Invariant Lagrangians and a new method of integration of nonlinear equations

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Abstract. A method for solving the inverse variational problem for differential equations admitting a Lie group is presented. The method is used for determining invariant Lagrangians and integration of second-order nonlinear differential equations admitting two-dimensional non-commutative Lie algebras. The method of integration suggested here is quite different from Lie's classical method of integration of second-order ordinary differential equations based on canonical forms of two-dimensional Lie algebras. The new method reveals existence and significance of one-parameter families of singular solutions to nonlinear equations of second order.

Keywords: Invariant Lagrangians, hypergeometric equation, conservation laws, singular solutions.

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1 Introduction

Many problems of mathematics, classical and relativistic mechanics, quantum field theory and other branches of theoretical and mathematical physics are connected with the calculus of variations. In these problems, one usually deals with the *direct variational problem* when one starts with a given Lagrangian $L(x, u, u', \dots)$ and investigates the dynamics of the physical system in question by considering the corresponding Euler-Lagrange equations $\delta L/\delta u = 0$. In the *inverse variational problem,* one starts with a given differential equation and looks for the corresponding Lagrangian. Sometimes, one can simply guess a Lagrangian, e.g. in the following examples.

Example 1. The simplest second-order ordinary differential equation $y''(x) = 0$ has the Lagrangian $L = -y'^2/2$ since it is identical with the Euler-Lagrange equation

$$\frac{\delta L}{\delta y} \equiv \frac{\partial L}{\partial y} - D_x\left(\frac{\partial L}{\partial y'}\right) = 0.$$

Indeed, $\partial L/\partial y = 0,\ \partial L/\partial y' = -y'$, and hence $\delta L/\delta y = -D_x(-y') = y''$. Likewise, we can easily find the Lagrangian for an arbitrary linear second-order equation

$$y'' + a(x)y' + b(x)y = f(x).$$

The previous simplest case suggests to seek the Lagrangian in the form

$$L = -\frac{p(x)}{2}y'^2 + \frac{q(x)}{2}y^2 - r(x)y.$$

Then

$$\frac{\delta L}{\delta y} = p(x)y'' + p'(x)y' + q(x)y - r(x).$$

Comparing the Euler-Lagrange equation $\delta L/\delta y = 0$ written in the form

$$y'' + \frac{p'(x)}{p(x)}y' + \frac{q(x)}{p(x)}y = \frac{r(x)}{p(x)}$$

with the second-order equation in question, we obtain

$$p(x) = \mathrm{e}^{\int a(x)dx}, \quad q(x) = b(x)\,\mathrm{e}^{\int a(x)dx}, \quad r(x) = f(x)\,\mathrm{e}^{\int a(x)dx}$$

and arrive at the following Lagrangian:

$$L = \left[-\frac{1}{2}\,y'^2 + \frac{b(x)}{2}\,y^2 - f(x)\,y\right]\mathrm{e}^{\int a(x)dx}.$$

Some twenty years ago, I sketched [1] a general method for constructing invariant Lagrangians for nonlinear partial differential equations and obtained, using the method, new conservation laws in fluid dynamics. Moreover, the group approach to the inverse variational problem allows one to find Lagrangians when the "guessing method" fails.

We can also use the fact that the Lagrangian is not uniquely determined and find several invariant Lagrangians starting with any operator X admitted by the Laplace equation. Recall that Lie's renowned methods for integration of ordinary differential equations admitting a group are basically based on reduction of Lie algebras to so-called *canonical forms* by proper changes of variables. Introduction of the *canonical variables* reduces the equation in question to an integrable form. Applying this idea, e.g. to second-order equations Lie found that there are *four* distinctly different canonical forms of two-dimensional Lie algebras, and accordingly any second-order equation with two infinitesimal symmetries can be reduced to one of four standard forms each of them being integrable by quadratures (see Lie's classical lectures [2] or the modern text [3]).

The objective of the present paper is to answer the question of how to find Lagrangians for nonlinear second-order ordinary differential equations $y'' = f(x, y, y')$ with two known non-commuting symmetries and integrate the equation using invariant Lagrangians. Our main concern is on developing practical devises for constructing Lagrangians for non-linear equations rather than on theoretical investigation of solvability of the inverse variational problem. The method of invariant Lagrangians is thoroughly illustrated by the following two nonlinear second-order ordinary differential equations:

$$y'' = \frac{y'}{y^2} - \frac{1}{xy}\,, \tag{1}$$

$$y'' = \mathrm{e}^y - \frac{y'}{x}\,\cdot \tag{2}$$

An attempt to find Lagrangians for equations (1) and (2) by the "natural" approach employed in Example 1 shows conclusively that the inverse variational problem can be rather complicated even for ordinary differential equations.

Furthermore, using equation (1) for illustration of Lie's integration method I came across the singular solutions

$$y = K\,x \qquad \text{and} \qquad y = \pm\sqrt{2x + C\,x^2}\,.$$

of equation (1) by chance, and it seemed that it was merely a result of a particular choice of variables. The new integration method presented here uncovers interesting connections of the singular solutions with singularities of the hypergeometric equation determining the Lagrangians. Moreover, the method furnishes an approach for obtaining one-parameter families of singular solutions that are otherwise not transparent.

2 Preliminaries

2.1 Invariance of nonlinear functionals and Noether's theorem

We will use the usual notation $x = \{x^i\},\ u = \{u^\alpha\},\ u_{(1)} = \{u_i^\alpha\},\ u_{(2)} = \{u_{ij}^\alpha\},\ \ldots$ for independent variables x and dependent variables u together with their partial derivatives $u_{(1)},\ u_{(2)}, \ldots$ of the first, second, etc. orders:

$$u_i^\alpha = D_i(u^\alpha),\ u_{ij}^\alpha = D_i(u_j^\alpha) = D_iD_j(u^\alpha), \ldots\,,$$

where

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + u_{ijk}^\alpha \frac{\partial}{\partial u_{jk}^\alpha} + \cdots$$

denotes the total differentiation.

An *action* (also termed a *variational integral*) used in Lagrangian mechanics is a nonlinear functional $l[u]$ (see, e.g. [4], Chapter IV):

$$l[u] = \int_V L(x, u, u_{(1)})\mathrm{d}x, \tag{3}$$

where $L(x, u, u_{(1)})$ is termed a *Lagrangian.* The necessary condition $l'[u] = 0$ for extrema of the functional $l[u]$ provides the Euler-Lagrange equations:

$$\frac{\partial L}{\partial u^\alpha} - D_i\left(\frac{\partial L}{\partial u_i^\alpha}\right) = 0, \quad \alpha = 1, \ldots, m. \tag{4}$$

Noether's theorem [6] states, e.g. in the case of first-order Lagranians L, that if the integral (3) is invariant under an infinitesimal transformation $\overline{x}^{\,i} \approx x^i + \varepsilon\xi^i$, $\overline{u}^{\,\alpha} \approx u^\alpha + \varepsilon\eta^\alpha$, in other words, under a continuous group generated by

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} \tag{5}$$

then the quantities

$$T^i(x, u, u_{(1)}) = L\xi^i + \big(\eta^\alpha - \xi^j u_j^\alpha\big)\frac{\partial L}{\partial u_i^\alpha}\,, \quad i = 1, \ldots, n, \tag{6}$$

define a conserved vector $T = (T^1, \ldots, T^n)$ for the Euler-Lagrange equations (4), i.e. $\mathrm{div}T \equiv D_i\big(T^i\big) = 0$ on the solutions of (4). If the integral (3) is invariant under r linearly independent operators $X_1, \ldots, X_r$ of the form (5), then the formula (6) provides r linearly independent conserved vectors $T_1, \ldots, T_r$.

