential invariant is a function of the normal derivatives $\mathcal{N}$ of $\mathcal{I}^{s+1}$.*

This result follows from an easy inductive argument on the following lemma.

Lemma 5.5 *For all $\beta \in \mathbb{N}^m$, $\beta \neq 0$, $\bar{\iota} u_\beta - \mathcal{D}^{\hat{\beta}}(\bar{\iota} u_{\bar{\beta}}) \in \mathcal{F}^{\mathcal{G}}(\mathrm{J}^{|\beta|-1})$.*

PROOF: This is trivially true for $|\beta| \leq s+1$ since then $\hat{\beta} = (0, \ldots, 0)$. We proceed by induction for $|\beta| > s+1$.

Assume the statement is true for all β with $s+1 \leq |\beta| \leq k$. Take β with $|\beta| = k+1$. Let $i = f(\beta)$ and $\beta' = \beta - \epsilon_i$. We have $\bar{\beta}' = \bar{\beta}$, $\hat{\beta}' = \hat{\beta} - \epsilon_i$ and $\mathcal{D}^{\hat{\beta}} = \mathcal{D}_i \mathcal{D}^{\hat{\beta}'}$ so that $\bar{\iota} u_\beta - \mathcal{D}^{\hat{\beta}}(\bar{\iota} u_{\bar{\beta}}) = \bar{\iota}(\mathrm{D}_i(u_{\beta'})) - \mathcal{D}_i \mathcal{D}^{\hat{\beta}'}(\bar{\iota} u_{\bar{\beta}'})$. Thus, by Theorem 3.6,

$$\bar{\iota} u_\beta - \mathcal{D}^{\hat{\beta}}(\bar{\iota} u_{\bar{\beta}}) = \mathcal{D}_i \left(\bar{\iota} u_{\beta'} - \mathcal{D}^{\hat{\beta}'}(u_{\bar{\beta}'}) \right) + \sum_{a=1}^{r} K_{ia}\, \bar{\iota}\left(\mathrm{V}_a(u_{\beta'})\right).$$

The entries of K are functions of $\mathcal{I}^{s+1}$, while the entries of $\bar{\iota}\left(\mathrm{V}(u_{\beta'})\right)$ are functions of $\mathcal{I}^k$. By induction hypothesis $\mathcal{D}^{\hat{\beta}'}(\bar{\iota} u_{\bar{\beta}'}) - \bar{\iota} u_{\beta'} \in \mathcal{F}^{\mathcal{G}}(\mathrm{J}^{k-1})$ and thus $\mathcal{D}_i \left(\mathcal{D}^{\hat{\beta}'}(u_{\bar{\beta}'}) - \bar{\iota} u_{\beta'} \right) \in \mathcal{F}^{\mathcal{G}}(\mathrm{J}^k)$. $\square$

Following the induction on Lemma 5.5, rewriting any $\bar{\iota} u_\beta$ in terms of the normal derivatives is an effective process. That the rewriting is unique, modulo P, is expressed as follows.

Proposition 5.6 *Assume $P = (p_1, \ldots, p_r)$ are the r independent functions of $\mathcal{F}(\mathrm{J}^s)$ that cut out the cross-section $\mathcal{P}^s$ to the orbits on J^s. Let $F \in \mathcal{F}(\mathrm{J}^{s+k})$ be a function such that $F(\bar{\iota} x, \mathcal{D}^{\hat{\beta}}(\bar{\iota} u_{\bar{\beta}})) = 0$. Then there exist $a_1, \ldots, a_r \in \mathcal{F}(\mathrm{J}^{s+k})$ such that $F = \sum_{i=1}^r a_i\, p_i$ on an open set that contains $\mathcal{P}$.*

PROOF: By Lemma 5.5, for $|\beta| \leq s+k$, there exists ϕ_β in $\mathcal{F}(\mathrm{J}^{|\beta|-1})$ such that $\mathcal{D}^{\hat{\beta}}(\bar{\iota} u_{\bar{\beta}}) - \bar{\iota} u_\beta = \bar{\iota} \phi_\beta$. The map $\psi : \mathrm{J}^{s+k} \to \mathrm{J}^{s+k}$ given by $\psi^* u_\beta = u_\beta + \phi_\beta$ is a diffeomorphism so that ψ^* is an automorphism of $\mathcal{F}(\mathrm{J}^{s+k})$. In particular, the restriction of ψ to J^{s+1} is the identity so that $\psi^* p_i = p_i$.

Note that for any $F \in \mathcal{F}(\mathrm{J}^{s+k})$ there is a $f \in \mathcal{F}(\mathrm{J}^{s+k})$ such that $F(\bar{\iota} x, \mathcal{D}^{\hat{\beta}} \bar{\iota} u_{\bar{\beta}}) = \psi^* f(\bar{\iota} x, \bar{\iota} u_\beta)$ and $F\left(\bar{\iota} x, \mathcal{D}^{\hat{\beta}}(\bar{\iota} u_{\bar{\beta}})\right) = 0$ implies that $f(\bar{\iota} x, \bar{\iota} u_\beta) = 0$. By Proposition 2.6 this implies that there exist $b_1, \ldots, b_r \in \mathcal{F}(\mathrm{J}^{s+k})$ such that $f = \sum_{i=1}^r b_i\, p_i$ on an open set that contains $\mathcal{P}^{s+k}$. Let $a_i = \psi^* b_i$. We have $F = \sum_{i=1}^r a_i\, p_i$. $\square$

We introduce the algebra of the smooth functions of the monotone derivatives of $\mathcal{I}^{s+1}$. We endow this algebra with derivations $\mathfrak{D}_1, \ldots, \mathfrak{D}_m$ in order to ensure a differential morphism onto $\mathcal{F}^{\mathcal{G}}(\mathrm{J})$.

