Symmetry analysis of initial-value problems

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Abstract

Symmetry analysis is a powerful tool that enables the user to construct exact solutions of a given differential equation in a fairly systematic way. For this reason, the Lie point symmetry groups of most well-known differential equations have been catalogued. It is widely believed that the set of symmetries of an initial-value problem (or boundary-value problem) is a subset of the set of symmetries of the differential equation. The current paper demonstrates that this is untrue; indeed, an initial-value problem may have no symmetries in common with the underlying differential equation. The paper also introduces a constructive method for obtaining symmetries of a particular class of initial-value problems.

Key words: Symmetry analysis, initial-value problem, ordinary differential equation
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1 Introduction

More than a century ago, the Norwegian mathematician Sophus Lie developed techniques for finding and using the continuous point symmetries of differential equations. Whilst symmetry analysis remains the most widely-applicable method of constructing exact solutions of differential equations, it has not been particularly successful in the treatment of initial- and boundary-value problems. Perhaps one reason for this is the generally-accepted view that a symmetry of a boundary-value problem must satisfy three criteria [1]. Namely, it must be

(1) a symmetry of the governing differential equation;
(2) a smooth bijective mapping of the domain to itself;

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(3) a mapping of the set of boundary data to itself.

The procedure for finding Lie point symmetries of a given differential equation is well-known (see [2–4] for an introduction and [1,5,6] for a more detailed description). This procedure has been implemented within many computer algebra packages [7]. Once the Lie point symmetries are known, they can be used to construct the discrete point symmetries [8]. Thus it is commonly quite easy to find all point symmetries that satisfy condition (1). The difficulty is that the extra conditions (2) and (3) may not be satisfied by any of these symmetries; in this case it seems that the boundary-value problem has no point symmetries. Even if some symmetries satisfy conditions (2) and (3), they are not usually sufficient in number to yield a complete solution of the given boundary-value problem.

So far, research has focused on the alternative problem of classifying all initial or boundary conditions that are consistent with symmetries of a particular differential equation. For integrable partial differential equations (PDEs), a test for consistency between higher symmetries and boundary conditions has been developed recently [9].

Solvable initial-value problems (IVPs) for evolution equations with two independent variables can be catalogued as follows [10]. First, conditional symmetries of the PDE are sought, and the invariance condition leads to an ordinary differential equation (ODE) for the dependent variable. Then consistency conditions are used to find out which initial conditions reduce the original IVP to a Cauchy problem for a system of ODEs.

For linear (or linearizable) PDEs, some IVPs can be solved by solving a simpler IVP [11,12]. The idea is to start with a solvable IVP, and then to use the symmetries associated with linear superposition to construct a hierarchy of related IVPs. In this way, one can construct a catalogue of problems that are solvable for that particular PDE. At present, there is no general procedure for dealing with a given IVP that lacks sufficient symmetries satisfying (1)–(3) and is neither linear nor evolutionary.

It is worth examining the conditions (1)–(3) with a critical eye; are they really necessary? Clearly, conditions (2) and (3) are necessary, for otherwise the boundary-value problem would be mapped to a different boundary-value problem. However, given any Lie point symmetry generator

$$\tilde{X} = \xi(x,y)\partial_x + \eta(x,y)\partial_y\,,$$

(where $\partial_x = \partial/\partial x$, etc.) there is an equivalent dynamical symmetry generator

$$X = Q(x,y,y')\,\partial_y\,, \qquad \text{where} \quad Q(x,y,y') = \eta(x,y) - y'\xi(x,y).$$

The independent variable x is invariant under the symmetries generated by X, so condition (2) is automatically satisfied. From now on, we shall write all generators in the evolutionary form X. However, we restrict attention to dynamical symmetries that are equivalent to point symmetries, so that the characteristic $Q(x, y, y')$ is linear in y'. We shall refer to such symmetries as point-equivalent symmetries.

Condition (1) is not necessary. To see this, let S denote the set of all solutions of a given differential equation, and let $T \subset S$ denote the set of all solutions that also satisfy the boundary conditions. Every symmetry of the equation maps S to itself. However, we are not interested in *all* solutions, but only in those that satisfy the boundary conditions. Therefore condition (1) can be replaced by the weaker requirement that a symmetry of the boundary-value problem should be a symmetry of an equation that is satisfied by all solutions in the subset T. Solutions that are in $S \setminus T$ need no longer be mapped to solutions in S. (This approach is similar in spirit to the method of nonclassical symmetries, which can be used to find invariant solutions of partial differential equations [13,14]). In a recent review of significant open problems in symmetry analysis, it was suggested that such an approach may yield extra symmetries [15]. The chief problem is that, until now, there has been no way of constructing an equation that is satisfied by all solutions of T, but not by all solutions of S. The aims of the current paper are to show how this can be done, and to outline the difficulties that are involved in using initial conditions to construct symmetries.