Noether's original proof was based on calculations involving variations of integrals $\int L\mathrm{d}x$. An alternative proof of Noether's theorem given in [7] (see also [1] or [3] for more detailed presentation) provides the following infinitesimal test for the invariance of nonlinear functionals $l[u]$ (3) under the group with the generator X :

$$X_{(1)}(L) + LD_i(\xi^i) = 0, \tag{7}$$

where $X_{(1)}$ denotes the first prolongation of the generator (5), i.e.

$$X_{(1)}(L) = \xi^i \frac{\partial L}{\partial x^i} + \eta^\alpha \frac{\partial L}{\partial u^\alpha} + \zeta_i^\alpha \frac{\partial L}{\partial u_i^\alpha}\,, \quad \zeta_i^\alpha = D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j).$$

Definition 1. The function $L(x, u, u_{(1)})$ satisfying equation (7) is termed an *invariant Lagrangian* with respect to the generator (5).

Remark 1. One should not confuse invariant Lagrangians with *differential invariants* the latter being defined by the equation $X_{(1)}(L) = 0$.

2.2 Laplace's method for integration of hyperbolic equations

Recall that any linear hyperbolic second-order differential equations with two independent variables x, y :

$$a^{11}u_{xx} + 2a^{12}u_{xy} + a^{22}u_{yy} + b^1 u_x + b^2 u_y + cu = 0, \tag{8}$$

can be written in characteristic variables in the standard form

$$L[u] \equiv u_{xy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u = 0. \tag{9}$$

The *Laplace invariants* for equation (9) are defined by

$$h = a_x + ab - c, \quad k = b_y + ab - c. \tag{10}$$

Let us discuss the integration method when one of the invariants (10) vanishes.

If $h = 0$, we rewrite the left-hand side of equation (9) in the form

$$L[u] = v_x + bv - hu, \quad \text{where} \quad v = u_y + au,$$

and arrive at the following integrable form of equation (9):

$$v_x + bv = 0.$$

Integration with respect to x yields

$$v = Q(y)\, \mathrm{e}^{-\int b(x,y)dx}$$

with an arbitrary function $Q(y)$. Substituting v in $u_y + au = v$, one obtains:

$$u_y + au = Q(y)\, \mathrm{e}^{-\int b(x,y)dx}$$

whence upon integration with respect to y :

$$u = \left[P(x) + \int Q(y)\mathrm{e}^{\int ady - bdx} dy\right] e^{-\int ady}. \tag{11}$$

Likewise, if $k = 0$ we rewrite the left-hand side of equation (9) in the form

$$Lu = w_y + aw - ku, \quad \text{where} \quad w = u_x + bu,$$

and obtain the following general solution of (9) with $k = 0$:

$$u = \left[Q(y) + \int P(x)e^{\int bdx - ady} dx\right] e^{-\int bdx}. \tag{12}$$

2.3 Integration of a class of hypergeometric equations

In the theory of hypergeometric functions, the main emphasis is on asymptotics of the hypergeometric equation near its singular points (see, e.g. [9]). For our purposes, we will need analytic expressions for the general solutions of certain types of the hypergeometric equation. Therefore, I determine in Theorem 1 a class of hypergeometric equations integrable by elementary functions or by quadrature. Particular cases of this class can be found in various books on special functions.

The second-order linear differential equation

$$x(1-x)\,y'' + [\gamma - (\alpha+\beta+1)x]\,y' - \alpha\beta\,y = 0 \tag{13}$$

with arbitrary parameters α, β, and γ is known as the *hypergeometric equation.* It has singularities at $x = 0,\ x = 1$ and $x = \infty$.

Any homogeneous linear second-order differential equation of the form

$$(x^2 + Ax + B)\,y'' + (Cx + D)\,y' + E\,y = 0 \tag{14}$$

is transformable to the hypergeometric equation (13), provided that the equation $x^2 + Ax + B = 0$ has two distinct roots x_1 and x_2. Indeed, rewriting equation (13) in the new independent variable t defined by

$$x = x_1 + (x_2 - x_1)t \tag{15}$$

one obtains

$$t(1-t)\,\frac{d^2y}{dt^2} + \left[\frac{Cx_1 + D}{x_1 - x_2} - C\,t\right]\frac{dy}{dt} - Ey = 0.$$

Whence setting

$$\frac{Cx_1 + D}{x_1 - x_2} = \gamma, \quad C = \alpha + \beta + 1, \quad E = \alpha\beta$$

and denoting the new independent variable t again by x, one arrives at Eq. (13).

If $\alpha\beta = 0$ the hypergeometric equation (13) is integrable by two quadratures. Indeed, letting, e.g. $\beta = 0$ and integrating the equation

$$\frac{dy'}{y'} = \frac{(\alpha+1)x - \gamma}{x(1-x)}\,dx,$$

we have

$$y' = C_1 \mathrm{e}^{q(x)}, \quad q(x) = \int \frac{(\alpha+1)x - \gamma}{x(1-x)}\,dx, \quad C_1 = \text{const.}$$

The second integration yields:

$$y = C_1 \int \mathrm{e}^{q(x)}\,dx + C_2, \quad C_1,\ C_2 = \text{const.}$$

The following theorem singles out the equations (13) with $\alpha\beta \neq 0$ that can be integrated by transforming them to equations not containing the term with w.

Theorem 1. The general solution of the hypergeometric equation (13) with $\beta = -1$ and two arbitrary parameters α and γ :

$$x(1-x)\,y'' + (\gamma - \alpha\,x)\,y' + \alpha\,y = 0 \tag{16}$$

is given by quadrature and has the form

$$y = C_1\Big(x - \frac{\gamma}{\alpha}\Big)\int\Big(|x|^{-\gamma}\,|x-1|^{\gamma-\alpha}\,[x - (\gamma/\alpha)]^{-2}\Big)\,dx + C_2\Big(x - \frac{\gamma}{\alpha}\Big), \tag{17}$$

where C_1 and C_2 are arbitrary constants.

Proof. Since the case $\alpha = 0$ was considered above, we assume in what follows that $\alpha \neq 0$. Let

$$y = \sigma(x)\,w. \tag{18}$$

Substitution of the expressions

$$y = \sigma(x)\,w, \quad y' = \sigma(z)\,w' + \sigma'(z)\,w, \quad y'' = \sigma(z)\,w'' + 2\sigma'(z)\,w' + \sigma''(z)\,w$$

into equation (13) yields:

$$\begin{aligned} &x(1-x)\sigma\,w'' + \{2x(1-x)\sigma' + [\gamma - (\alpha+\beta+1)x]\sigma\}\,w' \\ &+\{x(1-x)\sigma'' + [\gamma - (\alpha+\beta+1)x]\,\sigma' - \alpha\beta\,\sigma\}\,w = 0. \end{aligned} \tag{19}$$

To annul the term with w, we have to find $\sigma(x)$ satisfying the equation

$$x(1-x)\sigma'' + [\gamma - (\alpha+\beta+1)x]\,\sigma' - \alpha\beta\,\sigma = 0. \tag{20}$$

It seems that we did not make any progress since we have to solve the original equation (13) for the unknown function $\sigma(x)$. However, we will take a particular solution $\sigma(x)$ by letting $\sigma''(x) = 0$. Hence, we consider the transformation (18) of the form

$$y = (kx + l)\,w, \quad k, l = \text{const}.$$

Then equation (20) reduces to

$$\gamma\,k - \alpha\beta\,l = 0, \quad k(\alpha+1)(\beta+1) = 0.$$

Since $\alpha\beta \neq 0$, it follows from the above equations that $k \neq 0$, and hence $\beta = -1$ (or $\alpha = -1$, but since equation (13) is symmetric with respect to the substitution $\alpha \leftrightarrows \beta$ we shall consider only $\beta = -1$). In what follows, we can set $k = 1$. Then the first equation of the above system yields $l = -\gamma/\alpha$. Thus we arrive at equation (16). Furthermore, it follows from (19) that the substitution

$$y = \Big(x - \frac{\gamma}{\alpha}\Big)w \tag{21}$$

reduces (16) to the equation

$$x(1-x)\Big(x-\frac{\gamma}{\alpha}\Big)\,w'' + \Big[2x(1-x) - \alpha\Big(x-\frac{\gamma}{\alpha}\Big)^2\Big]\,w' = 0. \tag{22}$$

