Symmetries and first integrals of ordinary difference equations

P E Hydon

Department of Mathematics and Statistics
University of Surrey
Guildford, Surrey GU2 5XH
England

Abstract

This paper describes a new symmetry-based approach to the solution of ordinary difference equations. This approach makes it possible to devise techniques for solving difference equations, by adapting existing differential equation techniques. In particular, we obtain a new systematic method of determining one-parameter Lie groups of symmetries in closed form. This method enables the user to calculate the general solution of a given ordinary difference equation for which sufficiently many such symmetries can be obtained. Several examples are used to illustrate the technique for both transitive and intransive symmetry groups. It is shown that every linear second-order ordinary difference equation has a Lie algebra of symmetry generators that is isomorphic to $\mathfrak{sl}(3)$. The paper concludes with a new systematic method for constructing first integrals directly, which can be used even if no symmetries are known.

1. Introduction

Over a century ago, Sophus Lie introduced symmetry-based techniques for solving ordinary differential equations (ODEs). Lie's approach enables the user to determine Lie groups of symmetries of a given ODE. If a sufficiently large symmetry group can be found, it may be used to solve the ODE. For an introduction to symmetry methods for ODEs, see Olver (1993), Bluman & Kumei (1989), Stephani (1989), or Hydon (2000).

Recently, Maeda (1987) showed that autonomous systems of first-order ordinary difference equations (OΔEs) can be simplified or solved using an extension of Lie's method. Maeda also showed that the linearized symmetry condition for such OΔEs amounts to a set of functional equations. In general, these are hard to solve, but Maeda described two examples for which a very restrictive ansatz yields Lie symmetries. Gaeta (1993) used formal series expansions to derive some symmetries of those systems of OΔEs that are discretizations of continuous systems. Given an ODE with known Lie point symmetries, one may ask whether it is possible to discretize the ODE in a way that preserves at least some of the symmetries. Dorodnitsyn (1994) describes how this can be achieved, and lists some classes of OΔEs that have a given Lie group.

Maeda's ideas have been extended to nonautonomous systems and higher-order OΔEs by Quispel & Sahadevan (1993) and Levi *et al.* (1997). These papers describe different series-based methods for obtaining some solutions of the linearized symmetry condition. Series expansions can be calculated if the symmetry condition has a fixed point, although it is usually not obvious how to sum the series to obtain solutions in closed form. Unfortunately, the well-known method for calculating invariants requires the symmetry generator to be in closed form. This is a substantial limitation on the usefulness of series-based techniques.

In the current paper, we introduce a systematic method for obtaining Lie symmetries (in closed form) of a given OΔE. The new method uses the linearized symmetry condition, which is a functional equation, to a derive an associated system of linear partial differential equations. This system is similar to the system of determining equations for Lie symmetries of a given ODE. Moreover, having set up the mathematical framework for the new method, we find that it enables us to transfer all of the main symmetry methods for ODEs across to OΔEs; only minor modifications are needed. The paper describes some nontrivial applications of the underlying transfer principle (which will be discussed elsewhere).

Anco & Bluman (1998) have described a constructive method for obtaining first integrals of ODEs directly, without using Lie symmetries. Instead, the method uses the adjoint of the linearized symmetry condition. In §5 of the current paper, we introduce a technique for obtaining first integrals of OΔEs directly. Unlike the method described by Anco & Bluman, this technique does not use the adjoint of the linearized symmetry condition. Nevertheless, it has many features in common with the ODE method, and it is easy to use.

2. Symmetries of ordinary difference equations

In the following, we consider N^{th}-order OΔEs of the form

$$u_{n+N} = \omega\left(n, u_n, u_{n+1}, \ldots, u_{n+N-1}\right), \tag{2.1}$$

where ω is a given smooth function. Here the independent variable n is an integer. Some authors prefer to use x_n as the independent variable (particularly if the OΔE arises as a discretization of an ODE). It does not matter which notation is used, provided that there is a bijection that maps n to x_n. (N.B. The meshpoints, x_n, need not be uniformly spaced.)

For simplicity, attention is restricted to regions in which $\omega_{u_n} \neq 0$. A *first integral* of the OΔE (2.1) is a non-constant function,

$$\phi = \phi(n, u_n, \ldots, u_{n+N-1}),$$

that is constant on solutions of (2.1). In other words, a non-constant function ϕ is a first integral if

$$\phi\left(n+1, u_{n+1}, \ldots, u_{n+N-1}, \omega(n, u_n, \ldots, u_{n+N-1})\right) = \phi\left(n, u_n, \ldots, u_{n+N-2}, u_{n+N-1}\right). \tag{2.2}$$

This condition holds as an identity in the variables $n, u_n, \ldots, u_{n+N-1}$. To simplify the notation, we introduce the shift operator (restricted to solutions):

$$\mathcal{S} : \left(n, u_n, \ldots, u_{n+N-2}, u_{n+N-1}\right) \mapsto \left((n+1, u_{n+1}, \ldots, u_{n+N-1}, \omega(n, u_n, \ldots, u_{n+N-1})\right). \tag{2.3}$$

The action of this operator on any function is defined by the action on the function's arguments:

$$\mathcal{S}\left(F(n, u_n, \ldots, u_{n+N-1})\right) = F(\mathcal{S}n, \mathcal{S}u_n, \ldots, \mathcal{S}u_{n+N-1}) = F(n+1, u_{n+1}, \ldots, \omega).$$

