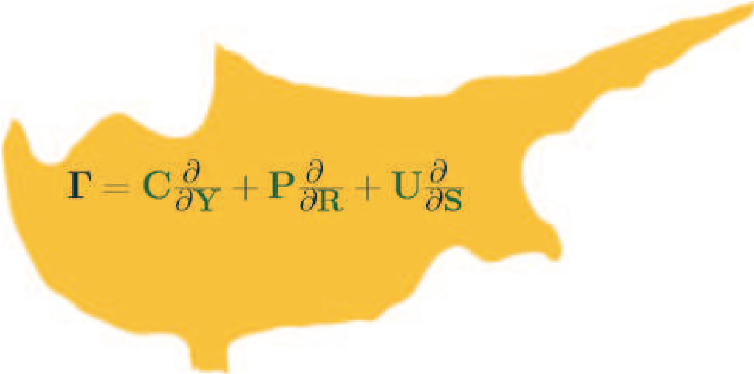
The 10th International Conference in

MOdern GRoup ANalysis (MOGRAN X)



Proceedings

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Preface

The 10th International Conference in Modern Group Analysis, MOGRAN X, took place in Larnaca, Cyprus, from October 24th, to October 31st, 2004. The aim of the meeting was to bring together leading scientists in group analysis and mathematical modelling for exchange of ideas and presentation of results. The main emphasis of the conference was on applications of group methods in investigating nonlinear wave and diffusion phenomena, Mathematical models in biology, Integrable systems as well as the Classical heritage, historical aspects and new theoretical developments in group analysis. The conference also highlighted educational aspects and introduced new software packages in group analysis.

This series of international conferences was intended to be a continuation of two conferences in group theoretic methods in mechanics held in Calgary (Canada) in 1974 and in Novosibirsk (USSR) in 1978. The subsequent MOGRAN III to MOGRAN IX conferences took place in 1991 (Ufa, USSR), 1992 (Catania, Italy), 1994 (Johananesburg, S.A.), 1996 (Johannesburg, S.A.), 1997 (Nordfjordeid, Norway), 2000 (Ufa, Russia) and 2002 (Moscow, Russia).

Approximately 60 scientists from 20 different countries participated in MOGRAN X. Forty five lectures on recent developments in traditional and modern aspects of group analysis were presented.

This book consists of selected papers presented at the conference. All 33 papers have been reviewed by two independent referees.

We would like to thank the team of support for their great help in organizing this conference.

We are grateful to the contributors for preparing their manuscript promptly. Furthermore we express our gratitude to all anonymous referees for their constructive suggestions for improvement of the papers that appear in this volume.

The conference was made possible by the financial support of several sponsors that are listed below.

It is finally a pleasure to thank our colleagues of the Institute of Mathematics of National Academic of Science of Ukraine, and in particular Nataliya Ivanova, Roman Popovych and Vyacheslav Boyko, for preparing this volume.

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Contents

On a Group Method for the Poisson Equation in a Polygon

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The group method is applied for the solution of a Poisson equation in an arbitrary convex polygon with N sides. We solve both the Laplace and Poisson equations in the regular hexagonal and sector plates with certain boundary conditions. The obtained results are presented graphically.

1 Introduction

In many physical problems, Poisson equation (1) occurs for instance when calculating an electric potential $T(x, y)$ at the point $(x, y) \in \Omega$ in the presence of a charge distribution prescribed by a function $h(x, y)$. Also occurs in heat conduction problems when there is a source of heat $h(x, y)$ inside the region in which the temperature distribution $T(x, y)$ is being calculated.

Although a large quantity of theoretical investigations relating to steady state temperature has appeared in the literature, approximately all the existing works have considered boundary conditions of specified temperature, specified heat flux in only specified regions such as rectangular or circular. Only few theoretical studies involving a multilateral regions exist in the literature, see Fokas and Kapaev [1]. This is perhaps due to the failure of standard analytical techniques for such problems.

The mathematical technique in the present analysis is the parameter-group transformation. The advantages of the method are due to simplicity, and in reducing the number of independent variables by one, consequently, yields complete results with less efforts. Hence it is applicable to solve a wider variety of nonlinear problems. The base of the group-theoretic method were introduced and treated extensively by Lie [2]. Birkhoff [3], made use of one-parameter transformation groups. Morgan [4], presented a theory which has led to improvements over earlier similarity methods. Recently, the method has been applied intensively by Abd-el-Malek [5] et al.

2 Mathematical Formulation of the Problem

The governing equation for the distribution of temperature $T(x, y)$, is given by

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = h(x, y); \quad (x, y) \in \Omega, \tag{1}$$

with the boundary conditions (see Fig. 1):

$$T(x, y) = \alpha x^n; \quad (x, y) \in \text{Ł}_1, \quad T(x, y) = \beta y^m; \quad (x, y) \in \text{Ł}_2. \tag{2}$$

where $n, m \in \{0, 1, 2, 3, \ldots\}$ and both α, β are constants.

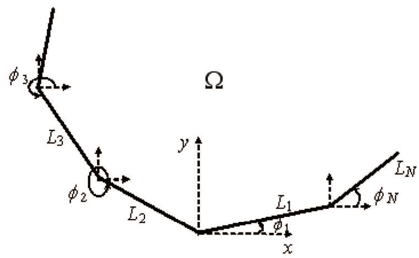

Figure 1. Geometrical of the problem

It is required to find the distribution $T(x, y)$ inside the domain Ω and the heat flux across L_k; $2 < k \leq N$ where: $L_1 : \ y = x \tan \phi_1$, $L_2 : \ y = x \tan \phi_2$, $L_k : \ y = x \tan \phi_k + b_k$, $b_k \neq 0$; $2 < k \leq N$. Let $T(x, y) = w(x, y)q(x)$, $q(x) \neq 0$ in Ω. Hence (1) and (2) reduce to

$$q(x) \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + 2 \frac{\partial w}{\partial x} \frac{dq}{dx} + w \frac{d^2 q}{dx^2} = h(x, y), \tag{3}$$

with the boundary conditions:

$$w(x, y) = \frac{\alpha x^n}{q(x)}; \quad (x, y) \in \text{Ł}_1, \quad w(x, y) = \frac{\beta y^m}{q(x)}; \quad (x, y) \in \text{Ł}_2. \tag{4}$$

3 Solution of the Problem

The method of solution depends on the application of one-parameter group transformation to the partial differential equation (3).

3.1 The Group Systematic Formulation

Consider the group **G**, of one parameter a of the form

$$\mathbf{G} : \ \bar{S} = C^s(a)S + P^s(a), \tag{5}$$

where S stands for x, y, w, q, h and the $C's$ and $P's$ are real-valued functions and at least differentiable in the real argument a.

3.2 The Invariance Analysis

Transformations of the derivatives are obtained from **G** via chain-rule operations:

$$\bar{S}_{\bar{i}} = \left(\frac{C^s}{C^i}\right) S_i, \quad \bar{S}_{\bar{i}\bar{j}} = \left(\frac{C^s}{C^i C^j}\right) S_{ij}, \quad i = x,\, y, \quad j = x,\, y, \tag{6}$$

where S stands for w, q and h, and $S_i = \partial S/\partial i$, $S_{ij} = \partial^2 S/(\partial i \partial j)$, ... Equation (3) is said to be invariantly transformed whenever

$$\begin{aligned}&\bar{q}(\bar{x})\left(\frac{\partial^2 \bar{w}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{w}}{\partial \bar{y}^2}\right) + 2\frac{\partial \bar{w}}{\partial \bar{x}}\frac{d\bar{q}}{d\bar{x}} + \bar{w}\frac{d^2\bar{q}}{d\bar{x}^2} - \bar{h}(\bar{x},\bar{y})\\ &= H(a)\left[q(x)\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + 2\frac{\partial w}{\partial x}\frac{dq}{dx} + w\frac{d^2 q}{dx^2} - h(x,y)\right],\end{aligned} \tag{7}$$

for some function $H(a)$ which may be a constant.

Substitution from (5) into (7) yields

$$\begin{aligned}&q(x)\left(\left[\frac{C^q C^w}{(C^x)^2}\right]\frac{\partial^2 w}{\partial x^2} + \left[\frac{C^q C^w}{(C^y)^2}\right]\frac{\partial^2 w}{\partial y^2}\right)\\ &+2\left[\frac{C^q C^w}{(c^x)^2}\right]\frac{\partial w}{\partial x}\frac{dq}{dx} + \left[\frac{c^q C^w}{(C^x)^2}\right] w\frac{d^2 q}{dx^2} - C^h h(x,y) + \zeta(a)\\ &= H(a)\left[q(x)\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + 2\frac{\partial w}{\partial x}\frac{dq}{dx} + w\frac{d^2 q}{dx^2} - h(x,y)\right],\end{aligned} \tag{8}$$

where

$$\zeta(a) = \left[\frac{P^q C^w}{(C^x)^2}\right]\frac{\partial^2 w}{\partial x^2} + \left[\frac{P^q C^w}{(C^y)^2}\right]\frac{\partial^2 w}{\partial y^2} + \left[\frac{P^w C^q}{(C^x)^2}\right]\frac{d^2 q}{dx^2} + P^h.$$

The invariance of (8) implies $\zeta(a) = 0$. This is satisfied by $P^q = P^w = P^h = 0$, and

$$\left[\frac{C^w C^q}{(C^x)^2}\right] = \left[\frac{C^w C^q}{(C^y)^2}\right] = C^h = H(a), \tag{9}$$

which yields

$$C^x = C^y \quad \texttt{and} \quad C^h = \frac{C^w C^q}{(C^x)^2}. \tag{10}$$

Moreover, the invariance of boundary conditions (4) implies $P^x = P^y = 0$ and $C^q = (C^x)^n/C^w = (C^y)^m/C^w$. From (9) and (10), we get $n = m$ and

$$C^h = (C^x)^{n-2}. \tag{11}$$

Finally, we get the one-parameter group **G** which transforms invariantly (8) and (4) is of the form:

$$\mathbf{G}: \bar{x} = C^x x, \quad \bar{y} = C^x y, \quad \bar{w} = C^w w, \quad \bar{q} = \frac{(C^x)^n}{C^w} q, \quad \bar{h} = (C^x)^{n-2} h.$$

3.3 The Complete Set of Absolute Invariants

Our aim is to make use of group method to represent the problem in the form of an ordinary differential equation in a single independent variable. In addition to the absolute invariant of the independent variable, there are three absolute invariants of the independent variables w, q and h.

If $\eta = \eta(x, y)$ is the absolute invariant of the independent variables, then

$$g_j(x, y; w, q, h) = F_j\left[\eta(x, y)\right], \quad j = 1,\, 2,\, 3 \tag{12}$$

which are the three absolute invariants corresponding to w, q and h. The function $g = g(x, y; w, q, h)$ is an absolute invariant of a one parameter group if it satisfies the following first-order linear differential equation

$$\sum_{i=1}^{5} \left(\alpha_i S_i + \beta_i\right) \partial g / \partial S_i = 0, \tag{13}$$

where S_i stands for x, y w, q and h, and $\alpha_i = \partial C^{S_i}(a^0)/\partial a$ and $\beta_i = \partial P^{S_i}(a^0)/\partial a$, $i = 1, 2, \ldots, 5$, where a^0 denotes the identity element of the group. From group (11) and using (13), we get $\alpha_1 = \alpha_2$ and $\beta_i = 0$; $i = 1, 2, \ldots, 5$.

At first, we seek the absolute invariant of the independent variable. From (12), $\eta(x, y)$ is an absolute invariant if it satisfies $x\partial\eta/\partial x + y\partial\eta/\partial y = 0$, which has a solution of the form

$$\eta(x, y) = f\left(y/x\right), \tag{14}$$

where f is an arbitrary function and can be selected to be the identity function. Thus we get

$$\eta(x, y) = y/x. \tag{15}$$

Similarly, the absolute invariants of the dependent variables q, w, and h are

$$q(x) = R(x)\theta(\eta), \quad w(x, y) = \Gamma(x)F(\eta), \quad h(x, y) = V(x)\psi(\eta). \tag{16}$$

It is clear $\theta(\eta)$ must be a constant, say $\theta(\eta) = 1$, hence

$$q(x) = R(x). \tag{17}$$

3.4 The Reduction to Ordinary Differential Equation

As the general analysis proceeds, the established forms of the dependent and independent absolute invariant are used to obtain ordinary differential equation. Generally, the absolute invariant $\eta(x, y)$ has the form given in (14).

Substituting from (15) and (16) into (3) yields

$$\begin{aligned}&\left(\eta^2 + 1\right) \frac{d^2F}{d\eta^2} - 2\eta \left[\left(\frac{1}{\Gamma}\frac{d\Gamma}{dx} + \frac{1}{R}\frac{dR}{dx}\right) x - 1\right] \frac{dF}{d\eta} \\ &+ \left[\frac{1}{\Gamma}\frac{d^2\Gamma}{dx^2} + \frac{2}{R\Gamma}\frac{dR}{dx}\frac{d\Gamma}{dx} + \frac{1}{R}\frac{d^2R}{dx^2}\right] x^2 F = \frac{x^2 V(x)\psi(\eta)}{\Gamma(x)R(x)}.\end{aligned} \tag{18}$$

For (17) to be reduced to an expression in a single independent variable η, the coefficients in (17) should be constants or functions of η only. Thus

$$\left(\frac{1}{\Gamma}\frac{d\Gamma}{dx} + \frac{1}{R}\frac{dR}{dx}\right) x = C_1, \tag{19}$$

$$\left(\frac{1}{\Gamma}\frac{d^2\Gamma}{dx^2} + \frac{2}{R\Gamma}\frac{dR}{dx}\frac{d\Gamma}{dx} + \frac{1}{R}\frac{d^2R}{dx^2}\right) x^2 = C_2. \tag{20}$$

It follows from (18) that $\Gamma(x)R(x) = C_3 x^{C_1}$. Integrating (19), and using (20), we get $C_2 = C_1(C_1 - 1)$. Take $C_3 = 1$ and $C_1 = n$, we get $\Gamma(x)R(x) = x^n$ and $C_2 = n(n-1)$. Also, from (17) we obtain only

$$V(x) = \frac{1}{x^2}\Gamma(x)R(x) = x^{n-2}. \tag{21}$$

Therefore, the ordinary differential equation (17) reduces to

$$\left(\eta^2 + 1\right)\frac{d^2F}{d\eta^2} - 2(n-1)\eta\frac{dF}{d\eta} + n(n-1)F = \psi(\eta). \tag{22}$$

Under the similarity variable η, the boundary conditions (4) take the form

$$F(\tan\phi_1) = \alpha, \quad F(\tan\phi_2) = \beta\tan^n\phi_2. \tag{23}$$

3.5 Analytical Solution

The solution of the corresponding homogenous differential equation of (21) can be expressed in the form of the following power series $F(\eta) = a_1\Psi_1(\eta) + a_2\Psi_2(\eta)$, where

$$\Psi_1(\eta) = \sum_{i=0}^{[\frac{n}{2}]}(-1)^i\binom{n}{2i}\eta^{2i}, \quad \Psi_2(\eta) = \frac{1}{n}\sum_{i=0}^{[\frac{n-1}{2}]}(-1)^i\binom{n}{2i+1}\eta^{2i+1}, \tag{24}$$

a_1 and a_2 are constants that can evaluated from (22).

The general solution of the differential equation (21), can be found if one of the solutions of the corresponding homogenous differential equation, say Ψ_1, is known, has the form

$$F(\eta) = \left\{\int_{\eta=\xi}\left[\frac{(1+\xi^2)^{n-1}}{{\Psi_1}^2(\xi)}\left(\int_{\xi=\nu}\frac{\psi(\nu)\Psi_1(\nu)}{(1+\nu^2)^n}d\nu\right)\right]d\xi + c_1\int_{\eta=\xi}\frac{(1+\xi^2)^{n-1}}{{\Psi_1}^2}d\xi + c_2\right\}\Psi_1(\eta),$$

where c_1 and c_2 are constants that be evaluated from (22). Uniqueness of the solution exists if $\phi_2 - \phi_1$ is not a multiple of π/n.

The temperature distribution can be calculated from the formula

$$T(x,y) = x^n F\left(\frac{y}{x}\right). \tag{25}$$

The heat flux, Q, across L_m is given by

$$Q = -\sigma \frac{\partial T(x,y)}{\partial M}, \tag{26}$$

where σ is the thermal conductivity and M is the outer normal direction to the side L_m. Therefore $Q = -\sigma\left(l_x \partial T/\partial x + l_y \partial T/\partial y\right)$, where l_x and l_y are the direction cosines of the normal to the side L_m with the x and y coordinate axes, respectively.

4 Problem of Regular Hexagonal Plate

A steady distribution temperature in a regular hexagonal, see Fig. 2, is considered for two cases:

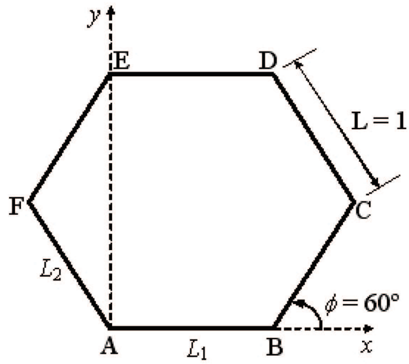

Figure 2. Geometrical of the regular hexagonal

4.1 Laplace Equation with Quadratic Boundary Conditions

The governing equation and the boundary conditions are given by

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0; \quad (x,y) \in \Omega,$$
$$T(x,y) = x^2;\ (x,y) \in \text{Ł}_1, \quad T(x,y) = y^2;\ (x,y) \in \text{Ł}_2.$$

Substituting $n = 2$ in the solution (23) we get $T(x,y) = x^2 - 5xy/\sqrt{3} - y^2$. The shape of the temperature distribution and the heat flux are plotted in Fig. 3a and Fig. 3b, respectively.

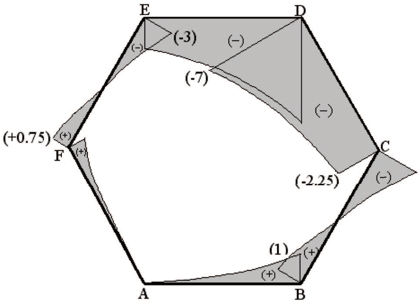

Figure 3a. Temperature distribution for $h(x, y) = 0$

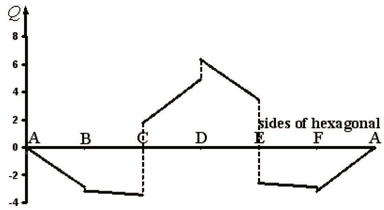

Figure 3b. Heat flux profiles for $h(x, y) = 0$

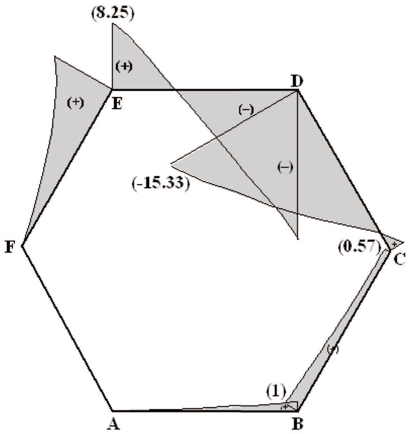

Figure 4a. Temperature distribution for $h(x, y) = x^4/(x^2 + y^2)$

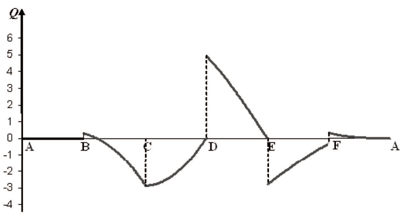

Figure 4b. Heat flux profiles for $h(x, y) = x^4/(x^2 + y^2)$

4.2 Poisson Equation with Quartic Boundary Conditions

The governing equation and the boundary conditions are given by

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{x^4}{x^2 + y^2}; \quad (x, y) \in \Omega,$$
$$T(x, y) = x^4; \; (x, y) \in Ł_1, \quad T(x, y) = 0; \; (x, y) \in Ł_2.$$

Substituting $n = 4$ in the solution (24) we get

$$T(x, y) = \left(\frac{1}{16} \tan^{-1}\left(\frac{y}{x}\right) + \frac{357\sqrt{3} + 6\pi}{288} \right) xy(x^2 - y^2) + \left(x^4 - \frac{89}{16} x^2 y^2 + \frac{11}{12} y^4 \right).$$

The shape of the temperature distribution and the heat flux are plotted in Fig. 4a and Fig. 4b, respectively.

5 Problem of Sector Plate

The plate of Fig. 5 has a boundary L_3 as a sector of circle with radius R.

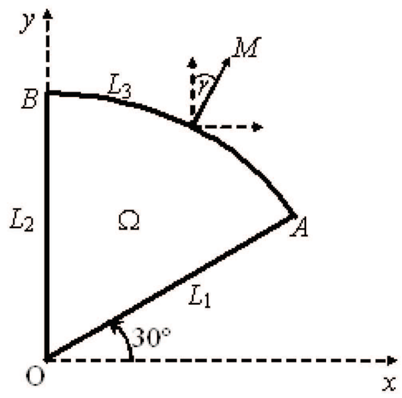

Figure 5. Geometrical of the curved plate

5.1 Laplace Equation with Quartic Boundary Conditions

The governing equation and the boundary conditions are given by

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0; \quad (x,y) \in \Omega,$$
$$T(x,y) = \alpha x^4;\ (x,y) \in \text{Ł}_1, \quad T(x,y) = \beta y^4;\ (x,y) \in \text{Ł}_2. \tag{27}$$

Substituting $n = 4$ in the homogenous solution (23) we get

$$T(x,y) = \beta\left(x^4 - 6x^2y^2 + y^4\right) + \frac{\sqrt{3}}{6}\left(9\alpha + 8\beta\right)\left(x^3y - xy^3\right). \tag{28}$$

The heat flux Q across L_3 is given by

$$Q = -\sigma\frac{\partial T}{\partial M}(x,y)\,|_{L_3}, \tag{29}$$

where

$$\left.\frac{\partial T}{\partial M}(x,y)\right|_{L_3} = \frac{\partial T}{\partial x}\sin\gamma + \frac{\partial T}{\partial y}\cos\gamma \qquad \text{and} \tag{30}$$

$$\tan\gamma = \frac{x}{\sqrt{R^2 - x^2}}. \tag{31}$$

Substituting (27) in (29), using (30), we get

$$\frac{\partial T}{\partial M} = \frac{4\beta}{R}\left(R^4 - 8R^2x^2 + 8x^4\right) + \frac{\sqrt{3}(9\alpha + 8\beta)}{3R}x(R^2 + x^2)\sqrt{R^2 - x^2}.$$

Upon substituting (31) in (28) we get the heat flux across L_3.

5.2 Poisson Equation with Quartic Boundary Conditions

Consider the previous case with $h(x,y) = x^2$ with the same boundary conditions. The calculated temperature distribution inside the plate is given by

$$T(x,y) = \frac{x^4}{12} + \beta\left(x^4 - 6x^2y^2 + y^4\right) + \frac{\sqrt{3}}{24}\left(36\alpha + 32\beta - 3\right)xy\left(x^2 - y^2\right).$$

The heat flux Q across L_3 is given by (28), where

$$\frac{\partial T}{\partial M} = \frac{4\beta}{R}\left(R^4 - 8R^2x^2 + 8x^4\right)$$
$$+\frac{\sqrt{3}}{8R}\left(36\alpha + 32\beta - 3\right)x(2x^2 - R^2)\sqrt{R^2 - x^2}.$$

6 Conclusions and Remarks

The group method transformation is presented to study steady temperature that satisfy a boundary-value-problem for the Poisson equation in an arbitrary convex polygon with N sides. The most difficult step in solving boundary-value-problems for linear elliptic equations is some parts of the boundary have no prescribed conditions!

The Poisson, and consequently Laplace, equation is characterized by that it can be investigated using conformal mapping. But conformal mappings fail for general boundary conditions, see [1] . However, the method presented here can be applied to elliptic equations for which conformal mappings are not applicable.

Acknowledgements

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Super-Integrability of the Non-Periodic Toda Lattice

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We prove that the classical, non-periodic Toda lattice is super-integrable. In other words, we show that it possesses $2N-1$ independent constants of motion, where N is the number of degrees of freedom. The main ingredient of the proof is some special action-angle coordinates introduced by Moser to solve the equations of motion.

1 Introduction

The Toda lattice is arguably the most fundamental and basic of all finite dimensional Hamiltonian integrable systems. It has various intriguing connections with other parts of mathematics and Physics.

The Hamiltonian of the Toda lattice is given by

$$H(q_1,\ldots,q_N,\,p_1,\ldots,p_N)=\sum_{i=1}^{N}\frac{1}{2}\,p_i^2+\sum_{i=1}^{N-1}e^{q_i-q_{i+1}}\ . \qquad (1)$$

Equation (1) is known as the classical, finite, non–periodic Toda lattice to distinguish the system from the many and various other versions, e.g., the relativistic, quantum, infinite, periodic etc. This system was investigated in [7–9, 11–14] and numerous other papers that are impossible to list here.

This type of Hamiltonian, sometimes called the Toda chain, was considered first by Morikazu Toda [14]. The original Toda lattice can be viewed as a discrete version of the Korteweg–de Vries equation. It is called a lattice as in atomic lattice since interatomic interaction was studied. This system also appears in Cosmology. It appears also in the work of Seiberg and Witten on supersymmetric Yang–Mills theories and it has applications in analog computing and numerical computation of eigenvalues. But the Toda lattice is mainly a theoretical mathematical model which is important due to the rich mathematical structure encoded in it.

Hamilton's equations become

$$\dot{q}_j=p_j,\quad \dot{p}_j=e^{q_{j-1}-q_j}-e^{q_j-q_{j+1}}.$$

The system is integrable. One can find a set of independent functions $\{H_1, \ldots, H_N\}$ which are constants of motion for Hamilton's equations. To determine the constants of motion, one uses Flaschka's transformation:

$$a_i = \frac{1}{2} e^{(q_i - q_{i+1})/2}, \quad b_i = -\frac{1}{2} p_i. \tag{2}$$

Then

$$\dot{a}_i = a_i (b_{i+1} - b_i), \quad \dot{b}_i = 2(a_i^2 - a_{i-1}^2). \tag{3}$$

These equations can be written as a Lax pair $\dot{L} = [B, L]$, where L is the Jacobi matrix

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & \cdots & 0 \\ a_1 & b_2 & a_2 & \cdots & & \vdots \\ 0 & a_2 & b_3 & \ddots & & \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & a_{N-1} \\ 0 & \cdots & & \cdots & a_{N-1} & b_N \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & a_1 & 0 & \cdots & \cdots & 0 \\ -a_1 & 0 & a_2 & \cdots & & \vdots \\ 0 & -a_2 & 0 & \ddots & & \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & a_{N-1} \\ 0 & \cdots & \cdots & & -a_{N-1} & 0 \end{pmatrix}$$

This is an example of an isospectral deformation; the entries of L vary over time but the eigenvalues remain constant. It follows that the functions $H_j = \operatorname{tr} L^j / j$ are constants of motion. This elegant integrability demonstration is due to Flaschka in 1974.

Consider $\mathbf{R}^{2N}$ with coordinates $(q_1, \ldots, q_N, p_1, \ldots, p_N)$, the standard symplectic bracket

$$\{f, g\}_s = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right),$$

and the mapping $F : \mathbf{R}^{2N} \to \mathbf{R}^{2N-1}$ defined by

$$F : (q_1, \ldots, q_N, p_1, \ldots, p_N) \to (a_1, \ldots, a_{N-1}, b_1, \ldots, b_N).$$

There exists a bracket on $\mathbf{R}^{2N-1}$ which satisfies $\{f,g\}\circ F = \{f\circ F, g\circ F\}_s$. It is a bracket which (up to a constant multiple) is given by

$$\{a_i, b_i\} = -a_i, \quad \{a_i, b_{i+1}\} = a_i \tag{4}$$

all other brackets are zero. $H_1 = b_1 + b_2 + \cdots + b_N$ is the only Casimir. The Hamiltonian in this bracket is $H_2 = \operatorname{tr} L^2/2$. The Lie algebraic interpretation of this bracket can be found in [10]. We denote this bracket by π_1.

The quadratic Toda bracket appears in conjunction with isospectral deformations of Jacobi matrices. First, let λ be an eigenvalue of L with normalized eigenvector v. Standard perturbation theory shows that

$$\nabla\lambda = (2v_1v_2, \ldots, 2v_{N-1}v_N, v_1^2, \ldots, v_N^2)^T := U^\lambda,$$

where $\nabla\lambda$ denotes $(\partial\lambda/\partial a_1, \ldots, \partial\lambda/\partial b_N)$. Some manipulations show that U^λ satisfies $\pi_2\, U^\lambda = \lambda\, \pi_1\, U^\lambda$, where π_1 and π_2 are skew-symmetric matrices. It turns out that π_1 is the matrix of coefficients of the Poisson tensor (4), and π_2, whose coefficients are quadratic functions of the a's and b's, can be used to define a new Poisson tensor. The quadratic Toda bracket appeared in a paper of Adler [1] in 1979. It is a Poisson bracket in which the Hamiltonian vector field generated by H_1 is the same as the Hamiltonian vector field generated by H_2 with respect to the π_1 bracket. The defining relations are

$$\begin{aligned} &\{a_i, a_{i+1}\} = \tfrac{1}{2}a_i a_{i+1}, \quad \{a_i, b_i\} = -a_i b_i, \\ &\{a_i, b_{i+1}\} = a_i b_{i+1}, \quad \{b_i, b_{i+1}\} = 2\,a_i^2, \end{aligned} \tag{5}$$

all other brackets are zero. This bracket has $\det L$ as Casimir and $H_1 = \operatorname{tr} L$ is the Hamiltonian. The eigenvalues of L are still in involution. Furthermore, π_2 is compatible with π_1. We also have

$$\pi_2 \nabla H_j = \pi_1 \nabla H_{j+1}\ . \tag{6}$$

These relations are similar to the Lenard relations for the KdV equation; they are generally called the Lenard relations. Taking $j = 1$ in (6), we conclude that the Toda lattice is bi–Hamiltonian.

The multi-Hamiltonian structure of the Toda lattice was first derived using master symmetries. We quote the results from refs. [3, 4].

Theorem 1. *There exists a sequence of Poisson tensors π_i and a sequence of master symmetries X_i such that*
i) π_j *are all Poisson.*

ii) The functions H_i are in involution with respect to all of the π_j.

iii) $X_i(H_j) = (i+j)H_{i+j}$.

iv) $L_{X_i}\pi_j = (j-i-2)\pi_{i+j}$.

v) $[X_i,\ X_j] = (j-i)X_{i+j}$.

vi) $\pi_j\ \nabla\ H_i = \pi_{j-1}\ \nabla\ H_{i+1}$, *where π_j denotes the Poisson matrix of the tensor π_j.*

The super-integrability of the Toda lattice was conjectured in [5] where a third integral was obtained for the special case $N = 2$. The integral in [5] was obtained using Noether's theorem.

In the case of two degrees of freedom the potential is simply $V(q_1, q_2) = e^{q_1 - q_2}$, and the procedure of Noether produces the following three integrals:

$$H_1 = -\frac{1}{2}(p_1 + p_2), \quad J_1 = (p_1 - p_2)^2 + 4e^{q_1 - q_2},$$
$$I_1 = \frac{p_1 - p_2 + \sqrt{J_1}}{p_1 - p_2 - \sqrt{J_1}} \exp\left(\sqrt{J_1}\frac{q_1 + q_2}{p_1 + p_2}\right). \tag{7}$$

Note that $H = H_1^2 + J_1/4$ and that the function $G = (q_1 + q_2)/(p_1 + p_2)$ which appears in the exponent of I_1 is a time function, i.e., it satisfies $\{G, H\} = 1$.

The existence of the integral I_1 shows that the two degrees of freedom Toda lattice is super-integrable with three integrals of motion $\{H_1, J_1, I_1\}$. As we will see, the complicated integral I_1 has a simple expression if one uses Moser's coordinates.

The super-integrability of this type of systems should be expected due to their dispersive asymptotic behavior. However, the construction of integrals is not typically a trivial task. In the case of the open Toda lattice, asymptotically the particles become free as time goes to infinity with asymptotic momenta being the eigenvalues of the Lax matrix. Therefore, the system behaves asympotically like a system of free particles which is super-integrable. For this reason we believe that the generalizations of the Toda lattice to other semi–simple Lie groups due to Bogoyavlensky should also be super-integrable. On the other hand, the periodic Toda lattice is clearly not super-integrable.

2 Moser's Solution of the Toda Lattice

Moser's beautiful solution of the open Toda lattice uses the Weyl function $f(\lambda)$ and an old (19th century) method of Stieltjes which connects the continued fraction of $f(\lambda)$ with its partial fraction expansion. The key ingredient is the map which takes the (a, b) phase space of tridiagonal Jacobi matrices to a new space of variables (λ_i, r_i) where λ_i is an eigenvalue of the Jacobi matrix and r_i^2 is the residue of rational functions that appear in the solution of the equations. We present a brief outline of Moser's construction.

Moser in [12] introduced the resolvent $R(\lambda) = (\lambda I - L)^{-1}$ and defined the Weyl function $f(\lambda) = (R(\lambda)e_1,\ e_1)$, where $e_1 = (1, 0, \dots, 0)$.

The function $f(\lambda)$ has a simple pole at $\lambda = \lambda_i$ and positive residue at λ_i equal to r_i^2:

$$f(\lambda) = \sum_{i=1}^{N} \frac{r_i^2}{\lambda - \lambda_i}.$$

Moreover, for λ large one has $\lambda f(\lambda) \to 1$ and therefore $\sum_{i=1}^N r_i^2 = 1$. Thus we have a mapping from the space of $(a_1, \ldots, a_{N-1}, b_1, \ldots, b_N)$ with $a_i > 0$ to the space $(\lambda_1, \ldots, \lambda_N, r_1, \ldots, r_N)$ with $\lambda_1 < \lambda_2 < \cdots < \lambda_N$ and $\sum_{i=1}^N r_i^2 = 1$. This mapping is one-to-one and onto. The inverse of this map is a discrete analogue of the inverse scattering method of spectral theory.

The variables (a, b) may be expressed as rational functions of λ_i and r_i using a continued fraction expansion of $f(\lambda)$ which dates back to Stieltjes. Since the computation of the continued fraction from the partial fraction expansion is a rational process the solution is expressed as a rational function of the variables $(\lambda_i,\, r_i)$.

Moser ignores the condition $\sum_{i=1}^N r_i^2 = 1$ and views r_i as projective coordinates. Under this modification the equations of the Toda lattice take the simple form

$$\dot{\lambda}_i = 0, \quad \dot{r}_i = \lambda_i r_i\ . \tag{8}$$

These equations show that the $(\lambda_i,\, \log r_i)$ are action–angle variables.

Finally, we comment on the Poisson brackets in the new coordinates. In [6] Faybusovich and Gekhtman find another method of generating the multi-Hamiltonian structure for the Toda lattice. The Poisson brackets of Theorem 1 project onto some rational brackets in the space of Weyl functions and in particular, the Lie–Poisson bracket π_1 corresponds to the Atiyah–Hitchin bracket [2]. The idea is to construct a sequence of Poisson brackets on the space (λ_i, r_i) whose image under the inverse spectral transform are the brackets π_i defined in Theorem 1. A rational function of the form $q(\lambda)/p(\lambda)$ is determined uniquely by the distinct eigenvalues of $p(\lambda)$, $\lambda_1, \ldots, \lambda_n$ and values of q at these roots. The residue r_i is equal to $q(\lambda_i)/p'(\lambda_i)$ and therefore we may choose

$$\lambda_1, \ldots, \lambda_n,\ q(\lambda_1), \ldots, q(\lambda_n)$$

as global coordinates on the space of rational functions (of the form q/p with p having simple roots and q, p coprime). We have to remark that the image of the Moser map is a much larger set.

The kth Poisson bracket is defined by

$$\{\lambda_i,\ q(\lambda_i)\} = -\lambda_i^k q(\lambda_i), \quad \{q(\lambda_i),\ q(\lambda_j)\} = \{\lambda_i,\ \lambda_j\} = 0.$$

The initial Poisson bracket, i.e., the image of the linear bracket (4) under the Moser map is given explicitly by

$$\begin{aligned} &\{\lambda_i, \lambda_j\} = 0, \quad \{r_i, r_j\} = 0, \\ &\{\lambda_i, r_j\} = \delta_{ij} r_j, \quad i, j = 1, \ldots, N. \end{aligned} \tag{9}$$

Similarly, the quadratic Toda bracket, π_2, corresponds to a bracket with only non-zero terms $\{\lambda_i, r_i\} = \lambda_i r_i$.

The Hamiltonian function in the new coordinates is $H_2 = \frac{1}{2}\sum_{i=1}^N \lambda_i^2$. More generally, the functions H_j take the form $H_j = \frac{1}{j}\sum_{i=1}^N \lambda_i^j$.

3 The Toda Lattice is Super-Integrable

We now come to the main result of this paper. We define

$$I_j = (r_j/r_{j+1})^2\, e^{F_{j,j+1}}, \quad j = 1, \dots, N-1, \tag{10}$$

where

$$F_{j,j+1} = \frac{2(\lambda_{j+1} - \lambda_j)}{H_1} \ln\Big(\prod_{i=1}^{N} r_i\Big) .$$

It is easily shown, using equation (8) that $dI_j/dt = 0$, for $j = 1, \dots, N-1$ and thus the functions I_j are constants of motion.

The functions H_i and I_j, $i = 1, \dots, N$, $j = 1, \dots, N-1$ are functionally independent. In fact, the Jacobian $(2N-1) \times 2N$ matrix of the functions H_i and I_j has a $(2N-1) \times (2N-1)$ subdeterminant, d_{N+1}, which is obtained by deleting the $(N+1)$-column which is not identically zero. A simple calculation gives

$$d_{N+1} = -2^{N-1} N \frac{r_1^2}{r_{N-2} r_{N-1} r_N^3} \frac{\lambda_1}{H_1}\, e^{F_{1,N}} \prod_{1 \le i < j \le N} (\lambda_i - \lambda_j) .$$

Since the eigenvalues of real Jacobi matrices are distinct, the functions H_i and I_j are independent. We summarize the results in the following:

Theorem 2. *The Toda lattice with N degrees of freedom possesses $2N-1$ independent constants of motion, H_i, $i = 1, \dots, N$, I_j, $j = 1, \dots, N-1$, and is therefore super-integrable.*

Remark 1. It is clear that the functions H_n, $n = 1, \dots, N$ are in involution. Moreover it can be shown that $\{I_i, I_j\} = 0$, $i, j = 1, \dots, N-1$. In addition, for $n = 1, \dots, N$, $j = 1, \dots, N-1$ $\{H_n, I_j\} = 2c_n(\lambda_{j+1} - \lambda_j)E_j I_j/H_1$, where $c_n = 1$ for $n = 2, \dots, N$, $c_1 = N/(N-2)$ and

$$E_j = \sum \lambda_i^{n-1} - w(n-2) \sum \lambda_i - 2\lambda_{j+1}\lambda_j\, w(n-3).$$

The sums are taken over all i from 1 to N where $i \neq j, j+1$ The function $w(n)$ symbolizes the full homogeneous polynomial in λ_j and λ_{j+1} that have total weight equal to n. For instance, $w(n) = 0, n \in -\mathbf{Z}^+$, $w(0) = 1$, $w(1) = \lambda_j + \lambda_{j+1}$, $w(2) = \lambda_j^2 + \lambda_{j+1}^2 + \lambda_j\lambda_{j+1}$, etc.

One can, of course, use the quadratic Toda bracket in (λ_i, r_i) coordinates. We must then take $\mathrm{Tr}L = \lambda_1 + \dots + \lambda_N$ as Hamiltonian. However, in this bracket the H_i, I_j do not form a finite dimensional algebra.

Remark 2. We clearly have $\{H_2, I_j\} = 0$, $j = 1, \dots, N-1$, since H_2 is the Hamiltonian and the functions I_j are constants of motion.

We define the sets $S_1 = \{H_1, \ldots, H_N\}$ and $S_2 = \{H_2, I_1, \ldots, I_{N-1}\}$. Then if $f, g \in S_1 \Rightarrow \{f, g\} = 0$ and if $f, g \in S_2 \Rightarrow \{f, g\} = 0$. In other words the sets S_1 and S_2 are both maximal sets of integrals in involution. We therefore have two different sets demonstrating the complete integrability of the Toda lattice.

Remark 3. We finally would like to comment on how the integrals I_j were guessed: The complicated integral (7) at the end of the introduction is quite simple in Moser's coordinates. For example, $\sqrt{J_1}$ is simply equal to $2(\lambda_2 - \lambda_1)$ and the expression

$$\frac{p_1 - p_2 + \sqrt{J_1}}{p_1 - p_2 - \sqrt{J_1}}$$

reduces to $-(r_1/r_2)^2$. The exponent is simplified as follows. On the one hand,

$$(q_1 + q_2)^{\cdot} = p_1 + p_2 = -2(b_1 + b_2) = -2(\lambda_1 + \lambda_2)\ .$$

On the other hand from $\dot{r}_i = -r_i\lambda_i$ one obtains that $(\ln r_i)^{\cdot} = -\lambda_i$ and therefore the exponent is simply $2(\lambda_1 - \lambda_2)\ln(r_1 r_2)/(\lambda_1 + \lambda_2)$. Therefore, up to a sign the integral (7) is precisely the same as the one in (10).

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The Pseudo-Potential Technique for Nonlinear Optical Equations

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We demonstrate that a reduction of a Maxwell–Bloch system, that includes the effects of a permanent dipole and inhomogeneous broadening, is completely integrable and we present the Lax pair. An appropriate Bäcklund transformation is employed to solve the equations exactly and produce a family of soliton solutions.

1 Introduction

We consider the optical phenomenon of propagation of an electric field through a quantized medium. We make use of the classical wave-equation of Maxwell for the dynamical evolution of the electric field, coupled with the quantum Bloch equations that describe the behavior of the induced polarization field. At the atomic level, the phenomenon of stimulated emission of radiation is made experimentally possible when an appropriate medium is used and the frequency of the electromagnetic light waves is close to resonance with the natural oscillatory modes of the medium [10]. The optical scales involved are those of the electric field and the relaxation times of the medium. When the latter is larger then the former, the result is a pulse containing a half up to only a few optical cycles, resulting to what is commonly referred to as an extremely short pulse or electromagnetic bubble.

Under certain physically motivated assumptions the Maxwell–Bloch (MB) system reduces to several models, each one derived at a different level of approximation. Two physical phenomena closely related to the interaction of light and matter are the effects of: (i) inhomogeneous broadening, and (ii) a permanent dipole. In reducing the MB equations either the first and/or the second phenomena were usually neglected. In several treatments of reduced Maxwell–Bloch models, for example [4, 6, 7, 9], the equations under consideration incorporate the effects of inhomogeneous broadening in the absence, however, of a permanent dipole. In [1] a model has been introduced that includes the effects of a permanent dipole in a homogeneous medium. The presence of the permanent dipole results to stronger coupling of the electric and polarization fields, and mathematically this translates to further nonlinearity in the equations. Also in [8], a homogeneous version of the MB equations with the permanent dipole effect present is studied. The goal of this paper is to demonstrate the integrability of a reduction of the MB system, in the case when the relevant equations retain the properties of *both* inhomogeneous broadening, and a permanent dipole. We present the Lax pair for the model and use a Bäcklund transformation to find explicit solutions in terms of elementary functions for both the one- and the two-soliton solutions. We note that the relatively small in amplitude backscattering of the electric field is neglected.

The method for constructing the new Lax pair involves the use of a pseudo-potential [3,5,11]. In section 2 we present the inhomogeneously broadened reduced Maxwell–Bloch (ib-rMB) equations and in section 3 the Lax pair representation is formulated. In section 4 we construct a Bäcklund transformation that enables us to obtain the soliton solutions family in section 5. In section 6 we discuss the physical implications of the analysis of the solutions, and in 7 we summarize the results of the paper and give directions for further research on the subject.

2 Inhomogeneously Broadened Model

We begin by introducing a set of Maxwell–Bloch equations that recount the interaction between light and matter as described in the introduction.

$$\frac{\partial E}{\partial x} + \frac{1}{c}\frac{\partial E}{\partial t} = \frac{\Delta d \mathcal{N}}{2c\epsilon_0}\langle \omega S_\omega \rangle_g, \qquad \hbar \frac{\partial R_\omega}{\partial t} = (\Delta h - \Delta d E) S_\omega, \tag{1}$$

$$\hbar \frac{\partial S_\omega}{\partial t} = -(\Delta h - \Delta d E) R_\omega + \frac{1}{2}\hbar\omega U_\omega, \qquad \frac{\partial U_\omega}{\partial t} = -2\omega S_\omega. \tag{2}$$

The dynamical variables are the electric field, E, and the elements of the Bloch vector $(R_\omega, S_\omega, U_\omega)$, which are linear combinations of the elements of the polarization matrix. The subscript ω for $(R_\omega, S_\omega, U_\omega)$ is meant to indicate the dependence of those quantities on the varying parameter ω, which portrays the different oscillation frequencies of the atoms of the medium. The function that gives the spread of the frequencies around a specific resonant frequency, ω_0, is $g(\omega)$ and in the sharp line case $g(\omega) = \delta(\omega - \omega_0)$. For a function F of ω, $\langle F(\omega)\rangle_g$ is the average of the function F over all possible frequencies:

$$\langle F(\omega)\rangle_g = \int_{-\infty}^{\infty} F(\omega) g(\omega) d\omega.$$

Time and space are represented by t and x. The permanent dipole effect is encoded in the parameters Δd and Δh. The absence of the permanent dipole is translated to $\Delta h = 0$. Δh has units of energy and Δd has units of charge $\times$ distance. c is the speed of light in the vacuum, $\mathcal{N}$ is the atomic density and ϵ_0 is the electric capacitance of the vacuum per unit length.

To nondimensionalize the equations above, we let $\omega' = \omega/\omega_a$, where ω_a is a typical atomic frequency such that ω' is of the order of unity. We note that ω_a is fixed. We define,

$$\begin{aligned} &\tau = \omega_a t, \qquad \zeta = \frac{\omega_a}{c}x, \qquad \beta = \frac{\Delta h}{\omega_a \hbar}, \\ &\gamma e = \frac{\Delta d}{\omega_a \hbar}E, \qquad \gamma = \frac{\Delta d^2 \mathcal{N}}{2\omega_a \epsilon_0 \hbar}, \qquad \Omega = \frac{\omega}{\omega_a}. \end{aligned} \tag{3}$$

β is a dimensionless constant since Δh has units of energy, and is of the order of unity. γ is also a dimensionless constant of the order of unity. For example, in SI units a typical value of Δd is 10^{-18}, of $\mathcal{N}/\epsilon_0$ is 10^{18} and of $\hbar\omega_a$ is $10^{-34} \times 10^{15} = 10^{-19}$. Using this scaling in equations (1)–(2), letting $\tau \mapsto \tau + \zeta$, $\zeta \mapsto \zeta$, and dropping the prime for ω' yields the ib-rMB equations,

$$\frac{\partial e}{\partial \zeta} = \langle \omega S_\omega \rangle, \qquad \frac{\partial R_\omega}{\partial \tau} = (\beta - \gamma e) S_\omega, \tag{4}$$

$$\frac{\partial S_\omega}{\partial \tau} = -(\beta - \gamma e) R_\omega + \frac{1}{2}\omega U_\omega, \qquad \frac{\partial U_\omega}{\partial \tau} = -2\omega S_\omega. \tag{5}$$

The permanent dipole is encoded in the dimensionless constants β and γ. It is note worthy that this set of equations can be thought of as an infinite, one-parameter family of equations, the parameter being the broadened transition frequency ω, which can be chosen randomly according to the distribution function $g(\omega)$.

3 Pseudo-Potential Technique

The inhomogeneously broadened reduced Maxwell–Bloch equations (4)–(5) are completely integrable. There exists a rational, one-parameter family of pairs of differential operators, that depend on the dynamical variables e, R_ω, S_ω, U_ω and commute in a Lie-bracket sense if and only if e, R_ω, S_ω, U_ω satisfy the ib-rMB equations (4)–(5).

The Lax pair representation is found using the pseudo-potential technique. We consider the following scalar equations,

$$\psi_\zeta = X(\vec{u}, \psi), \quad \psi_\tau = T(\vec{u}, \psi),$$

where $\vec{u}$ is a vector whose entries are the unknown dynamical quantities e, R_ω, S_ω, U_ω, usually called potentials. ψ is called a pseudo-potential. We derive necessary and sufficient conditions for the commutativity of the two flows, $(\psi_\zeta)_\tau = (\psi_\tau)_\zeta$, to be equivalent to the ib-rMB equations.

We first consider the case of the n-species and then we will take the limit when n goes to infinity. We have $3n + 1$ potentials: $R_1, \ldots, R_n$, $S_1, \ldots, S_n$, $U_1, \ldots, U_n$ and e, which we shall put in a vector

$$\vec{u} = (u_1, \ldots, u_{3n+1}) := (R_1, \ldots, R_n, S_1, \ldots, S_n, U_1, \ldots, U_n, e).$$

We let $\psi_\zeta = X(\vec{u},\psi) := \sum_{i=1}^{3n+1} \alpha_i(\psi) u_i$. Now, $(\psi_\zeta)_\tau = \sum_{i=1}^{3n+1} \alpha_i' T u_i + \alpha_i \frac{\partial u_i}{\partial \tau}$ and $(\psi_\tau)_\zeta = T_\psi \sum_{1=1}^{3n+1} \alpha_i u_i + \sum_{i=1}^{3n+1} \frac{\partial T}{\partial u_i}\frac{\partial u_i}{\partial \zeta}$.

Using the ib-rMB equations we rewrite $(\psi_\zeta)_\tau$,

$$(\psi_\zeta)_\tau = \sum_{i=1}^{3n+1} \alpha_i' T u_i + \sum_{i=n+1}^{2n} \alpha_i \Big(-(\beta-\gamma e) R_{i-n} + \frac{1}{2}\omega_{i-n} U_{i-n}\Big) + \sum_{i=1}^{n} \alpha_i (\beta - \gamma e) S_i + \sum_{i=2n+1}^{3n} \alpha_i(-2\omega_{i-2n} S_{i-2n}) + \alpha_{3n+1}\frac{\partial e}{\partial \tau}\,.$$

In the ib-rMB equations we have no $\zeta-$derivatives of R, S and U and no τ-derivative of e. Thus, the commutation of the flows gives that $a_{3n+1} = 0$, and $\partial T/\partial u_i = 0$ for $i = 1, \ldots, 3n$. Hence, $T = T(e,\psi)$. We take T to be linear in e and without loss of generality we take the coefficient of e to be 1 (if it is not 1 then we divide by the coefficient to make it 1). Therefore, the flows take the form,

$$X(\vec{u},\psi) = \sum_{i=1}^{3n} \alpha_i(\psi) u_i, \qquad T(e,\psi) = e + G_2(\psi).$$

We have,

$$(\psi_\zeta)_\tau = \sum_{i=1}^{3n} \alpha_i'(e + G_2(\psi)) u_i + \sum_{i=1}^{n} \alpha_i(\beta-\gamma e) S_i + \sum_{i=1}^{n} \alpha_{i+n}\Big(-(\beta-\gamma e)R_i + \frac{1}{2}\omega_i U_i\Big) + \sum_{i=1}^{n} \alpha_{i+2n}(-2\omega_i S_i)$$

and

$$(\psi_\tau)_\zeta = G_2' \sum_{1=1}^{3n} \alpha_i u_i + \sum_{i=1}^{n} \omega_i g(\omega_i) S_i d\omega.$$

Equating the two flows and demanding that the R_i, S_i, U_i, eR_i, eS_i, eU_i, $i = 1, \ldots, n$ vanish independently yields the following ordinary differential equations,

$$\begin{aligned}
eU_i:&\quad \alpha_{2n+i}' = 0,\\
eR_i:&\quad \alpha_i' + \alpha_{i+n}\gamma = 0,\\
eS_i:&\quad \alpha_{i+n}' - \gamma\alpha_i = 0,\\
R_i:&\quad \alpha_i'\, G_2 - \beta\alpha_{i+n} = G_2'\, \alpha_i,\\
U_i:&\quad \alpha_{2n+i}'\, G_2 + \frac{1}{2}\alpha_{i+n}\omega_i = G_2'\, \alpha_{2n+i},\\
S_i:&\quad \beta\alpha_i + \alpha_{n+i}'\, G_2 - 2\omega_i\alpha_{i+2n} = G_2'\, \alpha_{n+i} + \omega_i g(\omega_i) d\omega.
\end{aligned}$$

These can be integrated to give $\alpha_i = A_i \cos(\gamma\psi + \phi)$, $\alpha_{n+i} = A_i \sin(\gamma\psi + \phi)$, $\alpha_{2n+i} = c_{2n+i}$, for $i = 1, \ldots, n$ and, $G_2 = -\beta/\gamma + c\cos(\gamma\psi + \phi)$. c_{2n+i}, A_i and c are constants of integration. In total we have $2n + 1$ constants of integration. However, the unused equations are only $2n$. To determine the $2n$ constants we use the equations for the S_i and the U_i to obtain,

$$A_i = \frac{\lambda}{\omega_i^2 + \lambda^2}\omega_i g(\omega_i) d\omega, \quad c_{2n+i} = -\frac{1}{2}\frac{\omega_i^2 g(\omega_i)}{\omega_i^2 + \lambda^2} d\omega, \quad i = 1, \ldots, n,$$

where c was set to $c = \lambda/\gamma$. The two flows take the form,

$$\psi_\tau = e - \frac{\beta}{\gamma} + \frac{\lambda}{\gamma}\cos(\gamma\psi + \phi),$$

$$\psi_\zeta = \sum_{i=1}^{n} \frac{\omega_i g(\omega_i)}{\omega_i^2 + \lambda^2}\Big(\lambda(R_i \cos(\gamma\psi + \phi) + S_i \sin(\gamma\psi + \phi)) - \frac{1}{2}\omega_i U_i\Big) d\omega.$$

Let $x = \gamma\psi + \phi$. For any two functions, $f_1(x)$ and $f_2(x)$, we define the operation $[f_1(x), f_2(x)] := f_1'(x) f_2(x) - f_1(x) f_2'(x)$, where the differentiation is with respect to x. The operation is antisymmetric, bilinear and satisfies the Jacobi identity and consequently defines a Lie bracket. The generating functions appearing in the above flows are 1, $\sin(x)$, $\cos(x)$. Their Lie brackets are computed, $[\sin(x), \cos(x)] = 1$, $[\sin(x), 1] = \cos(x)$, $[\cos(x), 1] = -\sin(x)$. The Lie algebra defined by these brackets is isomorphic to the well known Lie algebra with basis elements

$$\sigma_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad i\sigma_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

that satisfy the following brackets, $[\sigma_z, \sigma_x] = i\sigma_y$, $[\sigma_z, i\sigma_y] = \sigma_x$, $[\sigma_x, i\sigma_y] = -\sigma_z$. Therefore the two Lie algebras are isomorphic via the Lie bracket preserving isomorphism: $\sin(x) \mapsto \sigma_z$, $\cos(x) \mapsto \sigma_x$, $1 \mapsto i\sigma_y$. Therefore the Lax pair takes the form,

$$\psi_\tau = \{(e - \frac{\beta}{\gamma})i\sigma_y + \frac{\lambda}{\gamma}\sigma_x\}\psi,$$

$$\psi_\zeta = \Big\{\sum_{i=1}^{n} \frac{1}{\omega_i^2 + \lambda^2}[\lambda\omega_i g(\omega_i)(R_i\sigma_x + S_i\sigma_z) - \frac{1}{2}\omega_i^2 g(\omega_i) U_i i\sigma_y] d\omega\Big\}\psi.$$

Finally, we take the limit when $n \mapsto \infty$, and write σ_x, σ_z, $i\sigma_y$ in terms of the basis elements of the Lie algebra $\mathfrak{su}(2)$, $\mathcal{H}$, $\mathcal{F}$, $\mathcal{E}$ given as,

$$\mathcal{H} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \mathcal{F} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathcal{E} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

We send $\lambda \mapsto i\lambda$ and then use the Lie algebra map (that preserves the bracket), $\mathcal{H} \mapsto \mathcal{F}$, $\mathcal{F} \mapsto \mathcal{E}$, $\mathcal{E} \mapsto \mathcal{H}$ to obtain,

$$\psi_\tau = \Big\{ \frac{1}{2}\Big(e - \frac{\beta}{\gamma}\Big)\mathcal{E} + \frac{\lambda}{2\gamma}\mathcal{H}\Big\}\, \psi,$$

$$\psi_\zeta = \Big\{ \frac{1}{4}\int_{-\infty}^{\infty} \frac{1}{\omega^2 - \lambda^2}\Big(\lambda 2\omega g(\omega)(R_\omega \mathcal{H} + S_\omega \mathcal{F}) - \omega^2 g(\omega) U_\omega \mathcal{E}\Big) d\omega \Big\}\, \psi,$$

and arrive at the final form of the Lax pair for the ib-rMB equations. Namely, we define two differential operators L and A, that depend on λ, the spectral parameter, such that $[L, A] := LA - AL = 0$ is equivalent to equations (4)–(5),

$$A = -\partial_\tau + Q^{(0)}, \quad L = \partial_\zeta + Q^{(1)},$$

where,

$$Q^{(0)} = \lambda(h_0\mathcal{H} + f_0\mathcal{F}) + e_0\mathcal{E},$$

$$Q^{(1)} = \int_{-\infty}^{\infty} \frac{1}{(\omega^2 - \lambda^2)}\Big(\lambda(h_1\mathcal{H} + f_1\mathcal{F}) + e_1\mathcal{E}\Big) d\omega\,,$$

and

$$\begin{aligned} &h_0 = \frac{1}{2}, \quad h_1 = -\frac{1}{2}\gamma\omega g(\omega), \quad f_0 = 0, \quad f_1 = -\frac{1}{2}\gamma\omega g(\omega) S_\omega R_\omega, \\ &e_0 = -\frac{1}{2}(\beta - \gamma e), \quad e_1 = \frac{1}{4}\gamma\omega^2 g(\omega) U_\omega. \end{aligned} \tag{6}$$

We call $Q^{(0)}$, and $Q^{(1)}$ loop elements due to their close relationship to loop algebras [2] and $h_j,\ f_j,\ e_j$ for $j = 0, 1$, the potentials. We note that the potentials depend on the solutions $e, R_\omega, S_\omega, U_\omega$ of equations (4)–(5). Therefore *a precise description of the loop element is equivalent to having a set of solutions for our system.*

4 Bäcklund Transformation

The existence of a Lax pair for the ib-rMB equations (4)–(5) has lead us to the search of an appropriate Bäcklund transform, that could iteratively produce new solutions of the system. In particular, we consider the eigenvalue problem,

$$\partial_\tau \Psi = Q^{(0)}\Psi, \quad \partial_\zeta \Psi = -Q^{(1)}\Psi.$$

We aim to find a new eigenfunction Ψ and the corresponding new loop elements $Q^{(0)}, Q^{(1)}$ that satisfy the eigenvalue problem. The loop elements are functions of $h_j,\ e_j,\ f_j,\ j = 0, 1$ and will in turn give rise to the new solutions of the ib-rMB system via expressions (6). This transformation theory leads to an analogue of superposition formulas that allows one to construct multisoliton solutions starting from single solitons by purely algebraic means.