One can readily integrate equation (22) in terms of elementary functions and one quadrature. Indeed, the equation

$$\frac{dw'}{w'} = -\frac{2x(1-x) - \alpha[x-(\gamma/\alpha)]^2}{x(1-x)[x-(\gamma/\alpha)]}$$

gives $w' = C_1 \mathrm{e}^{r(x)}$, where C_1 is an arbitrary constant and

$$r(x) = -\int \frac{2x(1-x) - \alpha[x-(\gamma/\alpha)]^2}{x(1-x)[x-(\gamma/\alpha)]}\,dx.$$

Evaluating the latter integral, one obtains

$$r(x) = \ln\Big(|x|^{-\gamma}\,|x-1|^{\gamma-\alpha}\,[x-(\gamma/\alpha)]^{-2}\Big),$$

and hence

$$w' = C_1\,\Big(x^{-\gamma}\,(x-1)^{\gamma-\alpha}\,[x-(\gamma/\alpha)]^{-2}\Big).$$

Thus, the solution of equation (22) is given by quadrature:

$$w = C_1 \int \Big(|x|^{-\gamma}\,|x-1|^{\gamma-\alpha}\,[x-(\gamma/\alpha)]^{-2}\Big)\,dx + C_2, \qquad C_1,\ C_2 = \text{const.} \tag{23}$$

Substituting the expression (23) in the formula (21), we obtain two independent solutions of the original equation (16):

$$y_1(x) = \Big(x-\frac{\gamma}{\alpha}\Big)\int \Big(|x|^{-\gamma}\,|x-1|^{\gamma-\alpha}\,[x-(\gamma/\alpha)]^{-2}\Big)\,dx, \quad y_2(x) = x - \frac{\gamma}{\alpha}\,\cdot$$

Taking linear combination of $y_1(x)$ and $y_2(x)$ with arbitrary constant coefficients, we obtain the general solution (17) to equation (16), thus completing the proof.

Remark 2. If γ and $\gamma-\alpha$ are rational numbers, one can reduce (23) to integration of a rational function by standard substitutions and represent the solution (23) in terms of elementary functions. See examples in Sections 4.2 and 7.

Corollary. Using the transformation (15) of equation (14) to the standard form (13), one can integrate the equation (14) with $E = -\,C$:

$$(x^2 + Ax + B)\,y'' + (Cx + D)\,y' - C\,y = 0. \tag{24}$$

3 Lagrangians for second-order equations

Definition 2. A differential function $L(x, y, y')$ is called a Lagrangian for a given second-order ordinary differential equation

$$y'' - f(x, y, y') = 0 \tag{25}$$

if equation (25) is equivalent to the Euler-Lagrange equation

$$\frac{\delta L}{\delta y} \equiv \frac{\partial L}{\partial y} - D_x\left(\frac{\partial L}{\partial y'}\right) = 0, \tag{26}$$

i.e. if $L(x, y, y')$ satisfies the equation

$$D_x\left(\frac{\partial L}{\partial y'}\right) - \frac{\partial L}{\partial y} = \sigma(x, y, y') \cdot [y'' - f(x, y, y')], \quad \sigma \neq 0, \tag{27}$$

with an undetermined multiplier $\sigma(x, y, y')$.

The expanded form of equation (27) is

$$y'' L_{y'y'} + y' L_{yy'} + L_{xy'} - L_y = \sigma y'' - \sigma f(x, y, y').$$

Noting that the equality of the coefficients for y'' in both sides of the above equation yields $\sigma = L_{y'y'}$, we reduce equation (27) to the following *linear second-order partial differential equation* for unknown Lagrangians[1]:

$$f(x, y, y') L_{y'y'} + y' L_{yy'} + L_{xy'} - L_y = 0, \qquad L_{y'y'} \neq 0. \tag{28}$$

Definition 3. The inverse variational problem for the second-order ordinary differential equation (25) consists in finding a solution $L(x, y, y')$ of the partial differential equation (28) with the independent variables x, y, y' and the given function $f(x, y, y')$.

Theorem 2. The inverse variational problem has a solution for any second-order ordinary differential equation (25). In other words, a Lagrangian exists for any equation $y'' = f(x, y, y')$, where $f(x, y, y')$ is an arbitrary differential function.

Proof. The proof follows almost immediately from the Cauchy-Kovalevski theorem. Let us first assume that $f(x, y, y') \neq 0$. Then the equation

$$f(x, y, y')\Omega_{y'}^2 + y'\Omega_y \Omega_{y'} + \Omega_x \Omega_{y'} = 0$$

for characteristics of the differential equation (28) is not satisfied by $\Omega = y'$, and hence the plane $y' = 0$ is not a characteristic surface. Consequently, the Cauchy-Kovalevski theorem guarantees existence of a solution to the Cauchy problem

$$f(x, y, y') L_{y'y'} + y' L_{yy'} + L_{xy'} - L_y = 0, \quad L\Big|_{y'=0} = P(x, y), \quad L_{y'}\Big|_{y'=0} = Q(x, y)$$

[1] The requirement $L_{y'y'} \neq 0$, known as the *Legendre condition*, guarantees that the Euler-Lagrange equation (26), upon solving for y'', is identical with equation (25). See, e.g. [4], Chapter IV.

with arbitrary $P(x,y)$ and $Q(x,y)$. This solution satisfies the required condition $L_{y'y'} \neq 0$. Indeed, otherwise it would have the form $L = A(x,y)y' + B(x,y)$. Substitution of the latter expression in equation (28) shows that the functions $A(x,y)$ and $B(x,y)$ cannot be arbitrary, but should be restricted by the equation

$$\frac{\partial A(x,y)}{\partial x} = \frac{\partial B(x,y)}{\partial y}\,\cdot$$

The initial conditions $L\big|_{y'=0} = P(x,y),\ L_{y'}\big|_{y'=0} = Q(x,y)$ require that $A(x,y)$ and $B(x,y)$ should be identical with $Q(x,y)$ and $P(x,y)$, respectively. Thus, Equation (28) has solutions satisfying the Legendre condition $L_{y'y'} \neq 0$, provided that $f(x,y,y') \neq 0$. In the singular case $f = 0$ the existence of the required solution is evident since the equation $y'' = 0$ has the Lagrangian $L = y'^2/2$. This completes the proof (cf. [8], pp. 37-39; see also [4]. Chapter IV, §12).

The above existence theorem does not furnish, however, simple practical devices for calculating Lagrangians of nonlinear differential equations (25). I suggested in 1983 ([1], Section 25.3, Remark 1) a method for determining invariant Lagrangians using the infinitesimal invariance test (7), and illustrated the efficiency of the method by second-order partial differential equations from fluid dynamics. I will give here a more detailed presentation of my method, develop a new integration theory based on invariant Lagrangians and illustrate it by nonlinear ordinary differential equations of the second order.