Therefore (2.2) amounts to

$$\mathcal{S}\phi = \phi. \tag{2.4}$$

The OΔE (2.1) has N functionally independent first integrals, $\phi^1, \ldots, \phi^N$, and the general solution of (2.1) is

$$\phi^i = c^i, \qquad i = 1, \ldots, N, \tag{2.5}$$

where $c^1, \ldots, c^N$ are arbitrary constants. Here "functionally independent" means that the Jacobian does not vanish, that is,

$$\frac{\partial(\phi^1, \ldots, \phi^N)}{\partial(u_n, \ldots, u_{n+N-1})} \neq 0. \tag{2.6}$$

This condition ensures that (in principle, at least) each of $u_n, \ldots, u_{n+N-1}$ can be written as a function of $n, \phi^1, \ldots, \phi^N$. In particular, if

$$u_n = F(n, \phi^1, \ldots, \phi^N),$$

the general solution of (2.1) is

$$u_n = F(n, c^1, \ldots, c^N).$$

Throughout this paper, we shall work in the space of variables $n, u_n, \ldots, u_{n+N-1}$; the condition (2.6) enables us to use $n, \phi^1, \ldots, \phi^N$ as an alternative set of (local) coordinates. We consider symmetries that are analogous to dynamical (or internal) symmetries of ODEs. Dynamical symmetries are the point symmetries of an equivalent system of n first-order ODEs. This is the most general class of symmetries, and there are infinitely many independent symmetry generators. To find any of these, some kind of restriction (ansatz) is usually needed. For further details, see Stephani (1989) or Anderson *et al.* (1993).

A symmetry, Γ, of (2.1) maps the set of solutions to itself. Therefore, if

$$\Gamma : (n, \phi^1, \ldots, \phi^N) \mapsto (\hat{n}, \hat{\phi}^1, \ldots, \hat{\phi}^N), \tag{2.7}$$

each $\hat{\phi}^i$ is a smooth function of $\phi^1, \ldots, \phi^N$ only. Symmetries are required to be sufficiently smooth; for example, point symmetries of ODEs are diffeomorphisms of the space of independent and dependent variables. However, the corresponding space for an OΔE consists of a set of disjoint fibres, because the independent variable is discrete. Therefore symmetries of OΔEs must be fibre-preserving, which means that $\hat{n}$ is a function of n only. Moreover, the fibres must not be shuffled (because shuffling transformations are not allowed in the continuous case); consequently, symmetries of OΔEs are required to be *neighbour-preserving.* This can happens in one of two ways: either a symmetry is order-preserving, in which case

$$\hat{n}(n+1) = \hat{n}(n) + 1,$$

or else the symmetry is order-reversing, in which case its action on n is equivalent to a reflection. Henceforth, we shall restrict attention to order-preserving symmetries (which are generally more useful than reflections).

A symmetry is *trivial* if every solution is mapped to itself, that is, if

$$\hat{\phi}^i = \phi^i, \qquad i = 1, \ldots, N. \tag{2.8}$$

Lemma 1

For each $k \in \mathbb{Z}$, the transformation generated by $\mathcal{S}^k$ is a trivial symmetry of (2.1). (N.B. Where k is negative, $\mathcal{S}^k$ denotes $(\mathcal{S}^{-1})^{-k}$.)

Proof

For $k \geq 0$, apply $\mathcal{S}$ repeatedly to obtain

$$\mathcal{S}^k : (n, \phi^1, \ldots, \phi^N) \mapsto (n+k, \phi^1, \ldots, \phi^N). \tag{2.9}$$

Hence every solution $\phi^i = c^i$ is mapped to itself. Equation (2.9) also holds for $k < 0$, because (2.4) implies that $\mathcal{S}^{-1}\phi^i = \phi^i$. (N.B. The operator $\mathcal{S}^{-1}$ is obtained by first using (2.1) to write u_n as a function of $n, u_{n+1}, \ldots, u_{n+N}$, then replacing n by $n-1$; the condition $\omega_{u_n} \neq 0$ ensures that this is possible.)

One consequence of Lemma 1 is that every nontrivial order-preserving symmetry can be regarded as the composition of a *vertical* (or *evolutionary*) symmetry, which acts only on the first integrals ϕ^i (leaving n unchanged), and a trivial symmetry. Just as for ODEs, it is only the nontrivial symmetries that can be used to solve OΔEs, so we lose nothing by concentrating on vertical symmetries.

Lemma 2

Every nontrivial order-preserving symmetry (2.7) is equivalent to a vertical symmetry,

$$\tilde{\Gamma} : (n, \phi^1, \ldots, \phi^N) \mapsto (n, \tilde{\phi}^1, \ldots, \tilde{\phi}^N),$$

modulo a trivial symmetry.

Proof

The proof is by construction:

$$\tilde{\Gamma} = \mathcal{S}^{n-\hat{n}(n)}\Gamma$$

is the unique vertical symmetry that is equivalent to Γ. (The condition that Γ is order-preserving ensures that $n - \hat{n}(n)$ is independent of n.)

In view of Lemma 2, we shall consider only vertical symmetries from now on. Accordingly we seek symmetries (2.7) with $\hat{n} = n$. In terms of the original variables,

$$\Gamma : (n, u_n, \ldots, u_{n+N-1}) \mapsto (n, \hat{u}_n, \ldots, \hat{u}_{n+N-1}). \tag{2.10}$$

The action of Γ on the variables u_{n+k} is determined by the action on u_n. To see this, suppose that

$$\hat{u}_n = g(n, u_n, \ldots, u_{n+N-1}) = G(n, \phi^1, \ldots, \phi^N).$$

Then, on the set of solutions of the OΔE (2.1),

$$\hat{u}_{n+k} = G(n+k, \phi^1, \ldots, \phi^N) = \mathcal{S}^k \hat{u}_n, \qquad k = 1, \ldots, N. \tag{2.11}$$

The conditions (2.11) are analogous to the prolongation formulae for dynamical symmetries of ODEs, which reflect the necessity for contact conditions to be satisfied on the set of solutions.