We define first a sequence $(\mathfrak{A}^k)_k$ of manifolds[4] that correspond to the spaces of the normal derivatives of $\mathcal{I}^{s+1}$ of order k. $\mathfrak{A}^0$ is isomorphic to J^{s+1} and therefore of dimension $N = m + n\binom{m+s+1}{s+1}$. The coordinate function on $\mathfrak{A}^0$ are noted $\{\mathfrak{x}_1^0, \ldots, \mathfrak{x}_m^0\} \cup \{\mathfrak{u}_\alpha^0 \,|\, |\alpha| \leq s+1\}$. Then, for each k, $\mathfrak{A}^k$ is a submanifold of $\mathfrak{A}^{k+1}$ and $\mathfrak{A}^k$ is of dimension $N\binom{k+m}{m}$. A coordinate system is given by $\{\mathfrak{x}^\beta \,|\, |\beta| \leq k\} \cup \{\mathfrak{u}_\alpha^\beta \,|\, |\beta| \leq k,\, |\alpha| \leq s+1\}$. We actually focus on the algebras of smooth functions $\mathcal{F}(\mathfrak{A}^k)$ and $\mathcal{F}(\mathfrak{A})$, where $\mathfrak{A} = \bigcup_{k\geq 0} \mathfrak{A}^k$.

Proposition 5.7 *The ring morphism $\phi : \mathcal{F}(\mathfrak{A}^k) \to \mathcal{F}^{\mathcal{G}}(\mathrm{J}^{s+k+1})$ defined by*

$$\phi(\mathfrak{x}^\alpha) = \mathcal{D}^\alpha(\bar{\iota}x) \quad \text{and} \quad \phi(\mathfrak{u}_\beta^\alpha) = \mathcal{D}^\alpha(\bar{\iota}u_\beta), \text{ for all } \alpha \in \mathbb{N}^m \text{ and } |\beta| \leq s+1,$$

is surjective.

This is nothing else than the statement that any differential invariants can be written in terms of the monotone derivatives of $\mathcal{I}^{s+1}$ (Lemma 5.4). Yet another equivalent statement is the following:

Proposition 5.8 *There exists a ring morphism $\psi : \mathcal{F}(\mathrm{J}^{s+1+k}) \to \mathcal{F}(\mathfrak{A}^k)$ such that $\phi \circ \psi(u_\alpha) = \bar{\iota}u_\alpha$.*

In other words, $\psi(u_\alpha)$ is a function that allows one to rewrite $\bar{\iota}u_\alpha$ in terms of the monotone derivatives of $\mathcal{I}^{s+1}$. In particular we can choose ψ such that $\psi(u_\alpha) = \mathfrak{u}_\alpha^0$, for $|\alpha| \leq s+1$.

We proceed now to define on $\mathcal{F}(\mathfrak{A})$ the derivations $\mathfrak{D}_1, \ldots, \mathfrak{D}_m$ that will turn ϕ into a differential morphism.

Definition 5.9 *Let $c_{ijk} = \psi(\Lambda_{ijl}) \in \mathcal{F}(\mathfrak{A}^1)$, for all $1 \leq i,j,l \leq m$, where ψ is the morphism of Proposition 5.8 and $\{\Lambda_{ijl}\}_{1\leq i,j,l\leq m}$ are the commutator invariants defined in Proposition 3.8. Define the derivations $\mathfrak{D}_1, \ldots, \mathfrak{D}_m$ from $\mathcal{F}(\mathfrak{A}^k)$ to $\mathcal{F}(\mathfrak{A}^{k+1})$ by the following inductive process:*

$$\mathfrak{D}_i(\mathfrak{z}^\beta) = \begin{cases} \mathfrak{z}^{\beta+\epsilon_i}, & \text{if } i \leq f(\beta) \\ \mathfrak{D}_f \mathfrak{D}_i(\mathfrak{z}^{\beta-\epsilon_f}) + \sum_{l=1}^m c_{ifl}\, \mathfrak{D}_l(\mathfrak{z}^{\beta-\epsilon_f}), & \text{where } f = f(\beta), \text{ otherwise,} \end{cases}$$

where $\mathfrak{z} \in \{\mathfrak{x}_1, \ldots, \mathfrak{x}_m\} \cup \{\mathfrak{u}_\alpha \,|\, |\alpha| \leq s+1\}$.

[4] We shall simply think of them as open subsets of $\mathbb{R}^k$ for the right k.

Taking the notation $\mathfrak{D}^\alpha = \mathfrak{D}_1^{\alpha_1} \dots \mathfrak{D}_m^{\alpha_m}$ of Definition 5.1 we have $\mathfrak{D}^\alpha(\mathfrak{z}^0) = \mathfrak{z}^\alpha$ but in general $\mathfrak{D}^\alpha(\mathfrak{z}^\beta) \neq \mathfrak{z}^{\alpha+\beta}$, unless $l(\alpha) \leq f(\beta)$. We nonetheless have the following property, that is expected for a differential elimination theory, and that here allows to show that ϕ is a differential morphism. The proofs of the two next results are reasonably straightforward inductions exploiting the definition of the derivations.

Lemma 5.10 *$\mathfrak{D}^\alpha(\mathfrak{z}^\beta) - \mathfrak{z}^{\alpha+\beta} \in \mathcal{F}(\mathfrak{A}^{|\alpha+\beta|-1})$, for any $\mathfrak{z} \in \{\mathfrak{x}_1, \dots, \mathfrak{x}_m\} \cup \{\mathfrak{u}_\alpha \,|\, |\alpha| \leq s+1\}$.*

PROOF: By definition of the derivations $\mathfrak{D}$, this is true whenever α or β is zero and when $l(\alpha) \leq f(\beta)$. It is in particular true when $l(\alpha) = 1$ or $f(\beta) = m$. The result is then proved by induction along the well-founded pre-order:

$$(\alpha', \beta') \prec (\alpha, \beta) \;\Leftrightarrow\; \begin{cases} \beta' \prec_f \beta \text{ or} \\ f(\beta') = f(\beta) = f \text{ and } \beta'_f = \beta_f \text{ and } \alpha' \prec_l \alpha \end{cases}$$

where

$$\beta' \prec_f \beta \;\Leftrightarrow\; \begin{cases} f(\beta') > f(\beta) \text{ or} \\ f(\beta') = f(\beta) = f \text{ and } \beta'_f < \beta_f \end{cases}$$

and

$$\alpha' \prec_l \alpha \;\Leftrightarrow\; \begin{cases} l(\alpha') < l(\alpha) \text{ or} \\ l(\alpha') = l(\alpha) = l \text{ and } \alpha'_l < \alpha_l. \end{cases}$$

