Generation properties of Maurer-Cartan invariants

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Abstract

For the action of a Lie group, which can be given by its infinitesimal generators only, we characterize a generating set of differential invariants of bounded cardinality and show how to rewrite any other differential invariants in terms of them. Those invariants carry geometrical significance and have been used in equivalence problem in differential geometry.

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Introduction

Group actions are ubiquitous in mathematics and arise in diverse fields of science and engineering. Whether algebraic or differential, one can distinguish two families of applications for invariants of group actions: equivalence problems and symmetry reduction. In equivalence problems, invariants come as functions whose values *separate* the orbits and thus distinguish inequivalent problems. Symmetry reduction postulates that invariants are the best coordinates in which to think a problem. Here a *generating* set of invariants is needed, so as to rewrite the problem in terms of those. In this paper we show that the set of differential invariants that has classically been used for equivalence problems in differential geometry is also a generating set, endowed with a simple rewriting algorithm.

In differential geometry equivalence problems are diverse though their resolutions often take their roots in the work of Elie Cartan [3, 4]. For the equivalence of submanifolds in a homogeneous space, an interpretation of Cartan's moving frame method [10, 22, 5, 21] exhibits separating invariants: they are are the coefficients of the pull-back of the Maurer-Cartan forms to the submanifold and we shall call those the *Maurer-Cartan invariants*. The finite generation of differential invariants, for the more general case of pseudo-groups, was addressed in [44, 26, 27, 28, 39] - see also [43, 41] for Lie groups. In their reinterpretation of Cartan's moving frame method, Fels and Olver [7, 29, 23] addressed equivalence problems as well as finite generation. *Normalized invariants* are the focus there. They form a generating set and the relationships (syzygies) they satisfy were given in [16].

In this paper we show that Maurer-Cartan invariants form a generating set. On one hand we shall give the explicit formulae for the Maurer-Cartan invariants in terms of the normalized invariants. Besides the methods in [7, 24], there is an algorithm for computing normalized invariants in the algebraic case [18, 19]. On the other hand we show how to rewrite any differential invariant in terms of the Maurer-Cartan invariants. This is a simple recursive process.

The generation property is a rather simple observation, yet meaningful and unifying. We draw a link between the reinterpretation of the moving frame by Griffiths [10, 22, 5, 21] on one hand and Fels and Olver [7, 23, 18, 16, 30] on the other hand. We also show how it links with the invariants arising in curve evolution [32, 33, 35, 36, 37, 31].

Beside their geometrical significance, Maurer-Cartan invariants have a number of desirable features over normalized invariants. First their number is bounded by the product of the submanifold dimension with the dimension of the group. Second, their syzygies can be written down explicitly from the structure equations of the group. Then, a meaningful item for symbolic computation, they allow to avoid denominators in the symmetry reduction.

In Section 1 we describe part of the moving frame method as reinterpreted by Griffiths to show the prominent role of Maurer-Cartan forms, and their pull-backs, have held for equivalence problems in differential geometry. The geometrical construction of normalized invariants based on a cross-section is reviewed in Section 2. The Maurer-Cartan invariants come in Section 3. They arise as the coefficients of the pullback of the Maurer-Cartan forms and define the difference between the invariantization of the total derivation and the invariant derivation of the invariantization. It is from this latter formulae that one deduces in Section 4 that the Maurer-Cartan invariants, along with the normalized and edge invariants, form a generating set. The differential relationships they satisfy are discussed in Section 5. In Section 6 we show how to recast our results in the context of the better known case of matrix groups. In place of running examples along the text, we have gathered those in Section 7.

1 Equivalence of submanifolds

The content of this section is rather independent from the rest of the paper. Its purpose is to link our approach to the interpretation of the moving frame method in [10, 22, 5, 21]. We show where the Maurer-Cartan invariants occur.

We consider a r-dimensional Lie group $\mathcal{G}$. We make use of a basis of invariant vector fields on $\mathcal{G}$, $\mathrm{v} = (\mathrm{v}_1, \ldots, \mathrm{v}_r)$. They form a Lie algebra with structure constants $\left\{C_{ijk}\right\}_{1\leq i,j,k\leq r}$, i.e.

$$[\mathrm{v}_i, \mathrm{v}_j] = \sum_{k=1}^{r} C_{ijk}\, \mathrm{v}_k. \tag{1}$$

The Maurer-Cartan forms[1] $\omega = (\omega_1, \ldots, \omega_r)$ are the invariant one-forms that are dual to the chosen basis of invariant vector fields $\mathrm{v} = (\mathrm{v}_1, \ldots, \mathrm{v}_r)$ on $\mathcal{G}$. The structure equations, dual to (1), are

$$\mathrm{d}\omega_k = -\sum_{1\leq i<j\leq r} C_{ijk}\, \omega_i \wedge \omega_j. \tag{2}$$

The Maurer-Cartan forms are at the heart of the Cartan's approach for equivalence of submanifolds. As observed in [45, 22] one can distinguish the fixed parameterization problem and the unparameterized problem. For the fixed parameterization problem Griffiths [10] proposed a reinterpretation of Cartan's moving frame method that works along the following lines. The monograph by Jensen [22] treats the unparameterized problem.

In Klein geometry one considers the action of $\mathcal{G}$ on a homogeneous space $\mathcal{G}/\mathcal{H}$, where $\mathcal{H}$ is a subgroup of $\mathcal{G}$. Given two immersions $f_1, f_2 : \mathcal{X} \to \mathcal{G}/\mathcal{H}$ one wishes to determine when there exists $\lambda \in \mathcal{G}$ s.t. $f_1(x) = \lambda \cdot f_2(x)$, for all $x \in \mathcal{X}$.

[1] In Section 6 we link this definition to the one used [10, 21]

A solution to the problem is based on the following result [10], [21, Theorem 1.6.10, Corolary 1.6.11].

Lemma 1.1 *Let $\tilde{f}_1, \tilde{f}_2 : \mathcal{X} \to \mathcal{G}$ be two smooth maps of a connected manifold $\mathcal{X}$ into the Lie group $\mathcal{G}$. Then $\tilde{f}_1(x) = \lambda \cdot \tilde{f}_2(x)$, for a fixed $\lambda \in \mathcal{G}$ and for all $x \in \mathcal{X}$, if and only if*

$$\tilde{f}_1^*\omega_1 = \tilde{f}_2^*\omega_1, \ldots, \tilde{f}_1^*\omega_r = \tilde{f}_2^*\omega_r,$$

where $\omega_1, \ldots, \omega_r$ are the Maurer-Cartan forms on $\mathcal{G}$.

The solution to the equivalence problem relies then on finding an equivariant *lift* i.e. a process that associates to each immersion $f : \mathcal{X} \to \mathcal{G}/\mathcal{H}$ an immersion $\tilde{f} : \mathcal{X} \to \mathcal{G}$ s.t. $\widetilde{\lambda \cdot f} = \lambda \cdot \tilde{f}$ and $\pi \circ \tilde{f} = f$, for the natural projection $\pi : \mathcal{G} \to \mathcal{G}/\mathcal{H}$. The lifts, the craft of which requires a good geometrical understanding, usually involve the jets of f. Differential invariants thus appear as coefficients when we write the invariant forms $f^*\omega^i$ in terms of an (invariant) coframe the for $\mathcal{X}$. The lemma thus shows that those invariants *separate* orbits. In the fixed parametrization case, the forms on $\mathcal{X}$ are invariant, but intrinsic parametrisations are to be determined and used for generality.

In the first lines of this section we spoke of invariance on $\mathcal{G}$ without specifying if it was under left or right multiplication. In line with [7, 16] we consider invariance under right multiplication. The prolongation to higher order jets combined with the moving frame as defined in Section 2.3 methodologically provide the equivariant lift, the art residing only in choosing the most appropriate cross-section.

The approach is not restricted to homogeneous spaces. In line with [7] and in view of our original interest in differential equations, we consider an action on a space of dependent and independent variables. On a manifold of jets of high enough order, a local cross-section to the orbits determines a *moving frame*. The Maurer-Cartan invariants are the entries of the matrix relating the pullback of the Maurer-Cartan forms to a basis of contact invariant differential forms, which is given by the moving frame.

2 Normalized invariants

We go through the construction of normalized invariants based on the choice of a cross-section to the orbit of the prolonged action. This is in line with [7, 18, 16] and we refer to those three papers for more details and examples.

2.1 Group action and prolongations

For a smooth manifold $\mathcal{M}$ we note $\mathcal{F}(\mathcal{M})$ the $\mathbb{R}$-algebra of smooth functions on $\mathcal{M}$. For a smooth map $\phi : \mathcal{M} \to \mathcal{N}$, we denote $\phi^* : \mathcal{F}(\mathcal{N}) \to \mathcal{F}(\mathcal{M})$ the pullback: $(\phi^* f)(z) = f(\phi(z))$ for $z \in \mathcal{M}$ and $f \in \mathcal{F}(\mathcal{N})$.