The symmetry condition for the OΔE (2.1) is

$$\hat{u}_{n+N} = \omega\left(n, \hat{u}_n, \ldots, \hat{u}_{n+N-1}\right), \qquad \text{when (2.1) holds.} \tag{2.12}$$

Lie symmetries are obtained by linearizing the symmetry condition about the identity, as follows. We seek one-parameter (local) Lie groups of symmetries of the form

$$\hat{u}_n = u_n + \epsilon Q(n, u_n, \ldots, u_{n+N-1}) + O(\epsilon^2).$$

The function Q is called the *characteristic* of the one-parameter group. The prolongation formulae (2.11) yield

$$\hat{u}_{n+k} = u_{n+k} + \epsilon \mathcal{S}^k Q + O(\epsilon^2), \qquad k = 1, \ldots, N.$$

Expanding (2.12) in powers of ϵ yields the linearized symmetry condition

$$\mathcal{S}^N Q - X\omega = 0, \tag{2.13}$$

where

$$X = Q\,\partial_{u_n} + (\mathcal{S}Q)\partial_{u_{n+1}} + \cdots + (\mathcal{S}^{N-1}Q)\partial_{u_{n+N-1}}\,. \tag{2.14}$$

Note that the symmetry generator X, when written in terms of the first integrals, is of the form

$$X = F^1(\phi^1,\ldots,\phi^N)\partial_{\phi^1} + \cdots + F^N(\phi^1,\ldots,\phi^N)\partial_{\phi^N}. \tag{2.15}$$

This is because each $\hat{\phi}^i$ is a function of $\phi = (\phi^1,\ldots,\phi^N)$ only. The most important consequence of (2.15) is that X and $\mathcal{S}$ commute as operators on functions. Given any sufficiently smooth function,

$$h(n, u_n, \ldots, u_{n+N-1}) = H(n, \phi),$$

we use (2.15) to obtain

$$\mathcal{S}(XH) = \mathcal{S}\left(F^i(\phi)H_{\phi^i}(n,\phi)\right) = F^i(\phi)H_{\phi^i}(n+1,\phi) = X(\mathcal{S}H).$$

Therefore

$$\mathcal{S}(Xh) = X(\mathcal{S}h). \tag{2.16}$$

Just as for ODEs, the linearized symmetry condition is both necessary and sufficient to obtain the local Lie group of symmetries generated by X. To find solutions of (2.13), we must impose some constraint upon Q, in order to be able to split (2.13) into an overdetermined system of equations. For example, if we seek characteristics that are independent of u_{n+N-1}, it may be possible to split the linearized symmetry condition by equating powers of u_{n+N-1}. Before this can be achieved, some work is needed to transform (2.13) from a functional equation into a differential equation for Q. The next section introduces a method for accomplishing this transformation.

3. How to construct the determining equations

Consider the OΔE

$$u_{n+2} = \frac{u_n u_{n+1}}{2u_n - u_{n+1}}. \tag{3.1}$$

We shall seek *point symmetries*, whose characteristics are of the form $Q = Q(n, u_n)$. The linearized symmetry condition (2.13) is

$$Q(n+2,\omega) - \frac{2u_n^2}{(2u_n - u_{n+1})^2}Q(n+1,u_{n+1}) + \frac{u_{n+1}^2}{(2u_n - u_{n+1})^2}Q(n,u_n) = 0, \tag{3.2}$$

where ω denotes the right-hand side of (3.1). The chief difficulty with (3.2) is that the function Q takes three separate pairs of arguments. To overcome this difficulty, we differentiate (3.2) with respect to u_n, keeping ω fixed. Here u_{n+1} is regarded as a function of n, u_n, and ω. Using a standard result from multivariable calculus, we obtain

$$\frac{\partial u_{n+1}(n,u_n,\omega)}{\partial u_n} = -\frac{\omega_{u_n}}{\omega_{u_{n+1}}} = \frac{u_{n+1}^2}{2u_n^2}\,.$$

Therefore (3.2) reduces to

$$-\frac{u_{n+1}^2}{(2u_n - u_{n+1})^2}Q'(n+1,u_{n+1}) + \frac{2u_{n+1}}{(2u_n - u_{n+1})^2}Q(n+1,u_{n+1})$$

$$+\frac{u_{n+1}^2}{(2u_n-u_{n+1})^2}Q'(n,u_n)-\frac{2u_{n+1}^2}{u_n(2u_n-u_{n+1})^2}Q(n,u_n)=0,$$

where $'$ denotes a derivative with respect to the continuous variable. Rescaling, we obtain

$$-Q'(n+1,u_{n+1})+\frac{2}{u_{n+1}}Q(n+1,u_{n+1})+Q'(n,u_n)-\frac{2}{u_n}Q(n,u_n)=0 \tag{3.3}$$

