Notes on the Topology of Metric Spaces

MIHAELA IANCU AND SCOTT W. WILLIAMS

Abstract. We study topological properties which affect, or are the consequence of metric constructions such as the box metric. We consider connected, compact, totally bounded, complete, zero-dimensional and cardinal functions. We isolate a notion we call taut, which in the presence of complete implies connected. We prove that each metric space is the minimal image of a taut metric space.

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0. Introduction

This paper has its genesis in the study of the still open box product problem:

Is the product of real lines a normal space when it is given the box topology ([6], [5], [9])?

We discovered that certain open covers in the box product of the reals always have a locally-finite refinement and this lead to considering the box metric, a construction with categorical connections. Several definitions are made to study topological properties of this construction, and these lead to other interesting results. Our sections and major results are:

1. Fundamentals
2. Connected I

Theorem 2.6. The box metric power of totally bounded connected metric spaces is connected.

3. Taut

Theorem 3.8. In a bounded complete taut metric space, each pair of points are joined by the image of a distance-decreasing mapping from an interval of the reals.

4. Connected II

Theorem 4.3. The box metric product of countably many connected separable taut metric spaces is connected.

5. Tautification

Theorem 5.11. Each metric has a tautification.
6. Complete and compact
Theorem 6.1. A compact metric space reaches its tautification in one step through the τ operator.
Example 6.6. There is a metric, topologically equivalent to the usual metric on [0,1], whose tautification is the discrete metric.
7. Zero-dimensional.

1. Fundamentals

In topology our maps are continuous functions; however, continuous functions do not reflect the metric of a space, or the uniformity of a uniform space, they only reflect the topology. In much of mathematics, the maps between objects reflect the object.

For a category of metric spaces, this is accomplished by *distance-decreasing (DD) mappings*. A DD mapping between metric spaces, $f : (X, d_X) \to (Y, d_Y)$, is defined by the property that

$$d_Y(f(x), f(x')) \leq d_X(x, x') \text{ for any } x,\ x' \in X.$$

In this category the product $X \times Y$ of two metric spaces can have but one metric, called the *box metric* because the open balls in $\mathbb{R}^n$ are n-cubes:

$$d((x, y), (x', y')) = max\{d_X(x, x'), d_Y(y, y')\}.$$

The metric above induces the usual product topology and topological interest is further increased through the following

Observation 1.1. A mapping $f : (X, d_X) \to (Y, d_Y)$ between metric spaces is continuous iff there is a metric d inducing the topology of X such that $f : (X, d) \to (Y, d_Y)$ is a DD mapping.

Proof. IF: Assume that $f : (X, d_X) \to (Y, d_Y)$ is continuous. The space X embeds into $X \times Y$ as the graph of f through the mapping $x \rightarrowtail (x, f(x))$.
The metric we want is induced by the box metric on $X \times Y$, so for $x, x' \in X$ we define $d(x, x') = \max\{d_X(x, x'), d_Y(f(x), f(x'))\}$, which makes $f : (X, d) \to (Y, d_Y)$ DD mapping.

ONLY IF: Any DD mapping is continuous. □

The question is how do we handle infinite products. We have some suggestions from literature:

Let $C(X, Y)$ denote the set of continuous functions from X to Y.

If (Y,d) is a metric space, we have the space of paths on Y, $C([0,1],Y)$ with the metric $\rho(f,g) = \sup\{d(f(t),g(t)) \,:\, t \in [0,1]\}$ ([4]).

Given a topological space X, $C(X,\mathbb{R})$ denotes the set of continuous functions from X to the reals. If $\mathbb{R}$ is given the bounded metric $d(r,s) = \min\{1, |r-s|\}$, then the well-known topology of uniform convergence on $C(X,\mathbb{R})$ is induced by the metric $d(f,g) = \sup\{|f(x)-g(x)| \,:\, x \in X\}$ ([2]). The box metric expands on both these ideas to the case when X has the discrete topology, i.e. when we consider a product.

Example 1.2. Consider $\prod_{n\in\mathbb{N}}[0,n]$, where each $[0,n]$ has the Euclidean metric. For the constant 0 function, $x = (0)_{n\in\mathbb{N}}$, and the identity, $y = (n)_{n\in\mathbb{N}}$, we have $sup\,\{d(x(n),y(n)) \,:\, n \in \mathbb{N}\} = \infty$.

This example suggests we allow our metrics to take the value infinity. *Henceforth, all metrics will have range contained in* $[0,\infty]$. The triangle inequality is still verified through the additional property that for any three points in the product such that the distance between two points becomes infinite, at least one of the other two distances also becomes infinite.

Algebraists have the same way to extend the box metric on infinite products, as in the category of metric spaces and DD mappings ([8], [3]) the universal laws of products require the following definition of the product of metric spaces:

Definition 1.3. For a family of metric spaces, $\{(X_i,d_i)\}_{i\in\mathcal{I}}$, the *box metric*, d^{sup}, on $\prod_{i\in\mathcal{I}} X_i$ is defined by

$$d^{sup}(x,y) = \sup\{d_i(x(i),y(i)) \,:\, i \in \mathcal{I}\} \text{ for } x,y \in X.$$

The product $\prod_{i\in\mathcal{I}} X_i$ with the topology of the box metric shall be called *the box metric product* of the spaces $\{(X_i,d_i)\}_{i\in\mathcal{I}}$.

Proposition 1.4. *The box metric product of metric spaces is a metric space with DD projection mappings.*

Notation 1.5. In a metric space (X,d), given $x \in X$, $A \subseteq X$, $\varepsilon > 0$, we use $B(x,\varepsilon) = \{y \in X \,:\, d(x,y) < \varepsilon\}$ and $B(A,\varepsilon) = \bigcup_{a\in A} B(a,\varepsilon)$.

The following properties are routine:

Lemma 1.6. *a) For $\{X_i\}_{i\in\mathcal{I}}$ family of metric spaces and the box metric topology on their product we have*
a1) If $x \in \prod_{i\in\mathcal{I}} X_i$ and $\varepsilon > 0$, then $B(x,\varepsilon) = \bigcup_{\delta<\varepsilon} \prod_{i\in\mathcal{I}} B(x(i),\delta)$.
a2) If $A_i \subseteq X_i$ for $i \in \mathcal{I}$, then $cl(\prod_{i\in\mathcal{I}} A_i) = \prod_{i\in\mathcal{I}} cl(A_i)$.

b) If X, Y are metric spaces, $\mathcal{I}$, $\mathcal{J}$ are index sets and we consider the box metric on products, then $\prod^{\mathcal{J}}(\prod^{\mathcal{I}} X)$ is isometric to $\prod^{\mathcal{I}\times\mathcal{J}} X$ and $\prod^{\mathcal{I}}(X \times Y)$ is isometric to $\prod^{\mathcal{I}} X \times \prod^{\mathcal{I}} Y$.

From 1.6a1) we observe that the box metric topology is contained in the box topology and that it contains the Tychonoff product topology. While the sets $\prod_{i\in\mathcal{I}} B(x(i),\varepsilon)$ are open in $\prod_{i\in\mathcal{I}} X_i$ with the box topology, they are generally not open with respect to the box metric.

Example 1.7. The box topology is the smallest topology on $\prod^{\mathbb{N}} \mathbb{R}$ containing all the sets $\prod_{n\in\mathbb{N}} B(x_n,\varepsilon)$.

Proof. Let $(x(n))_{n\in\mathbb{N}}$ be a point in $\prod^{\mathbb{N}} \mathbb{R}$ and $\prod_{n\in\mathbb{N}}(y(n), z(n))$ a neighbourhood of $(x(n))_{n\in\mathbb{N}}$.
Then $\prod_{n\in\mathbb{N}}(y(n), z(n)) = \prod_{n\in\mathbb{N}}(y(n), y(n)+1) \cap \prod_{n\in\mathbb{N}}(z(n)-1, z(n))$, where the sets $\prod_{n\in\mathbb{N}}(y(n), y(n) + 1)$ and $\prod_{n\in\mathbb{N}}(z(n) - 1, z(n))$ are products of balls of radius $\frac{1}{2}$.
□

To return full circle, we can see the box metric topology as a generalization of the Tychonoff product topology:

Proposition 1.8. *If $\{(X_n, d_n)\}_{n\in\mathbb{N}}$ is a family of metric spaces and $\{b_n\}_{n\in\mathbb{N}}$ is a sequence of positive reals converging to 0 such that d_n is bounded by b_n for $n \in \mathbb{N}$, then $\prod_{n\in\mathbb{N}} X_n$ with the box metric topology is homeomorphic to $\prod_{n\in\mathbb{N}} X_n$ with the Tychonoff product topology.*

Proof. We only need to see that the box metric topology is included in the Tychonoff product topology.
For fixed $x \in \prod_{n\in\mathbb{N}} X_n$ and $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $d_n < \varepsilon$ for $n \geq n_0$, so $B(x(n),\varepsilon) = X_n$ for $n \geq n_0$.
□

As we are expanding the topology, many Tychonoff product results (as compact, totally bounded, connected) are not preserved for infinite non-trivial products with the box metric. However, the next result should not be a surprise.