We consider a manifold $\mathcal{X} \times \mathcal{U}$ where $\mathcal{X}$ and $\mathcal{U}$ are covered by single coordinate systems, $x = (x_1, \ldots, x_m)$ and $u = (u_1, \ldots, u_n)$ respectively. The additional coordinate functions for the space of jets of order k, $\mathrm{J}^k = \mathrm{J}^k(\mathcal{X},\mathcal{U})$, are the $u_{i,\alpha}$, where $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$ is such that $|\alpha| = \sum_{i=1}^m \alpha_i \leq k$, that correspond to the derivatives $\frac{\partial^{|\alpha|} u_i}{\partial x^\alpha}$.

We consider an action g of $\mathcal{G}$ on $\mathrm{J}^0 = \mathcal{X} \times \mathcal{U}$ and its prolongations to the jet spaces J^k, $g : \mathcal{G} \times \mathrm{J}^k \to \mathrm{J}^k$. We note $\mathrm{V}^k = (\mathrm{V}_1^k, \ldots, \mathrm{V}_r^k)$ the infinitesimal generators associated to our choice of right invariant vector fields $\mathrm{v} = (\mathrm{v}_1, \ldots, \mathrm{v}_r)$ on $\mathcal{G}$:

$$\mathrm{V}_i^k(f)(z) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(g(e^{t\mathrm{v}}, z)), \quad z \in \mathrm{J}^k, f \in \mathcal{F}(\mathrm{J}^k).$$

The expression of the prolongations, of the action or of the infinitesimal generators, can be derived explicitly from the expression of the action of g on J^0, or of V^0 [40, 41].

The maximal dimension r_k of the orbits on J^k can only increase as the action is prolonged to higher order jets. It can not go beyond the dimension of the group though. The *stabilization order* is the order at which the maximal dimension of the orbits becomes stationary. If the action on J^0 is locally effective on subsets, i.e. the global isotropy group is discrete, then, for s bigger than the stabilization order, the action on J^s is locally free on an open subset of J^s [41, Theorem 5.11]. We shall make this assumption of a locally effective action. The maximal dimension of the orbits in J^s is then r, the dimension of the group. We have:

$$r_0 \leq r_1 \leq \ldots \leq r_{s-1} < r_s = r_{s+1} = \ldots = r.$$

Definition 2.1 *A differential invariant of order k is a function f of $\mathcal{F}(\mathrm{J}^k)$ s.t. $\mathrm{V}_1^k(f) = 0, \ldots, \mathrm{V}_r^k(f) = 0$. The $\mathbb{R}$-algebra of differential invariants of order k is noted $\mathcal{F}^{\mathcal{G}}(\mathrm{J}^k)$ while $\mathcal{F}^{\mathcal{G}}(\mathrm{J})$ denotes the $\mathbb{R}$-algebra of differential invariants of any order.*

2.2 Local cross-section

For any $k \in \mathbb{N}$, a local cross-section to the orbits of g in J^k defines an invariantization and a set of normalized invariants on an open set of J^k.

Definition 2.2 *An embedded submanifold $\mathcal{P}^k$ of J^k is a* local cross-section *to the orbits if there is a neighborhood $\mathcal{U}^k$ of $\mathcal{P}^k$ in J^k such that*

- *$\mathcal{P}^k$ is of complementary dimension to the orbits in $\mathcal{U}^k$,*
- *$\mathcal{P}^k$ intersects at a unique point and transversally the connected part of the orbit of any z in $\mathcal{U}^k$.*

The first condition implicitly implies that the action is semi-regular in $\mathcal{U}^k$, i.e. that all the orbits are of the same dimension. The results in this paper actually restrict to the open set $\mathcal{U}^k$. Though we shall keep the global notation, J^k is understood to be restricted to this neighborhood of $\mathcal{P}^k$.

An embedded submanifold of codimension r_k can be locally defined as the zero set of a map $P : \mathrm{J}^k \to \mathbb{R}^{r_k}$ where the components $(p_1, \ldots, p_{r_k})$ are independent functions of $\mathcal{F}(\mathrm{J}^k)$. The condition for P to define a local cross section on J^k is:

$$\text{the rank of the } r \times r_k \text{ matrix } \left(\mathrm{V}_i(p_j)\right)_{i=1..r}^{j=1..r_k} \text{ is } r_k \text{ on } \mathcal{P}^k. \tag{3}$$

When $\mathcal{G}$ acts semi-regularly on J^k there is thus a lot of freedom in choosing a cross-section. In particular we can always choose a coordinate cross-section [19, Theorem 5.6], i.e. the level set of an appropriate subset of the coordinate functions.

Let s be equal or greater than the stabilization order and let $\mathcal{P}^s$ be a cross-section to the orbits in J^s. Its pre-image $\mathcal{P}^{s+k}$ in J^{s+k} by the projection map $\pi_s^{s+k} : \mathrm{J}^{s+k} \to \mathrm{J}^s$ is a cross-section to the orbits in J^{s+k}. In other words, if $\mathcal{P}^s$ is the zero set in J^s of $p_1, \ldots, p_r : \mathrm{J}^s \to \mathbb{R}$ then those same functions cut out a local cross-section for J^{s+k}. Note though that the projection of $\mathcal{P}^s$ on J^k, for $k < s$, need not be a cross-section to the orbits in J^k.

Definition 2.3 *A cross-section $\mathcal{P}^k$ to the orbits in J^k is of* minimal order *if, for any $l \leq k$, its projection on J^l is a local cross-section.*

There is no harm in then assuming that a minimal order cross-section $\mathcal{P}^k$ is defined by $P = (p_1, \ldots, p_{r_k})$ where $(p_1, \ldots, p_{r_l})$ defines a local cross-section on J^l, for any $l \leq k$.

Definition 2.4 *Let $\mathcal{P}^k$ be a local cross-section to the orbits of the action $g : \mathcal{G} \times \mathrm{J}^k \to \mathrm{J}^k$. The* invariantization *$\bar{\iota} f$ of $f \in \mathcal{F}(\mathrm{J}^k)$ is the function defined by $\bar{\iota} f(z) = f(\bar{z})$ where $\bar{z}$ is the intersection of the orbit of $z \in \mathrm{J}^k$ with $\mathcal{P}^k$.*

Definition 2.5 *The normalized invariants of order k associated to a cross-section $\mathcal{P}^k$ in J^k are the invariantizations of the coordinate functions on J^k. The set they form is denoted*

$$\mathcal{I}^k = \{\bar{\iota} x_1, \ldots, \bar{\iota} x_m\} \cup \{\bar{\iota} u_\alpha \,|\, u \in \mathcal{U},\ |\alpha| \leq k\}.$$

If we chose a coordinate cross-section, then some of those normalized invariants are constants. More generally, the normalized invariants are subject to the functional relationships implied by the choice of the equations of the cross-section. If the cross-section is defined by $p_1(x, u_\alpha) = 0, \ldots, p_{r_k}(x, u_\alpha) = 0$ then $p_1(\bar{\iota} x, \bar{\iota} u_\alpha) = 0, \ldots, p_{r_k}(\bar{\iota} x, \bar{\iota} u_\alpha) = 0$. This is a particular case of the following result.

Theorem 2.6 *Assume $\bar{\iota}$ is the invariantization associated to a cross-section $\mathcal{P}^k$ in J^k. For $f \in \mathcal{F}(\mathrm{J}^k)$ $\bar{\iota} f$ is the unique differential invariant (of order k) whose restriction to $\mathcal{P}^k$ is equal to the restriction of f to $\mathcal{P}^k$.*

Corollary 2.7 *For $f \in \mathcal{F}(\mathrm{J}^k)$, $\bar{\iota} f(x, u_\alpha) = f(\bar{\iota} x, \bar{\iota} u_\alpha)$.*

In particular, if $f \in \mathcal{F}^{\mathcal{G}}(\mathrm{J}^k)$ then $\bar{\iota} f = f$ and $f(x, u_\alpha) = f(\bar{\iota} x, \bar{\iota} u_\alpha)$. Therefore $\mathcal{I}^k$ is a generating set for the differential invariants of order k in a functional sense. Note though that, unless $\mathcal{P}^k$ is of minimal order, $\mathcal{I}^k$ need not contain a set of generators for $\mathcal{F}^{\mathcal{G}}(\mathrm{J}^l)$, for $l < k$.

2.3 Moving frame

As noted already by [10, 8, 22, 21], the geometric idea of classical moving frames, like the Frenet frame for space curves in Euclidean geometry, can be understood as maps to the group. Accordingly Fels and Olver [7] defined a *moving frames* as an equivariant map $\rho : \mathrm{J}^s \rightarrow \mathcal{G}$, for s big enough, as they gave a new interpretation of Cartan's *repère mobile* [3, 4] .