We derive a formula for the loop element after n iterations of the BT, call it Q_n, in terms of the previous one, Q_{n-1} (see [1]).

Proposition 1.

$$Q_n(\lambda) = \lambda h_0^{n-1}\mathcal{H} + m_n h_0^{n-1}[\mathcal{H}, N_n\mathcal{H}N_n^{-1}] + e_0^{n-1}\mathcal{E} \tag{7}$$
$$+ \int_{-\infty}^{\infty} \frac{1}{(\omega^2-\lambda^2)} \frac{1}{(\omega^2+m_n^2)} \Big\{\lambda\Big(\omega^2(h_1^{n-1}\mathcal{H} + f_1^{n-1}\mathcal{F})$$
$$-(m_n)^2(N_n\mathcal{H}N_n^{-1})(h_1^{n-1}\mathcal{H} + f_1^{n-1}\mathcal{F})(N_n\mathcal{H}N_n^{-1}) + m_n e_1^{n-1}[\mathcal{E}, N_n\mathcal{H}N_n^{-1}]\Big)$$
$$+ m_n\omega^2\left(h_1^{n-1}[\mathcal{H}, N_n\mathcal{H}N_n^{-1}] + f_1^{n-1}[\mathcal{F}, N_n\mathcal{H}N_n^{-1}]\right)$$
$$+\omega^2 e_1^{n-1}\mathcal{E} - (m_n)^2 e_1^{n-1}(N_n\mathcal{H}N_n^{-1})\mathcal{E}(N_n\mathcal{H}N_n^{-1})\Big\}d\omega.$$

We have taken the specific value of the spectral parameter to be purely imaginary, $\nu_n = im_n \in i\mathbb{R}$, to ensure the reality of the potentials and consequently the solutions e, R_ω, S_ω, U_ω. However, we will demonstrate in section 5, that this condition can be relaxed when we iterate the BT twice (figure 1). Expression (7) gives the loop element at the n-th BT in terms of the potentials of the $(n-1)$ BT, h_0^{n-1}, f_0^{n-1}, e_0^{n-1}, h_1^{n-1}, f_1^{n-1}, e_1^{n-1} and the matrix N_n which can be constructed using data appearing at the $(n-1)$ BT. Therefore, formula (7) iteratively produces the n-soliton potentials for $n \in \mathbb{N}$. The upper indices of the potentials h_0^n, f_0^n, e_0^n, h_1^n, f_1^n, e_1^n, and the lower indices of the loop element Q_n are meant to indicate the level of the Bäcklund transformation. We note that the potentials h_0 and f_0 are invariant under the BT because they are constant quantities and thus $h_0^j = h_0$, $f_0^j = f_0 = 0$, for $j \in \mathbb{N}$. We also note that in the calculations if there is no upper index for the potentials, then it is meant to be 0.

5 Solutions

To initialize the Bäcklund transformation we start with a constant set of solutions to the system. Namely, $e = \beta/\gamma$, $S_\omega = 0$, $U_\omega = 0$ and $R_\omega = R^{\text{init}}$, a nonzero constant. The reader can easily verify that these constitute a set of solutions for equations (4)–(5).

Using Proposition 1 we obtain the one-soliton potentials, which in turn give the one-soliton solutions of the ib-rMB equations (4)–(5):

$$e^{1-\text{sol}}(\zeta,\tau) = \frac{2m_1}{\gamma}\text{sech}(x_1) + \frac{\beta}{\gamma},$$

$$S_\omega^{1-\text{sol}}(\zeta,\tau) = \frac{2m_1^2}{\omega^2+m_1^2}R^{\text{init}}\text{sech}(x_1)\tanh(x_1),$$

$$R_\omega^{1-\text{sol}}(\zeta,\tau) = \frac{1}{\omega^2+m_1^2}R^{\text{init}}(\omega^2 + m_1^2 - 2m_1^2\text{sech}^2(x_1)),$$

$$U_\omega^{1-\text{sol}}(\zeta,\tau) = \frac{-4m_1\omega}{\omega^2+m_1^2}R^{\text{init}}\text{sech}(x_1),$$

where, $x_1 = 2(\alpha_1 \zeta - m_1 h_0 \tau) + \ln(c_1)$,

$$\alpha_1 = \int_{-\infty}^{\infty} \frac{m_1}{\omega^2 + (m_1)^2} h_1 d\omega, \quad h_1 = -\frac{1}{2} \gamma \omega g(\omega) R^{\text{init}}$$

and c_1 is any positive constant.

An iteration of the procedure, which is equivalent to setting $n = 2$ in Proposition 1, produces the two-soliton solutions. In particular, the two-soliton potential e_0^2 takes the form,

$$e_0^2 = 2h_0 \frac{m_1^2 - m_2^2}{m_1^2 + m_2^2} \frac{m_1 \operatorname{sech} x_1 - m_2 \operatorname{sech} x_2}{1 - \dfrac{2m_1 m_2}{m_1^2 + m_2^2} (\tanh x_1 \tanh x_2 - \operatorname{sech} x_1 \operatorname{sech} x_2)},$$

where $x_j = 2(\alpha_j \zeta - m_j h_0 \tau) + \ln(c_j)$ and $\alpha_j = \displaystyle\int_{-\infty}^{\infty} \frac{m_j}{\omega^2 + m_j^2} h_1 d\omega$ for $j = 1, 2$, and $h_1 = -\gamma \omega g(\omega) R^{\text{init}}/2$. The two-soliton electric field, $e^{2-\text{sol}}$, is obtained via $e_0^{2-\text{sol}} = -(\beta - \gamma e^{2-\text{sol}})/2$. We note that for complex values of the spectral parameters, $m_2 = -\overline{m_1}$, the two-soliton solutions remain real (figure 1).

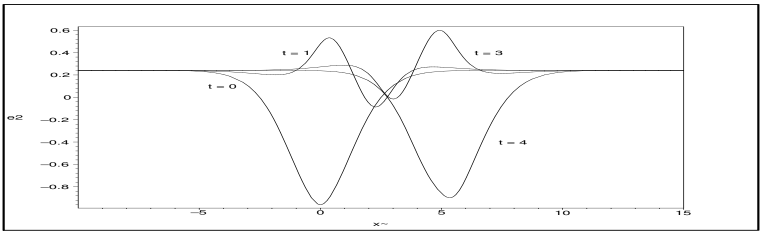

Figure 1. Self-similar propagation of the two-soliton electric field ($m_1 = 1 + 1.5i$, $m_2 = -m_1^*$, $\omega = 1.005$)

6 Discussion of the Solutions

Let us consider a distribution function of the form $g(\omega) = \sigma/(\pi((\omega - \omega_0)^2 + \sigma^2))$. Then, $\alpha_1 = -\gamma R^{\text{init}} m_1 \omega_0 / (2(m_1 + \sigma)^2 + \omega_0^2)$ and the velocity, v, of the one-soliton electric field is, $v = -((m_1 + \sigma)^2 + \omega_0^2)/(\gamma R^{\text{init}} \omega_0)$. The amplitude of the one-soliton electric field is analogous to the value of the spectral parameter m_1, whereas the amplitude of the Bloch fields $R^{1-\text{sol}}$, $S^{1-\text{sol}}$ and $U^{1-\text{sol}}$ depends on the ratio ω/m_1. If we assume that ω is very close to the resonant frequency $\omega_0 = 1$ and set the variance of the distribution to $\sigma = 0.01 = \epsilon$, then with probability 0.98, ω belongs in the interval $(0.5, 1.5)$. Within this range for the transition frequency we have three subcases depending on the amplitude of the electric field: (1a) order ϵ Bloch fields coexist with an order ϵ electric field, (1b) order 1 Bloch fields interact with an order 1 electric field, and (1c) in the large amplitude limit, the dielectric accommodates Bloch fields $S^{1-\text{sol}}$, $R^{1-\text{sol}}$ of order 1 and $U^{1-\text{sol}}$ of order ϵ. A less likely to happen case corresponds to a value of $\omega \in (0, 0.08)$. Here we have: (2a) in the small amplitude limit Bloch fields of order 1 interacting with an electric field of order ϵ and (2b) an electric field of order 1 or higher that occurs when $R^{1-\mathrm{sol}}, S^{1-\mathrm{sol}}$ are of order 1 and $U^{1-\mathrm{sol}}$ is of order ϵ. We note that in the rare case of the small amplitude limit (2a), the contribution of the Bloch fields is significantly bigger than (1a), and has the potential to affect the dynamics of the equations (figure 2).

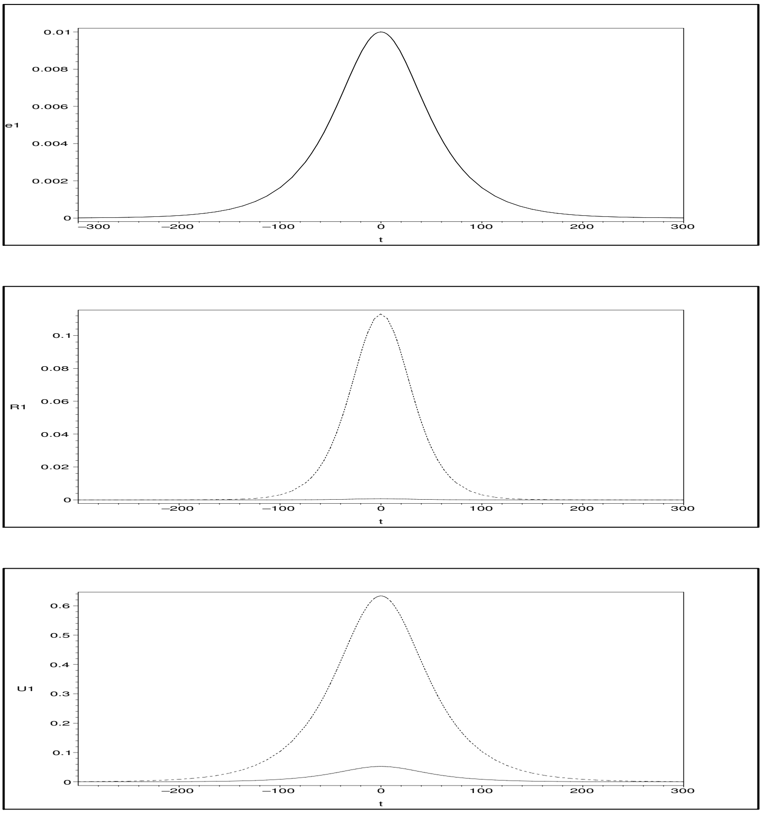

Figure 2. Simultaneous snapshots ($\zeta = 0$) of the one-soliton solutions with $m_1 = 0.025$ (small amplitude limit), for $\omega = 0.95$ (solid) and $\omega = 0.07$ (dotted).

7 Summary

We study the ib-rMB equations that describe the optical pulse propagation through a two-level atom medium. The effects of both inhomogeneous broadening, caused by the Doppler shifting of the resonant frequencies of the atoms, and a permanent dipole are present in the analysis. The equations are integrable and a Lax pair is presented. Using the Lax pair we develop a Bäcklund transformation in Darboux form so that it is easily iterated, and produces analytical expressions for the one- and the two-soliton solutions in terms of elementary functions.

The interplay between the randomly chosen transition frequency and the spectral parameter(s) produces several scenarios, that can be separated in two main categories: the most likely to happen cases that correspond to transition frequency values that are close to the resonant transition frequency value, and the unlikely to happen cases, that correspond to values of the probability function g that are much less than one. We have demonstrated that in the latter case the polarization fields can significantly influence the nonlinear dynamics of the equations.

Different avenues that we are currently exploring are related with an appropriate loop algebra construction. We would like to place the ib-rMB equations in a wider frame, that will enable us to view the system as one among an infinite family of systems in involution with respect to an appropriate Poisson bracket, and reveal an infinite number of conservation laws. We speculate that equations that belong in this hierarchy could possibly be considered as higher order corrections to the ib-rMB equations.

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On a Reaction–Diffusion System Arising in the Development of Bacterial Colonies

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We consider here a reaction diffusion model which arises in the development of swarm colonies of the *Proteus mirabilis*. After having obtained the equivalence transformations admitted by the model we give a classification of two special subclasses of this last one.

1 Introduction

In this paper, we consider an important example of a reaction–diffusion system of partial differential equations arising in the development of bacterial colonies, provided by the *Proteus mirabilis* swarm colonies.

Proteus mirabilis, part of the Enterobacteriaceae family, is a small Gram-negative facultative anerobic bacterium. Commonly, it is part of the normal flora of the human intestinal tract, but can also be found free living in water and soil [1]. *Proteus mirabilis* is an opportunistic pathogen that can colonize the bladder, surgical wounds, lungs and the urinary tract causing severe histological damage. It has the power to shift its shape and often appears in different forms: swarmer cell and swimmer cell. The organism, in fact, goes through a cycle of differentiation, migration and consolidation depending upon the level of nutrients available to it. When the bacillus is a nutrient-rich environment, it exists as swarmer cell, or in swarms. When the nutrients run out the cells form swimmer cells through dividing. Swimmer cells consist virtually exclusively of short oligo-flagellated cells comparable in their behaviour to motile *Escherichia coli*, and go through a prototypical bacterial cell growth and division cycle. However, a second channel of behavior appears: some cells cease septation but continue to grow and produce many lateral flagella to form elongated multinucleoid hyper-flagellated swarmer cells which aggregate in parallel arrays to form elongated motile multicellular *rafts*. Only swarmer cells in contact with other cells are capable of translocation, while swimmer cells and isolated swarmer cells are immobile. Thus, swarm motility is an inherently cooperative process resulting in non-linear transport of the population characterized by expansion dependent on bacterial density. After some time migrating, swarmer cells have been observed to cease movement, septate and produce groups of swimmer cells which can then undergo the typical cell division cycle.

Thus, the colonies of *Proteus mirabilis* are especially interesting because their morphogenesis involves periodic oscillation between phases of migration over the substrate (swarming) and phases of growth within stationary populations (consolidation). Proteus colonies present two key problems:

i to account for their deceptively simple circular symmetry and regular terracing;

ii to explain the robust periodicity of cyclic behaviour under conditions when the velocity and duration of swarming are variable.

Based on experimental observations of cellular differentiation and group motility some models [1–3] have been developed to describe the swarmer cell differentiation — dedifferentiation cycle and the spatial evolution cycle and the spatial evolution of swimmer and swarmer cells during the *Proteus mirabilis* swarm colonies development.

Here we consider the (1+1)-dimensional mathematical model of the *Proteus mirabilis* showed in [2]

$$u_t = \nu v + (\alpha - \mu)u + (D(u,\, v)u_x)_x, \quad v_t = (\alpha - \nu)v + \mu u. \tag{1}$$

The dependent variables $u(t, x)$ and $v(t, x)$ correspond to the surface densities of the populations of swarm cells and swimmer cells, respectively. The coefficients α, μ, ν are density dependent functions and characterize the cellular growth, division and differentiation, respectively. The diffusion appearing in the equation (1) models the migration of swarm cells.

Even though the $(1 + 1)$-model considered is, of course, quite restricted in its ability to demonstrate some observed results as, for instance, pattern formations, in this paper we begin the study of system (1) in the framework of the group analysis.

In particular the search for symmetries gives not only the generators of admitted invariance groups, but, at same time, offers informations about the functional forms of α, μ, ν, D, so that the model allows invariant solutions.

As in the phenomenological theories of continuum media we have in mind to use these results in order to select functional forms of constitutive parameter in agreement with experimental observations.

The plan of the paper is the following. In the section 2 we find, by using the Lie–Ovsiannikov infinitesimal criterion, the algebra of continuous equivalence transformations assuming α, μ, ν, D, arbitrary and without requiring the invariance of the socalled *auxiliary conditions*. In the section 3, we restrict to two special cases which usually can occur in the development of bacterial colonies of *Proteus mirabilis* and obtain a classification of the corresponding models. In the section 4, we verify that, if we require the invariance of the *auxiliary conditions*, in general, we get symmetry algebras smaller than that ones we obtained in the section 3. The conclusions are given in the section 5.

2 On the Equivalence Algebra

Here we look for continuous equivalence transformations for the system (1) whose infinitesimal generator is of the form:

$$Y = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta^1 \frac{\partial}{\partial u} + \eta^2 \frac{\partial}{\partial v} + \phi^1 \frac{\partial}{\partial \alpha} + \phi^2 \frac{\partial}{\partial \mu} + \phi^3 \frac{\partial}{\partial \nu} + \phi^4 \frac{\partial}{\partial D}, \qquad (2)$$

where ξ^1, ξ^2, η^1 and η^2 are sought depending on t, x, u and v, while ϕ^i ($i = 1, 2, 3, 4$) are sought depending on t, x, u, v, α, μ, ν and D. We apply the infinitesimal criterion of Lie–Ovsiannikov [4] by requiring the invariance of the system (1) with respect the suitable prolongation $Y^{(2)}$ of generator (2), that is

$$Y^{(2)}\left[u_t - \nu v - (\alpha - \mu)u - (D(u, v)u_x)_x\right] = 0,$$
$$Y^{(2)}\left[v_t - (\alpha - \nu)v - \mu u\right] = 0$$

under the constraints that variables u and v have to satisfy the Eqs. (1).

At this step, in view of further applications, and as suggested in [4], we do not require the invariance of the *auxiliary conditions* [5–7].

After having solved the *determining system* obtained from invariance conditions, by supposing that $\alpha - \nu \neq 0$ or $\mu \neq 0$, we get the equivalence algebra $\mathcal{L}_\mathcal{E}$ which is infinite-dimensional and is spanned by:

$$Y_0 = \partial_x, \quad Y_1 = x\partial_x + 2D\partial_D, \qquad (3)$$

$$Y_f = f(t)\partial_u + \frac{f' - \alpha f}{u + v}\partial_\alpha + \left[-\frac{f\mu}{u} + \frac{vf\alpha - vf'}{u(u + v)}\right]\partial_\mu, \qquad (4)$$

$$Y_g = ug(t)\partial_u + \frac{ug' + g(\nu v - \mu u)}{u + v}\partial_\alpha + \left[\frac{-u\mu g - vg'}{u + v} + \frac{-v^2 g\nu}{u(u + v)}\right]\partial_\mu, \qquad (5)$$

$$Y_h = h(t)\partial_t - \alpha h_t\partial_\alpha + \left(\frac{\nu v h_t}{u} - \mu h_t\right)\partial_\mu - Dh_t\partial_D, \qquad (6)$$

$$Y_l = l(t, v)\partial_v + \frac{l_t + l_v(\alpha - \nu)v + l_v\mu u - \alpha l}{u + v}\partial_\alpha +$$
$$+ \left[\frac{l_t + l_v(\alpha - \nu)v + l_v\mu u - \alpha l}{u + v} + \frac{\nu l}{u}\right]\partial_\mu, \qquad (7)$$

$$Y_n = \frac{vn(t, x, u, v, \alpha, \mu, \nu, D)}{u}\partial_\mu + n\partial_\nu, \qquad (8)$$

where f, g, h, l and n are arbitrary functions of their arguments.

We have not considered the special cases $\alpha - \nu = 0$ or $\mu = 0$ as they are not compatible with our model.

3 Special Cases

Here we restrict ourself to the following special cases which usually can occur in the development of bacterial colonies of *Proteus mirabilis*:

1. The functions α, μ and ν (cellular growth, division and differentiation) are assigned constants while the diffusion coefficient $D = \tilde{D}(u, v)$ is still considered as an arbitrary function.

2. The diffusion coefficient D has the form $D(u, v) = D_0 u/(u + kv)$ while α, μ and ν are considered arbitrary functions depending only on v.

In both cases in order to get equivalence algebras and symmetries we follow the procedures showed in [5–7].

3.1 Case 1

In this case, α, μ and ν being assigned constants, we must require $\phi_1 = \phi_2 = \phi_3 = 0$ so the equivalence algebra $\mathcal{L}_{\mathcal{E}}$ becomes six-dimensional and it is spanned by:

$$Y_0 = \partial_x, \quad Y_1 = x\partial_x + 2D\partial_D, \quad Y_2 = u\partial_u + v\partial_v, \tag{9}$$

$$Y_3 = \partial_t + (\alpha - \mu)u\partial_u + (\alpha - \mu)v\partial_v, \tag{10}$$

$$Y_4 = \frac{\nu e^{\alpha t}}{\mu + \nu}\partial_u + \frac{\mu e^{\alpha t}}{\mu + \nu}\partial_v, \quad Y_5 = -e^{(\alpha-\mu-\nu)t}\partial_u + e^{(\alpha-\mu-\nu)t}\partial_v. \tag{11}$$

In order to get the symmetries we apply the following theorem [6, 8]

Theorem 1. *Let the infinitesimal equivalence generator $Y = c_i Y_i$ $(i = 0, \ldots, 5)$ for the system considered in the Case 1. The projection of Y in the (t, x, u, v)-space $X = \xi^1\partial_t + \xi^2\partial_x + \eta^1\partial_u + \eta^2\partial_v$ is an infinitesimal symmetry generator if and only if the specializations of the function D are invariant with respect to Y.*

In our case, the invariance of the function $D = \tilde{D}(u, v)$ with respect to operator Y leads to $\phi^4 = \tilde{D}_u\eta^1 + \tilde{D}_v\eta^2$ which, taking (9)–(11) into account, reads

$$\begin{aligned} 2\tilde{D}c_1 = {} & \tilde{D}_u \left[c_4 \frac{\nu e^{\alpha t}}{\mu + \nu} - c_5 e^{(\alpha-\mu-\nu)t} + c_3 u(\alpha - \mu) + c_2 u \right] \\ & + \tilde{D}_v \left[c_4 \frac{\mu e^{\alpha t}}{\mu + \nu} + c_5 e^{(\alpha-\mu-\nu)t} + c_3 v(\alpha - \mu) + c_2 v \right]. \end{aligned}$$

From the previous classifying equation we get easily that the principal Lie algebra $\mathcal{L}_{\mathcal{P}}$ is two-dimensional and spanned by $X_1 = \partial_t$, $X_2 = \partial_x$, while the extensions with respect to $\mathcal{L}_{\mathcal{P}}$ are the following:

1. $D = \tilde{D}(\sigma)$ with $\tilde{D}$ arbitrary function of $\sigma = u+v$: $X_3 = e^{(\alpha-\mu-\nu)t}(-\partial_u+\partial_v)$.

2. $D = (u+v)^h D_0$ where h and D_0 are arbitrary constitutive constants:

$$X_3 = -e^{(\alpha-\mu-\nu)t}\partial_u + e^{(\alpha-\mu-\nu)t}\partial_v, \quad X_4 = \frac{hx}{2}\partial_x + u\partial_u + v\partial_v.$$

3. $D = \tilde{D}(\sigma)$ with $\tilde{D}$ arbitrary function of $\sigma = \mu u - \nu v$: $X_3 = \nu e^{\alpha t}\partial_u + \mu e^{\alpha t}\partial_v$.

4. $D = (\sigma + D_1)^h D_0$ where $\sigma = \mu u - \nu v$, while h, D_0 and D_1 are arbitrary constitutive constants:

$$X_3 = \nu e^{\alpha t}\partial_u + \mu e^{\alpha t}\partial_v,$$
$$X_4 = \frac{hx}{2}\partial_x + \left(u - \frac{D_1 e^{(\alpha-\mu-\nu)t}}{\mu+\nu}\right)\partial_u + \left(v + \frac{D_1 e^{(\alpha-\mu-\nu)t}}{\mu+\nu}\right)\partial_v.$$

5. $D = (u - D_0)^h \tilde{D}(\sigma)$ with $\tilde{D}$ arbitrary function of $\sigma = (u-D_0)/(v+D_0)$, while D_0 and h are arbitrary constitutive constants:

$$X_3 = \frac{hx}{2}\partial_x + \left(u - D_0 e^{(\alpha-\mu-\nu)t}\right)\partial_u + \left(v + D_0 e^{(\alpha-\mu-\nu)t}\right)\partial_v.$$

3.2 Case 2

The coefficient of diffusion D assumes the form $D(u,v) = D_0 u/(u+kv)$ and $\alpha = \tilde{\alpha}(v)$, $\mu = \tilde{\mu}(v)$ and $\nu = \tilde{\nu}(v)$ are arbitrary functions of v.

In this case, by applying the aforsaid procedures, we get that the equivalence algebra $\mathcal{L}_{\mathcal{E}}$ is infinite-dimensional and is spanned by:

$$\begin{aligned} &Y_0 = \partial_x, \quad Y_1 = \partial_t,\\ &Y_2 = 2t\partial_t + x\partial_x - 2\alpha\partial_\alpha + \left(\tfrac{2\nu v}{u} - 2\mu\right)\partial_\mu,\\ &Y_f = uf(t)\partial_u + vf(t)\partial_v + f'(t)\partial_\alpha,\\ &Y_n = \frac{v}{u}n(t,x,u,v,\alpha,\mu,\nu)\partial_\mu + n\partial_\nu, \end{aligned} \tag{12}$$

where f and n are arbitrary functions of their arguments.

Following the Theorem 1, in a similar way, we get easily the following classifying system

$$\phi^1 = \tilde{\alpha}_v \eta^2, \quad \phi^2 = \tilde{\mu}_v \eta^2, \quad \phi^3 = \tilde{\nu}_v \eta^2,$$

which, taking (12) into account, reads

$$f' - 2c_2\tilde{\alpha} = \tilde{\alpha}_v v f, \quad \frac{2c_2(\tilde{\nu}v - \tilde{\mu}u) + vn}{u} = \tilde{\mu}_v v f, \quad n = \tilde{\nu}_v v f.$$

From the previous system it follows that the principal Lie algebra $\mathcal{L}_{\mathcal{P}}$ is two-dimensional and spanned by $X_1 = \partial_t$, $X_2 = \partial_x$, while the extensions with respect to $\mathcal{L}_{\mathcal{P}}$ are the following:

1. $\alpha = \alpha_0 v^{k_0}$, $\mu = \mu_0 v^{k_0}$ and $\nu = \nu_0 v^{k_0}$: $X_3 = -\frac{k_0 x}{2}\partial_x - k_0 t\partial_t + u\partial_u + v\partial_v$.
2. $\alpha = ln\, v^{k_0}$, $\mu = \mu_0$ and $\nu = \nu_0$: $X_3 = e^{k_0 t}u\partial_u + e^{k_0 t}v\partial_v$.

4 A Remark on the Equivalence Transformations

In the previous section we have obtained a classification of two specializations (case 1 and case 2) of the system (1). We have used the equivalence algebras obtained in both cases from the equivalence algebra (3)–(8) by specializing some of the four arbitrary costitutive functions appearing in the system (1).

If we require the invariance of *auxiliary conditions* concerned with the functional dependence of arbitrary functions, i.e.

$$\alpha_t = \alpha_x = \mu_t = \mu_x = \nu_t = \nu_x = D_t = D_x = 0,$$

we get the following subset of the algebra (3)–(8)

$$Y_0 = \partial_x, \quad Y_1 = x\partial_x + 2D\partial_D, \quad Y_2 = \partial_u - \frac{\alpha}{u+v}\partial_\alpha + \left[-\frac{\mu}{u} + \frac{v\alpha}{u(u+v)}\right]\partial_\mu,$$

$$Y_3 = u\partial_u + \frac{\nu v - \mu u}{u+v}\partial_\alpha - \left[\frac{u\mu}{u+v} + \frac{v^2\nu}{u(u+v)}\right]\partial_\mu,$$

$$Y_4 = t\partial_t - \alpha\partial_\alpha + \left(\frac{\nu v}{u} - \mu\right)\partial_\mu - D\partial_D, \quad Y_5 = \partial_t,$$

$$Y_l = l(v)\partial_v + \frac{(\alpha - \nu)vl_v + \mu u l_v - \alpha l}{u+v}\partial_\alpha + \left[\frac{l_v(\alpha - \nu)v + l_v\mu u - \alpha l}{u+v} + \frac{\nu l}{u}\right]\partial_\mu,$$

$$Y_n = \frac{vn(u, v, \alpha, \mu, \nu, D)}{u}\partial_\mu + n\partial_\nu.$$

Then by considering for instance the Case 1, following the above procedures, we get two classifying equations:

- $2\tilde{D}c_1 = c_3 u\tilde{D}_u + c_3 v\tilde{D}_v$ if $\alpha \neq \mu + \nu$;
- $2\tilde{D}c_1 = \tilde{D}_u(c_2 + uc_3) + \tilde{D}_v(c_3 v - c_2)$ if $\alpha = \mu + \nu$.

From which we obtain, only, the following extensions of the principal Lie algebra $\mathcal{L}_\mathcal{P}$:

1. $D = (u - D_0)^h \tilde{D}(\sigma)$ with $\tilde{D}$ arbitrary function of $\sigma = (u - D_0)/(v + D_0)$, while h and D_0 are arbitrary constitutive constants:

$$X_3 = \frac{hx}{2}\partial_x + (u - D_0)\partial_u + (v + D_0)\partial_v.$$

2. $D = (u + v)^h D_0$ where h and D_0 are arbitrary constitutive constants and $\alpha = \mu + \nu$:

$$X_3 = -\partial_u + \partial_v, \quad X_4 = \frac{hx}{2}\partial_x + u\partial_u + v\partial_v.$$

These results verify that the symmetry algebra is smaller than that one we obtained for the case 1 in the previous section.

5 Conclusions

Starting from the search for the equivalence transformations of a general model in the development of bacterial colonies of *Proteus mirabilis*, we have classified two interesting specializations of the model and have obtained several extensions of $\mathcal{L}_{\mathcal{P}}$ algebra concerned with the different functional forms of the constitutive parameters α, μ, ν and D.

These results are, of course, only the first step of our study in this field. Further news about the agreement of the model with experimental observations, will be obtained from the analyses of the solutions of different systems of *reduced equations*. These ones are currently under consideration and will be matter of a future paper.

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Two Important Examples of Nonlinear Oscillators

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We discuss a classical nonlinear oscillator, which is proved to be a superintegrable system for which the bounded motions are quasiperiodic oscillations and the unbounded (scattering) motions are represented by hyperbolic functions. This oscillator can be seen as a position-dependent mass system and we show a natural quantization prescription admitting a factorization with shape invariance for the $n = 1$ case, and then the energy spectrum is found. Other isochronous systems which can also be considered as a generalization of the harmonic oscillator and admit a nonstandard Lagrangian description are also discussed.

1 Introduction

The harmonic oscillator is a system playing a privileged rôle both in classical and quantum mechanics. It is almost ubiquitous in Physics and appears in many physical applications running from condensed matter to semiconductors (see e.g. [1] for references to such problems). The dynamical evolution of the classical system in one dimension is given by

$$\frac{dq}{dt} = v, \quad \frac{dv}{dt} = -\omega^2 q,$$

and admits a Lagrangian formulation with $L = \left(v^2 - \omega^2 q^2\right)/2$, the general solution of the equations of motion being

$$q = q_0 \cos \omega t - \frac{v_0}{\omega} \sin \omega t = A \cos(\omega t + \varphi)$$

and therefore the solutions are periodic with angular frequency ω, while A and φ are arbitrary. This is the main characteristic of the classical system. As far as the quantum system is concerned, the eigenvalues of the Hamiltonian, which is given by $H = (p^2 + \omega^2 q^2)/2$, are equally spaced. We should also remark that the natural extensions to two dimensions, given by $H = (p_x^2 + p_y^2)/2 + (\omega_1^2 x^2 + \omega_2^2 y^2)/2$

admits two constants of motion in involution, $I_1 = E_x = (p_x^2+\omega_1^2 x^2)/2$, $I_2 = E_y = (p_v^2 + \omega_2^2 y^2)/2$, and therefore it is completely integrable in the sense of Liouville. Moreover it has been proved that, when ω_1 and ω_2 are rationally related, i.e. $\omega_1 = n_1\omega_0$, $\omega_2 = n_2\omega_0$, with $n_1, n_2 \in \mathbb{N}$, there exist a new constant of motion and the system is superintegrable. Actually the complex function, $J = K_x^{n_2}(K_y^*)^{n_1}$ with $K_x = p_x + in_1\omega_0\, x$ and $K_y = p_y + in_2\omega_0 y$, is a constant of the motion.

Our aim is to comment on some possible generalizations of this system from the perspective of the theory of the symmetry, i.e. trying to preserve the fundamental symmetry properties.

2 A Position-Dependent Mass Nonlinear Oscillator

A often used generalization was proposed by Mathews and Lakshmanan [2, 3] as a one-dimensional analogue of some models of quantum field theory [4, 5]. It is described by a Lagrangian

$$L = \frac{1}{2}\left(\frac{1}{1+\lambda x^2}\right)(\dot{x}^2 - \alpha^2 x^2), \tag{1}$$

which can be considered as an oscillator with a position-dependent effective mass $m = (1 + \lambda x^2)^{-1}$ (see e.g. [6, 7] and references therein). It was proved that the general solution is also $q(t) = A\cos(\omega t+\varphi)$, but now the amplitude A depends on the frequency. More explicitly $\omega^2(1 + \lambda A^2) = \alpha^2$. Note also that this Lagrangian is of mechanical type, the kinetic term being invariant under the tangent lift of the vector field

$$X_x(\lambda) = \sqrt{1+\lambda x^2}\,\frac{\partial}{\partial x}.$$

It was recently shown in [8] that there is a generalization to n dimensions preserving the symmetry characteristics. In particular the two-dimensional generalization studied in [8] was given by the Lagrangian

$$L(\lambda) = \frac{1}{2}\Big(\frac{1}{1+\lambda r^2}\Big)\Big[v_x^2 + v_y^2 + \lambda(xv_y - yv_x)^2 - \alpha^2 r^2\Big], \quad r^2 = x^2 + y^2, \tag{2}$$

and it was shown to be not only integrable but also superintegrable. This is the only generalization to n dimensions for which the kinetic term is a quadratic function in the velocities that is invariant under rotations and under the two vector fields generalizing the symmetries of the one-dimensional model, i.e.

$$X_1(\lambda) = \sqrt{1+\lambda r^2}\,\frac{\partial}{\partial x}, \quad X_2(\lambda) = \sqrt{1+\lambda r^2}\,\frac{\partial}{\partial y}.$$

This is valid for any value of λ. However, when $\lambda < 0$, $\lambda = -|\lambda|$, this function has a singularity at $1 - |\lambda| r^2 = 0$ and we restrict our dynamics to the interior of the circle $x^2 + y^2 < 1/|\lambda|$.

These two vector fields close with the generator of rotations, $X_J = x\partial/\partial y - y\partial/\partial x$, on a Lie algebra

$$[X_1(\lambda), X_2(\lambda)] = \lambda X_J, \quad [X_1(\lambda), X_J] = X_2(\lambda), \quad [X_2(\lambda), X_J] = -X_1(\lambda).$$

which is isomorphic either to $SO(3,\mathbb{R})$, when $\lambda > 0$, or to $SO(2,1)$, when $\lambda < 0$, or finally to the Euclidean group in two dimensions when $\lambda = 0$.

The important property shown in [8] is that this bidimensional nonlinear harmonic oscillator is completely integrable, because one can show that, if K_1 and K_2 are the functions

$$K_1 = P_1(\lambda) + i\alpha\,\frac{x}{\sqrt{1+\lambda r^2}}, \quad K_2 = P_2(\lambda) + i\alpha\,\frac{y}{\sqrt{1+\lambda r^2}},$$

with

$$P_1(\lambda) = \frac{v_x - \lambda J y}{\sqrt{1+\lambda r^2}}, \quad P_2(\lambda) = \frac{v_y + \lambda J x}{\sqrt{1+\lambda r^2}}, \quad J = x v_y - y v_x,$$

then the complex functions K_{ij} defined as $K_{ij} = K_i\,K_j^*$, $i,j = 1,2$, are constants of motion. In fact the time-evolution of the functions K_1 and K_2 is

$$\frac{d}{dt}\,K_1 = \frac{i\alpha}{1+\lambda\,r^2}\,K_1, \quad \frac{d}{dt}\,K_2 = \frac{i\alpha}{1+\lambda\,r^2}\,K_2\,,$$

from which we see that the complex functions K_{ij} are constants of the motion. Therefore the system is superintegrable with the following first integrals of motion

$$I_1(\lambda) = |\,K_1\,|^2\,, \quad I_2(\lambda) = |\,K_2\,|^2\,, \quad I_3 = \Im(K_{12}) = \alpha\,(x v_y - y v_x)\,.$$

The Legendre transformation for a two-dimensional Lagrangian system of mechanical type with kinetic term as in (2) is given by

$$p_x = \frac{(1+\lambda\,y^2)v_x - \lambda\,xyv_y}{1+\lambda\,r^2}\,, \quad p_y = \frac{(1+\lambda\,x^2)v_y - \lambda\,xyv_x}{1+\lambda\,r^2}\,,$$

(note that $xp_y - yp_x = xv_y - yv_x$) and the general expression for the corresponding λ-dependent Hamiltonian is

$$H(\lambda) = \frac{1}{2}\left[p_x^2 + p_y^2 + \lambda\,(xp_x + yp_y)^2\right] + \frac{1}{2}\,\alpha^2\,V(x,y)\,, \tag{3}$$

and hence the associated Hamilton–Jacobi equation is

$$\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \lambda\left(x\,\frac{\partial S}{\partial x} + y\,\frac{\partial S}{\partial y}\right)^2 + \alpha^2\,V(x,y) = 2\,E\,. \tag{4}$$

This equation is not separable in (x,y) coordinates, but it was shown in [8] that there exist three particular systems of orthogonal coordinates, and three particular families of associated potentials, in which such Hamiltonians admit a Hamilton–Jacobi separability. The first system of coordinates is given by

$$(z_x, y)\,, \qquad z_x = \frac{x}{\sqrt{1+\lambda\,y^2}}\,, \tag{5}$$

for which the Hamilton–Jacobi equation becomes:

$$(1+\lambda\,z_x^2)\left(\frac{\partial S}{\partial z_x}\right)^2 + (1+\lambda\,y^2)^2\left(\frac{\partial S}{\partial y}\right)^2 + \alpha^2\,(1+\lambda\,y^2)V = 2(1+\lambda\,y^2)E,$$

and therefore the Hamilton–Jacobi equation is separable if the potential $V(x,y)$ can be written on the form

$$V = \frac{W_1(z_x)}{1+\lambda\,y^2} + W_2(y). \tag{6}$$

The potential is therefore integrable with the following two quadratic integrals of motion

$$\begin{aligned} I_1(\lambda) &= (1+\lambda\,r^2)p_x^2 + \alpha^2 W_1(z_x)\,, \\ I_2(\lambda) &= (1+\lambda\,r^2)p_y^2 - \lambda\,J^2 + \alpha^2\left(W_2(y) - \frac{\lambda\,y^2}{1+\lambda\,y^2}\,W_1(z_x)\right). \end{aligned}$$

In a similar way, one can see, using coordinates (x, z_y) with $z_y = y(1+\lambda\,x^2)^{-1/2}$, that the Hamilton–Jacobi equation is separable when the potential $V(x,y)$ is of the form

$$V = W_1(x) + \frac{W_2(z_y)}{1+\lambda\,x^2}\,. \tag{7}$$

and the potential is integrable with the following two quadratic first integrals:

$$\begin{aligned} I_1(\lambda) &= (1+\lambda\,r^2)p_x^2 - \lambda\,J^2 + \alpha^2\Big(W_1(x) - \frac{\lambda\,x^2}{1+\lambda\,x^2}\,W_1(z_y)\Big), \\ I_2(\lambda) &= (1+\lambda\,r^2)p_y^2 + \alpha^2 W_2(z_y). \end{aligned}$$

Finally using polar coordinates (r,ϕ) the Hamiltonian $H(\lambda)$ is written

$$H(\lambda) = \frac{1}{2}\left[(1+\lambda\,r^2)p_r^2 + \frac{p_\phi^2}{r^2}\right] + \frac{\alpha^2}{2}\,V(r,\phi) \tag{8}$$

and the Hamilton–Jacobi equation is given by

$$(1+\lambda\,r^2)\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \phi}\right)^2 + \alpha^2\,V(r,\phi) = 2\,E\,.$$

Then the equation admits separability when the potential V is of the form

$$V = F(r) + G(\phi)/r^2. \tag{9}$$

Such a potential V is integrable with the following two quadratic first integrals:

$$\begin{aligned} I_1(\lambda) &= (1+\lambda\, r^2)p_r^2 + \frac{1-r^2}{r^2}\, p_\phi^2 + \alpha^2 \left[F(r) + \frac{1-r^2}{r^2}\, G(\phi) \right], \\ I_2(\lambda) &= p_\phi^2 + \alpha^2 G(\phi). \end{aligned}$$

Consequently, the potential

$$V = \frac{\alpha^2}{2} \Big(\frac{x^2+y^2}{1+\lambda\,(x^2+y^2)} \Big)$$

is super-separable since it is separable in three different systems of coordinates (z_x, y), (x, z_y), and (r, ϕ) because

$$\begin{aligned} V &= \frac{\alpha^2}{2} \left(\frac{1}{1+\lambda\, y^2} \right) \left[\frac{z_x^2}{1+\lambda\, z_x^2} + y^2 \right] = \frac{\alpha^2}{2} \left(\frac{1}{1+\lambda\, x^2} \right) \left[x^2 + \frac{z_y^2}{1+\lambda\, z_y^2} \right] \\ &= \frac{\alpha^2}{2} \left(\frac{r^2}{1+\lambda\, r^2} \right). \end{aligned}$$

3 The One-Dimensional Quantum Nonlinear Oscillator

We now consider the quantum case and restrict ourselves to the one-dimensional case. The first problem is to define the quantum operator describing the Hamiltonian of this position-dependent mass system, because the mass function and the momentum P do not commute and this fact gives rise to an ambiguity in the ordering of factors. It has recently been proposed to avoid the problem by modifying the Hilbert space of functions describing the system [9]. More explicitly we can consider the measure $d\mu = (1+\lambda\, x^2)^{-1/2}\, dx$, which is invariant under the vector field $X_x(\lambda) = \sqrt{1+\lambda\, x^2}\, \partial/\partial x$, for then the operator $P = -i\sqrt{1+\lambda\, x^2}\, \partial/\partial x$ is selfajdoint in the space $L^2(\mathbb{R}, d\mu)$. In the case of the nonlinear oscillator in which we are interested we can consider the Hamiltonian operator

$$\widehat{H}_1 = -\frac{1}{2}\,(1+\lambda\, x^2)\,\frac{d^2}{dx^2} - \frac{1}{2}\,\lambda\, x\,\frac{d}{dx} + \frac{1}{2}\,\frac{\alpha^2\, x^2}{1+\lambda\, x^2}\,. \tag{10}$$

The spectral problem of such operator can be solved by means of algebraic techniques. We first remark that if β is such that $\alpha^2 = \beta(\beta+\lambda)$, then $\widehat{H}_1' = \widehat{H}_1 - \beta/2$ can be factorized as a product $\widehat{H}_1' = A^\dagger(\beta)\, A(\beta)$ and

$$A = \frac{1}{\sqrt{2}} \left(\sqrt{1+\lambda\, x^2}\, \frac{d}{dx} + \frac{\beta\, x}{\sqrt{1+\lambda\, x^2}} \right), \tag{11}$$

for which its adjoint operator is

$$A^\dagger = \frac{1}{\sqrt{2}} \left(-\sqrt{1+\lambda\, x^2}\, \frac{d}{dx} + \frac{\beta\, x}{\sqrt{1+\lambda\, x^2}} \right). \tag{12}$$

The important point is that the partner Hamiltonian $\widehat{H}'_2 = A(\beta)\,A^\dagger(\beta)$ is related to $\widehat{H}'_1$ by $\widehat{H}'_2(\beta) = \widehat{H}'_1(\beta_1) + R(\beta_1)$ with $\beta_1 = f(\beta)$ and where f and R are the functions $f(\beta) = \beta - \lambda$ and $R(\beta) = \beta + 1/2$. Hamiltonians admitting such factorization [10] and related to its s partner in such a way are said to be shape invariant and their spectra and the corresponding eigenvectors can be found by using the method proposed by Gendenshteïn [11, 12] (see also [13] for a modern presentation based on the Riccati equation). Therefore, as the quantum nonlinear oscillator is shape invariant, we can develop the method proposed in [11, 12] for finding both the spectrum and the corresponding eigenvectors. The spectrum is given by [9] $E_n = n\,\beta - n^2\lambda/2 + \beta/2$. The existence of a finite or infinite number of bound states depends up on the sign of λ as also discussed in [9].

4 Periodic Motions and Another Nonlinear Oscillator

Another possible generalization of the harmonic oscillator would be to look for alternative isochronous systems. For instance one can consider a potential

$$U(x) = \begin{cases} U_1(x) & \text{if } x < 0, \\ U_2(x) & \text{if } x > 0\,, \end{cases}$$

where $U_2(x)$ is an increasing function and $U_1(x)$ is a decreasing function, and try to determine the explicit functions U_1 and U_2 in order to have an isochronous system. The problem of the determination of the potential when the period is known as a function of the energy was solved by Abel [14]. When the potential is symmetric the solution is unique. Therefore the only symmetric potential giving rise to isochronous motions around the origin is the harmonic oscillator. The isotonic oscillator is also symmetric and isochronous, but the origin is a singular point and not a minimum of the potential. Other nonsymmetric potentials can be used, for instance a potential given by

$$U_1(x) = \omega_1^2\,x^2, \quad U_2(x) = \omega_2^2\,x^2.$$

If we want to find more general solutions for the symmetric case we may consider Lagrangians of a nonstandard mechanical type, in which there is no potential term. These more general Lagrangians can also be relevant in other problems. For instance another interesting oscillator-like system has recently been studied by Chandrasekar *et al* [15]. As mentioned in that paper the oscillator-like system admits a Lagrangian formulation. We recall that there are systems admitting a Lagrangian formulation of a nonmechanical type. As an example we can consider $Q = \mathbb{R}$ as the configuration space and the Lagrangian function [16]

$$L(x, v) = (\alpha(x)\,v + U(x))^{-1}\,, \tag{13}$$

which is singular in the zero level set of the function $\varphi(x, v) = \alpha(x)\,v + U(x)$. Then the Euler–Lagrange equation is

$$\alpha'(x)\,v + U'(x) - \alpha'(x)\,v = -\frac{2\alpha(x)[\alpha'(x)v^2 + U'(x)v + \alpha(x)a]}{\alpha(x)v + U(x)}\,,$$

where v and a denote the velocity and the acceleration, respectively. This is a conservative system the equation of motion of which can be rewritten as

$$[\alpha(x)]^2\,\ddot{x} + \alpha(x)\,\alpha'(x)\,\dot{x}^2 + \frac{3}{2}\,\alpha(x)\,U'(x)\,\dot{x} + \frac{1}{2}\,U(x)\,U'(x) = 0.$$

The energy is given by $E_L(x,v) = -[2\,\alpha(x)\,v + U(x)][\alpha(x)\,v + U(x)]^{-2}$. In particular, when $\alpha(x) = 1$, the Lagrangian is $L(x,v) = [v + U(x)]^{-1}$ and the Euler–Lagrange equation reduces to

$$\ddot{x} + \frac{3}{2}\,U'(x)\,\dot{x} + \frac{1}{2}\,U(x)\,U'(x) = 0 \tag{14}$$

and the energy function turns out to be $E_L(x,v) = -[2\,v + U(x)][v + U(x)]^{-2}$. When $U(x) = k\,x^2$, the equation is

$$\ddot{x} + 3\,k\,x\,\dot{x} + k^2\,x^3 = 0,$$

and the energy is $E_L = -[2\,v + k\,x^2][v + k\,x^2]^{-1}$. It can be seen from the energy conservation law that the general solution is

$$x = \frac{2\,t}{k\,t^2 - E}\,.$$

The two-dimensional system described by $L(x,y,v_x,v_y) = [v_x + k_1\,x^2]^{-1} + [v_y + k_2\,y^2]^{-1}$ is superintegrable. Actually not only the energies of each degree of freedom are conserved but also the functions [16]

$$I_3 = \frac{x}{v_x + k_1\,x^2} - \frac{y}{v_y + k_2\,y^2},$$
$$I_4 = \frac{k_2}{v_x + k_1\,x^2} + \frac{k_1}{v_y + k_2\,y^2} - \frac{k_1\,k_2\,x\,y}{(v_x + k_1\,x^2)(v_y + k_2\,y^2)}.$$

Another example is that of a nonlinear oscillator for which we were looking. The following Lagrangian depending on the parameter ω

$$L(x,v;\omega) = \frac{1}{k\,v_x + k^2\,x^2 + \omega^2}, \tag{15}$$

produces the nonlinear Euler–Lagrange equation $\ddot{x}+3\,k\,x\,\dot{x}+k^2\,x^2+\omega^2 x = 0$, which is the nonlinear oscillator system recently studied by Chandrasekar *et al* [15], and the energy is $E_L = -[2\,k\,v_x + k^2\,x^2 + \omega^2][(k\,v_x + k^2\,x^2 + \omega^2)]^{-2}$. The general solution for the dynamics, which can be found from the energy conservation, is

$$x = \frac{\omega\,\sqrt{E}\,\sin(\omega t + \phi)}{1 - k\,\sqrt{E}\,\cos(\omega t + \phi)}.$$

We have recently been able to prove [16] that in the rational case of the two-dimensional problem, for which $\omega_1 = n_1\,\omega_0$ and $\omega_2 = n_2\,\omega_0$, the system is superintegrable as it was the case for the harmonic oscillator. To introduce the additional constants of motion we define

$$\mathbb{K}_1 = \frac{v_x + k_1\,x^2 + i\,n_1\,\omega_0\,x}{k_1\,v_x + k_1^2\,x^2 + n_1^2\,\omega_0^2}\,, \quad \mathbb{K}_2 = \frac{v_y + k_2\,y^2 + i\,n_2\,\omega_0\,y}{k_2\,v_y + k_2^2\,y^2 + n_2^2\,\omega_0^2}\,, \tag{16}$$

and then the complex function $\mathbb{K}_1^{n_2}\,(\mathbb{K}_2^*)^{n_1}$ is a constant of the motion.

In summary, not only position-dependent mass generalizations of the harmonic oscillator can be interesting but there exist also systems described by Lagrangians of non-mechanical type which preserve the property of superintegrability for the harmonic oscillator with rationally related frequencies. This example points out the importance of the study of such non-standard Lagrangians.

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Special Polynomials Associated with Rational Solutions of the Fourth Painlevé Equation and the Defocusing Nonlinear Schrödinger Equation

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Special polynomials associated with rational solutions of the fourth Painlevé equation and with rational and rational-oscillatory solutions of the defocusing nonlinear Schrödinger equation are studied. It is demonstrated that the roots of these polynomials have regular, symmetric structure in the complex plane.