In the case of ordinary differential equations, the invariance test (7) has the form

$$X_{(1)}(L) + D_x(\xi)L = 0. \tag{29}$$

We apply it to unknown Lagrangians $L(x,y,y')$ and known generators

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$$

admitted by the equation (1) in question, where $\zeta(x,y,y')$ is obtained by the usual prolongation formula:

$$\zeta = D_x(\eta) - y'D_x(\xi) = \eta_x + (\eta_y - \xi_x)y' - y'^2\xi_y. \tag{30}$$

4 Invariant Lagrangians for Equation (1)

We search for invariant Lagrangians for equation (1),

$$y'' = \frac{y'}{y^2} - \frac{1}{xy}\,,$$

using the following two known symmetries ([3], Section 12.2.4):

$$X_1 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}\,, \quad X_2 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\,\cdot \tag{31}$$

The determining equation (28) for the Lagrangians of equation (1) has the form

$$\left(\frac{y'}{y^2} - \frac{1}{xy}\right) L_{y'y'} + y' L_{yy'} + L_{xy'} - L_y = 0, \qquad L_{y'y'} \neq 0. \tag{32}$$

4.1 The Lagrangians admitting X_1

The invariance test (29) under the first generator (31),

$$X_1 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}$$

provides the non-homogeneous linear first-order partial differential equation

$$x^2 \frac{\partial L}{\partial x} + xy \frac{\partial L}{\partial y} + (y - xy') \frac{\partial L}{\partial y'} + 2xL = 0. \tag{33}$$

The implicit solution $V(x, y, y', L) = 0$ provides the homogeneous equation

$$x^2 \frac{\partial V}{\partial x} + xy \frac{\partial V}{\partial y} + (y - xy') \frac{\partial V}{\partial y'} - 2xL \frac{\partial V}{\partial L} = 0. \tag{34}$$

The characteristic system for the latter equation,

$$\frac{dx}{x^2} = \frac{dy}{xy} = \frac{dy'}{y - xy'} = -\frac{dL}{2xL}$$

gives three first integrals:

$$\lambda = \frac{y}{x}, \quad \mu = y - xy', \quad \nu = x^2 L,$$

and the implicit solution $V(\lambda, \mu, \nu) = 0$ yields the solution

$$L = \frac{1}{x^2} \Phi(\lambda, \mu) \tag{35}$$

to the partial differential equation (33).

We have by definition of λ and μ :

$$\lambda_x = -\frac{y}{x^2}, \quad \lambda_y = \frac{1}{x}, \quad \lambda_{y'} = 0, \quad \mu_x = -y, \quad \mu_y = 1, \quad \mu_{y'} = -x,$$

and therefore

$$L_{y'} = -\frac{1}{x} \Phi_\mu, \quad L_{y'y'} = \Phi_{\mu\mu}, \quad L_{yy'} = -\frac{1}{x^2} \Phi_{\lambda\mu} - \frac{1}{x} \Phi_{\mu\mu},$$

$$L_{xy'} = \frac{1}{x^2} \Phi_\mu + \frac{y}{x^3} \Phi_{\lambda\mu} + \frac{y'}{x} \Phi_{\mu\mu}, \quad L_y = \frac{1}{x^3} \Phi_\lambda + \frac{1}{x^2} \Phi_\mu \,.$$

Substitution of these expressions reduces the equation (32) to the following linear equation with two variables λ and μ :

$$\mu\,\Phi_{\lambda\mu} - \frac{\mu}{\lambda^2}\,\Phi_{\mu\mu} - \Phi_\lambda = 0, \qquad \Phi_{\mu\mu} \neq 0. \tag{36}$$

The characteristics $\Omega(\lambda,\mu) = C$ of equation (36) are determined by the equation

$$\mu\,\Omega_\lambda\Omega_\mu - \frac{\mu}{\lambda^2}\,\Omega_\mu^2 \equiv \frac{\mu}{\lambda^2}\,(\lambda^2\Omega_\lambda - \Omega_\mu)\,\Omega_\mu = 0$$

equivalent to the system of linear first-order equations

$$\Omega_\mu = 0, \qquad \lambda^2\Omega_\lambda - \Omega_\mu = 0.$$

Two independent first integrals of the latter system have the form

$$\lambda = C_1, \qquad \mu - \frac{1}{\lambda} = C_2$$

and provide the characteristic variables u and v :

$$u = \lambda, \qquad v = \mu - \frac{1}{\lambda}\,\cdot$$

In these variables equation (36) takes the form

$$\Phi_{uv} - \frac{u}{1+uv}\,\Phi_u - \frac{1}{u(1+uv)}\,\Phi_v = 0. \tag{37}$$

i.e. the canonical form

$$\Phi_{uv} + a(u,v)\,\Phi_u + b(u,v)\,\Phi_v + c(u,v)\Phi = 0 \tag{38}$$

with the coefficients

$$a(u,v) = -\frac{u}{1+uv}\,, \quad b(u,v) = -\frac{1}{u(1+uv)}\,, \quad c(u,v) = 0. \tag{39}$$

Equation (37) can be integrated by Laplace's cascade method. Indeed, calculating the Laplace invariants

$$h = a_u + ab - c, \qquad k = b_v + ab - c$$

for equation (37), one can readily see that one of these invariants vanishes, namely, $h = 0$. Therefore, one can obtain the solution to equation (37) using the known formula

$$\Phi(u,v) = \left[U(u) + \int V(v) \mathrm{e}^{\int a(u,v)dv - b(u,v)du}\,dv\right] \mathrm{e}^{-\int a(u,v)dv} \tag{40}$$

for the general solution of equation (38) with $h = 0$, where $U(u)$ and $V(v)$ are arbitrary functions.

Evaluating the integrals of the coefficients (39):

$$\int a(u,v)dv = -\ln|1+uv|, \qquad \int b(u,v)du = -\ln\left|\frac{1+uv}{u}\right|$$

and using the formula (40) one obtains the following general solution to equation (37):

$$\Phi(u,v) = \Big[U(u) + \int V(v)\frac{udv}{(1+uv)^2}\Big](1+uv). \tag{41}$$

Let us take a particular solution, e.g. by letting in (41) $U(u) = 0$, $V(v) = v$. Then

$$\Phi = \frac{1}{u} + \frac{1+uv}{u}\ln|1+uv| = \frac{1}{\lambda} + \mu\ln|\lambda\mu| = \frac{x}{y} + (y - xy')\ln\left|\frac{y^2}{x} - yy'\right|,$$

and the formula (35) provides the following Lagrangian for equation (1):

$$L = \frac{1}{xy} + \left(\frac{y}{x^2} - \frac{y'}{x}\right)\ln\left|\frac{y^2}{x} - yy'\right|. \tag{42}$$

We have with this Lagrangian:

$$\frac{\delta L}{\delta y} = \frac{1}{xy' - y}\left(y'' - \frac{y'}{y^2} + \frac{1}{xy}\right).$$

Note that the exceptional situation in our approach that occurs when $y - xy' = 0$, singles out the solution $y = Cx$ to equation (1). See Remark in Section 4.2.