Now differentiate (3.3) with respect to u_n, this time keeping u_{n+1} fixed, to obtain the ODE

$$\frac{d}{du_n}\left(Q'(n,u_n)-\frac{2}{u_n}Q(n,u_n)\right)=0. \tag{3.4}$$

Note that n appears only as a parameter at this stage. The general solution of (3.4) is

$$Q(n,u_n)=A(n)u_n+B(n)u_n^2. \tag{3.5}$$

Substituting (3.5) into (3.3) yields

$$A(n+1)=A(n),$$

and hence

$$A(n)=c_1.$$

[N.B. We use c_i to denote arbitrary constants.] The remaining unknown function, $B(n)$, is determined by substituting (3.5) into the original linearized symmetry condition (3.2); this yields

$$B(n+2)-2B(n+1)+B(n)=0.$$

Hence

$$B(n)=c_2n+c_3.$$

Summarizing these results, we have found a three-dimensional Lie algebra of symmetry generators, whose characteristics are linear combinations of

$$Q_1=u_n,\qquad Q_2=nu_n^2,\qquad Q_3=u_n^2. \tag{3.6}$$

The same method can be applied to any second-order OΔE

$$u_{n+2}=\omega(n,u_n,u_{n+1}),\qquad \omega_{u_{n+1}}\neq 0. \tag{3.7}$$

[N.B. The condition $\omega_{u_{n+1}}\neq 0$ ensures that the OΔE is genuinely second-order, not equivalent to a first-order problem with step length 2.] The linearized symmetry condition for point symmetries is

$$Q(n+2,\omega)-\omega_{u_{n+1}}Q(n+1,u_{n+1})-\omega_{u_n}Q(n,u_n)=0. \tag{3.8}$$

By eliminating $Q(n+2,\omega)$ and $Q(n+1,u_{n+1})$, we can transform (3.8) to an ODE of order three or less. First, we differentiate (3.8) with respect to u_n, keeping ω fixed, to obtain (after rescaling)

$$Q'(n+1,u_{n+1})+\lambda_{u_{n+1}}Q(n+1,u_{n+1})-Q'(n,u_n)+\lambda_{u_n}Q(n,u_n)=0, \tag{3.9}$$

where

$$\lambda=\ln|\omega_{u_{n+1}}|-\ln|\omega_{u_n}|.$$

Differentiating (3.9) with respect to u_n, keeping u_{n+1} fixed, we obtain

$$\lambda_{u_n u_{n+1}} Q(n+1, u_{n+1}) - Q''(n, u_n) + \lambda_{u_n} Q'(n, u_n) + \lambda_{u_n u_n} Q(n, u_n) = 0. \tag{3.10}$$

If $\lambda_{u_n u_{n+1}} = 0$, equation (3.10) is a second-order ODE for $Q(n, u_n)$. Otherwise, we must divide (3.10) by $\lambda_{u_n u_{n+1}}$ and differentiate once more with respect to u_n (keeping u_{n+1} fixed) to obtain a third-order ODE for $Q(n, u_n)$. Typically, the coefficients in the reduced ODE depend upon u_{n+1}. If this occurs, the ODE can be split by gathering together all terms with the same dependence upon u_{n+1}.

The solution of the reduced ODE contains arbitrary functions of n. It is substituted into the linearized symmetry condition, which can then be split into a system of determining OΔEs for the arbitrary functions (by grouping together all terms with the same dependence upon u_n and u_{n+1}). These determining equations are usually very easy to solve.

This approach is capable of yielding more symmetries than can be obtained by using fixed point summations. For example, Quispel & Sahadevan (1993) used a fixed-point method to look for symmetries of

$$u_{n+2} = \frac{2u_{n+1} - u_n(1 - u_{n+1}^2)}{1 - u_{n+1}^2 + 2u_n u_{n+1}}.$$

They found two independent characteristics of the form $Q = Q(n, u_n)$, namely

$$Q_1 = u_n^2 + 1, \qquad Q_2 = n(u_n^2 + 1).$$

However, the method described above yields Q_1, Q_2, and a third independent characteristic:

$$Q_3 = (u_n^2 + 1)\tan^{-1}(u_n).$$

The ansatz $Q = Q(n, u_n)$ yields only a few independent characteristics for second-order OΔEs. For instance, every linear homogeneous OΔE,

$$u_{n+2} = p(n)u_{n+1} + q(n)u_n, \tag{3.11}$$

has precisely three such characteristics, namely

$$Q_1 = u_n, \qquad Q_2 = U_1(n), \qquad Q_3 = U_2(n), \tag{3.12}$$

where $u_n = U_1(n)$ and $u_n = U_2(n)$ are linearly independent solutions of (3.11). Consequently, every OΔE that is linearizable by a point transformation

$$T : (n, u_n) \mapsto (n, \tilde{u}_n(n, u_n)) \tag{3.12}$$

also has three characteristics of the form $Q = Q(n, u_n)$. By contrast, every second-order ODE that is linear or linearizable by a point transformation has an eight-parameter Lie algebra of point symmetry generators, which is isomorphic to $\mathfrak{sl}(3)$. The characteristics of these symmetries are all linear in y', which suggests that we may obtain further symmetries of OΔEs by trying an ansatz of the form

$$Q = a(n, u_n)u_{n+1} + b(n, u_n). \tag{3.13}$$

Even though this ansatz is more complicated than before, our method can be used to obtain all such symmetries of a given OΔE. The linearized symmetry condition (2.13) amounts to

$$a(n+2, \omega)\, \mathcal{S}\omega + b(n+2, \omega) - \omega_{u_{n+1}} \left\{ a(n+1, u_{n+1})\, \omega + b(n+1, u_{n+1}) \right\}$$

$$-\omega_{u_n}\{a(n,u_n)u_{n+1}+b(n,u_n)\}=0. \tag{3.14}$$

This is reduced to a set of ODEs for $a(n,u_n)$ and $b(n,u_n)$ in essentially the same way as before. First differentiate with respect to u_n, keeping ω fixed, to eliminate $b(n+2,\omega)$. Then rescale to obtain an equation of the form

$$a(n+2,\omega)+\text{ other terms}=0.$$

Differentiate this with respect to u_n, keeping ω fixed, to eliminate $a(n+2,\omega)$. Then, by repeatedly rescaling and differentiating with respect to u_n (keeping u_{n+1} fixed), it is possible to eliminate all terms containing $a(n+1,u_{n+1})$, $b(n+1,u_{n+1})$, and their derivatives. Finally, the resulting ODE is split into a set of ODEs (by equating terms with the same dependence upon u_{n+1}). This approach readily generalizes to any ansatz for an OΔE of order $N\geq 2$. The calculations rapidly become too lengthy to be done by hand, but can be done with the aid of computer algebra. For the remainder of this paper, we shall state symmetries without describing the details of their derivation.