1 Introduction

In this paper our interest is in special polynomials associated with rational solutions of the fourth Painlevé equation ($\mathrm{P_{IV}}$)

$$w'' = \frac{(w')^2}{2w} + \frac{3}{2}w^3 + 4zw^2 + 2(z^2-\alpha)w + \frac{\beta}{w}, \tag{1}$$

where $' \equiv \mathrm{d}/\mathrm{d}z$, with α and β arbitrary constants, and special polynomials associated with rational and rational-oscillatory solutions of the defocusing nonlinear Schrödinger (NLS) equation

$$\mathrm{i}u_t = u_{xx} - 2|u|^2u, \tag{2}$$

which is a soliton equation solvable by inverse scattering [53].

The six Painlevé equations ($\mathrm{P_I}$–$\mathrm{P_{VI}}$), were discovered by Painlevé, Gambier and their colleagues whilst studying which second order ordinary differential equations (ODEs) have the property that the solutions have no movable branch points; now known as the *Painlevé property* [28, Chap. 14]. The general solutions of the Painlevé equations are transcendental in the sense that they cannot be expressed in terms of known elementary functions and so require the introduction of new transcendental functions. Further the Painlevé equations can be thought of as nonlinear analogues of the classical special functions (cf. [11, 29, 49]). However $\mathrm{P_{II}}$–$\mathrm{P_{VI}}$ have rational solutions, algebraic solutions, and solutions expressed in terms of the classical special functions (see, e.g. [4,6,18–20,25,31,34–37,39–42,46] and the references therein).

Vorob'ev [51] and Yablonskii [52] expressed the rational solutions of $\mathrm{P_{II}}$ in terms of certain special polynomials, the *Yablonskii–Vorob'ev polynomials*. Okamoto [39] obtained analogous special polynomials related to rational solutions of $\mathrm{P_{IV}}$, which were generalised by Noumi and Yamada [38] (see Section 2). Clarkson and Mansfield [16] investigated the locations of the roots of the Yablonskii–Vorob'ev polynomials in the complex plane and showed that these roots have a very regular, approximately triangular structure. The structure of the roots of the special polynomials associated with rational solutions of $\mathrm{P_{IV}}$ in [13], which either have an approximate rectangular structure and or are a combination of approximate rectangular and triangular structures.

Ablowitz and Segur [2] demonstrated a relationship between the Painlevé equations and completely integrable partial differential equations solvable by inverse scattering, the *soliton equations*, such as the Korteweg-de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0, \tag{3}$$

and the NLS equation (2). Airault, McKean and Moser [5] studied the motion of the poles of rational solutions of the KdV equation (3) and a related many-body problem; see also [1, 3, 10]. Subsequently there have been studies of the motion of poles of rational solutions of other soliton equations, e.g. the Boussinesq equation [23], the classical Boussinesq system [45], the Kadomtsev-Petviashvili equation [43, 44] and the NLS equation [26, 27].

This paper is organised as follows. In Section 2 we discuss the special polynomials associated with rational solutions of $\mathrm{P_{IV}}$ (1). In Section 3 we discuss the special polynomials associated with rational and rational-oscillatory solutions of the NLS equation equation (2). In Section 4 we discuss our results.

2 Special Polynomials Associated with Rational Solutions of $\mathrm{P_{IV}}$

Rational solutions of $\mathrm{P_{IV}}$ (1) are summarized in the following theorem.

Theorem 1. $\mathrm{P_{IV}}$ *has rational solutions if and only if the parameters* (α, β) *are given by either of the following*

$$\alpha = m, \quad \beta = -2(2n - m + 1)^2, \tag{4}$$

$$\alpha = m, \quad \beta = -2(2n - m + \tfrac{1}{3})^2, \tag{5}$$

with $m, n \in \mathbb{Z}$*. Further the rational solutions for these parameters are unique.*

Proof. See Lukashevich [32], Gromak [24] and Murata [36]; also [6, 25, 50]. ∎

Simple rational solutions of $\mathrm{P_{IV}}$ are

$$w_1(z; \pm 2, -2) = \pm 1/z, \quad w_2(z; 0, -2) = -2z, \quad w_3(z; 0, -\frac{2}{9}) = -\frac{2}{3}z. \tag{6}$$

It is known that there are three sets of rational solutions of $\mathrm{P_{IV}}$, which have the solutions (6) as the simplest members. These sets are known as the "$-1/z$ hierarchy", the "$-2z$ hierarchy" and the "$-2z/3$ hierarchy", respectively (cf. [6]). The "$-1/z$ hierarchy" and the "$-2z$ hierarchy" form the set of rational solutions of $\mathrm{P_{IV}}$ with parameters given by (4) and the "$-2z/3$ hierarchy" forms the set with parameters given by (5). The rational solutions of $\mathrm{P_{IV}}$ with parameters given by (4) lie at the vertexes of the "Weyl chambers" and those with parameters given by (5) lie at the centres of the "Weyl chambers" [50].

In a comprehensive study of $\mathrm{P_{IV}}$, Okamoto [39] defined two sets of polynomials associated with rational solutions of $\mathrm{P_{IV}}$, analogous to the Yablonskii–Vorob'ev polynomials. Noumi and Yamada [38] generalized Okamoto's results and introduced the *generalized Hermite polynomials* $H_{m,n}(z)$, defined in Theorem 2, and the *generalized Okamoto polynomials* $Q_{m,n}(z)$, defined in Theorem 3; see also [13].

Theorem 2. *Suppose $H_{m,n}(z)$ satisfies the recurrence relations*

$$\begin{aligned} 2mH_{m+1,n}H_{m-1,n} &= H_{m,n}H''_{m,n} - \left(H'_{m,n}\right)^2 + 2mH^2_{m,n}, \\ 2nH_{m,n+1}H_{m,n-1} &= -H_{m,n}H''_{m,n} + \left(H'_{m,n}\right)^2 + 2nH^2_{m,n}, \end{aligned} \tag{7}$$

with $H_{0,0} = H_{1,0} = H_{0,1} = 1$ and $H_{1,1} = 2z$, then

$$w^{(i)}_{m,n} = -\frac{\mathrm{d}}{\mathrm{d}z}\left\{\ln\left(\frac{H_{m,n+1}}{H_{m,n}}\right)\right\} \equiv -2m\frac{H_{m+1,n}\,H_{m-1,n+1}}{H_{m,n+1}\,H_{m,n}}, \tag{8a}$$

$$w^{(ii)}_{m,n} = \frac{\mathrm{d}}{\mathrm{d}z}\left\{\ln\left(\frac{H_{m+1,n}}{H_{m,n}}\right)\right\} \equiv 2n\frac{H_{m,n+1}\,H_{m+1,n-1}}{H_{m+1,n}\,H_{m,n}}, \tag{8b}$$

where $w^{(j)}_{m,n} = w(z;\alpha^{(j)}_{m,n},\beta^{(j)}_{m,n})$, for $j = i, ii$, are solutions of $\mathrm{P_{IV}}$, respectively, for the parameters $(\alpha^{(i)}_{m,n},\beta^{(i)}_{m,n}) = (-(m+2n+1),-2m^2)$ and $(\alpha^{(ii)}_{m,n},\beta^{(ii)}_{m,n}) = (2m+n+1,-2n^2)$.

Proof. See Noumi and Yamada [38]; also [13]. ■

The rational solutions of $\mathrm{P_{IV}}$ defined by (8) include all solutions in the "$-1/z$" and "$-2z$" hierarchies, i.e. the set of rational solutions of $\mathrm{P_{IV}}$ with parameters given by (4), and can be expressed in terms of determinants whose entries are Hermite polynomials [30, 38]. These rational solutions of $\mathrm{P_{IV}}$ are special cases of the special function solutions which are expressible in terms of parabolic cylinder functions $D_\nu(\xi)$ (cf. [13]). Each polynomial $H_{m,n}(z)$ has degree mn with integer coefficients [38]; in fact $H_{m,n}(\zeta/2)$ is a monic polynomial in ζ with integer coefficients. Examples of generalized Hermite polynomials and plots of the locations of their roots are given in [13]. Plots of the roots of $H_{6,6}(z)$ and $H_{7,7}(z)$ are given in Figure 1.

Next we consider the generalized Okamoto polynomials.

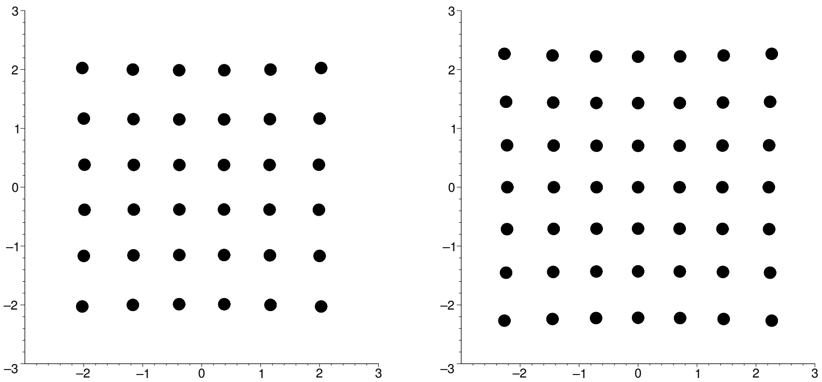


Figure 1. Roots of the generalized Hermite polynomials $H_{6,6}(z)$ and $H_{7,7}(z)$

Theorem 3. *Suppose $Q_{m,n}(z)$ satisfies the recurrence relations*

$$Q_{m+1,n}Q_{m-1,n} = \frac{9}{2}\left[Q_{m,n}Q''_{m,n} - \left(Q'_{m,n}\right)^2\right] + \left[2z^2 + 3(2m+n-1)\right]Q^2_{m,n}, \tag{9a}$$

$$Q_{m,n+1}Q_{m,n-1} = \tfrac{9}{2}\left[Q_{m,n}Q''_{m,n} - \left(Q'_{m,n}\right)^2\right] + \left[2z^2 + 3(1-m-2n)\right]Q^2_{m,n}, \tag{9b}$$

with $Q_{0,0} = Q_{1,0} = Q_{0,1} = 1$ and $Q_{1,1} = \sqrt{2}\,z$, then

$$\widetilde{w}^{(i)}_{m,n} = -\frac{2z}{3} - \frac{\mathrm{d}}{\mathrm{d}z}\left\{\ln\left(\frac{Q_{m,n+1}}{Q_{m,n}}\right)\right\} \equiv -\frac{\sqrt{2}}{3}\,\frac{Q_{m+1,n}\,Q_{m-1,n+1}}{Q_{m,n+1}\,Q_{m,n}}, \tag{10a}$$

$$\widetilde{w}^{(ii)}_{m,n} = -\frac{2z}{3} + \frac{\mathrm{d}}{\mathrm{d}z}\left\{\ln\left(\frac{Q_{m+1,n}}{Q_{m,n}}\right)\right\} \equiv -\frac{\sqrt{2}}{3}\,\frac{Q_{m,n+1}\,Q_{m+1,n-1}}{Q_{m+1,n}\,Q_{m,n}}, \tag{10b}$$

where $\widetilde{w}^{(j)}_{m,n} = w(z;\widetilde{\alpha}^{(j)}_{m,n},\widetilde{\beta}^{(j)}_{m,n})$, for $j = i, ii$, are solutions of $\mathrm{P_{IV}}$*, respectively, for the parameters $(\alpha^{(i)}_{m,n},\beta^{(i)}_{m,n}) = (-(m+2n), -2(m-\frac{1}{3})^2)$ and $(\alpha^{(ii)}_{m,n},\beta^{(ii)}_{m,n}) = (2m+n, -2(n-\frac{1}{3})^2)$.*

Proof. See Noumi and Yamada [38]; also [13]. ■

The rational solutions of $\mathrm{P_{IV}}$ defined by (10) include all solutions in the "$-2z/3$" hierarchy, i.e. the set of rational solutions of $\mathrm{P_{IV}}$ with parameters given by (5), which also can be expressed in the form of determinants [30, 38]. Each polynomial $Q_{m,n}(z)$ has degree $d_{m,n} = m^2 + n^2 + mn - m - n$ with integer coefficients [38]; in fact $Q_{m,n}(\zeta/\sqrt{2})$ is a monic polynomial in ζ with integer coefficients. Examples of generalized Okamoto polynomials and plots of the locations of their roots are given in [13]. Plots of the roots of $Q_{6,6}(z)$ and $Q_{7,7}(z)$ are given in Figure 2.

Next we express rational solutions of the ODE satisfied by the Hamiltonian for $\mathrm{P_{IV}}$ in terms of $H_{m,n}(z)$ and $Q_{m,n}(z)$.

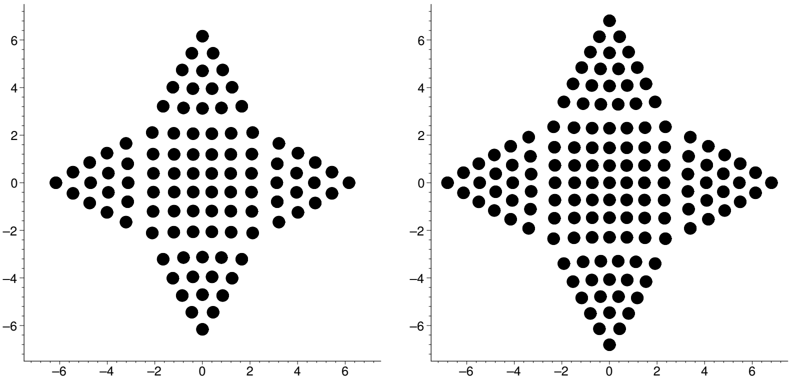


Figure 2. Roots of the generalized Okamoto polynomials $Q_{6,6}(z)$ and $Q_{7,7}(z)$

Example 1. The Hamiltonian for P_{IV} is [39] $\mathcal{H}_{IV}(q,p,z;\theta_0,\theta_\infty) = 2qp^2 - (q^2 + 2zq + 2\theta_0)p + \theta_\infty q$, then from Hamilton's equation we have

$$q' = 4qp - q^2 - 2zq - 2\theta_0, \quad p' = -2p^2 + 2pq + 2zp - \theta_\infty. \tag{11}$$

Eliminating p in (11), shows that $q = w$ satisfies P_{IV} with parameter values $(\alpha,\beta) = \left(1-\theta_0+2\theta_\infty, -2\theta_0^2\right)$, and eliminating q in (11), gives that $w = -2p$ satisfies P_{IV} with $(\alpha,\beta) = \left(-1+2\theta_0-\theta_\infty, -2\theta_\infty^2\right)$. The Hamiltonian function $\sigma(z;\theta_0,\theta_\infty) = \mathcal{H}_{IV}(q,p,z;\theta_0,\theta_\infty)$ satisfies

$$\left(\sigma''\right)^2 = 4\left(z\sigma'-\sigma\right)^2 - 4\sigma'\left(\sigma'+2\theta_0\right)\left(\sigma'+2\theta_\infty\right). \tag{12}$$

This is equivalent to equation SD-I.c in the classification of second-order, second-degree ODEs with the Painlevé property due to Cosgrove and Scoufis [17], an equation first derived and solved by Chazy [9] and rederived by Bureau [8]. Further (12) arises in various applications including random matrix theory (cf. [21, 48]). It is shown in [14] that rational solutions of (12) have the form

$$\sigma_{m,n}(z;-n,m) = H'_{m,n}(z)/H_{m,n}(z), \tag{13a}$$

$$\widetilde{\sigma}_{m,n}(z;-n+\frac{1}{3},m-\frac{1}{3}) = \frac{4}{27}z^3 - \frac{2}{3}(m-n)z + Q'_{m,n}(z)/Q_{m,n}(z). \tag{13b}$$

Using this Hamiltonian formalism, it is shown in [14] that $H_{m,n}(z)$ and $Q_{m,n}(z)$, which are defined by differential-difference equations (7) and (9) respectively, also satisfy fourth order bilinear ODEs and homogeneous difference equations.

3 Rational and Rational-Oscillatory Solutions of the Nonlinear Schrödinger Equation

The NLS equation (2) has the scaling reduction $u(x,t) = t^{-1/2}R(\zeta)\exp\{\mathrm{i}\Theta(\zeta)\}$, $\zeta = x/\sqrt{t}$, where $R(z)$ and $\Theta(\zeta)$ satisfy

$$R'' - R\left(\Theta'\right)^2 = \frac{1}{2}R\zeta\Theta' + 2R^3, \quad 2R'\Theta' + R\Theta'' + \frac{1}{2}\zeta R' + \frac{1}{2}R = 0 \tag{14}$$

(see [7, 22, 33] for details). Multiplying (14b) by R and integrating yields

$$\Theta'(\zeta) = -\frac{1}{4}\zeta + \frac{C}{R^2(\zeta)} - \frac{1}{4R^2(\zeta)}\int^{\zeta} R^2(s)\,\mathrm{d}s,$$

with C an arbitrary constant. Substituting this into (14a) and setting $V(\zeta) = \int^{\zeta} R^2(s)\,\mathrm{d}s - 4C$ yields a third-order equation which has first integral

$$\left(V''\right)^2 = -\left(V - \zeta V'\right)^2/4 + 4\left(V'\right)^3 + KV', \tag{15}$$

with K an arbitrary constant. This is solvable in terms of $\mathrm{P_{IV}}$ provided that $K = (\alpha+1)^2/9$ and $\beta = -2(\alpha + 1 + 2\mathrm{i}\mu)^2/9$ since making the transformation $V(\zeta) = -\mathrm{e}^{-\pi\mathrm{i}/4}W(z)/2$, with $\zeta = 2\mathrm{e}^{\pi\mathrm{i}/4}z$, in (15) yields

$$\left(W''\right)^2 = 4\left(zW' - W\right)^2 - 4\left(W'\right)^3 + 4\kappa^2 W', \tag{16}$$

with $\kappa^2 = 4K = 4(\alpha+1)^2/9$. Equation (16) is a special case of (12), with $\theta_0 = \pm\kappa/2$ and $\theta_\infty = \mp\kappa/2$, and so can be solved in terms of $\mathrm{P_{IV}}$. Therefore, from (13), rational solutions of (16) have the form

$$W_n(z; \pm 2n) = H'_{n,n}(z)/H_{n,n}(z), \tag{17a}$$

$$\widetilde{W}_n(z; \pm 2(n - \tfrac{1}{3})) = 4z^3/27 + Q'_{n,n}(z)/Q_{n,n}(z). \tag{17b}$$

Hirota and Nakamura [26] (see also [7, 27]) show that the NLS equation (2) has rational solutions, which decay algebraically as $|x| \to \infty$, in the form

$$u_n(x,t) = g_n(x,t)/f_n(x,t), \tag{18}$$

where $g_n(x,t)$ and $f_n(x,t)$ are polynomials in x of degrees $n^2 - 1$ and n^2, respectively. Hence it can be shown that

$$g_n(x,t) = n\exp\left\{\frac{1}{2}(n^2-1)(\ln t - \frac{1}{2}\pi\mathrm{i})\right\} H_{n+1,n-1}(z), \qquad z = \frac{x\,\mathrm{e}^{\pi\mathrm{i}/4}}{2\sqrt{t}},$$

$$f_n(x,t) = \exp\left\{\frac{1}{2}n^2(\ln t - \frac{1}{2}\pi\mathrm{i})\right\} H_{n,n}(z),$$

and so algebraically decaying rational solutions of the NLS equation (2) are given by

$$u_n(x,t) = \frac{n\,\mathrm{e}^{\pi\mathrm{i}/4}}{\sqrt{t}}\,\frac{H_{n+1,n-1}(z)}{H_{n,n}(z)}, \qquad z = \frac{x\,\mathrm{e}^{-\pi\mathrm{i}/4}}{2\sqrt{t}}. \tag{19}$$

The first of these are

$$u_1(x,t) = \frac{1}{x}, \qquad u_2(x,t) = \frac{2x(x^2 + 6\mathrm{i}t)}{x^4 - 12t^2},$$

$$u_3(x,t) = \frac{3(x^8 + 16\mathrm{i}tx^6 - 120t^2x^4 + 720t^4)}{x(x^8 - 72t^2x^4 - 2160t^4)}.$$

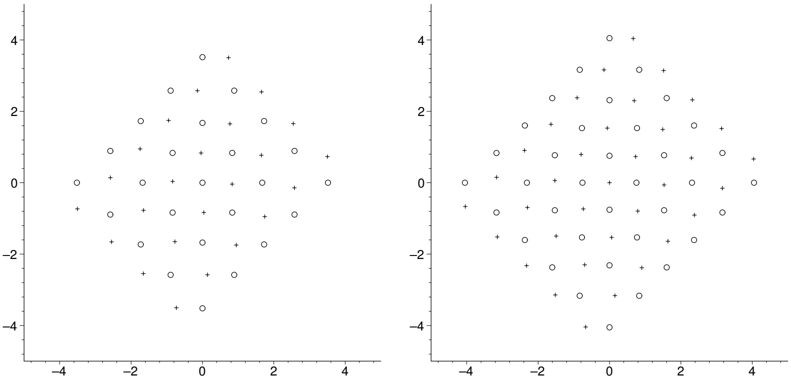


Figure 3. The zeroes ($\circ$) and poles ($+$) of the rational solutions $u_5(x,t)$ and $u_6(x,t)$

Plots of the zeroes ($\circ$) and poles ($+$) of $u_5(x,t)$ and $u_6(x,t)$ are given in Figure 3.

Hone [27] showed that more general algebraically decaying rational solutions of the NLS equation (2) have the form

$$u_n(x,t) = G_n(x,t;\boldsymbol{\kappa}_{2n-1})/F_n(x,t;\boldsymbol{\kappa}_{2n-1}), \tag{20}$$

where $G_n(x,t;\boldsymbol{\kappa}_{2n-1})$ and $F_n(x,t;\boldsymbol{\kappa}_{2n-1})$ are polynomials in x of degrees n^2-1 and n^2, respectively, with coefficients that are polynomials in t and the parameters $\boldsymbol{\kappa}_m = (\kappa_3, \kappa_4, \ldots, \kappa_m)$. The first few polynomials are

$$\begin{aligned}
F_2(x,t;\boldsymbol{\kappa}_3) &= x^4 - 12t^2 + \kappa_3 x, \qquad G_2(x,t;\boldsymbol{\kappa}_3) = 2x^3 + 12\mathrm{i}xt - \kappa_3,\\
F_3(x,t;\boldsymbol{\kappa}_5) &= x^9 + 6\kappa_3 x^6 - 72x^5t^2 + \kappa_5 x^4 + 120\kappa_4 x^3 t + 360\kappa_3 x^2 t^2\\
&\quad + (\kappa_3\kappa_5 - 15\kappa_4^2 - 2160t^4)x - 12\kappa_5 t^2 - 60\kappa_3\kappa_4 t - 5\kappa_3^3,\\
G_3(x,t;\boldsymbol{\kappa}_5) &= 3x^8 + 48\mathrm{i}x^6 t + 6\kappa_3 x^5 - 30(\mathrm{i}\kappa_4 + 12t^2)x^4 - 2(\kappa_5 + 60\mathrm{i}\kappa_3 t)x^3\\
&\quad + 30\kappa_3^2 x^2 - \left[\mathrm{i}(12\kappa_5 t + 30\kappa_3\kappa_4) + 360\kappa_3 t^2\right] x + 2160t^4 + \kappa_3\kappa_5\\
&\quad - 15\kappa_4^2 - 60\mathrm{i}(\kappa_3^2 + 6\kappa_4 t)t.
\end{aligned}$$

Note that when $\boldsymbol{\kappa}_{2n-1} = \mathbf{0}$ then $F_n(x,t;\mathbf{0}) = f_n(x,t)$ and $G_n(x,t;\mathbf{0}) = g_n(x,t)$.

We write the generalized rational solution (20) in the form

$$u_n(x,t) = \frac{G_n(x,t;\boldsymbol{\kappa}_{2n-1})}{F_n(x,t;\boldsymbol{\kappa}_{2n-1})} \equiv \sum_{j=1}^{n^2} \frac{\psi_j(t;\boldsymbol{\kappa}_{2n-1})}{x - \varphi_j(t;\boldsymbol{\kappa}_{2n-1})}, \tag{21}$$

to study the motion of the residues $\psi_j(t;\boldsymbol{\kappa}_{2n-1})$ and the poles $\varphi_j(t;\boldsymbol{\kappa}_{2n-1})$, for $j = 1,2,\ldots,n^2$. Preliminary numerical simulations suggest the following conjecture, which it is anticipated can be verified by developing the ideas in [27], though we shall not pursue this further here.

Conjecture 1. *Generalized rational solutions of the NLS equation (2) have the form*

$$u(x,t) = \sum_{j=1}^{n} \frac{\alpha_j(t)}{x - a_j(t)} + \sum_{k=1}^{n(n-1)/2} \left\{ \frac{\beta_k(t)}{x - b_k(t)} + \frac{\gamma_k(t)}{x - b_k^*(t)} \right\}, \tag{22}$$

where $a_j(t)$ are real, $b_k^*(t)$ is the complex conjugate of $b_k(t)$ and

$$|\alpha_j(t)| = 1, \quad j = 1, 2, \ldots, n, \quad \beta_k(t)\gamma_k^*(t) = 1, \quad k = 1, 2, \ldots, n(n-1)/2, \tag{23}$$

with $\gamma_k^*(t)$ the complex conjugate of $\gamma_k(t)$.

Analogously, using the rational solutions of (16) that are expressed in terms of $Q_{m,n}(z)$, i.e. (17b), it can be shown that the NLS equation (2) has rational-oscillatory solutions of the form

$$\widetilde{u}_n(x,t) = \frac{\mathrm{e}^{-\pi\mathrm{i}/4}}{3\sqrt{2t}} \frac{Q_{n+1,n-1}(z)}{Q_{n,n}(z)} \exp\left(-\frac{\mathrm{i}x^2}{6t}\right), \qquad z = \frac{x\,\mathrm{e}^{\pi\mathrm{i}/4}}{2\sqrt{t}}. \tag{24}$$

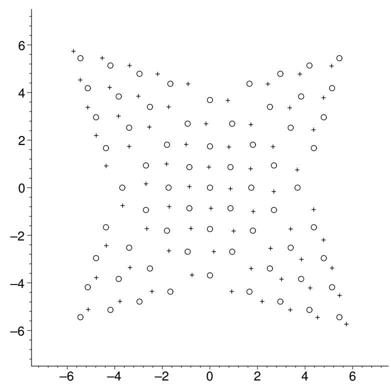
We believe that these are new solutions of the NLS equation (2). The first few of these are

$$\widetilde{u}_0(x,t) = \frac{x}{6t} \exp\left(-\frac{\mathrm{i}x^2}{6t}\right), \qquad \widetilde{u}_1(x,t) = \frac{x^2 - 6\mathrm{i}t}{6xt} \exp\left(-\frac{\mathrm{i}x^2}{6t}\right),$$
$$\widetilde{u}_2(x,t) = \frac{x(x^8 - 48\mathrm{i}tx^6 - 504t^2x^4 - 45360t^4)}{6t(x^8 + 504x^4t^2 - 9072t^4)} \exp\left(-\frac{\mathrm{i}x^2}{6t}\right).$$

Since $Q_{m,n}(z)$ has degree $d_{m,n} = m^2 + n^2 + mn - m - n$ then the solutions (24) have the form

$$\widetilde{u}_n(x,t) = \frac{\widetilde{g}_n(x,t)}{\widetilde{f}_n(x,t)} \exp\left(-\frac{\mathrm{i}x^2}{6t}\right),$$

where $\widetilde{g}_n(x,t)$ and $\widetilde{f}_n(x,t)$ are polynomials in x of degrees $3n^2-2n+1$ and $3n^2-2n$, respectively, with coefficients that are polynomials in t. Plots of the zeroes ($\circ$) and poles ($+$) of $\widetilde{u}_5(x,t)$ and $\widetilde{u}_6(x,t)$ are given in Figure 4.



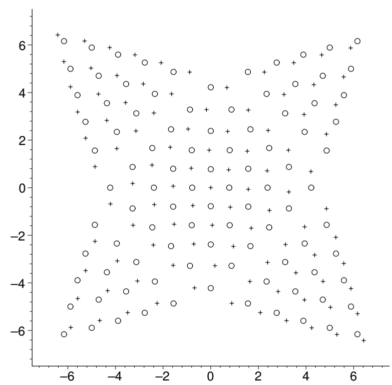


Figure 4. The zeroes ($\circ$) and poles ($+$) of the rational solutions $\widetilde{u}_5(x,t)$ and $\widetilde{u}_6(x,t)$

It is an open question whether, analogous to the generalized rational solutions (20), there are generalized rational-oscillatory solutions of the NLS equation (2) in the form

$$\widetilde{u}_n(x,t) = \frac{\widetilde{G}_n(x,t;\boldsymbol{\kappa}_m)}{\widetilde{F}_n(x,t;\boldsymbol{\kappa}_m)} \exp\left(-\frac{\mathrm{i}x^2}{6t}\right), \tag{25}$$

where $\widetilde{G}_n(x,t;\boldsymbol{\kappa}_m)$ and $\widetilde{F}_n(x,t;\boldsymbol{\kappa}_m)$ are polynomials in x of degrees $3n^2-2n+1$ and $3n^2-2n$, respectively, with coefficients that are polynomials in t and the parameters $\boldsymbol{\kappa}_m = (\kappa_3, \kappa_4, \ldots, \kappa_m)$, such that $\widetilde{G}_n(x,t;\mathbf{0}) = \widetilde{g}_n(x,t)$ and $\widetilde{F}_n(x,t;\mathbf{0}) = \widetilde{f}_n(x,t)$.

We remark that the NLS equation (2) has the rational-oscillatory solution

$$u(x,t) = \frac{1}{2}\rho \mathrm{e}^{\mathrm{i}(\kappa x - \omega t)} \left\{ 1 - \frac{4(1-\mathrm{i}\rho^2 t)}{1-\rho^2(x-2\kappa t)^2 + \rho^4 t^2} \right\}, \quad \omega = \kappa^2 + \frac{1}{2}\rho^2,$$

with ρ and κ arbitrary constants, which is not of the form (24), see Tajiri and Watanabe [47].

4 Discussion

In this paper we have studied properties of special polynomials associated with rational solutions of P_{IV} and with rational and rational-oscillatory solutions of the NLS equation (2). In particular the roots of these polynomials are shown numerically to have a very symmetric structure. There are similar results for special polynomials associated with rational solutions of P_{II} [16], rational and algebraic solutions of P_{III} [12], rational and algebraic solutions of P_V [15], and rational solutions of equations in the P_{II} hierarchy [16].

The poles of rational solutions of the KdV equation (3) satisfy a dynamical system, a constrained Calogero-Moser system [3, 5, 10]. The zeroes and poles of the rational solutions of the NLS equation (2) given by (20) satisfy an dynamical system [27] which warrants further investigation. It is anticipated that zeroes and poles of the rational-oscillatory solutions of the NLS equation (2) given by (24) will also satisfy an interesting dynamical system.

An explanation and interpretation of the numerical results for these special polynomials is an interesting open problem, as is whether they have applications, e.g. in numerical analysis?

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The Lie Derivative and Lie Symmetries

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Ordinarily Lie point symmetries, Noether symmetries, potential symmetries and so on, of differential equations have been calculated by determining the action of a vector field on solutions of the equation. An alternative method, devised by B. Kent Harrison and Frank Estabrook [4], calculates Lie symmetries of differential equations by calculating the Lie derivative of differential forms associated with the differential equation. In this paper the original method is modified slightly and then extended to incorporate potential symmetries and approximate symmetries. Examples are given.

1 Introduction

A method for writing a differential equation or system of differential equations in terms of differential forms is described. The method was devised by B. Kent Harrison and F. Estabrook and the reader is referred to [4] for a more complete description. For the properties of differential forms and their products see, e.g., Do Carmo [3].

A modification to the method is demonstrated on a wave equation with variable speed and the modified method is extended to calculate approximate and potential symmetries, and finally, Noether symmetries.

2 A Modification

In Harrison and Estabrook's original method differential equations were expressed in terms of differential forms and one requirement was that the differential forms should form a differential ideal. Another requirement was that the Lie derivative of these forms should remain in the ideal. Here, rather than ensuring that the Lie derivative of our forms stays within an ideal, we ensure that the Lie derivative of the forms is zero when the forms themselves are zero. There are some advantages to this method, one of which being that it is easy to extend the method to approximate symmetries. A wave equation with variable speed is used as an example.

Consider the equation

$$u_{tt} = e^{2x} u_{xx}. \tag{1}$$

Let $u_t = w$ and $u_x = z$. Then (1) becomes $w_t = e^{2x} z_x$, which is now a first order equation. Let

$$\alpha = \mathrm{d}u - w\mathrm{d}t - z\mathrm{d}x = u_x\mathrm{d}x + u_t\mathrm{d}t - w\mathrm{d}t - z\mathrm{d}x, \quad \text{(sectioning)}$$
$$\alpha = 0 \implies \quad u_t = w\,;\ u_x = z, \quad \text{(annulling)}$$

and

$$\begin{aligned}\beta &= \mathrm{d}w\mathrm{d}x + e^{2x}\mathrm{d}z\mathrm{d}t = (w_x\mathrm{d}x + w_t\mathrm{d}t)\mathrm{d}x + e^{2x}(z_x\mathrm{d}x + z_t\mathrm{d}t)\mathrm{d}t\\ &= w_t\mathrm{d}t\mathrm{d}x + e^{2x} z_x\mathrm{d}x\mathrm{d}t, \quad \text{(sectioning)}\end{aligned}$$
$$\beta = 0 \implies w_t = e^{2x} z_x. \quad \text{(annulling)}$$

We do not worry about $\mathrm{d}\alpha$ or $\mathrm{d}\beta$ because it is not necessary that they are members of an ideal. In fact, it turns out that imposing the condition $\mathrm{d}\alpha = 0$ (which corresponds to the equation $u_{tx} = u_{xt}$) actually *limits* the number of symmetries. This is because $\mathcal{L}_X\beta = 0$ does *not* imply that $\mathcal{L}_X\mathrm{d}\beta = 0$.

Now we calculate the Lie derivatives of α and β on solutions of (1), i.e. when $\alpha = \beta = 0$:

$$\begin{aligned}&\mathcal{L}_X\alpha|_{\alpha=0,\,\beta=0} = X\rfloor(\mathrm{d}\alpha) + \mathrm{d}(X\rfloor\alpha)|_{\alpha=0,\,\beta=0} = -X^z\mathrm{d}x - X^w\mathrm{d}t + \eta_x\mathrm{d}x\\ &+\eta_t\mathrm{d}t + \eta_u(z\mathrm{d}x + w\mathrm{d}t) - z\xi_x\mathrm{d}x - z\xi_t\mathrm{d}t - z\xi_u(z\mathrm{d}x + w\mathrm{d}t)\\ &-w\tau_x\mathrm{d}x - w\tau_t\mathrm{d}t - w\tau_u(z\mathrm{d}x + w\mathrm{d}t),\end{aligned}$$

where $\alpha = 0$ implies $\mathrm{d}u = w\mathrm{d}t + z\mathrm{d}x$, and $\beta = 0$ has no effect because β is a 2-form. Separating the coefficients of $\mathrm{d}t$ and $\mathrm{d}x$ shows that

$$\begin{aligned}&\mathcal{L}_X\alpha|_{\alpha=0,\,\beta=0} = 0 \quad \implies\\ &\mathrm{d}x : X^z = \eta_x + z(\eta_u - \xi_x) - z^2\xi_u - w\tau_x - wz\tau_u,\\ &\mathrm{d}t : X^w = \eta_t + w(\eta_u - \tau_t) - z\xi_t - zw\xi_u - w^2\tau_u.\end{aligned}$$

In other words, $\mathcal{L}_X\alpha|_{\alpha=0,\,\beta=0} = 0$ gives us the prolongation coefficients of X. Now, our wave equation (1) is a second-order equation, but with this technique, we need only calculate prolongation coefficients up to *first* order.

Next we turn our attention to β.

$$\begin{aligned}&\mathcal{L}_X\beta|_{\alpha=0,\,\beta=0} = X\rfloor(\mathrm{d}\beta) + \mathrm{d}(X\rfloor\beta)|_{\alpha=0,\,\beta=0}\\ &= (X^w_t - e^{2x}X^z_x - ze^{2x}X^z_u + wX^w_u)\mathrm{d}t\mathrm{d}x - (\xi_t + e^{2x}X^z_w + w\xi_u)\mathrm{d}t\mathrm{d}w\\ &- e^{2x}(X^z_z + \tau_t + w\tau_u - X^w_w - \xi_x - z\xi_u + 2\xi)\mathrm{d}t\mathrm{d}z\\ &- (X^w_z + e^{2x}\tau_x + e^{2x}z\tau_u)\mathrm{d}x\mathrm{d}z,\end{aligned}$$

where the substitutions $du = w\mathrm{d}t + z\mathrm{d}x$ and $\mathrm{d}w\mathrm{d}x = e^{2x}\mathrm{d}t\mathrm{d}z$ were made. Separating coefficients of $\mathrm{d}t\mathrm{d}x$, etc, gives

$$\begin{aligned}&\mathrm{d}t\mathrm{d}x : \quad X^w_t - e^{2x}X^z_x - ze^{2x}X^z_u + wX^w_u = 0,\\ &\mathrm{d}t\mathrm{d}w : \quad \xi_t + e^{2x}X^z_w + w\xi_u = 0,\\ &\mathrm{d}t\mathrm{d}z : \quad X^z_z + \tau_t + w\tau_u - X^w_w - \xi_x - z\xi_u + 2\xi = 0,\\ &\mathrm{d}x\mathrm{d}z : \quad X^w_z + e^{2x}\tau_x + e^{2x}z\tau_u = 0.\end{aligned}$$

We substitute the prolongation formulae, so that all equations are in terms of w, z and functions independent of w and z. The coefficients of w and z are split, to get the usual determining equations, and, eventually, the symmetries

$$X_1 = u\frac{\partial}{\partial u}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = \frac{\partial}{\partial x} - t\frac{\partial}{\partial t} + \frac{u}{2}\frac{\partial}{\partial u},$$
$$X_4 = t\frac{\partial}{\partial x} - \frac{1}{2}(t^2 + e^{-2x})\frac{\partial}{\partial t} + \frac{tu}{2}\frac{\partial}{\partial u}, \quad X_\infty = B(x,t)\frac{\partial}{\partial u},$$

where $B_{tt} = e^{2x}B_{xx}$. We note that the algebra is complete, with commutators as follows:

$$[X_1,\, X_2] = 0, \quad [X_1,\, X_3] = 0, \quad [X_1,\, X_4] = 0, \quad [X_1,\, X_\infty] = -X_\infty,$$
$$[X_2,\, X_3] = -X_2, \quad [X_2,\, X_4] = X_3, \quad [X_2,\, X_\infty] = B_t\frac{\partial}{\partial u} = X_\infty,$$
$$[X_3,\, X_4] = -X_4, \quad [X_3,\, X_\infty] = \left(B_x - tB_t - \frac{1}{2}B\right)\frac{\partial}{\partial u} = X_\infty,$$
$$[X_4,\, X_\infty] = \left(tB_x - \frac{1}{2}\left(t^2 + e^{-2x}\right)B_t - \frac{t}{2}B\right)\frac{\partial}{\partial u} = X_\infty.$$

3 Extensions

The ideas above may also be used to calculate potential symmetries, approximate symmetries and Noether symmetries.

3.1 Potential Symmetries

The method is demonstrated in the following example.

Consider Burgers' equation $u_{xx} - uu_x - u_t = 0$ which has the associated auxilliary system

$$v_x = 2u, \quad v_t = 2u_x - u^2. \tag{2}$$

We introduce the 2-forms

$$\alpha = \mathrm{d}v\mathrm{d}t - 2u\mathrm{d}x\mathrm{d}t = v_x\mathrm{d}x\mathrm{d}t - 2u\mathrm{d}x\mathrm{d}t,$$
$$\beta = \mathrm{d}v\mathrm{d}x + 2\mathrm{d}u\mathrm{d}t + u^2\mathrm{d}t\mathrm{d}x = v_t\mathrm{d}t\mathrm{d}x - 2u_x\mathrm{d}t\mathrm{d}x + u^2\mathrm{d}t\mathrm{d}x,$$

which return the system (2) when annulled. It is unnecessary to introduce new variables since the equations are already first-order. This means that *no* prolongation coefficients need be calculated. To calculate a symmetry $X = \tau\partial_t + \xi\partial_x + \phi\partial_u + \eta\partial_v$ of (2) we calculate the Lie derivatives of these forms. Firstly

$$\mathcal{L}_X\alpha = X\rfloor\mathrm{d}\alpha + \mathrm{d}(X\rfloor\alpha) = (2\phi - \eta_x + 2u\xi_x + 2u\tau_t)\mathrm{d}t\mathrm{d}x +$$
$$(2u\xi_u - \eta_u)\mathrm{d}t\mathrm{d}u + (2u\xi_v - \eta_v - \tau_t)\mathrm{d}t\mathrm{d}v - 2u\tau_u\mathrm{d}x\mathrm{d}u$$
$$+(-\tau_x - 2u\tau_v)\mathrm{d}x\mathrm{d}v - \tau_u\mathrm{d}u\mathrm{d}v.$$

When $\alpha = \beta = 0$, we have $\mathrm{d}t\mathrm{d}v = 2u\mathrm{d}t\mathrm{d}x$ and $\mathrm{d}x\mathrm{d}v = u^2\mathrm{d}t\mathrm{d}x - 2\mathrm{d}t\mathrm{d}u$ so that

$$\mathcal{L}_X\alpha|_{\alpha=\beta=0} = (2\phi - \eta_x + 2u\xi_x + 2u\tau_t - u^2\tau_x - 2u^3\tau_v + 4u^2\xi_v - 2u\eta_v$$
$$-2u\tau_t)\mathrm{d}t\mathrm{d}x + (2u\xi_u - \eta_u + 2\tau_x + 4u\tau_v)\mathrm{d}t\mathrm{d}u - 2u\tau_u\mathrm{d}x\mathrm{d}u - \tau_u\mathrm{d}u\mathrm{d}v$$

and we may now split the coefficients of $\mathrm{d}t\mathrm{d}x$, $\mathrm{d}t\mathrm{d}u$ etc to get

$$\mathrm{d}t\mathrm{d}x: \quad 2\phi - \eta_x + 2u\xi_x + 2u\tau_t - u^2\tau_x - 2u^3\tau_v + 4u^2\xi_v - 2u\eta_v - 2u\tau_t = 0,$$
$$\mathrm{d}t\mathrm{d}u: \quad 2u\xi_u - \eta_u + 2\tau_x + 4u\tau_v = 0,$$
$$\mathrm{d}u\mathrm{d}v: \quad \tau_u = 0,$$
$$\mathrm{d}x\mathrm{d}u: \quad \text{the same.}$$

Next

$$\mathcal{L}_X\beta = X\rfloor(\mathrm{d}\beta) + \mathrm{d}(X\rfloor\beta) = (2u\phi + \eta_t - 2\phi_x + u^2\tau_t + u^2\xi_x)\mathrm{d}t\mathrm{d}x$$
$$+(u^2\xi_u - 2\phi_u - 2\tau_t)\mathrm{d}t\mathrm{d}u + (u^2\xi_v - \xi_t - 2\phi_v)\mathrm{d}t\mathrm{d}v$$
$$+(-u^2\tau_u - \eta_u - 2\tau_x)\mathrm{d}x\mathrm{d}u + (-\eta_v - \xi_x - u^2\tau_v)\mathrm{d}x\mathrm{d}v + (2\tau_v - \xi_u)\mathrm{d}u\mathrm{d}v.$$

When $\alpha = \beta = 0$, we get

$$\mathcal{L}_X\beta|_{\alpha=\beta=0} = (2u\phi + \eta_t - 2\phi_x - u^2\tau_t + u^2\xi_x + 2u^3\xi_v - 2u\xi_t - 4u\phi_v$$
$$-u^2\eta_v - u^2\xi_x - u^4\tau_v)\mathrm{d}t\mathrm{d}x + (u^2\xi_u - 2\phi_u - 2\tau_t + 2\eta_v + 2\xi_x + 2u^2\tau_v)\mathrm{d}t\mathrm{d}u$$
$$-(\eta_u + 2\tau_x + u^2\tau_u)\mathrm{d}x\mathrm{d}u + (2\tau_v - \xi_u)\mathrm{d}u\mathrm{d}v,$$

which may be split into

$$u^2\xi_u - 2\phi_u - 2\tau_t + 2\eta_v + 2\xi_x + 2u^2\tau_v = 0,$$
$$\eta_u + 2\tau_x + u^2\tau_u = 0, \quad 2\tau_v - \xi_u = 0,$$
$$2u\phi + \eta_t - 2\phi_x - u^2\tau_t + u^2\xi_x + 2u^3\xi_v - 2u\xi_t - 4u\phi_v -$$
$$u^2\eta_v - u^2\xi_x - u^4\tau_v = 0.$$

From here onwards the calculations proceed in the standard way and eventually we arrive at the symmetries

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u} + 2x\frac{\partial}{\partial v}, \quad X_3 = \frac{\partial}{\partial v},$$
$$X_4 = \frac{\partial}{\partial x}, \quad X_\infty = e^{v/4}\left(2C_x + uC\right)\frac{\partial}{\partial u} + 4e^{v/4}C\frac{\partial}{\partial v},$$

where C is any solution of $C_t = C_{xx}$. The commutators are as follows:

$$[X_1,\, X_2] = X_4, \quad [X_1,\, X_3] = 0, \quad [X_1,\, X_4] = 0, \quad [X_2,\, X_3] = 0,$$
$$[X_1,\, X_\infty] = e^{v/4}(2C_{xt} + uC_t)\frac{\partial}{\partial u} + e^{v/4}C_t\frac{\partial}{\partial v} = X_\infty,$$
$$[X_2,\, X_4] = -2X_4, \quad [X_2,\, X_\infty] = e^{v/4}(2tC_{xx} + utC_x + C + xC_x$$
$$+\frac{ux}{2}C)\frac{\partial}{\partial u} + e^{v/4}(4tC_x + 2xC)\frac{\partial}{\partial v} = X_\infty, \quad [X_3,\, X_4] = 0,$$
$$[X_3,\, X_\infty] = \frac{1}{4}X_\infty, \quad [X_4,\, X_\infty] = e^{v/4}(2C_{xx} + uC_x)\frac{\partial}{\partial u} + 4e^{v/4}C_x\frac{\partial}{\partial v} = X_\infty.$$

3.2 Approximate Symmetries

We have shown that differential forms may be used to calculate ordinary Lie symmetries and potential symmetries of differential equations. The method may be extended to calculate approximate symmetries as well, as the next example demonstrates. Firstly, we recall some notation. Let

$$E(x^i, u^\alpha, u^\alpha_{(1)}, \ldots) + \epsilon F(x^i, u^\alpha, u^\alpha_{(1)}, \ldots) = 0 \tag{3}$$

be a perturbed equation, where $E = 0$ is the associated unperturbed equation. An approximate symmetry of (3) is a vector field X such that

$$X(E + \epsilon F)|_{E+\epsilon F=0} = O(\epsilon^2).$$

Now the perturbed equation gives rise to differential forms $\gamma_j = \alpha_j + \epsilon\beta_j$, where the α^j are forms arising from the unperturbed equation $E = 0$. We refer to the γ_j collectively as I and the α_j as I_0. The phrase $I = 0$ should be taken to mean that for each γ_j we have $\gamma_j = 0$ and similarly for $I_0 = 0$. The condition that X be an approximate symmetry of (3) can now be rewritten as the system

$$\mathcal{L}_X \gamma_j|_{I=0} = O(\epsilon^2).$$

3.2.1 A Perturbed Wave Equation

We adapt the algorithm due to Baikov *et al* in [1] and [2] to find the approximate symmetries of a perturbed wave equation

$$u_{tt} - e^{2x}u_{xx} + \epsilon F(t, x, u, u_t, u_x) = 0, \tag{4}$$

the unperturbed version of which we have already encountered (equation (1)) and calculated symmetries. Recall that we introduced new variables $w = u_t$ and $z = u_x$ and used the forms $\alpha = \mathrm{d}u - z\mathrm{d}x - w\mathrm{d}t$ and $\beta = \mathrm{d}w\mathrm{d}x + e^{2x}\mathrm{d}z\mathrm{d}t$, which gave rise to, among others, the symmetry

$$X_0 = \frac{\partial}{\partial x} - t\frac{\partial}{\partial t} + \frac{u}{2}\frac{\partial}{\partial u} \quad \left(+\frac{3w}{2}\frac{\partial}{\partial w} + \frac{z}{2}\frac{\partial}{\partial z}\right).$$

For the perturbed equation (4), we continue to use the 1-form α, which gives $w = u_t$ and $z = u_x$ when annulled, but β will not work without modification and so we introduce $\gamma = \mathrm{d}w\mathrm{d}x + e^{2x}\mathrm{d}z\mathrm{d}t + \epsilon F\mathrm{d}t\mathrm{d}x = \beta + \epsilon F\mathrm{d}t\mathrm{d}x$, which gives $u_{tt} - e^{2x}u_{xx} + \epsilon F = 0$ when annulled. Using the symmetry X_0 for the algorithm described above, we calculate $h_1 = \epsilon^{-1}\mathcal{L}_{X_0}\alpha|_{I=0} = 0$. Thus for α we must find a symmetry, X_1, such that

$$\mathcal{L}_{X_1}\alpha|_{I_0=0} + h_1 = 0 \implies \mathcal{L}_{X_1}\alpha|_{I_0=0} = 0,$$

which is no different to the unperturbed case and we end up finding that X_1 must have the usual prolongation coefficients although we note that, as before, they need only be calculated to first order.

Next $h_2 = \epsilon^{-1}\mathcal{L}_{X_0}\gamma|_{I=0} = (F_x - tF_t + (uF_u + 3wF_w + zF_z - 5F)/2)\,\mathrm{d}t\mathrm{d}x$. The next step in our algorithm is to find X_1 (which we call Y to avoid confusion with subscripts) such that $\mathcal{L}_Y\beta|_{\alpha=\beta=0} + h_2 = 0$, where we recall that $\beta = \gamma$ when $\epsilon = 0$.

$$\mathcal{L}_Y\beta|_{\alpha=\beta=0} + h_2 = \{Y\rfloor \mathrm{d}\beta + \mathrm{d}(Y\rfloor\beta)\}\,|_{\alpha=\beta=0} + h_2 = \Big(Y^w_t - e^{2x}Y^z_x$$
$$-ze^{2x}Y^z_u + wY^w_u + F_x - tF_t + \frac{u}{2}F_u + \frac{3w}{2}F_w + \frac{z}{2}F_z - \frac{5}{2}F\Big)\,\mathrm{d}t\mathrm{d}x$$
$$+ \left(-2\xi_t - e^{2x}Y^z_w - w\xi_u\right)\mathrm{d}t\mathrm{d}w + e^{2x}\left(-\xi - Y^z_z - \tau_t - w\tau_u + Y^w_w\right.$$
$$\left.+\xi_x + z\xi_u\right)\mathrm{d}t\mathrm{d}z + \left(-Y^w_z - e^{2x}\tau_x - ze^{2x}\tau_u\right)\mathrm{d}x\mathrm{d}z.$$

Thus $\mathcal{L}_Y\beta|_{\alpha=\beta=0} + h_2 = 0$ implies that

$$Y^w_t - e^{2x}Y^z_x - ze^{2x}Y^z_u + wY^w_u + F_x - tF_t + \frac{u}{2}F_u + \frac{3w}{2}F_w + \frac{z}{2}F_z - \frac{5}{2}F = 0,$$
$$\xi_t + e^{2x}Y^z_w + w\xi_u = 0, \qquad 2\xi + Y^z_z + \tau_t + w\tau_u - Y^w_w - \xi_x - z\xi_u = 0,$$
$$Y^w_z + e^{2x}\tau_x + ze^{2x}\tau_u = 0,$$

which is exactly the same set of determining equations that the ordinary method gives and so from here on the calculations are identical.

4 Conclusion

We see that the Lie derivative offers, in some ways, a more natural way of calculating symmetries of differential equations; with this method, fewer prolongation coefficients need be calculated than with the traditional method.

There are possible insights to be gained from the way the independent variables are handled using these methods. For example, we see that the exact role played by the equality (or lack of equality) of mixed derivatives of the dependent variables, and the way that this seems to limit the number of symmetries available, could perhaps be made clearer.

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SYM: A New Symmetry — Finding Package for *Mathematica*

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A new package for computing the symmetries of systems of differential equations using *Mathematica* is presented. Armed with adaptive equation solving capability and pattern matching techniques, this package is able to handle systems of differential equations of arbitrary order and number of variables with the least memory cost possible. By harnessing the capabilities of Mathematica's front end, all the intermediate mathematical expressions, as well as the final results apear in familiar form. This renders the package a very useful tool for introducing the symmetry solving method to students and non-mathematicians.

1 Introduction

The effectiveness of the method of symmetry analysis of differential equations first introduced by Sophus Lie is well established. The success of Lie's method is partly due to the fact that it allows one to find the symmetries of a given (system of) equation(s) algorithmically. However, as the number of variables and/or equations increases, the pertinent calculations become unmanageable. On the other hand, complex systems involving a large number of independent variables are frequently met in practice, e.g. in all areas of theoretical and applied physics. The Einstein field equations of general relativity and the Navier–Stokes equations of hydrodynamics could be cited as representative examples. In all such cases, the huge amount of calculations involved in applying the symmetry method render the use of computer algebra programs imperative.

In recent years, several symmetry–finding packages have been developed [1]. Most of them are based on the widely used computer algebra systems (CAS), such as REDUCE [2], MACSYMA, Maple and Mathematica [3]. The functionality of the above packages varies greatly. Some of them are effective only for differential equations of polynomial form. Others give only the determining equations in a reduced form and, then, the user must solve the latter interactively, at best. In any case, most of the packages developed so far fare well in practice only for determining the Lie point symmetries of scalar equations.

The purpose of this presentation is to introduce SYM, a new package for computing symmetries using Mathematica. SYM's main advantage over its predecessors is twofold. First, it provides the user with an easy to comprehend interface. This has been made possible by hiding effectively the cumbersome and awkward way the CAS itself represents mathematical expressions. In particular, all the expressions appearing in both the input and output of the package are represented in the familiar form encountered in the mathematical literature. Moreover, SYM is distinguished by its ability to handle and calculate the symmetries of complex systems of differential equations efficiently and without much intervention from the user. This has been achieved by making use of the powerful programming language of Mathematica. As a result the package is fast, reliable, and consumes less memory.

2 The *Sym* Package

The fundamental characteristic of SYM is its modularity. This means that it is based on a specific set of functions which are employed in the symmetry analysis of a given equation. They are functions defined using the well known algorithms of symmetry analysis which stems from Sophus Lie's theory ([5–8]). In this section we give some further details regarding the features of the program and a few examples that illustrate its effectiveness.

2.1 Main Features

The basic functions that any symmetry finding package has to perform are [4, 5]:

(i) To obtain the determining equations,

(ii) To reduce and simplify the system of determining equations, and

(iii) To integrate this overdetermined system

Besides complying to the above guidelines, SYM carries the following features.