Assume the result is true for all $(\alpha', \beta') \prec (\alpha, \beta)$. We only need to scrutinize the case $l = l(\alpha) > f(\beta) = f$. By definition of $\mathfrak{D}$ then:

$$\mathfrak{D}^\alpha(\mathfrak{z}^\beta) = \mathfrak{D}^{\alpha-\epsilon_l}\left(\mathfrak{D}_f \mathfrak{D}_l(\mathfrak{z}^{\beta-\epsilon_f})\right) + \sum_k c_{lfk} \mathfrak{D}_k(\mathfrak{z}^{\beta-\epsilon_f}).$$

We have $\beta - \epsilon_f \prec_f \beta$ and thus, by induction hypothesis, $\mathfrak{D}_k(\mathfrak{z}^{\beta-\epsilon_f}) = \mathfrak{z}^{\beta-\epsilon_f+\epsilon_k} + F$ where $F \in \mathcal{F}(\mathfrak{A}^{|\beta|})$, for all k, and in particular for $k = l$. We apply then the induction hypothesis on $\mathfrak{D}_f(\mathfrak{z}^{\beta-\epsilon_f+\epsilon_l})$ and on $\mathfrak{D}^{\alpha-\epsilon_l}(\mathfrak{z}^{\beta+\epsilon_l})$, observing that $\beta - \epsilon_f + \epsilon_l \prec_f \beta$ while $\alpha - \epsilon_l \prec_l \alpha$. □

Proposition 5.11 *The map $\phi : \mathcal{F}(\mathfrak{A}) \to \mathcal{F}^{\mathcal{G}}(\mathrm{J})$ of Proposition 5.7 is a morphism of differential algebras i.e. $\phi \circ \mathfrak{D}_i = \mathcal{D}_i \circ \phi$, for all $1 \leq i \leq m$.*

PROOF: We need to prove that

$$H(i, \alpha): \quad \phi(\mathfrak{D}_i(\mathfrak{z}^\alpha)) = \mathcal{D}_i(\phi(\mathfrak{z}^\alpha))$$

for all $\alpha \in \mathbb{N}^m$. If this is true for all $|\alpha| \leq k$ then $\phi(\mathfrak{D}_i(F)) = \mathcal{D}_i(\phi(F))$ for all

$F \in \mathcal{F}(\mathfrak{A}^k)$. The proof is an induction along the well founded pre-order:

$$(j,\beta) \prec (i,\alpha) \;\Leftrightarrow\; \begin{cases} |\beta| < |\alpha| \text{ or} \\ |\beta| = |\alpha| \text{ and } j < i. \end{cases}$$

$H(i,\alpha)$ is trivially true when α is zero or when $i \leq f(\alpha)$. It is therefore true whenever $i = 1$.

Assume $H(j,\beta)$ holds for any $(j,\beta) \prec (i,\alpha)$. Only the case $i > f(\alpha) = f$ needs scrutiny. We have $\mathfrak{D}_i(\mathfrak{z}^\alpha) = \mathfrak{D}_f\left(\mathfrak{D}_i(\mathfrak{z}^{\alpha-\epsilon_f})\right) + \sum_k c_{ifk}\mathfrak{D}_k(\mathfrak{z}^{\alpha-\epsilon_f})$. Since $\mathfrak{D}_i(\mathfrak{z}^{\alpha-\epsilon_f}) \in \mathcal{F}(\mathfrak{A}^{|\alpha|})$ while $f < i$, the induction hypothesis implies that $\phi\left(\mathfrak{D}_f\left(\mathfrak{D}_i(\mathfrak{z}^{\alpha-\epsilon_f})\right)\right) = \mathcal{D}_f\left(\phi\left(\mathfrak{D}_i(\mathfrak{z}^{\alpha-\epsilon_f})\right)\right)$. And since $|\alpha-\epsilon_f| < |\alpha|$, $\phi\left(\mathfrak{D}_k(\mathfrak{z}^{\alpha-\epsilon_f})\right) = \mathcal{D}_k(\phi\left(\mathfrak{z}^{\alpha-\epsilon_f}\right))$, for any k an in particular for $k = i$. Therefore

$$\phi\left(\mathfrak{D}_i(\mathfrak{z}^\alpha)\right) = \mathcal{D}_f\mathcal{D}_i\left(\phi(\mathfrak{z}^\alpha)\right) + \sum_k \Lambda_{ifk}\mathcal{D}_k\left(\phi(\mathfrak{z}^{\alpha-\epsilon_f})\right).$$

This is equal to $\mathcal{D}_i\left(\phi(\mathfrak{z}^\alpha)\right)$ by Proposition 3.8. □

5.3 Complete set of syzygies

As an immediate consequence of Theorem 3.6, the following differential relationships hold among the first order derivatives of $\mathcal{I}^{s+1}$:

$$\begin{aligned}
\mathcal{D}_i(\bar\iota x_j) &= \delta_{ij} - \textstyle\sum_{a=1}^r K_{ia}\bar\iota\left(\mathrm{V}_a(x_j)\right) && 1 \leq i, j, \leq m \\
\mathcal{D}_i(\bar\iota u_\alpha) &= \bar\iota u_{\alpha+\epsilon_i} - \textstyle\sum_{a=1}^r K_{ia}\bar\iota\left(\mathrm{V}_a(u_\alpha)\right), && |\alpha| \leq s \\
\mathcal{D}_i(\bar\iota u_\alpha) - \mathcal{D}_j(\bar\iota u_\beta) &= \sum_{a=1}^r K_{ja}\bar\iota\left(\mathrm{V}_a(u_\beta)\right) - K_{ia}\bar\iota\left(\mathrm{V}(u_\alpha)\right), && \alpha+\epsilon_i = \beta+\epsilon_j,\ |\alpha| = |\beta| = s+1
\end{aligned}$$

In this section we show that a subset of those relationships is a complete set of differential syzygies for $\mathcal{I}^{s+1}$. The subset is obtained by restricting the range of (i,j) for the third type of relationships which bears on $\mathcal{I}^{s+1} \setminus \mathcal{I}^s$. We use the setting introduced in the previous subsection to formalize and prove those results.