Proposition 1.9. *The box metric product of complete metric spaces is complete.*

Proof. Let $\{(X_i, d_i)\}_{i\in\mathcal{I}}$ be a family of complete metric spaces for which we consider the box metric topology on $\prod_{i\in\mathcal{I}} X_i$.
Let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence of points from $\prod_{i\in\mathcal{I}} X_i$. For any $\varepsilon > 0$ there is $n(\varepsilon) \in \mathbb{N}$ such that $m, n \geq n(\varepsilon)$ implies that $d_i(x_m(i), x_n(i)) \leq d^{sup}(x_m, x_n) \leq \varepsilon$ for $i \in \mathcal{I}$.
Thus $\{x_n(i)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X_i converging to some $y(i)$ for $i \in \mathcal{I}$.
Let $y = (y(i))_{i\in\mathcal{I}}$ and $m \in \mathbb{N}$ such that $m \geq n(\frac{\varepsilon}{2})$.
For a fixed $i \in \mathcal{I}$ there is some $k \geq n(\frac{\varepsilon}{2})$ such that $d_i(x_k(i), y(i)) \leq \frac{\varepsilon}{2}$.
Then $d_i(x_m(i), y(i)) \leq d_i(x_m(i), x_k(i)) + d_i(x_k(i), y(i)) \leq \varepsilon$ for $i \in \mathcal{I}$.
Thus if $m \geq n(\frac{\varepsilon}{2})$ we have $d^{sup}(x_m, y) \leq \varepsilon$, so $\{x_n\}_{n\in\mathbb{N}}$ converges to y. □

Corollary 1.10. *If $\{d_i\}_{i\in\mathcal{I}}$ is a family of complete metrics on a set X, then $sup\{d_i : i \in \mathcal{I}\}$ is a complete metric on X.*

Proof. It suffices to observe that $(X, sup\{d_i : i \in \mathcal{I}\})$ is an isometric to the diagonal, $\{(x)_{i\in\mathcal{I}} : x \in X\}$, closed subspace of $(\prod^{\mathcal{I}} X, d^{sup})$, and to apply Proposition 1.9.

□

Definitions 1.11. For a metric space (X, d) and a cardinal κ we define the *metric density* $\delta_\mu(X) \leq \kappa$ iff for any $\varepsilon > 0$ there is a subset A of X such that $|A| < \kappa$ and $B(A, \varepsilon) = X$.

Given a set X and an index set $\mathcal{I}$, we define

$$\Delta(\kappa, \mathcal{I}) := \left\{x \in \prod\nolimits^{\mathcal{I}} X \, : \, |\{x(i) \, : \, i \in \mathcal{I}\}| < \kappa\right\}.$$

The diagonal of $\prod^{\mathcal{I}} X$ is therefore denoted by $\Delta(1, \mathcal{I})$.

The next lemma shall be applied in Theorem 2.6 and in Corollary 4.5.

Lemma 1.12. *For a metric space (X, d) the following are true:*
i) (X, d) is totally bounded iff $\delta_\mu(X) \leq \aleph_0$;
ii) (X, d) is separable iff $\delta_\mu(X) \leq \aleph_1$;
iii) $\delta_\mu(X) \leq \kappa$ iff $\Delta(\kappa, \mathcal{I})$ is dense in $(\prod^{\mathcal{I}} X, d^{sup})$ for any index set $\mathcal{I}$.

Proof. The proofs of i) and ii) are immediate. We shall prove the equivalence in iii):

IF: If $\Delta(\kappa, \mathcal{I})$ is dense in $(\prod^{\mathcal{I}} X, d^{sup})$ for any index set $\mathcal{I}$, in particular $\Delta(\kappa, X)$ is dense in $(\prod^{X} X, d^{sup})$.
For $n \in \mathbb{N}$ fixed there is $(a_x)_{x\in X} \in \Delta(\kappa, X)$ such that
$d^{sup}((x)_{x\in X}, (a_x)_{x\in X}) < \varepsilon$ and so $d(x, a_x) < \varepsilon$ for any $x \in X$.
Then $A = \{a_x : x \in X\}$ has $|A| < \kappa$ and $B(A, \varepsilon) = X$.
As such a set A can be obtained for any ε positive, we conclude that $\delta_\mu(X) \leq \omega \cdot \kappa = \kappa$.

ONLY IF: Assume that $\delta_\mu(X) \leq \kappa$ and let $x \in \prod^{\mathcal{I}} X$.
Given $\varepsilon > 0$, let $A \subseteq X$ with $|A| < \kappa$ such that $X = B(A, \frac{\varepsilon}{2})$.
For $i \in \mathcal{I}$ we have $B(x(i), \frac{\varepsilon}{2}) \cap A \neq \emptyset$, so $\prod_{i\in\mathcal{I}} B(x(i), \frac{\varepsilon}{2}) \cap \prod^{\mathcal{I}} A \neq \emptyset$.
From a1) of Lemma 1.6, $B(x, \varepsilon) \supseteq \prod_{i\in\mathcal{I}} B(x(i), \frac{\varepsilon}{2})$. Also $\prod^{\mathcal{I}} A \subseteq \Delta(\kappa, \mathcal{I})$, therefore $B(x, \varepsilon) \cap \Delta(\kappa, \mathcal{I}) \neq \emptyset$, i.e. $\Delta(\kappa, \mathcal{I})$ is dense in $(\prod^{\mathcal{I}} X, d^{sup})$.

□

2. Connectedness I

In this section we start looking at the problem of finding conditions for connectedness of the product with the box metric topology.

Definitions 2.1. Let (X, d) be a metric space and x, y points in X.
A *chain of points from x to y* is a finite sequence of points in X, $Z_{xy} = (z_0, ..., z_n)$, such that $x = z_0$, $y = z_n$.
If in addition, for $\varepsilon > 0$ we have $d(z_i, z_{i+1}) \leq \varepsilon$ for $0 \leq i < n$, we call Z_{xy} an *ε-(d-)chain of points from x to y.*

We define $x \sim_\varepsilon y$ iff there is an *ε-chain of points from x to y* and

$$\mathcal{Z}_{xy,\varepsilon} = \{Z_{xy} : Z_{xy} \text{ is an } \varepsilon\text{-chain of points from } x \text{ to } y\},$$

$$\mathcal{Z}_\varepsilon(x) = \{z \in X \ : z \sim_\varepsilon x\}.$$

Observations 2.2. If (X, d) is a metric space, then

i) For ε fixed positive value, $\sim_\varepsilon$ is an equivalence relation with equivalence classes, $\{\mathcal{Z}_\varepsilon(x)\}_{x\in X}$, clopen subsets of X.

ii) Monotonicity: If $x \in X$ and $0 < \varepsilon < \varepsilon'$, then $\mathcal{Z}_\varepsilon(x) \subseteq \mathcal{Z}_{\varepsilon'}(x)$ and $\mathcal{Z}_{xy,\varepsilon} \subseteq \mathcal{Z}_{xy,\varepsilon'}$.

iii) Cantor's definition of connectedness appeared in [1] and it is nowadays known as uniform connectedness or as Cantor-connectedness ([7]). A metric space (X, d) is said to be *Cantor-connected* if $\mathcal{Z}_{xy,\varepsilon} \neq \emptyset$ for any $x, y \in X$, $\varepsilon > 0$. It is easy to observe that if (X, d) is connected, then (X, d) is Cantor-connected, while the reverse implication holds for special cases, as compact metric spaces.

Examples 2.3. For $n \in \mathbb{N}$ we consider $[0, n]$ with the Euclidean metric and on products, the box metric.

a) $\prod^{\mathbb{N}}[0, 1]$ is path-connected;

b) $\prod_{n\in\mathbb{N}}[0, n]$ is not connected (or even Cantor-connected).

Proof. a) It suffices to define a path from the point $(0)_{n\in\mathbb{N}}$ to a point $x = (x(n))_{n\in\mathbb{N}}$.
We define $f : [0, 1] \to \prod^{\mathbb{N}}[0, 1]$ by $f(t)(n) = t \cdot x(n)$.
As $d^{sup}(f(t_1), f(t_2)) = \sup\{|t_1 - t_2| \cdot x(n) : n \in \mathbb{N}\} \le |t_1 - t_2|$ for $t_1, t_2 \in [0, 1]$, f is continuous as a DD mapping.

b) We remark that $B((0)_{n\in\mathbb{N}}, \infty)$ is a proper clopen set in $\prod_{n\in\mathbb{N}}[0, n]$ since $d^{sup}((0)_{n\in\mathbb{N}}, (n)_{n\in\mathbb{N}}) = \infty$.
As ε-chains are finite, points at infinite distance admit no ε-chains, therefore Cantor-connectedness also fails.

□

The latter argument for the failure of Cantor-connectedness for box metric products leads us to introducing the following

Definitions 2.4. Given (X, d) metric space, the equivalence relation $x \sim_{fin} y$ defined by $d(x, y) < \infty$ determines a partition of X into clopen equivalence classes.

The metric space (X, d) is called *finitely (Cantor-/path-)connected* if each equivalence class determined by $\sim_{fin}$ is (Cantor-/path-)connected.