- Every infinitesimal generator and its prolongation are defined and used as operators. Hence, the action of an infinitesimal generator on any algebraic or differential equation can be easily manipulated. This is accomplished using the command `X[n,{x,y},{u}]` which turns the n-th extension of the infinitesimal operator $\xi_1(x,y,u)\partial_x+\xi_2(x,y,u)\partial_y+\eta(x,y,u)\partial_u$ into a "a pure function". Examples where this feature can be exploited are the analysis of the invariant surface condition and the supplementary equations involved in the symmetry analysis of an initial–boundary problem. Likewise, the command `X[n,{x,y},{u,h},2]` defines as a pure function the n-th extension of the infinitesimal generator in characteristic form:

$$\mathcal{Q}_1(x,y,u,h,u_x,u_y,u_{xy},u_{xx},u_{yy},h_y,h_{xy},h_{xx},h_{yy})\partial_u$$

$$+\mathcal{Q}_2(x,y,u,h,u_x,u_y,u_{xy},u_{xx},u_{yy},h_y,h_{xy},h_{xx},h_{yy})\partial_h,$$

which is needed in the investigation of generalized symmetries.

- The structural elements of the equations to be analyzed are automatically pinpointed and characterized. This is attained by making the program look at a differential equation in a human like fashion, using commands like `CharacterizeEq[*]`. The latter produces automatically several features of equation, such as its order, the independent and dependent variables, etc. This feature minimizes the input required, restricting it to the differential equation or the system of such equations under study, only. The above commands not only facilitate the substitutions needed in the process of automatically solving the linearized symmetry condition, but they render these substitutions easy to materialize in the case the user has to solve the above system interactively.

- An intelligent integrator of the system of overdetermined equations, which is incorporated in the fundamental command `SolveOverdeterminedEqs`. Enhancing Mathematica's internal one, SYM's differential solver mimics the human behavior by following a novel algorithm we call "Seek&Solve": it locates the appropriate equation to solve, substitutes the solution of the latter to the remaining equations and, after making the necessery simplifications, it repeats the previous cycle. Thanks to the various rules and tactics incorporated in the solver, the program will adapt its solving strategy to the system at hand. It terminates only when the complete solution is achieved, or when the remaining equations are not solvable. In this connection, we stress that the solver can deal with systems which include equations of non polynomial type. All possible differential constraints on arbitrary functions contained in the solutions are given explicitly. In addition, the package provides the option of printing all the steps followed in obtaining the solution. This feature allows the program's user to check all the intermediate steps at any time.

- Additional functions for manipulating the system's symmetries are included. SYM gives all the generators of the one–parameter subgroups, their commutator table and the structure constants of the corresponding algebra.

- All intermediate and final expressions are presented in a compact and elegant fashion. More specifically, by taking advandage of the expression masking capabilities of Mathematica, SYM presents both the equations to be solved as well as the intermediate and final results in the familiar form that one encounters in the mathematical literature. Moreover, these familiar expressions can be manipulated freely by the user himself.

2.2 Illustrative Examples

The package has been tested against a variety of differential equations, especially systems, from various sources [5–8]. It has also been tested by the interactive derivation of conditional symmetries — both point and generalized, of several equations of research interest. The following are characteristic examples of the equations against which SYM has been tested. The last one has been considered, up to now, as the benchmark for symmetry–finding packages.

(i) The modified Kadomtsev–Petviashvili equation

$$3u_{yy} - 4u_{xt} - 6u_y u_{xx} - 6u_x^2 u_{xx} + u_{xxxx} = 0 \tag{1}$$

(ii) The generalization of the Ernst equation derived in [9]

$$\begin{aligned}
&\partial_u \left(A_v + \frac{A^2}{\rho} U_v(u,v) + m\frac{A}{\rho} \right) + \partial_v \left(A_u + \frac{A^2}{\rho} U_u(u,v) - n\frac{A}{\rho} \right) = 0, \\
&\rho = \frac{1}{2}(v-u), \quad A = \frac{1}{2}\left(2\rho \frac{U_{uv}}{U_u U_v} + \frac{n}{U_u} - \frac{m}{U_v} \right).
\end{aligned} \tag{2}$$

(iii) The Einstein vacuum equations for the Bondi metric [10]

$$\begin{aligned}
\beta_r &= \frac{r\gamma_r^2}{2}, \\
U_{rr} &= \frac{2e^{-2\gamma(u,r,\theta)}}{r^3} \Big(-2e^{2\beta(u,r,\theta)}\beta_\theta - 2e^{2\gamma}r^2 U_r + e^{2\gamma} r^3 U_r \beta_r \\
&\quad - 2re^{2\beta}\gamma_r \cot\theta + 2e^{2\beta} r\gamma_r\gamma_\theta - e^{2\gamma}r^3 U_r \gamma_r + e^{2\beta} r\beta_{r\theta} - e^{2\beta} r\gamma_{r\theta} \Big), \\
\beta_{\theta\theta} &= -\frac{1}{4} e^{-4\beta} \Big(-4e^{4\beta} - 8e^{2(\beta+\gamma)} rU(u,r,\theta)\cot\theta - 8e^{2(\beta+\gamma)} rU_\theta + \\
&\quad 4e^{4\beta}\beta_\theta \cot\theta + 4e^{4\beta}\beta_\theta^2 - 12e^{4\beta}\gamma_\theta\cot\theta - 8e^{4\beta}\beta_\theta\gamma_\theta + 8e^{4\beta}\gamma_\theta^2 - \\
&\quad 4e^{4\beta}\gamma_{\theta\theta} - 2e^{2(\beta+\gamma)} r^2 U_r \cot\theta + e^{4\gamma} r^4 U_r^2 + 4e^{2(\beta+\gamma)} V_r - \\
&\quad 2e^{2(\beta+\gamma)} r^2 U_{r\theta} \Big), \\
\gamma_{\theta\theta} &= -e^{-2\beta} \Big(3e^{2\gamma} rU\cot\theta + e^{2\gamma} rU_\theta - 2e^{2\beta}\beta_\theta\cot\theta + 3e^{2\beta}\gamma_\theta\cot\theta \\
&\quad + e^{2\beta} - 2e^{2\gamma} rU\gamma_\theta + 2e^{2\beta}\beta_\theta\gamma_\theta - 2e^{2\beta}\gamma_\theta^2 + e^{2\gamma}r^2 U_r\cot\theta - e^{2\gamma} V_r \\
&\quad - e^{2\gamma} r^2 \gamma_\theta U_r - e^{2\gamma} r^2 U \cot\theta \gamma_r + e^{2\gamma} V(u,r,\theta)\gamma_r - e^{2\gamma} r^2 U_\theta \gamma_r \\
&\quad + e^{2\gamma} rV_r\gamma_r - 2e^{2\gamma} r^2 U\gamma_{r\theta} + e^{2\gamma} rV\gamma_{rr} - 2e^{2\gamma} r\gamma_u - 2e^{2\gamma} r^2 \gamma_{ur} \Big)
\end{aligned} \tag{3}$$

(iv) The Magneto–Hydro–Dynamics equations

$$\begin{aligned}
&\rho_t = -\nabla\cdot(\rho(x,y,z,t)\vec{v}), \\
&\vec{v}_t = -(\vec{v}\cdot\nabla)\vec{v} - \frac{1}{\rho}\left(\nabla\left(p(x,y,z,t) + \frac{1}{2}\vec{H}^2(x,y,z,t) \right) - (\vec{H}\cdot\nabla)\vec{H} \right), \\
&\vec{H}_t = (\vec{H}\cdot\nabla)\vec{v} - (\vec{v}\cdot\nabla)\vec{H} - \vec{H}\nabla\cdot\vec{v}, \\
&\nabla\cdot\vec{H} = 0, \quad p_t = -kp(\nabla\cdot\vec{v}) - (\vec{v}\cdot\nabla)p.
\end{aligned} \tag{4}$$

The Lie point symmetries of the above equations were obtained using SYM's `ClassicalSymmetries[]` function. In the table below we present the time and the amount of physical memory needed for the calculation. The PC used in the test was a Pentium IV laptop at 3.2GHz with 1GB of physical memory.

equation	time	physical memory
1	6.8 sec	6 MB
2	95.8 sec	374 MB
3	25.1 min	269 MB
4	19.4 min	32 MB

In way of comparison, we first mention that MathLie, the symmetry–finding package for Mathematica developed by G. Baumann [11], wasn't able to give *non–interactively* even the determining equations for examples (ii)–(iv). On the other hand, the MACSYMA based package SYMMGRP.MAX took 50 minutes of CPU time on a Digital VAX 4500 with 64MB of RAM for deriving only the (222) determining equations of example (iv).

The Lie point symmetries of the equations in examples (i) and (iv) are well documented [4]. Therefore we restrict ourselves to presenting the symmetry generators of the equations in examples (ii) and (iii). They are given by

$$X_1 = \partial_u + \partial_\upsilon, \quad X_2 = u\partial_\upsilon + \upsilon\partial_\upsilon, \quad X_3 = \partial_U, \quad X_4 = U\partial_U, \quad X_5 = U^2\partial_U$$

and

$$\begin{aligned}
X_1 &= -r\partial_r - 2V\partial_V + \frac{1}{2}\partial_\beta + \partial_\gamma, \quad X_2 = 2r\partial_r + 4V\partial_V + \partial_\beta, \\
X_{f_1} &= f_1(u)\partial_u - Uf_{1u}\partial_U - Vf_{1u}\partial_V - \frac{f_{1u}}{2}\partial_\beta, \\
X_{f_2} &= -\frac{r}{2}(f_2 \cot\theta x + f_{2\theta})\partial_r + f_2(u,\theta)\partial_\theta + \\
&\left(Uf_{2\theta} + f_{2u} + \frac{e^{2(\beta-\gamma)}}{2r}(f_{2\theta\theta} + f_{2\theta}\cot\theta - f_2\csc^2\theta)\right)\partial_U + \\
&\big(r^2(Uf_{2\theta\theta} + f_{2u}\cot\theta + f_{2u\theta}) + (r^2U\cot\theta - V)f_{2\theta} - \\
&(r^2U\csc^2\theta + V\cot\theta)f_2\big)\,\partial_V + \frac{f_2\cot\theta + f_{2\theta}}{4}\partial_\beta + \frac{f_2\cot\theta - f_{2\theta}}{2}\partial_\gamma,
\end{aligned}$$

respectively.

3 Applications in Education

Because of the familiar way it represents mathematical expressions, its easy to use interface and modular structure, SYM can be used effectively in courses on the symmetry analysis of differential equations. By using it, students can become familiar with the fundamental notions of symmetry analysis much more easily.

Because it presents the symmetry construction process in a step by step fashion and allows students to experiment on their own. In addition, the package can be exploited in the context of webMathematica. More specifically, everyone with an internet access can use SYM for getting introduced to modern group analysis of differential equations, without having to own the actual CAS.

4 Future Additions

The symmetry–finding package presented in this talk needs to be further developed and completed. The following are among the additions that would make SYM even more effective:

- High–level comands that would make it able to automatically calculate conditional, non–local and discrete symmetries,
- Tools for the construction of recursion operators and master symmetries,
- Functions concerning various aspects of the corresponding Lie algebras, such as their solvability, the optimal system etc., and the group classification of solutions.
- Differential algebra algorithms which determine the system of determining equations and specify its solution space [12, 13]

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SMM for an Equation in 2 + 1 Dimensions: Lax Pair and Darboux Transformations

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In a previous paper [1] the Singular Manifold Method was presented as an excellent tool to study the 2+1 dimensional equation $(h_{xxz} - 3(h_z)^{-1}h_{xz}^2/4 + 3h_xh_z)_x = h_{yz}$. In this paper a method to obtain iterated solutions is given and different solitonic solutions are presented.

1 Introduction

It is well-known that there are different ways to study non-linear partial differential equations, among them, the Singular Manifold Method (SMM), [6] based on the Painlevé property [7], playing a significant role. In fact, once the Painlevé test has been checked for a given partial differential equation, the SMM method gives Bäcklund transformations, Lax pairs, Darboux transformations and tau-functions for the partial differential equation. However, the procedure is not very straightforward, and some of the problems connected with Painlevé property, Painlevé test and SMM are listed in a previous paper [1]. It was shown there that a partial differential equation in 2+1 dimensions can be much better analyzed using the SMM than its reductions to 1+1 dimensions. This, apparently, strange behavior seems to be due to the excessive restrictions imposed by the SMM method when the number of dimensions is not high enough.

In the present work we deal with the same equation from [1] and give a rich number of iterated solutions, an aspect not studied in detail in that paper.

The paper is organized as follows: In section 2, the equation considered in [1] is given and the various results found are summarized. Section 3 offers a collection of solutions.

2 An Equation in 2+1 Dimensions

The equation addressed in our study is as follows

$$\left(h_{xxz} - \frac{3}{4}\left(\frac{h_{xz}^2}{h_z}\right) + 3h_xh_z\right)_x = h_{yz}, \tag{1}$$

where h is a field depending on 2+1 variables: x, y and z. By introducing a new dependent field $p(x,y,z)$, we can write (1) as the system

$$h_z + p^2 = 0, \quad -p_y + p_{xxx} + \frac{3}{2} p h_{xx} + 3 p_x h_x = 0.$$

2.1 Painlevé Test

As proved in [1], the above equation passes the Painlevé test. In fact, writing as in [1]

$$h = \sum_{j=0}^{\infty} h_j(x,y,z) \left(\phi(x,y,z)\right)^{j-a}, \tag{2}$$

we obtain $\sum_{j=0}^{\infty} C_j \left(\phi(x,y,z)\right)^{3j-3a-6}$, and it turns out that $a = 1$, and the equation has resonances in $j = 1, 3, 4$ while C_1, C_3 and C_4 are identically 0 for any value of h_1, h_3 and h_4.

2.2 Reductions

Obvious reductions of (1), as stated in [1], are:

1) $\partial h/\partial y = 0$ equivalent to $\partial h/\partial y = \partial h/\partial x$ (redefining h as $h + x/3$) gives us

$$\left(h_{xxz} - \frac{3}{4}\left(\frac{h_{xz}^2}{h_z}\right) + 3 h_x h_z \right)_x = 0, \tag{3}$$

$$h_z + p^2 = 0, \quad (2 p p_{xx} - p_x^2 + 3 p^2 h_x)_x = 0 \tag{4}$$

(4) is the Ermakov–Pinney equation [2] .

2) From $\partial h/\partial z = \partial h/\partial x$, we have the modified Korteweg-de Vries equation

$$p_y - p_{xxx} + 6 p^2 p_x = 0$$

3) $\partial h/\partial z = \partial h/\partial y$ affords the 1+1 equation

$$\left(h_{xxz} - \frac{3}{4}\left(\frac{h_{xz}^2}{h_z}\right) + 3 h_x h_z \right)_x = h_{zz}, \quad \text{or}$$

$$h_z + p^2 = 0, \quad -p_z + p_{xxx} + \frac{3}{2} p h_{xx} + 3 p_x h_x = 0.$$

The problems of these equations with respect to the SMM have been discussed in [1] and, we refer interested readers to that source.

2.3 The Singular Manifold Method

Writing equation (1) in non-local form (see [1]) as

$$h_y = n_x, \quad h_{xxz} h_z - \frac{3}{4} h_{xz}^2 + 3 h_x h_z^2 - h_z n_z = 0, \tag{5}$$

and using the truncated expansion of the Painlevé series (2) at the constant level $j = 1$, it follows that solutions $h^{(1)}$, $n^{(1)}$ of (5) can be written as

$$h^{(1)} = h + \frac{\phi_x}{\phi}, \quad n^{(1)} = n + \frac{\phi_y}{\phi}, \tag{6}$$

where h and n are also seed solutions of the system (5) and ϕ is the singular manifold associated with the solution (h, n). By substitution of (6) in (5), we obtain:

a) The seed solutions; that is, the seed field h can be written as

$$h_x = -\frac{V_x}{3} - \frac{V^2}{12} + \frac{Q}{3}, \quad h_z = -\frac{1}{4R}(R_x + RV)^2,$$

where V, R and Q are defined as $V = \phi_{xx}/\phi_x$, $R = \phi_z/\phi_x$, $Q = \phi_y/\phi_x$.

b) The Singular Manifold Equations, that is

$$Q_z = S_z - \frac{3}{2}R_x\left(S + \frac{R_{xx}}{R} - \frac{R_x^2}{2R^2}\right) \tag{7}$$

together with the compatibility conditions between the definitions of V, R and Q

$$V_z = (R_x + RV)_x, \quad V_y = (Q_x + QV)_x,$$

where S is the Schwartzian derivative, $S = V_x - V^2/2$.

c) The Lax pair

$$-\psi_y + \psi_{xxx} + 3h_x\psi_x + \frac{3}{2}h_{xx}\psi = 0 \tag{8}$$

$$2h_z\psi_{xz} - h_{xz}\psi_z + 2h_z^2\psi = 0, \tag{9}$$

where the eigenfunction ψ is related to the singular manifold by $\phi_x = \psi^2$.

d) If ψ_1 and ψ_2 are two different eigenfunctions for h, there must be two singular manifolds for h defined by

$$\phi_{1,x} = \psi_1^2 \quad \text{and} \quad \phi_{2,x} = \psi_2^2 \tag{10}$$

and we obtain a new solution $(h^{1)}, n^{1)})$ through the truncated expansion

$$h^{1)} = h + \frac{\phi_{1,x}}{\phi}, \tag{11}$$

$$n^{1)} = n + \frac{\phi_{1,y}}{\phi}, \tag{12}$$

and a singular manifold $\phi^{1)}$ for $h^{1)}$ defined through the expression $\phi_x^{1)} = (\psi^{(1)})^2$.

Thus, the truncated expression for $h^{1)}$ and $n^{1)}$ can be extended to $\psi^{1)}$ and $\phi^{1)}$ as

$$\psi^{1)} = \psi_2 + \frac{\Lambda}{\phi_1}, \quad \phi^{1)} = \phi_2 + \frac{\Delta}{\phi_1}, \tag{13}$$

where $\Lambda = -\psi_1\Omega$ and $\Delta = -\Omega^2$, and

$$d\Omega = \psi_1\psi_2 dx + (\psi_1\psi_{2,xx} + \psi_2\psi_{1,xx} - \psi_{1,x}\psi_{2,x} + 3h_x\psi_1\psi_2)dy - \frac{\psi_{1,z}\psi_{2,z}}{h_z}dz \tag{14}$$

Finally, it follows that (11) and (12), together with (13), are binary Darboux transformations, although as remarked in [1], they are not the usual binary transformations that appear, for instance, in references [4] and [5]. In reference [3] they are denominated Bäcklund-gauge transformations.

3 Iterated Solutions

The results of the previous section can be used as an iterative procedure of construction of solutions in the following way. According to (13), $\phi^{1)}$ is a singular manifold for $h^{1)}$. We can therefore construct a new solution $h^{2)}$ by iterating (11) as

$$h^{2)} = h^{1)} + \frac{\phi_x^{1)}}{\phi^{1)}}. \tag{15}$$

A combination of (11) and (15) provides $h^{2)} = h + \tau_x/\tau$, where (13) has been used to write $\tau = \phi^{1)}\phi_1 = \phi_1\phi_2 - \Omega^2$. Consequently, two elementary solutions ψ_1 and ψ_2 of the Lax pair (8)–(9) of a seed solution h allow us to construct a first and second iteration in the following way $h^{1)} = h + \phi_{1,x}/\phi_1$, $h^{2)} = h + \tau_x/\tau$, where

$$\phi_x^{1)} = (\psi^{(1)})^2, \quad \phi_x^{2)} = (\psi^{(2)})^2, \quad \tau = \phi^{1)}\phi_1 = \phi_1\phi_2 - \Omega^2,$$

and Ω is given in (14).

Let us to give some examples of how the method works.

3.1 Dromions: $h = 0$

Equation (9) is identically satisfied when $h = 0$. Elementary solutions of (8) are

$$\psi_1 = e^{k_1 x + k_1^3 y}, \quad \psi_2 = e^{k_2 x + k_2^3 y}.$$

Integration of (10) (with the aid of (7)) gives us

$$\phi_1 = \frac{1}{2k_1}\left(\alpha_1(y) + \beta_1(z) + \psi_1^2\right), \quad \phi_2 = \frac{1}{2k_2}\left(\alpha_2(y) + \beta_2(z) + \psi_2^2\right).$$

(14) provides $\Omega = \psi_1\psi_2/(k_1 + k_2)$, $\tau = \phi_1\phi_2 - \Omega^2$. A particular case can be obtained by choosing the arbitrary functions α_i and β_i as follows

$$\alpha_i = 0, \quad \beta_i = \frac{1 + \sum_{j=1}^n e^{2\omega_{ij}z}}{1 + \sum_{j=1}^n c_{ij}e^{2\omega_{ij}z}} \quad i = 1, 2.$$

In this case, we have

$$\phi_i = \frac{1}{2k_i}\left(\beta_i + \psi_i^2\right), \quad i = 1, 2.$$

$$\tau = \frac{1}{4k_1k_2}\left(\beta_1\beta_2 + \beta_2\psi_1^2 + \beta_1\psi_2^2 + \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 \psi_1^2\psi_2^2\right), \quad \text{and}$$

$$h^{1)} = \frac{\phi_{1,x}}{\phi_1}, \quad h^{2)} = \frac{\tau_x}{\tau}.$$

In figure 1, we show the behavior of $h_z^{1)}$ for $n = 2$. It corresponds to a dromion with two jumps.

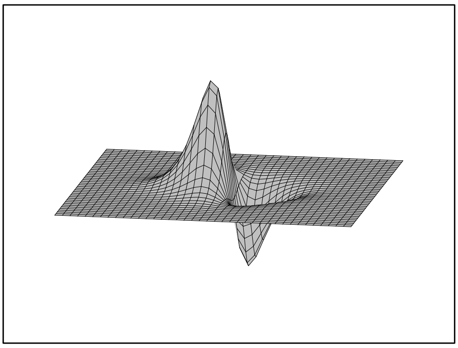

Figure 1. $h_z^{1)}$ *for* $n = 2$*: One-dromion solution with two jumps.*

The two-dromion solution can be obtained by setting $n = 1$ in $h_z^{2)}$. This case is shown in Figure 2.

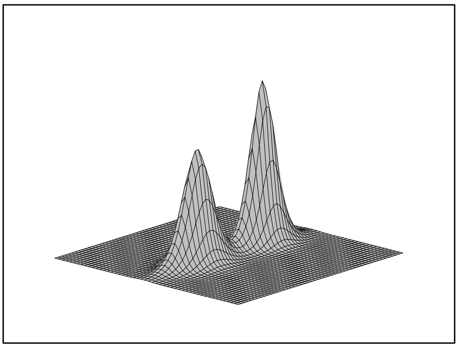

Figure 2. $h_z^{2)}$ *for* $n = 1$*: Two-dromion solution with one jump.*

3.2 Line Solitons: $h = \omega_0 z$

Solutions ψ_1 and ψ_2 of the Lax pair (8)–(9) are

$$\psi_1 = e^{k_1x + k_1^3y - \frac{\omega_0}{k_1}z}, \quad \psi_2 = e^{k_2x + k_2^3y - \frac{\omega_0}{k_2}z}.$$

The corresponding expressions for ϕ_1 and ϕ_2 are

$$\phi_1 = \frac{1}{2k_1}\left(1+\psi_1^2\right), \quad \phi_2 = \frac{1}{2k_2}\left(1+\psi_2^2\right),$$

and Ω and τ are

$$\Omega = \frac{1}{k_1+k_2}\psi_1\psi_2, \quad \tau = \frac{1}{4k_1k_2}\left(1+\psi_1^2+\psi_2^2+\left(\frac{k_1-k_2}{k_1+k_2}\right)^2\psi_1^2\psi_2^2\right).$$

The graphics corresponding to h_z^1 and h_z^2 are shown in figure 3.

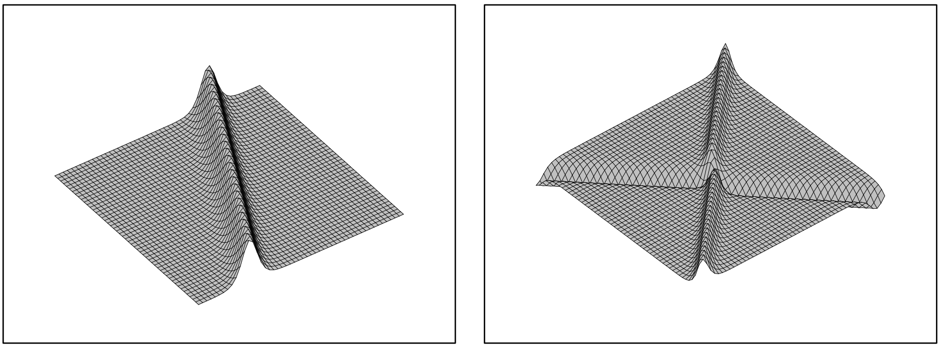

Figure 3. *Line solitons: $h_z^{1)}$ (left) and $h_z^{2)}$ (right).*

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A Note on Proper Poisson Actions

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We show that the fixed point set of a proper action of a Lie group G on a Poisson manifold M by Poisson automorphisms has a natural induced Poisson structure and we give several applications.

1 Introduction

In the present work, we consider a *Poisson action* $G \times M \to M$ of a Lie group G on a Poisson manifold M: this means that each element $g \in G$ acts by a Poisson diffeomorphism of M. We recall that the action is called *proper* if the map:

$$G \times M \to M \times M, \quad (g, p) \mapsto (p, g \cdot p),$$

is a proper map[1]. As usual, we will denote by M^G the *fixed point set* of the action:

$$M^G = \{p \in M : g \cdot p = p, \forall g \in G\}.$$

For proper actions, the connected components of the fixed point set M^G are (embedded) submanifolds of M. Notice that these components may have different dimensions.

The main result of this paper is the following:

Theorem 1. *Let $G \times M \to M$ be a proper Poisson action. Then the fixed point set M^G has a natural induced Poisson structure.*

This result is a generalization to Poisson geometry of a well-known proposition in symplectic geometry, due to Guillemin and Sternberg (see [6], Theorem 3.5), stating that fixed point sets of symplectic actions are symplectic submanifolds. We stress that the fixed point set *is not* a Poisson submanifold. This happens already in the symplectic case. In the general Poisson case, M^G will be a Poisson–Dirac submanifold in the sense of Crainic and Fernandes (see [1], Section 8) and Xu ([11]).

Proper symplectic/Poisson actions have been study intensively in the last 15 years. For example, the theory of (singular) reduction for Hamiltonian systems has been developed extensively for these kind of actions. We refer the reader

[1] A map $f : X \to Y$ between two topological spaces is called *proper* if for every compact subset $K \subset Y$, the inverse image $f^{-1}(K)$ is compact.

to the recent monograph by Ortega and Ratiu [7] for a nice survey of results in this area. Theorem 1 should have important applications in symmetry reduction, and this is one of our main motivations for this work. We refer the reader for an upcoming publication ([5]).

This paper is organized as follows. In Section 1, we recall the notion of a Poisson–Dirac submanifold, and some related results which are needed for the proof of Theorem 1. In Section 2, we prove our main result. In Section 3, we deduce some consequences and give some applications.

2 Poisson–Dirac Submanifolds

Let M be a Poisson manifold. For background in Poisson geometry we refer the reader to Vaisman's book [10]. We will denote by $\pi \in \mathfrak{X}^2(M)$ the Poisson bivector field so that the Poisson bracket is given by:

$$\{f,g\} = \pi(\mathrm{d}f, \mathrm{d}g), \quad \forall f,g \in C^\infty(M).$$

Recall that a Poisson submanifold $N \subset M$ is a submanifold which has a Poisson bracket and for which the inclusion $i : N \hookrightarrow M$ is a Poisson map:

$$\{f \circ i, g \circ i\}_M = \{f,g\}_N \circ i, \quad \forall f,g \in C^\infty(N).$$

Such Poisson submanifolds are, in a sense, extremely rare. In fact, they are collections of open subsets of symplectic leaves of M.

Example 1. Let M be a symplectic manifold with symplectic form ω. Recall that a *symplectic submanifold* is a submanifold $i : N \hookrightarrow M$ such that the restriction $i^*\omega$ is a symplectic form on N. For every even dimension $0 \le 2i \le \dim M$ there are symplectic submanifolds of dimension $2i$. On the other hand, the only Poisson submanifolds are the open subsets of M.

Crainic and Fernandes in [1] introduce the following natural extension of the notion of a Poisson submanifold:

Definition 1. Let M be a Poisson manifold. A submanifold $N \subset M$ is called a **Poisson–Dirac submanifold** if N is a Poisson manifold such that:

(i) the symplectic foliation of N is $N \cap \mathcal{F} = \{L \cap N : L \in \mathcal{F}\}$, and

(ii) for every leaf $L \in \mathcal{F}$, $L \cap N$ is a symplectic submanifold of L.

Note that if $(M, \{\cdot,\cdot\})$ is a Poisson manifold, then the symplectic foliation with the induced symplectic forms on the leaves, gives a smooth (singular) foliation with a smooth family of symplectic forms. Conversely, given a manifold M with a foliation $\mathcal{F}$ furnished with a smooth family of symplectic forms on the leaves, then we have a Poisson bracket on M defined by the formula[2]

$$\{f, g\} \equiv X_f(g),$$

for which the associated symplectic foliation is precisely $\mathcal{F}$. Hence, a Poisson structure can be defined by specifying its symplectic foliation. It follows that a submanifold N of a Poisson manifold M has at most one Poisson structure satisfying conditions (i) and (ii) above, and this Poisson structure is completely determined by the Poisson structure of M.

Example 2. If M is a symplectic manifold, then there is only one symplectic leave, and the Poisson–Dirac submanifolds are precisely the symplectic submanifolds of M.

Therefore, we see that the notion of a Poisson–Dirac submanifold generalizes to the Poisson category the notion of a symplectic submanifold.

Example 3. Let L be a symplectic leaf of a Poisson manifold, and $N \subset M$ a submanifold which is transverse to L at some x_0:

$$T_{x_0}M = T_{x_0}L \oplus T_{x_0}N.$$

Then one can check that conditions (i) and (ii) in Definition 1 are satisfied in some open subset in N containing x_0. In other words, if N is small enough then it is a Poisson–Dirac submanifold. Sometimes one calls the Poisson structure on N the transverse Poisson structure to L at x_0 (up to Poisson diffeomorphisms, this structure does not depend on the transversal N).

The two conditions in Definition 1 are not very practical to use. Let us give some alternative criteria to determine if a given submanifold is a Poisson–Dirac submanifold.

Observe that condition (ii) in the definition means that the symplectic forms on a leaf $L \cap N$ are the pull-backs $i^*\omega_L$, where $i : N \cap L \hookrightarrow L$ is the inclusion into a leaf and $\omega_L \in \Omega^2(L)$ is the symplectic form. Denoting by $\# : T^*M \to TM$ the bundle map determined by the Poisson bivector field, we conclude that we must have[3]:

$$TN \cap \#(TN^0) = \{0\}, \tag{1}$$

since the left-hand side is the kernel of the pull-back $i^*\omega_L$. If this condition holds, then at each point $x \in N$ we obtain a bivector $\pi_N(x) \in \wedge^2 T_xN$, and one can prove (see [1]):

[2]In a Poisson (or symplectic) manifold, we will denote by X_f the Hamiltonian vector field associated with a function $f : M \to \mathbb{R}$.

[3]For a subspace W of a vector space V, we denote by $W^0 \subset V^*$ its annihilator. Similarly, for a vector subbundle $E \subset F$, we denote by $E^0 \subset F^*$ its annihilator subbundle.

Proposition 1. *Let N be a submanifold of a Poisson manifold M, such that*

(a) equation (1) holds, and

(b) the induced tensor π_N is smooth.

Then π_N is a Poisson tensor and N is a Poisson–Dirac submanifold.

Notice that, by the remarks above, the converse of the proposition also holds.

Remark 1. Equation (1) can be interpreted in terms of the Dirac theory of constraints. This is the reason for the use of the term "Poisson–Dirac submanifold". We refer the reader to [1] for more explanations.

On the other hand, from Proposition 1, we deduce the following sufficient condition for a submanifold to be a Poisson–Dirac submanifold:

Corollary 1. *Let M be a Poisson manifold and $N \subset M$ a submanifold. Assume that there exists a subbundle $E \subset T_N M$ such that:*

$$T_N M = TN \oplus E$$

and $\#(E^0) \subset TN$. Then N is a Poisson–Dirac submanifold.

Proof. Under the assumptions of the corollary, one has a decomposition

$$\pi = \pi_N + \pi_E,$$

where $\pi_N \in \Gamma(\wedge^2 TN)$ and $\pi_E \in \Gamma(\wedge^2 E)$ are both smooth bivector fields. On the other hand, one checks easily that (1) holds. By Proposition 1, we conclude that N is a Poisson–Dirac submanifold. ■

There are Poisson–Dirac submanifolds which do not satisfy the conditions of this corollary. Also, the bundle E may not be unique. For a detailed discussion and examples we refer to [1].

Under the assumptions of the corollary, the Poisson bracket on the Poisson–Dirac submanifold $N \subset M$ is quite simple to describe: Given two smooth functions $f, g \in C^\infty(N)$, to obtain their Poisson bracket we pick extensions $\widetilde{f}, \widetilde{g} \in C^\infty(M)$ such that $\mathrm{d}_x \widetilde{f}, \mathrm{d}_x \widetilde{g} \in E^0_x$. Then the Poisson bracket on N is given by:

$$\{f, g\}_N = \{\widetilde{f}, \widetilde{g}\}|_N. \tag{2}$$

It is not hard to check that this formula does not depend on the choice of extensions.

Remark 2. Let M be a Poisson manifold and $N \subset M$ a submanifold. Assume that there exists a subbundle $E \subset T_N M$ such that E^0 is a *Lie subalgebroid* of T^*M (equivalently, E is a co-isotropic submanifold of the tangent Poisson manifold TM). Then E satisfies the assumptions of the corollary, so N is a Poisson–Dirac submanifold. This class of Poisson–Dirac submanifolds have very special geometric properties. They where first study by Xu in [11], which calls them **Dirac submanifolds**. They are further discussed by Crainic and Fernandes in [1], where they are called **Lie–Dirac submanifolds**.

3 Fixed Point Sets of Proper Poisson Actions

In this section we will give a proof of Theorem 1, which we restate now as follows:

Theorem 2. *Let $G \times M \to M$ be a proper Poisson action. Then the fixed point set M^G is a Poisson–Dirac submanifold.*

Since the action is proper, the fixed point set M^G is an embedded submanifold of M. Its connected components may have different dimensions, but our argument will be valid for each such component, so we will assume that M^G is a connected submanifold. The proof will consist in showing that there exists a subbundle $E \subset T_{M^G}M$ satisfying the conditions of Corollary 1.

First of all, given any action $G \times M \to M$ (proper or not) there exists a lifted action $G \times TM \to TM$. For proper actions we have the following basic property:

Proposition 2. *If $G \times M \to M$ is a proper action then there exists a G-invariant metric on TM.*

For a proof of this fact and other elementary properties of proper actions, we refer to [3]. Explicitly, the G-invariance of the metric means that:

$$\langle g \cdot v, g \cdot w \rangle_{g \cdot p} = \langle v, w \rangle_p, \quad \forall v, w \in T_pM.$$

where $g \in G$ and $p \in M$.

We fix, once and for all, a G-invariant metric $\langle\ ,\ \rangle$ for our proper Poisson action $G \times M \to M$. Let us consider the subbundle $E \subset T_{M^G}M$ which is orthogonal to TM^G:

$$E = \{v \in T_{M^G}M : \langle v, w \rangle = 0, \forall w \in TM^G\}.$$

We have:

Lemma 1. *$T_{M^G}M = TM^G \oplus E$ and $\#(E^0) \subset TM^G$.*

Proof. Since $E = (TM^G)^\perp$, the decomposition $T_{M^G}M = TM^G \oplus E$ is obvious. Now for a proper action, we have $(TM)^G = TM^G$ so this decomposition can also be written as:

$$T_{M^G}M = (TM)^G \oplus E, \tag{3}$$

On the other hand, we have the lifted cotangent action $G \times T^*M \to T^*M$, which is related to the lifted tangent action by $g \cdot \xi(v) = \xi(g^{-1} \cdot v)$, $\xi \in T^*M, v \in TM$. We claim that:

$$E^0 \subset (T^*M)^G. \tag{4}$$

In fact, if $v \in TM$ we can use (3) to decompose it as $v = v_G + v_E$, where $v_G \in (TM)^G$ and $v_E \in E$. Hence, for $\xi \in E^0$ we find:

$$\begin{aligned} g \cdot \xi(v_G + v_E) &= \xi(g^{-1} \cdot v_G + g^{-1} \cdot v_E) = \xi(v_G) + \xi(g^{-1} \cdot v_E) = \xi(v_G) \\ &= \xi(v_G) + \xi(v_E) = \xi(v_G + v_E). \end{aligned}$$

We conclude that $g \cdot \xi = \xi$ and (4) follows.

Since $G \times M \to M$ is a Poisson action, we see that $\# : T^*M \to TM$ is a G-equivariant bundle map. Hence, if $\xi \in E^0$, we obtain from (4) that:

$$g \cdot \#\xi = \#(g \cdot \xi) = \#\xi.$$

This means that $\#\xi \in (TM)^G = TM^G$, so the lemma holds. ■

This lemma shows that the conditions of Corollary 1 are satisfied, so M^G is a Poisson–Dirac submanifold and the proof of Theorem 2 is completed.

Remark 3. If one works further with the decomposition (3) and its transposed version, it is not hard to show that E^0 is actually a Lie subalgebroid of T^*M. Therefore, the fixed point set M^G of a proper Poisson action is, in fact, a Lie–Dirac submanifold of M (see Remark 2).

Remark 4. Special cases of Theorem 2 where obtained by Damianou and Fernandes in [2] for a compact Lie group G, and by Fernandes and Vanhaecke in [4] for a reductive algebraic group G. Xiang Tang also proves a version of this theorem in his PhD thesis [9].

Notice that the Poisson bracket of functions $f, g \in C^\infty(M^G)$ can be obtained simply by choosing G-invariant extensions $\widetilde{f}, \widetilde{g} \in C^\infty(M)^G$, and setting:

$$\{f, g\}_{M^G} = \{\widetilde{f}, \widetilde{g}\}|_{M^G}.$$

This follows from equation (2) and the remark that for any such G-invariant extensions we have $\mathrm{d}_{M^G}\widetilde{f}, \mathrm{d}_{M^G}\widetilde{g} \in E^0$. It is an instructive exercise to prove directly that the bracket on M^G does not depend on the choice of extensions.

4 Applications and Further Results

Every compact Lie group action is proper. In particular, a finite group action is always a proper. The case $G = \mathbb{Z}_2$ leads to the following result:

Corollary 2. *Let $\phi : M \to M$ be an involutive Poisson automorphism of a Poisson manifold M. The fixed point set $\{p \in M : \phi(p) = p\}$ has a natural induced Poisson structure.*

Proof. Apply Theorem 2 to the Poisson action of the group $G = \{\mathrm{Id}, \phi\}$. ■

This result is known in the literature as the Poisson Involution Theorem (see [2, 4, 11]). It has been applied in [2, 4] to explain the relationship between the geometry of the Toda and Volterra lattices, and there should be similar relations between other known integrable systems. In this respect, it should be interesting to find extensions of our results to infinite dimensional manifolds and actions.

Recall that if an action $G \times M \to M$ is proper and free then the space of orbits M/G is a smooth manifold. For general non-free actions the orbit space can be a very pathological topological space. However, for proper actions the singularities of the orbit space are very much controlled, and M/G is a nicely stratified topological space. For proper symplectic actions there is a beautiful theory of singular symplectic quotients due to Lerman and Sjamaar [8] which describes the geometry of M/G. For proper Poisson actions one should expect that the orbit space still exhibits some nice Poisson geometry. In fact, we will explain in [5] that Theorem 2 leads to the following result that generalizes a theorem due to Lerman and Sjamaar:

Theorem 3. *Let $G \times M \to M$ be a proper Poisson action. Then the quotient M/G is a Poisson stratified space.*

Note that if a Poisson action is proper *and* free then the orbit space is a smooth Poisson manifold. In this case one can identify the smooth functions on the quotient M/G with the G-invariant functions on M:

$$C^\infty(M/G) \simeq C^\infty(M)^G.$$

In the non-free case, the smooth structure of M/G as a stratified space also leads to such an identification. Rather than explaining in detail the notion of a Poisson stratified space (see the upcoming paper [5]), we will illustrate this result with an example.

Example 4. Let $\mathbb{C}^{n+1}$ be the complex $n+1$-dimensional space with holomorphic coordinates $(z_0, \ldots, z_n)$ and anti-holomorphic coordinates $(\overline{z}_0, \ldots, \overline{z}_n)$. On the (real) manifold $\mathbb{C}^{n+1} - 0$ we will consider a (real) quadratic Poisson bracket of the form:

$$\{z_i, z_j\} = a_{ij} z_i z_j, \quad \{z_i, \overline{z}_j\} = \{\overline{z}_i, \overline{z}_j\} = 0.$$

where $A = (a_{ij})$ is a skew-symmetric matrix.

The group $\mathbb{C}^*$ of non-zero complex numbers acts on $\mathbb{C}^{n+1} - 0$ by multiplication of complex numbers. This is a free and proper Poisson action, so the quotient $\mathbb{C}P(n) = \mathbb{C}^{n+1} - 0/\mathbb{C}^*$ inherits a Poisson bracket.

Let us consider now the action of the n-torus $\mathbb{T}^n$ on $\mathbb{C}^{n+1} - 0$ defined by:

$$(\theta_1, \ldots, \theta_n) \cdot (z_0, z_1, \cdots, z_n) = (z_0, e^{i\theta_1} z_1, \cdots, e^{i\theta_n} z_n).$$

This is a Poisson action that commutes with the $\mathbb{C}^*$-action. It follows that the $\mathbb{T}^n$-action descends to a Poisson action on $\mathbb{C}P(n)$. Note that the action of $\mathbb{T}^n$ on $\mathbb{C}P(n)$ is proper but not free. The quotient $\mathbb{C}P(n)/\mathbb{T}^n$ is not a manifold but it can be identified with the standard simplex

$$\Delta^n = \Big\{(\mu_0, \ldots, \mu_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^n \mu_i = 1, \mu_i \geq 0\Big\}.$$

This identification is obtained via the map $\mu : \mathbb{C}P(n) \to \Delta^n$ defined by:

$$\mu([z_0 : \cdots : z_n]) = \left(\frac{|z_0|^2}{|z_0|^2 + \cdots + |z_n|^2}, \cdots, \frac{|z_n|^2}{|z_0|^2 + \cdots + |z_n|^2} \right).$$

Let us describe the Poisson stratification of $\Delta^n = \mathbb{C}P(n)/\mathbb{T}^n$. The Poisson bracket on Δ^n is obtained through the identification:

$$C^\infty(\Delta^n) \simeq C^\infty(\mathbb{C}P(n))^{\mathbb{T}^n}.$$

For that, we simply compute the Poisson bracket between the components of the map μ. A more or less straightforward computation will show that:

$$\{\mu_i, \mu_j\} = \left(a_{ij} - \sum_{l=0}^{n} (a_{il} + a_{lj}) \mu_l \right) \mu_i \mu_j, \quad (i, j = 0, \ldots, n). \tag{5}$$

Now notice that (5) actually defines a Poisson bracket on $\mathbb{R}^{n+1}$. For this Poisson bracket, the interior of the simplex and its faces are Poisson submanifolds: a face $\Delta_{i_1,\ldots,i_{n-d}}$ of dimension $0 \le d \le n$ is given by equations of the form:

$$\sum_{i=0}^{n} \mu_i = 1, \quad \mu_{i_1} = \cdots = \mu_{i_{n-d}} = 0, \quad \mu_i > 0 \text{ for } i \notin \{i_1, \ldots, i_{n-d}\}.$$

These equations define Poisson submanifolds since:

(a) the bracket $\{\mu_i, \mu_l\}$ vanishes whenever $\mu_l = 0$, and

(b) the bracket $\{\mu_i, \sum_{l=0}^{n} \mu_l\}$ vanishes whenever $\sum_{l=0}^{n} \mu_l = 1$.

Therefore, the Poisson stratification of Δ^n consists of strata formed by the faces of dimension $0 \le d \le n$, which are smooth Poisson manifolds.

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Nonclassical Potential Symmetries for a Third Order Nonlinear Diffusion Equation

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In this paper we consider a class of third order diffusion equations which are of interest in mathematical physics. For some of these equations nonlocal potential symmetries are derived. These *nonclassical potential* symmetries allow us to increase the number of solutions. These solutions are neither solutions arising from nonclassical symmetries nor solutions arising from classical potential symmetries.

1 Introduction

In the past years we can observe a significant progress in application on symmetries to the study of linear and nonlinear partial differential equations of physical importance, as well as in finding exact solutions for these equations.

Motivated by the fact the symmetry reductions for many PDE's are known that are not obtained by using the classical Lie method there have been several generalizations of the classical Lie group method for symmetry reductions.

Bluman and Cole [3] developed the nonclassical method to study the symmetry reductions of the heat equation. The basic idea of the method is to require that the N order PDE

$$\Delta = \Delta\left(x, t, u, u^{(1)}(x,t), \ldots, u^{(N)}(x,t)\right) = 0$$

where $(x,t) \in \mathbb{R}^2$, are the independent variables, $u \in \mathbb{R}$ is the dependent variable and $u^{(l)}(x,t)$ denote the set of all partial derivatives of l order of u and the invariance surface condition

$$\xi u_x + \tau u_t - \phi = 0$$

which is associated with the vector field

$$\mathbf{v} = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \phi(x,t,u)\partial_u, \tag{1}$$

are both invariant under the transformation with infinitesimal generator (1). Since then, a great number of papers have been devoted to the study of nonclassical symmetries of nonlinear PDE's in both one and several dimensions.

An obvious limitation of group-theoretic methods based in local symmetries, in their utility for particular PDE's, is that many of these equations does not have local symmetries.

Akhatov, Gazizov and Ibragimov [1] gave nontrivial examples of nonlocal symmetries generated by heuristic procedures.

In [4, 5] Bluman introduced a method to find a new class of symmetries for a PDE. By writing a given PDE, denoted by R$\{x, t, u\}$ in a conserved form a related system denoted by S$\{x, t, u, v\}$ as additional dependent variables is obtained. Any Lie group of point transformations admitted by S$\{x, t, u, v\}$ induces a symmetry for R$\{x, t, u\}$; when at least one of the generators of the group depends explicitly of the potential, then the corresponding symmetry is neither a point nor a Lie-Bäcklund symmetry. These symmetries of R$\{x, t, u\}$ are called *potential* symmetries.

Knowing that an associated system to the Boussinesq equation has the same classical symmetries as the Boussinesq equation, Clarkson [7] proposed as an open problem if an auxiliary system of the Boussinesq equation does posses more or less nonclassical symmetries than the equation itself. Bluman claims [2] that the ansatz to generate nonclassical solutions of the associated system could yield solutions of the original equation which are neither nonclassical solutions nor solutions arising from potential symmetries.

However as far as we know these new class of potential symmetries, which we have called *nonclassical potential* symmetries, were first derived in [11] for the Burgers equation and in [10] for the porous medium equation. After that were have derived *nonclassical potential symmetries*, in different way for some interesting equations. We have obtained *nonclassical potential* symmetries for the Burgers equation [11] as nonclassical symmetries of the integrated equation and in [13] as nonclassical symmetries of the potential associated system. Nonclassical potential symmetries were derived in [12] for a nonlinear diffusion equation which arises in modelling two-phase flow in porous media and have multiple applications.

For a dissipative KdV equation *nonclassical potential* symmetries were derived in [14] by considering the integrated equation.

In previous works [9, 15], we have obtained respectively nonclassical symmetries for a porous medium equation with absorption and a porous medium with convection. The classical potential symmetries were classified in [8] for the porous medium equation when it can be written in a conserved form

$$u_t = [(u^n)_x + m^{-1} f(x) u^m]_x, \tag{2}$$

and lead for some special values of the parameters and some functions $f(x)$ to the linearization of (2) by a non-invertible mapping which transforms any solution of the a linear equation

$$w_{z_2} - w_{z_1 z_1} - cw = 0 \tag{3}$$

to a solution of (2). The *nonclassical potential* symmetries were derived [10] as nonclassical symmetries of the associated equation (3).

In [6] P.A. Clarkson found that the solutions arising from the nonclassical symmetries of the associated potential system of the shallow water equation were obtainable by the nonclassical symmetries of the shallow water equation. Consequently, it remain as an open problem the existence of nonclassical potential symmetries, in the sense that they lead to new solutions.

We were able to solve this problem for the Fokker–Planck equation

$$u_t = u_{xx} + [f(x)u]_x. \tag{4}$$

The classical symmetries for (4) were derived in [5]. The classical potential symmetries were derived by Pucci and Saccomandi in [20]. We have studied in [17,18] the nonclassical symmetries of the Fokker–Planck equation, as well as the *nonclassical potential symmetries*. We were able to find a class of functions $f(x)$ for which equation (4) does not admit, classical Lie symmetries, nonclassical symmetries nor classical potential symmetries but it admits nonclassical potential symmetries.

In [16], for equations that model diffusion processes

$$u_t = [u^n u_x]_x,$$

we have derived *nonclassical potential* symmetries, which are realized as nonclassical symmetries of an associated system. The significance of these symmetries were be pointed out by the fact that for this diffusion equations that model fast processes (with $n = -1$) classical potential symmetries were not admitted.

In [21] the authors have derived nonclassical potential symmetries for Richard's equation and in [19] the authors have derived nonclassical potential symmetries for some linear wave equations in $(1 + 1)$-dimensions.

The aim of this work is to derive classical, classical potential and nonclassical potential symmetries for a third order diffusion equation Although in this case the infinitesimals of these *nonclassical potential* symmetries do not depend on $v = \int u(x)dx$ they do not project on to any of the infinitesimals corresponding to the classical or nonclassical generators. Consequently the new exact solutions that we found *can not* be obtained by using classical Lie symmetries, nonclassical symmetries nor classical potential symmetries.

2 Lie Symmetries

One of the mathematical models for diffusion processes is the third order nonlinear diffusion equation

$$u_t = [u^n u_{xx}]_x, \tag{5}$$

where $u(x,t)$ is a function of position x and time t and may represent the temperature. In order to apply the classical method to equation (5), we consider the one-parameter Lie group of infinitesimal transformations in (x,t,z,u) given by

$$\begin{aligned}
x^* &= x + \varepsilon\xi(x,t,u) + \mathcal{O}(\varepsilon^2),\\
t^* &= t + \varepsilon\tau(x,t,u) + \mathcal{O}(\varepsilon^2),\\
u^* &= u + \varepsilon\eta(x,t,u) + \mathcal{O}(\varepsilon^2),
\end{aligned}$$

where ε is the group parameter. Then one requires that this transformation leaves invariant the set of solutions of (5). This yields to an overdetermined, linear system of equations for the infinitesimals $\xi(x,t,u)$, $\tau(x,z,t,u)$ and $\phi(x,z,t,u)$. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form (1). Having determined the infinitesimals, the symmetry variables are found by solving the invariant surface condition

$$\Phi_1 \equiv \xi u_x + \tau u_t - \phi = 0.$$

The classical Lie method applied to the the PDE (5) leads to a three-parameter Lie group. Associated with this Lie group we have the Lie algebra which can be represented by the generators, these generators are

$$\mathbf{v}_1 = \partial_t, \quad \mathbf{v}_2 = \partial_x, \quad \mathbf{v}_3 = (n+1)x\partial_x + 3t\partial_t + 3u\partial_u,$$

with n an arbitrary constant.

2.1 Classical Potential Symmetries

As equation (5) is a conservation law, in order to derive the classical potential symmetries of (5), we consider the associated potential system

$$v_x = u, \quad v_t = u^n u_{xx}. \tag{6}$$

In this case the classical Lie analysis is based upon the infinitesimal transformations

$$\begin{aligned} x^* &= x + \varepsilon\xi(x,t,u,v) + \mathcal{O}(\varepsilon^2), \quad & t^* &= t + \varepsilon\tau(x,t,u,v) + \mathcal{O}(\varepsilon^2), \\ u^* &= u + \varepsilon\phi(x,t,u,v) + \mathcal{O}(\varepsilon^2), \quad & v^* &= v + \varepsilon\psi(x,t,u,v) + \mathcal{O}(\varepsilon^2). \end{aligned}$$

The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form $\mathbf{v} = \xi\partial_x + \tau\partial_t + \phi\partial_u + \psi\partial_v$.

The classical Lie method applied to system (6) leads to a five-parameter Lie group. Associated with this Lie group we have the Lie algebra which can be represented by the generators, these generators are

$$\begin{aligned} &\mathbf{v}_1 = \partial_t, \quad \mathbf{v}_2 = \partial_x, \quad \mathbf{v}_3 = \partial_v, \quad \mathbf{v}_4 = nx\partial_x + 3\partial_u - (n+3)v\partial_v, \\ &\mathbf{v}_5 = nt\partial_x - u\partial_u + v\partial_v, \end{aligned}$$

where n is an arbitrary constant. We can easily see that none of these generators depend explicitly on v, that is condition $\xi_v^2 + \tau_v^2 + \phi_v^2 \neq 0$ is not satisfied. Consequently, we can assure that point symmetries of (6) project onto local symmetries of equation (5) and equation (5) does not admit classical potential symmetries by considering the associated potential system (6).

Nevertheless, when $n = 3$ we can also consider the following associated potential system

$$v_x = u^{-1}, \quad v_t = -uu_{xx} - u_x^2. \tag{7}$$

The classical Lie method applied to system (7) leads to an infinite-parameter Lie group of point transformations. Associated with this Lie group we have the Lie algebra which can be represented by the generators, these generators are

$$\mathbf{v}_1 = \partial_t, \quad \mathbf{v}_2 = \partial_v, \quad \mathbf{v}_3 = x\partial_x + u\partial_u, \quad \mathbf{v}_4 = 3t\partial_t - u\partial_u + v\partial_v,$$

and the following potential generator

$$\mathbf{v}_\alpha = \alpha(t,v)\partial_x + \alpha_v(t,v)\partial_u$$

here $\alpha(t,v)$ is an arbitrary function of t and v satisfying the linear equation $\alpha_t - \alpha_{vvv} = 0$. This generator $\mathbf{v}_\alpha$ allow us to linearize (5) when $n = 3$.

2.2 Nonclassical Potential Symmetries

In order to compute the nonclassical-potential symmetries of equation (5) we apply the nonclassical method to system (6). In the case $\tau \neq 0$ we can set $\tau \equiv 1$ without loss of generality. Then the nonclassical method applied to (6) give rise to six nonlinear determining equations for the infinitesimals that lead to

$$\xi = \xi(x,t), \quad \psi = \alpha(x,t)v + \beta(x,t), \quad \phi = (\psi_v - \xi_x)u + \psi_x,$$
$$\alpha = \xi_x + \delta(t), \quad \xi = -\delta x + \nu(t),$$

with β, δ and ν related by a nonlinear equation.