Definition 5.12 *Let $\phi : \mathcal{F}(\mathfrak{A}^k) \to \mathcal{F}^{\mathcal{G}}(\mathrm{J}^{s+k+1})$ be as in Proposition 5.7. An element of $\mathcal{F}(\mathfrak{A}^k)$ is a* (differential) syzygy *on the monotone derivatives of $\mathcal{I}^{s+1}$ if its image by ϕ is zero on the cross-section in J^k.*

Since differential invariants are locally determined by their restriction to the cross-section, this is the same as requesting that the image is zero on an open set that contains the cross-section. Furthermore, by Proposition 5.11, the set of syzygies is a differential ideal: if f is a syzygy then so is $\mathfrak{D}_i(f)$, for all $1 \leq i \leq m$.

Theorem 5.13 *Define the following sets of functions in* $\mathcal{F}(\mathfrak{A}^0)$ *and* $\mathcal{F}(\mathfrak{A}^1)$.

- $\mathfrak{R} = \{p(\mathfrak{x}^0, \mathfrak{u}^0_\alpha) \,|\, p \in P\}$
- $\mathfrak{S} = \{S^i_{x_j} \,|\, 1 \leq i, j \leq m\} \cup \{S^i_{u_\alpha} \,|\, |\alpha| \leq s,\ 1 \leq i \leq m\}$ *where*

$$S^i_{x_j} = \mathfrak{x}^{\epsilon_i}_j - \delta_{ij} - \sum_{a=1}^{r} \psi\left(K_{ia}\mathrm{V}_a(x_j)\right)$$

and

$$S^i_{u_\alpha} = \mathfrak{u}^{\epsilon_i}_\alpha - \mathfrak{u}^0_{\alpha+\epsilon_i} - \sum_{a=1}^{r} \psi\left(K_{ia}\mathrm{V}_a(u_\alpha)\right)$$

- $\mathfrak{T} = \{T^i_{u_\beta} \,|\, |\beta| = s+1$ *and* $f(\beta) < i \leq m\}$ *where, with* $f = f(\beta)$ *and* $\alpha = \beta + \epsilon_i - \epsilon_f$,

$$T^i_{u_\beta} = \mathfrak{u}^{\epsilon_i}_\beta - \mathfrak{u}^{\epsilon_f}_\alpha - \sum_{a=1}^{r} \psi\left(K_{ia}\mathrm{V}_a(u_\alpha) - K_{fa}\mathrm{V}_a(u_\beta)\right)$$

An element of the kernel of $\phi : \mathcal{F}(\mathfrak{A}) \to \mathcal{F}^{\mathcal{G}}(\mathrm{J})$ *is, locally, an element of the ideal generated by* $\mathfrak{R} \cup \mathfrak{S} \cup \mathfrak{T}$.

The result is deduced from the following lemma. It shows that any monotone derivative of $\mathcal{I}^{s+1}$ can be rewritten in terms of the normal derivatives modulo $\mathfrak{S} \cup \mathfrak{T}$.

Lemma 5.14 *For any* $\alpha \in \mathbb{N}^m$ *and* $|\gamma| \leq s+1$ *there exists a linear operator* $L^\alpha_{u_\gamma}$ *of order* $|\alpha| - 1$ *in* $\mathfrak{D}_1, \ldots, \mathfrak{D}_m$ *such that, for* $\beta = \alpha + \gamma$,

$$\mathfrak{u}^\alpha_\gamma - \mathfrak{u}^{\hat{\beta}}_{\bar{\beta}} - L^\alpha_{u_\gamma}(\mathfrak{S}, \mathfrak{T}) \in \mathcal{F}(\mathfrak{A}^{|\alpha|-1}).$$

PROOF: We consider first the case where $|\gamma| = s+1$ and prove that there exists a homogeneous linear operator $H^\alpha_{u_\gamma}$ of order $|\alpha| - 1$ in $\mathfrak{D}_1, \ldots, \mathfrak{D}_m$ such that $\mathfrak{u}^\alpha_\gamma - \mathfrak{u}^{\hat{\beta}}_{\bar{\beta}} - H^\alpha_{u_\beta}(\mathfrak{T}) \in \mathcal{F}(\mathfrak{A}^{|\beta|-1})$. The proof is by induction along the following well founded pre-order on $\mathbb{N}^m$:

$$\gamma \prec \gamma' \;\Leftrightarrow\; \begin{cases} |\gamma| < |\gamma'| \\ \text{or } |\gamma| = |\gamma'| \text{ and } l(\gamma) < l(\gamma') \\ \text{or } |\gamma| = |\gamma'| \text{ and } l = l(\gamma) = l(\gamma') \text{ and } \gamma_l < \gamma'_l \end{cases}$$

Let $E_\beta = \{\gamma' \mid |\gamma'| = s+1,\ \exists \alpha' \text{ such that } \alpha' + \gamma' = \beta\}$. Note that $\gamma \in E_\beta$ and that $\hat{\beta}$ is the minimal element of E_β according to $\prec$.

If $l(\alpha) \leq f(\gamma)$ then $\hat{\beta} = \alpha$ and $\bar{\beta} = \gamma$ and the result needs no further argument.

