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Discussion of Lie's nonlinear superposition theory

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Abstract. The basic results on nonlinear superposition principles were published by E. Vessiot [1], A. Guldberg [2] and S. Lie [3]. I formulate here the main theorem and illustrate it by several examples. For a detailed presentation of Lie's theory of nonlinear superposition principles, see [4].

1 Introduction

It is a very interesting problem to seek, together with E. Vessiot [1] and A. Guldberg [2], all systems

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = f^i(t,x), \quad i = 1,\dots,n, \tag{1}$$

whose general solutions $x = (x^1,\dots,x^n)$ can be expressed via m particular solutions $x_1 = (x_1^1,\dots,x_1^n),\dots,x_m = (x_m^1,\dots,x_m^n)$ in the form

$$x^i = \varphi^i(x_1,\dots,x_m;\ C_1,\dots,C_n), \quad i = 1,\dots,n. \tag{2}$$

S. Lie, 1893

Lie solved the problem by proving the following [3](ii)-(iv)

Theorem. Equations (1) possess a nonlinear superposition if and only if they have the form (discovered by Lie [3](i))

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = T_1(t)\xi_1^i(x) + \cdots + T_r(t)\xi_r^i(x), \quad i = 1,\dots,n, \tag{3}$$

whose coefficients $\xi_\alpha^i(x)$ satisfy the condition that the operators

$$X_\alpha = \xi_\alpha^i(x)\frac{\partial}{\partial x^i}\,, \quad \alpha = 1,\dots,r, \tag{4}$$

span a Lie algebra L_r of a finite dimension r termed the Vessiot-Guldberg-Lie algebra for equation (1). The number m of necessary particular solutions is estimated by $nm \geq r$. Superposition formulae (2) are defined implicitly by the equations

$$J_i(x, x_1,\dots,x_m) = C_i,\ i = 1,\dots,n, \tag{5}$$

where J_i are functionally independent (with respect to $x^1,\dots,x^n$) invariants of the $(m+1)$-point representation

$$V_\alpha = X_\alpha + X_\alpha^{(1)} + \cdots + X_\alpha^{(m)} \tag{6}$$

of the operators (4).

The present talk is aimed at illistrating Lie's theorem by non- trivial examples.

2 Examples

Example 1. Consider a single homogeneous linear equation written in the form $\mathrm{d}x/\mathrm{d}t = A(t)x$. Here $r = 1$ and $X = x\mathrm{d}/\mathrm{d}x$. We take the two-point representation (6) of X :

$$V = x\frac{\partial}{\partial x} + x_1\frac{\partial}{\partial x_1}$$

and its invariant $J(x,x_1) = x/x_1$. Equation (5) has the form $x/x_1 = C$. Hence, $m = 1$ and the formula (2) is the linear superposition $x = Cx_1$.

Lie's generalization (3) of this simplest example is the equation with separated variables:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = T(t)h(x).$$

Here $r = 1$ and $X = h(x)\mathrm{d}/\mathrm{d}x$. Taking the two-point representation V of X,

$$V = h(x)\frac{\partial}{\partial x} + h(x_1)\frac{\partial}{\partial x_1}\,,$$

and integrating the characteristic system $\mathrm{d}x/h(x) = \mathrm{d}x_1/h(x_1)$, one obtains the invariant $J(x,x_1) = H(x) - H(x_1)$, where $H(x) = \int(1/h(x))\mathrm{d}x$. Equation (5) has the form $H(x) - H(x_1) = C$. Hence, $m = 1$ and the formula (2) provides the nonlinear superposition $x = H^{-1}(H(x_1) + C)$.

Example 2. The non-homogeneous linear equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x + B(t)$$

has the form (3) with $T_1 = B(t)$ and $T_2 = A(t)$. The Vessiot-Guldberg-Lie algebra (4) is an L_2 spanned by the operators

$$X_1 = \frac{\mathrm{d}}{\mathrm{d}x}\,, \quad X_2 = x\frac{\mathrm{d}}{\mathrm{d}x}\,\cdot$$