For simplicity, we shall restrict attention to IVPs for third-order ODEs. In §2, we use a very simple example to demonstrate how an ODE that is constrained by initial conditions can have more symmetries than an unconstrained ODE. Following a discussion of some basic theory in $3, we derive a method for calculating symmetries of a class of initial-value problems. Several examples are given in $4 that illustrate how the method works in practice, and the paper concludes with a discussion of some extensions and open problems.

2 A simple example

We start with a short summary of the method for obtaining dynamical symmetries of a given third-order ODE,

$$y''' = \omega(x, y, y', y''). \tag{1}$$

First define the total derivative with respect to x, restricted to the set of solutions of the ODE, as follows:

$$D = \partial_x + y'\partial_y + y''\partial_{y'} + \omega(x, y, y', y'')\partial_{y''}\,.$$

The second prolongation of the dynamical symmetry generator $X = Q\,\partial_y$ is

$$X^{(2)} = Q\,\partial_y + DQ\,\partial_{y'} + D^2Q\,\partial_{y''}\,.$$

The characteristics Q of the dynamical symmetry generators are determined from the linearized symmetry condition

$$D^3Q = X^{(2)}\Big(\omega(x,y,y',y'')\Big). \tag{2}$$

In particular if $Q = \eta(x,y) - y'\xi(x,y)$, the linearized symmetry condition can be split into an overdetermined system of partial differential equations for $\xi(x,y)$ and $\eta(x,y)$ by using the fact that these functions do not depend on derivatives of y.

In this way, it is easy to show that the simplest third-order ODE,

$$y''' = 0, \tag{3}$$

has a seven-dimensional Lie algebra of point-equivalent symmetry generators, spanned by

$$X_1 = \partial_y\,, \qquad X_2 = x\partial_y\,, \qquad X_3 = y\partial_y\,, \qquad X_4 = y'\partial_y, \qquad X_5 = xy'\partial_y\,,$$

$$X_A = x^2\partial_y\,, \qquad X_B = (2xy - x^2y')\partial_y\,.$$

Clearly, the set of all solutions of (3) that also satisfy the initial condition

$$y''(0) = 0 \tag{4}$$

is the set of all solutions of the second-order ODE

$$y'' = 0. \tag{5}$$

The 'reduced' ODE (5) has an *eight*-dimensional Lie algebra of point-equivalent symmetry generators, spanned by

$$X_1 = \partial_y\,, \qquad X_2 = x\partial_y\,, \qquad X_3 = y\partial_y\,, \qquad X_4 = y'\partial_y, \qquad X_5 = xy'\partial_y\,,$$

$$X_6 = yy'\partial_y\,, \qquad X_7 = (xy - x^2y')\partial_y\,, \qquad X_8 = (y^2 - xyy')\partial_y\,.$$

The reduced ODE has the symmetry generators X_1–X_5 in common with the original ODE, whereas X_6–X_8 are new. The original symmetry generators X_A and X_B are not symmetries of the reduced equation.

To see why this is so, it is helpful to work in terms of the first integrals of the ODEs (3) and (5). A non-constant function $\alpha(x,y,y',y'')$ is a first integral of (1) if and only if $D\alpha = 0$. In other words, a first integral is a non-constant function that is constant on every solution of the ODE. Every first integral of (3) is a non-constant function of the first integrals

$$\alpha = y - xy' + \frac{1}{2}x^2 y'', \qquad \beta = y' - xy'', \qquad \gamma = y''. \tag{6}$$

The solutions of (3) that satisfy the initial condition $y''(0) = 0$ are precisely those for which $\gamma = 0$. Therefore the reduced equation is $\gamma = 0$, whose first integrals are

$$\alpha_0 = \alpha|_{\gamma=0} = y - xy', \qquad \beta_0 = \beta|_{\gamma=0} = y'.$$

By using the chain rule, we can write any generator X of dynamical symmetries of (3) in terms of the first integrals:

$$X = (X^{(2)}\alpha)\,\partial_\alpha + (X^{(2)}\beta)\,\partial_\beta + (X^{(2)}\gamma)\,\partial_\gamma\,.$$