After solving the determining system a complete classification of the nonclassical system of the governing equation has been performed for $\tau \neq 0$ and we can state:

Case 1: $n \neq -1$. The nonclassical method applied to system (6) does not yield any new symmetry different from the ones obtained by Lie classical method.

Case 2: $n = -1$. By applying the nonclassical method we get that $\alpha = 0$ and β, δ, ν are related by the following conditions

$$\delta_t x - 2\delta^2 x + 2\delta\nu = 0, \quad \beta_x\delta x - \beta_x\nu - 2\beta\delta - \beta_t = 0, \quad \beta\beta_x - \beta_{xxx} = 0.$$

We find that for $\delta = 0$, $\nu = c$ and $c = \text{const}$ the infinitesimals become

$$\xi = c, \quad \psi = \beta(x,t), \quad \phi = \beta_x$$

and β must satisfy

$$\beta_{xxx} - \beta\beta_x = 0, \quad \beta_t - c\beta_x = 0 \quad \text{i.e.} \quad \beta = \beta(w), \quad w = x + ct.$$

From the characteristic equation we obtain the independent similarity variable $z = x - ct$ and the similarity solution

$$v = \frac{1}{2c}\int \beta(w)dw + g(z) = f(x+ct) + g(x-ct), \tag{8}$$

where f satisfies the condition

$$f''' - c(f')^2 = k, \tag{9}$$

with k being an arbitrary constant. Finally, by introducing (8) into (6) we arrive at the reduced equation for the dependent similarity variable g:

$$g''' + c(g')^2 = -k. \tag{10}$$

Now, from a couple of solutions of equations (9) and (10) we have that a solution of (5) is given by

$$u(x,t) = v_x(x,t) = f'(x+ct) + g'(x-ct). \tag{11}$$

These solutions represent a linear superposition of tho waves propagating with velocities c and $-c$.

An explicit solution for (5) is the two soliton solution

$$u = 3c^{-1}\sqrt{-c\,k}\left(\operatorname{sech}^2\left(\frac{\sqrt[4]{-c\,k}(x-ct)}{\sqrt{2}} + \mathrm{k}_1\right) - \frac{1}{3}\right)$$
$$- 3c^{-1}\sqrt{-c\,k}\left(\operatorname{sech}^2\left(\frac{\sqrt[4]{-c\,k}(x-ct)}{\sqrt{2}} + \mathrm{k}_2\right) - \frac{1}{3}\right).$$

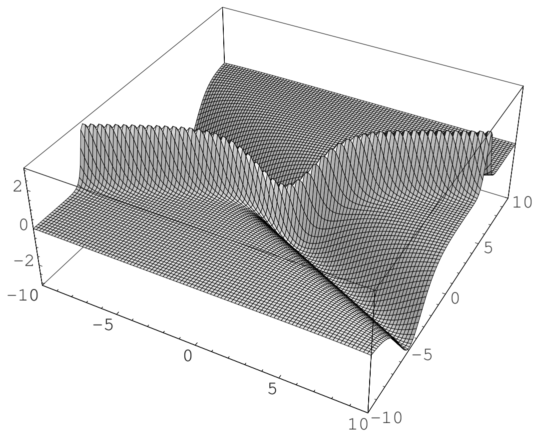

Figure 1. Two soliton solution

We must remark that although the infinitesimals do not depend on potential $v = \int u(x)dx$, they do no project on to any of the infinitesimals corresponding to the classical or nonclassical generators of (5). Indeed, it is easy to see that for $\tau \neq 0$, every nonclassical symmetry of (5) corresponds to a classical one. Consequently, solutions (11) are solutions of (5) which *can not* be obtained by using classical or nonclassical symmetries of (5).

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Ovsyannikov Vortex in Magnetohydrodynamics

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Partially invariant solution on the group of rotations of ideal magnetohydrodynamics equations is observed. It is shown that in nonreducible solution the velocity and magnetic field vectors of the particle are coplanar to the radius-vector of the particle. The initial system is reduced to invariant subsystem and involutive equations for non-invariant function. Description of the stationary solution is given.

1 Introduction

In the space $\mathbb{R}^3(x, y, z) \times \mathbb{R}^3(u, v, w)$ it is given a group $O(3)$ of simultaneous rotations of the subspaces $\mathbb{R}^3(x, y, z)$ and $\mathbb{R}^3(u, v, w)$. The corresponding Lie algebra is generated by the following operators:

$$\begin{aligned} X_1 &= z\partial_y - y\partial_z + w\partial_v - v\partial_w, \\ X_2 &= x\partial_z - z\partial_x + u\partial_w - w\partial_u, \\ X_3 &= y\partial_x - x\partial_y + v\partial_u - u\partial_v. \end{aligned} \tag{1}$$

In order to check the necessary conditions of existence of $O(3)$-invariant solution it is necessary to write the matrix of coefficients of operators (1).

$$M(\xi|\eta) = \left(\begin{array}{ccc|ccc} 0 & z & -y & 0 & w & -v \\ -z & 0 & x & -w & 0 & u \\ y & -x & 0 & v & -u & 0 \end{array} \right)$$

One can easily check that

$$\operatorname{rank} M(\xi) < \operatorname{rank} M(\xi|\eta). \tag{2}$$

Here $M(\xi)$ is a matrix of the first three columns of $M(\xi|\eta)$. Relation (2) proves a well-known fact [2], that nonsingular $O(3)$-invariant solution of any system of equations for sought functions $\mathbf{u} = (u, v, w)$ and independent variables $\mathbf{x} = (x, y, z)$ does not exist. However, one can use an ansatz

$$\mathbf{u} = f(|\mathbf{x}|)\,\mathbf{x}, \tag{3}$$

which corresponds to a singular $O(3)$-invariant solution. Solutions of the type (3) are usually called the rotationally-invariant ones.

Group $O(3)$ gives rise to another type of solution, namely, a partially invariant one. Let us observe a spherical coordinate system

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta. \tag{4}$$

The decomposition of a vector field $\mathbf{u}$ on the basis of a spherical coordinate system gives

$$\begin{aligned} u_r &= u\sin\theta\cos\varphi + v\sin\theta\sin\varphi + w\cos\theta, \\ u_\theta &= u\cos\theta\cos\varphi + v\cos\theta\sin\varphi - w\sin\theta, \\ u_\varphi &= -u\sin\varphi + v\cos\varphi. \end{aligned} \tag{5}$$

With these notations invariants of the $O(3)$ can be written as follows

$$r = |\mathbf{x}|, \quad u_r, \quad u_\theta^2 + u_\varphi^2.$$

The representation of invariant part of solution is distinguished by the following conditions

$$u_r = U(r), \quad u_\theta^2 + u_\varphi^2 = M^2(r). \tag{6}$$

Only two of three functions, which determine a vector field $\mathbf{u}$, are specified by the equalities (6). The third value is assumed to be an arbitrary function of (r, θ, φ), namely

$$u_\theta = M\cos\omega, \quad u_\varphi = M\sin\omega, \quad \omega = \omega(r, \theta, \varphi). \tag{7}$$

For any system of equations, which admits the Lie group $O(3)$, relations (6), (7) define a representation of $O(3)$-partially invariant solution. Functions U and M will be called further the invariant ones, since they depend only on invariant variable r. On the contrary, function ω, which depends on all the independent variables (r, θ, φ), will be called the non-invariant one.

Notice, that any other sought functions, which are not transformed under $O(3)$ action must be assumed as the invariant ones, i.e. dependent only on the invariant variable r. On the other hand, if equations involve some additional independent variables, for example, time t, the dependence on these variables must be added to both invariant and non-invariant functions.

The substitution of obtained representation of solution into the investigated system of equations usually gives equations of two types: invariant ones, which involve only invariant functions and variables, and non-invariant ones, which involve the non-invariant function ω. The latter equations should be observed as an overdetermined system for non-invariant function ω. Its compatibility conditions enlarge the invariant subsystem. Solution of the invariant subsystem and consequent determination of the non-invariant function provides the solution of the initial equations.

First, solution of the type (6), (7) was investigated by L.V. Ovsyannikov [1] for Euler equations for ideal compressible and incompressible fluid. In his work the overdetermined system for function ω was completed to involution. Its general implicit solution, which involve an arbitrary function of two arguments, was also given. All the invariant functions were determined from the well-defined system of PDEs with two independent variables. The main features of the fluid flow, governed by the obtained solution, were pointed out. Namely, it was shown, that trajectories of particles are flat curves in 3D space. The position and orientation of the plane, which contains the trajectory, depends on the particle's initial location. Another noted feature is that the continuous solution can be determined not in the whole 3D space, but in some moving or stationary channels.

The title of Ovsyannikov's article "singular vortex" is related to the special choice of non-invariant function, which guarantees the continuous initial data for the solution. Afterwards, the name "singular vortex" was awarded to all solutions, which are partially invariant with respect to the group $O(3)$.

Independent investigation of the singular vortex for ideal incompressible fluid is performed by H.V. Popovych [3]. The article includes the investigation of the overdetermined system for non-invariant function and also investigation of symmetry reductions of invariant subsystem. Further analysis of singular vortex for ideal compressible fluid can be found in [4, 5].

The general concept of singular vortex was proposed by L.V. Ovsyannikov at his lecture on the conference "New mathematical models at mechanics: construction and investigation", which was held in May 10–14, 2004 in Novosibirsk, Russia. Ovsyannikov has also shown the examples of acoustic singular vortex and irrotational singular vortex. According to the suggestion by corresponding member of Russian Academy of Science, S.I. Pohozhaev, the singular vortex is called now as "Ovsyannikov vortex".

In the present work we investigate a singular vortex for the mathematical model of ideal compressible magnetohydrodynamics. The analysis is complicated by simultaneous presence of two vector fields: velocity and magnetic. The system for non-invariant function ω is strongly overdetermined but it is possible to find a condition, under which the system is in involution and has a functional arbitrariness of the solution. The latter condition is that for any particle of fluid it's radius vector, velocity and magnetic field vectors are coplanar. In this case the non-invariant function is determined from the implicit finite (not differential) equation, which involves one arbitrary function of one argument.

The main features of the magnetic fluid flow, governed by the singular vortex are similar to those, obtained for ideal gas dynamics. Namely, trajectories and magnetic field lines are also flat curves. The solution is defined not in the whole space, but in some channel. The description of stationary solution is given.

2 Magnetohydrodynamics Equations

The equations for ideal fluid with infinite conductivity are the following:

$$\begin{aligned}
&D\,\rho + \rho\,\mathrm{div}\,\mathbf{u} = 0,\\
&D\,\mathbf{u} + \rho^{-1}\nabla p + \rho^{-1}\mathbf{H}\times\mathrm{rot}\,\mathbf{H} = 0,\\
&D\,p + A(p,\rho)\,\mathrm{div}\,\mathbf{u} = 0,\\
&D\,\mathbf{H} + \mathbf{H}\,\mathrm{div}\,\mathbf{u} - (\mathbf{H}\cdot\nabla)\mathbf{u} = 0,\\
&\mathrm{div}\,\mathbf{H} = 0, \quad D = \partial_t + \mathbf{u}\cdot\nabla.
\end{aligned} \tag{8}$$

Here $\mathbf{u} = (u, v, w)$ is the velocity vector, p, ρ are pressure and density, $\mathbf{H} = (H^1, H^2, H^3)$ is the magnetic field. All functions depend on time t and coordinates (x, y, z). Function $A(p, \rho)$ depends on the state equation of the fluid. Note that system (8) is overdetermined, it contains 9 equations for 8 sought functions. However, the system (8) is in involution since the last equation can be observed as a restriction for initial data. According to the induction equation if the last equation is satisfied at some moment of time, then it will also be valid for all times of solution existence.

The admitted group for the system (8) for the case of polytropic state equation $A(p, \rho) = \gamma p$ (γ is the adiabatic exponent) is known [6]. It is a 13-dimensional extension of the Euclidean group via the time-translation and dilatations.

The admitted group includes a simple subgroup $O(3)$ of simultaneous rotations in the spaces $\mathbb{R}^3(\mathbf{x})$, $\mathbb{R}^3(\mathbf{u})$ and $\mathbb{R}^3(\mathbf{H})$. Construction of the singular vortex for equations (8) demands calculation of invariants of $O(3)$ in the space of functions and variables.

2.1 The Representation of the Solution

For the convenience we introduce the spherical coordinate system (4). Vectors $\mathbf{u}$ and $\mathbf{H}$ are decomposed by spherical frame according to (5). The following individual notations of components of velocity and magnetic field vectors are introduced (see fig. 1)

$$\begin{aligned}
&v_r = U, \quad v_\theta = M\cos\Omega, \quad v_\varphi = M\sin\Omega;\\
&H_r = H, \quad H_\theta = N\cos\Sigma, \quad H_\varphi = N\sin\Sigma.
\end{aligned}$$

Here U and H are radial components of $\mathbf{u}$ and $\mathbf{H}$. Values M and N denote an absolute value of it's tangential to spheres $r = \mathrm{const}$ components. Functions Ω and Σ are the angles between tangential components of $\mathbf{u}$ and $\mathbf{H}$ and meridional direction.

In these notations the invariants of the group $O(3)$ could be chosen as follows:

$$t, \quad r, \quad U, \quad M, \quad H, \quad N, \quad \Omega - \Sigma, \quad p, \quad \rho.$$

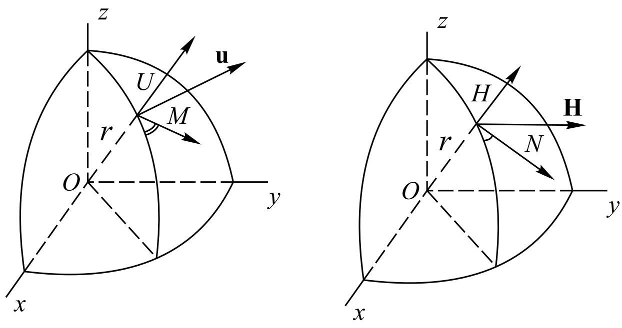

Figure 1. The decomposition of the velocity and magnetic field vectors

According to the described algorithm the representation of partially invariant solution can be constructed in the form

$$\begin{aligned} &U = U(t,r), \quad M = M(t,r), \quad H = H(t,r), \quad N = N(t,r), \\ &\Sigma = \sigma(t,r) + \omega(t,r,\theta,\varphi), \quad \Omega = \omega(t,r,\theta,\varphi), \quad p = p(t,r), \quad \rho = \rho(t,r). \end{aligned} \tag{9}$$

2.2 Equations of the Ovsyannikov Vortex

Substitution of representation (9) into MHD (8) provides a system Π of 9 equations for invariant functions U, H, N, M, ρ, p and non-invariant function ω. This system should be observed as an overdetermined system of the first-order PDEs for function $\omega(t,r,\theta,\varphi)$ under assumption that all the invariant functions are known. The compatibility conditions for this system give the equations for invariant functions. This procedure is illustrated by the following diagram

$MHD \xrightarrow{(9)}$ [Π — equations for ω] ↗ [CS — compatible system for ω]
↘ [IS — equations for invariant functions]

In order to omit trivial situations, we observe only the case, when function ω is determined with functional arbitrariness. Function ω has only constant arbitrariness if it is possible to express all the first-order derivatives of ω from the system Π. To impose a ban on this situation we calculate a matrix of coefficient of derivatives of function ω and demand it to be of rank 3 or less. The demand is satisfied only in the following 3 cases:

- $M = 0$ — radial velocity field;
- $N = 0$ — radial magnetic field;
- $\sigma = 0$ — coincidence of deviation angles of the tangential component of the velocity and magnetic vector fields.

All these 3 cases signify that velocity vector $\mathbf{u}$ and magnetic field vector $\mathbf{H}$ in each point are coplanar to the radius-vector of the point. Further we observe the most general case $\sigma = 0$, i.e. $\Sigma = \Omega = \omega(t, r, \theta, \varphi)$. New notations are introduced:

$$M_1 = r^{-1} M, \quad H_1 = r^2 H, \quad N_1 = r\, N, \quad H_1 = \cos^{-1} \tau. \tag{10}$$

The invariant subsystem IS can be presented as

$$\begin{aligned}
&D_0\, M_1 + \frac{2}{r}\, U M_1 - \frac{1}{r^4 \rho \cos\tau}\, N_{1r} = 0,\\
&D_0\, N_1 + N_1 U_r - \frac{1}{\cos\tau}\, M_{1r} - M_1 N_1 \tan\tau = 0,\\
&D_0\, p + A(p,\rho)\left(U_r + \frac{2}{r}\, U - M_1 \tan\tau\right) = 0,\\
&D_0\, U + \frac{1}{\rho}\, p_r + \frac{N_1 N_{1r}}{r^2 \rho} - r M_1^2 = 0, \qquad \tau_r = N_1 \cos\tau,\\
&D_0\, \rho + \rho\left(U_r + \frac{2}{r}\, U - M_1 \tan\tau\right) = 0, \qquad D_0\, \tau = M_1,\\
&D_0 = \partial_t + U \partial_r.
\end{aligned} \tag{11}$$

This overdetermined system of 7 equations for 6 functions is in involution (compatible and locally solvable) since the compatibility condition of last two equations of (11) (equations for τ) coincide with the second equation of (11).

Equations CS for non-invariant function ω are the following:

$$\begin{aligned}
&N_1\, \omega_t + (N_1 U - H_1 M_1)\, \omega_r = 0,\\
&H_1 \cos\omega\, \omega_r + N_1\, \omega_\theta - h N_1 \sin\omega = 0,\\
&\sin\theta \sin\omega\, \omega_\theta - \cos\omega\, \omega_\varphi - h \sin\theta - \cos\theta \cos\omega = 0.
\end{aligned} \tag{12}$$

The latter system is also in involution on the solutions of equations (11). The arbitrariness of the general solution of (12) is 1 function of 1 argument. The general solution of (12) can be implicitly represented as

$$F(\eta, \zeta) = 0, \tag{13}$$

where F is an arbitrary function of the invariants η and ζ, which are

$$\begin{aligned}
&\eta = \sin\theta \cos\omega \cos\tau - \cos\theta \sin\tau,\\
&\zeta = \varphi + \arctan\, \frac{\sin\omega \cos\tau}{\cos\theta \cos\omega \cos\tau + \sin\theta \sin\tau}\,.
\end{aligned} \tag{14}$$

The invariant system (11) determines the dependence of all the sought functions on radial coordinate, while equations (13), (14) define a distribution of tangential components of vectors $\mathbf{u}$ and $\mathbf{H}$ on spheres $r = \text{const}$.

3 Stationary Solution

Let us observe a stationary solution of invariant subsystem (11). It is assumed, that all the invariant functions depend only on r. Equations (11) are reduced to the following system of ODEs

$$UM_1' + \frac{2}{r}UM_1 - \frac{N_1'}{r^4\rho\cos\tau} = 0, \tag{15}$$

$$UN_1' + N_1U' - \frac{M_1'}{\cos\tau} - M_1N_1\tan\tau = 0, \tag{16}$$

$$Up' + \gamma p\left(U' + \frac{2}{r}U - M_1\tan\tau\right) = 0, \tag{17}$$

$$UU' + \frac{1}{\rho}p' + \frac{N_1N_1'}{r^2\rho} - rM_1^2 = 0, \tag{18}$$

$$U\rho' + \rho\left(U' + \frac{2}{r}U - M_1\tan\tau\right) = 0, \tag{19}$$

$$\tau' = N_1\cos\tau, \quad U\tau' = M_1. \tag{20}$$

Compatibility of equations (20) gives

$$UN_1\cos\tau = M_1. \tag{21}$$

According to (10) it is equivalent to $UN = MH$. This means, that under the considered assumptions the vectors $\mathbf{u}$ and $\mathbf{H}$ are collinear. Equation (16) is satisfied due to (21). From (17) and (19) we obtain the entropy conservation: $S = \text{const}$. Transformation of continuity equation (19) taking into account equation (20) gives

$$r^2\rho U\cos\tau = n^{-1}, \quad n = \text{const}.$$

From the latter and (21) it follows that $N_1 = nr^2\rho M_1$. According to involution transformation $U \to -U$, $M_1 \to -M_1$ one can assume $n > 0$. Equation (15) under condition (21) can be integrated as

$$r^2M_1 = nN_1 + m, \quad m = \text{const}.$$

Using the involution $r \to -r$, $M_1 \to -M_1$, $N_1 \to -N_1$ one can make $m > 0$. In the case $\rho \neq n^{-2}$ one can express M_1 and N_1 by means of ρ. Substitution of these relations into (18) allows us to integrate it as

$$U^2 + \frac{2\gamma}{\gamma-1}\rho^{\gamma-1} + \frac{m^2}{r^2(1-n^2\rho)^2} = b^2, \quad b = \text{const}. \tag{22}$$

Relation (22) is an analogue of the Bernoulli integral. The only equation left is (20). Let us introduce an auxiliary function $\sigma(r)$ by the following formula

$$\sigma = \int N_1(r)dr.$$

The first equation of (20) is integrated as $\tau = 2\arctan[\tanh(\sigma/2)]$. Expressing all the other functions in terms of σ gives

$$M_1 = \frac{m + n\sigma'}{r^2}, \quad N_1 = \sigma', \quad \rho = \frac{\sigma'}{mn + n^2\sigma'}, \quad U = \frac{(mn + n^2\sigma')\cosh\sigma}{nr^2\sigma'}.$$

Substitution of obtained representations into the Bernoulli integral (22) gives an equation for determining of σ:

$$\frac{(mn + n^2\sigma')^2\cosh^2\sigma}{n^2r^4\sigma'^2} + \frac{2\gamma}{\gamma - 1}\left(\frac{\sigma'}{mn + n^2\sigma'}\right)^{\gamma-1} + \frac{m^2}{r^2}\left(1 + \frac{n}{m}\sigma'\right)^2 = b^2. \quad (23)$$

The similar analysis in the case $\rho = n^{-2}$ gives the following. All invariant functions have a representation in terms of function σ:

$$M_1 = \frac{n\sigma'}{r^2}, \quad N_1 = \sigma', \quad U = \frac{n\cosh\sigma}{r^2}.$$

The function σ is determined from the equation

$$\sigma'^2 = \frac{b^2r^2}{n^2} - \frac{\cosh^2\sigma}{r^2}. \quad (24)$$

Thus, the solution is reduced to the set of first integrals and one first-order ODEs. Both ODEs (23), (24) are not solved with respect to derivative $\sigma'(r)$. Solutions of the equations are not unique. Different branches of solutions may be joined by means of weak or strong discontinuity. This fact gives extra possibilities for construction of solution.

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Linearization of Third-Order Ordinary Differential Equations

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We present here the complete solution to the problem on linearization of third-order equations by means of general point transformations. We also formulate the criteria for reducing third-order equations to the equation $y''' = 0$ by contact transformations.

1 Introduction

The problem of linearization of differential equations by a change of variables is a particular case of a more general equivalence problem. S.Lie made a significant contribution to this problem by solving in 1883 the linearization problem for second-order equations [1] and giving in 1896 the general form of third-order equations linearizable by contact transformations [2]. A. Tresse [3] treated Lie's result on linearization of the second-order equations in the framework of the equivalence problem using relative invariants of the equivalence group of point transformations. An infinitesimal technique for obtaining relative invariants of infinite equivalence groups was developed by N.H. Ibragimov [4] and subsequently applied [5] to the linearization problem.

A geometric approach for tackling the equivalence problem of second-order ordinary differential equations was developed by E.Cartan [6]. The idea of his approach was to associate with every differential equation a uniquely defined geometric structure of a certain form. S.S. Chern [7], using Cartan's approach, developed a geometric approach to third-order equations. In the series of subsequent articles [8–12], Chern's results were formulated in an explicit form more convenient for using as a test for linearization of third-order equations by contact transformations. Linearization by means of a restricted class of point transformations was studied in [13]. In [14], the linearization of third-order equations by means of non-local transformations (namely, so-called *generalized Sundman transformations*) was investigated.

The solution to the problem on linearization of third-order equations by general point transformations was given recently in [15].

2 Linearization by Point Transformations

Let us take the general linear third-order equation with the independent variable t and the dependent variable u in Laguerre's canonical form:

$$u''' + \alpha(t)u = 0. \tag{1}$$

The change of the independent and dependent variables

$$t = \varphi(x, y), \quad u = \psi(x, y) \tag{2}$$

leads to an equation of the form

$$y''' - \frac{3\varphi_y}{(\varphi_x + y'\varphi_y)}\,(y'')^2 + a(x, y, y')y'' + b(x, y, y') = 0.$$

More specifically, considering separately the transformations (2) with $\varphi_y = 0$ and $\varphi_y \neq 0$, we obtain two distinctly different candidates for linearization. Namely,

$$y''' + (A_1\, y' + A_0)y'' + B_3\, y'^3 + B_2\, y'^2 + B_1\, y' + B_0 = 0 \tag{3}$$

when $\varphi_y = 0$ and

$$\begin{aligned} &y''' + (y' + r)^{-1}\big(-3(y'')^2 + (C_2\, y'^2 + C_1\, y' + C_0)y'' + D_5\, y'^5 \\ &\quad + D_4\, y'^4 + D_3\, y'^3 + D_2\, y'^2 + D_1\, y' + D_0\big) = 0 \end{aligned} \tag{4}$$

when $\varphi_y \neq 0$, where we set $r = \varphi_x/\varphi_y$. Here $A_i = A_i(x, y)$, $B_i = B_i(x, y)$, $r = r(x, y)$, $C_i = C_i(x, y)$ and $D_i = D_i(x, y)$ are written through the functions φ and ψ and their derivatives. The following theorems are proved in [15].

Theorem 1. *Equation* (3) *is linearizable if and only if*

$$A_{0y} - A_{1x} = 0, \quad (3B_1 - A_0^2 - 3A_{0x})_y = 0, \tag{5}$$

$$3B_2 = 3A_{1x} + A_0A_1, \quad 9B_3 = 3A_{1y} + A_1^2, \tag{6}$$

$$\Omega_y = 0. \tag{7}$$

Provided that the conditions (5)–(7) *are satisfied, the linearizing transformation* $t = \varphi(x), \quad u = \psi(x, y)$ *is defined by the third order ordinary differential equation for the function* $\varphi(x)$*:*

$$3(2\chi_x - \chi^2) = 3B_1 - A_0^2 - 3A_{0x}, \quad \text{where} \quad \chi = \varphi_x^{-1}\varphi_{xx}, \tag{8}$$

and by the following involutive system of partial differential equations for $\psi(x, y)$*:*

$$3\psi_{yy} = A_1\, \psi_y, \qquad 3\psi_{xy} = (3\chi + A_0)\psi_y, \tag{9}$$

$$\psi_{xxx} = 3\chi\, \psi_{xx} + B_0\, \psi_y - \frac{1}{6}\,(3A_{0x} + A_0^2 - 3B_1 + 9\chi^2)\, \psi_x - \Omega\psi. \tag{10}$$

The coefficient α *of the resulting linear equation* (1) *is given by* $\alpha = \Omega\, \varphi_x^{-3}$*, where*

$$\Omega = \frac{1}{54}\left(9A_{0xx} + 18A_{0x}A_0 + 54B_{0y} - 27B_{1x} + 4A_0^3 - 18A_0B_1 + 18A_1B_0\right).$$

Theorem 2. *Equation (4) is linearizable if and only if its coefficients obey the following 8 equations:*

$$C_0 = 6r\,r_y - 6\,r_x + r\,C_1 - r^2\,C_2,$$

$$6\,r_{yy} = \frac{\partial C_2}{\partial x} - \frac{\partial C_1}{\partial y} + r\,\frac{\partial C_2}{\partial y} + C_2\,r_y,$$

$$\begin{aligned}18D_0 = {}& 3r^3\frac{\partial\,C_1}{\partial\,y} - 6r^2\frac{\partial C_1}{\partial x} - 3r^3\frac{\partial C_2}{\partial x} + 9r^4\frac{\partial C_2}{\partial y} - 36r^2 r_{xy} + 18r\,r_{xx}\\ & + 90r\,r_x r_y - 36r^2{r_y}^2 + 6r(3C_1 - rC_2)r_x + 9r^2(rC_2 - 2C_1)r_y\\ & - 54{r_x}^2 - 2r^2C_1^2 + 2r^3C_1C_2 + 4r^4C_2^2 + 18r^4D_4 - 72r^5D_5,\end{aligned}$$

$$\begin{aligned}18D_1 = {}& 9r^2\frac{\partial C_1}{\partial y} - 12r\frac{\partial C_1}{\partial x} - 27r^2\frac{\partial C_2}{\partial x} + 33r^3\frac{\partial C_2}{\partial y} - 36rr_{xy}\\ & + 18r_{xx} + 6(3C_1 + 4rC_2)r_x - 3r(6C_1 + 7rC_2)r_y + 18r{r_y}^2\\ & - 18r_xr_y - 4rC_1^2 - 2r^2C_1C_2 + 20r^3C_2^2 + 72r^3D_4 - 270r^4D_5,\end{aligned}$$

$$\begin{aligned}9D_2 = {}& 3r\frac{\partial C_1}{\partial y} - 3\frac{\partial C_1}{\partial x} - 21r\frac{\partial C_2}{\partial x} + 21r^2\frac{\partial C_2}{\partial y} + 15C_2r_x\\ & - 15\,r\,C_2r_y - C_1^2 - 5rC_1C_2 + 14r^2C_2^2 + 54r^2D_4 - 180r^3D_5,\end{aligned}$$

$$3D_3 = 3r\frac{\partial C_2}{\partial y} - 3\frac{\partial C_2}{\partial x} - C_1C_2 + 2rC_2^2 + 12rD_4 - 30r^2D_5,$$

$$\begin{aligned}54\frac{\partial D_4}{\partial x} = {}& 18\frac{\partial^2 C_1}{\partial y^2} + 3C_2\frac{\partial C_1}{\partial y} - 72\frac{\partial^2 C_2}{\partial x\partial y} - 39C_2\frac{\partial C_2}{\partial x} + 18r\frac{\partial^2 C_2}{\partial y^2} - 3rC_2\frac{\partial C_2}{\partial y}\\ & + 72\frac{\partial C_2}{\partial y}r_y + 378\,r\,\frac{\partial D_5}{\partial x} - 108r^2\frac{\partial D_5}{\partial y} + 270D_5r_x + 33C_2^2r_y + 108D_4r_y\\ & - 540rD_5r_y + 36\,r\,C_1D_5 - 8\,r\,C_2^3 - 36\,r\,C_2D_4 + 108\,r^2\,C_2D_5 + 54\,r\,H,\end{aligned}$$

and $H_x = 3Hr_y + rH_y$, *where*

$$\begin{aligned}H = {}& \frac{\partial D_4}{\partial y} - 2\frac{\partial D_5}{\partial x} - 3r\frac{\partial D_5}{\partial y} - 5D_5r_y - 2rC_2D_5\\ & + \frac{1}{3}\Big[\frac{\partial^2 C_2}{\partial y^2} + 2C_2\frac{\partial C_2}{\partial y} - 2C_1D_5 + 2C_2D_4\Big] + \frac{4}{27}C_2^3.\end{aligned}$$

The transformations (2) with $\varphi_y(x,y) \neq 0$ *mapping equation (4) into a linear equation (1) are obtained by solving the following compatible system of equations for the functions* $\varphi(x,y)$ *and* $\psi(x,y)$*:*

$$\varphi_x = r\,\varphi_y\,, \quad \psi_x = -W\,\varphi_y + r\,\psi_y\,,$$

$$6\varphi_y\,\varphi_{yyy} = 9{\varphi_{yy}}^2 + \left(15rD_5 - 3D_4 - C_2^2 - 3\frac{\partial C_2}{\partial y}\right){\varphi_y}^2,$$

$$\begin{aligned}\psi_{yyy} = {}& W\,D_5\,\varphi_y + \frac{1}{6}\Big[15\,r\,D_5 - C_2^2 - 3D_4 - 3\frac{\partial C_2}{\partial y}\Big]\psi_y\\ & - \frac{1}{2}H\psi + 3\varphi_{yy}\psi_{yy}{\varphi_y}^{-1} - \frac{3}{2}{\varphi_{yy}}^2\psi_y{\varphi_y}^{-2},\end{aligned}$$

where the function W *is defined by the equations*

$$3W_x = (C_1 - rC_2 + 6r_y)\, W, \quad 3W_y = C_2 W. \tag{11}$$

The coefficient α *of the resulting linear equation* (1) *is given by* $2\alpha = \varphi_y^{-3} H$*.*

3 Linearization by Contact Transformations

S.Lie [2] noticed that any third order ordinary differential equation related with the simplest linear equation $u''' = 0$ by a contact transformation

$$t = \varphi(x, y, p), \quad u = \psi(x, y, p), \quad q = g(x, y, p), \tag{12}$$

where $p = y'$, should be at most cubic in the second order derivative:

$$y''' + a(x, y, y')y''^{\,3} + b(x, y, y')y''^{\,2} + c(x, y, y')y'' + d(x, y, y') = 0. \tag{13}$$

Recall that the contact transformations (12) satisfy the conditions

$$g\varphi_p = \psi_p\,, \quad \psi_x + p\psi_y = g(\varphi_x + p\varphi_y),$$
$$(\varphi_y g - \psi_y)\,(\varphi_p(g_x + g_y p) - g_p(\varphi_x + \varphi_y p)) \neq 0.$$

Note that the contact transformation (12) of the general linear equation (1) also leads to the equations of the form (13). We will discuss here only the case $\alpha = 0$. We will assume that $\varphi_p \neq 0$, since the equation $\varphi_p = 0$ leads to $\psi_p = 0$, and hence corresponds to point transformations.

Theorem 3. *Equation* (13) *is linearizable to the equation* (1) *with* $\alpha = 0$ *if and only if its coefficients obey the equations*

$$J_1 = 0, \quad J_2 = 0, \quad J_3 = 0, \quad J_4 = 0, \tag{14}$$

where $J_1, J_2, J_3,$ *and* J_4 *are relative invariants defined by*

$$\begin{aligned}
J_1 = {} & 27a_{px} + 27a_{py}p - 18a_pc - 18a_xb - 18a_ybp + 81a_y + 18b_pb - 9b_{pp} + \\
& 18b_xa + 18b_yap - 36c_pa - 54a^2d + 18abc - 4b^3, \\
J_2 = {} & -18a_pd - 18a_xc + 9a_{xx} - 18a_ycp + 18a_{yx}p + 9a_{yy}p^2 + 6b_pc + 3b_{px} + \\
& 3b_{py}p + 6b_xb + 6b_ybp + 24b_y - 6c_{pp} - 36d_pa - 18abd + 12ac^2 - 2b^2c, \\
J_3 = {} & 36a_xd + 36a_ydp - 6b_{xx} - 12b_{yx}p - 6b_{yy}p^2 - 6c_pc + 3c_{px} + 3c_{py}p - 6c_xb - \\
& 6c_ybp - 21c_y + 18d_pb + 9d_{pp} + 18d_xa + 18d_yap - 18acd + 12b^2d - 2bc^2, \\
J_4 = {} & -36b_xd - 36b_ydp + 18c_pd + 18c_xc + 9c_{xx} + 18c_ycp + 18c_{yx}p + 9c_{yy}p^2 - \\
& 18d_pc - 27d_{px} - 27d_{py}p - 18d_xb - 18d_ybp + 54d_y + 54ad^2 - 18bcd + 4c^3.
\end{aligned}$$

Provided that the conditions (14) *are satisfied, the transformation* (12) *mapping equation* (13) *to a linear equation* (1) *with* $\alpha = 0$ *is obtained by solving the following compatible system of equations for the functions* $\varphi(x,y,p)$, $\psi(x,y,p)$, $g(x,y,p)$, $k(x,y,p)$, *and* $H(x,y,p)$*:*

$$
\begin{aligned}
& D\psi = Hg, \quad \psi_y = \varphi_y g + k, \quad \psi_p = \varphi_p g, \quad \varphi_p g_{pp} = (\varphi_{pp} g_p + ak),\\
& \varphi_p Dg = Hg_p - k, \quad 3g_y\varphi_p^2 = 3\varphi_{pp}k + 3aHk + \varphi_p(3\varphi_y g_p - bk),\\
& 3\varphi_p^2 Dk = 3\varphi_{pp}Hk + 3aH^2k - H\varphi_p bk + 3\varphi_p\varphi_y k,\\
& 9\varphi_p^2 k_y = 18\varphi_{py}\varphi_p k - 9\varphi_{pp}\varphi_y k + 3aHk(2\varphi_p c - 3\varphi_y) - 9a\varphi_p^2 dk -\\
& \quad 9DaH\varphi_p k + 3Db\varphi_p^2 k + H\varphi_p k(3b_p - 2b^2) + \varphi_p k(-3c_p\varphi_p + \varphi_p bc + 3\varphi_y b),\\
& 3\varphi_p k_p = 3\varphi_{pp}k + 3aHk - \varphi_p bk,\\
& 54\varphi_p^2\varphi_{yy} = 108\varphi_{py}\varphi_p\varphi_y - 54\varphi_{pp}\varphi_y^2 + 6aH(3\varphi_p^2 bd - 2\varphi_p^2 c^2 + 3\varphi_p^2 Dc +\\
& \quad 6\varphi_p\varphi_y c - 9\varphi_y^2) + 18a\varphi_p^2(2\varphi_p cd - 3\varphi_p Dd - 3\varphi_y d) +\\
& \quad 18DaH\varphi_p(2\varphi_p c - 3\varphi_y) - 108Da\varphi_p^3 d - 18DbH\varphi_p^2 b + 6Db\varphi_p^2(\varphi_p c + 3\varphi_y) +\\
& \quad H\varphi_p(-6b_p\varphi_p c + 18b_p\varphi_y - 27b_y\varphi_p + 6c_p\varphi_p b + 9(Db)_p\varphi_p + 2\varphi_p b^2 c -\\
& \quad 27\varphi_p D^2 a - 12\varphi_y b^2) + 2\varphi_p(9b_p\varphi_p^2 d - 9c_p\varphi_p\varphi_y + 27c_y\varphi_p^2 - 9(Dc)_p\varphi_p^2 -\\
& \quad 9d_p\varphi_p^2 b - 12\varphi_p^2 b^2 d + 2\varphi_p^2 bc^2 + 6\varphi_p^2 bDc + 9\varphi_p^2 D^2 b + 3\varphi_p\varphi_y bc + 9\varphi_y^2 b),\\
& 54\varphi_p^2\varphi_{ppy} = 108\varphi_{py}\varphi_{pp}\varphi_p + 54\varphi_{py}aH\varphi_p - 18\varphi_{py}\varphi_p^2 b - 27\varphi_{pp}^2\varphi_y -\\
& \quad 54\varphi_{pp}aH\varphi_y + 18\varphi_{pp}\varphi_p\varphi_y b + 9a^2H^2(-2\varphi_p c - 3\varphi_y) + 54a^2H\varphi_p^2 d +\\
& \quad 27aDaH^2\varphi_p - 18aDbH\varphi_p^2 + 3aH^2\varphi_p(-3b_p + 2b^2) +\\
& \quad 6aH\varphi_p(3c_p\varphi_p - \varphi_p bc + 3\varphi_y b) + 9a\varphi_p(-3d_p\varphi_p^2 - 4\varphi_p^2 bd + 2\varphi_p^2 c^2 -\\
& \quad \varphi_p^2 Dc - 2\varphi_p\varphi_y c - 3\varphi_y^2) + 9Da\varphi_p^2(-4\varphi_p c + 3\varphi_y) + 18Db\varphi_p^3 b - 54Ha_y\varphi_p^2 +\\
& \quad \varphi_p^2(6b_p\varphi_p c - 9b_p\varphi_y + 45b_y\varphi_p - 6c_p\varphi_p b - 9(Db)_p\varphi_p - 2\varphi_p b^2 c +\\
& \quad 27\varphi_p D^2 a + 3\varphi_y b^2),\\
& 6\varphi_p\varphi_{ppp} = 9\varphi_{pp}^2 - 9a^2H^2 + 6aH\varphi_p b - 12a\varphi_p\varphi_y - 3Da\varphi_p^2 -\\
& \quad 6Ha_p\varphi_p + \varphi_p^2(3b_p - b^2),\\
& 3\varphi_p^2 DH = 3\varphi_{pp}H^2 + 3aH^3 - H^2\varphi_p b + H\varphi_p(-\varphi_p c + 3\varphi_y) + 3\varphi_p^3 d,\\
& 18\varphi_p H_y = (18\varphi_{py}H + 6aH^2 c - 18aH\varphi_p d - 9DaH^2 +\\
& \quad 6DbH\varphi_p + H^2(3b_p - 2b^2) + 2H\varphi_p(-3c_p + bc) + 9d_p\varphi_p^2 +\\
& \quad 6\varphi_p^2 bd - 2\varphi_p^2 c^2 - 3\varphi_p^2 Dc + 9\varphi_y^2),\\
& \quad 3\varphi_p H_p = 3\varphi_{pp}H + 3aH^2 - 2H\varphi_p b + \varphi_p(\varphi_p c + 3\varphi_y).
\end{aligned}
$$

where $H = \varphi_x + p\varphi_y$, $k = \varphi_p g_x + \varphi_p g_y p - \varphi_x g_p - \varphi_y g_p p \neq 0$.

4 Examples

Example 1. It is known from [2] (see also [16], Section 8.3.3)) that the equations

$$(C)\ y''' = \frac{3y'\,y''^{\,2}}{1+y'^{\,2}} \quad \text{and} \quad (H)\ y''' = \frac{3y''^{\,2}}{2y'} \tag{15}$$

describing the families of circles and hyperbolas, respectively, are connected by a complex transformation, and that Equation (15) (H) can be linearized to the equation $u''' = 0$ by a contact transformation (specifically, by the Legendre transformation). One can readily check that Equation (15) (C) also satisfies the conditions (14), and hence can be reduced to $u''' = 0$ by a (real valued) contact transformation. The reckoning yields the following linearizing transformation:

$$\varphi = \frac{y(1+\sqrt{p^2+1})}{p} - x, \quad \psi = -y(1+\frac{1+\sqrt{p^2+1}}{p^2}), \quad g = -\frac{1+\sqrt{p^2+1}}{p}.$$

An alternative transformation is

$$\varphi = -\left(p+\sqrt{1+p^2}\right), \ \psi = (px-y)\left(p+\sqrt{1+p^2}\right), \ g = y - x\left(p+\sqrt{1+p^2}\right).$$

Remark 1. It is erroneously stated in [17] that the contact transformation

$$t = -2xg(x,y,p), \ u = y + xp, \ u' = g(x,y,p), \quad \text{where } g^2 = -p, \tag{16}$$

maps the equation $u''' = 0$ to the equation for circles (15) (C). In fact, the transformation (16) relates the equation $u''' = 0$ with the equation (15) (H), not (C).

Example 2. Consider again the equations of the form (3). One can readily verify that two of conditions for linearization by a contact transformation are satisfied, namely $J_1 = 0$ and $J_2 = 0$. Equating to zero two other invariants, J_3 and J_4, we conclude that equation (3) can be mapped by a contact transformation to the equation $u''' = 0$ if and only if the following equations hold:

$$\begin{aligned}
&2(3B_2 - 3A_{1x} - A_0A_1) = 7(A_{0y} - A_{1x}), \quad 9B_3 = 3A_{1y} + A_1^2,\\
&3(A_{0y} - A_{1x})_y - A_1(A_{0y} - A_{1x}) = 0,\\
&6(A_{0y} - A_{1x})_x + 2A_0(A_{0y} - A_{1x}) - 3(3B_1 - A_0^2 - 3A_{0x})_y = 0,\\
&9A_{0xx} + 18A_{0x}A_0 + 54B_{0y} - 27B_{1x} + 4A_0^3 - 18A_0B_1 + 18A_1B_0 = 0.
\end{aligned} \tag{17}$$

The last equation (17) yields $\Omega = 0$ (for the definition of Ω, see Theorem 1). Invoking Theorem 1 and noting that Equations (5)–(6) imply the first four equations (17), we conclude that Equation (3) is linearizable simultaneously by contact and point transformations if and only if its coefficients satisfy the equation $\Omega = 0$ and the equations (5)–(6).

For example, the equation $y''' + 3y'y''/y - 3y'' - 3y'^2/y + 2y' - y = 0$ is linearizable by a point transformation, but it is not linearizable by a contact transformation.

On the other hand, the equation $y''' + yy'' + (63p^2 + 24y^2p + y^4)/54 = 0$ can be linearized by a contact transformation, but cannot be linearized by a point transformation since $A_{0y} - A_{1x} \neq 0$.

The equation $y''' + yy'' + \beta(1 - y'^{\,2}) = 0$ widely used in hydrodynamics (it is called the Blasius equation when $\beta = 0$, the Hiemenz flow when $\beta = 1$, and also known as the Falkner–Skan equation [18]) is linearizable neither by point nor contact transformation.

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Conservation Laws of Variable Coefficient Diffusion–Convection Equations

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We study local conservation laws of variable coefficient diffusion–convection equations of the form $f(x)u_t = (g(x)A(u)u_x)_x + h(x)B(u)u_x$. The main tool of our investigation is the notion of equivalence of conservation laws with respect to the equivalence groups. That is why, for the class under consideration we first construct the usual equivalence group $G^\sim$ and the extended one $\hat{G}^\sim$ including transformations which are nonlocal with respect to arbitrary elements. The extended equivalence group $\hat{G}^\sim$ has interesting structure since it contains a non-trivial subgroup of gauge equivalence transformations. Then, using the most direct method, we carry out two classifications of local conservation laws up to equivalence relations generated by $G^\sim$ and $\hat{G}^\sim$, respectively. Equivalence with respect to $\hat{G}^\sim$ plays the major role for simple and clear formulation of the final results.

1 Introduction

In this paper we study local conservation laws of PDEs of the general form

$$f(x)u_t = (g(x)A(u)u_x)_x + h(x)B(u)u_x, \tag{1}$$

where $f = f(x)$, $g = g(x)$, $h = h(x)$, $A = A(u)$ and $B = B(u)$ are arbitrary smooth functions of their variables, and $f(x)g(x)A(u)\neq 0$.

Conservation laws were investigated for some subclasses of class (1). In particular, Dorodnitsyn and Svirshchevskii [2] (see also [4, Chapter 10]) constructed the local conservation laws for the class of reaction–diffusion equations of the form $u_t = (A(u)u_x)_x + C(u)$, which has non-empty intersection with the class under consideration. The first-order local conservation laws of equations (1) with $f = g = h = 1$ were constructed by Kara and Mahomed [6]. Developing the results obtained in [1] for the case $hB = 0$, $f = 1$, in the recent papers [5, 10] we completely classified potential conservation laws (including arbitrary order local ones) of equations (1) with $f = g = h = 1$ with respect to the corresponding equivalence group.

For class (1) in Section 2 we first construct the usual equivalence group $G^{\sim}$ and the extended one $\hat{G}^{\sim}$ including transformations which are nonlocal with respect to arbitrary elements. We discuss the structure of the extended equivalence group $\hat{G}^{\sim}$ having non-trivial subgroup of gauge equivalence transformations. Then we carry out two classifications of local conservation laws up to the equivalence relations generated by $G^{\sim}$ and $\hat{G}^{\sim}$, respectively, using the most direct method (Section 3).

The main tool of our investigation is the notion of equivalence of conservation laws with respect to equivalence groups, which was introduced in [10]. Below we adduce some necessary notions and statements, restricting ourselves to the case of two independent variables. See [9, 10] for more details and general formulations.

Let $\mathcal{L}$ be a system $L(t,x,u_{(\rho)}) = 0$ of PDEs for unknown functions $u = (u^1,\dots,u^m)$ of independent variables t (the time variable) and x (the space variable). Here $u_{(\rho)}$ denotes the set of all the partial derivatives of the functions u of order no greater than ρ, including u as the derivatives of the zero order.

Definition 1. A *conservation law* of the system $\mathcal{L}$ is a divergence expression

$$D_tF(t,x,u_{(r)}) + D_xG(t,x,u_{(r)}) = 0 \tag{2}$$

which vanishes for all solutions of $\mathcal{L}$. Here D_t and D_x are the operators of total differentiation with respect to t and x, respectively; F and G are correspondingly called the *density* and the *flux* of the conservation law.

Two conserved vectors (F,G) and (F',G') are *equivalent* if there exist functions $\hat{F}$, $\hat{G}$ and H of t, x and derivatives of u such that $\hat{F}$ and $\hat{G}$ vanish for all solutions of $\mathcal{L}$ and $F' = F + \hat{F} + D_xH$, $G' = G + \hat{G} - D_tH$.

Lemma 1. *[10] Any point transformation g between systems $\mathcal{L}$ and $\tilde{\mathcal{L}}$ induces a linear one-to-one mapping g_* between the corresponding linear spaces of conservation laws.*

Consider the class $\mathcal{L}|_S$ of systems $L(t,x,u_{(\rho)},\theta(t,x,u_{(\rho)})) = 0$ parameterized with the parameter-functions $\theta = \theta(t,x,u_{(\rho)})$. Here L is a tuple of fixed functions of t, x, $u_{(\rho)}$ and θ. θ denotes the tuple of arbitrary (parametric) functions $\theta(t,x,u_{(\rho)}) = (\theta^1(t,x,u_{(\rho)}),\dots,\theta^k(t,x,u_{(\rho)}))$ satisfying the additional condition $S(t,x,u_{(\rho)},\theta_{(q)}(t,x,u_{(\rho)})) = 0$.

Let $P = P(L,S)$ denote the set of pairs each from which consists of a system from $\mathcal{L}|_S$ and a conservation law of this system. Action of transformations from an equivalence group $G^{\sim}$ of the class $\mathcal{L}|_S$ together with the pure equivalence relation of conserved vectors naturally generates an equivalence relation on P. Classification of conservation laws with respect to $G^{\sim}$ will be understood as classification in P with respect to the above equivalence relation. This problem can be investigated in the way that it is similar to group classification in classes of systems of differential equations. Specifically, we firstly construct the conservation laws that are defined for all values of the arbitrary elements. (The corresponding conserved vectors may depend on the arbitrary elements.) Then we classify, with respect to the equivalence group, arbitrary elements for each of the systems that admits additional conservation laws.

2 Equivalence Transformations and Choice of Investigated Class

In order to classify the conservation laws of equations of the class (1), firstly we have to investigate equivalence transformations of this class.

The usual equivalence group $G^{\sim}$ of class (1) is formed by the nondegenerate point transformations in the space of (t, x, u, f, g, h, A, B), which are projectible on the space of (t, x, u), i.e. they have the form

$$\begin{aligned} &(\tilde t, \tilde x, \tilde u) = (T^t, T^x, T^u)(t, x, u), \\ &(\tilde f, \tilde g, \tilde h, \tilde A, \tilde B) = (T^f, T^g, T^h, T^A, T^B)(t, x, u, f, g, h, A, B), \end{aligned} \tag{3}$$

and transform any equation from the class (1) for the function $u = u(t, x)$ with the arbitrary elements (f, g, h, A, B) to an equation from the same class for function $\tilde u = \tilde u(\tilde t, \tilde x)$ with the new arbitrary elements $(\tilde f, \tilde g, \tilde h, \tilde A, \tilde B)$.

Theorem 1. *$G^{\sim}$ consists of the transformations*

$$\begin{aligned} &\tilde t = \delta_1 t + \delta_2, \quad \tilde x = X(x), \quad \tilde u = \delta_3 u + \delta_4, \\ &\tilde f = \frac{\varepsilon_1 \delta_1 f}{X_x(x)}, \quad \tilde g = \varepsilon_1 \varepsilon_2^{-1} X_x(x) g, \quad \tilde h = \varepsilon_1 \varepsilon_3^{-1} h, \quad \tilde A = \varepsilon_2 A, \quad \tilde B = \varepsilon_3 B, \end{aligned}$$

where δ_j $(j = \overline{1,4})$ and ε_i $(i = \overline{1,3})$ are arbitrary constants, $\delta_1\delta_3\varepsilon_1\varepsilon_2\varepsilon_3 \neq 0$, X is an arbitrary smooth function of x, $X_x \neq 0$.

It appears that class (1) admits other equivalence transformations which do not belong to $G^{\sim}$ and form, together with usual equivalence transformations, an extended equivalence group. We demand for these transformations to be point with respect to (t, x, u). The explicit form of the new arbitrary elements $(\tilde f, \tilde g, \tilde h, \tilde A, \tilde B)$ is determined via (t, x, u, f, g, h, A, B) in some non-fixed (possibly, nonlocal) way. We construct the complete (in this sense) extended equivalence group $\hat G^{\sim}$ of class (1), using the direct method.

Existence of such transformations can be explained in many respects by features of representation of equations in the form (1). This form leads to an ambiguity since the same equation has an infinite series of different representations. More exactly, two representations (1) with the arbitrary element tuples (f, g, h, A, B) and $(\tilde f, \tilde g, \tilde h, \tilde A, \tilde B)$ determine the same equation iff

$$\begin{aligned} &\tilde f = \varepsilon_1 \varphi f, \quad \tilde g = \varepsilon_1 \varepsilon_2^{-1} \varphi g, \quad \tilde h = \varepsilon_1 \varepsilon_3^{-1} \varphi h, \\ &\tilde A = \varepsilon_2 A, \quad \tilde B = \varepsilon_3 (B + \varepsilon_4 A), \end{aligned} \tag{4}$$

where $\varphi = \exp\left(-\varepsilon_4 \int \frac{h(x)}{g(x)} dx\right)$, ε_i $(i = \overline{1,4})$ are arbitrary constants, $\varepsilon_1\varepsilon_2\varepsilon_3 \neq 0$ (the variables t, x and u do not transform!).

The transformations (4) act only on arbitrary elements and do not really change equations. In general, transformations of such type can be considered as trivial [9] ("gauge") equivalence transformations and form the "gauge" (normal) subgroup $\hat{G}^{\sim g}$ of the extended equivalence group $\hat{G}^{\sim}$. Application of "gauge" equivalence transformations is equivalent to rewriting equations in another form. In spite of really equivalence transformations, their role in group classification comes not as a choice of representatives in equivalence classes but as a choice of the form of these representatives.

Let us note that transformations (4) with $\varepsilon_4 \neq 0$ are nonlocal with respect to arbitrary elements, otherwise they belong to $G^{\sim}$ and form the "gauge" (normal) subgroup $G^{\sim g}$ of the equivalence group $G^{\sim}$.

The factor-group $\hat{G}^{\sim}/\hat{G}^{\sim g}$ coincides for class (1) with $G^{\sim}/G^{\sim g}$ and can be assumed to consist of the transformations

$$\begin{aligned}&\tilde{t} = \delta_1 t + \delta_2, \quad \tilde{x} = X(x), \quad \tilde{u} = \delta_3 u + \delta_4,\\ &\tilde{f} = \frac{\delta_1 f}{X_x(x)}, \quad \tilde{g} = X_x(x) g, \quad \tilde{h} = h, \quad \tilde{A} = A, \quad \tilde{B} = B,\end{aligned} \tag{5}$$

where δ_i $(i = \overline{1,4})$ are arbitrary constants, $\delta_1\delta_3 \neq 0$, X is an arbitrary smooth function of x, $X_x \neq 0$.