4.2 The Lagrangians admitting X_1 and X_2

The prolongation of the second operator (31) is

$$X_2 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - y'\frac{\partial}{\partial y'}$$

hence the invariance test (29) under X_2 is written

$$2x\frac{\partial L}{\partial x} + y\frac{\partial L}{\partial y} - y'\frac{\partial L}{\partial y'} + 2L = 0. \tag{43}$$

By the same reasoning that led to equation (34), one obtains the equation

$$2x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} - y'\frac{\partial V}{\partial y'} - 2L\frac{\partial V}{\partial L} = 0. \tag{44}$$

Thus, the Lagrangians that are invariant under both X_1 and X_2 should solve simultaneously the equations (33) and (43):

$$\begin{aligned} x^2\frac{\partial L}{\partial x} + xy\frac{\partial L}{\partial y} + (y - xy')\frac{\partial L}{\partial y'} + 2xL &= 0, \\ 2x\frac{\partial L}{\partial x} + y\frac{\partial L}{\partial y} - y'\frac{\partial L}{\partial y'} + 2L &= 0. \end{aligned} \tag{45}$$

Looking for the solution in the implicit form $V(x, y, y', L) = 0$, one arrives at the system of two homogeneous equations (34) and (44):

$$\begin{aligned} Z_1(V) \equiv x^2\frac{\partial V}{\partial x} + xy\frac{\partial V}{\partial y} + (y - xy')\frac{\partial V}{\partial y'} - 2xL\frac{\partial V}{\partial L} &= 0, \\ Z_2(V) \equiv 2x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} - y'\frac{\partial V}{\partial y'} - 2L\frac{\partial V}{\partial L} &= 0, \end{aligned} \tag{46}$$

where Z_1 and Z_2 are the following first-order linear differential operators:

$$\begin{aligned} Z_1 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + (y - xy')\frac{\partial}{\partial y'} - 2xL\frac{\partial}{\partial L}\,, \\ Z_2 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - y'\frac{\partial}{\partial y'} - 2L\frac{\partial}{\partial L}\,\cdot \end{aligned} \tag{47}$$

The invariants for Z_1 are $\lambda = y/x,\ \mu = y - xy',\ \nu = x^2 L$. Furthermore, we have $Z_2(\lambda) = -\lambda,\ Z_2(\mu) = \mu,\ Z_2(\nu) = 2\nu$, and hence

$$Z_2 = -\lambda\frac{\partial}{\partial \lambda} + \mu\frac{\partial}{\partial \mu} + 2\nu\frac{\partial}{\partial \nu}\,\cdot$$

The characteristic equations

$$-\frac{d\lambda}{\lambda} = \frac{d\mu}{\mu} = \frac{d\nu}{2\nu}$$

yield the following common invariants of Z_1 and Z_2 :

$$z = \lambda\mu \equiv \frac{y^2}{x} - yy'\,, \quad q = \lambda^2\nu \equiv y^2 L.$$

Thus, the general solution of the system (46) is $V = V(z, q)$, and the equation $V(z, q) = 0$ provides the following solution to the system (45):

$$L = \frac{1}{y^2}\Psi(z), \quad z = \frac{y^2}{x} - yy'\,\cdot \tag{48}$$

We have:

$$z_x = -\frac{y\ 2}{x^2}\,, \quad z_y = 2\frac{y}{x} - y'\,, \quad z_{y'} = -y$$

and therefore

$$L_{y'} = -\frac{1}{y}\,\Psi'\,, \quad L_{y'y'} = \Psi''\,, \quad L_{yy'} = \frac{1}{y^2}\,\Psi' - \left(\frac{2}{x} - \frac{y'}{y}\right)\,\Psi''\,,$$

$$L_{xy'} = \frac{y}{x^2}\,\Psi''\,, \quad L_y = -\frac{2}{y^3}\,\Psi + \left(\frac{2}{xy} - \frac{y'}{y^2}\right)\,\Psi'. \tag{49}$$

Substitution of these expressions reduces the equation (32) to a linear ordinary differential equation of the second order, namely to the hypergeometric equation

$$z(1-z)\Psi'' + 2z\Psi' - 2\Psi = 0, \qquad \Psi'' \neq 0. \tag{50}$$

Equation (50) has singularities at points $z = 0,\ z = 1$ and the infinity, $z = \infty$. The singular points $z = 0$ and $z = 1$ define singular solutions to equations (1). This relationship between the singular points and singular solutions will be discussed in the next Section. Let us consider now the solutions to equation (50) at regular points z.

The substitution $\Psi = zw$ reduces (50) to the integrable form (see Section 2.3)

$$z(1-z)w'' + 2w' = 0. \tag{51}$$

Equation (51) can be solved in terms of elementary functions. Indeed, we have

$$\frac{dw'}{w'} = \frac{2dz}{z(z-1)} \equiv \frac{2dz}{z-1} - \frac{2dz}{z}$$

and obtain upon integration:

$$w' = C_1\Big(1 - \frac{1}{z}\Big)^2,$$

whence[2]

$$w = C_1\left(z - \frac{1}{z} - 2\ln|z|\right) + C_2. \tag{52}$$

We let $C_1 = 1, C_2 = 0$ and obtain the following solution $\Psi = zw$ to equation (50):

$$\Psi(z) = z^2 - 1 - 2z\ln|z|. \tag{53}$$

Substituting (53) in (48), we obtain the following function $L(x, y, y')$:

$$L = -\frac{1}{y^2} + \frac{y^2}{x^2} - 2\frac{yy'}{x} + y'^2 - 2\Big(\frac{1}{x} - \frac{y'}{y}\Big)\ln\Big|\frac{y^2}{x} - yy'\Big|. \tag{54}$$

Using (48), (49) and (53), one readily obtains

$$L_{y'} = \frac{2}{y} - 2\frac{y}{x} + 2y' + \frac{2}{y}\ln\Big|\frac{y^2}{x} - yy'\Big|, \qquad L_{y'y'} = 2 - \frac{2x}{x(y - xy'}\,,$$

$$L_{yy'} = -\frac{2}{x} + \frac{2}{y(y - xy')} - \frac{2}{y^2}\ln\Big|\frac{y^2}{x} - yy'\Big|, \qquad L_{xy'} = 2\frac{y}{x^2} - \frac{2}{x(y - xy'}\,,$$

$$L_y = \frac{2}{y^3} + 2\frac{y}{x^2} - 2\frac{y'}{x} - \frac{4}{xy} + 2\frac{y'}{y^2} - 2\frac{y'}{y^2}\ln\Big|\frac{y^2}{x} - yy'\Big|.$$

[2] The same result can be obtained from the general formula (23) with $\alpha = -2, \gamma = 0$.

and hence the variational derivative of the function (54):

$$\frac{\delta L}{\delta y} = 2\frac{1-z}{z}\Big(y'' - \frac{y'}{y^2} + \frac{1}{xy}\Big) \equiv 2\frac{x - y^2 + xyy'}{y(y - xy')}\,\Big(y'' - \frac{y'}{y^2} + \frac{1}{xy}\Big).$$

Hence, the function $L(x, y, y')$ is a Lagrangian for equation (1) with the exclusion of the singular points $z = 0$ and $z = 1$ of the hypergeometric equation (50).

Remark 3. According to (52) the general solution of equation (50) is given by

$$\Psi(z) = C_1(z^2 - 1 - 2z \ln|z|) + C_2 z.$$

It is spanned by the singular solution (53) and the regular solution $\Psi_* = z$. We eliminated the regular solution because it leads to the trivial Lagrangian (48):

$$L_* = \frac{z}{y^2} \equiv \frac{1}{x} - \frac{y'}{y}\,\cdot$$

Its variational derivative $\delta L_*/\delta y$ vanishes identically.

5 Method of invariant Lagrangians

In the case of ordinary differential equations (25),

$$y'' = f(x, y, y'), \tag{55}$$

the infinitesimal symmetries (5) are written

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} \tag{56}$$

and the conserved quantities (6) have the form

$$T = \xi L + \big(\eta - \xi y'\big)\frac{\partial L}{\partial y'}\,\cdot \tag{57}$$

The method of integration suggested here is quite different from Lie's classical methods (i.e. consecutive integration and utilization of canonical forms of two-dimensional Lie algebras, see, e.g. [3], Section 12.2) and comprises the following steps.

First step: Calculate the symmetries (56). Let equation (55) admit two linearly independent symmetries, X_1 and X_2.

Second step: Find an invariant Lagrangian $L(x, y, y')$ using the invariance test (29) under the operators X_1 and X_2 and then solving the defining equation (28) for L.

