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A THEOREM OF ÉLIE CARTAN

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André Weil [1] and Hopf and Samelson [2] have given a topological proof of the following theorem of Élie Cartan.

Two maximal Abelian subgroups of a compact connected Lie group G are conjugate within G .

I present a simple metric proof.

LEMMA. *If x and y are elements of the Lie algebra \mathfrak{g} of G then $[x, A_\sigma y]$ vanishes for some inner automorphism A_σ of G .*

PROOF. Because G is compact one can define on \mathfrak{g} a nonsingular bilinear form (u, v) which is invariant: $([u, v], w) + (v, [u, w]) = 0$. We choose ϵ in G so that $(x, A_\sigma y)$ attains its minimum for $\sigma = \epsilon$; without loss of generality we may assume ϵ to be the neutral element of G , and then $A_\sigma y = y$. If now z is any element of \mathfrak{g} the function $(x, A_{\exp(tz)} y)$ has a minimum for $t=0$, so that its derivative vanishes there. Thus, keeping in mind that

$$\left. \frac{d}{dt} A_{\exp(tz)} y \right|_{t=0} = [z, y],$$

we have $(x, [z, y]) = 0$. From this equation and from the invariance of the bilinear form it follows that $([x, y], z) = 0$ for all z ; this can happen only if $[x, y]$ vanishes, for the bilinear form is nondegenerate.

Before proving Cartan's theorem I recall some well-known facts: A maximal Abelian subgroup \mathcal{H} of G is a torus group; there is an element x in the Lie algebra \mathfrak{h} of \mathcal{H} such that the one parameter group $\exp tx$ is dense in \mathcal{H} ; if y belongs to \mathfrak{g} and $[x, y] = 0$, then y must lie in \mathfrak{h} .

Matters being so, let \mathcal{H}' be a second maximal Abelian subgroup of G and x' an element of its Lie algebra bearing the same relation

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to \mathcal{H}' as x does to \mathcal{H} . Now choose σ in \mathcal{G} so that $[x, A_\sigma x']$ vanishes. Then $A_\sigma x'$ lies in \mathfrak{h} ; consequently $A_\sigma(\exp tx') \equiv \exp(tA_\sigma x')$ lies in \mathcal{H} for every t . So \mathcal{H} , being closed, includes the closure $A_\sigma(\mathcal{H}')$ of the one-parameter group $A_\sigma(\exp tx')$. Finally $A_\sigma(\mathcal{H}) = \mathcal{H}$, because both are maximal Abelian subgroups of \mathcal{G} .

Since every element of \mathcal{G} can be written as $\exp y$, the argument shows that every element of \mathcal{G} can be moved into \mathcal{H} by an inner automorphism of \mathcal{G} .

The referee has pointed out that the argument of the lemma above is very like one used by R. Bott [3] in another context.

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