Examples 2.5. a) The product $\prod_{n\in\mathbb{N}}[0, n]$ of Example 2.3 b) is finitely path-connected:
For $x \in \prod_{n\in\mathbb{N}}[0, n]$ let

$$E_x = \{y \in \prod_{n\in\mathbb{N}}[0, n] \,:\, y \sim_{fin} x\}.$$

Then $E_x = \bigcup_{m\in\mathbb{N}} B(x, m)$. Each $B(x, m)$ contains x and is path-connected as in a) of Example 2.3, therefore E_x is path-connected.

b) Nevertheless, *the box metric product of path-connected metric spaces is not finitely Cantor-connected in general*:

For the same $\prod_{n\in\mathbb{N}}[0, n]$ we now consider the box metric product induced by metrics on $[0, n]$ given through $d_n(x, y) = min\{|x - y|, 1\}$. On $\prod_{n\in\mathbb{N}}[0, n]$ the equivalence relation $\sim_{fin}$ determines the whole space to be the unique equivalence class. But $\mathcal{Z}_{(0)_{n\in\mathbb{N}}\,(n)_{n\in\mathbb{N}},\,\frac{1}{2}} = \emptyset$.

Theorem 2.6. *If (X, d) is a totally bounded connected metric space, then $(\prod^{\mathcal{I}} X, d^{sup})$ is connected for any index set $\mathcal{I}$.*

Proof. For each finite partition $\mathcal{P}$ of $\mathcal{I}$ we let

$$\prod_{\mathcal{P}} = \{x \in \prod^{\mathcal{I}} X \,:\, x_{\restriction P} \text{ is constant for each } P \in \mathcal{P}\}$$

As $|\mathcal{P}| = n$ for some $n \in \mathbb{N}$, $\prod_{\mathcal{P}}$ is isometric to the finite product X^n and therefore $\prod_{\mathcal{P}}$ is connected in $\prod^{\mathcal{I}} X$.
As $\prod_{\{\mathcal{I}\}} \subseteq \prod_{\mathcal{P}}$ for any finite partition $\mathcal{P}$ of $\mathcal{I}$, we conclude that $\Delta(\aleph_0, \mathcal{I}) = \bigcup\{\prod_{\mathcal{P}} : \mathcal{P} \text{ is a finite partition of } \mathcal{I}\}$ is connected in $\prod^{\mathcal{I}} X$. By i) and iii) of Lemma 1.12, $\Delta(\aleph_0, \mathcal{I})$ is dense in $\prod^{\mathcal{I}} X$ and therefore $\prod^{\mathcal{I}} X$ is connected.

□

Example 2.7. There is a non-totally bounded metric space (X, d) such that $(\prod^{\mathbb{N}} X, d^{sup})$ is path-connected.

Proof. We consider the hedgehog (X, d) defined by letting X consist of the points of the unit disk, $(r, \theta) = r \cdot e^{i\theta}$ with $0 \leq r \leq 1$ and $0 \leq \theta < 2\pi$.
For $x = (r_0, \theta_0)$ and $y = (r_1, \theta_1)$ we have

$$d(x, y) = \begin{cases} |r_0 - r_1| & \text{if} \quad \theta_0 = \theta_1; \\ r_0 + r_1 & \text{if} \quad \theta_0 \neq \theta_1. \end{cases}$$

(X, d) is not totally bounded since $\{(\frac{1}{2}, 1] \times \{\theta\} \,:\, 0 \leq \theta < 2\pi\}$ is an infinite family of pairwise disjoint open balls.

On the other hand, $\prod^{\mathcal{I}} X$ is path-connected, as each of its points belongs to an isometric copy of $\prod^{\mathcal{I}}[0, 1]$, where $[0, 1]$ has the usual metric. Each of these copies contains the constant point $(0)_{i\in\mathcal{I}}$, so $\prod^{\mathcal{I}} X$ is path-connected.

□

We want to extend Theorem 2.6 from powers to arbitrary products and Example 2.7 indicates that the converse to the Theorem 2.6 does not hold. However, Example 2.5b) suggests a different tact.

The following is an adaptation of an open cover equivalence of connectedness, which shows that a connected product must satisfy some "simultaneous" version of Cantor-connectedness:

Observation 2.8. Let $\{(X_i, d_i)\}_{i\in\mathcal{I}}$ be a family of metric spaces and assume that $(\prod_{i\in\mathcal{I}} X_i, d^{sup})$ is connected.
Then for every two points $(x(i))_{i\in\mathcal{I}}$,$(y(i))_{i\in\mathcal{I}}$ in $\prod_{i\in\mathcal{I}} X_i$ and $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that each x_i and y_i admit an ε-chain of points $Z_{x_iy_i}$ with $|Z_{x_iy_i}| \leq n$.

Proof. The observation is clearly true when the index set is finite.
For $|\mathcal{I}| \geq \aleph_0$ suppose not. There is then an $\varepsilon > 0$ and an infinite set $\mathbb{J} = \{j_n : n \in \mathbb{N}\} \subseteq \mathcal{I}$ such that for any $n \in \mathbb{N}$ there are $x(j_n)$ and $y(j_n)$ in X_{j_n} which cannot be joined by an ε-chain of points $Z_{x(j_n)y(j_n)}$ with $|Z_{x(j_n)y(j_n)}| \leq n$.
Consider the subproduct $P := \prod_{n\in\mathbb{N}} X_{j_n}$ and let $x = (x(j_n))_{n\in\mathbb{N}}$ and $y = (y(j_n))_{n\in\mathbb{N}}$ be points in P.
By defining

$C = \{p \in P$: there is $m(p) \in \mathbb{N}$ such that each $x(j_n)$ and $p(j_n)$ admit an ε-chain of points, $Z_{x(j_n)p(j_n)}$, with $|Z_{x(j_n)p(j_n)}| \leq m(p)\}$,

we have $x \in C$ and $y \notin C$.
C is open in P : If $p \in C$, since $B(p, \varepsilon) \subseteq \prod_{n\in\mathbb{N}} B(p(j_n), \varepsilon)$, we find that for each $t \in B(p, \varepsilon)$ and for each $n \in \mathbb{N}$ we can join $x(j_n)$ and $t(j_n)$ by an ε-chain of points,$Z_{x(j_n)t(j_n)}$, with $|Z_{x(j_n)t(j_n)}| \leq m(p) + 1$.
C is closed in P: If $p \in cl(C)$, then $B(p, \varepsilon) \cap C \neq \emptyset$. For $t \in B(p, \varepsilon) \cap C$ we have $d^{sup}(p, t) < \varepsilon$, so $d(p(j_n), t(j_n)) < \varepsilon$ for $n \in \mathbb{N}$.
We conclude that $p \in C$ as $p(j_n)$ and $x(j_n)$ admit an ε-chain of points, $Z_{p(j_n)x(j_n)}$, with $|Z_{p(j_n)x(j_n)}| \leq m(t) + 1$.
Then $\prod_{i\in\mathcal{I}} X_i$ admits a proper clopen partition through $C \times \prod_{i\in\mathcal{I}\setminus\mathbb{J}} X_i$ and $(P \setminus C) \times \prod_{i\in\mathcal{I}\setminus\mathbb{J}} X_i$ and therefore $\prod_{i\in\mathcal{I}} X_i$ is not connected. □

3. Taut metric spaces

In this section we make formal the idea suggested in Observation 2.8.

Definitions 3.1. If (X, d) is a metric space and $Z_{xy} = (z_0, ..., z_n)$, is a chain of points from x to y, we call $l(Z_{xy}) := \Sigma_{0\leq i<n} d(z_i, z_{i+1})$ *the length of the chain of points* Z_{xy}.

We define the function $\chi : X^2 \times (0, \infty) \to [0, \infty]$ by

$$\chi(x, y, \varepsilon) = \begin{cases} inf\{\, l(Z_{xy}) \,:\, Z_{xy} \in \mathcal{Z}_{xy,\varepsilon}\} & \text{if } \mathcal{Z}_{xy,\varepsilon} \neq \emptyset \\ \infty & \text{if } \mathcal{Z}_{xy,\varepsilon} = \emptyset. \end{cases}$$

Notice that Observation 2.2ii) implies that $\chi(x, y, \varepsilon_1) \geq \chi(x, y, \varepsilon_2)$ when $\varepsilon_1 \leq \varepsilon_2$.

We define the function $\tau d : X^2 \to [0, \infty]$ by

$$\tau d(x,y) = lim_{\varepsilon \to 0}\chi(x,y,\varepsilon).$$

Definition 3.2. We call a metric space (X, d) with the property that $d = \tau d$ a *taut* metric space.

Lemma 3.3. *Let x and y be points in a metric space (X, d) with $\tau d(x, y) < v < \infty$.*
There is $\delta_0 > 0$ such that for any $\delta \le \delta_0$ we can find a δ-chain of points from x to y, $Z_{xy} = (x = z_0, z_1, ..., z_n = y)$, which contains a point z such that $Z_{xz} = (x = z_0, ..., z_k = z)$ and $Z_{zy} = (z = z_k, ..., z_n = y)$ have

$$max\{l(Z_{xz}), l(Z_{zy})\} < \frac{v}{2}.$$

Proof. Assume that $\frac{v}{2} > \tau d(x, y)$. As $\tau d(x, y) < \frac{2\tau d(x,y)+v}{4}$, for any $\delta > 0$ we can find a δ-chain of points Z_{xy} with $l(Z_{xy}) < \frac{2\tau d(x,y)+v}{4}$. Letting $z = x$ determines the required inequality as $\frac{2\tau d(x,y)+v}{4} < \frac{v}{2}$.