Otherwise assume the result is true for all $\gamma' \in E_\beta$ with $\gamma' \prec \gamma$. Let $l = l(\alpha) > f(\gamma) = f$. We have:

$$\begin{aligned}\mathfrak{u}_\gamma^\alpha &= \mathfrak{D}^{\alpha-\epsilon_l}(\mathfrak{u}_\gamma^{\epsilon_l}) \\ &= \mathfrak{D}^{\alpha-\epsilon_l}\left(\mathfrak{u}_{\gamma-\epsilon_f+\epsilon_l}^{\epsilon_f} + T_{u_\gamma}^l + \textstyle\sum_{a=1}^r \psi\left(K_{la}\mathrm{V}(u_{\gamma-\epsilon_f+\epsilon_l}) - K_{fa}\mathrm{V}(u_{\gamma-\epsilon_f+\epsilon_l})\right)\right).\end{aligned}$$

On one hand, the argument of ψ belongs to $\mathcal{F}(\mathrm{J}^{s+1})$ so that its image belongs to $\mathcal{F}(\mathfrak{A}^0)$. On the other hand $\mathfrak{D}^{\alpha-\epsilon_l}\left(\mathfrak{u}_{\gamma-\epsilon_f+\epsilon_l}^{\epsilon_f}\right) - \mathfrak{u}_{\gamma-\epsilon_f+\epsilon_l}^{\alpha+\epsilon_f-\epsilon_l} \in \mathcal{F}(\mathfrak{A}^{|\alpha|-1})$ according to Lemma 5.10. Thus

$$\mathfrak{u}_\gamma^\alpha - \mathfrak{u}_{\gamma-\epsilon_f+\epsilon_l}^{\alpha+\epsilon_f-\epsilon_l} - \mathfrak{D}^{\alpha-\epsilon_l}\left(T_{u_\gamma}^l\right) \in \mathcal{F}(\mathfrak{A}^{|\alpha|-1}).$$

Since $\gamma - \epsilon_f + \epsilon_l \prec \gamma$ we can conclude our induction argument.

We are left to prove that, for all $|\gamma| \leq s$ and $\alpha \in \mathbb{N}^m$, there is a $\mu \in \mathbb{N}^m$ with $|\mu| = s + 1 - |\gamma|$ and a differential operator $L_{u_\gamma}^\alpha$ such that

$$\mathfrak{u}_\gamma^\alpha - \mathfrak{u}_{\gamma+\mu}^{\alpha-\mu} - L_{u_\gamma}^\alpha(\mathfrak{S}) \in \mathcal{F}(\mathfrak{A}^{|\alpha|-1}).$$

For that it is sufficient to lead an inductive argument on the fact that

$$\mathfrak{u}_\gamma^\alpha = \mathfrak{D}^{\alpha-\epsilon_l}\left(\mathfrak{u}_\gamma^{\epsilon_l}\right) = \mathfrak{u}_{\gamma+\epsilon_l}^{\alpha-\epsilon_l} + \mathfrak{D}^{\alpha-\epsilon_l}\left(S_{u_\gamma}^l + \sum_{a=1}^r \psi K_{la}\mathrm{V}(u_\gamma))\right),$$

where $l = l(\alpha)$. □

PROOF: (of the theorem). Taylor's formula with integral remainder shows the following (Bourbaki, 1967, Paragraph 2.5). For a smooth function f on an open set $U \times I_1 \times \ldots \times I_l \subset \mathbb{R}^k \times \mathbb{R}^l$, where the I_i are intervals of $\mathbb{R}$ that contain zero, there are smooth functions f_0 on U, and f_i on $U \times I_1 \times \ldots \times I_i$, $1 \leq i \leq l$ such that $f(x, t_1, \ldots, t_l) = f_0(x) + \sum_{j=1}^l t_j\, f_j(x, t_1, \ldots, t_j)$.

Let us restrict the $\mathfrak{A}^k$ to appropriate neighborhoods of the zero set of $\mathfrak{S}$, $\mathfrak{T}$ and their derivatives. Take $f \in \mathcal{F}(\mathfrak{A}^{k+1})$. By first applying Lemma 5.14 for $|\alpha + \gamma| = k + 1$, we can first write it as:

$$f(\mathfrak{u}_\gamma^\alpha, \mathfrak{u}_{\gamma'}^{\alpha'}) = f_1(\mathfrak{u}_{\bar{\beta}}^{\hat{\beta}}, \mathfrak{u}_{\gamma'}^{\alpha'}) + \sum_{|\alpha+\gamma|=k+1} L_{u_\gamma}^\alpha(\mathfrak{S}, \mathfrak{T})\, F_{u_\gamma}^\alpha$$

where (γ, α) range over $|\alpha + \gamma| = k + 1$ while (γ', α') range over $|\alpha' + \gamma'| \leq k$ and $F_{u_\gamma}^\alpha \in \mathcal{F}(\mathfrak{A}^{k+1})$. We can iterate this process on the $\mathfrak{u}_{\gamma'}^{\alpha'}$, with $|\alpha' + \gamma'| = k$, in f_1. Induction then shows that

$$f(\mathfrak{u}_\gamma^\alpha) = F(\mathfrak{u}_{\bar{\beta}}^{\hat{\beta}}) + \sum_{|\alpha+\gamma|\leq k+1} L_{u_\gamma}^\alpha F_{u_\gamma}^\alpha$$

where now (α, γ) range over $|\alpha + \gamma| \leq k + 1$ and β over $|\beta| \leq k + 1$.

Thus $\phi(f) = \phi(F)$. By Lemma 5.6, if f belongs to the kernel of ϕ then F is a linear combination of elements of $\mathfrak{R}$. □

Example 5.15 *We carry on with Example 3.10 and 4.3.*

The non constant normalized invariants in the generating set $\mathcal{I}^2$ *are* $\{\bar\iota u_{00}, \bar\iota u_{10}, \bar\iota u_{01}, \bar\iota u_{20}, \bar\iota u_{11}, \bar\iota u_{02}\}$. *According to Theorem 5.13 a complete set of syzygies among those consist of the non trivial elements of*

$\mathfrak{R}$, *the functional relationships implied by the choice of the cross-section.*

$$\frac{1}{2} - \frac{1}{2}\left((\mathfrak{u}_{10}^{00})^2 + (\mathfrak{u}_{01}^{00})^2\right)$$