Theorem 2.8 *A Lie group $\mathcal{G}$ acts locally freely on J^s if and only if every point of J^s has an open neighborhood $\mathcal{U}^s$ where there exists a locally equivariant map $\rho\colon \mathcal{U}^s \rightarrow \mathcal{G}$ in the sense that, for any $z \in \mathcal{U}^s$, $\rho(\lambda \cdot z) = \rho(z) \cdot \lambda^{-1}$ for $\lambda \in \mathcal{G}$ sufficiently close to the identity.*

As before, we shall use the global notation J^s but we shall keep in mind that we are restricting to $\mathcal{U}^s$.

If the action g of $\mathcal{G}$ is locally free on J^s then a local cross-section $\mathcal{P}^s$ to the orbits defines a moving frame. Indeed the equation

$$g(\rho(z), z) \in \mathcal{P}^s \text{ for } z \in \mathrm{J}^s \text{ and } \rho(z) = e, \forall z \in \mathcal{P}^s$$

uniquely defines a smooth map $\rho : \mathrm{J}^s \rightarrow \mathcal{G}$ in a sufficiently small neighborhood of any point of the cross-section. This map is seen to be equivariant. If $\mathcal{P}$ is the zero set of the map $P = (p_1, \ldots, p_r)$ then $p_1(g(\rho, z)) = 0, \ldots, p_r(g(\rho, z)) = 0$ are implicit equations for the moving frame. If we can solve those, the moving frame ρ provides an explicit construction for the invariantization process. To spell it out let us introduce the following maps.

$$\begin{array}{rccc} \sigma : & \mathrm{J}^{s+k} & \rightarrow & \mathcal{G} \times \mathrm{J}^{s+k} \\ & z & \mapsto & (\rho(z),\, z) \end{array} \quad \text{and} \quad \begin{array}{rccc} \pi = g \circ \sigma : & \mathrm{J}^{s+k} & \rightarrow & \mathrm{J}^{s+k} \\ & z & \mapsto & g(\rho(z),\, z) \end{array} \tag{4}$$

Proposition 2.9 *For $f \in \mathcal{F}(\mathrm{J}^{s+k})$, $\bar{\iota} f = \pi^* f$, that is $\bar{\iota} f(z) = f(g(\rho(z), z))$ for all $z \in \mathrm{J}^{s+k}$.*

This is a restatement of [19, Proposition 1.16]. An invariantization of forms is defined in [23]. Besides the map σ and g it involves a projection on the jet differentials.

3 Invariant derivations and forms

The moving frame defined by a cross-section allows to construct a frame of invariant derivations, and, equivalently, their dual contact invariant coframe. The coordinates of the horizontal part of pullback of the Maurer-Cartan forms in this coframe are the*Maurer-Cartan invariants*. They are the entries of the $m \times r$ matrix K that can be explicitely written in terms of the normalized invariants with the only knowledge of the infinitesimal generators and the equations of the cross-section. The Maurer-Cartan invariants arise in then in the explicit formula for the invariant derivation of normalized invariants. The material of this section draws in an essential way on [7, 23, 16] where examples can be found.

The vector $\mathrm{D} = (\mathrm{D}_1, \dots, \mathrm{D}_m)^T$ of total derivations

$$\mathrm{D}_i = \frac{\partial}{\partial x_i} + \sum_{u,\alpha} u_{\alpha+\epsilon_i} \frac{\partial}{\partial u_\alpha}$$

is dual to the horizontal one forms $\mathrm{d}x = (\mathrm{d}x_1, \dots, \mathrm{d}x_m)^T$. Let $\mathcal{D} = (\mathcal{D}_1, \dots, \mathcal{D}_m)^T$ be a vector of total derivations defined by $\mathcal{D} = A^{-1}\mathrm{D}$ where $A \in \mathcal{F}(\mathrm{J}^{k+1})$ is an invertible matrix. The dual horizontal forms $\mathfrak{x} = (\mathfrak{x}_1, \dots, \mathfrak{x}_m)^T$ are then given by $\mathfrak{x} = A^T \mathrm{d}x$.

For $\lambda \in \mathcal{G}$, let $g_\lambda : \mathrm{J}^k \to \mathrm{J}^k$ be the map defined by $g_\lambda(z) = g(\lambda, z)$. A differential form ν on J^k is invariant if $g_\lambda^*(\nu|_{g_\lambda(z)}) = \nu|_z$, for all $\lambda \in \mathcal{G}$. It is *contact invariant* if $g_\lambda^*(\nu|_{g_\lambda(z)}) - \nu|_z$ is a contact form [41, Definition 5.25]. Dual to contact-invariant forms, *invariant derivations* are total derivations that commute with the infinitesimal generators of the group action. It maps differential invariants of order k to differential invariants of order $k+1$, for k large enough.

For instance, if f is a differential invariant then $\mathrm{d}f$ is an invariant differential form while its projection on the horizontal coframe, $\mathrm{d}_H f$, is contact invariant. A frame of invariant derivations can thus be constructed from sufficiently many independent differential invariants. Additionally, if $\rho : \mathrm{J}^s \to \mathcal{G}$ is a moving frame, then the pullback of the Maurer-Cartan forms, $\rho^*\omega_i$, $1 \le i \le r$ are invariant forms.

The construction of a contact-invariant coframe $\mathfrak{x} = (\mathfrak{x}[1], \dots, \mathfrak{x}[m])$ proposed by [7] is based on a moving frame. Let $P = (p_1, \dots, p_d)$ defines a cross-section $\mathcal{P}$ to the orbits in J^s, where s is greater than the stabilization order. Consider $\rho : \mathrm{J}^s \to \mathcal{G}$ the associated moving frame and $\bar\iota : \mathcal{F}(\mathrm{J}) \to \mathcal{F}^{\mathcal{G}}(\mathrm{J})$ the associated invariantization. Define $\sigma : \mathrm{J}^s \to \mathcal{G} \times \mathrm{J}^s$ by $\sigma(z) = (\rho(z), z)$.

Theorem 3.1 *The vector of derivations* $\mathcal{D} = (\sigma^* A)^{-1}\,\mathrm{D}$*, where* A *is the* $m \times m$ *matrix* $\left(\mathrm{D}_i(g^* x_j)\right)_{ij}$*, is a vector of invariant derivations. Equivalently,* $\mathfrak{x} = (\sigma^* A)^T dx$ *is a vector of contact invariant forms.*

Though those invariant derivations do not commute in general, their formidable benefit is an explicit formula for the derivation of a normalized invariants. This formula involves the Maurer-Cartan invariants.

We denote by $\mathrm{D}(P)$ the $m \times r$ matrix $\left(\mathrm{D}_i(p_j)\right)_{i,j}$ with entries in $\mathcal{F}(\mathrm{J}^{s+1})$ while $\mathrm{V}(P)$ is the $r \times r$ matrix $\left(\mathrm{V}_i(p_j)\right)_{i,j}$ with entries in $\mathcal{F}(\mathrm{J}^s)$. As $\mathcal{P}$ is transverse to the orbits in J^s, the matrix $\mathrm{V}(P)$ has non zero determinant along $\mathcal{P}$ and therefore in a neighborhood of each of its points.

Theorem 3.2 *Consider* $\mathcal{D} = (\mathcal{D}_1, \ldots, \mathcal{D}_m)^T$*, and* $\mathfrak{x} = (\mathfrak{x}_1, \ldots, \mathfrak{x}_m)^T$*, the vector of invariant derivations, and their dual contact-invariant forms, constructed in Theorem 3.1. Let* K *be the* $m \times r$ *matrix obtained by invariantizing the entries of* $\mathrm{D}(P)\,\mathrm{V}(P)^{-1}$*. Then*

$$\rho^* \omega = -K^T \mathfrak{x} \text{ modulo contact forms}$$

and

$$\mathcal{D}(\bar{\iota} f) = \bar{\iota}(\mathrm{D} f) - K\, \bar{\iota}(\mathrm{V}(f)).$$

As they relate the horizontal part of the pullback of the Maurer-Cartan forms $\rho^*\omega$ to the contact invariant coframe $\mathfrak{x}$ of Theorem 3.1, the entries of the matrix K are the *Maurer-Cartan invariants*

The statement can be deduced from [7, Section 13]. We extend the proof of [16] that takes the dual approach.