Returning to linear OΔEs of the form (3.11), it turns out that the linear ansatz (3.13) does provide us with more symmetries.

Theorem 3

Every second-order linear homogeneous OΔE has an eight-dimensional Lie algebra of symmetry generators whose characteristics are linear in u_{n+1}. This Lie algebra is isomorphic to $\mathfrak{sl}(3)$.

Proof

For a given linear homogeneous OΔE (3.11), with two linearly independent solutions $u_n=U_1(n)$ and $u_n=U_2(n)$, there are two functionally independent first integrals that are linear in u_n and u_{n+1}:

$$\phi^1(n,u_n,u_{n+1})=\frac{u_n\mathcal{S}U_2-U_2u_{n+1}}{U_1\mathcal{S}U_2-U_2\mathcal{S}U_1},\qquad \phi^2(n,u_n,u_{n+1})=\frac{U_1u_{n+1}-u_n\mathcal{S}U_1}{U_1\mathcal{S}U_2-U_2\mathcal{S}U_1}. \tag{3.15}$$

From (2.15), every symmetry generator

$$X=Q\,\partial_{u_n}+\mathcal{S}Q\,\partial_{u_{n+1}}$$

can be rewritten in the form

$$X=F^1(\phi^1,\phi^2)\,\partial_{\phi^1}+F^2(\phi^1,\phi^2)\,\partial_{\phi^2}, \tag{3.16}$$

where (by the chain rule)

$$F^i(\phi^1,\phi^2)=X(\phi^i(n,u_n,u_{n+1}))=\phi^i(n,Q,\mathcal{S}Q).$$

In particular, setting $Q=U_j(n)$ gives

$$X=\partial_{\phi^j}.$$

Therefore every one-parameter Lie group of symmetries of (3.11) has a characteristic of the form

$$Q(n,u_n,u_{n+1})=F^1(\phi^1,\phi^2)U_1(n)+F^2(\phi^1,\phi^2)U_2(n). \tag{3.17}$$

To find all characteristics that are linear in u_{n+1}, differentiate (3.17) twice with respect to u_{n+1} and (using the fact that $U_1(n)$ and $U_2(n)$ are independent) obtain constraints on the functions F^i. A basis for the space of such charactistics is

$$Q_1=U_1(n),\quad Q_2=U_2(n),\quad Q_3=\phi^1U_1(n),\quad Q_4=\phi^2U_1(n),$$

$$Q_5 = \phi^1 U_2(n), \quad Q_6 = \phi^2 U_2(n), \quad Q_7 = \phi^1 u_n, \quad Q_8 = \phi^2 u_n.$$

It is easy to check that the corresponding generators form a Lie algebra isomorphic to $\mathfrak{sl}(3)$.

N.B. It is *not* true that every OΔE that is linearizable by a point transformation has an eight-dimensional Lie algebra whose characteristics are linear in u_{n+1}. For example, the OΔE (3.1) can be linearized by the point transformation (3.12) with $\tilde{u}_n = 1/u_n$. However, there are no characteristics that are linear in u_{n+1} other than those that we found earlier, which are independent of u_{n+1}.

Theorem 3 generalizes a result of Levi *et al.* (1997), who showed that

$$u_{n+2} = 2u_{n+1} - u_n$$

has a Lie algebra that is isomorphic to $\mathfrak{sl}(3)$.

Just as for ODEs, it usually not easy to find more than one characteristic of a given second-order linear homogeneous OΔE, namely

$$Q = u_n = Q_3 + Q_6.$$

To obtain any other characteristic, one must find at least one solution of the OΔE (or its adjoint). This severely limits the usefulness of symmetry methods for such equations. For nonlinear OΔEs, however, symmetries usually can be found without too much difficulty. They may be used in the same ways as dynamical symmetries of ODEs.

For simplicity, we have focused on second-order OΔEs. However, the same method can also be used to obtain symmetries of higher-order OΔEs. If one uses a more general ansatz, such as $Q = Q(n, u_n, u_{n+1})$, the method leads to a system of partial differential equations for Q. So far, we have chosen to eliminate $\mathcal{S}^k Q,\ k \geq 1$, to obtain a system that involves only Q and its derivatives. This is not always the best strategy; sometimes it is better to obtain a system for $\mathcal{S}^{k_0} Q$ for some $k_0 > 0$. For example,

$$u_{n+4} = \frac{u_{n+1}^2}{u_n} + u_n$$

has only one characteristic of the form $Q = Q(n, u_n, u_{n+1})$, namely $Q = c_1 u_n$. This result is easy to obtain if differential elimination is used to derive a system for $S^4 Q$, whereas the system for Q is apparently intractable. This demonstrates that some experimentation may be needed if the standard reduction in favour of Q leads to a system that is too hard to solve.

4. How to use symmetries of OΔEs

It appears that almost any symmetry method for ODEs has a counterpart for OΔEs. Usually, only slight modification is needed to obtain the OΔE methods. In the following, we use second-order OΔEs to demonstrate various methods. The generalization to higher-order problems is straightforward.