Substituting $n = 1$ and $r = 2$ in $nm \geq r$, we see that the expression (2) for the general solution requires at least two ($m = 2$) particular solutions. In fact, this number is sufficient. Indeed, let us take the three-point representation (6) of the basic operators X_1 and X_2 :

$$V_1 = \frac{\partial}{\partial x}+\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\,, \quad V_2 = x\frac{\partial}{\partial x}+x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}\,,$$

and show that they admit one invariant. To find it, we first solve the characteristic system for the equation $V_1(J) = 0$, namely, $\mathrm{d}x = \mathrm{d}x_1 = \mathrm{d}x_2$. Integration yields two independent invariants, e.g. $u = x - x_1$ and $v = x_2 - x_1$. Hence, the common invariant $J(x, x_1, x_2)$ for two operators, V_1 and V_2, can be obtained by taking it in the form $J = J(u, v)$ and solving the equation $\widetilde{V}_2(J(u, v)) = 0$, where the action of V_2 is restricted to the space of the variables u, v by using the formula $\widetilde{V}_2 = V_2(u)\partial/\partial u + V_2(v)\partial/\partial v$. Noting, that $V_2(u) = x - x_1 \equiv u$ and $V_2(v) = x_2 - x_1 \equiv v$, we have

$$\widetilde{V}_2 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\,\cdot$$

Hence the invariant is $J(u, v) = u/v$, or $J(x, x_1, x_2) = (x - x_1)/(x_2 - x_1)$. Thus, equation (5) is written $(x-x_1)/(x_2-x_1) = C$. Hence, (2) is the linear superposition $x = x_1+C(x_2-x_1) \equiv (1-C)x_1+Cx_2$.

Example 3. Consider the Riccati equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = P(t) + Q(t)x + R(t)x^2. \tag{7}$$

Here the Vessiot-Guldberg-Lie algebra is L_3 spanned by

$$X_1 = \frac{\mathrm{d}}{\mathrm{d}x}\,, \quad X_2 = x\frac{\mathrm{d}}{\mathrm{d}x}\,, \quad X_3 = x^2\frac{\mathrm{d}}{\mathrm{d}x}\,\cdot \tag{8}$$

We take the four-point representation of the operators (8),

$$\begin{aligned} V_1 &= \frac{\partial}{\partial x} + \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}, \\ V_2 &= x\frac{\partial}{\partial x} + x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}\,, \\ V_3 &= x^2\frac{\partial}{\partial x} + x_1^2\frac{\partial}{\partial x_1} + x_2^2\frac{\partial}{\partial x_2} + x_3^2\frac{\partial}{\partial x_3}\,, \end{aligned} \tag{9}$$

and find its invariant

$$J = \frac{(x - x_2)(x_3 - x_1)}{(x_1 - x)(x_2 - x_3)}.$$

The equation $J = C$ gives the well-known nonlinear superposition.

Example 4. Lie's theorem associates with any Lie algebra a system of differential equations admitting a superposition of solutions. Consider, as an illustrative example, the three-dimensional algebra spananed by

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x}\,, \quad X_2 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\,, \\ X_3 &= x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}\cdot \end{aligned} \tag{10}$$

It is a subalgebra of the eight-dimensional Lie algebra of the projective group on the plane. Accordingly, the first equation of the associated system (3),

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= T_1(t) + 2T_2(t)x + T_3(t)x^2, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= T_2(t)y + T_3(t)xy, \end{aligned} \tag{11}$$

is the Riccati equation (7) with $P = T_1, Q = 2T_2, R = T_3$. The operators (10) span the Vessiot-Guldberg-Lie algebra L_3 for the system (11). The estimation $nm \geq r$ with $n = 2, r = 3$ determines the minimum $m = 2$ of necessary particular solutions. Consequently, we take the three-point representation of the operators (10):

$$\begin{aligned} V_1 &= \frac{\partial}{\partial x} + \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\,, \\ V_2 &= 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2x_1\frac{\partial}{\partial x_1} \\ &+ y_1\frac{\partial}{\partial y_1} + 2x_2\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial y_2}\,, \\ V_3 &= x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + x_1^2\frac{\partial}{\partial x_1} \\ &+ x_1y_1\frac{\partial}{\partial y_1} + x_2^2\frac{\partial}{\partial x_2} + x_2y_2\frac{\partial}{\partial y_2}\,\cdot \end{aligned}$$

The operator V_1 provides five invariants, viz. $y, y_1, y_2, z_1 = x_1-x, z_2 = x_2-x_1$. Restricting V_2 to these invariants, one obtains the dilation generator

$$\widetilde{V}_2 = 2z_1\frac{\partial}{\partial z_1} + 2z_2\frac{\partial}{\partial z_2} + y\frac{\partial}{\partial y} + y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2}\,\cdot$$

Its indepedent invariants are $u_1 = z_2/z_1$, $u_2 = y^2/(x_1-x)$, $u_3 = y_1^2/(x_1-x)$, and $u_4 = y_2^2/(x_1-x)$.