PROOF: From the definition of $\sigma : z \mapsto (\,\rho(z), z\,)$ and the chain rule we have

$$\mathcal{D}(\bar{\iota} f)(z) = \mathcal{D}(\sigma^* g^* f)(z) = \mathcal{D}(g^* f)(\rho(z), z) + (\rho_* \mathcal{D})(g^* f)(\rho(z), z). \tag{5}$$

Consider the matrix A of Theorem 3.1. The vector of derivations $\tilde{\mathrm{D}} = A^{-1}\mathrm{D}$ has the following property: $\tilde{\mathrm{D}}_j(g^* f) = g^*(\mathrm{D}_j f)$ for all $f \in \mathcal{F}(\mathrm{J})$. It is indeed used to compute the prolongations [41, Chapter 4], [16, Section 1.2]. We have $\mathcal{D}(g^* f)(\rho(z), z) = (\sigma^* \tilde{\mathrm{D}}(g^* f))(z) = \sigma^* g^* (\mathrm{D} f)(z) = \bar{\iota}(\mathrm{D} f)(z)$ and (5) becomes

$$\mathcal{D}(\bar{\iota} f)(z) = \bar{\iota}(\mathrm{D} f)(z) + \sigma^*(\rho_* \mathcal{D})(g^* f)(z). \tag{6}$$

Let $\hat{\mathrm{v}} = (\hat{\mathrm{v}}_1, \ldots, \hat{\mathrm{v}}_d)$ be the infinitesimal generators of the action of $\mathcal{G}$ on $\mathcal{G}$ by the right multiplication. Besides being left invariant, they form a basis for the derivations on $\mathcal{G}$. There is thus a matrix $\tilde{K}$ with entries in $\mathcal{F}(\mathcal{G} \times \mathrm{J}^s)$ s.t. $\rho_* \mathcal{D} = \tilde{K}\, \hat{\mathrm{v}}$.

We can write (6) as $\mathcal{D}(\bar{\iota} f)(z) = \bar{\iota}(\mathrm{D} f)(z) + \sigma^* \left(\tilde{K} \hat{\mathrm{v}}(g^* f)\right)(z)$ so that, by [16, Proposition 1.1],

$$\mathcal{D}(\bar{\iota} f)(z) = \bar{\iota}(\mathrm{D} f)(z) - \sigma^* \left(\tilde{K} \mathrm{V}(g^* f)\right)(z) \tag{7}$$

This latter equation shows that $\sigma^*\left(\tilde{K}\mathrm{V}(g^*f)\right) = \bar{\iota}(\mathrm{D}f) - \mathcal{D}(\bar{\iota}f)$ is a differential invariant. As such it is equal to its invariantization and thus

$$\sigma^*\left(\tilde{K}\mathrm{V}(g^*f)\right) = \bar{\iota}(\sigma^*\tilde{K})\,\bar{\iota}(\sigma^*\mathrm{V}(g^*f)).$$

For all $z \in \mathcal{P}$, $\rho(z) = e$ and therefore $\sigma^*\mathrm{V}(g^*f)$ and $\mathrm{V}(f)$ agree on $\mathcal{P}$: for all $z \in \mathcal{P}$, $\sigma^*\mathrm{V}(g^*f)\,(z) = \mathrm{V}(g^*f)\,(e,z) = \mathrm{V}(f)(z)$ by [16, Proposition 1.1]. It follows that $\bar{\iota}(\sigma^*\mathrm{V}(g^*f)) = \bar{\iota}(\mathrm{V}(f))$ so that (7) becomes

$$\mathcal{D}(\bar{\iota}f)(z) = \bar{\iota}(\mathrm{D}f)(z) - \bar{\iota}(\sigma^*\tilde{K})\,\bar{\iota}(\mathrm{V}(f)). \tag{8}$$

To find the matrix $K = \bar{\iota}(\sigma^*\tilde{K})$ we use the fact that $\bar{\iota}p_i = 0$ for all $1 \leq i \leq r$. Applying $\mathcal{D}$ and (8) to this equality we obtain: $\bar{\iota}(\mathrm{D}p_i) = K\,\bar{\iota}(\mathrm{V}(p_i))$ so that $\bar{\iota}(\mathrm{D}(P)) = K\,\bar{\iota}(\mathrm{V}(P))$. The transversality of $\mathcal{P}$ imposes that $\mathrm{V}(P)$ is invertible along $\mathcal{P}$, and thus so is $\bar{\iota}(\mathrm{V}(P))$.

We thus have proved that $\mathcal{D}(\bar{\iota}f) = \bar{\iota}(\mathrm{D}f) - K\,\bar{\iota}(\mathrm{V}(f))$ where $K = \bar{\iota}(\sigma^*\tilde{K}) = \bar{\iota}(\mathrm{D}(P)V(P)^{-1})$.

Since $\rho^*\omega$ is a vector of invariant forms while $\mathfrak{x}$ is a vector of contact invariant forms whose entries span the horizontal coframe there is a matrix A with differential invariants as entries such that $\rho^*\omega = A\mathfrak{x}$ modulo contact forms . We prove that $A = -K^T$.

Since $\mathfrak{x}$ and $\mathcal{D}$ are dual we have $A_{ai}(z) = \langle \rho^*\omega_a, \mathcal{D}^i\rangle = \langle \omega_a, \rho_*\mathcal{D}^i\rangle$ which can be written in matrix form as $A^T = \omega^T(\rho_*\mathcal{D})$ and thus, taking forms and vector fields where they should be taken, $A^T = \omega|_{\rho(z)}^T \tilde{K}(\rho(z), z)\hat{\mathrm{v}}|_{\rho(z)}$. In particular for $z \in \mathcal{P}$ we have $A(z)^T = \omega_e^T \tilde{K}(e,z)\hat{\mathrm{v}}|_e$. As $\hat{\mathrm{v}}|_e = -\mathrm{v}|_e$ and v is dual to ω we have $A(z)^T = -\tilde{K}(e,z)$ for all $z \in \mathcal{P}$. Therefore $A = -\bar{\iota}(\sigma^*\tilde{K})^T$ and we proved that $\sigma^*\tilde{K} = K$. □

The commutators of the invariant derivations is obtained explicitly by deriving a recurrence formula for forms in [7, Section 13]. It can also be derived directly from Theorem 3.2 through the use of *formal invariant derivations* [17].

Proposition 3.3 *For all* $1 \leq i, j \leq m$, $\left[\mathcal{D}_i, \mathcal{D}_j\right] = \sum_{k=1}^{m} \Lambda_{ijk}\,\mathcal{D}_k$ *where*

$$\Lambda_{ijk} = \sum_{c=1}^{d} K_{ic}\,\bar{\iota}(\mathrm{D}_j(\xi_{ck})) - K_{jc}\,\bar{\iota}(\mathrm{D}_i(\xi_{ck})) \;\in\; \mathcal{F}^{\mathcal{G}}(\mathrm{J}^{s+1}),$$

$K = \bar{\iota}\left(D(P)\,V(P)^{-1}\right)$, *and* $\xi_{ck} = \mathrm{V}_c(x_k)$.

4 Generation

The formula for the derivation of invariantized function of previous section is the key to the generation property of three different sets of differential invariants, among which the Maurer-Cartan invariants.

If we specialize Theorem 3.2 for $f = x_i$ of $f = u_\alpha$ we obtain the so called *recurrence formulae* [7, Section 13]:

$$\begin{aligned}\mathcal{D}_i(\bar\iota x_j) &= \delta_{ij} - \sum_{a=1}^{r} K_{ia}\bar\iota\left(\mathrm{V}(x_j)\right) && 1 \leq i, j, \leq m\\ \mathcal{D}_i(\bar\iota u_\alpha) &= \bar\iota u_{\alpha+\epsilon_i} + \sum_{a=1}^{r} K_{ia}\bar\iota\left(\mathrm{V}(u_\alpha)\right), && \forall \alpha \in \mathbb{N}^m,\ 1 \leq i \leq m\end{aligned}$$

As the entries of K belong to $\mathcal{F}^{\mathcal{G}}(\mathrm{J}^{s+1})$, it immediately follows from those recurrence formulae that the normalized invariants of order $s+1$ and less form a generating set of differential invariants. This set has cardinality $m+n\binom{m+s+1}{s+1}$. If we restrict to cross-sections of minimal order we can find a generating set of cardinality bounded by $m\,r + d_0$ where d_0 is the codimension of the orbits in J^0. In the case of coordinate cross-section it is a subset of $\mathcal{I}^{s+1}$ and the result appeared in [42]. The general case appeared in [16, Theorem 4.2].

Theorem 4.1 *If $P = (p_1, \ldots, p_r)$ defines a cross-section for the action of g on* J *such that $P_k = (p_1, \ldots, p_{r_k})$ defines a cross-section for the action of g on J^k, for all k, then the* edge invariants $\mathcal{E} = \{\bar\iota(\mathrm{D}_i(p_j))\,|\,1 \leq i \leq m,\ 1 \leq j \leq r\}$ *together with $\mathcal{I}^0$ form a generating set of differential invariants.*

As discussed in [16, Section 4 and 5] the rewriting in terms of the normalized invariants of $\mathcal{I}^{s+1}$ and their derivatives is effective and relies on a simple inductive process. The proof of [16, Theorem 4.2] shows that the rewriting in terms of $\mathcal{E}$ can be performed with linear algebra operations.