Third step: Use the invariant Lagrangian L and apply the formula (57) to the symmetries X_1 and X_2 to find two independent conservation laws:

$$T_1(x, y, y') = C_1, \quad T_2(x, y, y') = C_2. \tag{58}$$

Equations (58) mean only that the functions $T_1(x, y, y')$ and $T_2(x, y, y')$ preserve constant values along each solution of equation (55). However, if T_1 and T_2 are functionally independent, one can treat C_1 and C_2 as arbitrary parameters, since the Cauchy-Kovalevski theorem guarantees that equation (55) has solutions with any initial values of y and y', and hence C_1, C_2 in (58) can assume, in general, arbitrary values.

Fourth step: Eliminate y' from two equations (58) to obtain the solution of equation (55) in the implicit form:

$$F(x, y, C_1, C_2) = 0, \tag{59}$$

where C_1 and C_2 are two arbitrary parameters.

6 Application of the method to Equation (1)

Let us illustrate the new method of integration by applying it to equation (1):

$$y'' - \frac{y'}{y^2} + \frac{1}{xy} = 0 \tag{60}$$

For this equation, we already know two symmetries (31),

$$X_1 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}\,, \quad X_2 = 2x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}\,, \tag{61}$$

and the invariant Lagrangian (54),

$$L = -\frac{1}{y^2} + \frac{y^2}{x^2} - 2\frac{yy'}{x} + y'^2 - 2\Big(\frac{1}{x} - \frac{y'}{y}\Big) \ln\Big|\frac{y^2}{x} - yy'\Big|. \tag{62}$$

Since the Lagrangian (62) has singularites, we will begin with singling out the associated singular solutions of equation (60).

6.1 Singular solutions associated with singularities of the Lagrangian

Recall that the hypergeometric equation (50) has the singular points

$$z = 0 \qquad \text{and} \qquad z = 1. \tag{63}$$

According to definition of z given in (48), the singular points (63) provide two first-order differential equations,

$$y' = \frac{y}{x} \tag{64}$$

and

$$y' = \frac{y}{x} - \frac{1}{y}\,, \tag{65}$$

respectively. The Lagrangian (62) collapses at both singular points. Namely, L is not defined at the point $z = 0$, i.e. at (64), and vanishes identically at the point $z = 1$, i.e. at (65). See also Remark in Section 4.2. Equation (64) is readily solved and yields $y = Kx, \ K = \text{const}$.

Equation (65) can be integrated, e.g. by noting that it admits the operators (61). Indeed, equation (65) is identical with $z = 1$, where z is a differential invariant of X_1 and X_2. Hence, one can employ either *canonical variables* or *Lies's integrating factors.*

Method of canonical variables. Let us use, e.g. the operator X_1. The equations $X_1(\tau) = 1$ and $X_1(v) = 0$ provide the following canonical variables:

$$\tau = -\frac{1}{x}\,, \quad v = \frac{y}{x}\,\cdot$$

In this variables, equation (65) is written

$$\frac{dv}{d\tau} + \frac{1}{v} = 0$$

and yields

$$v = \mp\sqrt{-2t + C}.$$

Returning to the original variables, one obtains the following solution of equation (65):

$$y = \pm\sqrt{2x + Cx^2}\,. \tag{66}$$

One can also make use of the second symmetry (61). Then, solving the equations $X_2(\tau) = 1, \ X_2(v) = 0$ and assuming $x > 0$ for simplicity, one obtains

$$\tau = \ln x, \quad v = \frac{y}{\sqrt{x}}\,\cdot$$

In these canonical variables equation (65) becomes

$$\frac{dv}{d\tau} = \frac{v^2 - 2}{v}\,,$$

whence

$$v(\tau) = \pm\sqrt{2 + C\mathrm{e}^{2\tau}}\,\cdot$$

The substitution $y = \sqrt{x}\, v(\tau), \ \tau = \ln x$ yields the previous solution (66).

Method of integrating factors. Recall that a first-order differential equation

$$M(x, y)dx + N(x, y)dy = 0 \tag{67}$$

with a known infinitesimal symmetry

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}$$

has the following integrating factor known as *Lie's integrating factor:*

$$\mu(x, y) = \frac{1}{\xi M + \eta N} \cdot \tag{68}$$

Furthermore, if one knows two linearly independent integrating factors $\mu_1(x, y)$ and $\mu_1(x, y)$, one can obtain the general solution of equation (67) from the algebraic relation

$$\frac{\mu_1(x, y)}{\mu_2(x, y)} = C \tag{69}$$

with an arbitrary constant C. Applying these two principles to equations (67) with two known infinitesimal symmetries, one obtains the general solution without integration.

Let us return to our equation (65). We rewrite it in the form (67):

$$(x - y^2)dx + xydy = 0$$

and use its two known infinitesimal symmetries (61):

$$X_1 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}, \quad X_2 = 2x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \cdot$$

Lie's integrating factors (68) corresponding to X_1 and X_2, respectively, have the form:

$$\mu_1(x, y) = \frac{1}{x^2(x - y^2) + x^2 y^2} = \frac{1}{x^3}, \quad \mu_2(x, y) = \frac{1}{2x(x - y^2) + xy^2} = \frac{1}{2x^2 - xy^2} \cdot$$

Therefore the algebraic relation (69) has the form

$$\frac{2x - y^2}{x^2} = C.$$

Upon solving it with respect to y, one obtains the singular solution (66).

We summerize: *The Lagrangian* (62) *has singularities only at the following singular solutions of equation* (60):

$$y = K\,x, \quad y = \pm\sqrt{2x + C\,x^2}\,, \qquad K,\, C = \text{const.} \tag{70}$$

Remark 4. The singular point $z = 1$ provides two singular solutions,

$$y = \sqrt{2x + C\,x^2}$$

and

$$y = -\sqrt{2x + C\,x^2},$$

because equation (60) is invariant under the reflection $y \to -y$ of the dependent variable.

6.2 The general solution

We will find now the regular solutions by means of our integration method discussed in Section 5. Since we already know the symmetries (61) and an invariant Lagrangian (62) for equation (60), we can proceed to the *third step.*

Application of the formula (57) to the operators (61) yields:

$$T_1 = x^2\, L + x(y - xy')\, L_{y'}\,, \qquad T_2 = 2x\, L + (y - 2xy')\, L_{y'}\,.$$

Let us substitute here L and $L_{y'}$ from equations (48) and (49), respectively:

$$L = \frac{1}{y^2}\Psi(z), \quad L_{y'} = -\frac{1}{y}\Psi'(z), \qquad z = \frac{y^2}{x} - yy',$$

where, according to (53),

$$\Psi(z) = z^2 - 1 - 2z\ln|z|, \quad \Psi'(z) = 2(z - 1 - \ln|z|).$$

Thus, we arrive at the following two linearly independent conserved quantities:

$$\begin{aligned} T_1 &= \frac{x^2}{y^2}\,\Psi(z) - \frac{x(y-xy')}{y}\,\Psi'(z) \equiv -\frac{x^2}{y^2}\,(1-z)^2, \\ T_2 &= \frac{2x}{y^2}\,\Psi(z) - \frac{y-2xy'}{y}\,\Psi'(z) \equiv -2\frac{x}{y^2}\,(1-z)^2 - 2(z - 1 - \ln|z|). \end{aligned}$$

In the original variables x, y, and y' the conserved quantities are written

$$\begin{aligned} T_1 &= 2x - 2\frac{x^2y'}{y} - \frac{x^2}{y^2} - y^2 + 2xyy' - x^2y'^2\,, \\ T_2 &= 2\left(1 - \frac{x}{y^2} - 2\frac{xy'}{y} + yy' - xy'^2 - \ln\left|\frac{y^2}{x} - yy'\right|\right). \end{aligned} \tag{71}$$

However, the form (71) of the conserved quantities is not convenient for eliminating y' from the conservation laws. Therefore, we will use the following representation of T_1 and T_1 in terms of the differential invariant z :

$$T_1 = -\frac{x^2}{y^2}(1-z)^2, \qquad T_1 - \frac{x}{2}\,T_2 = x(1 - z + \ln|z|). \tag{72}$$

Then one can readily eliminate the variable z instead of y'.