The dynamical symmetries of any ODE act as point transformations in the space of first integrals; when the symmetry generators are written in terms of first integrals, they are independent of x. For the ODE (3), the point-equivalent symmetry generators amount to

$$X_1 = \partial_\alpha, \qquad X_2 = \partial_\beta, \qquad X_3 = \alpha\partial_\alpha + \beta\partial_\beta + \gamma\partial_\gamma, \qquad X_4 = \beta\partial_\alpha + \gamma\partial_\beta,$$

$$X_5 = \beta\partial_\beta + 2\gamma\partial_\gamma, \qquad X_A = 2\partial_\gamma, \qquad X_B = 2\alpha\partial_\beta + 2\beta\partial_\gamma.$$

The submanifold $\gamma = 0$ is invariant under the transformations generated by X_1–X_5. However X_A generates translations in the γ-direction, so the plane $\gamma = 0$ is mapped to $\gamma = c$, where c is a nonzero constant. Similarly, the group generated by X_B maps almost every solution satisfying $\gamma = 0$ to a solution with $\gamma \neq 0$. This explains why X_A and X_B do not generate symmetries of the reduced ODE (5).

In terms of the first integrals of (3) the new symmetries of the reduced ODE are

$$X_6 = (\alpha\beta + \frac{1}{2}x^3\gamma^2)\partial_\alpha + (\alpha\gamma + \beta^2 - \frac{3}{2}x^2\gamma^2)\partial_\beta + 3\gamma(\beta + x\gamma)\partial_\gamma,$$

$$X_7 = -\frac{1}{2}x^3\gamma\partial_\alpha + (\alpha + \frac{3}{2}x^2\gamma)\partial_\beta - 3x\gamma\partial_\gamma,$$

$$X_8 = (\alpha^2 - \frac{1}{2}x^3\beta\gamma - \frac{3}{4}x^4\gamma^2)\partial_\alpha + (\alpha\beta + \frac{3}{2}x^2\beta\gamma + 2x^3\gamma^2)\partial_\beta - 3x\gamma(\beta + x\gamma)\partial_\gamma.$$

These are not symmetries of (3) because they depend on x. However, on the submanifold $\gamma = 0$, they reduce to

$$X_6 = \alpha_0\beta_0\,\partial_{\alpha_0} + \beta_0^2\,\partial_{\beta_0}\,, \qquad X_7 = \alpha_0\,\partial_{\beta_0}\,, \qquad X_8 = \alpha_0^2\,\partial_{\alpha_0} + \alpha_0\beta_0\,\partial_{\beta_0}\,,$$

which are independent of x. Therefore they are generators of dynamical symmetries of the reduced ODE.

To summarize, if we impose an initial condition, we are interested only in the submanifold T of solutions that satisfy the condition. Symmetries of the original ODE that do not leave T invariant are not symmetries of the initial-value problem. However, transformations that are not symmetries of the original ODE may act as symmetries when they are restricted to T.

It is natural to ask what happens when further initial conditions are applied. After all, an initial-value problem is generally posed with a complete set of initial conditions. For a third-order ODE, the imposition of one initial condition restricts the set of solutions to a surface in the space of first integrals; this is why the reduced ODE is second-order. By imposing two initial conditions, we restrict attention to a curve in the space of first integrals, which gives a first-order reduced ODE. Similarly, by imposing three initial conditions, we do not have to solve a reduced differential equation at all. However, to find the particular solution, we need to know all first integrals; in other words, we need to obtain the general solution of the original ODE.

This presents no difficulty for $y''' = 0$, which has seven point-equivalent symmetries. However, the aim of this paper is to tackle problems where the original ODE cannot be solved using point-equivalent symmetries, but the reduced ODE can be. For all point-equivalent symmetries of the reduced ODE to be found using Lie's method, the reduced ODE must be of order two or more. Therefore, if the original ODE is third-order, we will never apply more than one initial condition. Of course, once the reduced ODE has been solved, any remaining initial conditions can be used.

There is one fundamental obstacle to be overcome. Given an ODE whose first integrals are entirely unknown, no techniques have been developed that enable a reduced ODE to be constructed from a given initial condition. Therefore we must work directly with the original ODE, and find a way to restrict attention to the submanifold T.