A new observation is that, without the assumption of minimal order on the cross-section, we can always find a generating set of cardinality bounded by $m\,r + d_0$, where d_0 is the dimension of the projection of the cross-section on J^0. Indeed the Maurer-Cartan invariants form a generating set. The proof of the following theorem provides the inductive process by which the rewriting is made effective.

Theorem 4.2 *Take s to be equal to the stabilization order or greater. Let $\mathcal{P}^s$ be a cross-section on J^s defined by the map $P = (p_1, \ldots, p_r)$. The union of 0^{th} order normalized invariants, $\mathcal{I}^0$, and $\mathcal{K} = \{K_{ia}\,|\,1 \leq i \leq m,\ 1 \leq a \leq r\}$, the entries of the matrix $K = \bar\iota(\mathrm{D}(P)V(P)^{-1})$, form a generating set of differential invariants.*

PROOF: From Theorem 3.2, $\bar\iota(\mathrm{D}u_\alpha) = \mathcal{D}(\bar\iota u_\alpha) + K\,\bar\iota(\mathrm{V}(u_\alpha))$ for any $\alpha \in \mathbb{N}^m$ and $u \in \mathcal{U}$. $\mathrm{V}(u_\alpha)$ is a vector of function of $\mathcal{F}(\mathrm{J}^{|\alpha|})$. Therefore the invariantization of all the coordinate functions of order $k+1$ can be written in terms of the differential invariants of order k, and their derivatives, together with the differential invariants coming as entries of the matrix K. An inductive argument brings the conclusion. □

The case of transitive action on J^0 has a special importance. In this case the cross-section on J^0 can be chosen as a single point and the zero-th order normalized invariants are then constant. We obtain then the striking result that the Maurer-Cartan invariants form a generating set.

Corollary 4.3 *Assume the action is transitive on* J^0 *and choose a cross-section the projection of which provides a cross-section on* J^0*. The Maurer-Cartan invariants form then a generating set of differential invariants.*

5 Syzygies

The Maurer-Cartan invariants, as well as the edge invariants, are differential invariants of order $s+1$, where s is the order of the chosen moving frame. They need not to be functionally independent. Yet we know all the functional relationships on the normalized invariants: they are given by the functions the invariantization of which is zero. They are the linear combination of the r functions cutting out the cross-section [16, Proposition 2.12]. If the coefficients of infinitesimal operators are rational functions while the cross-section is cut out by polynomial functions, we can algorithmically compute the functional relationships among the Maurer-Cartan invariants by algebraic elimination [1, 6, 25, 9].

Similarly we know a complete set of differential syzygies for $\mathcal{I}^{s+1}$ [16, Theorem 5.13]. By differential elimination [13, 15, 2, 14] we can determine a complete set of differential syzygies for the edge invariants $\mathcal{E}$ and for the Maurer-Cartan invariants $\mathcal{K}$. Yet the Maurer-Cartan invariants carry the structure equations that provides their syzygies.

The Maurer-Cartan forms satisfy the structure equations provided by the *structure constants* C_{abc} of the Lie algebra of right invariant vector fields on $\mathcal{G}$.

$$\mathrm{d}\omega_c = -\sum_{a<b} C_{abc}\,\omega_a \wedge \omega_b \quad \text{where} \quad [\mathrm{v}_a, \mathrm{v}_b] = C_{abc}\,\mathrm{v}_c \tag{9}$$

It follows that their pull-backs $\rho^*\omega$ satisfy the same closed exterior differential system. Let Θ be the contact ideal and μ_a, $1 \leq a \leq r$, be the horizontal part of $\rho^*\omega_a$, i.e. $\rho^*\omega = \mu_a \mod \Theta$. It satisfy thus:

$$\mathrm{d}\mu_c \mod \Theta = -\sum_{a<b} C_{abc}\,\mu_a \wedge \mu_b.$$

Rewriting this equation in terms of $\mathfrak{x}$ thus provides differential relationships satisfied by the Maurer-Cartan invariants. We give them explicitly.

Proposition 5.1 *The Maurer-Cartan invariants are subject to the relationships*

$$\Delta_{cij} : \mathcal{D}_j(K_{ic}) - \mathcal{D}_i(K_{jc}) + \sum_{1\leq a<b\leq r} C_{abc}\,(K_{ia}K_{jb} - K_{ja}K_{ib}) + \sum_{k=1}^{m} \Lambda_{ijk}\,K_{kc} = 0$$

where C_{abc} are the structure constants (9) and Λ_{ijk} are the coefficients of the commutations rules for the invariant derivations (Theorem 3.3).

PROOF: From Theorem 3.2 we know that $\mu_c \equiv -\sum_k K_{kc}\,\mathfrak{x}_k$. Since $\mathrm{d}\mathfrak{x}_k = -\sum_{i<j} \Lambda_{ijk}\mathfrak{x}_i \wedge \mathfrak{x}_j \mod \Theta$ we have

$$\mathrm{d}\mu_c \mod \Theta = \sum_{i<j}\left(\mathcal{D}_j(K_{ic}) - \mathcal{D}_i(K_{jc}) + \sum_{k=1}^{m} \Lambda_{ijk}\,K_{kc}\right)\mathfrak{x}_i \wedge \mathfrak{x}_j$$

while

$$\mu_a \wedge \mu_b = \sum_{i<j}(K_{ia}K_{jb} - K_{ib}K_{ja})\,\mathfrak{x}_i \wedge \mathfrak{x}_j$$

Equating the coefficients of $\mathfrak{x}_i \wedge \mathfrak{x}_j$ in $\mathrm{d}\mu_c = -\sum C_{abc}\mu_a \wedge \mu_b$ we obtain Δ_{cji}. □

In the case of curves, when there is only one independent variable, Proposition 5.1 predicts no syzygies. In Section 7.1 we exhibit a simple case where the non minimality of the cross-section implies a non trivial syzygies on the Maurer-Cartan invariants. In Section 7.2 we exhibit an example involving two independent variables where Proposition 5.1 does not provide a complete set of syzygies when the cross-section is not chosen of minimal order. We conjecture that in the case of an action transitive on J^0 and with the choice of a cross-section of minimal order, the syzygies of Proposition 5.1 form a complete set, i.e. any other differential relationships among the *monotone* derivatives of the Maurer-Cartan invariants belongs to the differential ideal they generate. The conjecture has been verified on classical examples by applying differential elimination [13, 2, 15, 14] on the complete set of differential syzygies for the normalized invariants $\mathcal{I}^{s+1}$ given in [16].

6 Matrix group

Because of the explicit form that can be achieved for matrix groups, the Maurer-Cartan form is often defined to be the one-form $\Omega = \sum_{a=1}^{r} \omega_a \otimes \mathrm{v}_a|_e$ with value in the Lie algebra of $\mathcal{G}$. The structure equation can then be written compactly $\mathrm{d}\Omega = -\frac{1}{2}[\Omega, \Omega]$ (see for instance [21]). In this section we specialize the results about Maurer-Cartan invariants to the case where the group is known by a representation. It is in particular intensely used in the evolution of curves in Klein geometries and their relationships to integrable systems (see [32, 34, 35, 31, 36, 37] and references therein). As a way of making the connection, we set as goal to show how the results we have obtained are linked with the results of the first part of [31]. In Section 7.3 we discuss a classical example in both the general formalism and in the matricial setting.

Let $\tau : \mathcal{G} \to \mathrm{Gl}(d)$ be a representation of the Lie group $\mathcal{G}$. The Lie algebra of $\tau(\mathcal{G})$ can be understood as a subspace of the $d \times d$ matrices. Let $(\mathfrak{a}_1, \ldots, \mathfrak{a}_r)$ be the $d \times d$ matrices that are images by τ_* at identity of the right invariant vector fields $(\mathrm{v}_1, \ldots, \mathrm{v}_r)$ on $\mathcal{G}$. The Maurer-Cartan form $\tilde{\Omega}$ on $\tau(\mathcal{G})$ can then be interpreted as a $d \times d$ matrix with one-form as entries. If $\omega_1, \ldots, \omega_r$ are the right invariant one-forms that are dual to $\mathrm{v}_1, \ldots, \mathrm{v}_r$ then

$$\tilde{\Omega} = \sum_{a=1}^{r} \omega_a \, \mathfrak{a}_a.$$

One shows that[2] $\tilde{\Omega} = (\mathrm{d}\tau)\,(\tau)^{-1}$ and the structure equation is given by $\mathrm{d}\tilde{\Omega} = -\tilde{\Omega} \wedge \tilde{\Omega}$.