Given a symmetry generator for a second-order OΔE,

$$X = Q\,\partial_{u_n} + \mathcal{S}Q\,\partial_{u_{n+1}}, \tag{4.1}$$

there exists an invariant,

$$v_n = v(n, u_n, u_{n+1}), \tag{4.2}$$

satisfying

$$X v_n = 0, \qquad \frac{\partial v_n}{\partial u_{n+1}} \neq 0. \tag{4.3}$$

This invariant is found by the method of characteristics; it is a first integral of

$$\frac{du_n}{Q} = \frac{du_{n+1}}{\mathcal{S}Q}.$$

Moreover, every invariant function of n, u_n, and u_{n+1} is a function of n and v_n only. For later use, we shall suppose that (4.2) can be inverted to obtain

$$u_{n+1} = w(n, u_n, v_n) \tag{4.4}$$

for some function w.

From (2.16),

$$X(\mathcal{S}v_n) = \mathcal{S}(Xv_n) = 0,$$

so $\mathcal{S}v_n$ is invariant: it is a function of n and v_n only. Thus the solutions of

$$u_{n+2} = \omega(n, u_n, u_{n+1}) \tag{4.5}$$

satisfy a first-order OΔE of the form

$$v_{n+1} = \mathcal{S}v_n = \Omega(n, v_n). \tag{4.6}$$

If (4.6) can be solved (perhaps by exploiting further symmetries of (4.5) – see below) then the general solution,

$$v_n = v(n, u_n, u_{n+1}) = f(n; c_1), \tag{4.7}$$

is equivalent to the first-order OΔE

$$u_{n+1} = w(n, u_n, f(n; c_1)), \tag{4.8}$$

which admits the symmetries generated by X. To solve (4.8), we need to obtain a canonical coordinate,

$$s_n = s(n, u_n), \tag{4.9}$$

that satisfies

$$Xs_n = 1.$$

The most obvious choice of canonical coordinate is

$$s(n, u_n) = \int \frac{du_n}{Q(n, u_n, w(n, u_n, f(n; c_1)))}. \tag{4.10}$$

Note that

$$Xs_{n+1} = X(\mathcal{S}s_n) = \mathcal{S}(Xs_n) = \mathcal{S}(1) = 1 = Xs_n,$$

so $s_{n+1} - s_n$ is an invariant. Consequently

$$s_{n+1} = s_n + g(n, v_n)$$

for some function g, and therefore (4.8) is equivalent to

$$s_{n+1} = s_n + g(n, f(n; c_1)). \tag{4.11}$$

The general solution of (4.11) is

$$s_n = c_2 + \sum_{r=n_0}^{n-1} g(r, f(r; c_1)), \tag{4.12}$$

where n_0 is any convenient integer.

If an OΔE has an N-dimensional solvable Lie (sub)algebra of symmetry generators, the solvable structure can be exploited in exactly the same way as for ODEs. Consider the nonlinear OΔE

$$u_{n+2} = \frac{2u_{n+1}^3}{u_n^2} - u_{n+1}. \tag{4.13}$$

The set of characteristics that are linear in u_{n+1} is spanned by

$$Q_1 = \frac{u_{n+1}}{u_n}, \qquad Q_2 = u_n. \tag{4.14}$$

The commutator $[X_1, X_2]$ has the characteristic

$$[Q_1, Q_2] \equiv X_1 Q_2 - X_2 Q_1 = \frac{u_{n+1}}{u_n} = Q_1.$$

Therefore the generators X_1, X_2, form a basis for a nonabelian solvable Lie algebra, whose derived subalgebra is spanned by X_1. Consequently X_1 should be used for the first reduction of order, so that the reduced OΔE inherits the symmetries generated by X_2. The invariant v_n of the group generated by X_1 satisfies

$$X_1 v_n = \left(\frac{u_{n+1}}{u_n} \partial_{u_n} + \left\{ \frac{2u_{n+1}^2}{u_n^2} - 1 \right\} \partial_{u_{n+1}} \right) v_n = 0.$$

Using the method of characteristics, we obtain

$$v_n = \frac{(u_{n+1}^2 - u_n^2)}{u_n^4}. \tag{4.15}$$

This reduces the OΔE (4.13) to

$$v_{n+1} = 4v_n, \tag{4.16}$$

which inherits the scaling symmetry generated by X_2. The general solution of (4.16) is

$$v_n = c_1 4^n, \tag{4.17}$$

which is equivalent to

$$u_{n+1} = \pm u_n \sqrt{1 + c_1 4^n u_n^2}. \tag{4.18}$$

However, the negative root is inconsistent with (4.13). Therefore the canonical coordinate is

$$s_n = \int \frac{du_n}{\sqrt{1 + c_1 4^n u_n^2}} = \frac{1}{\sqrt{c_1}\, 2^n} \sinh^{-1}\left(\sqrt{c_1}\, 2^n u_n\right). \tag{4.19}$$

Then (4.18) is equivalent to

$$s_{n+1} = s_n,$$

whose general solution is $s_n = c_2$. In the original variables, the general solution of (4.13) is

$$u_n = \frac{1}{\sqrt{c_1}\,2^n} \sinh\left(c_2\sqrt{c_1}\,2^n\right). \tag{4.20}$$

The above technique fails if X_2 is a scalar multiple of X_1, in which case the symmetry group generated by X_1 and X_2 is intransitive. Then if v_n satisfies $X_1 v_n = 0$, it also satisfies $X_2 v_n = 0$. A single reduction of order can be achieved, but the remaining one-parameter Lie group acts trivially on the reduced OΔE and cannot be used to solve it.