$\mathfrak{S}$, *the relationships describing the derivations of the elements of* $\mathcal{I}^s$*:*

$$\begin{aligned} S^1_{u_{00}} &: \mathfrak{u}_{00}^{10} - \mathfrak{u}_{10}^{00}, & S^2_{u_{00}} &: \mathfrak{u}_{00}^{01} - \mathfrak{u}_{01}^{00}, \\ S^1_{u_{10}} &: \mathfrak{u}_{10}^{10} - \mathfrak{u}_{20}^{00} + \mathfrak{u}_{10}^{00}\,\mathfrak{v}, & S^2_{u_{10}} &: \mathfrak{u}_{10}^{01} - \mathfrak{u}_{11}^{00} + \mathfrak{u}_{10}^{00}\,\mathfrak{w}, \\ S^1_{u_{10}} &: \mathfrak{u}_{01}^{10} - \mathfrak{u}_{11}^{00} + \mathfrak{u}_{01}^{00}\,\mathfrak{v}, & S^2_{u_{10}} &: \mathfrak{u}_{01}^{01} - \mathfrak{u}_{02}^{00} + \mathfrak{u}_{01}^{00}\,\mathfrak{w}, \end{aligned}$$

where

$$\mathfrak{v} = -\mathfrak{u}_{10}^{00}\,\mathfrak{u}_{20}^{00} - \mathfrak{u}_{01}^{00}\,\mathfrak{u}_{11}^{00} = \psi(v), \quad \mathfrak{w} = -\mathfrak{u}_{10}^{00}\,\mathfrak{u}_{11}^{00} - \mathfrak{u}_{01}^{00}\,\mathfrak{u}_{02}^{00}, = \psi(w)$$

recalling that v *and* w *are the two derivatives of* $\frac{1}{2} - \frac{1}{2}(u_{01}^2 + u_{10}^2)$.

$\mathfrak{T}$, *the relationships obtained by cross-differentiating the elements* $\mathcal{I}^{s+1} \setminus \mathcal{I}^s$*:*

$$\begin{aligned} T^2_{u_{20}} &: \mathcal{D}_2(\mathfrak{u}_{20}^{00}) - \mathcal{D}_1(\mathfrak{u}_{11}^{00}) - 2\,\mathfrak{u}_{20}^{00}\,\mathfrak{w} + 2\,\mathfrak{u}_{11}\mathfrak{v} \\ T^2_{u_{11}} &: \mathcal{D}_2(\mathfrak{u}_{11}^{00}) - \mathcal{D}_1(\mathfrak{u}_{02}^{00}) - 2\,\mathfrak{u}_{11}^{00}\,\mathfrak{w} + 2\,\mathfrak{u}_{02}^{00}\,\mathfrak{v}. \end{aligned}$$

Yet we saw that $\{\bar\iota u, \bar\iota v, \bar\iota w\}$ *form a generating set. As* $\bar\iota v$ *and* $\bar\iota w$ *are the coefficients of the commutation rules, we can perform a differential elimination to obtain a complete set of syzygies bearing on* $\{\bar\iota u, \bar\iota v, \bar\iota w\}$ *(Hubert, 2003, 2005b). As can be expected, we obtain:*

$$\mathfrak{D}_1(\mathfrak{w}) - \mathfrak{D}_2(\mathfrak{v}) = 0, \qquad \mathfrak{D}_1(\mathfrak{u})^2 + \mathfrak{D}_2(\mathfrak{u})^2 = 1.$$

6 Examples

We treat two very classical and well known geometries, curves and surfaces in Euclidean 3-space, in order to illustrate the general theory of this paper. With the knowledge of infinitesimal generators of the action and a choice of cross-section only, we can select a set of generators and compute their syzygies. This section is thus meant for understanding. Non trivial applications are presented by Hubert and Olver (2007).

For the benefit of a lighter notation system, we skip the Gothic notation of the formalism introduced in Section 5 when formalizing the notion of syzygies.

Therefore $\bar\iota u_\alpha$ will in turn represent a local invariant, i.e. an element of $\mathcal{F}^{\mathcal{G}}(\mathrm{J})$, or the coordinate function $\mathfrak{u}_\alpha^0$ of $\mathfrak{A}$. In the second example we furthermore use the conventional notation of derivatives so as to make it more familiar.

6.1 Curves in Euclidean geometry

For this example we will first work with a cross-section of minimal order, in which case there is no non trivial syzygies for the generating edge invariants. When we then use a cross-section that is not of minimal order, a non trivial syzygie appears.

We consider the classical action of SE(3) on space curves. We have $\mathrm{J}^0 = \mathcal{X}^1 \times \mathcal{U}^2$ with coordinate (x, u, v). The infinitesimal generators of the action are:

$$\mathrm{V}_1^0 = \frac{\partial}{\partial x}, \qquad \mathrm{V}_2^0 = \frac{\partial}{\partial u}, \qquad \mathrm{V}_3^0 = \frac{\partial}{\partial v}$$
$$\mathrm{V}_4^0 = v\,\frac{\partial}{\partial u} - u\,\frac{\partial}{\partial v}, \; \mathrm{V}_5^0 = x\frac{\partial}{\partial u} - u\frac{\partial}{\partial x}, \; \mathrm{V}_6^0 = x\frac{\partial}{\partial v} - v\frac{\partial}{\partial x}$$

so that their prolongations are given by

$$\mathrm{V}_1 = \frac{\partial}{\partial x}, \quad \mathrm{V}_2 = \frac{\partial}{\partial u}, \quad \mathrm{V}_3 = \frac{\partial}{\partial v}, \quad \mathrm{V}_4 = \sum_k v_k\,\frac{\partial}{\partial u_k} - u_k\,\frac{\partial}{\partial v_k},$$
$$\mathrm{V}_5 = -u_0\,\mathrm{D}_x + \sum_k \mathrm{D}^k(x - u_0\,u_1)\frac{\partial}{\partial u_k} - \sum_k \mathrm{D}^k(u_0\,v_1)\frac{\partial}{\partial v_k},$$
$$\mathrm{V}_6 = -v_0\mathrm{D} - \sum_k \mathrm{D}^k(v_0\,u_1)\frac{\partial}{\partial u_k} + \sum_k \mathrm{D}^k(x - v_0\,v_1)\frac{\partial}{\partial v_k}.$$

The action is transitive on J^1 and becomes locally free on J^2 with generic orbits of codimension 1.

Minimal order cross-section

We choose a non classical cross-section of minimal order: $P = (x, u_0, v_0, u_1, v_1, v_2 - u_2)$. Then:

$$\bar\iota(\mathrm{D}(P)) = \begin{pmatrix} 1 & 0 & 0 & \bar\iota u_2 & \bar\iota u_2 & \bar\iota(v_3 - u_3) \end{pmatrix}.$$

Theorem 4.2 implies that $\mathfrak{P} = \{\bar\iota u_2,\, \bar\iota w\}$, where $w = v_3 - u_3$, is a generating set. For the purpose of rewriting any other differential invariants we write every element of $\mathcal{I}^3$ in terms of $\mathfrak{P}$.