Consider an action $g : \mathcal{G} \times \mathrm{J}^0 \to \mathrm{J}^0$ effective on subsets, and its prolongations to the higher order jet spaces J^k, for any $k \in \mathbb{N}$. Assume that s is greater than the stabilization order and consider a cross-section $\mathcal{P}$ to the orbits on J^s cut out by $P = (p_1, \ldots, p_r)$, $p_i \in \mathcal{F}(\mathrm{J}^s)$. Let $\rho : \mathrm{J}^s \to \mathcal{G}$ be the associated moving frame and $\tilde{\rho} = \tau \circ \rho : \mathrm{J}^s \to \mathrm{Gl}(d)$.

The matrices $\mathcal{Q}_i = \mathcal{D}_i(\tilde{\rho})\,\tilde{\rho}^{-1}$, $1 \leq i \leq m$, were defined in [31] as the *curvature matrices* extending the concepts of [12]. In the case of curves ($m = 1$) it is the Serret-Frenet matrix [38, 32, 33, 37].

Proposition 6.1 *The curvature matrices satisfy* $\mathcal{Q}_i = \sum_{a=1}^{r} K_{ia}\, \mathfrak{a}_a$ *where* K *is the matrix of Theorem 3.2 i.e.* $K = \bar{\iota}\left(\mathrm{D}(P)\mathrm{V}(P)^{-1}\right)$.

The result was proved in [31] by calculus and we show how it can be recovered by Theorem 3.2.

PROOF: On one hand $\rho^*\omega_a = -\sum_{i=1}^{m} K_{ia}\, \mathfrak{x}_i \mod \Theta$, according to Theorem 3.2, and therefore

$$\tilde{\rho}^*\tilde{\Omega} = \sum_{a=1}^{r} \rho^*\omega_a\, \mathfrak{a}_a = \sum_{a=1}^{r} \left(\sum_{i=1}^{m} K_{ia}\, \mathfrak{x}_i \right) \mathfrak{a}_a \mod \Theta.$$

On the other hand, $\mathcal{D}$ and $\mathfrak{x}$ are dual so that $\mathrm{d}\tilde{\rho} = \sum_{i=1}^{m} \mathcal{D}_i(\tilde{\rho})\, \mathfrak{x}_i \mod \Theta$ and thus

$$\tilde{\rho}^*\tilde{\Omega} = (\mathrm{d}\tilde{\rho})\, \tilde{\rho}^{-1} = \sum_{i=1}^{m} \mathcal{Q}_i\, \mathfrak{x}_i \mod \Theta.$$

We thus obtain the announced result. □

[2] Many reference books, as [21], prone the left invariant version rather than the right invariant one used here. In that setting $\tilde{\Omega} = (\tau)^{-1}\,(\mathrm{d}\tau)$.

Corollary 4.3 of Theorem 4.2 furthermore informs us that the invariants in the curvature matrices $\mathcal{Q}$ form a generating set when the action is transitive on J^0.

To conclude let us note that by substituting $\tilde{\rho}^*\tilde{\Omega} = \mathcal{Q}_i\,\mathfrak{x}_i$ in the structure equation $\mathrm{d}(\tilde{\rho}^*\tilde{\Omega}) = -\tilde{\rho}^*\tilde{\Omega} \wedge \tilde{\rho}^*\tilde{\Omega}$ we obtain the syzygies on the curvature matrices $\mathcal{Q}_i$:

$$\mathcal{D}_j(\mathcal{Q}_i) - \mathcal{D}_i(\mathcal{Q}_j) = \Lambda_{ijk}\,\mathcal{Q}_k + \mathcal{Q}_j\,\mathcal{Q}_i - \mathcal{Q}_i\,\mathcal{Q}_j$$

given in [31, proposition 9] to generalize the *zero-curvature equation* [12] that applies only when the derivations $\mathcal{D}$ commute. This is thus a matricial form of the syzygies of Proposition 5.1.

7 Examples

We first give two examples where computations are easily performed, one for curves and and one for surfaces. Those examples are variations of the running examples in [16]. We then treat the two very classical geometries, curves and surfaces in Euclidean 3-space, in order to illustrate the general theory of this paper. With the knowledge of the infinitesimal generators of the action and a choice of cross-section only, we can characterize a set of generators and compute their syzygies. While non trivial applications are presented in [20], this section is only meant for understanding.

7.1 Curves

This example illustrates first the material in Section 2 and 4 for a minimal order cross-section. We then move to a cross-section that is not of minimal order. The Maurer-Cartan invariants still form a generating set, but there is a differential relationship among them that is not predicted by Proposition 5.1.

Group action

We consider the group $\mathcal{G} = \mathbb{R}^* \ltimes \mathbb{R}^2$ with multiplication $(\lambda_1, \lambda_2, \lambda_3)\cdot(\mu_1, \mu_2, \mu_3) = (\lambda_1\mu_1, \lambda_2 + \lambda_1\,\mu_2, \lambda_3 + \mu_3)$.

A basis of right invariant vector fields is given by [41, Example 2.46]

$$\mathrm{v}_1 = \lambda_1 \frac{\partial}{\partial\lambda_1} + \lambda_2 \frac{\partial}{\partial\lambda_2}, \quad \mathrm{v}_2 = \frac{\partial}{\partial\lambda_2}, \quad \mathrm{v}_3 = \frac{\partial}{\partial\lambda_3}$$

so that the dual Maurer-Cartan forms are:

$$\omega_1 = \frac{1}{\lambda_1}\,\mathrm{d}\lambda_1, \quad \omega_2 = -\frac{\lambda_2}{\lambda_1}\,\mathrm{d}\lambda_1 + \mathrm{d}\lambda_2, \quad \omega_3 = \mathrm{d}\lambda_3$$

We consider the action g of $\mathcal{G}$ on $\mathcal{X}^1 \times \mathcal{U}^1$ given by:

$$g^*x = \lambda_1\,x + \lambda_2, \quad g^*u = u + \lambda_3.$$

It extends to derivatives by $g^* u_k = \frac{u_k}{\lambda_1^k}$, for $k > 0$. The infinitesimal generators of this prolonged action are given by:

$$\mathrm{V}_1 = x\,\frac{\partial}{\partial x} - k\,u_k \frac{\partial}{\partial u_k}, \quad \mathrm{V}_2 = \frac{\partial}{\partial x}, \quad \mathrm{V}_3 = \frac{\partial}{\partial u_0}.$$

Minimal order cross-section

Let us consider first the cross-section defined by $P = (x, u_0, u_1 - 1)$. The Maurer-Cartan matrix is

$$K = \bar{\iota}\left(D(P)\,V(P)^{-1}\right) = \begin{pmatrix} -\bar{\iota}u_2 & 1 & 1 \end{pmatrix}.$$

By Theorem 4.2 (or Theorem 4.1) we deduce that $\{\bar{\iota}u_2\}$ is a generating set of differential invariants. By Theorem 3.2 we know that:

$$\mathcal{D}(\bar{\iota}u_i) = \bar{\iota}u_{i+1} + \bar{\iota}u_2\,\bar{\iota}(\mathrm{V}_1(u_i)) - \bar{\iota}(\mathrm{V}_2(u_i)) - \bar{\iota}(\mathrm{V}_3(u_i))$$

so that, for $i > 0$,

$$\mathcal{D}(\bar{\iota}u_i) = \bar{\iota}u_{i+1} - i\,\bar{\iota}u_2\,\bar{\iota}u_i.$$

We can verify the recurrence formulae above with the explicit form of the moving frame associated to $P = (x, u_0, u_1 - 1)$. It is indeed given by

$$\rho^*\lambda_1 = u_1, \quad \rho^*\lambda_2 = -x\,u_1, \quad \rho^*\lambda_3 = -u_0$$

so that the normalized invariants and invariant derivation are:

$$\bar{\iota}u_i = \frac{u_i}{u_1^i} \quad \text{and} \quad \mathcal{D} = \frac{1}{u_1}\mathrm{D}.$$

Non minimal order cross-section

Let us consider now the cross-section defined by $P = (x, u_0, u_2 - 1)$. The Maurer-Cartan matrix is

$$K = \bar{\iota}\left(D(P)\,V(P)^{-1}\right) = \begin{pmatrix} \psi & 1 & \phi \end{pmatrix} \text{ where } \psi = -\frac{1}{2}\bar{\iota}u_3,\ \phi = \bar{\iota}u_1.$$

By Theorem 3.2 we know that:

$$\mathcal{D}(\bar{\iota}u_i) = \bar{\iota}u_{i+1} - \psi\,\bar{\iota}(\mathrm{V}_1(u_i)) - \bar{\iota}(\mathrm{V}_2(u_i)) - \phi\,\bar{\iota}(\mathrm{V}_3(u_i))$$

so that, for $i > 0$,

$$\mathcal{D}(\bar{\iota}u_i) = \bar{\iota}u_{i+1} - i\,\psi\,\bar{\iota}u_i \tag{10}$$

By Theorem 4.2 we deduce that $\{\phi, \psi\}$ is a generating set of differential invariants. Applying (10) for $i = 1$ we obtain $\mathcal{D}(\bar{\iota}u_1) = 1 + \psi\,\bar{\iota}u_1$ so that

$$\phi\,\psi + \mathcal{D}(\phi) - 1 = 0.$$

On one hand this proves that $\{\phi\}$ is a generating set. On the other hand this latter equation is a syzygie that is not predicted by Proposition 5.1.