3 How to construct symmetries of a class of IVPs

For simplicity, we focus on a single class of IVPs, namely third-order ODEs

$$y''' = \omega(x, y, y', y'') \tag{7}$$

subject to the initial condition

$$y''(0) = 0. \tag{8}$$

The method presented in this section uses Taylor series about $x = 0$, so we shall restrict attention to problems for which such series are well-defined. In particular, we avoid dealing with singularities at $x = 0$ by supposing that $y(x)$ is analytic about $x = 0$, and that ω is polynomial in x and y''. (In §5, the possibility of generalizing the method is discussed.) On solutions of (7), the total derivative with respect to x is

$$D = \partial_x + y'\partial_y + y''\partial_{y'} + \omega(x, y, y', y'')\partial_{y''} \,.$$

From (2), the linearized symmetry condition for (7) is $\Gamma = 0$, where

$$\Gamma = D^3Q - \omega_{y''}D^2Q - \omega_{y'}DQ - \omega_y Q \tag{9}$$

Without loss of generality, choose the first integrals of (7) to satisfy

$$\alpha(x, y, y', y'') = y(0), \qquad \beta(x, y, y', y'') = y'(0), \qquad \gamma(x, y, y', y'') = y''(0)$$

on solutions of (7). Therefore the reduced ODE is $\gamma = 0$, as in the example of §2; solving this for y'' gives

$$y'' = F(x, y, y'), \tag{10}$$

where

$$\gamma(x, y, y', F(x, y, y')) \equiv 0.$$

On solutions of (10), the total derivative D is replaced by

$$D_0 = \partial_x + y'\partial_y + F(x, y, y')\partial_{y'} \,,$$

and the linearized symmetry condition for (10) is $\Gamma_0 = 0$, where

$$\Gamma_0 = D_0^2Q - F_{y'}D_0Q - F_yQ. \tag{11}$$

We shall show that it is possible to solve this linearized symmetry condition by using Taylor series, even though F, and hence D_0, are unknown.

Substituting the reduced ODE (10) into the original ODE (7) leads to the useful identity

$$\omega(x, y, y', F(x, y, y')) = D_0F(x, y, y'). \tag{12}$$

Given an arbitrary differentiable function $H(x, y, y', y'')$, let

$$h(x, y, y') = H(x, y, y', F(x, y, y')).$$

Then, by using the identity (12), it is easy to show that

$$D_0h(x, y, y') = \Big(DH(x, y, y', y'')\Big)\Big|_{y''=F(x,y,y')} \,. \tag{13}$$

In particular, if $\alpha_0(x, y, y') = \alpha\big|_{\gamma=0}$ then

$$D_0\alpha_0 = \Big(D\alpha(x, y, y', y'')\Big)\Big|_{y''=F(x,y,y')} = 0,$$

so α_0 (which is not constant) is a first integral of the reduced ODE. Similarly, $\beta_0(x,y,y') = \beta|_{\gamma=0}$ is another (independent) first integral of the reduced ODE.

By induction, it follows from (13) that

$$D_0^k h(x,y,y') = \left(D^k H(x,y,y',y'')\right)\Big|_{y''=F(x,y,y')}, \qquad \forall\, k \in \mathbb{N}. \tag{14}$$

This leads to the following relationships between the linearized symmetry conditions for the original and reduced ODEs.

Theorem 1 *Using the notation introduced above,*

$$\Gamma\Big|_{y''=F(x,y,y')} = D_0\Gamma_0 + \left(F_{y'} - (\omega_{y''})\Big|_{y''=F(x,y,y')}\right)\Gamma_0, \tag{15}$$

and, for each $k \in \mathbb{N}$*,*

$$\left(D^k\Gamma\right)\Big|_{y''=F(x,y,y')} = D_0^k\left\{D_0\Gamma_0 + \left(F_{y'} - (\omega_{y''})\Big|_{y''=F(x,y,y')}\right)\Gamma_0\right\}. \tag{16}$$

The proof of (15) is straightforward: differentiate (12) with respect to y and y' in turn, and use these expressions to eliminate ω_y and $\omega_{y'}$ in Γ (on solutions of the reduced ODE). To prove (16), apply (14) to (15).