Assume now that $\frac{v}{2} < \tau d(x, y)$.
There is an $\varepsilon_0 > 0$ such that $\chi(x, y, \varepsilon) > \frac{v}{2}$ for any $0 < \varepsilon \le \varepsilon_0$.
We choose $\delta_0 \le min\{\varepsilon_0, \frac{v-\tau d(x,y)}{2}\}$ and for $0 < \delta \le \delta_0$ we find a δ-chain of points, $Z_{xy} = (x = z_0, z_1, \ldots, z_n = y)$, with

$$l(Z_{xy}) < \tfrac{v+\tau d(x,y)}{2}. \qquad (1)$$

As $l(Z_{xy}) > \frac{v}{2}$, there is $k < n$ satisfying the following inequalities

$$\Sigma_{0\le i<k}\, d(z_i, z_{i+1}) < \tfrac{v}{2} \qquad (2)$$
$$\Sigma_{0\le i\le k}\, d(z_i, z_{i+1}) \ge \tfrac{v}{2} \qquad (3)$$

By (1) and (3) we obtain $\Sigma_{k<i<n} d(z_i, z_{i+1}) < \frac{v+\tau d(x,y)}{2} - \frac{v}{2} = \frac{\tau d(x,y)}{2}$, so

$$\Sigma_{k\le i<n} d(z_i, z_{i+1}) \le \delta + \Sigma_{k<i<n} d(z_i, z_{i+1}) < \tfrac{v-\tau d(x,y)}{2} + \tfrac{\tau d(x,y)}{2} = \tfrac{v}{2} \qquad (4).$$

By (2) and (4), we can take $z = z_k$.

If $\frac{v}{2} = \tau d(x, y)$, we take $v' = \frac{\tau d(x,y)+v}{2}$ and we use the previous case all the way to finding z such that $max\{l(Z_{xz}), l(Z_{zy})\} < \frac{v'}{2} < \frac{v}{2}$.

□

Theorem 3.4. *A metric space (X, d) is taut iff for every pair of points x, $y \in X$ with $d(x, y) < \infty$ and for each $v > d(x, y)$ there is $z \in X$ such that*

$$max\{d(x, z), d(z, y)\} < \frac{v}{2}.$$

Proof. ONLY IF: Assume that (X,d) is taut and let $x,\ y \in X$ and $v > d(x,y) = \tau d(x,y)$ be given.
We make use of Lemma 3.3 to find a chain of points, Z_{xy} which can be decomposed into two chains, Z_{xz} and Z_{zy} such that

$$max\{l(Z_{xz}), l(Z_{zy})\} < \frac{v}{2}.$$

By the triangle inequality we obtain

$$max\{d(x,z)), d(z,y)\} < \frac{v}{2}.$$

IF: Assume now that (X,d) is a metric space satisfying the given inequality.
Fix $v > d(x,y)$ and let $z(0) = x$ and $z(1) = y$. Choose $z(\frac{1}{2})$ such that $max\{d(z(0), z(\frac{1}{2})), d(z(\frac{1}{2}), z(1))\} < \frac{v}{2}$.
For $n \in \mathbb{N}$ and $k \leq 2^n$ we can define inductively points via the dyadic rationals, $z(\frac{k}{2^n})$, such that

$$max\{d(z(\tfrac{k-1}{2^n}), z(\tfrac{k}{2^n})), d(z(\tfrac{k}{2^n}), z(\tfrac{k+1}{2^n}))\} < \tfrac{v}{2^n} \text{ for } 1 \leq k < 2^n.$$

Thus, $\Sigma_{0\leq i<n} d(z(\frac{k}{2^n}), z(\frac{k+1}{2^n})) < v$ and further $\chi(x, y, \frac{1}{2^n}) < v$ for $n \in \mathbb{N}$.
Therefore $\tau d(x,y) < v$ for any $v > d(x,y)$ and so $\tau d(x,y) = d(x,y)$.

□

Definition 3.5. In Geometry, where the geodesic metric is useful, a metric with the property in Theorem 3.4 is said to have *the approximate midpoint property* .

Corollary 3.6. *A metric space (X,d) is taut iff for every $0 < v < \infty$ and for any $x, y \in X$ such that $d(x,y) < v$ there is a set V such that $\{0, v\} \subseteq V \subseteq cl(V) = [0, v]$ and a DD mapping, $\varphi : V \to X$ such that $\varphi(0) = x$ and $\varphi(v) = y$.*

Proof. ONLY IF: Assume that (X,d) is taut and $d(x,y) < v$ for fixed $x, y \in X$, v positive. Starting from $z(0) = x$ and $z(1) = y$, we make use of the approximate midpoint property to define inductively for $n \in \mathbb{N}$ and $k \leq 2^n$ points via the dyadic rationals, $z(\frac{k}{2^n})$, such that

$$max\{d(z(\tfrac{k-1}{2^n}), z(\tfrac{k}{2^n})), d(z(\tfrac{k}{2^n}), z(\tfrac{k+1}{2^n}))\} \leq \tfrac{v}{2^n} \text{ for } 1 \leq k < 2^n.$$

Thus, $d(z(\frac{k_0}{2^n}), z(\frac{k_1}{2^n})) \leq \Sigma_{k_0\leq i<k_1} d(z(\frac{k}{2^n}), z(\frac{k+1}{2^n})) < \frac{|k_1-k_0|\cdot v}{2^n}$ and moreover, following appropriate paths through the points indexed by the dyadic rationals, we find that

$$d(z(\tfrac{k_0}{2^{n_0}}), z(\tfrac{k_1}{2^{n_1}})) < |\tfrac{k_1}{2^{n_1}} - \tfrac{k_0}{2^{n_0}}| \cdot v \qquad (1)$$

Let $V=\{\frac{k\cdot v}{2^n} : n\in\omega,\ 0\le k\le 2^n\}$ and define the mapping $\varphi: V\to X$ by $\varphi(\frac{k\cdot v}{2^n}):=z(\frac{k}{2^n})$, which is DD by (1).

IF: Assume now that (X,d) is such that for every $x,y\in X$ and v positive with $d(x,y)<v$ there is a DD mapping φ as ennounced in a). For fixed $x,y\in X$ and v positive such that $d(x,y)<v$ choose v' such that $d(x,y)<v'<v$ and let $\varphi: V'\to X$ be a DD mapping such that $\{0,v'\}\subseteq V'\subseteq cl(V')=[0,v']$ and $\varphi(0)=x$, $\varphi(v')=y$. Using the density of V' in $[0,v']$, we find $v_0\in V'$ such that $v_0<\frac{v}{2}$ and $v'-v_0<\frac{v}{2}$. Then $\varphi(v_0)$ is a point in X such that $d(x,\varphi(v_0))<\frac{v}{2}$ and $d(\varphi(v_0),y)<\frac{v}{2}$. Therefore (X,d) has the approximate midpoint property, so it is taut by Theorem 3.4.

□

Definition 3.7. Given a metric space (X,d) and $x,y\in X$, a *DD arc from x to y* is the image of an interval $[a,b]\in\mathbb{R}$ with the usual metric under a DD mapping, $f:[a,b]\to X$ such that $f(a)=x$ and $f(b)=y$.

Theorem 3.8. *Suppose (X,d) is a complete taut metric space and that $x,y\in X$ are such that $d(x,y)<\infty$. Then there is DD arc from x to y.*

Proof. Having $d(x,y)<v$ for some positive v, we have the DD mapping $\varphi: V\to X$ constructed in Corollary 3.6. We further create an extension, $\tilde\varphi:[0,v]\to X$ through

$$\tilde\varphi(r):=\bigcap_{n\in\mathbb{N}} cl(\varphi(B(r,\tfrac{1}{n})\cap V)) \text{ for } r\in[0,v].$$

As φ is a DD mapping, $diam(\varphi(B(r,\frac{1}{n})\cap V))<\frac{1}{n}$ and therefore $diam(cl(\varphi(B(r,\frac{1}{n})\cap V)))\le\frac{1}{n}$. Hence the definition of $\tilde\varphi$ gives at most one point.

As X is a complete metric space, any countable decreasing intersection of closed sets is nonempty, so the definition of $\tilde\varphi$ gives exactly one point. It is now obvious that $\tilde\varphi$ extends φ. It is also easy to observe that $\tilde\varphi$ remains DD mapping, therefore $\tilde\varphi$ witnesses the property that x and y can be connected by a DD arc.

□

Lemma 3.9. *Let (X,d) be a metric space with (Y,d_Y) dense in X and $d_Y=d_{\restriction Y^2}$. Then (X,d) is taut iff (Y,d_Y) is taut.*

Proof. ONLY IF: Assume (X,d) is taut. For $y,y'\in Y$ with $d(y,y')<v<\infty$ we use the approximate midpoint property to find $z\in X$ such that $\max\{d(y,z),d(y',z)\}<\frac{d(y,y')+v}{4}$. By the density of Y, there is

$w \in B(z, \frac{v-d(y,y')}{4}) \cap Y$ and then $max\{d(w,y), d(w,y')\} < \frac{d(y,y')+v}{4} + \frac{v-d(y,y')}{4} = \frac{v}{2}$, hence (Y, d_Y) has the approximate midpoint property.