We can verify the recurrence formula above as the explicit form of the moving frame associated to $P = (x, u_0, u_2 - 1)$ is given by

$$\rho^*\lambda_1 = \sqrt{u_2}, \quad \rho^*\lambda_2 = -x\sqrt{u_2}, \quad \rho^*\lambda_3 = -u_0.$$

The normalized invariants and invariant derivation are thus

$$\bar\iota u_i = \frac{u_i}{u_2^{i/2}} \quad \text{and} \quad \mathcal{D} = \frac{1}{\sqrt{u_2}}\mathrm{D}.$$

7.2 Surfaces

We move now to the case of more than one independent variable. This example illustrates first the material discussed in Section 2, 4, and 5 for a minimal order cross-section. We then move to a cross-section that is not of minimal order.

Group action

We consider the group $\mathcal{G} = \mathbb{R}^* \ltimes \mathbb{R}^3$ with action on $\mathcal{X}^2 \times \mathcal{U}^1$ given by

$$g^*x = \lambda_1\, x + \lambda_2, \quad g^*y = \lambda_1\, y + \lambda_3, \quad g^*u = u + \lambda_4.$$

As mentionned earlier we only need the infinitesimal generators of the action as input data. Their prolongations to J are:

$$\mathrm{V}_1 = x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y} - \sum_{i,j\geq 0} (i+j)\, u_{ij}\,\frac{\partial}{\partial u_{ij}}, \quad \mathrm{V}_2 = \frac{\partial}{\partial x}, \quad \mathrm{V}_3 = \frac{\partial}{\partial y}, \quad \mathrm{V}_4 = \frac{\partial}{\partial u}.$$

Minimal order cross-section

The cross-section cut out by $P = (x, y, u_{00}, u_{10} - 1)$ is of minimal order. The associated Maurer-Cartan matrix is

$$K = \begin{pmatrix} \tau & 1 & 0 & 1 \\ \sigma & 0 & 1 & \psi \end{pmatrix} \text{ where } \psi = \bar\iota u_{01},\ \tau = -\bar\iota u_{20},\ \sigma = -\bar\iota u_{11}.$$

Theorem 3.2 implies that for $i + j > 0$

$$\mathcal{D}_1(\bar\iota u_{ij}) = \bar\iota u_{i+1,j} + (i+j)\,\tau\,\bar\iota u_{ij}, \quad \mathcal{D}_2(\bar\iota u_{ij}) = \bar\iota u_{i,j+1} + (i+j)\,\sigma\,\bar\iota u_{ij}$$

while, by Theorem 3.3,

$$[\mathcal{D}_1, \mathcal{D}_2] = \sigma\,\mathcal{D}_1 - \tau\mathcal{D}_2$$

Theorem 4.2 implies that $\{\psi, \tau, \sigma\}$ is a generating set of differential invariants. Proposition 5.1 indicates furthermore that

$$\mathcal{D}_1(\tau) - \mathcal{D}_2(\sigma) = 0, \quad \mathcal{D}_1(\psi) = \tau\,\psi - \sigma.$$

From the second equation we deduce that $\sigma = \tau\,\psi - \mathcal{D}_1(\psi)$ so that $\{\psi, \tau\}$ is already a generating set. It is subject to a differential syzygy of order 2 that is obtained by substituting τ in the first equation.

Non minimal cross-section

The cross-section cut out by $P = (x, y, u_{00}, u_{20} - 1)$ is of minimal order. The associated Maurer-Cartan matrix is

$$K = \begin{pmatrix} \tau & 1 & 0 & \phi \\ \sigma & 0 & 1 & \psi \end{pmatrix} \text{ where } \phi = \bar{\iota}u_{10},\ \psi = \bar{\iota}u_{01},\ \tau = -\frac{1}{2}\,\bar{\iota}u_{30},\ \sigma = -\frac{1}{2}\,\bar{\iota}u_{21}.$$

Theorem 4.2 implies that $\{\phi,\ \psi,\ \tau,\ \sigma\}$ is a generating set of differential invariants and Proposition 5.1 indicates

$$\mathcal{D}_1(\tau) - \mathcal{D}_2(\sigma) = 0, \quad \mathcal{D}_1(\psi) - \mathcal{D}_2(\phi) = \sigma\,\phi - \tau\,\psi.$$

From the second equation we deduce that $\{\phi, \psi, \tau\}$ can be taken as generating set.

Theorem 3.2 implies that for $i + j > 0$

$$\mathcal{D}_1(\bar{\iota}u_{ij}) = \bar{\iota}u_{i+1,j} + (i+j)\,\tau\,\bar{\iota}u_{ij}, \quad \mathcal{D}_2(\bar{\iota}u_{ij}) = \bar{\iota}u_{i,j+1} + (i+j)\,\sigma\,\bar{\iota}u_{ij}$$

while, by Theorem 3.3,

$$[\mathcal{D}_1, \mathcal{D}_2] = \tau\,\mathcal{D}_1 - \sigma\mathcal{D}_2.$$

Applying the recurrence formula for $i = 1, j = 0$ we obtain $\mathcal{D}_1(\bar{\iota}u_{10}) = 1 + \tau\bar{\iota}u_{10}$ i.e.

$$\mathcal{D}_1(\phi) = 1 + \tau\,\phi$$

from which we can draw τ in terms of ϕ. Therefore $\{\phi, \psi\}$ is already a generating set subject to a differential syzygy of order 2.

7.3 Curves in Euclidean geometry

Group action

We consider the classical action of SE(3) on space curves. We have $\mathrm{J}^0 = \mathcal{X}^1 \times \mathcal{U}^2$ with coordinate (x, u, v). The infinitesimal generators of the action are:

$$\mathrm{V}_1^0 = \frac{\partial}{\partial x}, \quad \mathrm{V}_2^0 = \frac{\partial}{\partial u}, \quad \mathrm{V}_3^0 = \frac{\partial}{\partial v},$$
$$\mathrm{V}_4^0 = v\,\frac{\partial}{\partial u} - u\,\frac{\partial}{\partial v}, \quad \mathrm{V}_5^0 = x\frac{\partial}{\partial u} - u\frac{\partial}{\partial x}, \quad \mathrm{V}_6^0 = x\frac{\partial}{\partial v} - v\frac{\partial}{\partial x}$$

so that their prolongations are given by

$$
\begin{aligned}
\mathrm{V}_1 &= \frac{\partial}{\partial x}, \quad \mathrm{V}_2 = \frac{\partial}{\partial u}, \quad \mathrm{V}_3 = \frac{\partial}{\partial v}, \quad \mathrm{V}_4 = \sum_k v_k \frac{\partial}{\partial u_k} - u_k \frac{\partial}{\partial v_k}, \\
\mathrm{V}_5 &= -u_0\,\mathrm{D}_x + \sum_k \mathrm{D}^k(x - u_0\,u_1)\frac{\partial}{\partial u_k} - \sum_k \mathrm{D}^k(u_0\,v_1)\frac{\partial}{\partial v_k}, \\
\mathrm{V}_6 &= -v_0\mathrm{D} - \sum_k \mathrm{D}^k(v_0\,u_1)\frac{\partial}{\partial u_k} + \sum_k \mathrm{D}^k(x - v_0\,v_1)\frac{\partial}{\partial v_k}
\end{aligned}
$$

The action is transitive on J^1 and becomes locally free on J^2 with generic orbits of codimension 1.