We assume that the linearized symmetry condition for the reduced ODE can be written as a Taylor series in x:

$$\Gamma_0 = \sum_{k=0}^{\infty} \frac{x^k}{k!}\left(D_0^k\Gamma_0\right)\Big|_0 = 0,$$

where the notation $|_0$ means that x, y, y' and y'' are replaced by $0, \alpha_0, \beta_0$ and 0 respectively. Consequently

$$\left(D_0^k\Gamma_0\right)\Big|_0 = 0, \quad \forall\, k \in \mathbb{N}_0.$$

By Theorem 1, these conditions can be replaced by

$$\Gamma_0\Big|_0 = 0, \qquad \text{and} \qquad \left(D^k\Gamma\right)\Big|_0 = 0, \quad \forall\, k \in \mathbb{N}_0.$$

As $F(0,\alpha_0,\beta_0) = 0$, projecting (11) and (14) onto the submanifold T at $x = 0$ yields

$$\Gamma_0\Big|_0 = (D_0^2 Q)\Big|_0 = (D^2 Q)\Big|_0.$$

This condition expresses the invariance of the initial condition $y''(0) = 0$ under the group of symmetries of the reduced ODE. Thus, we have been able to write the linearized symmetry condition for the unknown reduced ODE in terms of the original ODE. The results are summarized in the following theorem.

Theorem 2 *Provided that the linearized symmetry condition for the reduced ODE is analytic in x, the point-equivalent symmetry generators $X = Q\partial_y$ are obtained by solving the system*

$$(D^2 Q)\Big|_0 = 0, \qquad \text{and} \qquad \left(D^k \Gamma\right)\Big|_0 = 0, \quad \forall\, k \in \mathbb{N}_0, \tag{17}$$

where Γ *is given by* (9).

Therefore, in principle, it is possible to obtain characteristics $Q(x, y, y')$ without knowing the reduced ODE in advance. The chief difficulty is that it seems to be necessary to solve an infinite series of partial differential equations. Furthermore, even if characteristics can be expressed in closed form, their series may not be finite or easily recognizable. These difficulties can be overcome if we can find even *one* characteristic for the reduced ODE that does not satisfy the linearized symmetry condition for the original ODE. If $Q = \eta(x, y) - y'\xi(x, y)$ is such a characteristic, then

$$\tilde{X} = \xi(x, y)\partial_x + \eta(x, y)\partial_y$$

generates point symmetries of the reduced ODE (10). Let $r(x, y)$ and $v(x, y, y')$ be fundamental differential invariants of the group generated by $\tilde{X}$, so that

$$\tilde{X} r = 0, \qquad \tilde{X}^{(1)} v = 0.$$

Then there exists a function $g(r, v)$ such that (10) is equivalent to a first-order ODE of the form

$$\frac{\mathrm{d}v}{\mathrm{d}r} = g(r, v).$$

Consequently,

$$F(x, y, y') = \frac{g(r, v)(r_x + y' r_y) - v_x - y' v_y}{v_{y'}}. \tag{18}$$

Substituting (18) into the identity (12) leads to a partial differential equation for the unknown function $g(r, v)$. This splits into an overdetermined system, because $\tilde{X}$ does not generate symmetries of the original ODE (7). Once $g(r, v)$ has been found, the reduced ODE is easily reconstructed from (18), and any remaining symmetries can be found by standard symmetry analysis; these are typically in closed form. Therefore, we need only to be able to identify one new point-equivalent symmetry generator to complete the point symmetry analysis of the reduced equation.

4 Some examples

In this section, we use the conditions in Theorem 2 to construct the symmetries of the reduced ODE in each example. For brevity, details of the calculations will be included only for the first example. In the other two examples, attention is focused on ways of using symmetries.

The first example is the Blasius equation

$$y''' = yy'', \tag{19}$$

which has a two-dimensional Lie algebra of point-equivalent symmetry generators spanned by

$$X_A = (y + xy')\partial_y, \qquad X_B = y'\partial_y\,.$$

These symmetry generators may be used to reduce (19) to a first-order ODE, but they do not enable the general solution to be obtained. To find the symmetries of the set of solutions that satisfy $y''(0) = 0$, we must solve (17), with

$$Q(x, y, y') = \eta(x, y) - y'\xi(x, y) \quad \text{and} \quad \Gamma(x, y, y', y'') = D^3Q - yD^2Q - y''Q.$$

Taking the condition $y''(0) = 0$ into account, we obtain

$$\begin{aligned}
\Gamma\Big|_0 &= (D^3Q - yD^2Q)\Big|_0 \\
(D\Gamma)\Big|_0 &= (D^4Q - yD^3Q - y'D^2Q)\Big|_0 \\
(D^2\Gamma)\Big|_0 &= (D^5Q - yD^4Q - 2y'D^3Q)\Big|_0 \\
(D^3\Gamma)\Big|_0 &= (D^6Q - yD^5Q - 3y'D^4Q)\Big|_0 \\
&\text{etc.}
\end{aligned}$$