Using the transformation $\tilde{t} = t$, $\tilde{x} = \int \frac{dx}{g(x)}$, $\tilde{u} = u$ from $G^{\sim}/G^{\sim g}$, we can reduce equation (1) to

$$\tilde{f}(\tilde{x})\tilde{u}_{\tilde{t}} = (A(\tilde{u})\tilde{u}_{\tilde{x}})_{\tilde{x}} + \tilde{h}(\tilde{x})B(\tilde{u})\tilde{u}_{\tilde{x}},$$

where $\tilde{f}(\tilde{x}) = g(x)f(x)$, $\tilde{g}(\tilde{x}) = 1$ and $\tilde{h}(\tilde{x}) = h(x)$. (Likewise any equation of form (1) can be reduced to the same form with $\tilde{f}(\tilde{x}) = 1$.) That is why, without loss of generality we restrict ourselves to investigation of the equation

$$f(x)u_t = (A(u)u_x)_x + h(x)B(u)u_x. \tag{6}$$

Any transformation from $\hat{G}^{\sim}$, which preserves the condition $g = 1$, has the form

$$\begin{aligned}&\tilde{t} = \delta_1 t + \delta_2, \quad \tilde{x} = \delta_5 \textstyle\int e^{\delta_8 \int h\, dx} dx + \delta_6, \quad \tilde{u} = \delta_3 u + \delta_4,\\ &\tilde{f} = \delta_1\delta_5^{-1}\delta_9 f e^{-2\delta_8 \int h\, dx}, \quad \tilde{h} = \delta_9\delta_7^{-1} h e^{-\delta_8 \int h\, dx},\\ &\tilde{A} = \delta_5\delta_9 A, \quad \tilde{B} = \delta_7(B + \delta_8 A),\end{aligned} \tag{7}$$

where δ_i $(i = \overline{1,9})$ are arbitrary constants, $\delta_1\delta_3\delta_5\delta_7\delta_9 \neq 0$. The set $\hat{G}_1^{\sim}$ of such transformations is a subgroup of $\hat{G}^{\sim}$. It can be considered as a generalized equivalence group of class (6) after admitting dependence of (3) on arbitrary elements [10] and additional supposition that such dependence can be nonlocal. The group $G_1^{\sim}$ of usual (local) equivalence transformations of class (6) coincides with the subgroup singled out from $\hat{G}_1^{\sim}$ via the condition $\delta_8 = 0$. The transformations (7) with non-vanishing values of the parameter δ_8 are nonlocal and are compositions of (nonlocal) gauge and usual equivalence transformations from $G_1^{\sim}$.

There exists a way to avoid operations with nonlocal in (t, x, u) equivalence transformations. More exactly, we can assumed that the parameter-function B is determined up to an additive term proportional to A and subtract such term from B before applying equivalence transformations (5).

3 Local Conservation Laws

We search local conservation laws of equations from class (6).

Lemma 2. *Any conservation law of form* (2) *of any equation from class* (6) *is equivalent to a conservation law that has the density depending on* t*,* x*, and* u *and the flux depending on* t*,* x*,* u *and* u_x*.*

Note 1. A similar statement is true for an arbitrary (1+1)-dimensional evolution equation $\mathcal{L}$ of the even order $r = 2\bar{r}$, $\bar{r} \in \mathbb{N}$. For example [3], for any conservation law of $\mathcal{L}$ we can assume up to equivalence of conserved vectors that F and G depend only on t, x and derivatives of u with respect to x, and the maximal order of derivatives in F is not greater than $\bar{r}$.

Theorem 2. *A complete list of* $G_1^{\sim}$*-inequivalent equations* (6) *having nontrivial conservation laws is exhausted by the following ones*

1. $h = 1: \quad (\,fu,\ -Au_x - \textstyle\int B\,).$
2. $h = x^{-1}: \quad (\,xfu,\ -xAu_x + \textstyle\int A - \int B\,).$
3. $B = \varepsilon A: \quad (yfe^{-\varepsilon\int h}u,\ -yAe^{-\varepsilon\int h}u_y + \textstyle\int A),\ (fe^{-\varepsilon\int h}u,\ -Ae^{-\varepsilon\int h}u_y).$
4. $B = \varepsilon A + 1, \quad f = -hZ^{-1}, \quad h = Z^{-1/2}\exp\left(-\displaystyle\int \frac{a_{00}+a_{11}}{2Z}dy\right):$

   $(\,(\sigma^{k1}y + \sigma^{k0})fe^{-\varepsilon\int h}u,\ -(\sigma^{k1}y + \sigma^{k0})(Ae^{-\varepsilon\int h}u_y + hu) + \sigma^{k1}\textstyle\int A\,)$
5. $B = \varepsilon A + 1, \quad f = h_y: \quad (\,e^{t-\varepsilon\int h}h_y u,\ -e^t(Ae^{-\varepsilon\int h}u_y + hu)\,).$
6. $B = \varepsilon A + 1, \quad f = h_y + hy^{-1}:$

   $(\,e^{t-\varepsilon\int h}yfu,\ -e^t(yAe^{-\varepsilon\int h}u_y + yhu - \textstyle\int A)\,).$
7. $A = 1, \quad B_u \neq 0, \quad f = -h(h^{-1})_{xx}:$

   $(\,e^t(h^{-1})_{xx}u,\ e^t(h^{-1}u_x - (h^{-1})_x u + \textstyle\int B)\,).$
8. $A = 1, \quad B = 0: \quad (\,\alpha fu,\ -\alpha u_x + \alpha_x u\,).$

Here y *is implicitly determined by the formula* $x = \int e^{\varepsilon\int h(y)dy}dy$*;* $\varepsilon, a_{ij} = \mathrm{const}$*,* $i,j = \overline{0,1}$*;* $(\sigma^{k1},\sigma^{k0}) = (\sigma^{k1}(t),\sigma^{k0}(t))$*,* $k = \overline{1,2}$*, is a fundamental solution of the system of ODEs* $\sigma^{\nu}_t = a_{\mu\nu}\sigma^{\mu}$*;* $Z = a_{01}y^2 + (a_{00} - a_{11})y - a_{10}$*;* $\alpha = \alpha(t,x)$ *is an arbitrary solution of the linear equation* $f\alpha_t + \alpha_{xx} = 0$ *(which is an adjoint equation to the initial one). (Together with constraints on* A*,* B*,* f *and* h *we also adduce complete lists of linear independent conserved vectors.)*

In Theorem 2 we classify conservation laws with respect to the usual equivalence group $G_1^{\sim}$. The results that are obtained can be formulated in an implicit form only, and indeed Case 4 is split into a number of inequivalent cases depending on values of a_{ij}. At the same time, using the extended equivalence group $\hat{G}_1^{\sim}$, we can present the result of classification in a closed and simple form with a smaller number of inequivalent equations having nontrivial conservation laws.

Theorem 3. *A complete list of $\hat{G}_1^{\sim}$-inequivalent equations* (6) *having nontrivial conservation laws is exhausted by the following ones*

1. $h=1:\quad (\,fu,\ -Au_x-\smallint B\,).$

2a. $B=0:\quad (\,fu,\ -Au_x\,),\ (\,xfu,\ -xAu_x+\smallint A\,).$

2b. $B=1,\quad f=1,\quad h=1:\quad (\,u,\ -Au_x-u\,),$
$(\,(x+t)u,\ -(x+t)(Au_x+u)+\smallint A\,).$

2c. $B=1,\quad f=e^x,\quad h=e^x:\quad (\,e^{x+t}u,\ -e^t(Au_x+e^xu)\,),$
$(\,e^{x+t}(x+t)u,\ -e^t(x+t)(Au_x+e^xu)+e^t\smallint A\,).$

2d. $B=1,\quad f=x^{\mu-1},\quad h=x^\mu:\quad (\,x^{\mu-1}e^{\mu t}u,\ -e^{\mu t}(Au_x+x^\mu u)\,),$
$(\,x^\mu e^{(\mu+1)t}u,\ e^{(\mu+1)t}(-xAu_x-x^{\mu+1}u+\smallint A)\,).$

3. $B=1,\quad f=e^{\mu/x}x^{-3},\quad h=e^{\mu/x}x^{-1},\quad \mu\in\{0,1\}:$
$(\,fe^{-\mu t}xu,\ -e^{-\mu t}x(Au_x+hu)+e^{-\mu t}\smallint A\,),$
$(\,fe^{-\mu t}(tx-1)u,\ -e^{-\mu t}(tx-1)(Au_x+hu)+te^{-\mu t}\smallint A\,).$

4. $B=1,\ f=|x-1|^{\mu-3/2}|x+1|^{-\mu-3/2},\ h=|x-1|^{\mu-1/2}|x+1|^{-\mu-1/2}:$
$(\,fe^{(2\mu+1)t}(x-1)u,\ -e^{(2\mu+1)t}(x-1)(Au_x+hu)+e^{(2\mu+1)t}\smallint A\,),$
$(\,fe^{(2\mu-1)t}(x+1)u,\ -e^{(2\mu-1)t}(x+1)(Au_x+hu)+e^{(2\mu-1)t}\smallint A\,).$

5. $B=1,\quad f=e^{\mu\arctan x}(x^2+1)^{-3/2},\quad h=e^{\mu\arctan x}(x^2+1)^{-1/2}:$
$(\,fe^{\mu t}(x\cos t+\sin t)u,\ -e^{\mu t}(x\cos t+\sin t)(Au_x+hu)+e^{\mu t}\cos t\smallint A\,),$
$(\,fe^{\mu t}(x\sin t-\cos t)u,\ -e^{\mu t}(x\sin t-\cos t)(Au_x+hu)+e^{\mu t}\sin t\smallint A\,).$

6. $B=1,\quad f=h_x:\quad (\,e^th_xu,\ -e^t(Au_x+hu)\,).$

7. $B=1,\quad f=h_x+hx^{-1}:\quad (\,e^txfu,\ -e^t(xAu_x+xhu-\smallint A)\,).$

8. $A=1,\quad B_u\neq0,\quad f=-h(h^{-1})_{xx}:$
$(\,e^t(h^{-1})_{xx}u,\ e^t(h^{-1}u_x-(h^{-1})_xu+\smallint B)\,).$

9. $A=1,\quad B=0:\quad (\,\alpha fu,\ -\alpha u_x+\alpha_xu\,).$

Here $\mu=\mathrm{const}$, $\alpha=\alpha(t,x)$ *is an arbitrary solution of the linear equation* $f\alpha_t+\alpha_{xx}=0$ *(which is an adjoint equation to the initial one). (Together with constraints on A, B, f and h we also adduce complete lists of linear independent conserved vectors.)*

Note 2. The cases 2b–2d can be reduced to the case 2a by means of additional equivalence transformations:

2b→2a: $\tilde t=t,\ \tilde x=x+t,\ \tilde u=u;$

2c→2a: $\tilde t=e^t,\ \tilde x=x+t,\ \tilde u=u;$

2d $(\mu+1\neq0)$→2a: $\tilde t=(\mu+1)^{-1}(e^{(\mu+1)t}-1),\ \tilde x=e^tx,\ \tilde u=u;$

2d $(\mu+1=0)$→2a: $\tilde t=t,\ \tilde x=e^tx,\ \tilde u=u.$

4 Conclusion

The present paper is the beginning for further studies on this subject. For the class under consideration we intend to perform a complete classification of potential conservation laws and construct an exhaustive list of locally inequivalent potential systems corresponding to them. These results can be developed and generalized in a number of different directions. So, studying different kinds of symmetries (Lie, nonclassical, generalized ones) of constructed potential systems, we may obtain the corresponding kinds of potential symmetries (usual potential, nonclassical potential, generalized potential). Analogously, local equivalence transformations between potential systems constructed for different initial equations result in nonlocal (potential) equivalence transformations for the class under consideration. In such way it is possible to find new nonlocal connections between variable coefficient diffusion–convection equations. We believe that the same approach used in this article, can be employed for investigation of wider classes of differential equations.

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Nonclassical Potential Symmetries and Exact Solutions of Partial Differential Equations

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In this paper, a method to compute symmetries merging the notions of nonclassical and potential symmetries, which we call nonclassical potential symmetries, for partial differential equations (PDEs) written in conserved forms is presented. We determine a number of new such symmetry generators for a wave equation in inhomogeneous media as an illustrative example. The corresponding group-invariant solutions are also constructed.

1 Introduction

The classical symmetry group methods based on local symmetries provide a systematic method for obtaining *group-invariant* solutions of partial differential equations (PDEs) see [1–5] and the references therein. Motivated by the fact that for many PDEs group-invariant solutions are known that are not obtained by using the classical symmetry group method, there have been several generalisations of these methods for finding group-invariant solutions, the nonclassical method of Bluman and Cole [6], the direct method of Clarkson and Kruskal [7] and the differential constraint approach by Olver and Rosenau [8].

In nonclassical method the original PDE is augmented with the invariant surface conditions, a system of first-order differential equations satisfied by all functions invariant under a certain vector field [9]. The number of determining equations for the nonclassical method is smaller than for the classical method therefore the set of solutions is larger than for the classical method.

In [10], Bluman and Kumei introduced a method which yields new classes of symmetries of a given PDE that are neither Lie point nor Lie–Bäcklund symmetries. They are nonlocal symmetries called *potential* symmetries. The potential symmetries of a given PDE are realized as a local symmetries of a system of PDEs, obtained by replacing the PDE by an equivalent conserved form, with additional dependent variables [1]. A Lie point symmetry generator of the system, acting on the space consisting of the independent and dependent variables of the given PDE and the (potential) variables, yields a potential symmetry of the given PDE if atleast one of the infinitesimals of the generator depends explicitly on the potential variables.

The *nonclassical potential symmetry* method is a combination of potential symmetry and nonclassical method and was studied in, see [11–13] and the references therein. By combining the nonclassical method with potential symmetry method it was shown that useful nonclassical potential symmetries can be found.

We adopt the notation presented in [13] wherein the method is elaborately detailed.

2 Applications

We consider the quasi-linear second order hyperbolic PDE in two independent variables x, t and the dependent variable u, the wave equation

$$u_{tt} = c^2(x)\, u_{xx}, \tag{1}$$

which charaterise small transverse vibrations of a string with variable density with wave speed $c(x)$ in an inhomogeneous medium. If $c(x) = x$, then the PDE (1) can be written in a conserved form

$$D_t\big(x^{-2}u_t\big) - D_x(u_x) = 0. \tag{2}$$

Then the auxiliary covering system of first-order PDEs obtained from the equation (2) with additional dependent variable v (potential) is

$$v_t = u_x, \quad v_x = x^{-2}u_t. \tag{3}$$

The symmetry generator

$$X = \tau(t,x,u,v)\frac{\partial}{\partial t} + \xi(t,x,u,v)\frac{\partial}{\partial x} + \phi(t,x,u,v)\frac{\partial}{\partial u} + \zeta(t,x,u,v)\frac{\partial}{\partial v} \tag{4}$$

that leaves invariant the equation (3) and the following system of first-order PDEs

$$\tau u_t + \xi u_x - \phi = 0, \quad \tau v_t + \xi v_x - \zeta = 0 \tag{5}$$

is a non-trivial nonclassical potential symmetry of the equation (1) if the infinitesimals τ, ξ, ϕ depend on v explicitly.

The system (3) admits the following potential symmetry

$$\widetilde{X_3} = 2tx\frac{\partial}{\partial x} + 2\log x\frac{\partial}{\partial t} + (tu - xv)\frac{\partial}{\partial u} - (tv + x^{-1}u)\frac{\partial}{\partial v}$$

of equation (1) (see [1]).

To obtain the nonclassical potential symmetries of (1), we apply the nonclassical method to the system of equations (3). To apply the nonclassical method (3) and (5) to be invariant under the infinitesimal generator (4).

Case 1. $\tau = 1$. We determine ξ, ϕ and ζ by giving specific form for them, namely, $\xi = A(t,x)$, $\phi = B(t,x)u + C(t,x)v$ and $\zeta = D(t,x)u + E(t,x)v$. The following determining equations for ξ, ϕ and ζ are obtained by substitution and then separating by the coefficients of u and v

$$\begin{aligned} u:\ \ & D_t + \frac{DB}{1-A^2x^{-2}} - \frac{AD^2}{1-A^2x^{-2}} + \frac{DE}{1-A^2x^{-2}} - \frac{ABEx^{-2}}{1-A^2x^{-2}} \\ & + \frac{AA_tDx^{-2}}{1-A^2x^{-2}} - \frac{A_tBx^{-2}}{1-A^2x^{-2}} - B_x + \frac{AB^2x^{-2}}{1-A^2x^{-2}} - \frac{BD}{1-A^2x^{-2}} \\ & + \frac{ACDx^{-2}}{1-A^2x^{-2}} - \frac{BCx^{-2}}{1-A^2x^{-2}} + \frac{A_xD}{1-A^2x^{-2}} - \frac{ABA_xx^{-2}}{1-A^2x^{-2}} = 0, \end{aligned} \tag{6}$$

$$\begin{aligned} v:\ \ & E_t + \frac{DC}{1-A^2x^{-2}} - \frac{ADE}{1-A^2x^{-2}} + \frac{E^2}{1-A^2x^{-2}} - \frac{ACEx^{-2}}{1-A^2x^{-2}} \\ & + \frac{AA_tEx^{-2}}{1-A^2x^{-2}} - \frac{A_tCx^{-2}}{1-A^2x^{-2}} - C_x + \frac{ABCx^{-2}}{1-A^2x^{-2}} - \frac{BE}{1-A^2x^{-2}} \\ & + \frac{ACEx^{-2}}{1-A^2x^{-2}} - \frac{C^2x^{-2}}{1-A^2x^{-2}} + \frac{A_xE}{1-A^2x^{-2}} - \frac{AA_xCx^{-2}}{1-A^2x^{-2}} = 0 \end{aligned} \tag{7}$$

and

$$\begin{aligned} u:\ \ & D_x + \frac{D^2}{1-A^2x^{-2}} - \frac{ABDx^{-2}}{1-A^2x^{-2}} + \frac{BEx^{-2}}{1-A^2x^{-2}} - \frac{ADEx^{-2}}{1-A^2x^{-2}} \\ & + \frac{AA_xDx^{-2}}{1-A^2x^{-2}} - \frac{A_xBx^{-2}}{1-A^2x^{-2}} + \frac{2ABx^{-3}}{1-A^2x^{-2}} - \frac{2A^2Dx^{-3}}{1-A^2x^{-2}} \\ & - x^{-2}B_t + \frac{ABDx^{-2}}{1-A^2x^{-2}} - \frac{B^2x^{-2}}{1-A^2x^{-2}} + \frac{ABCx^{-4}}{1-A^2x^{-2}} - \frac{CDx^{-2}}{1-A^2x^{-2}} \\ & + \frac{A_tDx^{-2}}{1-A^2x^{-2}} - \frac{AA_tBx^{-4}}{1-A^2x^{-2}} = 0, \end{aligned} \tag{8}$$

$$\begin{aligned} v:\ \ & E_x + \frac{DE}{1-A^2x^{-2}} - \frac{ACDx^{-2}}{1-A^2x^{-2}} + \frac{CEx^{-2}}{1-A^2x^{-2}} - \frac{AE^2x^{-2}}{1-A^2x^{-2}} \\ & + \frac{AA_xEx^{-2}}{1-A^2x^{-2}} - \frac{A_xCx^{-2}}{1-A^2x^{-2}} + \frac{2ACx^{-3}}{1-A^2x^{-2}} - \frac{2A^2Ex^{-3}}{1-A^2x^{-2}} \\ & - x^{-2}C_t + \frac{ABEx^{-2}}{1-A^2x^{-2}} - \frac{BCx^{-2}}{1-A^2x^{-2}} + \frac{AC^2x^{-4}}{1-A^2x^{-2}} - \frac{CEx^{-2}}{1-A^2x^{-2}} \\ & + \frac{A_tEx^{-2}}{1-A^2x^{-2}} - \frac{AA_tCx^{-4}}{1-A^2x^{-2}} = 0. \end{aligned} \tag{9}$$

If $A = B = E = 0$, then solving the equations (6)–(9) yields $C = (\alpha - 1)x$, $D = \alpha/x$, $\alpha \neq 1$ which results in obtaining the following nonclassical potential symmetry

$$X_1 = \frac{\partial}{\partial t} + (\alpha - 1)xv\frac{\partial}{\partial u} + \frac{\alpha u}{x}\frac{\partial}{\partial v}.$$

If $A = B = D = 0$, as a result of solving the determining equations we obtain the following nonclassical potential symmetries

$$X_2 = \frac{\partial}{\partial t} + \frac{xv}{x-1}\frac{\partial}{\partial u} + \frac{v}{t+1}\frac{\partial}{\partial v}, \quad X_3 = \frac{\partial}{\partial t} - xv\frac{\partial}{\partial u} + \frac{v}{t}\frac{\partial}{\partial v}.$$

Case 2. $\tau = 0$, $\xi = 1$. Again, after substitutions and then separating by the coefficients of u and v we obtain the following determining equations

$$u: \ D_t + x^2D^2 + BE - B_x - B^2 - CD = 0, \tag{10}$$

$$v: \ E_t + x^2DE + CE - C_x - BC - CE = 0, \tag{11}$$

$$u: \ x^2D_x + x^2BD + x^2DE + 2xD - B_t - x^2BD - BC = 0, \tag{12}$$

$$v: \ x^2E_x + x^2CD + x^2E^2 + 2xE - C_t - x^2BE - C^2 = 0. \tag{13}$$

If $B = D = 0$, then solving the equations (10)–(13) results in obtaining the following nonclassical potential symmetries for the equation (1)

$$X_4 = \frac{\partial}{\partial x} + \frac{v}{t+1}\frac{\partial}{\partial u} + \frac{v}{x^2 - x}\frac{\partial}{\partial v}, \quad X_5 = \frac{\partial}{\partial x} + \frac{v}{t+1}\frac{\partial}{\partial u} - \frac{v}{x}\frac{\partial}{\partial v}.$$

3 Group-Invariant Solutions Corresponding to the Nonclassical Potential Symmetries

1. For the nonclassical potential symmetry generator X_1 the characteristic system related to the invariant surface conditions (5) admits the following three invariants

$$y = x, \quad z = \beta x^2v^2 - \alpha u^2,$$

$$w = -\sqrt{\alpha\beta}\,t + \log\left(\sqrt{\beta}\,xv + \sqrt{\alpha}\,u\right) - \frac{1}{2}\log\left(\beta\,x^2v^2 - \alpha\,u^2\right), \tag{14}$$

where $\beta = \alpha - 1$.

Differentiating the equation (14b) w.r.t. t we obtain $v = (\alpha u u_t)/(\beta x^2 u_x)$. Now, differentiate the equation (14c) w.r.t. t again and substituting $v = (\alpha u u_t)/(\beta x^2 u_x)$ into the resulting equation and solving it yields $u(t,x) = x^\alpha \Phi(t)$. Now, using $v_t = u_x$ we obtain $\Phi''(t) - \alpha\beta\,\Phi(t) = 0$. Hence the solution to the equation (1) is

$$u(t,x) = x^\alpha\,\Big(C_1 \exp\left(\sqrt{\alpha\beta}\,t\right) + C_2 \exp\left(-\sqrt{\alpha\beta}\,t\right)\Big)$$

where C_1 and C_2 are arbitrary constants.

2. The nonclassical potential symmetry generators X_2 and X_4 of the equation (1) yield the group-invariant solution

$$u(t,x) = C_3(x - \log x) + C_3\frac{(t+1)^2}{2} + C_4$$

and X_3 and X_5 give rise to the group-invariant solutions

$$u(t,x) = C_5 \log x - C_5\,\frac{t^2}{2} + C_6 \quad \text{and} \quad u(t,x) = C_7 \log x - C_7\,\frac{(t+1)^2}{2} + C_8$$

respectively, where C_3, C_4, C_5, C_6, C_7 and C_8 are arbitrary constants.

4 Concluding Remarks

In this paper, we extend the theory of potential symmetry method of Bluman and Kumei [1] to nonclassical potential symmetries and we have shown that these new symmetries yield some interesting group-invariant solutions for a quasi-linear PDE which is considered differently in literature. We found that different class of solutions may be obtained when one studies given PDEs through nonclassical potential symmetries.

The authors have learnt that the method used here has been used by Gandarias [12]. However, we believe that the results, viz., the exact solutions, obtained here differs. This is possible as the choice of nonclassical symmetry generators to determine exact solutions are numerous. Also, the set of nonclassical symmetry generators do not form a Lie algebra. The reader is also referred to the paper by Bluman and Yan [14].

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Symmetries for the Euclidean Non-Linear Schrödinger Equation and Related Free Equations

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We compare certain infinite dimensional Lie algebras of conserved quantities for the free Newton equation $\ddot{q} = 0$, the free heat system and the Euclidean non-linear Schrödinger equation. There is a natural differential operator defined for all polynomials of the conservation laws I_0, I_1, ... in the NLS hierarchy. We discuss the invariant polynomials and point out a connection to the free classical equation. The basic ingredient is the presence of an extra 'Heisenberg' element in addition to I_0, I_1, ...

1 Introduction

This work is related to the studies of invariance properties of Schrödinger (Bernstein) and related diffusion processes in [1-4, 8, 10, 14, 15], in particular, the case of Gaussian processes [1, 2].

Going back to a paper [11] by Sophus Lie from 1881, we know that the Lie algebra for the free heat equation in 1+1 dimensions is, except for a 'trivial' infinite dimensional stemming from linearity, of dimension six. It is a general fact, see [3], that this Lie algebra has a classical counterpart of constants of motion. In fact, in a certain sense they differ at most by one element which needs a "quantum correction". In particular, they have the same dimension. It is shown in the papers referred to above, how to obtain martingales, or stochastic constants of motion, from the heat Lie algebra.

The classical counterpart to the free heat equation is the free Newton equation $\ddot{q} = 0$, which has a six dimensional Poisson–Lie algebra given by the functions 1, p, $pt - q$, p^2, $p(pt - q)$ and $(pt - q)^2$, where $p := \dot{q}$. It consists of all functions in t, q and p which are of order at most two w.r.t. p. With $I_n = p^n$ we get an infinite sequence of constants of motion. They commute w.r.t. the natural Poisson bracket.

In the case of the free heat Lie algebra, it is clear that partial derivation w.r.t. the space variable q preserves the heat equation: ∂_q is a recursion operator [6, 12]. We can express this in a more symmetric way by looking at the free heat system $\dot{u} + u''/2 = 0$, $-\dot{v} + v''/2 = 0$. (Here and below $\dot{u} = u_t$ and $u' = u_q$.) Then all the functionals $I_n := (u^{(n)}v + (-1)^n uv^{(n)})/2$, for $n = 0, 1, \ldots$ are conservation laws in involution w.r.t. a (well known) natural Lie bracket defined below. It is an elementary but deep fact that

$$v\frac{\delta I_n}{\delta v} - u\frac{\delta I_n}{\delta u} = u^{(n)}v - (-1)^n uv^{(n)} = \frac{\partial}{\partial q}\left(u^{(n-1)}v - u^{(n-2)}v' + \ldots\right)$$

and that $u^{(n-1)}v - u^{(n-2)}v' + \ldots$, as a conservation law, is equivalent to $\mathsf{D}I_n = nI_{n-1}$. This operator D can be extended to a derivation on the space of polynomials obtained from I_n, $n \geq 0$.

In the last part of the paper we compare this example with the classical case and a non-linear system of heat equations, viz the *Euclidean non-linear heat equation* $\dot{u} + u''/2 = u^2 v$, $-\dot{v} + v''/2 = uv^2$. We also study more generally the structure on the polynomials obtained from a derivation D. This is sketched below, details will appear elsewhere.

2 The Heat Lie Algebra in 1+1 Dimensions

For $u = u(t, q)$, t, q real, we define the (backward) free *heat operator* by $Ku := \dot{u} + u''/2$. Consider all linear partial differential operators Λ of order at most one in (t, q): $\Lambda = T\partial/\partial t + Q\partial/\partial q + U$, where T, Q and U are functions of (t, q), and where U acts as multiplication operator.

Definition 1. Λ belongs to the *heat Lie algebra* if, for some function $\Phi = \Phi_\Lambda(t, q)$ it holds that $[K, \Lambda] = K\Lambda - \Lambda K = \Phi \cdot K$.

Simple calculations lead to the following well-known facts: The heat Lie algebra consists of two parts, of which the first is generated by the operators $\Lambda_0 = 1$ (the centre), $\Lambda_1 = \partial/\partial q$, $\Lambda^* = \Lambda_1^* = t\partial/\partial q - q$, forming the Heisenberg algebra (since $[\Lambda_1^*, \Lambda_1] = 1$), whereas the second, generated by

$$\Xi_1 = \Lambda_2 = \frac{\partial}{\partial t}, \quad \Xi_2 = t\frac{\partial}{\partial t} + \frac{1}{2}q\frac{\partial}{\partial q}\left(+\frac{1}{4}\right), \quad \Xi_3 = \frac{t^2}{2}\frac{\partial}{\partial t} + \frac{t}{2}q\frac{\partial}{\partial q} - \frac{1}{4}(q^2 - t)$$

form the Lie algebra sl_2. (We remark that Λ^* is not intended to suggest adjoint.)

3 The Classical Counterpart

Consider a two-dimensional phase space with coordinates p and q, and the usual Poisson bracket

$$\{\phi, \psi\} = \frac{\partial\phi}{\partial p}\frac{\partial\psi}{\partial q} - \frac{\partial\phi}{\partial q}\frac{\partial\psi}{\partial p}.$$

Regarding t as a parameter, the functions 1, p and $pt - q$, form the Heisenberg algebra, whereas p^2, $p(pt - q)$ and $(pt - q)^2$ form sl_2.

We now turn to the *free Newton equation* $\ddot{q} = 0$, in which case $p := \dot{q}$ and $pt - q$ are obvious constants of motion (CMs). The functions $1, \dots, (pt-q)^2$ is a basis for the CMs which are of order at most two in the variable p. We call it the *classical algebra*.

The first five functions correspond to the same operators as in the heat case. The sixth function corresponds to the operator

$$\frac{t^2}{2}\frac{\partial}{\partial t} + \frac{t}{2}q\frac{\partial}{\partial q} - \frac{q^2}{4} \quad \text{(no } t\text{-term)}$$

The zero-order term $q^2 - t$ in the heat case is the "quantum version", $q^2 - t =: q^2 :$ in physicist notation. Readers familiar with stochastic analysis will no doubt recognise the extra term, e.g. from Ito's formula.

The heat Lie algebra and the classical algebra satisfy the same commutator relations. The Heisenberg algebra is an ideal. Clearly $I_0 = 1$, $I_1 = p$, $I_2 = p^2, \dots$ all commute. With $I^* = I_1^* = pt - q$, we get $\{I^*, I_n\} = nI_{n-1} = dI_n/dp$. We see that $I_n \to I_{n+1}$ is the "creation operator" multiplication with p, whereas $I_n \to I_{n-1}$ is the "annihilation operator" d/dp.

4 The Free Heat System

We consider the system of two equations

$$\dot{u} + \frac{1}{2}u'' = 0, \quad -\dot{v} + \frac{1}{2}v'' = 0, \tag{1}$$

obtained from the Lagrangian $L = (u\dot{v} - \dot{u}v)/2 + u'v'/2$. The symmetry Lie algebra contains the vector fields

$$\Lambda_0 = v\frac{\partial}{\partial v} - u\frac{\partial}{\partial u}, \quad \Lambda_1 = \frac{\partial}{\partial q}, \quad \Lambda_1^* = t\frac{\partial}{\partial q} - q\left(v\frac{\partial}{\partial v} - u\frac{\partial}{\partial u}\right), \quad \Lambda_2 = \frac{\partial}{\partial t}.$$

By Noether's theorem we get the conservation laws $I_0 = uv$, $I_1 = \frac{1}{2}(u'v - uv')$, $I^* = tI_1 - qI_0$, $I_2 = (u''v + uv'')/2$, where I_0, I_1 and $I^* = I_1^*$ form a Heisenberg algebra with respect to the (field theory) bracket

$$\{F, G\} := \int \left(\frac{\delta F}{\delta u}\frac{\delta G}{\delta v} - \frac{\delta F}{\delta v}\frac{\delta G}{\delta u}\right) dq,$$

t being looked upon as a parameter. Here, the variational derivative refers to the space variable:

$$\frac{\delta F}{\delta u} = \frac{\partial F}{\partial u} - \frac{d}{dq}\frac{\partial F}{\partial u'} + \frac{d^2}{dq^2}\frac{\partial F}{\partial u''} - \dots .$$

There are two more vector fields corresponding to the remaining elements of sl_2, viz.

$$\Xi_2 = t\frac{\partial}{\partial t} + \frac{q}{2}\frac{\partial}{\partial q}, \quad \text{and} \quad \Xi_3 = \frac{t^2}{2}\frac{\partial}{\partial t} + \frac{tq}{2}\frac{\partial}{\partial q} - \frac{q^2}{4}\Lambda_0 - \frac{t}{4}\left(u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\right)$$

but our main interest is with I_0, I_1, I^* and I_2. We remark however, that in this more symmetric setting the quantum correction disappears: the conservation law corresponding to Ξ_3 is $1/2$ times $t^2 I_2 + tqI_1 - q^2 I_0/2$.

Now, defining $I_n := \left(u^{(n)}v + (-1)^n uv^{(n)}\right)/2$, $n \geq 0$, i.e. $I_{n+1} = CI_n = D/2 \cdot I_n$ (Hirota bilinear derivative) one finds $\{I_m, I_n\} = 0$ and $\{I^*, I_n\} = nI_{n-1}$, m, $n \geq 0$ (with $I_{-1} = 0$). Exactly as in the classical case, C is the creation and $\{I^*, \cdot\}$ the annihilation operator.

5 Constants of Motion for the Free Heat System

Assume that u and v satisfy (1) (we could add interaction terms here). Then

$$\frac{d}{dt}\int uv\, dq = 0.$$

Let $f = f(t,q)$. It is easy to show that $\frac{d}{dt}\int f\, uv\, dq = \int Df\, uv\, dq = \int D^* f\, uv\, dq$, where

$$Df := \frac{1}{u}K(fu) = \dot{f} + \frac{1}{2}f'' + \frac{u'}{u}f, \quad D^* f := -\frac{1}{v}K^\dagger(fv) = \dot{f} - \frac{1}{2}f'' - \frac{v'}{v}f.$$

Then, if $\Lambda = T\partial_t + Q\partial_q + U$ belongs to the heat Lie algebra, we have

$$D(\Lambda u/u) = K\Lambda u = ([K,\Lambda] + \Lambda K)u = (\Phi_\Lambda + \Lambda)Ku = 0.$$

This is an alternative way to express that $\Lambda u \cdot v$ is the density of a conservation law. In more detail, the preceding equation may be written $D(\Lambda u/u) = T\dot{u}/u + Qu'/u + U = 0$, very much as in the classical case. The coefficients $\dot{u}/u$ and u'/u are, respectively, the *energy density* and the *momentum density* in a form that emphasizes the backward motion. The density is $I_0 = uv$, and e.g. $u'v = u'/u \cdot I_0$ is an equivalent form for I_1.

6 Euclidean Non-Linear Schrödinger System

ENS may be looked upon as an extension of the free heat system with a 'potential' V that depends on u and v. We start somewhat more generally with $\dot{u} + u''/2 = Vu$ and $-\dot{v} + v''/2 = Vv$, obtained from the Lagrangian $L = (u\dot{v} - \dot{u}v)/2 + u'v'/2 + \Phi(uv)$, provided $V = \phi(uv)$, with $\phi = \Phi'$.

The following are always conservation laws: $I_0 = uv$, $I_1 = \frac{1}{2}(u'v - uv')$, $I^* = tI_1 - qI_0$, $I_2 = (u''v + uv'')/2 - 2\Phi(uv)$ for the same reasons as in the free heat case. They also commute. One can prove that there is a third order conservation law,

$$(u'''v - uv''')/2 + \text{ terms of lower order},$$

only when $\Phi''' = 0$. Leaving the linear case ($\Phi'' = 0$) aside, we choose $\Phi(s) = s^2/2$ so that our heat equations become $\dot{u} + u''/2 = u^2 v$ and $-\dot{v} + v''/2 = uv^2$, corresponding to $V = uv$. This can be seen as a two-dimensional field theory with quartic interaction. One can prove [8] that there is an operator C such that $I_n := C^n I_0$, $n \geq 0$, satisfy the same relations as in the free case: $\{I_m, I_n\} = 0$ and $\{I^*, I_n\} = nI_{n-1}$, m, $n \geq 0$. After I_2, the next two are

$$I_3 = \tfrac{1}{2}(u'''v - uv''') - \tfrac{3}{2}uv(u'v - uv'),$$
$$I_4 = \tfrac{1}{2}(u^{\text{iv}}v + uv^{\text{iv}}) + u'^2v^2 + u^2v'^2 + 6uu'vv' + 2u^3v^3.$$

Since all I_n commute with I_0, we have $v\delta I_n/\delta v - u\delta I_n/\delta u = da_n/dq$ for some functional a_n. More generally, one sees that for each I that commutes with I_0, the operator

$$\mathsf{D}I := \left(\frac{d}{dq}\right)^{-1}\left(v\frac{\delta I}{\delta v} - u\frac{\delta I_n}{\delta u}\right)$$

is well defined. One can show that for functions of I_0, I_1, $I_2, \ldots$, D is a *derivation* in that $\mathsf{D}\{f(I_0, I_1, \ldots, I_n)\} = \sum_{\mu=0}^{n} \partial_\mu f(I_0, I_1, \ldots, I_n)\mathsf{D}I_\mu$, for any C^1 function f. This holds also in the free heat case.

7 Invariant Polynomials

We assume, without reference to the particular cases considered above, that we are given variables I_0, I_1, I_2, ..., and an operator D such that $\mathsf{D}I_\mu = \mu I_{\mu-1}$ for all $\mu = 0, 1, \ldots$ We also assume that $\mathsf{D}f(I_0, I_1, \ldots, I_n) = \sum_{\mu=0}^{n} \partial_\mu f(I_0, I_1, \ldots, I_n)\mu I_{\mu-1}$ for any $n \in \mathbb{N}$ and for any function $f \in C^1(\mathbb{R}^{n+1})$.

Definition 2. A function $M = M_\alpha$ of the form $M = I_0^{\alpha_0} I_1^{\alpha_1} \cdots I_n^{\alpha_n}$, where $\alpha_0, \alpha_1, \ldots, \alpha_n \in \mathbb{N}$, is a *monomial of order* $N = \sum \mu\alpha_\mu = ||\alpha||$.

Definition 3. A function P of the form $P = \sum_{||\alpha||=N} c_\alpha M_\alpha$, where c_α are constants, is a *polynomial of order* N.

Definition 4. A polynomial P is *invariant* if $\mathsf{D}P = 0$.

For $N = 0$ every polynomial, in fact, every differentiable function, of I_0 is invariant. These functions should be looked upon as scalars. For $N = 1$ there are no invariant polynomials. For $N = 2$, $K_2 = I_1^2 - I_0I_2$ is invariant, and for $N = 3$, $K_3 = 2I_1^3 - 3I_0I_1I_2 + I_0^2I_3$ is invariant. Up to multiplication with functions of I_0, K_2 and K_3 are unique. For $N = 4$, of course K_2^2 is invariant. There is another one, unique up to multiplication with functions of I_0, viz. $K_4 = 4I_1I_3 - 3I_2^2 - I_0I_4$. K_4 is *irreducible*. For $N = 5$ we get the obviously invariant polynomial K_2K_3 and a new, irreducible, invariant polynomial, K_5. For $N = 6$ we get K_2^3, K_3^2 and K_2K_4 in addition to the new, irreducible, invariant polynomial K_6.

Theorem 1. *For each $N \geq 2$ there is an irreducible invariant polynomial K_N, unique up to multiplication with functions of I_0.*

Denote by $\mathcal{P}$ all polynomials and by $\mathcal{M}$ the quotient space $\mathcal{M} = \mathcal{P}/(K_2, K_3, \ldots)$. That $K_2 \equiv 0$ means that $I_0 I_2 \equiv I_1^2$ or $I_2/I_0 \equiv (I_1/I_0)^2$. Using also $K_3 \equiv 0$ we find $I_3/I_0 \equiv (I_1/I_0)^3$, and so on: $I_n/I_0 \equiv (I_1/I_0)^n$, $n = 2, 3, \ldots$. Note that

$$\mathsf{D}\frac{I_1}{I_0} = 1, \quad \text{and} \quad \mathsf{D}\frac{I_n}{I_0} \equiv \mathsf{D}\left(\frac{I_1}{I_0}\right)^n = n\left(\frac{I_1}{I_0}\right)^{n-1}, \quad n = 0, 1, \ldots.$$

Hence I_1/I_0 correspond to p in our first example, the free equation $\ddot{q} = 0$. We remark that I_1/I_0 is the momentum density mentioned at the end of Section 5 above.

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Symmetries of Integral Equations in Plasma Kinetic Theory

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The report reviews the recent advances in modern group analysis approach to constructing solutions of self-consistent Vlasov-Poisson system of integral and differential equations in plasma. The solutions obtained describe different physical phenomena, such as nonlinear plasma oscillations, the ion acceleration in the adiabatic expansion of a plasma bunch and the Coulomb explosion of cluster plasma.

1 Introduction

The self-consistent Vlasov–Poisson system of equations is used for a kinetic treatment of plasma expansion into a vacuum. Theoretically the process has been studied for almost 40 years, since the work by Gurevich et al. [1]. However, until last decade this problem have been treated mainly by using hydrodynamic models [2]. In the 90-s the kinetic aspects of plasma expansion already prevailed. The need for kinetic treatment was aimed to better understanding the mechanisms and characteristics of ions triggered by the interaction of a short-laser-pulse with plasma. Developments in laser technology have enabled high power lasers to produce multi-terrawatt femtosecond pulses, which allow the examination of the fundamental physics of ion acceleration at multi-MeV energies. At high focal intensities laser-triggered ion acceleration results in the formation of a multi-MeV beam propagating in the forward direction [3,4]. Experiments have already proven the possibility of transforming the laser energy into collimated ultra-fast ion bunches with high efficiency when focusing ultra-short laser pulses on solid targets [5, 6]. The commonly recognized effect responsible for ion acceleration is charge separation in the plasma due to high-energy electrons, driven by the laser inside the target [3–6]. During the plasma expansion, the kinetic energy of the fast electrons transforms into the energy of electrostatic field, which accelerates ions and their energy is expected to be at the level of the hot-electron energy. The mechanisms and characteristics of ions triggered by the interaction of a short-laser-pulse with plasma are of current interest because of their possible applications to the novel-neutron-source development and isotope production. In the near future ultra-intense laser pulses will be used for ion beam generation with energies useful for proton therapy, fast ignition inertial confinement fusion, radiography, neutron-sources.

Along with interactions with solid targets there exists the other interesting field of application of high-intensity ultrashort laser pulses, namely experiments with cluster plasmas. A characteristic feature of cluster plasma is its ability to strongly (almost entirely) absorb laser radiation. This property makes it possible to built high-brightness X-ray sources [7]. Moreover, the expansion of clusters results in ion acceleration to high energies [8–11]. In the case of deuteron clusters, ion–ion collisions produce fusion neutrons; this provides an opportunity to create sub-nanosecond neutron sources for use in materials science. If the laser field is strong enough, it almost instantaneously knocks electrons out of a cluster, thereby creating conditions for the subsequent Coulomb explosion of a positively charged microplasma. The ions of the exploding clusters are accelerated to high energies and give rise to a macroplasma with a high effective ion temperature. The maximum energy of the accelerated ions, the ion energy spectrum and the relation between this spectrum and the initial ion density distribution are of current theoretical interest.

The mathematical model for phenomena of cluster explosion and laser-plasma interactions with solid foil targets, which demonstrate ion acceleration at multi-MeV energies, is based on the unified system of Vlasov–Maxwell (differential and integral) equations of the plasma kinetic theory. Up to recently theoretical investigations were based mainly on a simplified phenomenological hydrodynamic theory and numerical modelling, and only few analytical approaches to the complete system of Vlasov–Maxwell equations were known. Here we describe a substantial progress made recently in analytical investigations of plasma kinetic theory equations that is based on methods of the modern group analysis. As an illustration we indicate several approaches to construction of new solutions of Vlasov–Maxwell and Vlasov–Poisson equations, which describe nonlinear plasma oscillations, the expansion of a cluster plasma and acceleration of ions in quasi-neutral approximation.

2 Basic Model: Vlasov–Maxwell Equations

Different physical phenomena for short pulse laser-plasma interactions are treated on basis of the same mathematical model, i.e. the Vlasov equations for the particle distribution functions $f_\alpha(t, \boldsymbol{r}, \boldsymbol{v})$ for the electrons ($\alpha = e$) and ions ($\alpha = 1, 2, \ldots$),

$$\begin{aligned} & f_t^\alpha + \boldsymbol{v} f_{\boldsymbol{r}}^\alpha + \frac{e_\alpha}{m_\alpha \gamma} \Big\{ \boldsymbol{E} + \frac{1}{c} [\boldsymbol{v}\boldsymbol{B}] - \frac{1}{c^2} \boldsymbol{v} \, (\boldsymbol{v}\boldsymbol{E}) \Big\} f_{\boldsymbol{v}}^\alpha = 0; \\ & \gamma = \frac{1}{\sqrt{1 - (\boldsymbol{v}/c)^2}} \, , \end{aligned} \tag{1}$$

supplemented by Maxwell equations for the electric and magnetic fields, $\boldsymbol{E}$ and $\boldsymbol{B}$,

$$\begin{aligned} & \boldsymbol{B}_t + c \, \mathrm{rot} \, \boldsymbol{E} = 0, \quad \mathrm{div} \, \boldsymbol{E} = 4\pi\rho, \\ & \boldsymbol{E}_t - c \, \mathrm{rot} \, \boldsymbol{B} + 4\pi \boldsymbol{j} = 0, \quad \mathrm{div} \, \boldsymbol{B} = 0, \end{aligned} \tag{2}$$

and the nonlocal material equations for the charge and current densities, ρ and $\boldsymbol{j}$,

$$\rho = \sum_\alpha e_\alpha m_\alpha^3 \int \mathrm{d}\boldsymbol{v}\, f^\alpha(\gamma)^5\,, \quad \boldsymbol{j} = \sum_\alpha e_\alpha m_\alpha^3 \int \mathrm{d}\boldsymbol{v}\, f^\alpha(\gamma)^5 \boldsymbol{v}. \tag{3}$$

The symmetry group admitted by Vlasov–Maxwell (VM) equations (1)–(3) is given by the 10-th dimensional Poincare group

$$\begin{aligned} &L_{10} = \langle X_0,\, \boldsymbol{X}_1,\, \boldsymbol{X}_2,\, \boldsymbol{X}_3 \rangle, \\ &X_0 = \partial_t, \quad \boldsymbol{X}_1 = \partial_{\boldsymbol{r}}, \quad \boldsymbol{X}_2 = \boldsymbol{r}\partial_t + c^2 t \partial_{\boldsymbol{r}} + c^2 \partial_{\boldsymbol{v}} - \boldsymbol{v}\,(\boldsymbol{v},\, \partial_{\boldsymbol{v}}) \\ &\qquad - c\,[\boldsymbol{B},\, \partial_{\boldsymbol{E}}] + c\,[\boldsymbol{E},\, \partial_{\boldsymbol{B}}] + c^2 \rho \partial_{\boldsymbol{j}} + \boldsymbol{j}\partial_\rho, \\ &\boldsymbol{X}_3 = [\boldsymbol{r},\, \partial_{\boldsymbol{r}}] + [\boldsymbol{v},\, \partial_{\boldsymbol{v}}] + [\boldsymbol{E},\, \partial_{\boldsymbol{E}}] + [\boldsymbol{B},\, \partial_{\boldsymbol{B}}] + [\boldsymbol{j},\, \partial_{\boldsymbol{j}}]\,, \end{aligned} \tag{4}$$

that describes time and space translations, X_0 and $\boldsymbol{X}_1$, Lorentz transformations $\boldsymbol{X}_2$, and rotations $\boldsymbol{X}_3$. These operators are supplemented by the generator of dilations

$$X_4 = t\partial_t + \boldsymbol{r}\partial_{\boldsymbol{r}} - 2\sum_\alpha f^\alpha \partial_{f^\alpha} - \boldsymbol{E}\partial_{\boldsymbol{E}} - \boldsymbol{B}\partial_{\boldsymbol{B}} - 2\boldsymbol{j}\partial_{\boldsymbol{j}} - 2\rho\partial_\rho, \tag{5}$$

and $(k-1)$ generators (omitted here) of pairwise translations in the space of distribution functions when there are more than two particle species in plasma [12, 13].

For the plane geometry and in non-relativistic limit ($c \to \infty$) equations (1)–(3) are simplified

$$f_t^\alpha + v f_x^\alpha + (e_\alpha/m_\alpha) E(t,x) f_v^\alpha = 0, \tag{6}$$

$$E_x - 4\pi \sum_\alpha \int \mathrm{d}v e_\alpha f^\alpha = 0, \quad E_t + 4\pi \sum_\alpha \int \mathrm{d}v v e_\alpha f^\alpha = 0. \tag{7}$$

For this case the Poincare group and the dilation operator are reduced to the following generators

$$\begin{aligned} &X_0 = \partial_t, \quad X_1 = \partial_x, \quad X_3 = x\partial_x + v\partial_v - \sum_\alpha f^\alpha \partial_{f^\alpha} + E\partial_E, \\ &X_2 = t\partial_x + \partial_v, \quad X_4 = 2t\partial_t + x\partial_x - v\partial_v - 3\sum_\alpha f^\alpha \partial_{f^\alpha} - 2E\partial_E, \end{aligned} \tag{8}$$

describing time and space translations, X_0 and X_1, Galilean boosts, X_2, and dilations, X_3 and X_4. The joint system of equations (6) and the first equation in (7) is often referred to as Vlasov-Poisson (VP) equations. We are interested in a solution to the Cauchy problem to equations (6) with the initial conditions that correspond to the electron and ion distribution functions specified at the initial time, $f^\alpha\big|_{t=0} = f_0^\alpha(x,v)$.

3 Expansion of the Plasma Bunch

VP equations (6)–(7) seem to be the simplest one-dimensional mathematical model, which is commonly used to describe the expansion of a plasma slab. Even so the modern group analysis methods fail to create the spatially symmetric solution of (6)–(7) for the distribution functions with initial zero mean velocity. Thus, with the goal to find physically reasonable solution we are forced to simplify the basic system of VP equations and consider two limiting cases.

I. The first case corresponds to small time scales, when the term proportional to the electric field in (6) is small and can be considered as a perturbation that violates the free particle movement. In this case we calculate the symmetry group as the approximate symmetry starting from the zero-order approximation that describes the flow of particles when the electric field is neglected. The approximate symmetry group operator thus constructed depends on the initial space-velocity particle distribution functions. In the particular case of the Maxwellian velocity and the Gaussian space distribution it is presented in the following form

$$
\begin{aligned}
R^{appr} &= \kappa^e \partial_{f^e} + \kappa^q \partial_{f^q} + \dots\,, \quad \kappa^j = \kappa^{j,0} + \kappa^{j,1}\,, \quad j = 0,1,2,\dots \\
\kappa^{1,0} &= -v f^e_x + \frac{x - vt}{d^2} V^2_{Te}(f^e_v + t f^e_x)\,, \quad \Leftrightarrow \quad f^e_0 \sim e^{-v^2/2V^2_{Te} - x^2/2d^2}\,, \\
\kappa^{2,0} &= -v f^i_x + \frac{x - vt}{d^2} V^2_{Ti}(f^i_v + t f^i_x)\,, \quad \Leftrightarrow \quad f^i_0 \sim e^{-v^2/2V^2_{Ti} - x^2/2d^2}\,, \\
\kappa^{1,1} &= \frac{e}{m} \Big\{ E\left(1 + (t^2 V^2_{Te}/d^2)\right) f^e_v - \left(D^0_t + v D^0_x\right)^{-1} f^e_v \times \\
&\quad \left(3tE(V^2_{Te}/d^2) + (4\pi/d^2)(V^2_{Ti} - V^2_{Te})(t^2 j^i - tx\rho^i)\right) \Big\}\,, \\
\kappa^{2,1} &= \frac{e^i}{M} \Big\{ E\left(1 + (t^2 V^2_{Ti}/d^2)\right) f^i_v - \left(D^0_t + v D^0_x\right)^{-1} f^i_v \times \\
&\quad \left(3tE(V^2_{Ti}/d^2) + (4\pi/d^2)(V^2_{Te} - V^2_{Ti})(t^2 j^e - tx\rho^e)\right) \Big\}\,.
\end{aligned}
\tag{9}
$$

Here we use standard notations for the operators of the total differentiation evaluated on the VP manifold when electric field is neglected.

The approximate symmetry group obtained describes the evolution of particles distribution functions and the electric field on the initial stage of plasma dynamics when nonlinear oscillations of plasma density are excited. Particularly, this approach made it possible to calculate the plasma densities disturbances and the related space distribution of the electrostatic field [14] for the expanding electron-ion plasma on small time scales. The initial electron and ion distribution functions were taken Maxwellian with the thermal velocities V_{Te} and V_{Ti} and the initial density space distribution were described by the Gaussian curve with the same density scale d.

II. The second case describes the physical situation on long time scales, when the laser pulse terminates and the plasma trends toward quasi-neutrality. It means that one can neglect the field terms in Poisson and Maxwell equations (7) and consider the total charge and current densities equal to zero. Hence, particle distribution functions $f^\alpha(t,x,v)$ for the electrons ($\alpha = e$) and ions $\alpha = 1, 2, \ldots$) are assumed to satisfy the quasi-neutrality conditions,

$$\int \mathrm{d}v \sum_\alpha e_\alpha f^\alpha = 0\,, \quad \int \mathrm{d}v\, v \sum_\alpha e_\alpha f^\alpha = 0\,. \tag{10}$$

and the electric field is expressed in terms of moments of distribution functions

$$E(t,x) = \int \mathrm{d}v\, v^2\, \partial_x \sum_\alpha e_\alpha f^\alpha \Big\{ \int \mathrm{d}v \sum_\alpha \frac{e_\alpha^2}{m_\alpha} f^\alpha \Big\}^{-1}. \tag{11}$$

To find the symmetry group, admitted by (6) and (10), the electric field $E(t,x)$ is considered to be unknown function of the coordinate x and time t. This case of finding the symmetry logically follows from the simpler, quasi-neutral model of plasma description, in contrast to the complete system of VM or VP equations. The principal difference of the admitted symmetry group, as compared to the symmetry group generators for VP equations, is that the two dilation operators X_3, X_4 in (8) are replaced by the three dilation operators X_3, X_4, and X_5, supplemented by the new, projective group generator X_6,

$$\begin{aligned} &X_3 = t\partial_t - v\partial_v, \quad X_4 = x\partial_x + v\partial_v, \quad X_5 = \sum_\alpha f^\alpha \partial_{f^\alpha}, \\ &X_6 = t^2\partial_t + tx\partial_x + (x - vt)\partial_v. \end{aligned} \tag{12}$$

The linear combination of time translations generator X_1 and the generator X_6,

$$R^{quasi} = (1 + \Omega^2 t^2)\partial_t + \Omega^2 tx\partial_x + \Omega^2(x - vt)\partial_v, \tag{13}$$

allows to derive an entire class of solutions to the Cauchy problem for different initial distributions of the particles [15]; the generalization of these results to the spherically symmetric case is straightforward [16].