IF: If (Y, d_Y) is taut, then for $x, x' \in X$ such that $d(x,x') < v < \infty$ we first use the density of Y to find $y \in B(x, \frac{v-d(x,y)}{8}) \cap Y$ and $y' \in B(x', \frac{v-d(x,y)}{8}) \cap Y$. As $d(x,x') < \frac{v+d(x,y)}{2}$, then $d(y,y') < \frac{v+d(x,y)}{2} + \frac{v-d(x,y)}{4} = \frac{3v+d(x,y)}{4}$. By the approximate midpoint property, we find $z \in Y$ such that $\max\{d(z,y), d(z,y')\} < \frac{3v+d(x,y)}{8}$.
Then $\max\{d(z,x), d(z,x')\} < \frac{3v+d(x,y)}{8} + \frac{v-d(x,y)}{8} = \frac{v}{2}$ and so the approximate midpoint property holds for the pairs of points from X at finite distance.

□

Example 3.10. A taut connected non-arcwise connected metric space:

$$X := \big([0,1] \times \{0,1\}\big) \bigcup_{n\in\mathbb{N}} \Big(\bigcup_{1\le k<2^n} \Big\{\frac{k}{2^n}\Big\} \times \Big[\frac{1}{2^n}, 1\Big]\Big).$$

We consider X with the usual metric, d. Since (X,d) is dense in the unit square, Lemma 3.9 shows that (X,d) is taut. X is connected since $X \setminus ([0,1] \times \{0\})$ is arcwise connected and $cl(X \setminus ([0,1] \times \{0\})) = X$. However, there is no arc from $(0,0)$ to $(1,1)$.

4. Connectedness II

Lemma 4.1. *The box metric product of taut metric spaces is taut.*

Proof. Suppose $\{(X_i, d_i)\}_{i\in\mathcal{I}}$ is a family of taut metric spaces and consider x, y points in $\prod_{i\in\mathcal{I}} X_i$ such that $d^{sup}(x,y) < v < \infty$.
We can see that d^{sup} has the approximate midpoint property:
By Lemma 3.4, for each $i \in \mathcal{I}$ we can find $z(i) \in X_i$ such that $max\{d_i(x(i), z(i)), d_i(z(i), y(i))\} < \frac{2d_i(x(i),y(i))+v}{6}$.
Then $z := (z(i))_{i\in\mathcal{I}}$ has $max\{d^{sup}(x,z), d^{sup}(z,y)\} \le \frac{2d^{sup}(x,y)+v}{6} < \frac{v}{2}$. □

Through Proposition 1.9, Theorem 3.8 and Lemma 4.1 we obtain

Proposition 4.2. *The box metric product of taut complete metric spaces is finitely arcwise-connected.*

The next theorem represents an attempt to remove completeness from the hypothesis of 4.2:

Theorem 4.3. *The box metric product of countably many taut finitely connected metric spaces is finitely connected.*

Proof. We shall make use of the following result used in the Tychonoff product of connected spaces:

Remark 4.4. For $x \in \prod_{i\in\mathcal{I}} X_i$ and $F \subseteq \mathcal{I}$ we let
$E(F,x) := \{y \in \prod_{i\in\mathcal{I}} X_i : \{i \in \mathcal{I} : y(i) \neq x(i)\} \subseteq F\}$ and further
$E(x) := \bigcup_{F\subseteq\mathcal{I}, |F|<\infty} E(F,x)$.
As each $E(F,x)$ is isometric to $\prod_{i\in F} X_i$, if (X_i, d_i) are connected for $i \in \mathcal{I}$, then $E(x)$ is connected.

Proof of 4.3: Suppose that there are $x, y \in \prod_{n\in\mathbb{N}} X_n$ and C clopen in $\prod_{n\in\mathbb{N}} X_n$ such that $x \sim_{fin} y$, $x \in C$ and $y \notin C$.
Applying Lemma 3.4, we define two sequences $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ of points in C, respectively not in C:
Let $x_1 := x$ and $y_1 := y$. Suppose that we have defined x_n and y_n. Choose $z \in \prod_{n\in\mathbb{N}} X_n$ such that $max\{d(x_n,z), d(z,y_n)\} < d(x_n,y_n)\cdot\frac{2}{3}$. If $z \in C$ let $x_{n+1} := z$ and $y_{n+1} := y_n$. Otherwise define $x_{n+1} := x_n$ and $y_{n+1} := z$.
Clearly, for each $\varepsilon > 0$ there is n so large that

$$sup\{d(x_m,y_m), d(x_n,x_m), d(y_n,y_m) : m \geq n\} < d(x,y)\cdot\left(\tfrac{2}{3}\right)^n \leq \varepsilon \quad (\star)$$

Define now $u, v \in \prod_{n\in\mathbb{N}} X_n$ by $u(n) := x_n(n)$, $v(n) := y_n(n)$ for $n \in \mathbb{N}$. We show $u \in C$, the proof that $v \notin C$ is analogous.
For each $n \in \mathbb{N}$ we define $q_n \in \prod_{n\in\mathbb{N}} X_n$ by

$$q_n(i) = \begin{cases} x_i(i) & \text{if} \quad i \leq n; \\ x_n(i) & \text{if} \quad i \geq n. \end{cases}$$

We know that $E(x_n)$ is connected by Remark 4.4, so $E(x_n) \subseteq C$ for any $n \in \mathbb{N}$.
As $q_n \in E(\{1,2,..,n\}, x_n)$, we conclude that $q_n \in C$ for $n \in \mathbb{N}$.
According to $(\star)$, $d^{sup}(q_n,u) < d(x,y)\cdot\left(\frac{2}{3}\right)^n \leq \varepsilon$ for $n \in \mathbb{N}$, thus u is a limit point of $\{q(n)\}_{n\in\mathbb{N}} \subseteq C$. As C is closed, $u \in C$.
Also by $(\star)$, $lim_{n\to\infty} d_n(u(n), v(n)) = 0$, so $v \in cl(E(u))$.
From Remark 4.4, $E(u)$ is connected, so $cl(E(u)) \subseteq C$. Therefore $v \in C$ - a contradiction.

□

Corollary 4.5. *Arbitrary powers of taut separable finitely connected metric spaces are finitely connected with respect to the box metric.*

Proof. Let (X, d) be taut separable finitely connected metric space and $\mathcal{I}$ an index set.
By Theorem 4.3 $(\Delta(\aleph_1, \mathcal{I}), d^{sup})$ is finitely connected. By ii) and iii) of Lemma 1.12, $\Delta(\aleph_1, \mathcal{I})$ is dense in $(\prod^{\mathcal{I}} X, d^{sup})$, hence $(\prod^{\mathcal{I}} X, d^{sup})$ is finitely connected.

□

5. Tautification

The principal aim of this section is to prove that each metric space is the DD image of a "minimal" taut metric space, which we call its tautification.
There are various roads to the tautification. The τ process introduced in 3.1, an approach derived from the idea of extending connectedness in the box metric product, is developing at "intermediate" speed with respect to the others, that shall be described together with properties.

Definitions 5.1. Suppose (X, d) is a metric space. Given x, $y \in X$ and $r > 0$, we say that x and y are *r-linked* provided that for every $a \in [0, r]$ there is $z \in X$ such that $d(x, z) \leq a$ and $d(z, y) \leq r - a$.

We define the function $\lambda d : X^2 \to [0, \infty)$ by

$$\lambda d(x, y) = \begin{cases} \infty & \text{if } x \text{ and } y \text{ are not } r\text{-linked for any } r \in [0, \infty) \\ inf\{r : x \text{ and } y \text{ are } r\text{-linked}\} & \text{otherwise.} \end{cases}$$

The following notions are inspired by Corollary 3.6.

Definitions 5.2. Suppose (X, d) is a metric space. Given x, $y \in X$ and $r > 0$, we say that x and y are *r-densely linked* provided that there is $\{0, r\} \subseteq R \subseteq cl(R) = [0, r]$ and a DD mapping $\varphi : R \to (X, d)$ such that $\varphi(0) = x$ and $\varphi(r) = y$.

We define the function $\Upsilon d : X^2 \to [0, \infty)$ by

$$\Upsilon d(x, y) = \begin{cases} \infty \text{ if } x \text{ and } y \text{ are not } r\text{-densely linked for any } r \in [0, \infty) \\ inf\{r : x \text{ and } y \text{ are } r\text{-densely linked}\} \text{ otherwise.} \end{cases}$$

Lemma 5.3. *Given (X, d) metric space, λd, τd and Υd are metrics on X. Moreover, $(X, \Upsilon d)$ is taut.*

Proof. It is easy to see by the definitions 3.1, 5.1 and 5.2 that λd, τd and Υd are symmetric functions on X^2 and that they are null exactly on the diagonal Δ_X.

We further show the triangle inequality.

For λd:

Let $x, y, z \in X$, $\delta > 0$ and $a \in [0, \lambda d(x,y) + \lambda d(y,z) + \delta]$.

Assuming that $a \leq \lambda d(x,y) + \delta$ holds, there is $w \in X$ such that $d(x,w) \leq a$ and $d(w,y) \leq \lambda d(x,y) + \delta - a$.

Then $d(w,z) \leq d(w,y) + d(y,z) \leq \lambda d(x,y) + d(y,z) + \delta - a$.

Assuming that $a > \lambda d(x,y) + \delta$ holds, we first write this as $\lambda d(y,z) < \lambda d(x,y) + \lambda d(y,z) + \delta - a$.