From Theorem 3.6 we have $\mathcal{D}(\bar\iota u_2) = \bar\iota u_3 - \frac{1}{2}\,\bar\iota w$ since

$$K = \begin{pmatrix} 1 & 0 & 0 & \dfrac{\bar\iota w}{2\,\bar\iota u_2} & \bar\iota u_2 & \bar\iota u_2 \end{pmatrix}$$

while $\bar\iota\,(\mathrm{V}(u_2)) = \begin{pmatrix} 0 & 0 & 0 & \bar\iota u_2 & 0 & 0 \end{pmatrix}^T$. Thus

$$\bar\iota v_2 = \bar\iota u_2,\ \bar\iota u_3 = \mathcal{D}(\bar\iota u_2) - \frac{\bar\iota w}{2}, \quad \text{and } \bar\iota v_3 = \mathcal{D}(\bar\iota u_2) + \frac{\bar\iota w}{2}.$$

Note that $\bar\iota u_2$ is a differential invariant of order 2 and is therefore a function of the curvature, while $\bar\iota(u_3 - v_3)$, as a differential invariant of order 3 is a function of the curvature κ and the torsion τ. There are several ways to compute the algebraic expression for $\bar\iota u_2, \bar\iota u_3$ and $\bar\iota v_3$ (Fels and Olver, 1999; Hubert and Kogan, 2007a,b). But conversely, given the analytic expression for the curvature and the torsion (Wikipedia, 2007; Guggenheimer, 1963) it is easy to write them in terms of $\bar\iota u_2, \bar\iota u_3$ and $\bar\iota v_3$ thanks to Theorem 2.11.

$$\kappa = \sqrt{2\,\bar\iota u_2^2}, \quad \tau = \frac{\bar\iota u_3 - \bar\iota v_3}{2\,\bar\iota u_2}.$$

Non minimal cross-section

We consider now the third order cross-section $P = (x, u_0, v_0, v_1, v_2, v_3 - 1)$. Olver (2007b) introduced it to show that the minimal order condition is necessary for Theorem 4.2.

As a consequence of Theorem 4.1, $\{\bar\iota u_1,\ \bar\iota u_2,\ \bar\iota u_3,\ \bar\iota u_4, \bar\iota v_4\}$ is a generating set. According to Theorem 5.13 they are subject to the following non trivial differential syzygies.

$$\begin{aligned}
\mathcal{D}(\bar\iota u_1) &= \bar\iota u_2 + \tfrac{1}{3}\tfrac{1+\bar\iota u_1{}^2}{\bar\iota u_1\,\bar\iota u_2}\,\bar\iota u_3 - \tfrac{1}{3}\tfrac{1+\bar\iota u_1^2}{\bar\iota u_1}\,\bar\iota v_4\\
\mathcal{D}(\bar\iota u_2) &= 2\,\bar\iota u_3 - \bar\iota u_2\,\bar\iota v_4\\
\mathcal{D}(\bar\iota u_3) &= \bar\iota u_4 - \left(\tfrac{4}{3}\,\bar\iota u_3 + \tfrac{\bar\iota u_2{}^2}{\bar\iota u_1}\right)\bar\iota v_4 + \tfrac{\bar\iota u_1{}^2+1}{\bar\iota u_2} + \tfrac{4}{3}\,\tfrac{\bar\iota u_3{}^2}{\bar\iota u_2} + \tfrac{\bar\iota u_2\,\bar\iota u_3}{\bar\iota u_1}
\end{aligned}$$

From the two first equations we can deduce $\bar\iota u_3$ and $\bar\iota v_4$ in terms of $\{\bar\iota u_1,\ \bar\iota u_2\}$ and their derivatives. Substituting in the last equation we can do the same for $\bar\iota u_4$ so that $\{\bar\iota u_1, \bar\iota u_2\}$ is a generating set. Indeed, by Theorem 2.11, we can write the curvature and the torsion in terms of those:

$$\kappa = \sqrt{\frac{\bar\iota u_2^2}{(1+\bar\iota u_1^2)^3}}, \quad \tau = \frac{1}{\bar\iota u_2(1+\bar\iota u_1^2)}.$$

6.2 Surfaces in Euclidean geometry

We shall show how to retrieve the Codazzi equation as the syzygie between the two generators for the differential invariants.

We choose coordinate functions (x, y, u) for $\mathbb{R}^2 \times \mathbb{R}$. The infinitesimal generators of the classical action of the Euclidean group $SE(3)$ on $\mathbb{R}^3$ are:

$$\mathrm{V}_1 = \frac{\partial}{\partial x},\ \mathrm{V}_2 = \frac{\partial}{\partial y},\ \mathrm{V}_3 = \frac{\partial}{\partial u},$$
$$\mathrm{V}_4 = x\,\frac{\partial}{\partial u} - u\,\frac{\partial}{\partial x}, \mathrm{V}_5 = y\,\frac{\partial}{\partial u} - u\,\frac{\partial}{\partial y}, \mathrm{V}_6 = x\,\frac{\partial}{\partial y} - y\,\frac{\partial}{\partial x}$$

We consider x, y as the independent variables and u as the dependent variable, and choose the classical cross-section defined by $P = (x, y, u, u_x, u_y, u_{xy})$.