Hence, a basis of the common invariants of V_1 and V_2 :

$$\begin{aligned} u_1 &= \frac{x_2 - x_1}{x_1 - x}\,, \quad u_2 = \frac{y^2}{x_1 - x}\,, \\ u_3 &= \frac{y_1^2}{x_1 - x}\,, \quad u_4 = \frac{y_2^2}{x_1 - x}\,\cdot \end{aligned}$$

It remains to find the resrtiction $\widetilde{V}_3$ of V_3 to the above invariants by the formula

$$\widetilde{V}_3 = V_3(u_1)\frac{\partial}{\partial u_1} + \cdots + V_3(u_4)\frac{\partial}{\partial u_4}\,\cdot$$

The reckoning shows that

$$\begin{aligned} V_3(u_1) &= \frac{(x_2 - x_1)(x - x_2)}{x - x_1} \equiv (x_1 - x)(1 + u_1)u_1, \\ V_3(u_3) &= y_1^2 \equiv (x_1 - x)u_3, \\ V_3(u_2) &= -y^2 \equiv -(x_1 - x)u_2, \\ V_3(u_4) &= \frac{x + x_1 - 2x_2}{x - x_1}\; y_2^2 \equiv (x_1 - x)(1 + 2u_1)u_4. \end{aligned}$$

Hence,

$$\begin{aligned} \widetilde{V}_3 &= \Big(x_1 - x\Big)\Bigg((1 + u_1)u_1\frac{\partial}{\partial u_1} - u_2\frac{\partial}{\partial u_2} + u_3\frac{\partial}{\partial u_3} \\ &\quad + (1 + 2u_1)u_4\frac{\partial}{\partial u_4}\Bigg). \end{aligned}$$

Consequently, the equation $\widetilde{V}_3\psi(u_1, \ldots, u_4) = 0$ is equivalent to

$$(1+u_1)u_1\frac{\partial\psi}{\partial u_1} - u_2\frac{\partial\psi}{\partial u_2} + u_3\frac{\partial\psi}{\partial u_3} + (1+2u_1)u_4\frac{\partial\psi}{\partial u_4} = 0,$$

whence, by solving the characteristic system

$$\frac{\mathrm{d}u_1}{(1 + u_1)u_1} = -\frac{\mathrm{d}u_2}{u_2} = \frac{\mathrm{d}u_3}{u_3} = \frac{\mathrm{d}u_4}{(1 + 2u_1)u_4}\,,$$

one obtains the following three independent invariants:

$$\begin{aligned} \psi_1 &= u_2u_3 \equiv \frac{y^2y_1^2}{(x_1 - x)^2}\,, \\ \psi_2 &= \frac{u_1u_2}{1 + u_1} \equiv \frac{(x_2 - x_1)y^2}{(x_1 - x)(x_2 - x)}\,, \\ \psi_3 &= \frac{u_4}{(1 + u_1)u_1} \equiv \frac{(x_1 - x)y_2^2}{(x_2 - x_1)(x_2 - x)\cdot} \end{aligned}$$

Hence, the general nonlinear superposition (5), involving two particular solutions, (x_1, y_1) and (x_2, y_2), is written

$$J_1(\psi_1, \psi_2, \psi_3) = C_1, \quad J_2(\psi_1, \psi_2, \psi_3) = C_2, \quad (12)$$

where J_1 and J_2 are arbitrary functions of three variables such that their Jacobian with respect to x, y does not vanish identically. Letting, e.g. $J_1 = \sqrt{\psi_1}$ and $J_2 = \sqrt{\psi_2\psi_3}$, i.e. specifying (12) in the form

$$\frac{yy_1}{x_1 - x} = C_1, \quad \frac{yy_2}{x_2 - x} = C_2,$$

one arrives at the following representation of the general solution via two particular solutions:

$$x = \frac{C_1x_1y_2 - C_2x_2y_1}{C_1y_2 - C_2y_1}\,, \quad y = \frac{C_1C_2(x_2 - x_1)}{C_1y_2 - C_2y_1}\,\cdot$$

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