Therefore, by induction, the system (17) for this example amounts to

$$(D^kQ)\Big|_0 = 0, \qquad \forall\, k \geq 2. \tag{20}$$

It is convenient to introduce the notation

$$\eta_k(\alpha_0) = \frac{\partial^k \eta(x, \alpha_0)}{\partial x^k}\bigg|_{x=0}, \qquad \xi_k(\alpha_0) = \frac{\partial^k \xi(x, \alpha_0)}{\partial x^k}\bigg|_{x=0}, \qquad k \in \mathbb{N}_0.$$

Derivatives of $\eta_k(\alpha_0)$ with respect to α_0 will be denoted by η_k', η_k'', etc; a similar notation is used for derivatives of $\xi_k(\alpha_0)$. For brevity, the argument α_0 is omitted. Thus

$$Q\Big|_0 = \eta_0 - \beta_0\xi_0 \tag{21}$$

and

$$(DQ)\Big|_0 = \eta_1 + \beta_0(\eta_0' - \xi_1) - \beta_0^2\xi_0'. \tag{22}$$

The first of the determining equations (20) is

$$(D^2Q)\Big|_0 = \eta_2 + \beta_0(2\eta_1' - \xi_2) + \beta_0^2(\eta_0'' - 2\xi_1') - \beta_0^3\xi_0'' = 0.$$

By equating powers of β_0, this can be split into the following system:

$$\eta_2 = 0, \quad 2\eta_1' - \xi_2 = 0, \quad \eta_0'' - 2\xi_1' = 0, \quad \xi_0'' = 0. \tag{23}$$

In a similar way, the second determining equation

$$(D^3Q)\Big|_0 = \eta_3 + \beta_0(3\eta_2' - \xi_3) + 3\beta_0^2(\eta_1'' - \xi_2') + \beta_0^3(\eta_0''' - 3\xi_1'') - \beta_0^4\xi_0''' = 0$$

splits to give the extra conditions

$$\eta_3 = 0, \quad 3\eta_2' - \xi_3 = 0, \quad \eta_1'' - \xi_2' = 0, \quad \eta_0''' - 3\xi_1'' = 0. \tag{24}$$

The general solution of (23) and (24) is

$$\eta_0(\alpha_0) = c_1\alpha_0^2 + c_2\alpha_0 + c_3, \qquad \eta_1(\alpha_0) = c_4\alpha_0 + c_5, \qquad \eta_2(\alpha_0) = \eta_3(\alpha_0) = 0,$$

$$\xi_0(\alpha_0) = c_6\alpha_0 + c_7, \qquad \xi_1(\alpha_0) = c_1\alpha_0 + c_8, \qquad \xi_2(\alpha_0) = 2c_4, \qquad \xi_3(\alpha_0) = 0,$$

where each c_i is an arbitrary constant. The remaining determining equations do not constrain these constants in any way, and they provide the information that

$$\eta_k = \xi_k = 0, \qquad \forall\, k \geq 4.$$

Then $Q(x, y, y')$ can be reconstructed from the Taylor series

$$Q(x, y, y') = \sum_{k=0}^{\infty} \frac{x^k}{k!}\Big(\eta_k(y) - y'\xi_k(y)\Big).$$

Hence the point-equivalent symmetry generators of the reduced ODE have characteristics of the form

$$Q(x, y, y') = (c_4y + c_5)x + c_1y^2 + c_2y + c_3 - y'(c_4x^2 + (c_1y + c_8)x + c_6y + c_7).$$

The set of all such generators forms the same eight-dimensional Lie algebra as in the example of §2. Indeed, the reduced ODE is, once again, $y'' = 0$. To see this, note that $X = \partial_y$ is a symmetry of the reduced ODE, but not of the Blasius equation. Substituting the differential invariants $r = x$, $v = y'$ into (18) yields

$$F(x, y, y') = g(r, v) = g(x, y').$$

Then (12) amounts to

$$yg(x, y') = g_x(x, y') + g(x, y')g_{y'}(x, y').$$

Splitting this into an overdetermined system by equating powers of y, one obtains $g(x, y') = 0$, and hence $y'' = 0$.