Fourth step: Let us write the conservation laws (58) in the form

$$T_1 = -\,C_1^2, \qquad T_2 = 2C_2.$$

Using the expression of T_1 given in (72), we have

$$\frac{x^2}{y^2}\,(1-z)^2 = C_1^2,$$

whence

$$1 - z = C_1 \frac{y}{x}\,, \quad z = 1 - C_1 \frac{y}{x}\,\cdot$$

Substitution of the above expressions in the second equation (72) yields:

$$-C_1^2 - C_2 x = x\Big(C_1\,\frac{y}{x} + \ln\Big|C_1\,\frac{y}{x} - 1\Big|\Big).$$

Hence, the two-parameter solution (59) is given in the implicit form:

$$C_1 y + C_2 x + C_1^2 + x \ln\Big|C_1\,\frac{y}{x} - 1\Big| = 0.$$

Invoking two singular solutions (70), we see that the complete set of solutions to equation (60) is given by the following *distinctly different formulae* (cf. [3]):

$$\begin{aligned} &y = K\,x, \qquad y = \pm\sqrt{2x + C\,x^2}\,,\\ &C_1 y + C_2 x + C_1^2 + x \ln\Big|C_1\,\frac{y}{x} - 1\Big| = 0. \end{aligned} \tag{73}$$

Remark 5. The representation of the solution by the different formulae (73) does not conflict with uniqueness of the solution to the Cauchy problem. Indeed, any initial data $x = x_0,\ y(x_0) = y_0,\ y'(x_0) = y_0'$ is compatible only with one formula (73) chosen in accordance with the initial value $z_0 = (y_0^2/x_0) - y_0 y_0'$ of the invariant z. Namely, the solution with the initial data $x = x_0,\ y(x_0) = y_0,\ y'(x_0) = y_0'$ is given by the first or the second formula (73) if $z_0 = 0$ or $z_0 = 1$, respectively. Otherwise it is given by the third formula (73). The constants K, C, C_1, C_2 are found by substituting $x = x_0,\ y = y_0,\ y' = y_0$ in the formulae (73) together with their differential consequences:

$$\begin{aligned} &y' = K, \qquad y' = \pm\frac{1 + Cx}{\sqrt{2x + C\,x^2}}\,,\\ &C_1\,y' + C_2 + \ln\Big|C_1\,\frac{y}{x} - 1\Big| + \frac{C_1(xy' - y)}{C_1\,y - x} = 0, \end{aligned} \tag{74}$$

Examples of initial value problems.

(i) Let $x_0 = 1,\ y_0 = 1,\ y_0' = 1$. Then $z_0 = 0$, and hence the solution belongs to the first formula (73). The substitution $x = 1,\ y = 1,\ y' = 1$ in (73) and (74) yields $K = 1$. Hence, the solution of equation (60) with the initial data $x_0 = 1,\ y_0 = 1,\ y_0' = 1$ has the form $y = x$.

(ii) For the initial data $x_0 = 1,\ y_0 = 1,\ y_0' = 2$, one has $z_0 = -1$. Therefore the solution belongs to the third formula (73). Substituting the initial values of x, y, y' (73)-(74), one obtains $C_1 = 2,\ C_2 = -6$, and hence $2y - 6x + 4 + x\ln|(2y/x) - 1| = 0$.

(iii) For the initial data $x_0 = 1,\ y_0 = 1,\ y_0' = -1$, one has $z_0 = 2$. Accordingly, the solution is given by the third formula (73) with $C_1 = -1,\ C_2 = -\ln 2$.

(iv) If $x_0 = 1,\ y_0 = 1,\ y'_0 = 0$, one has $z_0 = 1$. Hence, we should use the second formula (73), where we have to specify the sign and determine the constat C by substituting the initial values $x_0 = 1,\ y_0 = 1,\ y'_0 = 0$ in (73)-(74). The reckoning shows that the solution is given by the second formula (73) with the positive sign and $C = -1$, i.e. it has the form $y = \sqrt{2x - x^2}$.

(v) Likewise, taking $x_0 = 1,\ y_0 = -1,\ y'_0 = 0$, one can verify that the solution of the Cauchy problems is given by the second formula (73) with the negative sign and $C = -1$, i.e. $y = -\sqrt{2x - x^2}$.

7 The Lagrangian and integration of Equation (2)

Consider now Equation (2):

$$y'' = \mathrm{e}^y - \frac{y'}{x}\cdot$$

It has the following two symmetries ([3], Section 9.3.1):

$$X_1 = x\ln|x|\frac{\partial}{\partial x} - 2(1 + \ln|x|)\frac{\partial}{\partial y}\,, \quad X_2 = x\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y}\cdot \tag{75}$$

The determining equation (28) for the Lagrangians of equation (2) has the form

$$\left(\mathrm{e}^y - \frac{y'}{x}\right) L_{y'y'} + y'L_{yy'} + L_{xy'} - L_y = 0, \qquad L_{y'y'} \neq 0. \tag{76}$$

7.1 Calculation of the invariant Lagrangian

Let us find the invariant Lagrangian for our equation using both symmetries (75). The invariance test (29) under the operators (75) yields:

$$\begin{aligned} x\ln|x|\frac{\partial L}{\partial x} - 2(1 + \ln|x|)\frac{\partial L}{\partial y} - \left[\frac{2}{x} + (1 + \ln|x|)\,y'\right]\frac{\partial L}{\partial y'} + (1 + \ln|x|)L &= 0,\\ x\frac{\partial L}{\partial x} - 2\frac{\partial L}{\partial y} - y'\frac{\partial L}{\partial y'} + L &= 0. \end{aligned} \tag{77}$$

Looking for the solution of the system (77) in the implicit form $V(x, y, y', L) = 0$, one arrives at the system of two homogeneous equations

$$Z_1(V) = 0, \qquad Z_2(V) = 0 \tag{78}$$

with the linear differential operators

$$\begin{aligned} Z_1 &= x\ln|x|\frac{\partial}{\partial x} - 2(1 + \ln|x|)\frac{\partial}{\partial y} - \left(\frac{2}{x} + (1 + \ln|x|)y'\right)\frac{\partial}{\partial y'} - (1 + \ln|x|)L\frac{\partial}{\partial L}\,,\\ Z_2 &= x\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y} - y'\frac{\partial}{\partial y'} - L\frac{\partial}{\partial L}\cdot \end{aligned}$$

The invariants for Z_1 are

$$\lambda = x\,\mathrm{e}^{y/2}\ln|x|, \quad \mu = (2 + xy')\ln|x|, \quad \nu = L\,\mathrm{e}^{-y/2}.$$

Hence, the first equation $Z_1(V) = 0$ yields that $V = V(\lambda, \mu, \nu)$. Furthermore, we have

$$Z_2(\lambda) = x\,\mathrm{e}^{y/2} = \frac{\lambda}{\ln|x|}\,, \quad Z_2(\mu) = 2 + xy' = \frac{\mu}{\ln|x|}\,, \quad Z_2(\nu) = 0.$$

Thus, we have for $V = V(\lambda, \mu, \nu)$:

$$Z_2(V) = \frac{1}{\ln|x|}\left(\lambda\frac{\partial V}{\partial\lambda} + \mu\frac{\partial V}{\partial\mu}\right).$$

Hence, the second equation (78) is identical with the equation $\widetilde{Z}_2(V) = 0$, where

$$\widetilde{Z}_2 = \lambda\frac{\partial}{\partial\lambda} + \mu\frac{\partial}{\partial\mu}\,\cdot$$