The generator (13) is the only which selects the spatially symmetric initial distribution functions with zero mean velocity. The value Ω can be treated as the ratio of the ion acoustic velocity to the gradient length L_0. Distribution functions are group invariants of the transformations, defined by (13),

$$f^\alpha = f_0^\alpha(I^{(\alpha)})\,,\; I^{(\alpha)} = \frac{1}{2}\left(v^2 + \Omega^2(x - vt)^2\right) + \frac{e_\alpha}{m_\alpha}\Phi_0(x'), \tag{14}$$

where the dependence of Φ_0 on self-similar variable $x' = x/\sqrt{1 + \Omega^2 t^2}$ is defined by quasi-neutrality conditions (10).

Distribution functions (14) give exhaustive information on the kinetics of plasma bunch expansion. However, for practical applications rough integral characteristics, such as partial ion density, $n^q(t,x)$, ($q = 1, 2, \ldots$) and ion energy spectra, $dN_q/d\epsilon$, might be more useful,

$$n^q = \int_{-\infty}^{\infty} \mathrm{d}v f^q(t,x,v)\,, \quad \frac{dN_q}{d\epsilon} = \frac{1}{m_q v} \int_{-\infty}^{\infty} \mathrm{d}x\, (f^q(t,x,v) + f^q(t,x,-v))\,. \tag{15}$$

Here n^q is the linear functional of f^q, hence we prolong [17] the generator R^{quasi} to get the following generator that describes the density transformations

$$R^{\text{density}} = (1+\Omega^2 t^2)\partial_t + \Omega^2 tx\partial_x + \Omega^2 tn^q \partial_{n^q} \,. \tag{16}$$

Finite group transformations defined by (16) yield the behavior of n^q for arbitrary $t \neq 0$,

$$n^q = \frac{n_{q0}}{\sqrt{1+\Omega^2 t^2}} \mathcal{N}_q(\chi) \,, \quad \mathcal{N}_q = \int_{-\infty}^{\infty} \mathrm{d}v f_0^q \,, \quad \chi = \frac{x}{\sqrt{1+\Omega^2 t^2}} \,. \tag{17}$$

The general form of $dN_q/d\epsilon$ is rather complicated but its asymptotic behavior at $\Omega t \to \infty$ is described by the same function $\mathcal{N}_q$,

$$\frac{dN_q}{d\epsilon} \approx \sqrt{\frac{2}{m_q \epsilon}} \frac{n_{q0}}{\Omega} \mathcal{N}_q \left(\epsilon = \frac{m_q U^2}{2}\right) \,, \quad 2\epsilon/T_q \gg (\Omega t)^{-2} \,, \quad U = \Omega\chi \,. \tag{18}$$

These results are applied to plasma that contains ions of several, say two, types (the index $q = 1, 2$ corresponds to heavy and light ions, respectively) with initial Maxwellian velocity distribution functions, and the electrons obeying a two-temperature Maxwellian distribution function with densities and temperatures of the cold and hot components n_{c0} and n_{h0} ($n_{c0} + n_{h0} = \sum_q Z_q n_{q0}$) and T_e and T_h, respectively. In this case the density distribution and, hence, the ion energy spectrum is expressed as

$$\mathcal{N}_q = \exp\left[\mathcal{E}\left(\frac{Z_q T_c}{T_q}\right) - \frac{U^2}{2v_{T_q}^2}\left(1 + \frac{Z_q m_e}{m_q}\right)\right] \,, \quad q = 1, 2 \,, \quad v_{Tq}^2 = \frac{T_q}{m_q} \,, \tag{19}$$

where the function $\mathcal{E}$ is defined in the implicit form

$$\begin{aligned} n_{c0} = \sum_{q=1,2} Z_q n_{q0} \exp\left[(1 + (Z_q T_c/T_q))\,\mathcal{E} - (U^2/2v_{Tq}^2)\right. \\ \left. \times (1 + (Z_q m_e/m_q))\right] - n_{h0} \exp\left[(1 - (T_c/T_h))\,\mathcal{E}\right] . \end{aligned} \tag{20}$$

The high end of the energy spectrum for light ions ($q = 2$) in a plasma bunch with hot electrons, defined by (19)–(20), typically has a sharp decrease, so that the value $(2-3)Z_2 T_h$ can be referred to as the characteristic ion energy cutoff. Acceleration of light impurity ions is of current interest in the experiments on high-energy proton generation by short laser pulses with thin foil targets and the measured proton energy cutoff (see, e.g., [18]) is in qualitative agreement with the above estimation.

4 Coulomb Explosion in a Cluster Plasma

Up to recently the Coulomb explosion of a cluster and the related spectrum of the accelerated ions have been described in the simplest "ideal cluster" model, in which the exploding cluster is treated as an exploding homogeneous spherical bunch with a given initial radius r_c. It predicts a square-root ion energy spectrum $\sqrt{\epsilon}$ with a sharp energy cutoff at the maximal energy $\epsilon_{max} \propto r_c^2$. Similar energy spectra were obtained in three-dimensional particle-in-cell simulations [10] and also were observed in experiments. At the same time, in some experiments, ion energy spectra were found to differ from the above square-root spectrum with a sharp energy cutoff. This difference may be attributed to, e.g., the radial non-uniformity of the ion density $n_c(r)$ in a cluster or to the initial spread in the cluster radii. In cluster plasma theory, however, the question of how to describe the ion energy spectrum as a function of the initial ion density profile up to recently has remained open. We just indicate a feasible routine to the problem, based on modern group analysis.

The motion of the cluster plasma governed by the electrostatic field is defined by a single kinetic equation (1) for cluster ions. For spherically symmetric case, $f = f^r(t, r, v) f^{\perp}(\boldsymbol{v}_{\perp}^2)$ we integrate f over transverse velocities, $F(t, r, v) = \int f \mathrm{d}\boldsymbol{v}_{\perp}$ and get the kinetic equation, which we will write down neglecting transverse thermal motion,

$$g_t + v g_r + \frac{e^i}{M} E g_v = 0\,, \quad (r^2 E)_r - 4\pi e^i \int_{-\infty}^{\infty} \mathrm{d}v\, g = 0, \quad g \equiv r^2 F, \tag{21}$$

which should be solved with the initial condition $F\mid_{t=0} = f_0(v, r)$. The boundary conditions for the electric field imply that it vanishes at $r = 0$ and decreases to zero at infinity. The cluster ion spectrum, defined by the equality, akin to (18), is the linear functional of the distribution function F. Therefore, using the suitable symmetry that defines the solution of the initial value problem we can prolong the symmetry operator on this functional and restore the behavior of the ion energy spectrum at arbitrary moment $t \neq 0$. In the "cold" ion plasma limit, $f_0(v, r) = \delta(v) n_c(r)$ the form of ion spectra and its dependence on the initial profile of ion density and the initial spread in the cluster radii were discribed in [19] (the admitted symmetry group for this case was reported in [21]). However, even for the case when thermal motion of ions is neglected the problem of calculation of the ion spectrum is not trivial as the "cold" solution exists over a finite time interval, followed by the multi-flux regime [19, 20].

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Newtonian Economics

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Some important equations in Financial Mathematics such as the Black–Scholes equation arise from an application of the Feynman–Kac Theorem to the stochastic processes underlying the mathematical modelling of financial instruments. This is in close parallel to the mathematical modelling of physical phenomena based on Brownian motion. We examine certain of the equations which arise in Financial Mathematics from the point of view of their symmetries. These include both linear and nonlinear models. We note certain peculiarities of the nonlinear models. We relate the symmetry structure of $(1+1)$ evolution equations to what is known in mechanical systems, both Classical and Quantal. In particular we consider the relationship with the Noetherian symmetries of a classical Lagrangian. The connection with mechanical systems prompts some speculation about the behaviour of the solutions of the equations of Financial Mathematics in $(1+2)$ dimensions.

1 Some Equations of Financial Mathematics

The best known equation of Financial Mathematics is undoubtedly the Black–Scholes equation [2, 14]. It and related equations arise from an application of the Feynman–Kac Theorem to the stochastic processes underlying the mathematical modelling of financial instruments. The Feynman–Kac Theorem comes from the study of Brownian Motion in Statistical Mechanics which is not generally regarded as having any connection with Finance. However, the stochastic processes are based on Brownian Motion as a model.

This connection in itself demonstrates the power of Mathematical Modelling since it manages to bring two disparate areas of inquiry under the same mathematical umbrella.

We list a number of partial differential equations which arise in Financial Mathematics. We give some emphasis to the Black–Scholes Equation for the very reason of its familiarity to the broader scientific community.

1.1 The Black–Scholes Equation

The practice of taking and exercising put and call options became common and acceptable in the Sixties of the last century. Earlier opinion had been that these were somewhat akin to gambling and were not appropriate activities for the Stock Exchange, perhaps in reaction to some of the excesses associated with the Great Crash of 1929. In 1973 two seminal papers appeared which were devoted to the mathematical theory of option pricing. Black and Scholes developed a parabolic partial differential equation to describe the evolution in time of the value of what is known as an 'European option'. Black and Scholes obtained the equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} + rx\frac{\partial u}{\partial x} - ru = 0, \qquad (1)$$

where $u(t, x)$ is the value of the option as a function of the time t and the stock price x. The parameters r and σ^2 represent the risk-free interest rate and the variance of the rate of the return on the stock respectively. In the model as it stands these parameters are constants. The rigorous derivation of the Black–Scholes equation was given by Harrison and Pliska in 1981 [5].

At about the same time as the appearance of the paper by Black and Scholes a mathematically somewhat more sophisticated paper was presented by Merton which was a substantial revision of some earlier work. The paper by Black and Scholes was received for publication towards the end of 1970 so that it would seem that both Merton and Black and Scholes were working closely in parallel. Nevertheless Merton acknowledges the superiority of the Black–Scholes model in its provision of supplementary assumptions which enable a precision to be attached to the predictions of the model which were not available in his more general construct on mathematical principles and theorems.

In his Introduction Merton observes that, 'since options are specialised and relatively unimportant financial securities, the amount of time and space devoted to the development of a pricing theory might be questioned.'

Already in their Conclusion Black and Scholes had observed that their results could be extended to many other situations and, in a sense, that virtually every financial instrument could be regarded in terms of an option.

By way of contrast to Merton's comment Kwok [12] observes that the revolution in derivate securities, which began in the early Seventies of the last century, has led to growth in the field which can only be described as phenomenal. The widespread growth of hedging as an attempt, not entirely with complete success, to protect assets since then is indicative of this phenomenal growth.

Underlying the development of the theory is the principle of riskless hedging. Black and Scholes made the following assumptions [17] on the workings of the financial markets. They are

1. trading takes place continuously in time;

2. the riskless interest rate r is known and is constant over time;

3. the asset pays no dividend;
4. there are no transaction costs in buying or selling the asset for the option and no taxes or other imperfections;
5. the assets are perfectly divisible;
6. there are no penalties to short selling and the full use of proceeds is permitted and
7. there are no riskless arbitrage opportunities.

This list of preconditions or assumptions is enough to make the moderately aware reader wonder about what sort of a Dream World Black and Scholes indwelled. Nevertheless they provide a framework in which a mathematical model may be constructed. The value of the mathematical model is to be measured in its predictions and their correlations with observations. Black and Scholes reported the results of empirical tests of their formula on a large body of call-option data and the results of the tests indicated that there were systematic variations of reality from the prediction. Most of the deviation could be attributed to the large transaction costs of the options market. Of slighter importance was a difference for low-risk stocks compared with high-risk stocks.

Subsequent modifications to the Black–Scholes model have demonstrated that it is quite robust with respect to many of the above assumptions.

We recall that Nail Ibragimov [10] reported the group analysis of the Black–Scholes Equation in 1996.

1.2 Simplification of the Black–Scholes Equation

Equation (1) is an evolution equation with time as the evolution variable and the price of the underlying stock being, as it were, the spatial variable. Strictly speaking the Black–Scholes and similar equations are backwards evolution equations since they have a *terminus a quo*. However, the relative locations of terminal time and current time can be reversed by a reflection of the time variable. Thus mathematically these equations are evolution equations even though in their practical manifestations the expression 'backwards evolution equation' is naturally suggested. The equation is also linear. Thus it is reasonable to seek a transformation which renders it as the archetypal evolution equation in $1+1$ dimensions. Under the transformation

$$\tau = -\sigma^2 t, \quad \rho = \log x, \quad \psi(\rho,\tau) = \exp\left[\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)\rho + \Omega\tau\right] u(x,t),$$

where $\Omega = \left(1 + 2r/\sigma^2\right)^2/8$, (1) becomes $-\partial\psi/\partial t + \frac{1}{2}\partial^2\psi/\partial\rho^2 = 0$ which is the archetypal evolution equation in $1+1$ dimensions!

The equation, (1), is the simplest form of the Black–Scholes equation. In its original conception the equation dealt with the pricing of options, but it has become apparent that the equation has a much wider applicability.

1.3 Some Other Equations of Mathematical Finance

There is a number of equations related to the Mathematics of Finance. We list a few other equations which arise in various models [11].

Goldys and Musiela [4] (p. 325) present the backwards Kolmogorov equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} + rx\frac{\partial u}{\partial x} - ru = 0,$$

which is just a Black–Scholes equation (actually one should reverse the comment since Kolmogorov considerably preceded Black and Scholes), with the terminal condition $u(T, x) = (K - x)^+$.

Heath *et al* [7] (p. 516) give an equation for mean-variance hedging as

$$\frac{\partial J}{\partial t} + a\frac{\partial J}{\partial y} + \frac{b}{2}^2 \frac{\partial^2 J}{\partial y^2} - \frac{1}{2}\left(\frac{\partial J}{\partial y}\right)^2 + \left(\frac{\mu}{y}\right)^2 = 0 \tag{2}$$

with the terminal condition $J(T, y) = 0$.

When (2) is solved, there remains the further equation

$$\frac{\partial v_{\hat{p}}}{\partial t} + \left(a - b^2\frac{\partial J}{\partial y}\right)\frac{\partial v_{\hat{p}}}{\partial y} + \frac{1}{2}x^2 y^2 \frac{\partial^2 v_{\hat{p}}}{\partial x^2} + \frac{1}{2}\frac{\partial^2 v_{\hat{p}}}{\partial y^2} = 0.$$

We return to (2) below.

The list is not restricted to equations of the $(1+1)$ type. Cvitanić [3] (p. 595) informs us that the K-hedging price for $V(t, s)$ is the solution of the d-dimensional Black–Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij} s_i s_j \frac{\partial^2 V}{\partial s_i \partial s_j} + r\left(\sum_{i=1}^{d} s_i \frac{\partial V}{\partial s_i} - V\right) = 0,$$

which is an equation of Hamilton-Jacobi-Bellman type.

From an entirely different source we have another form for the Hamilton-Jacobi-Bellman equation which is [8, 9]

$$J_t - \frac{1}{2}\frac{\mu^2}{\sigma^2}\frac{J_x^2}{J_{xx}} - \frac{\mu s v \rho}{\sigma}\frac{J_{xs} J_x}{J_{xx}} + \mu s J_s - \frac{1}{2}s^2 v^2 \rho^2 \frac{J_{xs}^2}{J_{xx}} + \frac{1}{2}s^2 v^2 J_{ss} = 0,$$

where μ, σ, v and ρ are constant parameters. For a detailed discussion of this equation see Naicker *et al* [15].

This is just a small sampling of the variety of equations to be found in the Mathematics of Finance. Evidently we cannot discuss all of these equations in the context of a brief communication and here we are very selective to the extent of considering a single example and then just one case at that. One hopes that the selection gives some idea of the role which symmetry has in the analysis of equations which arise in the area of Financial Mathematics.

2 Mean-Variance Hedging

The system which we consider, at least in part, is given by the pair of equations

$$\frac{\partial J}{\partial t} + a\frac{\partial J}{\partial y} + \frac{b}{2}^2\frac{\partial^2 J}{\partial y^2} - \frac{1}{2}\left(\frac{\partial J}{\partial y}\right)^2 + \nu(y) = 0 \tag{3}$$

with the terminal condition $J(T, y) = 0$ and

$$\frac{\partial v_{\hat{p}}}{\partial t} + \left(a - b^2\frac{\partial J}{\partial y}\right)\frac{\partial v_{\hat{p}}}{\partial y} + \frac{1}{2}x^2y^2\frac{\partial^2 v_{\hat{p}}}{\partial x^2} + \frac{1}{2}\frac{\partial^2 v_{\hat{p}}}{\partial y^2} = 0. \tag{4}$$

Not surprisingly Program LIE [6, 18] returns results which depend upon the explicit expression for the function $\nu(y)$. In a simplified form, *i.e.* up to point transformations in the variables, the possible results are

$$\nu(y) = \begin{cases} \phi(y) & 1 + 1 + \infty \\ \dfrac{\mu}{y^2} & 3 + 1 + \infty \\ \mu & 5 + 1 + \infty \\ \mu y & 5 + 1 + \infty \\ \mu y^2 & 5 + 1 + \infty \\ \text{third or fifth} \ + \dfrac{k}{y^2} & 3 + 1 + \infty \end{cases}.$$

We do not examine all of these possible cases for the present, but report a single case to give a flavour for the discussion. In the case that $\nu(y) = \mu^2$ the Lie point symmetries are

$$\begin{aligned} &\Gamma_1 = f(t, y)\exp[-J/b^2]\partial_J, \quad \Gamma_2 = \partial_J, \quad \Gamma_3 = \partial_t, \\ &\Gamma_4 = t\partial_t + \frac{1}{2}(y + at)\partial_y + \mu^2 t\partial_u, \\ &\Gamma_5 = t^2\partial_t + ty\partial_y + \frac{1}{2}\left(b^2t - 2\mu^2ty - (y - at)^2\right)\partial_J, \\ &\Gamma_6 = \partial_y, \quad \Gamma_7 = t\partial_y - (y - at)\partial_J, \end{aligned} \tag{5}$$

where $f(t, x)$ is a solution of

$$\frac{\partial f}{\partial t} + \frac{1}{2}b^2\frac{\partial^2 f}{\partial y^2} + a\frac{\partial f}{\partial y} - \frac{\mu^2}{b^2}f = 0.$$

The compatibility of any of the symmetries listed in (5) with the terminal condition $J(T, y) = 0$ must firstly be established before one attempts to construct a similarity solution.

The general Lie point symmetry of (3) is $\Gamma = \sum_{i=1}^{6} a_i\Gamma_i$, where the $a_i,\ i = 1, 6$ are constants to be determined.

The application of Γ to the terminal, *i.e.* $t = T$, gives

$$a_3 + a_4 T + a_5 T^2 = 0 \tag{6}$$

and to the terminal condition itself gives

$$a_2 + a_4\mu^2 T + \frac{1}{2} a_5 \left(b^2 T - 2\mu^2 T y - (y - aT)^2\right) - a_7(y - aT) = 0. \tag{7}$$

From (7) we obtain three relationships, *videlicet*

$$\begin{array}{ll} y^2 & a_5 = 0, \\ y & a_7 = 0, \\ - & a_2 + a_4\mu^2 T = 0, \end{array}$$

so that with (6) we have

$$a_2 = -a_4\mu^2 T, \quad a_3 = -a_4 T. \tag{8}$$

With the combination of (7) and (8) we find that the system of equation and terminal condition are invariant under the two Lie point symmetries

$$\Lambda_1 = \partial_y, \quad \Lambda_2 = (T - t)\partial_t + \mu^2(T - t)\partial_J.$$

We can now seek a similarity solution of (3) which is compatible with the given terminal condition.

Invariance under Λ_1 means that $J = J(t)$ only. The associated Lagrange's system of Λ_2 is

$$\frac{\mathrm{d}t}{T - t} = \frac{\mathrm{d}J}{\mu^2(T - t)},$$

i.e. the characteristic is $\omega = J - \mu^2 t$. From (3) we obtain the similarity solution $J = K - \mu^2 t$ so that the solution which is consistent with the terminal condition is $J = \mu^2(T - t)$.

We note that this is the unique solution [19] to (3).

3 An Absence of an Infinite Number of Lie Point Symmetries

The application of Program LIE to the equation

$$u_t + u_{xx} + (x + u)\, u_x - (Dx + Eu) = 0 \tag{9}$$

for general values of the parameters D and E gives three Lie point symmetries. The algebra of the symmetries is $A_1 \oplus_s 2A_1$, which is a representation of $D \otimes_s T_2$, the group of dilatations and translations in the plane.

In the particular case that $E = -1$ we find that there is additional symmetry. In this case the symmetries are

$$\Lambda_1 = \partial_t, \quad \Lambda_{2\pm} = \exp\left[\pm Bt\right]\left\{\partial_x \pm (B \mp 1)\partial_u\right\}, \\ \Lambda_{3\pm} = \exp\left[\pm 2Bt\right]\left\{\partial_t \pm Bx\partial_x + \left(2B^2x \mp 2Bx \mp Bu\right)\partial_u\right\}, \tag{10}$$

where $B^2 = D + 1$, with the Lie Brackets

$$[\Lambda_1, \Lambda_{2\pm}]_{LB} = \pm B\Lambda_{2\pm}, \quad [\Lambda_{2+}, \Lambda_{2-}]_{LB} = 0, \quad [\Lambda_{3+}, \Lambda_{3-}]_{LB} = -4B\Lambda_1, \\ [\Lambda_1, \Lambda_{3\pm}]_{LB} = \pm 2B\Lambda_{3\pm}, \quad [\Lambda_{2\pm}, \Lambda_{3\pm}]_{LB} = 0, \quad [\Lambda_{2+}, \Lambda_{3-}]_{LB} = -\Lambda_{2-}, \\ [\Lambda_{2-}, \Lambda_{3+}]_{LB} = 2B\Lambda_{2+}.$$

The algebra is $sl(2,R) \oplus_s 2A_1$ with the set $\{\Lambda_1,\ \Lambda_{3\pm}\}$ constituting the $sl(2,R)$ and $\Lambda_{2\pm}$ the abelian subalgebra.

If in (9) with $E = -1$ we set $w = u + x$, we obtain the equation

$$w_t + w_{xx} + ww_x - (D-1)x = 0, \tag{11}$$

which we recognise as a not quite Burgers equation. Under the transformation $w = 2v_x/v$ and a subsequent integration with respect to x (11) becomes

$$v_t = -v_{xx} + \left(K(t) + \frac{1}{4}(D+1)x^2\right)v, \tag{12}$$

which is basically the Schrödinger equation for the simple harmonic oscillator in imaginary time. As such it has $5 + 1 + \infty$ Lie point symmetries. The second and third are due to the linearity of (12). It is tempting to believe that the first set of five symmetries is derived from the set given in (10), but one would need to verify that this be the case. The transformation from (11) to (12) is not a point transformation and so one can understand the lack of preservation of point symmetries. Indeed given that there is also an integration the connection between the two is even more tenuous.

4 The Classical Connection

The results regarding the number of Lie point symmetries of a linear evolution equation – indeed linearisable – remind one of the possible cases for the number of the Noetherian symmetries [16] of a classical Lagrangian of the form $L = \dot{x}/2 - V(t,x)$.

The possibilities, up to nontrivial time-dependent affine transformations, are

V	no sym	algebra
$V(t,x)$	0	–
$V(x)$	1	A_1
$\omega^2x^2 + \dfrac{h^2}{x^2}$	3	$sl(2,R)$
ω^2x^2	5	$sl(2,R) \oplus_s 2A_1$

and this relates to the Lie point symmetries of the corresponding time-dependent Schrödinger Equation [13] $2i\partial u/\partial t + \partial^2 u/\partial x^2 - V(t,x)u = 0$ for which we have

V	no sym	algebra
$V(t,x)$	$0+1+\infty$	$A_1 \oplus_s \infty A_1$
$V(x)$	$1+1+\infty$	$2A_1 \oplus_s \infty A_1$
$\omega^2 x^2 + \dfrac{h^2}{x^2}$	$3+1+\infty$	$\{A_1 \oplus sl(2,R)\} \oplus_s \infty A_1$
$\omega^2 x^2$	$5+1+\infty$	$\{sl(2,R) \oplus_s W\} \oplus_s \infty A_1$,

where W is the Weyl algebra with

$$[\Sigma_1, \Sigma_2]_{LB} = 0, \ [\Sigma_1, \Sigma_3]_{LB} = 0, \ [\Sigma_2, \Sigma_3]_{LB} = \Sigma_1.$$

Naturally the time-dependent Schrödinger Equation is transformed to the heat equation by a point transformation. Subsequent transformation, as we have seen, brings us to the various forms of the Black–Scholes equation, not necessarily with $5 + 1 + \infty$ Lie point symmetries which may well have been the attraction of the symmetry analysis of the Black–Scholes equation in the first place.

In this context we have so far mentioned only the $(1+1)$-dimensional equations of Financial Mathematics. However, there is no necessity to restrict the considerations to such problems. Consequently we must look at the equivalent in Classical Mechanics. Immediately we are in two spatial dimensions there is the possibility of chaos if the potential departs even only modestly from the quadratic. Classical chaos has its counterpart in the so-called Quantum Chaos which has been the object of so much investigation and discussion over the last few decades. Given the connection already indicated several times above between the Schrödinger equation and the equations of Financial Mathematics we must ask whether there is an equivalent of quantum chaos in Financial Mathematics.

If this be the case, the immediate question is whether we would have Chaotic Economics or Economical Chaos?

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On Some Aspects of Ordinary Differential Equations Invariant under Translation in the Independent Variable and Rescaling

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We study the class of general second-order ordinary differential equations invariant under translation in the independent variable and rescaling from the Lagrangian perspective and show that the differential equation, $y''+yy'+ky^3 = 0$, is a unique member of this class. Other aspects of equations arising from this analysis are also discussed.

1 Introduction

The study of second-order ordinary differential equations invariant under translation in the independent variable and rescaling has received an incredible amount of attention [1–3]. For example the differential equation,

$$y'' + yy' + ky^3 = 0, \tag{1}$$

arises in the study of univalent functions [4], the study of the stability of gaseous spheres [5], the Riccati equation [6] and in the modeling of the fusion of pellets [7]. Furthermore, for rational values of $k \in (1/9, 1/8)$ the solution can be expressed in parametric form [8] and (1) passes the weak Painlevé test. For $k = 1/9$ the equation possesses eight Lie-point symmetries [3] with the algebra $sl(3, R)$ which implies that equation (1) is equivalent to $Y'' = 0$ under a point transformation given by $X = x - 1/y$, $Y = x^2/2 - x/y$.

For $k \neq 1/9$ the equation has only the two point symmetries

$$G_1 = \partial_x, \quad G_2 = -x\partial_x + y\partial_y \tag{2}$$

with the algebra A_2. Leach *et al* [1] pointed out that the value of $k = 1/8$ was critical in that the solution of (1) passes from nonoscillatory to oscillatory. The main aim of this paper is to give a method to determine Lagrangians of second-order ordinary differential equations invariant under (2) and to analyse the equations in terms of the Painlevé analysis. Once a Lagrangian is known, Noether's approach can be used to determine the corresponding integrals [9].

2 The Self-Similarity Symmetry and the Lagrangian

The general form of the second-order ordinary differential equation invariant under the symmetries (2) is

$$\frac{y''}{y^3} + f\left(\frac{y'}{y^2}\right) = 0. \tag{3}$$

Proposition 1. *Equation* (3) *has a Lagrangian of the form* $L = y^n F\left(y'/y^2\right)$.

This choice emanates from the facts that the term y'/y^2 occurs arbitrarily in (3), where $f = f(y'/y^2)$, and that one cannot expect the Lagrangian to possess the similarity symmetry.

Proof. Assume a Lagrangian of the form $L = g(y)F\left(y'/y^2\right)$. Note that g can also be taken to be function of y'. The corresponding Euler–Lagrange equation is then given by

$$\left(2u^2 + f(u)\right) F'' - y\frac{g'(y)}{g(y)}uF' + \frac{g'(y)}{g(y)}yF = 0, \tag{4}$$

where we recall that $u = y'/y^2$ and $F = F(u)$. The integration of (4) forces us to eliminate g so that $g'(y)/g(y) = n/y \iff g = y^n$, where n is a constant. Then (4), which is the differential equation for F, simplifies to the form

$$\left(2u^2 + f(u)\right) F'' - nuF' + nF = 0. \tag{5}$$

■

3 A Hierarchy of Equations

A hierarchy of equations can be developed from the coefficient of F'' in (5). In the case of (5) the coefficient of F'' is $2u^2 + f(u)$. Here we examine the behaviour of the equation for F for which the coefficient function is a square, that is, $2u^2 + f(u) = m(u+\alpha)^2$, where α is a constant. Then F in (5) satisfies

$$m(u+\alpha)^2 F'' - nuF' + nF = 0. \tag{6}$$

If we set $\alpha = 0$, then (6) is $u^2F'' - nm^{-1}uF' + nm^{-1}F = 0$ with characteristic roots 1 and n/m. The solution is

$$F = \begin{cases} Au + Bu^{n/m} = A\dfrac{y'}{y^2} + B\left(\dfrac{y'}{y^2}\right)^{n/m}, & n \neq m \\[2ex] Au + Bu\log u = A\dfrac{y'}{y^2} + B\dfrac{y'}{y^2}\log\dfrac{y'}{y^2}, & n = m \end{cases}$$

In both cases we may set $A = 0$ without loss of generality. For $F = (y'/y^2)^{(n/m)}$,

$n \neq 0$, the Lagrangian takes the form $L = y^{n-2n/m} y'^{n/m}$ and the Euler–Lagrange equation is

$$yy'' + (m-2)y'^2 = 0. \tag{7}$$

If we use $F = y'/y^2 \log\left(y'/y^2\right)$, i.e. $n = m$, the Euler–Lagrange equation is also (7).

If $\alpha \neq 0$ in (6) we set $v = u + \alpha$ and $F(u) = G(v)$ so that the equation becomes

$$v^2 G'' - \frac{n}{m}(v-\alpha)G' + \frac{n}{m}G = 0. \tag{8}$$

After a resubstitution for the variables in (8) we obtain

$$F = C_1 u + C_2(u+\alpha)\exp\left(-\alpha/(u+\alpha)\right), \quad \text{where } u = y'/y^2.$$

Therefore we have the Lagrangian $L = y^{-m}(y'/y^2 + \alpha)\exp\left(-\alpha/(y'/y^2 + \alpha)\right)$. The corresponding Euler–Lagrange equation is given by

$$yy'' + (m-2)y'^2 + 2m\alpha y'y^2 + m\alpha^2 y^4 = 0. \tag{9}$$

Remark 1. Equation (9) can be rescaled to

$$y''y + (m-2)y'^2 + y'y^2 + \frac{1}{4m}y^4 = 0. \tag{10}$$

If $m = 2$, (10) reduces to $y'' + yy' + y^3/8 = 0$ which is just the special case of (1) with $k = 1/8$. The Painlevé test of (10) gives $p = -1$ and $\alpha = 2m$ (twice) for the leading order behaviour. The resonances are $r = -1, 0$. Thus there is a logarithmic singularity at the χ^{-1} term and the equation does not possess the Painlevé property.

3.1 Quadratic Coefficients

If the coefficient of F'' is quadratic, that is $m(u+\alpha)^2 + \beta$, then the example of (1) suggests that the value of β and its sign may have a critical impact on the number of point symmetries, the Noetherian structure and the possibility of possessing the Painlevé Property. The Euler–Lagrange equation corresponding to $f(u) = m(u+\alpha)^2 + \beta - 2u^2$ is

$$yy'' + (m-2)y'^2 + 2m\alpha y'y^2 + (m\alpha^2 + \beta)y^4 = 0 \tag{11}$$

which differs from (9) by the presence of the term βy^4.

The rescaled form of (11) is

$$yy'' + (m-2)y'^2 + y'y^2 + \left(\frac{1}{4m} + k\right)y^4 = 0, \tag{12}$$

where $k = \beta/4m^2\alpha^2$ so that the signs of k and β are the same and there is no real need to pass from (12) to (11) in the final analysis. The leading order behaviour of (12) gives $p = -1$ and α is a root of

$$m - \alpha + \left(\frac{1}{4m} + k\right)\alpha^2 = 0. \tag{13}$$

(Here the α is the one for the leading order behaviour and not to be confused with the α above.) We solve for k in (13) so that

$$k = -\frac{1}{4m}\left(\frac{2m}{\alpha} - 1\right)^2 \tag{14}$$

and $\alpha \neq 0$. The resonances for (12) are $r = -1$ and $2m - \alpha$, which fits in with the $4-\alpha$ in the Painlevé analysis of (1). We therefore require (for a Right Painlevé Series) that $4 - \alpha$ be a nonnegative integer which means that α take the values $(3, 2, 1, -1, ...)$. This in turn specifies the value of k. For instance, if $\alpha = 1$ and $m = 2$, then $k = -1/72$. This value of k permits (12) to be linearised as it just reduces to (1) with $k = 1/9$. For $m = 3$ in (14) we have $k = -\left(6/\alpha - 1\right)^2/12$. Since $k \neq 0$, we need $\alpha > 6$ or $\alpha < 6$. This gives separate values of α. For example, when $\alpha = 5$, then $k = -1/300$ and (12) takes the form $yy'' + y'^2 + y'y^2 + 2y^4/25 = 0$. The resonances are $r = -1$ and 1. The equation passes the Painlevé test and still has only the two symmetries (2). If the coefficient of F'' is quadratic, then (6) is now

$$\left[(u+\alpha)^2 + \frac{\beta}{m}\right]F'' - \frac{n}{m}uF' + \frac{n}{m}F = 0. \tag{15}$$

Equation (15) is an hypergeometric equation. Once y is determined from (15) the Lagrangian follows. This gives a Lagrangian which is not particularly useful in general since the hypergeometric function is a convenient label for an infinite series. The formal expression for the first integral follows from the autonomous Lagrangian obtained in terms of the hypergeometric function.

In the context of the relationship between coefficient functions and Lagrangians we now look to other possibilities which the coefficient function can take.

Proposition 2. *Suppose that the coefficient of F'' in (6) is $mu^2 + k$ then there exists a Lagrangian of the form*

$$L = y^n\left(\frac{y'}{y^2} + 1\right)^2 \quad \textit{or} \quad L = y^n\left(\frac{y'}{y^2} - 1\right)^2, \tag{16}$$

where $n = 2m$.

Proof. Starting with equation (6) we have

$$\left(u^2 + \frac{k}{m}\right)F'' - \frac{n}{m}uF' + \frac{n}{m}F = 0. \tag{17}$$

Let $u = \gamma x$, $F(u) = y(x)$ and put $k/m = -\gamma^2$. Then the solution of (17) from Kamke [10, 2.247] is

$$F = C_1(u+\gamma)^2 + C_2(u-\gamma)^2$$

with the corresponding Lagrangian given by (16). ■

The differential equation corresponding to (16) is

$$yy'' + (m-2)y'^2 + ky^4 = 0. \tag{18}$$

Under a suitable rescaling we can put $k = +1$ or -1 in (18). We consider for the Painlevé analysis the case for which $k = 1$ so that (18) becomes

$$yy'' + (m-2)y'^2 + y^4 = 0. \tag{19}$$

The Painlevé analysis gives for the leading order behaviour, $p = -1$ and $\alpha^2 = -m$. The resonances are $r = -1$ and $2m$. The resonance condition is satisfied and hence the equation does pass the Painlevé test. For a Right Painlevé Series we need $2m$ to be a positive integer. We can represent (19) as

$$\left(y^{m-1}\right)'' + (m-1)y^{m+1} = 0. \tag{20}$$

For $z = y^{m-1}$, (20) is now equivalent to

$$z'' + z^{\frac{m+1}{m-1}} = 0. \tag{21}$$

Note that equation (21) is never linear homogeneous, but for $m = -1$ it is a linear nonhomogeneous equation and has eight Lie point symmetries. Equation (21) is the Emden–Fowler equation in specific form (see reference [13] and references cited therein) and has the two symmetries $G_1 = \partial_x$ and $G_2 = (2/(1-m))\, x\partial_x + 2z\partial_z$ except when the index is 1 (impossible in this case) and -3. The latter value of the index gives $m = 1/2$ and (21) becomes an instance of the Ermakov–Pinney equation, $z'' + z^{-3} = 0$ [14, 15].

4 The Riccati Transformation

The insertion of the well-known Riccati transformation, $y = \alpha\vartheta'/\vartheta$, where α is a constant, into (12) puts the equation into the form

$$\frac{\vartheta'\vartheta'''}{\vartheta^2} + (m-2)\frac{\vartheta''^2}{\vartheta^2} + (\alpha+1-2m)\frac{\vartheta''\vartheta'^2}{\vartheta^3} + \left[m - \alpha + \alpha^2\left(\frac{1}{4m} + k\right)\right]\frac{\vartheta'^4}{\vartheta^4} = 0. \tag{22}$$

We note that the symmetry associated with the Riccati transformation is the homogeneity symmetry. Using (13) in (22) and $m = 1/2$ reduces to the Kummer-Schwartz equation $2\vartheta'\vartheta''' - 3\vartheta''^2 = 0$ which plays an important role in the Painlevé analysis of differential equations. However, we cannot have $m = 1/2$ as this would mean that $\alpha = 0$ which is not permitted. Furthermore, if $\alpha = 2m - 1$ in (13) and σ is given by

$$\sigma = \frac{y'}{y} + \frac{y}{2m-1}, \tag{23}$$

in (10), then we obtain

$$\sigma' + (m-1)\sigma^2 = 0, \tag{24}$$

which is a Riccati equation with solution

$$\sigma = ((m-1)x - K)^{-1}. \tag{25}$$

Equation (23) together with (25) leads to another Riccati equation

$$y' + \frac{y^2}{2m-1} - \frac{y}{(m-1)x - K} = 0. \tag{26}$$

We substitute for $y = \alpha\omega'/\omega$ in (26) and put $\alpha = 2m - 1$ as before to give

$$\frac{\omega''}{\omega'} = \frac{1}{(m-1)x - K}. \tag{27}$$

The integration of (27) leads to $\omega = (A/m)\left[(m-1)x - K\right]^{m/(m-1)} + C$ and y immediately follows from $y = \alpha\omega'/\omega$. We have moved from a first-order equation to a second-order equation which is easily integrated to give a solution to the first-order equation. The insertion of $\sigma = \alpha\omega'/\omega$ into (24) leads to

$$\frac{\omega''}{\omega} - \frac{\omega'^2}{\omega^2} + (m-1)\alpha\frac{\omega'^2}{\omega^2} = 0$$

and, with $\alpha = 1/(m-1)$, this is just equivalent to $\omega'' = 0$ and has eight Lie-point symmetries. We observe that the Painlevé analysis seems to pick out a parameter which results in an equation with interesting properties different from those of the general class of the original equation. As a result it is easy to show that we have the eight Lie-point symmetries of $\omega'' = 0$ as seven (one is used in the transformation) nonlocal symmetries of the original equation. We now move to the third-order level to check what happens to the number of symmetries there. We recall that the value of $\alpha = 2m - 1$ gives the differential equation $\vartheta'''/\vartheta'' + (m-2)\,\vartheta''/\vartheta' = 0$ which can be represented as $\left(\vartheta'^{m-1}\right)'' = 0$ suggesting the presence of more symmetries in the third-order equation. The contact symmetries are $G_1 = \vartheta'^{m-1}\partial_x + ((m-1)/m)\vartheta'^m\partial_\vartheta$, $G_2 = \partial_\vartheta$, $G_3 = \partial_x$, $G_4 = x\partial_x$ and $G_5 = \vartheta\partial_\vartheta$. To investigate further what happens to these symmetries under the reduction of order we look at the general equation (9) with $y = \alpha\omega'/\omega$, that is,

$$\begin{aligned}&\omega'\omega''' + (m-2)\omega''^2 + (\alpha - 2m + 1)\,\frac{\omega'^2\omega''}{\omega}\\&\quad + \left[m - \alpha + \alpha^2\left(k + \frac{1}{4m}\right)\right]\frac{\omega'^4}{\omega^2} = 0,\end{aligned} \tag{28}$$

which has the three point symmetries $G_1 = \partial_x, G_2 = x\partial_x$ and $G_3 = \omega\partial_\omega$. If we reduce (28) using the point symmetry ∂_x, we have the invariants $u = \omega$ and $v = \omega'$ so that the reduced equation becomes

$$z'' + (\alpha - 2m)z' + \left[m - \alpha + \alpha^2\left(\frac{1}{4m} + k\right)\right] z = 0, \tag{29}$$

with $z = v^m$ and $\eta = \log u$. Equation (29) has the characteristic roots

$$l_\pm = \frac{1}{2}\left[2 - \alpha \pm \sqrt{(-4m\alpha^2 k)}\right].$$

In general we have equation (29) and α has not been specified. In a spirit of simplification and connection with the Painlevé property we take the value of α that satisfies condition (13) in (29). Then we have

$$z'' + (\alpha - 2m)z' = 0, \tag{30}$$

the characteristic roots are now 0 and $2m - \alpha$ and the symmetries are

$$\begin{aligned}
&G_1 = \partial_z, \quad G_2 = \exp((2m-\alpha)\eta)\partial_z,\\
&G_3 = \exp((2m-\alpha)\eta)\left[\partial_\eta - (\alpha - 2m)z\partial_z\right], \quad G_4 = \partial_\eta,\\
&G_5 = \exp((\alpha - 2m)\eta)\partial_\eta, \quad G_6 = 2\partial_\eta - (\alpha - 2m)z\partial_z,\\
&G_7 = z\exp((\alpha - 2m)\eta)\partial_\eta, \quad G_8 = z\partial_\eta - (\alpha - 2m)z^2\partial_z.
\end{aligned}$$

In terms of y with $\sigma_1 = \exp\left(1/\alpha\int y\mathrm{d}x\right)$, $\sigma_2 = \exp\left(m/\alpha\int y\mathrm{d}x\right)$, $\sigma_3 = \exp(2m-\alpha)$ and $\sigma_4 = \left(1/\alpha\int y\mathrm{d}x\right)$ we have

$$\begin{aligned}
G_1 &= \left[\int\left[m(y/\alpha)^m\sigma_2\right]^{-1}\mathrm{d}x\right]\partial_x - \alpha\left[m(y/\alpha)^{m-1}\sigma_2\right]^{-1}\partial_y,\\
G_2 &= \exp((2m-\alpha)\sigma_4)\left[\int\left[m(y/\alpha)^m\sigma_2\right]^{-1}\mathrm{d}x\right]\partial_x\\
&\quad - \sigma_3\left(\int y\mathrm{d}x\right)\left[m(y/\alpha)^{m-1}\sigma_2\right]^{-1}\partial_y,\\
G_3 &= \exp((2m-\alpha)\sigma_4)\left(x + (\alpha - 2m)\left[(y/\alpha)\sigma_1\right]^m\int\left[m(y/\alpha)^m\sigma_2\right]^{-1}\mathrm{d}x\right)\partial_x\\
&\quad - \exp((2m-\alpha)\sigma_4)\left[y + (\alpha - 2m)\frac{y}{m}\right]\partial_y,\\
G_4 &= x\partial_x - y\partial_y, \quad G_5 = \exp((\alpha - 2m)\sigma_4)\left[x\partial_x - y\partial_y\right],\\
G_6 &= \left[2x + (\alpha - 2m)\left[(y/\alpha)\sigma_1\right]^m\int\left[m(y/\alpha)^m\sigma_2\right]^{-1}\mathrm{d}x\right]\partial_x - \frac{\alpha}{m}\partial_y,\\
G_7 &= \left[(y/\alpha)\sigma_1\right]^m\exp((\alpha - 2m)\sigma_4)\left[x\partial_x - y\partial_y\right],\\
G_8 &= \left[(y/\alpha)\sigma_1\right]^m\left(x + (\alpha - 2m)\left[(y/\alpha)\sigma_1\right]^m\int\left[m(y/\alpha)^m\sigma_2\right]^{-1}\mathrm{d}x\right)\partial_x\\
&\quad - \left[y/\alpha\exp(\sigma_4)\right]^m\left[y + (\alpha - 2m)\frac{y}{m}\right]\partial_y.
\end{aligned}$$

We observe that these symmetries are somewhat nonlocal except for G_4 which is preserved as a point symmetry of the original equation (12). Under an increase of order of (30) by putting $z = s'$ we obtain $s''' + (\alpha - 2m)s'' = 0$ with five contact symmetries, $G_1 = \partial_x$, $G_2 = \partial_s$, $G_3 = \exp(-Ax)\left[\partial_s - A\partial_{s'}\right]$, $G_4 = -x\partial_s - \partial_{s'}$ and $G_5 = -s\partial_s - s'\partial_{s'}$, where $A = \alpha - 2m$. We note that all of the five are point symmetries. It is well-known that the number of point symmetries that a linear third-order ordinary differential equation can have is 7, 5 or 4. The value of k which gives the Painlevé property in (12) corresponds to the increase in the amount of symmetry. For instance, $\alpha = 2m - 1$ leads to the generalised Kummer-Schwartz equation which is equivalent to $V''' = 0$ under a nonlocal transformation [11]. For any other values of α we obtain the three point symmetries ∂_x, $x\partial_x$ and $\omega\partial_\omega$ in (28).

5 Conclusion

We have presented some properties of second-order ordinary differential equations invariant under translation in the independent variable and rescaling in terms of the Lagrangian. The representation of the Lagrangian was taken to be $L = y^n F(y'/y^2)$ since the variable y'/y^2 appears as an arbitrary argument in the second-order ordinary differential equation. The multiplier for F in the expression of the Lagrangian can be chosen to be any function of y or y'. However, the differential equation for F imposes an effective restriction on the permissible form of the multiplier of F due to the problems of integrating the equation. The associated Euler–Lagrange equation is $(2u^2 + f(u))F'' - nuF' + nF = 0$ with $u = y'/y^2$. A feature associated with the class of differential equations for F with a quadratic $f = f(u)$ is that in general they are hypergeometric functions. The Painlevé analysis of the equations arising in the different cases was performed and it was found that the differential equation $y'' + yy' + ky^3 = 0$ appears to have a distinct behaviour for specific values of the parameters involved. A wider class is seen also to have interesting properties characterised by nonlocal transformations rather than point transformations. The number of symmetries following an increase and decrease in order is another aspect to take into account. We have dealt with the particular case of a parameter k which gives the Painlevé property and corresponds to the increase in the amount of symmetry. We again remark that the Painlevé analysis seems to pick out certain values of the parameters for which the 'integrable' equations possess interesting properties.

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Solving the Camassa–Holm Equation

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A direct method for solving the Camassa–Holm shallow water wave equation is presented and is used to find multisoliton solutions in explicit form.

1 Introduction

The eponymous Camassa–Holm (CH) equation [1]

$$u_t + 2ku_x + 3uu_x - u_{xxt} = 2u_x u_{xx} + uu_{xxx}, \quad k = \text{const}, \tag{1}$$

has attracted considerable interest, and acquired a substantial pedigree, over the last decade. Although its origin can be traced back to the work of Fuchssteiner and Fokas [2], the equation was "rediscovered" by Camassa and Holm [1] as a model for shallow water waves. The excitement that greeted this equation can, arguably, be ascribed to two attributes: the CH equation is *completely integrable* and it possesses non-standard (solitonic) properties. Among the latter, we note that, for the special case $k=0$, equation (1) admits peaked soliton solutions (*multipeakons*) which exhibit a 'corner' at the wave crests.

Yet attempts to find explicit solutions of the general CH equation and, in particular, the hallmark *analytic* N-soliton solutions, have met with limited success. Schiff [3] used a Bäcklund transformation (BT) to derive the solitary wave ($N = 1$) and an (incomplete) two-soliton solution, while Johnson [4] has implemented the inverse scattering transform (IST) procedure for equation (1) [5] to obtain multisoliton solutions as far as $N = 3$. However, their results were restricted to $k \neq 0$ in (1). Significantly, these authors showed that the analytic soliton solutions of CH equation can be represented explicitly only in *parametric* form (see Section 2).

We present an alternative method for finding exact solutions of equation (1) that is based on Hirota's bilinear transformation method [6], and show how it may be used to elicit the erstwhile elusive N-solitons for *any* value of k.

2 Bilinear Form of the Camassa–Holm Equation

The work of Schiff [3], Johnson [4] and Constantin [5] suggests that we cannot find the Hirota form of the CH equation (1) directly, but must first transform the equation appropriately. Without loss of generality, we may assume that $k \geq 0$ and rewrite (1) as

$$u_t + 2\kappa^2 u_x + 3uu_x - u_{xxt} = 2u_x u_{xx} + uu_{xxx}, \quad \kappa \geq 0 \text{ const.} \tag{2}$$

In what follows, we will take $\kappa > 0$ and treat the special case $\kappa = 0$ of equation (2) separately. Now, introducing the quantity

$$r(x,t) = \sqrt{u - u_{xx} + \kappa^2}, \tag{3}$$

equation (2) may be written in the conserved form

$$r_t + (ur)_x = 0.$$

Then, following Fuchssteiner [7], one can define a *reciprocal* co-ordinate transformation $(x,t) \to (y,t)$ by

$$\mathrm{d}y = r\,\mathrm{d}x - ur\,\mathrm{d}t, \quad \mathrm{d}t = \mathrm{d}t, \tag{4}$$

which transforms the CH equation (2) to the the *associated Camassa–Holm* (ACH) equation

$$u = r^2 - r\,\partial^2_{yt} \ln r - \kappa^2, \quad r_t + r^2 u_y = 0. \tag{5}$$

Schiff's [3] strategy for solving the CH equation is now apparent: first solve for $r(y,t)$ and use (5) to find a solution $u(y,t)$ of the ACH equation. One can then use the inverse of (4),

$$\frac{\partial x}{\partial y} = \frac{1}{r(y,t)}, \quad \frac{\partial x}{\partial t} = u(y,t), \tag{6}$$

to obtain the co-ordinate transformation $x(y,t)$. This gives a solution $u(x,t)$ of the CH equation in *parametric* form in terms of y, with the proviso that $r > 0$ (equation (3)). Whereas Schiff sought solutions for r by using BTs, we shall do so by means of Hirota's bilinear method. Accordingly, we introduce the Hirota function $f(y,t)$ such that

$$r = \kappa - 2\,\partial^2_{yt} \ln f, \tag{7}$$

which yields the bilinear form of the ACH equation [8]

$$\Big[\mathrm{D}_y(\mathrm{D}_t + 2\kappa^3 \mathrm{D}_y - \kappa^2 \mathrm{D}_y^2 \mathrm{D}_t) + \frac{1}{3}\kappa^2 \mathrm{D}_t(\mathrm{D}_\tau + \mathrm{D}_y^3)\Big] f \cdot f = 0, \tag{8}$$

$$\mathrm{D}_y(\mathrm{D}_\tau + \mathrm{D}_y^3)\, f \cdot f = 0, \tag{9}$$

where D_y, D_t are the Hirota derivatives defined by [6]

$$\mathrm{D}_y \mathrm{D}_t\, a \cdot b = (\partial_y - \partial_{y'})(\partial_t - \partial_{t'})\, a(y,t)\, b(y',t')|_{y'=y,\, t'=t}. \tag{10}$$

We remark that the auxiliary variable τ that appears in (8) and (9) is needed to effect the bilinearisation of the ACH equation in terms of the D-operators (10); it plays no role in the final form of the solution.

3 Solitary Waves

To obtain a solitary-wave solution, we take [8] $f(y,t) = 1+\mathrm{e}^\theta$, $\theta = py+\omega t+\sigma\tau+\eta$, where p, ω, σ and η are (real) constants. This solves the bilinear equations (8) and (9) if $\omega = -2\kappa^3 p/(1-\kappa^2p^2)$, $\sigma = -p^3$, with p and η arbitrary. Inserting f into (7), we deduce $r(y,t) = (2\kappa + cp^2\mathrm{sech}^2(\theta/2))/2$, where we have introduced the wave speed $c = 2\kappa^3/(1-\kappa^2p^2)$ in (y,t)-space. From (5), one then obtains the solitary-wave solution [8]

$$u(y,t) = \frac{c^2p^2}{\kappa}\cdot\frac{\mathrm{sech}^2(\theta/2)}{2\kappa + cp^2\,\mathrm{sech}^2(\theta/2)}, \qquad \theta = p(y-ct+\alpha), \tag{11}$$

of the ACH equation, where the τ-dependence has been subsumed in the arbitrary phase constant α. (The auxiliary variable τ can always be absorbed in this way and, henceforth, will be omitted whenever it is convenient to do so, without further comment.)

A solitary wave $u(x,t)$ of equation (2) follows by integrating (6) to get the inverse coordinate transformation

$$p(x-\tilde{c}t+x_0) = \frac{\theta}{\kappa} + p\ln\left[\frac{(1+\kappa p)+(1-\kappa p)\,\mathrm{e}^\theta}{(1-\kappa p)+(1+\kappa p)\,\mathrm{e}^\theta}\right], \tag{12}$$

in which the wave speed $\tilde{c}$ in (x,t)-space is given by $\tilde{c} = c/\kappa = 2\kappa^2/(1-\kappa^2p^2)$. Equations (11) and (12) give a parametric representation (in θ) for the *analytic* solitary wave (or one-soliton) solution of the Camassa–Holm equation (2) for *all* $\kappa > 0$. The restriction $r(y,t) > 0$ for all (y,t) requires that

$$0 < \kappa p < 1, \tag{13}$$

or, equivalently, $\tilde{c} > 2\kappa^2$; i.e. the CH solitary waves propagate at supercritical speed.

It is well-known that the *reduced Camassa–Holm* (RCH) equation ($\kappa = 0$),

$$u_t + 3uu_x - u_{xxt} = 2u_xu_{xx} + uu_{xxx}, \tag{14}$$

admits the *peakon* solitary wave that is given by [1]

$$u(x,t) = \tilde{c}\exp(-|\,x-\tilde{c}t+x_0|). \tag{15}$$

The peakon wave speed $\tilde{c}$ in (15) is now arbitrary, which means that equation (14) also admits *anti*-peakons ($\tilde{c} < 0$). These curious non-analytic solutions have a “corner” at the crest of the wave. The peakon (15) can be recovered from the analytic CH solitary wave $u(x,t)$ in the limit [8]

$$\kappa \to 0, \quad \kappa p \to 1, \quad \tilde{c} = \text{const.}$$

However, the RCH equation also possesses an *analytic* solitary wave that is obtained by transforming the CH solution $u(x,t)$ under the mapping

$$x \to x - \kappa^2 t, \quad t \to t, \quad u \to u - \kappa^2, \tag{16}$$

that reduces the CH equation (2) to (14).