By the definition of $\lambda d(y,z)$, there is $w \in X$ such that $d(w,z) \leq \lambda d(x,y) + \lambda d(y,z) + \delta - a$ and such that $d(y,w) \leq \lambda d(y,z) + \delta - (\lambda d(x,y) + \lambda d(y,z) + \delta - a) = a - \lambda d(x,y)$.

Further $d(w,x) \leq d(w,y) + d(x,y) \leq a - \lambda d(x,y) + d(x,y) \leq a$.

As for every $a \in [0, \lambda d(x,y) + \lambda d(y,z) + \delta]$ we find a point $w \in X$ such that $d(x,w) \leq a$ and $d(w,z) \leq \lambda d(x,y) + \lambda d(y,z) + \delta - a$, we conclude that $\lambda d(x,z) \leq \lambda d(x,y) + \lambda d(y,z) + \delta$ for every $\delta > 0$ and the triangle inequality is verified.

For τd:

Assume that $\tau d(x,y)$ and $\tau d(y,z)$ are finite and fix $n \in \mathbb{N}$.

There are ε_1, $\varepsilon_2 > 0$ such that $\chi(x,y,\varepsilon) < \tau d(x,y) + \frac{1}{n}$ for $0 < \varepsilon \leq \varepsilon_1$ and $\chi(y,z,\varepsilon) < \tau d(y,z) + \frac{1}{n}$ for $0 < \varepsilon \leq \varepsilon_2$.

Then for $0 < \varepsilon \leq min\{\varepsilon_1, \varepsilon_2\}$ we find ε-chains of points from x to z of d-length no greater than $\tau d(x,y) + \tau d(y,z) + \frac{2}{n}$.

As $\chi(x,z,\varepsilon) \leq \tau d(x,y) + \tau d(y,z) + \frac{2}{n}$ for $\varepsilon \leq min(\varepsilon_1, \varepsilon_2)$, we conclude that $\tau d(x,z) \leq \tau d(x,y) + \tau d(y,z) + \frac{2}{n}$ and as n was arbitrarily fixed, we obtain the triangle inequality.

For Υd:

We let $n \in \mathbb{N}$, $\varphi_1 : R_1 \to X$ DD mapping with $cl(R_1) = [0, \Upsilon d(x,y) + \frac{1}{n}]$, $\varphi_1(0) = x$, $\varphi_1(\Upsilon d(x,y) + \frac{1}{n}) = y$ and $\varphi_2 : R_2 \to X$ DD mapping with $cl(R_2) = [0, \Upsilon d(y,z) + \frac{1}{n}]$, $\varphi_2(0) = y$, $\varphi_2(\Upsilon d(y,z) + \frac{1}{n}) = z$.

Letting $R_2 + \Upsilon d(x,y) + \frac{1}{n}$ denote the right translation of the set R_2 by $\Upsilon d(x,y) + \frac{1}{n}$, we define the mapping $\varphi : R_1 \cup (R_2 + \Upsilon d(x,y) + \frac{1}{n}) \to X$ by

$$\varphi(r) = \begin{cases} \varphi_1(r) \text{ if } r \in R_1 \\ \varphi_2(r) \text{ if } r \in R_2 + \Upsilon d(x,y) + \frac{1}{n}. \end{cases}$$

φ is DD, with $cl(R_1 \cup (R_2 + \Upsilon d(x,y) + \frac{1}{n})) = [0, \Upsilon d(x,y) + \Upsilon d(y,z) + \frac{2}{n}]$, $\varphi(0) = x$ and $\varphi(\Upsilon d(x,y) + \Upsilon d(y,z) + \frac{2}{n}) = z$.

As $\Upsilon d(x,z) \leq \Upsilon d(x,y) + \Upsilon d(y,z) + \frac{2}{n}$ for $n \in \mathbb{N}$ arbitrarily fixed, we obtain the triangle inequality for Υd.

The tautness of $(X, \Upsilon d)$ follows easily through observing that it is a metric space with the approximate midpoint property.

□

Lemma 5.4. *Let (Y, ρ) be a taut metric space. If $f : (Y, \rho) \to (X, d)$ is a DD mapping, to a metric space (X, d), then $f : (Y, \rho) \to (X, \tau d)$ is also DD mapping.*

Proof. Let $\rho(y, y')$ be finite and let $n \in \mathbb{N}$. By Corollary 3.6, y and y' are $\rho(y, y') + \frac{1}{n}$-densely linked, so let $\varphi : R \to Y$ be a DD mapping with $cl(R) = [0, \rho(y, y') + \frac{1}{n}]$, $\varphi(0) = y$ and $\varphi(\rho(y, y') + \frac{1}{n}) = y'$. By the density of R, for fixed $\varepsilon > 0$ there is an ε-chain of points from 0 to $\rho(y, y') + \frac{1}{n}$, $Z_{0\,\rho(y,y')+\frac{1}{n}} = (r_0 = 0, r_1, \ldots, r_k = \rho(y, y') + \frac{1}{n})$, where $k \in \mathbb{N}$ and $r_i \in R$ for $0 \le i \le k$.
As φ and f are DD, the mapping $f \circ \varphi$ is DD and
$Z_{\varphi(y)\varphi(y')} := (f \circ \varphi(r_0) = f(y), f \circ \varphi(r_1), \ldots, f \circ \varphi(r_k) = f(y'))$ is an ε-chain of points from $f(y)$ to $f(y')$ with

$$l(Z_{f(y)f(y')}) = \sum_{0 \le i < n} d(f \circ \varphi(r_i), f \circ \varphi(r_{i+1})) \le \sum_{0 \le i < n} |r_{i+1} - r_i| = \rho(y, y') + \frac{1}{n}.$$

Since $\chi(f(y), f(y'), \varepsilon) \le \rho(y, y') + \frac{1}{n}$ for arbitrary ε and n, we conclude that $\tau d(f(y), f(y')) \le \rho(y, y')$ and therefore $f : (Y, \rho) \to (X, \tau d)$ is DD.

□

Lemma 5.5. *Given (X, d) metric space, the metrics λd, τd and Υd satisfy the inequalities*

$$d \le \lambda d \le 2d \text{ and } \lambda d \le \tau d \le \Upsilon d.$$

Proof. To show the inequalities fix $x, y \in X$.

If $\lambda d(x, y) = \infty$, then $\lambda d(x, y) \ge d(x, y)$.
By definition, if $\lambda d(x, y) < r$ for $r > 0$, then there is $z \in X$ such that $d(x, z) \le r$ and $d(z, y) \le 0$, therefore $d(x, y) \le r$. Hence $d(x, y) \le \lambda d(x, y)$.

Any points x and y are $2d(x, y)$-linked as for $a \in [0, 2d(x, y)]$ we can choose $z = x$ if $a \le d(x, y)$ and $z = y$ if $a \ge d(x, y)$. Therefore $\lambda d \le 2d$.

If x, $y \in X$ are such that $\tau d(x, y)$ is finite, then for a fixed $n \in \mathbb{N}$ we have $\chi(x, y, \varepsilon) < \tau d(x, y) + \frac{1}{n}$ for any $\varepsilon > 0$.
Given an $a \in [0, \tau d(x, y) + \frac{1}{n}]$, for an ε positive sufficiently small we can find an ε-chain of points from x to y, Z_{xy} of d-length under $\tau d(x, y) + \frac{1}{n}$ such that there is $z \in Z_{xy}$ which decomposes Z_{xy} into two chains of points, Z_{xz} and Z_{zy} of length no greater than a and respectively no greater than $\tau d(x, y) + \frac{1}{n} - a$. This gives $d(x, z) \le a$ and $d(z, y) \le$

$\tau d(x,y) + \frac{1}{n} - a$ and so $\lambda d(x,y) \leq \tau d(x,y) + \frac{1}{n}$.
As n is arbitrarily chosen, then $\lambda d(x,y) \leq \tau d(x,y)$.

We apply Lemma 5.4 to the DD identity from $(X, \Upsilon d)$ to (X,d) and obtain $\tau d \leq \Upsilon d$.

□

Remark 5.6. While (X,d) and $(X,\lambda d)$ are topologically equivalent, the topology of $(X,\tau d)$ is in general finer than the topology of (X,d).

Examples 5.7. The inequalities of Lemma 5.5 cannot be straightened:
a) If we consider the unit circle with d the Euclidean distance, the points $p = (-1,0)$ and $q = (1,0)$, then $\lambda d(p,q) = 2\sqrt{2}$, while $\tau d(p,q) = \pi$. Therefore $d(p,q) < \lambda d(p,q) < 2d(p,q)$ and $\lambda d(p,q) < \tau d(p,q)$.

b) For $\tau d(p,q) \neq \Upsilon d(p,q)$ we can look at the following example:
For each $n \in \mathbb{N}$ let f_n be the graph of $y(x) = x^n$, where $x \in [0,1]$.
On f_n we choose the points $p = (0,0) = p_{n,0}$, $p_{n,1}$,...,$p_{n,n} = (1,1) = q$ such that they partition G_n into n arcs of equal lengths. Let $X = \bigcup_{n\in\mathbb{N}, 0\leq k\leq n}\{p_{n,k}\}$ with the Euclidean distance d.
It is easy to observe that $\tau d(p,q) = 2$ and as all other points from X distinct from p and q are isolated, $\Upsilon d(p,q) = \infty$.