Solutions $z = \mu/\lambda$ and ν to the characteristic equation

$$\frac{d\lambda}{\lambda} = \frac{d\mu}{\mu}$$

provide the following common invariants of Z_1 and Z_2 :

$$z = \left(\frac{2}{x} + y'\right)\mathrm{e}^{-y/2}\,, \quad \nu = L\,\mathrm{e}^{-y/2}.$$

Thus, $V = V(z, \nu)$. The equation $V(z, \nu) = 0$ when solved for ν yields $\nu = \Psi(x)$, or:

$$L = \mathrm{e}^{y/2}\,\Psi(z), \quad z = \left(\frac{2}{x} + y'\right)\mathrm{e}^{-y/2}\,. \tag{79}$$

We have:

$$z_x = -\frac{2}{x^2}\mathrm{e}^{-y/2}\,, \quad z_y = -\frac{z}{2}\,, \quad z_{y'} = \mathrm{e}^{-y/2}\,,$$

and therefore

$$L_{y'} = \Psi'\,, \quad L_{y'y'} = \mathrm{e}^{-y/2}\,\Psi''\,, \quad L_{yy'} = -\frac{z}{2}\Psi''\,,$$

$$L_{xy'} = -\frac{2}{x^2}\,\mathrm{e}^{-y/2}\,\Psi''\,, \quad L_y = \frac{1}{2}\,\mathrm{e}^{y/2}\left(\Psi - z\,\Psi'\right). \tag{80}$$

Substitution of these expressions reduces the equation (76) to the integrable linear ordinary differential equation of the form (24) with the coefficients $C = -1$, $B = -2$ and $A = D = 0$:

$$(z^2 - 2)\Psi'' - z\Psi' + \Psi = 0. \tag{81}$$

In accordance with Section 2.3, we rewrite equation (81) in the new independent variable t defined by (15), where we replace x and x_1, x_2 by z and $z_1 = \sqrt{2}$, $z_2 = -\sqrt{2}$, respectively. Thus, we let

$$z = \sqrt{2}\,(1 - 2\,t) \tag{82}$$

and arrive at the hypergeometric equation of the form (16) with $\alpha = -1$ and $\gamma = -1/2$:

$$t(1-t)\Psi'' + \Big(t - \frac{1}{2}\Big)\Psi' - \Psi = 0, \tag{83}$$

where Ψ' denotes the differentiation with respect to t. The solution to equation (83) is given by the formula (17) and has the form

$$\Psi(t) = (2t-1)(M\mathcal{J} + N), \tag{84}$$

where M, N are arbitrary constants and $\mathcal{J}$ is the following integral:

$$\mathcal{J} = 4\int \frac{\sqrt{|t(t-1)|}}{(2t-1)^2}\,dt. \tag{85}$$

Let us evaluate the integral $\mathcal{J}$ and express the solution (84) in elementary functions.

We will first assume that $t(t-1) > 0$, i.e. either $t > 1$ or $t < 0$. According to (82), it means that

$$z^2 - 2 > 0. \tag{86}$$

Using this assumption, let us rewrite the integral $\mathcal{J}$ in the form

$$\mathcal{J} = 4\int \frac{t\sqrt{(t-1)/t}}{(2t-1)^2}\,dt. \tag{87}$$

The standard substitution $(t-1)/t = s^2$ together with the expressions

$$\sqrt{\frac{t-1}{t}} = s, \quad t = \frac{1}{1-s^2}\,, \quad dt = \frac{2s ds}{(1-s^2)^2}\,, \quad 2t - 1 = \frac{1+s^2}{1-s^2}\,, \tag{88}$$

transforms the integral (85) to the form

$$\mathcal{J} = 8\int \frac{s^2 ds}{(1-s^2)(1+s^2)^2} = \int \Big(\frac{1}{s+1} - \frac{1}{s-1} + \frac{2}{s^2+1} - \frac{4}{(s^2+1)^2}\Big) ds\,.$$

Since

$$\int \frac{ds}{(s^2+1)^2} = \frac{s}{2(s^2+1)} + \frac{1}{2}\int \frac{ds}{s^2+1}\,,$$

the integral $\mathcal{J}$ reduces to

$$\mathcal{J} = \ln\left|\frac{s+1}{s-1}\right| - \frac{2s}{s^2+1}\,.$$

Invoking the definition (88) of s and using the assumption (86), one obtains the following expression for the integral (85):

$$\mathcal{J} = \ln\left(\frac{\sqrt{|t|}+\sqrt{|t-1|}}{\sqrt{|t|}-\sqrt{|t-1|}}\right) - 2\frac{\sqrt{t(t-1)}}{2t-1} \equiv \ln\Big(\sqrt{|t|}+\sqrt{|t-1|}\Big)^2 - 2\frac{\sqrt{t(t-1)}}{2t-1}\,\cdot$$

Substituting $\mathcal{J}$ in (84) one obtains the following solution of equation (83):

$$\Psi(t) = M\Big[(2t-1)\,\ln\Big(|t|+|t-1|+2\sqrt{t(t-1)}\Big) - 2\sqrt{t(t-1)}\Big] + (2t-1)N.$$

Using the definition (82) of variable z, the inequality (86) and the equations

$$2t-1 = -\frac{z}{\sqrt{2}}\,,\quad |t|+|t-1| = \frac{|z|}{\sqrt{2}}\,,\quad 2\sqrt{t(t-1)} = \frac{\sqrt{z^2-2}}{\sqrt{2}}\,,$$

we have

$$\Psi(z) = -\frac{M}{\sqrt{2}}\left[\sqrt{z^2-2} + z\ln(\sqrt{z^2-2}+|z|)\right] + \frac{M\ln\sqrt{2}-N}{\sqrt{2}}\,z\,.$$

We simplify the above expression by taking $M = -\sqrt{2}$, $N = -\sqrt{2}\ln\sqrt{2}$ and obtain

$$\Psi(z) = \sqrt{z^2-2} + z\ln(\sqrt{z^2-2}+|z|). \tag{89}$$

Finally, substituing (89) in (79), we arrive at the following Lagrangian:

$$L = \sqrt{\Big(\frac{2}{x}+y'\Big)^2 - 2\mathrm{e}^y} + \Big(\frac{2}{x}+y'\Big)\left\{\ln\left(\sqrt{\Big(\frac{2}{x}+y'\Big)^2 - 2\mathrm{e}^y} + \Big|\frac{2}{x}+y'\Big|\right) - \frac{y}{2}\right\}. \tag{90}$$

Introducing in (90) the notation

$$\mathcal{B} = \sqrt{\Big(\frac{2}{x}+y'\Big)^2 - 2\mathrm{e}^y}$$

and using (80) and (89), one obtains:

$$L = \mathcal{B} + \Big(\frac{2}{x}+y'\Big)\left\{\ln\left(\mathcal{B} + \Big|\frac{2}{x}+y'\Big|\right) - \frac{y}{2}\right\},$$

$$L_y = \frac{1}{2}\mathcal{B},\ L_{y'} = -\frac{y}{2} + \ln\left(\mathcal{B} + \Big|\frac{2}{x}+y'\Big|\right),$$

$$L_{y'y'} = -\frac{1}{\mathcal{B}}\,,\quad L_{yy'} = \frac{1}{2\mathcal{B}}\Big(\frac{2}{x}+y'\Big),\quad L_{xy'} = \frac{2}{x^2\mathcal{B}}\,\cdot \tag{91}$$

Thus, one has with the Lagrangian (90):

$$\frac{\delta L}{\delta y} = \frac{1}{\mathcal{B}}\left(y'' + \frac{y'}{x} - \mathrm{e}^y\right).$$

Remark 6. By reducing $\Psi(z)$ to (89) we eliminated the regular solution $\Psi = z$ of equation (81). The reason for the elimination is that $\Psi = z$ leads to the trivial Lagrangian (79), $L^* = y' + (2/x)$.

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