Intransitive two-dimensional Lie subgroups of point symmetries also occur for some second-order ODEs. They are of little consequence, because there is always a transitive two-dimensional subgroup of point symmetries as well (Stephani, 1989). However, for second-order OΔEs, the usual ansätze may not yield a transitive group. The group generated by X_1 and X_2 is intransitive if

$$\frac{Q_2}{Q_1} = \frac{\mathcal{S}Q_2}{\mathcal{S}Q_1} = \mathcal{S}\left(\frac{Q_2}{Q_1}\right),$$

that is, if the ratio of the characteristics is a first integral:

$$\frac{Q_2}{Q_1} = \Phi.$$

We now show how to construct another (functionally independent) first integral. The method depends upon whether or not X_1 and X_2 commute.

For now, it is most convenient to write the generators in terms of first integrals, with X_1 in normal form. Thus

$$X_1 = \partial_{\phi^1}, \qquad X_2 = \Phi\partial_{\phi^1}, \tag{4.21}$$

for some first integral ϕ^1, and there is an independent first integral, ϕ^2, that is mapped to itself by the group action. From (4.21), we obtain

$$[X_1, X_2] = \Phi_{\phi^1} X_1.$$

If $[X_1, X_2] \neq 0$ then Φ depends nontrivially on ϕ^1. Now construct an invariant v_n of X_1 as described earlier. Clearly, v_n is a function of n and ϕ^2 only, so

$$\phi^2 = F(n, v_n),$$

for some function F. To obtain ϕ^2, we must find a solution of

$$F(n+1, \Omega(n, v_n)) = F(n, v_n).$$

(In practice, this is usually easy.) Then Φ and ϕ^2 are functionally independent first integrals.

If X_1 and X_2 commute then Φ is a function of ϕ^2 only. Indeed, without loss of generality, we can set

$$\phi^2 = \Phi.$$

To obtain ϕ^1, first note that

$$X_1\phi^1 = 1,$$

and so ϕ^1 is a canonical coordinate. Therefore

$$\phi^1 = \int \frac{du_n}{Q_1\left(n, u_n, u_{n+1}(n, u_n, \phi^2)\right)} + G(n, \phi^2), \tag{4.22}$$

for some function G. To obtain G (up to an arbitrary function of ϕ^2), we apply the condition

$$\mathcal{S}\phi^1 - \phi^1 = 0,$$

and solve the resulting first-order linear OΔE using the standard method.

To illustrate this technique, consider the OΔE

$$u_{n+2} = \frac{u_{n+1}^2}{u_n} + u_{n+1}. \tag{4.23}$$

The symmetry generators whose characteristics are linear in u_{n+1} form an abelian Lie algebra; the characteristics are linear combinations of

$$Q_1 = u_n, \qquad Q_2 = u_{n+1} - nu_n. \tag{4.24}$$

It is easy to verify that

$$\mathcal{S}\left(\frac{Q_2}{Q_1}\right) = \frac{Q_2}{Q_1},$$

and because the generators commute, we choose

$$\phi^2 = \frac{Q_2}{Q_1} = \frac{u_{n+1}}{u_n} - n. \tag{4.25}$$

From (4.22),

$$\phi^1 = \ln|\,u_n| + G(n, \phi^2),$$

where

$$G(n+1, \phi^2) - G(n, \phi^2) = \ln|\,u_n| - \ln|\,u_{n+1}| = -\ln|\,n + \phi^2|. \tag{4.26}$$

The general solution of (4.26) is

$$G(n, \phi^2) = A(\phi^2) - \ln|\,\Gamma(n + \phi^2)|,$$

where $\Gamma(z)$ is the Gamma function and A is an arbitrary function. Without loss of generality, we can set $A(\phi^2) = 0$ and replace ϕ^1 by its exponential,

$$\tilde{\phi}^1 = \frac{u_n}{\Gamma(n + \phi^2)}.$$

Therefore the general solution of (4.23) is

$$u_n = c_1 \Gamma(n + c_2). \tag{4.27}$$

5. Direct construction of first integrals

Anco & Bluman (1998) describes a method for obtaining first integrals of a given ODE directly, whether or not any Lie symmetries are known. A simplified version of this method is given in Hydon (2000).

It is also possible to construct first integrals of OΔEs directly, even if no symmetries are known. The starting point for this approach is the equation

$$\mathcal{S}\phi = \phi, \qquad \phi_{u_{n+N-1}} \neq 0. \tag{5.1}$$

For second-order OΔEs, (5.1) amounts to

$$\phi(n+1, u_{n+1}, \omega(n, u_n, u_{n+1})) = \phi(n, u_n, u_{n+1}), \qquad \phi_{u_{n+1}} \neq 0. \tag{5.2}$$

(For brevity, we shall consider only second-order problems; the generalization to higher-order OΔEs is entirely straightforward.)

It is convenient to introduce the functions

$$P_1(n, u_n, u_{n+1}) = \phi_{u_n}(n, u_n, u_{n+1}), \tag{5.3}$$

$$P_2(n, u_n, u_{n+1}) = \phi_{u_{n+1}}(n, u_n, u_{n+1}). \tag{5.4}$$

By differentiating (5.2) with respect to u_n and u_{n+1} in turn, we obtain

$$P_1 = \omega_{u_n} \mathcal{S} P_2, \tag{5.5}$$

$$P_2 = \mathcal{S} P_1 + \omega_{u_{n+1}} \mathcal{S} P_2. \tag{5.6}$$

Therefore P_2 satisfies the second-order linear functional equation

$$(\mathcal{S}\omega_{u_n})\mathcal{S}^2 P_2 + \omega_{u_{n+1}} \mathcal{S} P_2 - P_2 = 0. \tag{5.7}$$

Just as for the linearized symmetry condition, we obtain solutions of (5.7) by first choosing an ansatz, then differentiating repeatedly to obtain a differential equation for P_2. Given a solution P_2 of (5.7), it is straightforward to construct P_1. At this stage, it is necessary to check that the integrability condition

$$\frac{\partial P_1}{\partial u_{n+1}} = \frac{\partial P_2}{\partial u_n} \tag{5.8}$$

is satisfied. (This is because some solutions of (5.7) are not derived from any first integral.) If (5.8) holds then the first integral ϕ is of the form

$$\phi = \int (P_1 \, du_n + P_2 \, du_{n+1}) + F(n), \tag{5.9}$$

where $F(n)$ is determined (up to an arbitrary constant) by substituting (5.9) into (5.2) and solving the resulting first-order linear OΔE.