Corollary 5.8. *A metric space (X,d) is taut iff*

$$d = \lambda d = \tau d = \Upsilon d.$$

Proof. Corollary 3.6 reads now as (X,d) is taut iff $d = \Upsilon d$. As $d \leq \lambda d \leq \tau d \leq \Upsilon d$ by Lemma 5.5, (X,d) is taut iff the four metrics coincide.

□

Definitions 5.9. Let (X,d) be a metric space and β an ordinal.

We define $\lambda_0 d = d = \tau_0 d$ and $\lambda_\beta d = sup\{\lambda(\lambda_\alpha d) : \alpha < \beta\}$, $\tau_\beta d = sup\{\tau(\tau_\alpha d) : \alpha < \beta\}$.

We let Λd be the metric on X such that $\Lambda d = \lambda_\gamma d = \lambda(\lambda_\gamma)d$ from an ordinal γ.

We let Td be the metric on X such that $Td = \tau_\nu d = \tau(\tau_\nu)d$ from an ordinal ν.

Lemma 5.10. *If d_1 and d_2 are metrics on X such that $d_1 \leq d_2$, then $\tau d_1 \leq \tau d_2$, $\lambda d_1 \leq \lambda d_2$, $\Upsilon d_1 \leq \Upsilon d_2$.*

Proof. For x, y points in X and $\varepsilon > 0$ it suffices to remark that all ε-d_2-chains of points from x to y are ε-d_1-chains of points, if x and y

are ε-d_2 (densely) linked then they also are ε-d_1 (densely) linked.

□

Theorem 5.11. *Given a metric space (X, d), the following are true:*
i) (X, Td) and $(X, \Lambda d)$ are taut and $\Lambda d = Td = \Upsilon d$.
ii) If (Y, ρ) is a taut metric space and $f : (Y, \rho) \to (X, d)$ is a DD mapping to a metric space (X, d), then $f : (Y, \rho) \to (X, Td)$ is also DD.

Proof. i) (X, Td) is taut by definition. From the definition of Λd, it is easy to observe that $(X, \Lambda d)$ has the approximate midpoint property. For the equality of the three metrics we make use of Corollary 5.8 and of the monotonicity properties from Lemma 5.10:
As $d \leq \Lambda d$, $\Upsilon d \leq \Upsilon(\Lambda d)$. As Λd is taut, $\Upsilon(\Lambda d) = \Lambda d$, hence $\Upsilon d \leq \Lambda d$.
As $d \leq \Upsilon d$, $\Lambda d \leq \Lambda(\Upsilon d)$. As Υd is taut, $\Lambda(\Upsilon d) = \Upsilon d$, hence $\Lambda d \leq \Upsilon d$.
We conclude that $\Lambda d = \Upsilon d$ and we can show similarly that $Td = \Upsilon d$.

ii) can be seen as a consequence of Lemma 5.4 together with the property that if $f : (Y, \rho) \to (X, d_i)$ is DD mapping for $i \in \mathcal{I}$, then $f : (Y, \rho) \to (X, sup\{d_i : i \in \mathcal{I}\})$ is also DD.

□

Definition 5.12. Henceforth, we call (X, Td) the *tautification* of (X, d).

We extend the idea of Lemma 3.9 to the tautification process:

Lemma 5.13. *Let (X, d) be a metric space with (Y, d_Y) dense in X and $d_Y = d_{\restriction Y^2}$. Then $\tau d_Y = (\tau d)_{\restriction Y^2}$.*

Proof. Let χ_X and χ_Y represent the infimum functions calculated through chains of points in X and respectively through chains of points in Y.
For $y, y' \in Y$ and $\varepsilon > 0$ we have $\chi_Y(y, y', \varepsilon) \geq \chi_X(y, y', \varepsilon)$ as all the ε-chains of points in Y are ε-chains of points in X.
For $\delta > 0$ we can find a chain of points, $(y = z_0, \ldots, z_n = y')$, such that

$$\Sigma_{0 \leq i < n} d(z_i, z_{i+1}) \leq \chi_X(y, y', \varepsilon) + \frac{\delta}{2} \qquad (1)$$

By the density of Y in X, for each $1 \leq i \leq n-1$ we can find points $u_i \in B(z_i, \frac{\delta}{4n})$. We let $u_0 = y$ and $u_n = y'$.
Using the triangle inequality,

$$\Sigma_{0 \leq i < n} d(u_i, u_{i+1}) \leq \Sigma_{0 \leq i < n} \Big(d(z_i, z_{i+1}) + \frac{\delta}{2n} \Big) \qquad (2)$$

From (1) and (2) we obtain

$$\Sigma_{0 \leq i < n} d(u_i, u_{i+1}) \leq \chi_X(y, y', \varepsilon) + \delta$$

As $(u_0, ..., u_n)$ is a chain of points in Y from y to y', we conclude that $\chi_Y(y, y', \varepsilon) \le \chi_X(y, y', \varepsilon) + \delta$.
As δ was chosen arbitrary positive value, $\chi_Y(y, y', \varepsilon) = \chi_X(y, y', \varepsilon)$ and therefore $\tau d_Y = (\tau d_X)_{\restriction Y^2}$.

□

6. Complete and Compact

We have seen in Example 5.7a) that the λ operator takes longer iteration to attain the tautification, even when the metric space is compact. On the other hand, the τ operator gives the tautification of a compact metric space in only one step:

Theorem 6.1. *Suppose (X, d) is a compact metric space. Then $\tau d = Td$; thus $(X, \tau d)$ is taut.*

Proof. We want to see that for every $x, y \in X$ with $\tau d(x, y) < v$ there is $z \in X$ such that $\tau d(x, z) < \frac{v}{2}$ and $\tau d(z, y) < \frac{v}{2}$.
As $\tau d(x, y) < \frac{\tau d(x,y)+v}{2}$, by Lemma 3.3, we find $n_0 \in \mathbb{N}$ such that for every $n \ge n_0$ there is a point z_n of Z_{xy} such that the $\frac{1}{n}$-chains of points from x to z_n, Z_{x,z_n}, and from z_n to y, $Z_{z_n,x}$ have

$$max\{l(Z_{x,z_n}), l(Z_{z_n,x})\} < \frac{\tau d(x, y) + v}{4}.$$

Without loss of generality, we can assume that the sequence of points $\{z_n\}_{n\ge n_0}$ is convergent to some z in X.
For every $n \ge n_0$ let $m_n \ge n$ be such that $z_{m_n} \in B(z, \frac{1}{n})$.
Then $Z_{x,z_{m_n}} \cup \{z\}$ and $Z_{z_{m_n},y} \cup \{z\}$ are $\frac{1}{n}$-chains of points from x to z and respectively from z to y of d-lengths no greater than $\frac{\tau d(x,y)+v}{4} + \frac{1}{n}$ for $n \ge n_0$.
Having then $max\{\chi(x, z, \frac{1}{n}), \chi(z, y, \frac{1}{n})\} \le \frac{\tau d(x,y)+v}{4} + \frac{1}{n}$ for any $n \ge n_0$, we conclude that $max\{\tau d(x, z), \tau d(y, z)\} \le \frac{\tau d(x,y)+v}{4} < \frac{v}{2}$.

□

Example 6.2. Theorem 6.1 does not extend to complete metric spaces. We modify 5.7 b) in order to introduce an example of a complete metric space (Y, ρ) with $\tau\rho \ne T\rho$:

Let (H, d) denote the hedgehog with countably many spikes, $H = \bigcup_{n\in\omega} I_n$ and of center $\{o\} = I_m \cap I_n$ for $m \ne n$.
For $n \in \omega$ and $k \in \overline{0, 2^n}$ we let $p_n(k)$ denote the point from I_n such

that $d(o, p_n(k)) = \frac{k}{2^n}$.
We let (Y, ρ) be the quotient space given by $Y = X_{/\sim}$, where

$$p_m(k) \sim p_n(l) \quad \text{iff} \quad (m,k) = (n,l) \text{ or } \tfrac{k}{2^m} = \tfrac{l}{2^n} \in \{0,1\}$$

We denote by $q := \widehat{p_0(1)} = \{p_n(2^n) : n \in \omega\}$ and let

$$\rho(\widehat{x}, \widehat{y}) = min\{d(x,y), d(x,q) + d(y,q)\}.$$

The Cauchy sequences in (Y, ρ) are those converging to either o or q. As for any $n \in \omega$ we have along I_n $\frac{1}{2^n}$-ρ-chains of points from o to q of length 1, $\tau\rho(o,q) = 1$.
All other points of Y are isolated with respect to ρ, so $T\rho(o,q) = \infty$.

A metric space (X, d_1) with $d_1 \geq d_0$ for some (X, d_0) complete is not necessarily complete. Nevertheless, the metrics defined in this section inherit completness:

Proposition 6.3. *a) If (X,d) is complete and ξ is an ordinal, then $(X, \lambda_\xi d)$ and $(X, \tau_\xi d)$ are complete.*
b) If (X,d) is complete, then $(X, \Upsilon d)$ is complete.

Proof. a) From the first inequalities in Lemma 5.5, d and λd have the same convergent sequences and Cauchy sequences. Thus $(X, \lambda d)$ inherits the completness from (X,d).
Assuming that $(X, \lambda_\alpha d)$ is complete for $\alpha < \xi$, where ξ is a limit ordinal, then $\lambda_\xi d = sup\{\lambda_\alpha d : \alpha < \xi\}$ is a metric preserving the completeness of X according to Corollary 1.10.