Generation with non mininimal cross-section

We consider the non-minimal coordinate cross-section proved to fail Theorem 4.1 in [42, Section 6]:

$$P = (x, u_0, v_0, v_1, v_2, v_3 - 1)$$

We have

$$K = \begin{pmatrix} 1 & \bar{\iota}u_1 & 0 & -\dfrac{1}{\bar{\iota}u_2} & \dfrac{1}{3}\dfrac{\bar{\iota}(u_2\,v_4 - u_3)}{\bar{\iota}(u_1 u_2)} & -\dfrac{\bar{\iota}(u_1)}{\bar{\iota}(u_2)} \end{pmatrix}.$$

By Theorem 4.2 we expect that $\{\,\bar{\iota}u_1, \bar{\iota}u_2, \bar{\iota}w\,\}$, where $w = u_2\,v_4 - u_3$, to be a generating set. And indeed, working with Theorem 3.2 as indicated in the proof of Theorem 4.2 we find:

$$
\begin{gathered}
\bar{\iota}u_3 = \bar{\iota}w + \mathcal{D}(\bar{\iota}u_2), \qquad \bar{\iota}v_4 = \frac{\mathcal{D}(\bar{\iota}u_2) + 2\,\bar{\iota}w}{\bar{\iota}u_2} \\
\bar{\iota}u_4 = \mathcal{D}^2(\bar{\iota}u_2) + \mathcal{D}(\bar{\iota}w) + \frac{\bar{\iota}u_2\,\bar{\iota}w}{\bar{\iota}u_1} - \frac{3 + 3\,\bar{\iota}u_1{}^2 + 4\,\bar{\iota}w^2 + 4\,\bar{\iota}w\,\mathcal{D}(\bar{\iota}u_2)}{3\,\bar{\iota}u_2}
\end{gathered}
$$

With the complete set of syzygies on $\mathcal{I}^{s+1}$ [16, Theorem 5.13], we find by differential elimination [15, 14] that $\{\bar{\iota}u_1, \bar{\iota}u_2, \bar{\iota}w\}$ are algebraically dependent. More precisely we have:

$$\bar{\iota}u_2^2 - \bar{\iota}u_1\,\bar{\iota}u_2 - \frac{1}{3}\frac{\bar{\iota}w\,(1 + \bar{\iota}u_1^2)}{\bar{\iota}u_1} = 0$$

Therefore $\{\bar{\iota}u_1, \bar{\iota}u_2\}$ form a generating set of differential invariants. Indeed, given the analytic expression for the curvature and the torsion [11] we can rewrite them in terms of the normalized invariants by a simple substitution (Theorem 2.7). We obtain:

$$\kappa = \sqrt{\frac{\bar{\iota}u_2^2}{(1 + \bar{\iota}u_1^2)^3}}, \quad \tau = \frac{1}{\bar{\iota}u_2(1 + \bar{\iota}u_1^2)}.$$

Minimal order cross-section and matrix representation

The cross-section cut out by $P = (x, u_0, v_0, u_1, v_1, v_2)$ is of minimal order. By Theorem 4.1 we know that $\{\bar{\iota}v_2, \bar{\iota}u_3\}$ is a generating set of differential invariants. On one hand, applying the replacement property of normalized invariants (Corollary 2.7) to the explicit expression of the curvature and the torsion we obtain:

$$\kappa = \bar{\iota}u_2, \quad \tau = \frac{\bar{\iota}v_3}{\bar{\iota}u_2}.$$

On the other hand the Maurer-Cartan matrix can be made explicit in terms of the normalized invariants, so that we can now recognize its expression in terms of the curvature and the torsion:

$$K = \begin{pmatrix} 1 & 0 & 0 & -\dfrac{\bar{\iota}v_3}{\bar{\iota}u_2} & \bar{\iota}u_2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -\tau & \kappa & 0 \end{pmatrix}$$

A representation of $SE(3)$ is given by the matrix group

$$\left\{ \begin{pmatrix} 1 & 0 \\ \lambda & R \end{pmatrix} \middle| \, \lambda \in \mathbb{R}^3, \; R \in SO(3) \right\}.$$

The Lie algebra of this matrix group is then spanned by

$$\mathfrak{a}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; \mathfrak{a}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; \mathfrak{a}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathfrak{a}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \; \mathfrak{a}_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; \mathfrak{a}_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

where $\mathfrak{a}_i$ and V_i are associated to the same right invariant vector field v_i on $\mathcal{G}$. We can determine the curvature matrix Q from Proposition 6.1, and recognize there the equations for the Serret-Frenet frame [21, Chapter 1]:

$$Q = \mathfrak{a}_1 + \kappa\,\mathfrak{a}_5 - \tau\,\mathfrak{a}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -\kappa & 0 \\ 0 & \kappa & 0 & \tau \\ 0 & 0 & -\tau & 0 \end{pmatrix}.$$

7.4 Surfaces in Euclidean geometry

Group action

We choose coordinate functions (x, y, u) for $\mathbb{R}^2 \times \mathbb{R}$, i.e. we consider x, y as the independent variables and u as the dependent variable. The infinitesimal generators of the classical action of the Euclidean group $E(3)$ on $\mathbb{R}^3$ are:

$$\mathrm{V}_1 = \frac{\partial}{\partial x},\ \mathrm{V}_2 = \frac{\partial}{\partial y},\ \mathrm{V}_3 = \frac{\partial}{\partial u},$$
$$\mathrm{V}_4 = x\,\frac{\partial}{\partial u} - u\,\frac{\partial}{\partial x}, \mathrm{V}_5 = y\,\frac{\partial}{\partial u} - u\,\frac{\partial}{\partial y}, \mathrm{V}_6 = x\,\frac{\partial}{\partial y} - y\,\frac{\partial}{\partial x}.$$

The non zero structure constants are given by the following commutators of the infinitesimal generators:

$$[\mathrm{V}_1,\mathrm{V}_4] = \mathrm{V}_3,\ [\mathrm{V}_1,\mathrm{V}_6] = \mathrm{V}_2,\ [\mathrm{V}_2,\mathrm{V}_5] = \mathrm{V}_3,\ [\mathrm{V}_2,\mathrm{V}_6] = -\mathrm{V}_1,\ [\mathrm{V}_3,\mathrm{V}_4] = -\mathrm{V}_1,$$
$$[\mathrm{V}_3,\mathrm{V}_5] = -\mathrm{V}_2,\ [\mathrm{V}_4,\mathrm{V}_5] = -\mathrm{V}_6,\ [\mathrm{V}_4,\mathrm{V}_6] = \mathrm{V}_5,\ [\mathrm{V}_5,\mathrm{V}_6] = -\mathrm{V}_4.$$

Minimal order cross-section and syzygies

Let us choose the classical cross-section defined by $P = (x, y, u, u_{10}, u_{01}, u_{11})$. The Maurer-Cartan matrix is then

$$K = \begin{pmatrix} 1 & 0 & 0 & \kappa & 0 & \phi \\ 0 & 1 & 0 & 0 & \tau & \psi \end{pmatrix}$$

where

$$\kappa = \bar\iota u_{20}, \quad \tau = \bar\iota u_{02}, \quad \phi = \frac{\bar\iota u_{21}}{\bar\iota u_{20} - \bar\iota u_{02}}, \ \text{ and }\ \psi = \frac{\bar\iota u_{12}}{\bar\iota u_{20} - \bar\iota u_{02}}.$$

By Theorem 3.3 we have

$$[\mathcal{D}_2, \mathcal{D}_1] = \phi\,\mathcal{D}_1 + \psi\,\mathcal{D}_2.$$

The non zero syzygies of Proposition 5.1 are:

$$\begin{array}{lcl} \Delta_{412} & : & \mathcal{D}_2(\kappa) - \phi\,(\kappa - \tau) = 0 \\ \Delta_{512} & : & -\mathcal{D}_1(\tau) - \psi\,(\tau - \kappa) = 0 \\ \Delta_{612} & : & \mathcal{D}_2(\phi) - \mathcal{D}_1(\psi) - \kappa\,\tau - \phi^2 - \psi^2 = 0 \end{array}$$

The first two syzygies imply that

$$\phi = \frac{\mathcal{D}_2(\kappa)}{\kappa - \tau}, \quad \psi = \frac{\mathcal{D}_1(\tau)}{\kappa - \tau}. \tag{11}$$

It follows that $\{\kappa, \tau\}$ form a generating set. Indeed, from their analytic expression [11], [21, Section 1.1] we can write the Gauss and mean curvatures in terms of the normalized invariants, thanks to Theorem 2.7, and eventually in terms of the Maurer-Cartan invariants by Theorem 3.2 and 4.2:

$$\sigma = \frac{u_{20}u_{02} - u_{11}}{(1 + u_{10}^2 + u_{01}^2)^2} = \bar\iota u_{20}\,\bar\iota u_{02} = \kappa\,\tau,$$
$$\pi = \frac{1}{2}\,\frac{(1 + u_{01}^2)u_{20} - 2_{10}u_{01}u_{11} + (1 + u_{10}^2)u_{20}}{(1 + u_{10}^2 + u_{01}^2)^{\frac{3}{2}}} = \frac{1}{2}(\bar\iota u_{20} + \bar\iota u_{02}) = \frac{1}{2}(\kappa + \tau).$$

Our generating invariants $\{\kappa, \tau\}$ are thus the principal curvatures. Substituting (11) in Δ_{612}, we retrieve the Codazzi equation as their syzygy:

$$\mathcal{D}_2\left(\frac{\mathcal{D}_2(\kappa)}{\kappa-\tau}\right)-\mathcal{D}_2\left(\frac{\mathcal{D}_2(\tau)}{\kappa-\tau}\right)=\left(\frac{\mathcal{D}_2(\kappa)}{\kappa-\tau}\right)^2+\left(\frac{\mathcal{D}_2(\tau)}{\kappa-\tau}\right)^2+\kappa\tau.$$

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