To illustrate the method, consider the OΔE

$$u_{n+2} = \frac{n}{n+1} u_n + \frac{1}{u_{n+1}}. \tag{5.10}$$

We use the ansatz $P_2 = P_2(n, u_n)$; then (5.7) amounts to

$$\frac{n+1}{n+2} P_2(n+2, \omega) - \frac{1}{u_{n+1}^2} P_2(n+1, u_{n+1}) - P_2(n, u_n) = 0. \tag{5.11}$$

Using the symmetry-finding algorithm of §3, we obtain a single solution (up to an arbitrary constant factor):

$$P_2 = nu_n. \tag{5.12}$$

Therefore

$$P_1 = nu_{n+1}, \tag{5.13}$$

and the integrability condition is satisfied. From (5.9),

$$\phi = nu_n u_{n+1} + F(n),$$

and hence

$$\mathcal{S}\phi - \phi = F(n+1) - F(n) + n + 1 = 0.$$

(N.B. No matter how complicated the original OΔE is, the function F always satisfies a first-order linear OΔE which is easily solved.) In this example,

$$F(n) = -\frac{n(n+1)}{2},$$

(up to an irrelevant constant). Therefore we have obtained the first integral

$$\phi = nu_n u_{n+1} - \frac{n(n+1)}{2}. \tag{5.14}$$

The general solution to this particular problem can be found by rewriting $\phi = c_1$ as a first-order linear OΔE for $v_n = \ln|\,u_n|$:

$$v_{n+1} + v_n = \ln\left|\frac{n+1}{2} + \frac{c_1}{n}\right|.$$

By using the standard method for such OΔEs, we obtain the general solution,

$$v_n = (-1)^n \left(c_2 + \sum_{k=n_0}^{n} (-1)^k \ln\left|\frac{k}{2} + \frac{c_1}{k-1}\right|\right);$$

here n_0 is a suitably-chosen integer.

Any pair of functions (P_1, P_2) that satisfies (5.5) and (5.6) can be combined with characteristics of symmetries to yield first integrals, as follows.

Theorem 4

Given a second-order OΔE (3.7), suppose that (P_1, P_2) solves (5.5), (5.6), and that Q is the characteristic of a one-parameter Lie group of symmetries. Then

$$\Phi = P_1 Q + P_2 \mathcal{S}Q \tag{5.15}$$

is either a first integral or a constant.

Proof

We use the linearized symmetry condition to show that $\mathcal{S}\Phi = \Phi$, as follows:

$$\begin{aligned}\mathcal{S}\Phi &= (\mathcal{S}P_1)(\mathcal{S}Q) + (\mathcal{S}P_2)(\mathcal{S}^2 Q)\\ &= (\mathcal{S}P_1)(\mathcal{S}Q) + (\mathcal{S}P_2)(\omega_{u_{n+1}}\mathcal{S}Q + \omega_{u_n}Q)\\ &= P_1 Q + P_2 \mathcal{S}Q\\ &= \Phi.\end{aligned}$$

Note that Theorem 4 does not require the integrability condition (5.8) to hold. If (5.8) is satisfied then it may be possible to construct two functionally independent first integrals from one pair (P_1, P_2) and one characteristic.

6. Conclusion

We have seen that many symmetry-based methods for ODEs are easily adapted to OΔEs. It is often possible to obtain symmetries or first integrals sytematically. The method for obtaining symmetries can be generalized to partial difference equations (PΔEs), as will be described in a separate paper. (For PΔEs, the chief obstacle to obtaining symmetries is the complexity of the calculations in the differential elimination stage.)

References

Anco, S. C. & Bluman, G. 1998 Integrating factors and first integrals for ordinary differential equations. *Eur. J. Appl. Math.* **9**, 245–259.

Anderson, I. M., Kamran, N. & Olver, P. J. 1993 Internal, external and generalized symmetries. *Adv. in Math.* **100**, 53–100.

Bluman, G. W. & Kumei, S. 1989 *Symmetries and differential equations.* New York: Springer.

Dorodnitsyn, V. A. 1994 Symmetry of finite-difference equations. In *CRC handbook of Lie group analysis of differential equations, vol. 1.* (ed. N. H. Ibragimov), pp. 365–403. Boca Raton: CRC Press.

Gaeta, G. 1993 Lie-point symmetries of discrete versus continuous dynamical systems. *Phys. Lett.* **178A**, 376–384.

Hydon, P. E. 2000 *Symmetry methods for differential equations.* Cambridge: Cambridge University Press. (In press)

Levi, D., Vinet, L. & Winternitz, P. 1997 Lie group formalism for difference equations. *J. Phys. A: Math. Gen.* **30**, 633–649.

Maeda, S. 1987 The similarity method for difference equations. *IMA J. Appl. Math.* **38**, 129–134.

Olver, P. J. 1993 *Applications of Lie groups to differential equations*, 2nd edn. New York: Springer.

Quispel, G. R. W. & Sahadevan, R. 1993 Lie symmetries and the integration of difference equations. *Phys. Lett.* **184A**: 64–70.

Stephani, H. 1989 *Differential equations: their solution using symmetries.* Cambridge: Cambridge University Press.