If $\{x_n\}_{n \in \mathbb{N}}$ is a τd-Cauchy sequence, then the second inequality of 5.5 shows it is also d-Cauchy and so it d-converges to some $x \in X$.
As $\{x_n\}_{n \in \mathbb{N}}$ is a τd-Cauchy sequence, given $\varepsilon > 0$, there is n_ε such that $\tau d(x_{n_\varepsilon}, x_n) \leq \varepsilon$ for any $n \geq n_\varepsilon$.
This implies that $\chi(x_{n_\varepsilon}, x_n, \frac{\varepsilon}{k}) \leq \varepsilon$ for any $n \geq n_\varepsilon$ and for any $k \in \mathbb{N}$.
Also, as $\{x_n\}_{n \in \mathbb{N}}$ is d-convergent to x, we have that $d(x_n, x) \leq \frac{\varepsilon}{k}$ for n no smaller than some $n_{\varepsilon,k}$.
In particular we can form $\frac{\varepsilon}{k}$-chains of points from x_{n_ε} to x through points x_n with $n \geq max\{n_\varepsilon, n_{\varepsilon,k}\}$ such that we obtain $\chi(x_{n_\varepsilon}, x, \frac{\varepsilon}{k}) \leq \varepsilon + \frac{\varepsilon}{k}$.
We conclude that $\tau d(x_{n_\varepsilon}, x) \leq \varepsilon$ and further that the sequence $\{x_n\}_{n \in \mathbb{N}}$ is τd-convergent to x. Therefore $(X, \tau d)$ is complete.
The same kind of argument as for $\lambda_\xi d$ completes the inductive argument that $\tau_\xi d$ is complete.

b) The completeness of $(X, \Upsilon d)$ is a consequence of a) in view of the fact that Υd is an iteration of the λ or of the τ operator.

□

Lemma 6.4. *a) Given a metric space (X, d), a mapping $f : I \to (X, d)$ from an interval $I \subseteq \mathbb{R}$ is a DD arc iff $f : I \to (X, Td)$ is a DD arc.*

b) The tautification Td of a complete metric space (X, d) is discrete iff all the DD arcs in (X, d) are constant.

Proof. a) *IF:* As I with the usual metric is taut, Theorem 5.11 ii) implies that $f : I \to (X, Td)$ is a DD arc.

ONLY IF: If $f : I \to (X, Td)$ is a DD arc, then we use the fact that the identity from (X, Td) to (X, d) is DD to argue that $f : I \to (X, d)$ is also a DD arc.

b) *IF:* Assume that $Td(x, y) = \infty$ for $x \neq y \in X$. Then x and y can be separated by disjoint clopen balls, $B(x, \infty)$ and $B(y, \infty)$, so the points cannot be Td-arc connected and in particular there are no d-DD arcs from x to y by a).

ONLY IF: If (X, d) is complete, then so is (X, Td) by b) of Proposition 6.3.

According to Theorem 3.8, $Td(x, y) < \infty$ implies that there is a Td-DD arc from x to y, which is also a d-DD arc from x to y by a).

□

Remark 6.5. For a general metric space (X, d) with tautification $Td(x, y) < \infty$ we do not generally have x and y d-arcwise connected, nor Td-arcwise connected, as we can observe in the case of the rationals.

Example 6.6. There is a homeomorphic image (X, d) of $[0, 1]$ such that (X, d) has no d-DD arcs and therefore $(X, \tau d)$ is discrete.

We need the following lemma, which shows that there are no space-filling DD curves, and which can be extended to analogous results for n-dimensional manifolds:

Lemma 6.7. *Consider $(\mathbb{R}, d)$, where d is the usual metric and $(\mathbb{R}^2, d_2)$, where d_2 is the induced box metric of d or the Euclidean distance.*
The range of any DD mapping from a closed interval of $(\mathbb{R}, d)$ to $(\mathbb{R}^2, d_2)$ has empty interior.

Proof. Suppose not. As the mapping can be scaled, we can assume without loss of generality that there is an interval, $I \subseteq \mathbb{R}$ and a DD mapping, $f : I \to \mathbb{R}^2$ such that $[0, 1]^2 \subseteq f(I)$.

Fix $n \in \mathbb{N}$ and choose $r_{ij} \in f^{-1}((\frac{i}{n}, \frac{j}{n}))$ for $1 \leq i, j \leq n$.
We first observe that if there is a point $x \in f(B(r_{ij}, \frac{1}{2n})) \cap f(B(r_{kl}, \frac{1}{2n}))$, then $d_2(x, (\frac{i}{n}, \frac{j}{n})) < \frac{1}{2n}$ and $d_2(x, (\frac{k}{n}, \frac{l}{n})) < \frac{1}{2n}$ since f is DD mapping, so $d_2((\frac{i}{n}, \frac{j}{n}), (\frac{k}{n}, \frac{l}{n})) < \frac{1}{n}$, which implies $(i, j) = (k, l)$.
Therefore $f(B(r_{ij}, \frac{1}{2n}))$ are disjoint sets and further $\{B(r_{ij}, \frac{1}{2n})\}_{1 \leq i,j \leq n}$ is a set of n^2 disjoint intervals of length $\frac{1}{n}$.
As $n \in \mathbb{N}$ was arbitrarily fixed, this contradicts the boundedness of the interval I.

□

Proof of 6.6: Let now $\sigma : [0, 1] \to [0, 1]^2$ be a space-filling curve with the property

$$\sigma(G) \neq \emptyset \text{ for any open } \emptyset \neq G \subseteq [0, 1]. \qquad (\star)$$

Let (X, d) be the graph of σ as a subset of $[0, 1]^3$ with the box metric. As the box metric and the Euclidean metric are equivalent in 3-space, (X, d) is a homeomorph of $[0, 1]$.
Suppose that there is a non-constant DD arc, A, in (X, d) and let J_1 be the projection of A onto the domain of σ. As A is connected, J_1 contains an open set $G \subseteq [0, 1]$. Thus, $(\star)$ shows that the projection J_2 of A onto $[0, 1]^2$ has $int(A) \neq \emptyset$. However, when $[0, 1]^2$ is given the (equivalent) box metric, the projection is a DD mapping. Hence $int(A) = \emptyset$ - a contradiction.

7. Zero-Dimensional

Recall that for (X, d) metric space, x, $y \in X$ and $\varepsilon > 0$ we have $x \sim_\varepsilon y$ iff there is an ε-chain of points from x to y.
We introduce a definition relating to the notion of measure of connectedness used by Lowen in [7]:

Definition 7.1. Given (X, d) metric space and $x, y \in X$, we define

$$sd(x, y) = inf\{\varepsilon > 0 \,:\, x \sim_\varepsilon y\}$$

We call a metric space (X, d) *slack* if $d = sd$.

Recall that a (pseudo)metric d on X is said to be an ultra(pseudo)metric provided that $d(x, y) \leq max\{d(x, z), d(z, y)\}$ for any $x, y \in X$.

Remarks 7.2. For (X, d) metric space the following are true:
i. sd is an ultrapseudometric on X such that the identity map from (X, d) to (X, sd) is DD;
ii. (X, d) has $sd = 0$ iff (X, d) is Cantor-connected.

iii. (X, d) is slack iff d is an ultrametric:

IF: If d is an ultrametric, let $Z_{xy} = (x = z_0, z_1, \ldots, z_n = y)$ be an ε-chain of points from x to y. Then $d(z_0, z_1) \leq \varepsilon$ and $d(z_1, z_2) \leq \varepsilon$ implies $d(z_0, z_2) \leq \varepsilon$. Further $d(z_0, z_i) \leq \varepsilon$ and $d(z_i, z_{i+1}) \leq \varepsilon$ implies $d(z_0, z_{i+1}) \leq \varepsilon$ for $i < n$. Therefore we obtain $d(x, y) = d(z_0, z_n) \leq \varepsilon \leq sd(x, y) \leq d(x, y)$, so $d = sd$.

ONLY IF: If $d = sd$, then the ultrapseudometric sd is actually an ultrametric.

In particular this implies that *slack metric spaces are 0-dimensional.*

Proposition 7.3. *If (X, d) is a compact 0-dimensional metric space, then there is a compatible metric ρ on X such that (X, ρ) is slack.*

Proof. Given x, $y \in X$ we let

$\mathcal{P}_{xy} = \{(A, B) \subseteq X^2 : \{A, B\}$ clopen partition of X with $x \in A, y \in B\}$.

The ultrametric ρ defined by

$$\rho(x, y) = \begin{cases} sup\{d(A, B) : (A, B) \in \mathcal{P}_{xy}\} & \text{if} \quad x \neq y; \\ 0 & \text{otherwise} . \end{cases}$$

has $\rho \leq d$ and therefore it generates a topology contained in the topology generated by d. The topology generated by ρ is also Hausdorff and therefore it agrees with the topology of the compact (X, d).

By Proposition 7.2, (X, ρ) is a slack metric space.

□

Proposition 7.4. *Products of slack metric spaces are slack with respect to the box metric topology.*

Therefore products of slack metric spaces are 0-dimensional with respect to the box metric topology.

Proof. The supremum on a family of ultrametrics is an ultrametric. Further we invoque Remark 7.2.

□

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Mihaela Iancu miancu2@buffalo.edu
Scott W. Williams sww@buffalo.edu

State University of New York at Buffalo
14260-2900