As a second example, consider the ODE

$$y''' = \frac{2x}{y^5} - \frac{5y'y''}{y}, \tag{25}$$

whose only point-equivalent symmetries are scalings, with the characteristic

$$Q_1 = 2y - 3xy'.$$

By using the computer algebra package Maple [16], equations (17) for $k \leq 11$ have been solved. These yield the symmetries Q_1 and

$$Q_2 = xy - x^2 y'.$$

The fundamental differential invariants of the Lie group of point symmetries generated by

$$X_2 = x^2 \partial_x + xy\partial_y$$

are

$$r = \frac{y}{x}, \qquad v = xy' - y.$$

Hence the reduced ODE,

$$\frac{\mathrm{d}v}{\mathrm{d}r} = g(r, v),$$

amounts to

$$y'' = \frac{vg(r, v)}{x^3}.$$

Substituting (18) into (12) yields

$$\frac{2}{x^4 r^5} - \frac{5(v + rx)vg}{x^5 r} = -\frac{3vg}{x^4} + \frac{v^2 g_r}{x^5} + \frac{(vg_v + g)vg}{x^5}.$$

By equating powers of x, we obtain the split system

$$\frac{2}{r^5} - 5vg = -3vg, \qquad -\frac{5v^2 g}{r} = v^2 g_r + (vg_v + g)vg,$$

whose only solution is

$$g(r, v) = \frac{1}{r^5 v}.$$

Consequently, the reduced ODE is

$$y'' = \frac{x^2}{y^5}.$$

This can be reduced to quadrature by using the two point symmetries with characteristics Q_1 and Q_2 in the usual way. Therefore, although the original

ODE (3) cannot be solved, the set of solutions that satisfy the initial condition $y''(0) = 0$ can be found.

In the previous examples, the series for the symmetries have contained finitely many terms. This does not generally happen. Consider the ODE

$$y''' = y''^{\,2} + \left(\frac{y'}{y} - xy\right) y'' + y, \tag{26}$$

which has no point-equivalent symmetries. The general solution of the system (17) has infinitely many terms. With the aid of Maple, the first few terms have been calculated. It appears that there are eight independent point equivalent symmetries, whose characteristics (up to order x^8) are as follows.

$$\begin{aligned}
Q_1 &= y\\
Q_2 &= 1 + \frac{1}{6}\,x^3 + \frac{1}{180}\,x^6\\
Q_3 &= x + \frac{1}{12}\,x^4 + \frac{1}{504}\,x^7\\
Q_4 &= \left(x^3 + \frac{3}{20}\,x^6\right) y - \left(2x + \frac{1}{2}\,x^4 + \frac{3}{70}\,x^7\right) y'\\
Q_5 &= \left(x + \frac{5}{12}\,x^4 + \frac{11}{252}\,x^7\right) y - \left(x^2 + \frac{1}{6}\,x^5 + \frac{11}{1008}\,x^8\right) y'\\
Q_6 &= \left(x^2 + \frac{7}{30}\,x^5 + \frac{13}{720}\,x^8\right) y - \left(2 + \frac{2}{3}\,x^3 + \frac{7}{90}\,x^6\right) y'\\
Q_7 &= \left(1 + \frac{1}{3}\,x^3 + \frac{1}{72}\,x^6\right) y^2 - \left(x + \frac{1}{12}\,x^4 + \frac{1}{504}\,x^7\right) yy'\\
Q_8 &= \left(x^2 + \frac{1}{15}\,x^5 + \frac{1}{720}\,x^8\right) y^2 - \left(2 + \frac{1}{3}\,x^3 + \frac{1}{90}\,x^6\right) yy'
\end{aligned}$$

As only the first few terms are calculated, we should bear in mind that a later equation in (17) might remove one or more of these characteristics. The calculations have been extended to terms of order x^{14} without destroying any characteristics, but this does not prove that there are eight point-equivalent symmetries. Moreover, the above series are not very easy to recognize.

This problem can be solved by using the point-equivalent symmetries generated by Q_1, which is the only characteristic that seems to have finitely many terms. The fundamental differential invariants of the corresponding point symmetries are

$$r = x, \qquad v = \frac{y'}{y}\,.$$

Substituting (18) into (12) and splitting the resulting equation by equating powers of y (as in the first example) gives the following system of equations for $g(r, v)$:

$$(g + v^2)(g + v^2 - r) = 0, \qquad g_r + g(g_v + 2v) = 1,$$

whose only solution is

$$g(r, v) = r - v^2.$$

Consequently, the reduced ODE is the Airy equation

$$y'' = xy.$$

This does indeed have eight point-equivalent symmetries; all characteristics except Q_1 can be written in terms of the Airy functions $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ and their derivatives (see [2] for a way of doing this).