The Maurer-Cartan matrix of Theorem 3.6 is:

$$K = \begin{pmatrix} 1 & 0 & 0 & \bar{\iota}u_{xx} & 0 & \frac{\bar{\iota}u_{xxy}}{\bar{\iota}u_{xx}-\bar{\iota}u_{yy}} \\ 0 & 1 & 0 & 0 & \bar{\iota}u_{yy} & \frac{\bar{\iota}u_{xyy}}{\bar{\iota}u_{xx}-\bar{\iota}u_{yy}} \end{pmatrix}$$

and applying Proposition 3.8 we have

$$[\mathcal{D}_2, \mathcal{D}_1] = \frac{\bar{\iota}u_{xxy}}{\bar{\iota}u_{xx} - \bar{\iota}u_{yy}}\,\mathcal{D}_1 + \frac{\bar{\iota}u_{xyy}}{\bar{\iota}u_{xx} - \bar{\iota}u_{yy}}\,\mathcal{D}_2. \tag{6.1}$$

Given that $\bar{\iota}x, \bar{\iota}y, \bar{\iota}u, \bar{\iota}u_{10}, \bar{\iota}u_{01}, \bar{\iota}u_{xy} = 0$ the non zero elements of $\mathfrak{S}$ are

$$S^1_{u_{xx}} = \mathcal{D}_1(\bar{\iota}u_{xx}) - \bar{\iota}u_{xxx}, \qquad S^2_{u_{xx}} = \mathcal{D}_2(\bar{\iota}u_{xx}) - \bar{\iota}u_{xxy},$$
$$S^1_{u_{yy}} = \mathcal{D}_1(\bar{\iota}u_{yy}) - \bar{\iota}u_{xyy}, \qquad S^2_{u_{yy}} = \mathcal{D}_2(\bar{\iota}u_{yy}) - \bar{\iota}u_{yyy},$$

while the elements of $\mathfrak{T}$ are

$$T^2_{u_{xyy}} = \mathcal{D}_2(\bar{\iota}u_{xyy}) - \mathcal{D}_1(\bar{\iota}u_{yyy}) - \tfrac{\bar{\iota}u_{xyy}}{\bar{\iota}u_{xx}-\bar{\iota}u_{yy}}\,(\bar{\iota}u_{xxy} + \bar{\iota}u_{yyy}),$$
$$T^2_{u_{xxx}} = \mathcal{D}_2(\bar{\iota}u_{xxx}) - \mathcal{D}_1(\bar{\iota}u_{xxy}) - \tfrac{\bar{\iota}u_{xxy}}{\bar{\iota}u_{xx}-\bar{\iota}u_{yy}}\,(\bar{\iota}u_{xyy} + \bar{\iota}u_{xxx}),$$
$$T^2_{u_{xxy}} = \mathcal{D}_2(\bar{\iota}u_{xxy}) - \mathcal{D}_1(\bar{\iota}u_{xyy}) - \tfrac{\bar{\iota}u_{xxy}\,\bar{\iota}u_{yyy}+\bar{\iota}u_{xyy}\,\bar{\iota}u_{xxx}-2\,\bar{\iota}u_{xxy}{}^2-2\,\bar{\iota}u_{xyy}{}^2}{\bar{\iota}u_{xx}-\bar{\iota}u_{yy}} + (\bar{\iota}u_{xx} - \bar{\iota}u_{yy})\,\bar{\iota}u_{yy}\,\bar{\iota}u_{xx}.$$

Theorem 4.2 predicts that $\{\bar{\iota}u_{xx}, \bar{\iota}u_{yy}, \bar{\iota}u_{xxy}, \bar{\iota}u_{xyy}\}$ form a generating set. From $\mathfrak{S}$ we see furthermore that all the third order normalized invariants can be expressed as derivatives of $\{\bar{\iota}u_{xx}, \bar{\iota}u_{yy}\}$. This latter set therefore already forms a generating set of invariants. Indeed, with Theorem 2.11, we can write the Gauss and mean curvatures in terms of $\{\bar{\iota}u_{xx}, \bar{\iota}u_{yy}\}$ (Guggenheimer, 1963), (Ivey and Landsberg, 2003, Section 1.1):

$$\sigma = \tfrac{u_{xx}u_{yy}-u_{xy}}{(1+u_x^2+u_y^2)^2} = \bar{\iota}u_{xx}\,\bar{\iota}u_{yy},$$
$$\pi = \tfrac{1}{2}\,\frac{(1+u_y^2)u_{xx}-2_x u_y u_{xy}+(1+u_x^2)u_{yy}}{(1+u_x^2+u_y^2)^{\frac{3}{2}}} = \tfrac{1}{2}(\bar{\iota}u_{xx} + \bar{\iota}u_{yy}).$$

Our generators are thus the principal curvatures. Let us write $\kappa = \bar{\iota}u_{xx}$ and $\tau = \bar{\iota}u_{yy}$. From $\mathfrak{S}$ we have

$$\bar{\iota}u_{xxx} = \mathcal{D}_1(\kappa),\ \bar{\iota}u_{xxy} = \mathcal{D}_2(\kappa),\ \bar{\iota}u_{xyy} = \mathcal{D}_1(\tau),\ \text{and}\ \bar{\iota}u_{yyy} = \mathcal{D}_2(\tau).$$

Making the substitution in $\mathfrak{T}$ we obtain

$$\mathcal{D}_2\mathcal{D}_1(\tau) - \mathcal{D}_1\mathcal{D}_2(\tau) - \frac{\mathcal{D}_1(\tau)}{\kappa - \tau}\left(\mathcal{D}_2(\kappa) + \mathcal{D}_2(\tau)\right)$$
$$\mathcal{D}_2\mathcal{D}_1(\kappa) - \mathcal{D}_1\mathcal{D}_2(\kappa) - \frac{\mathcal{D}_2(\kappa)}{\kappa - \tau}\left(\mathcal{D}_1(\kappa) + \mathcal{D}_1(\tau)\right)$$
$$\mathcal{D}_2^2(\kappa) - \mathcal{D}_1^2(\tau) - \frac{\mathcal{D}_1(\kappa)\,\mathcal{D}_1(\tau) + \mathcal{D}_2(\kappa)\,\mathcal{D}_2(\tau) - 2\,\mathcal{D}_2(\kappa)^2 - 2\,\mathcal{D}_1(\tau)^2}{\kappa - \tau} + (\kappa - \tau)\,\kappa\,\tau.$$

The two first functions vanish when one rewrites $\mathcal{D}_2\mathcal{D}_1(\tau)$ and $\mathcal{D}_2\mathcal{D}_1(\kappa)$ in terms of monotone derivatives using (6.1). The last function provides the Codazzi equation (Guggenheimer, 1963), (Ivey and Landsberg, 2003, Exercise 2.3.1).

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