5 Discussion and open problems

The method proposed in this paper is the first systematic technique for determining the point-equivalent symmetries of a given initial value problem. However, it is not without difficulties, as illustrated by the last of the above examples. These difficulties are entirely due to the need for series expansions. First, it is not usually easy to recognize series that have infinitely many terms. At present, there is no way to obtain differential invariants (and hence the reduced ODE) from a series; the characteristic needs to be written in closed form. However, many IVPs have at least some symmetries whose characteristics can be written as a low-order polynomial in x, so there are grounds for hope that the reduced ODE might be obtained by this method.

A fundamental difficulty is that one can only calculate the first few terms, as the complexity of the calculation becomes prohibitive as k becomes large. For some IVPs, the inclusion of higher-order terms can remove what at first sight appears to be a characteristic. For instance, the ODE

$$y''' = \frac{1}{y^3}$$

(which occurs in lubrication theory) has only one point-equivalent characteristic, namely

$$Q_1 = 3y - 4xy'.$$

If one solves (17) for $k \leq 2$, it appears that there are three extra characteristics for the reduced ODE, namely:

$$\begin{aligned}
Q_2 &= x - \frac{x^4}{8y^4} - \frac{3x^5 y'}{10y^5}\,, \\
Q_3 &= xy + \frac{5x^4}{24y^3} - \left(x^2 - \frac{9x^5}{20y^4}\right) y', \\
Q_4 &= y^2 - \frac{x^3}{3y^2} - \left(xy + \frac{5x^4}{24y^3}\right) y'.
\end{aligned}$$

However, these are all removed when the $k = 3$ equation of (17) is taken into account. (The only characteristic for the reduced equation is Q_1; as this is a characteristic for the original ODE, the reduced ODE cannot be found.) In the same way, it is conceivable that some ODEs might appear to have symmetries that are not removed within the limits of the calculations that can be done by computer algebra. Nevertheless if a (likely) new symmetry can be identified then it may be used to try and obtain the reduced ODE. If (18) and (12) do not produce a solvable split system then the putative 'symmetry' is not really a symmetry. Indeed, once the reduced ODE is found, its point-equivalent symmetries can often be determined in closed form; they will include the symmetry that has been used to derive the reduced ODE.

For simplicity, we have restricted attention to third-order problems that can be solved with the aid of Taylor series. The generalization to higher-order ODEs is obvious, although the complexity of the calculations increases with the order of the ODE. To deal with regular singularities, it is necessary to use the Frobenius method. For example, the IVP $y''' = 0$ subject to $y'(0) = 0$ gives the reduced ODE

$$y'' = \frac{y'}{x},$$

whose point-equivalent characteristics include $Q = y'/x$, which cannot be obtained directly with Taylor series. Extensions of the method to singular problems will be described in a separate paper.

An initial-value problem is usually posed with a complete set of initial conditions, yet only one condition is needed to reduce the order of the ODE by one. It would be useful to know whether or not there is a systematic way to choose this condition so that the reduced ODE admits new point-equivalent symmetries. For example, the Blasius boundary layer is described by the solution of the Blasius equation that satisfies the boundary conditions

$$y(0) = 0, \qquad y'(0) = 0, \qquad \lim_{x\to\infty} y'(x) = 1.$$

The solution to this problem can be obtained from the solution of the IVP

$$y''' = (y - x^2/2)(y'' - 1), \qquad y(0) = y'(0) = y''(0) = 0.$$

The above ODE subject to the single condition $y''(0) = 0$ has no point-equivalent symmetries. Is there a choice of constants c_1, c_2, c_3 such that

$$y''' = (y - x^2/2)(y'' - 1), \qquad c_1 y(0) + c_2 y'(0) + c_3 y''(0) = 0,$$

yields a reduced ODE with new point-equivalent symmetries? If so, then it might be possible to obtain a closed-form solution for the Blasius boundary layer.

It remains to be seen how far the method introduced in this paper can be generalized to initial- and boundary-value problems for PDEs. Whilst the Lie point symmetries of many PDEs have been classified (see [17,18]), our results indicate that it may be possible to calculate other point symmetries that appear when initial or boundary conditions are imposed.

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