4 Two-soliton solution of the Camassa–Holm equation

In order to find an explicit two-soliton solution of the CH equation (2), we take [9]

$$f(y,t) = 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + A_{12}\mathrm{e}^{\theta_1+\theta_2}, \tag{17}$$

where $\theta_i = p_i y + \omega_i t + \sigma_i \tau + \eta_i$, $i = 1, 2$, and p_i, ω_i, σ_i, η_i and A_{12} are real constants. This is a solution of the bilinear form (8) and (9) if

$$A_{12} = \left(\frac{p_1 - p_2}{p_1 + p_2}\right)^2, \quad \omega_i = -\frac{2\kappa^3 p_i}{1-\kappa^2 p_i^2}, \quad \sigma_i = -p_i^3, \quad i = 1, 2. \tag{18}$$

Substituting (17) into (7), we find that

$$\begin{aligned} r(y,t) = \kappa + \frac{2}{f^2}\Big[&c_1 p_1^2 \mathrm{e}^{\theta_1} + c_2 p_2^2\,\mathrm{e}^{\theta_2} + \frac{4\kappa^3(p_1-p_2)^2}{(1-\kappa^2p_1^2)(1-\kappa^2p_2^2)}\mathrm{e}^{\theta_1+\theta_2} \\ &+ A_{12}\big(c_1p_1^2\mathrm{e}^{\theta_1+2\theta_2} + c_2p_2^2\,\mathrm{e}^{2\theta_1+\theta_2}\big)\Big], \end{aligned} \tag{19}$$

where we have defined 'wave speeds' c_i in (y,t)-space by

$$c_i = \frac{2\kappa^3}{1-\kappa^2 p_i^2}, \quad i = 1, 2.$$

Then (5) yields the ACH two-soliton solution

$$u(y,t) = \frac{2}{\kappa}\cdot\frac{\omega_1^2\mathrm{e}^{\theta_1} + \omega_2^2\,\mathrm{e}^{\theta_2} + b_{12}\,\mathrm{e}^{\theta_1+\theta_2} + A_{12}\left(\omega_1^2\mathrm{e}^{\theta_1+2\theta_2} + \omega_2^2\,\mathrm{e}^{2\theta_1+\theta_2}\right)}{rf^2}, \tag{20}$$

where

$$b_{12} = \frac{8\kappa^6(p_1-p_2)^2(1-\kappa^4p_1^2p_2^2)}{(1-\kappa^2p_1^2)^2(1-\kappa^2p_2^2)^2}.$$

This solution can be reformulated in various ways; in particular, it can be made to agree with the two-soliton results that were reported by Schiff [3] and Johnson [4].

Finally, upon inserting (19) and (20) into (6), and integrating, we obtain the inverse coordinate mapping

$$x(y,t) = \frac{y}{\kappa} + \ln\left[\frac{a_1a_2 + b_1a_2\mathrm{e}^{\theta_1} + b_2a_1\mathrm{e}^{\theta_2} + b_1b_2A_{12}\mathrm{e}^{\theta_1+\theta_2}}{b_1b_2 + a_1b_2\mathrm{e}^{\theta_1} + a_2b_1\mathrm{e}^{\theta_2} + a_1a_2A_{12}\mathrm{e}^{\theta_1+\theta_2}}\right] + \alpha, \tag{21}$$

where α is an arbitary constant and

$$a_i = 1 + \kappa p_i, \quad b_i = 1 - \kappa p_i, \quad i = 1, 2. \tag{22}$$

Taken together, equations (20) and (21) give an explicit expression — albeit parametrically in terms of y — for the *analytic* two-soliton solution of the CH equation (2) for any $\kappa > 0$. A corresponding analytic two-soliton solution of the RCH equation (14) ($\kappa = 0$) now follows by using the transformation (16).

A notable feature of the ACH two-soliton solution (20) is the additional parameter b_{12} that is needed to formulate $u(y,t)$ explicitly. In fact, the 'extra' parameter turns out to be a recurrent feature of the multisoliton solutions of the Camassa–Holm equation (see Section 5).

5 Three-Soliton Solution

Following the standard procedure, we start with the Hirota ansatz [9]

$$f(y,t) = 1 + \sum_{i=1}^{3} \mathrm{e}^{\theta_i} + \sum_{i<j}^{3} A_{ij}\,\mathrm{e}^{\theta_i+\theta_j} + A_{12}A_{13}A_{23}\,\mathrm{e}^{\theta_1+\theta_2+\theta_3}, \tag{23}$$

with $\theta_i = p_i y + \omega_i t + \sigma_i \tau + \eta_i$, and $i<j$ denotes summation over the ordered pairs of $(1,2)$, $(1,3)$, and $(2,3)$. It is then straightforward to show that (23) is a solution of the bilinear form (8) and (9) if

$$A_{ij} = \left(\frac{p_i - p_j}{p_i + p_j}\right)^2, \quad 1 \le i < j \le 3, \tag{24}$$

and ω_i, σ_i $(i = 1,\, 2,\, 3)$ are the dispersion laws in (18). Then, from (5) and (23), we deduce the three-soliton solution of the ACH equation

$$u(y,t) = \frac{2}{\kappa}\frac{R(y,t)}{rf^2}, \tag{25}$$

where

$$\begin{aligned}R(y,t) &= \sum_{i=1}^{3} \omega_i^2 \mathrm{e}^{\theta_i} + \sum_{i<j}^{3} b_{ij}\,\mathrm{e}^{\theta_i+\theta_j} + \sum_{i<j}^{3} A_{ij}\left(\omega_i^2 \mathrm{e}^{\theta_i+2\theta_j} + \omega_j^2 \mathrm{e}^{2\theta_i+\theta_j}\right)\\ &+ b_{123}\mathrm{e}^{\theta_1+\theta_2+\theta_3} + \sum_{<i>}^{3} b_{ij}A_{ik}A_{jk}\,\mathrm{e}^{\theta_i+\theta_j+2\theta_k} + \prod_{i<j}^{3} A_{ij} \sum_{<i>}^{3} \omega_i^2 A_{jk}\,\mathrm{e}^{\theta_i+2(\theta_j+\theta_k)},\end{aligned}$$

with

$$b_{ij} = 8\kappa^6\,\frac{C_{ij}}{D_{ij}}, \quad b_{123} = 16\kappa^6\,\frac{C_{123}}{D_{12}D_{13}D_{23}},$$

$$C_{ij} = (p_i^2 - p_j^2)^2(1-\kappa^4 p_i^2 p_j^2), \quad D_{ij} = \left[(p_i+p_j)(1-\kappa^2 p_i^2)(1-\kappa^2 p_j^2)\right]^2$$

and

$$\begin{aligned}C_{123} &= \sum_{i<j}^{3} p_i^2 p_j^2\, C_{ij} - 8\kappa^2\left[\ll p_i^6 p_j^2 p_k^2 \gg - \ll p_i^4 p_j^4 p_k^2 \gg\right] + 2\kappa^4\left[\ll p_i^8 p_j^2 p_k^2 \gg\right.\\ &\left. + 2 \ll p_i^6 p_j^4 p_k^2 \gg - 15\, p_1^4 p_2^4 p_3^4\right] - 8\kappa^6\left[\ll p_i^6 p_j^6 p_k^2 \gg - \ll p_i^6 p_j^4 p_k^4 \gg\right]\\ &+ \kappa^8\left[\ll p_i^8 p_j^6 p_k^2 \gg - 2 \ll p_i^8 p_j^4 p_k^4 \gg\right].\end{aligned}$$

The symbol $<i>$ here means that the summation is strictly over the three cyclic permutations of $(1\,2\,3)$, and $\ll\gg$ denotes the sum over all *distinct* products of the wave numbers p_i obtained from the permutations $(i\,j\,k)$ of $(1\,2\,3)$. The function $r(y,t)$ in (25) may be found by inserting (23) into (7); the calculation is quite routine and we omit its detailed expression here. However, to ensure that our solutions are well-defined, we will require $0 < \kappa p_i < 1$, $i = 1, 2, 3$, so that $r > 0$ (cf. equation (13)).

In order to find the CH three-soliton $u(x,t)$, it only remains to determine the inverse coordinate transformation $(y,t) \to (x,t)$ which follows by integrating (6). After some careful manipulation, we obtain

$$x(y,t) = \frac{y}{\kappa} + \ln\left(\frac{P}{Q}\right) + \alpha, \tag{26}$$

$$P(y,t) = a_1a_2a_3 + \sum_{<i>}^{3} b_i a_j a_k \mathrm{e}^{\theta_i} + \sum_{i<j}^{3} A_{ij}\, b_i b_j a_k \mathrm{e}^{\theta_i+\theta_j} + \prod_{i<j}^{3} A_{ij}\, b_1 b_2 b_3 \mathrm{e}^{\theta_1+\theta_2+\theta_3},$$

$$Q(y,t) = b_1b_2b_3 + \sum_{<i>}^{3} a_i b_j b_k \mathrm{e}^{\theta_i} + \sum_{i<j}^{3} A_{ij}\, a_i a_j b_k \mathrm{e}^{\theta_i+\theta_j} + \prod_{i<j}^{3} A_{ij}\, a_1 a_2 a_3 \mathrm{e}^{\theta_1+\theta_2+\theta_3},$$

where a_i, b_i $(i = 1, \ldots, 3)$ are defined as in (22). Thus, equations (25) and (26) provide an explicit representation of the three-soliton solution of the Camassa–Holm equation (2) — albeit in *parametric* form in y — for all values of $\kappa > 0$. To get an analytic three-soliton solution of the RCH equation (14) with $\kappa = 0$, one simply resorts to the mapping (16).

6 *N*-Soliton Solutions

Further multisoliton solutions of the Camassa–Holm equation can be constructed by following the procedure described in Section 4. Indeed, to find the N-soliton solution, one uses the celebrated Hirota ansatz [6]

$$f(y,t) = \sum_{\mu=0,1} \exp\Big[\sum_{i=1}^{N} \mu_i\theta_i + \sum_{i<j}^{N} \mu_i\mu_j \ln A_{ij}\Big], \quad \theta_i = p_i y + \omega_i t + \sigma_i \tau + \eta_i, \tag{27}$$

where we have omitted the τ-dependence in $f(y,t)$ as before (see Section 3). This is a solution of the bilinear form (8) and (9) provided that $\omega_i(p_i)$, $\sigma_i(p_i)$ satisfy the dispersion laws (18), and the interaction coefficients $A_{ij}, 1 \le i < j \le N$ are given by (24).

In principle, at least, we can now find the CH N-soliton to any order N. Thus, we substitute (27) into equation (5) to obtain the ACH N-soliton $u(y,t)$, where $r(y,t)$ is found from (7). Once u and r are known, we use equation (6) to determine the inverse co-ordinate transformation $x(y,t)$. This yields the *analytic* N-soliton solution $u(x,t)$ of the CH equation (2) parametrically in y, for any $\kappa > 0$. The N-soliton solution of the RCH equation (14) ($\kappa = 0$) follows from (16) in the usual way. However, as it stands, the final steps in the procedure are less than straightforward. Unfortunately, as the order N increases, the calculation for

$u(y,t)$ in (5) becomes extremely complicated and arduous, so making the integration in (6) all but intractable. Further, if one wants a generic representation for the CH N-soliton, then the solution must surely be sought in the simplest possible form. Work on these aspects of the method and solutions is still ongoing, and we shall report on this elsewhere. Nevertheless, we have shown that Hirota's bilinear transformation method provides a systematic means of extracting the erstwhile elusive analytic multisoliton solutions of the general Camassa–Holm equation.

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"Accelerating" Self-Similar Solutions and their Stability

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The variable $\chi \equiv x - 2Vz - \kappa z^2$, with V, κ constants, is a similarity variable for many equations and systems of nonlinear Schrödinger type arising in optoelectronics, with z the evolution (time-like) variable. For equations which lack the $x \leftrightarrow -x$ symmetry, localized 'pulses' exist as special solutions, with $\kappa \neq 0$. Although the governing equations are not completely integrable, these solutions are called *accelerating solitons*. Their decay as $\chi \to \pm\infty$ has Airy-function asymptotics, but, in practice, the pulses exhibit only small asymmetry.
By adapting the Evans function technique (widely used for reaction–diffusion systems) stability for photorefractive (self-bending) solitons and for accelerating solitons of the sliding frequency (SFF) equation is analysed. In each system the asymptotics must be expressed in terms of Airy functions, rather than in terms of the decaying exponentials used in the usual Evans function technique. Parameter regimes for which these *accelerating solitons* are (linearly) stable are identified and regimes allowing (oscillatory) *internal modes* are identified and confirmed by numerical integration of the original PDEs (with initial conditions chosen as perturbations from the self-similar profiles).
Equations possessing the same symmetry reductions as the photorefractive and SFF equations are identified. They include other practical examples.

1 Introduction

While many evolution equations, both integrable and non-integrable, possess travelling wave solutions, a number of these equations also possess solutions which are self-similar but with a similarity variable associated either with uniform acceleration or with propagation along a parabolic path. The existence and stability of localized solutions of this type is the subject of this investigation.

For systems of coupled nonlinear Schrödinger equations of the type

$$\mathrm{i}\partial_z u_m + D_m {\partial_x}^2 u_m + u_m f_m(|u_1|^2, |u_2|^2, \ldots, |u_N|^2) = 0, \quad m = 1, 2, \ldots, N \quad (1)$$

(with $u_r \in \mathbb{C}$, constants D_m real) the occurrence amongst the possibilities revealed by Lie symmetry analysis of a similarity variable $\chi = x - 2Vz - \kappa z^2$ involving constant parameters V, κ has been known since the 1980's (see e.g. [1]). However, the resulting defining ODEs typically do not possess solutions bounded throughout $-\infty < \chi < \infty$, so that the similarity reduction then has limited significance. However, more recently, a number of equations from nonlinear optics and electronics have been shown to predict accelerating pulses or self-deflecting confined beams of light. The complex amplitude $A(x,z)$ of an ultra-short laser-generated pulse travelling through an active medium was described by Vanin *et al.* [2] by

$$-\mathrm{i}A_z + A_{xx} + |A|^2 A + \mathrm{i}A\left\{1 - \int_{-\infty}^{x} |A|^2\,\mathrm{d}x\right\} = 0\,, \tag{2}$$

where x is a travelling wave coordinate and z is propagation distance. The similarity reduction was noted in [2] and accelerating pulses were computed numerically. In a number of papers, Christodoulides and co-workers investigated one-dimensional light beams travelling through photorefractive media, showing in [3] by direct numerical integration that localized solutions of

$$\mathrm{i}u_z + u_{xx} - \frac{u}{1+|u|^2} + \gamma\frac{(|u|^2)_x u}{1+|u|^2} = 0\,, \qquad \gamma = \text{ const} \tag{3}$$

follow a parabolic path (consistent with solutions $u = \mathrm{e}^{\mathrm{i}\theta(\chi,x)}F(\chi)$).

After Mollenauer *et al.* [4] proposed that, to overcome noise-induced 'jitter' in long-distance transmission systems, amplifiers should have successively shifted central frequencies, an averaged treatment of such a *sliding-frequency filter* (SFF) system was given by Hasegawa and Kodama [5] in terms of

$$\mathrm{i}u_z + \tfrac{1}{2}u_{xx} + |u|^2 u \;=\; \mathrm{i}\delta u + \mathrm{i}\beta(\partial_x \;+ \mathrm{i}\hat{\alpha}z)^2 u\,. \tag{4}$$

Here, x is retarded time, z is propagation distance and $\delta > 0$, $\beta > 0$ and $\hat{\alpha}$ are constants. It is readily verified that the choice $\kappa = -\hat{\alpha}/2$ allows the complex amplitude to have the form $u = \mathrm{e}^{\mathrm{i}\Theta(z,\chi)}W(\chi)$ with $\Theta \equiv (2\kappa\chi + c_0)z + 2\kappa^2 z^3/3 + 2\kappa V z + c_1$ with $W(\chi)$ complex-valued [6]. Equations (3) and (4) are the main examples considered in this paper, but others arising in recent opto-electronics literature are the coupled system for the amplitudes u^+ and u^- of two modes in a fibre with Raman scattering [7]

$$\mathrm{i}{u^{\pm}}_z + {u^{\pm}}_{xx} + |u^{\pm}|^2 u^{\pm} = C(|u^{\pm}|^2)_x u^{\pm} - K u^{\mp}\,, \quad C,\ K \text{ constants} \tag{5}$$

and a generalization of (3) to include higher-order effects of space-charge distributions [8].

For (3) and (4), this account summarizes symmetry reductions, determines restrictions which permit the existence of 'accelerating' localized pulses (or beams) and outlines a recent generalization of the Evans function method for determining regimes in which the resulting pulses are (linearly) stable. It also seeks broader classes of equations permitting the accelerating similarity reduction, so showing that the behaviour studied in the two examples may occur much more widely.

2 Self-Deflecting Similarity Solutions for (3)

Equation (3), written as a pair for u and its formally independent complex conjugate v has a 4-parameter space of invariances spanned by the generators

$$X_1 = \partial_z \,, \quad X_2 = \partial_x \,, \quad X_3 = u\partial_u - v\partial_v \,, \quad X_4 = 2z\partial_x + \mathrm{i}xX_3 \,,$$

as is readily found by standard Lie symmetry procedures. While X_1, X_2 and X_3 correspond simply to shifts in z, x and phase respectively, the generator X_4 is responsible for self-bending. The general similarity reduction is

$$u(z,x) = \mathrm{e}^{\mathrm{i}\theta(z,\chi)}F(\chi) \,, \quad \text{with} \quad \chi \equiv x - 2Vz - \kappa z^2 \,, \tag{6}$$

with real *amplitude* $F = F(\chi)$ governed by the ordinary differential equation

$$F''(\chi) + \left\{ B - \kappa\chi - \frac{1}{1+F^2} + \gamma\frac{2FF'}{1+F^2} \right\} F = 0 \,, \tag{7}$$

provided that the *phase* $\theta = \theta(z,\chi)$ has the form $\theta \equiv (V+\kappa z)\chi + \kappa^2 z^3/3 + \kappa V z^2 + (V^2 - B)z + c$. Here, B and c are parameters associated with translations in x and in phase, respectively.

In the case $\kappa = 0$, the reduction (6) is the travelling wave reduction and (7) becomes autonomous and possesses localized solutions symmetric about their peak, which has arbitrary value $F_{\max} \equiv \mu$ [3, 9]. However, these *rectilinear beam* solutions can exist only when the parameter γ (associated physically with diffusivity of the induced space-charge) vanishes. This follows from the identity

$$4\gamma \int_{-\infty}^{\infty} \frac{(FF')^2}{1+F^2} \mathrm{d}\chi + \kappa \int_{-\infty}^{\infty} F^2 \mathrm{d}\chi = 0 \tag{8}$$

readily obtained from (7) and showing that $\kappa < 0$ when $\gamma > 0$. For $\kappa \neq 0$, localized solutions are computed numerically over a wide range of values of $F_{\max}$ (see [9]), using a shooting method which matches decaying solutions of the linearization $F''(\chi) + (B - 1 - \kappa\chi)F = 0$ (transformable to Airy's equation). This procedure allows γ and $F_{\max}$ to be specified and then determines B, the curvature parameter κ and the location $\chi = \chi_0$ of the solution maximum $F = F_{\max}$.

Stability of the self-similar photorefractive beam profiles is analysed by writing $u(z,x) = \mathrm{e}^{\mathrm{i}\theta(z,\chi)}\{F(\chi) + w(z,\chi)\}$, linearizing the resulting equations for w and its complex conjugate $w^*(z,\chi)$ and then using the Evans function method [10]. This utilizes solutions of the form $w = c\tilde{u}(\chi)\mathrm{e}^{\mathrm{i}\lambda z} + c^*\tilde{v}^*(\chi)\mathrm{e}^{-\mathrm{i}\lambda^* z}$ (with $w^* = c\tilde{v}(\chi)\mathrm{e}^{\mathrm{i}\lambda z} + c^*\tilde{u}^*(\chi)\mathrm{e}^{-\mathrm{i}\lambda^* z}$, correspondingly) and leads to a 4th-order ODE system for $\tilde{u}(\chi)$ and $\tilde{v}(\chi)$. As $\chi \to \pm\infty$, a basis in the diffusionless case ($\gamma = 0$) is $\exp[\rho_i(\lambda)\chi]$, with ρ_i known ($i = 1,2,3,4$). The relevant localized solutions have decay both as $\chi \to -\infty$ and $\chi \to +\infty$. This *connection problem* selects the eigenvalue λ. The Evans function procedure involves evaluation of a 4×4 determinant $D(\lambda)$ associated with two (numerically computed) basis functions spanning the two-dimensional space of solutions decaying as $\chi \to -\infty$ (an unstable manifold) and another two basis functions for the space of solutions decaying as $\chi \to \infty$ (a stable manifold). A *connection* exists if and only if these two spaces intersect non-trivially so that $D(\lambda)$ vanishes. Since the defining properties ensure that $D(\lambda)$ is analytic, occurrence of zeros of D in a region of the λ-plane is identified (using an argument principle) by tracking changes in $\arg D$ as λ traces around the boundary of a region. If zeros appear anywhere within the half-plane $\operatorname{Im}\lambda < 0$, then a mode of instability exists.

The 4-parameter similarity reduction (6) implies that $\lambda = 0$ is an eigenvalue of algebraic multiplicity 4. Calculations confirm that $D(\lambda)$ has a 4th-order zero at $\lambda = 0$. As $F_{\max}$ is varied (at least for $F_{\max} < 6$) no zeros of $D(\lambda)$ occur in the lower half-plane. It is, however, found that around $F_{\max} = 1.25$ a pair of zeros with $\operatorname{Re}\lambda = 0$ arise, leading to a mode which is an even function of $\chi - \chi_0$, but oscillatory in z. Further increase in $F_{\max}$ introduces a second such *internal mode* [11], asymmetric in $\chi - \chi_0$. However, since the amplitudes of such modes do not grow with z, it is deduced that localized solutions for $\gamma = 0$ are stable.

In [9], a similar conclusion is shown to hold for $\gamma \neq 0$. Since the linearized equations for $\tilde{u}$ and $\tilde{v}$ have asymptotic forms (as $\chi \to \pm\infty$) which contain terms $-\kappa\chi w$ and $-\kappa\chi w^*$, the appropriate basis functions providing a *modified Evans function* are identified by matching to suitable *recessive* branches of the Airy function. In fact, the limiting forms (as $\chi \to \pm\infty$) of the variables $\tilde{u}$ and $\tilde{v}$ satisfy

$$(\partial_{z^+z^+} - z^+)\tilde{u} = 0\,, \quad (\partial_{z^-z^-} - z^-)\tilde{v} = 0 \qquad \text{with} \quad \kappa^{2/3} z^{\pm} \equiv \kappa\chi + 1 - B \pm \lambda\,. \quad (9)$$

$\tilde{u}$ and $\tilde{v}$ are constructed by numerical integration over the interval $\chi_1 \leq \chi \leq \chi_2$ (the *profile domain*) used for computation of $F(\chi)$ (i.e. where $|F| \geq 10^{-3}$, say). At $\chi = \chi_1, \chi_2$, boundary conditions fix $\tilde{u}'(\chi)/\tilde{u}(\chi)$ and $\tilde{v}'(\chi)/\tilde{v}(\chi)$ to match to decaying solutions to (9). Figure 1 shows images in the $z^{\pm}$ planes of the profile domain, for typical $\lambda \equiv \alpha - \mathrm{i}\beta$.

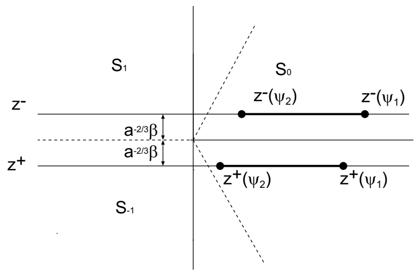

Figure 1. Geometrical of the problem

Since standard Airy function theory [12] shows that $\mathrm{Ai}_0(z^{\pm}) \equiv \mathrm{Ai}(z^{\pm})$ and $\mathrm{Ai}_{\pm 1}(z^{\pm}) \equiv \mathrm{Ai}(z^{\pm}\mathrm{e}^{\mp 2\pi\mathrm{i}/3})$ are the solutions recessive in the 120^0 sectors S_0 and $\mathrm{S}_{\pm 1}$ respectively, the boundary conditions at χ_1, χ_2 use the branches of the Airy function recessive in the sectors within which $\operatorname{Re} z^{\pm} \to \infty, -\infty$ along the respective images of the real χ-axis (recall that $\kappa < 0$, for $\gamma > 0$. Also note that the strategy requires modification for $\beta = 0$, using the algebraically decaying $\mathrm{Bi}(z^{\pm})$).

Evaluating the 4×4 determinant whose columns are $[\tilde{u}\ \tilde{u}'\ \tilde{v}\ \tilde{v}']^T$, with solutions $\tilde{u}(\chi)$, $\tilde{v}(\chi)$ corresponding to fixed basis sets of the unstable and stable manifolds yields the modified Evans function $D_{\mathrm{ai}}(\lambda)$. Invoking analyticity of $D_{\mathrm{ai}}(\lambda)$, noting that eigenvalues (zeros of $D_{\mathrm{ai}}(\lambda)$) occur in the pattern $\lambda, -\lambda, \lambda^*, -\lambda^*$ and using the argument principle for convenient contours in the complex λ-plane yields the results: (i) No eigenvalues exist off the real axis. (ii) The origin $\lambda = 0$ is a four-fold zero of $D_{\mathrm{ai}}(\lambda)$ (as confirmed by numerical evaluation for λ real). (iii) As for $\gamma = 0$, pairs of eigenvalues emerge from the *continuous spectrum* (the portions $|\lambda| \geq \sqrt{1-B}$ of the real axis) as $F_{\max}$ increases. The corresponding eigenfunctions describe mildly asymmetric *internal modes*, i.e. perturbations to the profile $F(\chi)$, with increasingly many maxima and oscillatory in the evolution variable z.

Numerical computation of (3) with initial profiles slightly perturbed from $u(0,x) = \mathrm{e}^{\mathrm{i}\theta(0,x)}F(x)$ confirms not just that perturbations do not grow, but also, for suitably large $F_{\max}$, exhibits oscillations about the self-deflecting profile with frequency matching to the predicted eigenvalues λ.

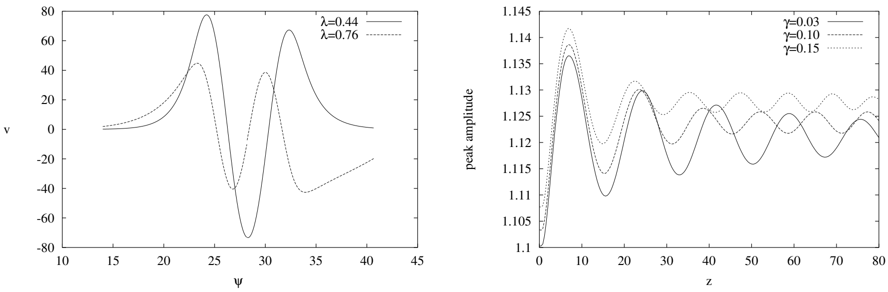

Figure 2. a) Profiles $\tilde{v}$ for two eigenvalues; $\gamma = 0.03, F_{\max} = 4.0$,
b) Persistent oscillations resulting when $F_{\max} = 1.0$ is perturbed by 10%.

3 Accelerating SFF Solitons

For the SFF equation (4), although the similarity reduction $u = \mathrm{e}^{\mathrm{i}\Theta(z,\chi)}W(\chi)$ leads to the ODE

$$(1+4\beta^2)W''(\chi) + 2(1+2\mathrm{i}\beta)[-2\mathrm{i}VW'(\chi) + (\hat{\alpha}\chi - c_0 - \mathrm{i}\delta + |W|^2)W] = 0 \quad (10)$$

with $\Theta = (2\kappa\chi + c_0)z + 2V\kappa z^2 + \frac{2}{3}\kappa^2 z^3 + c_1$, the acceleration coefficient κ is not adjustable. It must equal $-\frac{1}{2}\hat{\alpha}$. Even though this reduction involves the free parameters c_0, c_1 and V, it is found that imposing the (*connection*) condition $W \to 0$ as $\chi \to \infty$ and as $\chi \to -\infty$ imposes a relation between c_0 and V. Equation (10) is analysed [13] in its renormalized form

$$w''(y) + 2(1+2\mathrm{i}\beta)\{\mathrm{i}Bw'(y) + (y - \mathrm{i}\Delta + |w|^2)w\} = 0\,, \quad (11)$$

where $w(y) = [\hat{\alpha}^2(1+4\beta^2)]^{-1/6}W(\chi)$, $y = (\hat{\alpha}\chi - c_0)[\hat{\alpha}^2(1+4\beta^2)]^{-1/3}$, with $B \equiv -2V\hat{\alpha}^{1/3}(1+4\beta^2)^{-2/3}$ and $\Delta \equiv \hat{\alpha}^{-2/3}(1+4\beta^2)^{-1/3}\delta$. Previous approximate treatments, for (β, Δ) small [14], have suggested the existence of two families of solutions — one narrow and with large peak amplitude $|W|_{\max}$ and one broader and of lower amplitude. The former was found to be stable, the latter unstable. Numerical search using (11) has confirmed existence of these two families, over a wide range of β and Δ. For each family, B is determined by the choice of (β, Δ).

Analysing linear stability using

$$u(z,x) = [\hat{\alpha}^2(1+4\beta^2)]^{1/6} \mathrm{e}^{\mathrm{i}\Theta}\{w(y) + \tilde{u}(y)\mathrm{e}^{-\lambda\delta z} + \tilde{v}^*(y)\mathrm{e}^{\lambda^*\delta z}\}$$

(adapting the approach used for (3)) again leads to an eigenvalue problem for $[\tilde{u}\ \ \tilde{u}'\ \ \tilde{v}\ \ \tilde{v}']^T$ having Airy function asymptotics (factors $\mathrm{e}^{\pm \mathrm{i}B(1\pm 2\mathrm{i}\beta)y}$ having been removed). A modified Evans function may be defined and constructed much as before, the crucial difference being that the images of the profile domain in the $z^{\pm}$-planes are lines inclined at angles $\pm\frac{1}{3}\arg(1+2\mathrm{i}\beta)$, so interchanging the choices of Airy-function branches needed in boundary conditions at $z^{\pm}(\chi_2)$.

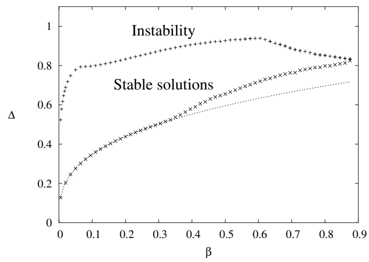

Figure 3. Stability region ($\hat{\alpha} = 0.05$). Existence boundary of [14] shown dotted.

For the larger-amplitude family, existence and stability of accelerating solitons is summarized in Figure 3 (numerical search has been undertaken only above the curve $\Delta = 3\beta^{1/3}/4$, the existence boundary predicted in [14] as $\beta \to 0$). The boundary $\Delta = \Delta_{\lim}(\beta)$ above which pulses become unstable corroborates earlier numerical investigations [15, 16]. Moreover, as for (3) the modified Evans function method constructs internal modes (at suitable soliton amplitude). As Δ is increased above the upper stability boundary, these oscillatory modes become modes with growing amplitude, as illustrated in Figure 4, where direct numerical integration with perturbed initial conditions shows emergence of (small) periodic oscillation in peak amplitude for $\Delta = 0.93$, but growing oscillations for $\Delta = 0.95$ (respectively below and above $\Delta = \Delta_{\lim}(\beta)$, for $\beta = 0.5$ and $\hat{\alpha} = 0.05$).

4 Equation Classes Having the Accelerating Symmetry

To determine equations of the type $u_z = L(u, v, p, q, r, s)$, $v_z = M(u, v, p, q, r, s)$ (where $p \equiv u_x$, $q \equiv v_x$, $r \equiv u_{xx}$, $s \equiv v_{xx}$) sharing with (3) the symmetry generators $X_1 = \partial_z$, $X_2 = \partial_x$, $X_3 = u\partial_u - v\partial_v$ and $X_4 = 2z\partial_x + \mathrm{i}xX_3$ the procedure in [17] is followed. Solving the resulting determining equations corresponding to $X = X_1$,

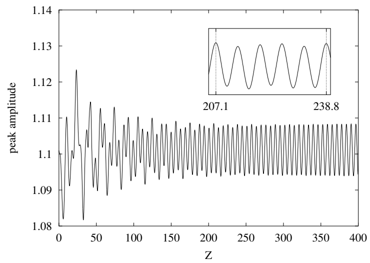

Figure 4. Emerging oscillations (a) $\Delta = 0.93$ (stable), (b) $\Delta = 0.95$ (unstable).

X_2 and X_3 yields the forms $L = u\hat{f}(I,P,Q,R,S)$, $M = v\hat{g}(I,P,Q,R,S)$ with $I \equiv uv$, $P \equiv vu_x$, $Q \equiv uv_x$, $R \equiv vu_{xx}$ and $S \equiv uv_{xx}$. Considering also the generator X_4 gives the result that both $\mathrm{i}\hat{f} + R/I$ and $-\mathrm{i}\hat{g} + S/I$ should depend only upon I, $P+Q$, $P^2 - IR$ and $Q^2 - IS$. After rearrangement, this leads to the statement that the accelerating symmetry reduction for the photorefractive equation (3) is shared by all equations of the form

$$\mathrm{i}u_z + u_{xx} + u\tilde{f}[uu^*, (uu^*)_x, (uu^*)_{xx}, (\arg u)_{xx}] = 0\,. \tag{12}$$

For these equations, the reduction $u = \mathrm{e}^{\mathrm{i}\theta(\chi,z)}F(\chi)$ (F real) leads to $\theta(\chi,z)$ as for (3) and to the ordinary differential equation

$$F''(\chi) - \kappa\chi F'(\chi) + \tilde{f}(F^2, 2FF', 2[FF'' + (F')^2], 0) = 0\,,$$

(when $c_1 = 0$, without loss of generality). A special case is equation (9) of [8] for which $\tilde{f}$ takes the complicated, specific form $\tilde{f} = (\alpha uu^* - \beta)(1+uu^*)^{-1} + \{\sqrt{2}[\gamma_1 - \gamma_2 uu^* + \gamma(1+uu^*)^2](uu^*)_x - 2\gamma_3[(uu^*)_x]^2\}(1+uu^*)^{-3} + 2\gamma_4(uu^*)_{xx}(1+uu^*)^{-2}$, with $z = \sqrt{2}s$. Self-deflecting solutions computed in [8] are therefore consistent with the similarity variable χ.

Moreover, more general classes of equation share this symmetry reduction, since it is readily checked that the coupled pair (5) including Raman scattering effects [7] and more general coupled systems

$$\mathrm{i}(u_m)_z + (u_m)_{xx} + u_m N_m\{|u_1|^2, \ldots, |u_N|^2; (|u_1|^2)_x, \ldots, (|u_N|^2)_x\} = 0$$

(for $m = 1, \ldots, N$) share this reduction. It may readily be verified that the class (12) of equations possessing this same family of symmetries may be extended, by including higher order derivatives, to the class

$$\mathrm{i}u_z + u_{xx} + uf[uu^*, (uu^*)_x, (uu^*)_{xx}, (uu^*)_{xxx}, (\mathrm{arg}u)_{xx}, (\mathrm{arg}u)_{xxx}] = 0,$$

but does not include the equation $u_z + u_{xxx} + 6|u|^2u_x = \mathrm{i}\alpha|u|^2u - \gamma(|u|^2)_x u$ for which Yang [18] demonstrates numerically stable accelerating pulses.

Generalizing (4) as $u_z = L(u,v,p,q,r,s,z,x)$, $v_z = M(u,v,p,q,r,s,z,x)$ and insisting that $X_2 \equiv \partial_x$, $X_3 \equiv u\partial_u - v\partial_v$ and $X_5 \equiv \partial_z - \hat{\alpha}zX_2 - \mathrm{i}xX_3$ are generators, then solving the defining equations corresponding to X_2 and X_3 yields

$$L = u\hat{f}(I,P,Q,R,S,z)\,, \quad M = v\hat{g}(I,P,Q,R,S,z)\,,$$

while consideration also of the generator X_5 specializes these further to give

$$\mathrm{i}u_z + \tfrac{1}{2}u_{xx} + u\tilde{f}\{uv, (uv)_x, u^{-1}(u_x + \mathrm{i}\hat{\alpha}zu), \\ u^{-1}[(\partial_x + \mathrm{i}\hat{\alpha}z)^2u], v^{-1}[(\partial_x - \mathrm{i}\hat{\alpha}z)^2v]\} = 0\,.$$

Equivalently, the function $\tilde{f}$ may be written as $\tilde{f} = \bar{f}\{uu^*, [N_1(uu^*)]_x, [N_2(uu^*)]_{xx}, u^{-1}u_x + \mathrm{i}\hat{\alpha}z, u^{-1}(\partial_x + \mathrm{i}\hat{\alpha}z)^2u\}$. These examples show the importance of dependence upon $|u|^2$ within the nonlinear terms and of the operator $\partial_x + \mathrm{i}\hat{\alpha}z$. For these generalized SFF equations the governing differential equation for $W(\chi) = u\mathrm{e}^{-\mathrm{i}\Theta}$ becomes

$$W''(\chi) - 4\mathrm{i}VW' + 2(\hat{\alpha}\chi - c_0)W \\ +2W\bar{f}\{|W|^2, (N_1(|W|^2))_\chi, (N_2(|W|^2))_{\chi\chi}, W'/W, W''/W\} = 0\,.$$

For many of these functions $\bar{f}$, existence and stability of accelerating solitons may be expected to follow from use of the methods applied to the special case (4).

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Admissible Point Transformations of Nonlinear Schrödinger Equations

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The notion of a normalized class of differential equations is introduced. Using it, we describe admissible point transformations in the class of nonlinear (1+1)-dimensional Schrödinger equations with modular nonlinearities and potentials, which have the form $i\psi_t+\psi_{xx}+f(|\psi|)\psi+V(t,x)\psi=0$, where f is an arbitrary complex-valued nonlinearity depending only on $\rho=|\psi|$ and V is an arbitrary complex-valued potential depending on t and x. We also carry out complete group classification for the subclass $\rho f_{\rho\rho}/f_\rho\neq\mathrm{const}\in\mathbb{R}$.

1 Introduction

Before mathematical notions are defined in rigorous and precise form, they can be implicitly used a long time. This commonplace is true also for the notion of a normalized class of differential equations introduced below. The most known classical group classification problems such as the Lie's classifications of second order ordinary differential equations [8] and of second order two-dimensional linear partial differential equations [7] were solved with essential using strong normalization of the above classes of differential equations. Similar classification technics based on the properties of normalized classes was recently applied in solving group classification problems by a number of authors (see e.g. [2, 11–14]).

In this paper we give rigorous definitions of sets of admissible point transformations and normalized classes of differential equations. Then we describe admissible point transformations in the class of nonlinear (1+1)-dimensional Schrödinger equations with modular nonlinearities and potentials, which have the form

$$i\psi_t+\psi_{xx}+f(|\psi|)\psi+V\psi=0, \tag{1}$$

where f is an arbitrary complex-valued nonlinearity depending only on $\rho=|\psi|$ and V is an arbitrary complex-valued potential depending on t and x. Using proposed technics, we also carry out complete group classification for the subclass $\rho f_{\rho\rho}/f_\rho\neq\mathrm{const}\in\mathbb{R}$.

2 Admissible Transformations

Let $\mathcal{L}(\theta)$ be a system $L(x,u_{(p)},\theta(x,u_{(p)}))=0$ of l differential equations for m unknown functions $u=(u^1,\ldots,u^m)$ of n independent variables $x=(x_1,\ldots,x_n)$. Here $u_{(p)}$ denotes the set of all the derivatives of u with respect to x of order no greater than p, including u as the derivatives of the zero order. $L=(L^1,\ldots,L^l)$ is a tuple of l fixed functions depending on x, $u_{(p)}$ and θ. θ denotes the tuple of arbitrary (parametric) functions $\theta(x,u_{(n)})=(\theta^1(x,u_{(p)}),\ldots,\theta^k(x,u_{(p)}))$ running the set $\mathcal{S}$ of solutions of the system $S(x,u_{(p)},\theta_{(q)}(x,u_{(p)}))=0$. This system consists of differential equations with respect to θ, where x and $u_{(p)}$ play the role of independent variables and $\theta_{(q)}$ stands for the set of all the partial derivatives of θ of order no greater than q. In what follows we call the functions θ as arbitrary elements. Denote the class of systems $\mathcal{L}(\theta)$ with the arbitrary elements θ running $\mathcal{S}$ as $\mathcal{L}|_\mathcal{S}$.

For $\theta,\tilde\theta\in\mathcal{S}$ we will call the set of point transformations which maps the system $\mathcal{L}(\theta)$ into the system $\mathcal{L}(\tilde\theta)$ as the set of admissible transformations from $\mathcal{L}(\theta)$ into $\mathcal{L}(\tilde\theta)$ and will denote it $\mathrm{T}(\theta,\tilde\theta)$. The maximal Lie symmetry group $G^{\max}(\theta)$ of the system $\mathcal{L}(\theta)$ coincides with $\mathrm{T}(\theta,\theta)$. If the systems $\mathcal{L}(\theta)$ and $\mathcal{L}(\tilde\theta)$ are equivalent with respect to point transformations then $\mathrm{T}(\theta,\tilde\theta)=G^{\max}(\theta)\circ\varphi^0=\varphi^0\circ G^{\max}(\tilde\theta)$, where φ^0 is a fixed transformation from $\mathrm{T}(\theta,\tilde\theta)$. Otherwise, $\mathrm{T}(\theta,\tilde\theta)=\varnothing$. The set $\mathrm{T}(\theta,\mathcal{L}|_\mathcal{S})=\{\,(\tilde\theta,\varphi)\mid\tilde\theta\in\mathcal{S},\ \mathrm{T}(\theta,\tilde\theta)\neq\varnothing,\ \varphi\in\mathrm{T}(\theta,\tilde\theta)\,\}$ is called as the *set of admissible transformations of the equation* $\mathcal{L}(\theta)$ *in the class* $\mathcal{L}|_\mathcal{S}$. The set $\mathrm{T}(\mathcal{L}|_\mathcal{S})=\{\,(\theta,\tilde\theta,\varphi)\mid\theta,\tilde\theta\in\mathcal{S},\ \mathrm{T}(\theta,\tilde\theta)\neq\varnothing,\ \varphi\in\mathrm{T}(\theta,\tilde\theta)\,\}$ is called as the *set of admissible transformations in* $\mathcal{L}|_\mathcal{S}$.

Note 1. First the set of admissible transformations was described by J.G. Kingston and C. Sophocleous in [4] for a class of generalised Burgers equations. These authors call transformations of such type as *form-preserving ones* [5, 6].

Note 2. All the notions and results adduced in this and the next sections can be reformulated in the infinitesimal terms by means of using the notions of operators, Lie algebras instead of transformations, Lie groups etc. For instance, see [1] for the definition of "cones of tangent equivalences" which is the infinitesimal analogue of the definition of $\mathrm{T}(\theta,\mathcal{L}|_\mathcal{S})$. Ibid a non-trivial example of semi-normalized class of differential equations is investigated in the framework of infinitesimal approach.

Note 3. In the case of one dependent variable ($m=1$) we can extend all the above and below notions to contact transformations.

We can define the usual equivalence group in a rigorous way using the notion of admissible transformations. Namely, any element Φ from the usual equivalence group $G^\sim(\mathcal{L}|_\mathcal{S})$ is a point transformation in the space of $(x,u_{(p)},\theta)$, which is projectible on the space of $(x,u_{(p')})$ for any $0\le p'\le p$, and $\Phi|_{(x,u_{(p')})}$ being the p'-th order prolongation of $\Phi|_{(x,u)}$, and $\forall\theta\in\mathcal{S}$: $\Phi\theta\in\mathcal{S}$ and $\Phi|_{(x,u)}\in\mathrm{T}(\theta,\Phi\theta)$.

Let us remind that the local transformation φ: $\tilde z=\varphi(z)$ in the space of the variables $z=(z_1,\ldots,z_k)$ is called projectible on the space of variables $z'=(z_{i_1},\ldots,z_{i_{k'}})$, where $1\le i_1<\cdots<i_{k'}\le k$, if expressions for $\tilde z'$ depend only on z'. We denote the restriction of φ on the space of z' as $\varphi|_{z'}$: $\tilde z'=\varphi|_{z'}(z')$.

If the arbitrary elements θ explicitly depend on x and u only (one always can do it, assuming derivatives as new dependent variables), we may consider the generalized equivalence group $G^{\sim}_{\rm gen}(\mathcal{L}|_{\mathcal{S}})$ [10], admitting dependence of transformations of (x,u) on θ. More rigorously, any element Φ from $G^{\sim}_{\rm gen}(\mathcal{L}|_{\mathcal{S}})$ is a point transformation in the space of (x,u,θ) such that $\forall\theta\in\mathcal{S}$: $\Phi\theta\in\mathcal{S}$ and $\Phi(\cdot,\cdot,\theta(\cdot,\cdot))|_{(x,u)}\in\mathrm{T}(\theta,\Phi\theta)$.

Roughly speaking, $G^{\sim}(\mathcal{L}|_{\mathcal{S}})$ is the set of admissible transformations which can be applied to any $\theta\in\mathcal{S}$ and $G^{\sim}_{\rm gen}(\mathcal{L}|_{\mathcal{S}})$ is formed by the admissible transformations which can be separated to classes parameterized with θ running whole $\mathcal{S}$.

It is possible to consider other generalizations of equivalence groups, e.g. with nonlocal dependence on arbitrary elements [3].

We can formulate the problem of description of $\mathrm{T}(\mathcal{L}|_{\mathcal{S}})$ similarly to the group classification problem. The steps of investigation are the following:

1. Construction of $G^{\sim}(\mathcal{L}|_{\mathcal{S}})$ (or $G^{\sim}_{\rm gen}(\mathcal{L}|_{\mathcal{S}})$).
2. Description of *conditional* equivalence transformations in the class $\mathcal{L}|_{\mathcal{S}}$, i.e. searching for additional conditions $\{S_\gamma\,|\,\gamma\in\Gamma\}$ which determine the sets $\mathcal{S}_\gamma$ of arbitrary elements such that $G^{\sim}(\mathcal{L}|_{\mathcal{S}\cap\mathcal{S}_\gamma})\not\subset G^{\sim}(\mathcal{L}|_{\mathcal{S}})$ (or $G^{\sim}_{\rm gen}(\mathcal{L}|_{\mathcal{S}\cap\mathcal{S}_\gamma})\not\subset G^{\sim}_{\rm gen}(\mathcal{L}|_{\mathcal{S}})$).
3. Finding admissible transformations which do not belong groups obtained on the previous steps.

3 Normalized Classes of Differential Equations

Solving group classification problems is essentially simpler if a class of differential equations under consideration has an additional property of normalization with respect to point transformations.

Definition 1. The class $\mathcal{L}|_{\mathcal{S}}$ of differential equations is called *normalized* (with respect to point transformations) if $\forall(\theta,\tilde\theta,\varphi)\in\mathrm{T}(\mathcal{L}|_{\mathcal{S}})$ $\exists\Phi\in G^{\sim}(\mathcal{L}|_{\mathcal{S}})$: $\tilde\theta=\Phi\theta$ and $\varphi=\Phi|_{(x,u)}$.

The class $\mathcal{L}|_{\mathcal{S}}$ is called *normalized in generalized sense* if $\forall(\theta,\tilde\theta,\varphi)\in\mathrm{T}(\mathcal{L}|_{\mathcal{S}})$ $\exists\Phi\in G^{\sim}_{\rm gen}(\mathcal{L}|_{\mathcal{S}})$: $\tilde\theta=\Phi\theta$ and $\varphi=\Phi(\cdot,\cdot,\theta(\cdot,\cdot))|_{(x,u)}$.

Proposition 1. *If $\mathcal{L}|_{\mathcal{S}}$ is a normalized class of differential equations (in usual or generalized sense) then for any $\theta^0\in\mathcal{S}$ the Lie symmetry group $G^{\max}(\theta^0)$ coincides with restriction, on the space of (x,u), of the subgroup of $G^{\sim}(\mathcal{L}|_{\mathcal{S}})$ (or $G^{\sim}_{\rm gen}(\mathcal{L}|_{\mathcal{S}})$) preserving the value $\theta=\theta^0(x,u_{(p)})$.*

Definition 2. The class $\mathcal{L}|_{\mathcal{S}}$ of differential equations is called *strongly normalized* if $\mathcal{L}|_{\mathcal{S}}$ is normalized and $G^{\sim}(\mathcal{L}|_{\mathcal{S}})|_{(x,u)}=\prod_{\theta\in\mathcal{S}}G^{\max}(\theta)$.

The class $\mathcal{L}|_{\mathcal{S}}$ of differential equations is called *strongly normalized* with respect to point transformations *in generalized sense* if $\mathcal{L}|_{\mathcal{S}}$ is normalized in generalized sense and $\forall\theta^0\in\mathcal{S}$: $G^{\sim}_{\rm gen}(\mathcal{L}|_{\mathcal{S}})|^{\theta=\theta^0}_{(x,u)}=\prod_{\theta\in\mathcal{S}_{\theta^0}}G^{\max}(\theta)$, where

$$\mathcal{S}_{\theta^0}=\{\theta'\in\mathcal{S}\,|\,G^{\sim}_{\rm gen}(\mathcal{L}|_{\mathcal{S}})|^{\theta=\theta'}_{(x,u)}=G^{\sim}_{\rm gen}(\mathcal{L}|_{\mathcal{S}})|^{\theta=\theta^0}_{(x,u)}\}.$$

Definition 3. The class $\mathcal{L}|_\mathcal{S}$ of differential equations is called *semi-normalized* (with respect to point transformations) if $\forall(\theta,\tilde\theta,\varphi)\in\mathrm{T}(\mathcal{L}|_\mathcal{S})$ $\exists\tilde\varphi\in G^{\max}(\theta)$, $\exists\Phi\in G^\sim(\mathcal{L}|_\mathcal{S})$: $\varphi=\tilde\varphi\circ\Phi|_{(x,u)}$, i.e.

$$\mathrm{T}(\mathcal{L}|_\mathcal{S})=\{(\theta,\Phi\theta,\tilde\varphi\circ\Phi|_{(x,u)})\,|\,\theta\in\mathcal{S},\ \tilde\varphi\in G^{\max}(\theta),\ \Phi\in G^\sim(\mathcal{L}|_\mathcal{S})\}.$$

($\mathrm{T}(\mathcal{L}|_\mathcal{S})=\{(\theta^0,\Phi\theta^0,\tilde\varphi\circ\Phi|^{\theta=\theta^0}_{(x,u)})\,|\,\theta^0\in\mathcal{S},\ \tilde\varphi\in G^{\max}(\theta),\ \Phi\in G^\sim_{\rm gen}(\mathcal{L}|_\mathcal{S})\}$ if $\mathcal{L}|_\mathcal{S}$ is *semi-normalized in generalized sense.*)

Roughly speaking, the class $\mathcal{L}|_\mathcal{S}$ is normalized if any admissible transformation in this class belongs to $G^\sim(\mathcal{L}|_\mathcal{S})$ and is strongly normalized if additionally $G^\sim(\mathcal{L}|_\mathcal{S})|_{(x,u)}$ is generated by elements from $G^{\max}(\theta)$, $\theta\in\mathcal{S}$. The set of admissible transformations of a semi-normalized class is generated by the transformations from the equivalence group of the whole class and the transformations from the Lie symmetry groups of equations of this class.

Proposition 2. *Let G^i, $i=1,2$, be local groups of point transformations in the space of (x,u), for which $\mathcal{S}^i=\{\theta\in\mathcal{S}\,|\,G^{\max}(\theta)=G^i\}\neq\varnothing$. Then $\mathcal{S}^1\sim\mathcal{S}^2$ $\mathrm{mod}\ G^\sim(\mathcal{L}|_\mathcal{S})$ iff $G^1\sim G^2$ $\mathrm{mod}\ G^\sim(\mathcal{L}|_\mathcal{S})$.*

Proposition 3. *Two systems from a semi-normalized class are pointwise equivalent iff they are equivalent with respect to the equivalence group of this class.*

Proposition 4. *Any normalized class of differential equations is semi-normalized.*

In view of the above propositions, the group classification problem in any normalized class of differential equations is reduced to subgroup analysis of the corresponding equivalence group. The property of strong normalization allows us to hope that essential part of subgroups will be Lie symmetry groups of systems from the class under consideration.

Investigation of normalization of the class $\mathcal{L}|_\mathcal{S}$ or its subclasses is necessary for description of $\mathrm{T}(\mathcal{L}|_\mathcal{S})$ and can be included as a step in studying $\mathrm{T}(\mathcal{L}|_\mathcal{S})$.

There exist a number of examples of implicit using the notion of normalized classes in group classification of differential equations.

4 Covering Class of Nonlinear Schrödinger Equations

Consider the more general class of NSchEs

$$i\psi_t+\psi_{xx}+S(t,x,|\psi|)\psi=0, \tag{2}$$

which cover class (1) and is more convenient, in some sense, for preliminary group classification. Here S is an arbitrary complex-valued function depending on t, x and $\rho=|\psi|$, and we additionally assume $S_\rho\neq0$. The latter condition is invariant under any local transformation which transforms a fixed equation from class (2) to an equation from the same class. The auxiliary system for the arbitrary element S has the form

$$\psi S_\psi-\psi^*S_{\psi^*}=0,\quad \psi S_\psi+\psi^*S_{\psi^*}\neq0. \tag{3}$$

Theorem 1. *Class* (2) *is strongly normalized. The equivalence group* $G^{\sim}_{\{S\}}$ *of the class* (2) *is formed by the transformations*

$$\tilde t = T,\quad \tilde x = \varepsilon x|T_t|^{1/2} + X,\quad \tilde\psi = \hat\psi R(t)\exp\left(\frac{i}{8}\frac{T_{tt}}{|T_t|}\,x^2 + \frac{i}{2}\frac{\varepsilon\varepsilon_T X_t}{|T_t|^{1/2}}\,x + i\Psi\right),$$

$$\tilde S = \frac{1}{|T_t|}\left(S + \frac{1}{8}\left(\frac{T_{tt}}{T_t}\right)_t x^2 + \frac{\varepsilon}{2}\left(\frac{X_t}{|T_t|^{1/2}}\right)_t x\right) - \left(\frac{\varepsilon}{4}\frac{T_{tt}}{|T_t|^{3/2}}\,x + \frac{1}{2}\frac{X_t}{T_t}\right)^2 \qquad (4)$$
$$+\frac{1}{T_t}\left(\Psi_t - i\frac{R_t}{R} - \frac{i}{4}\frac{T_{tt}}{T_t}\right).$$

Here T, X, R *and* Ψ *are arbitrary smooth real-valued functions of* t, $T_t \neq 0$, $R > 0$, $\varepsilon = \pm 1$, $\varepsilon_T = \operatorname{sign} T$, *and for any complex value* β

$$\hat\beta = \beta \quad \textit{if}\quad T_t > 0 \quad \textit{and}\quad \hat\beta = \beta^* \quad \textit{if}\quad T_t < 0.$$

Note 4. Indeed, the equivalence group $G^{\sim}_{\{S\}}$ is generated by the continuous family of transformations of form (4), where $T_t > 0$ and $\varepsilon = 1$, and two discrete transformations: the space reflection I_x ($\tilde t = t$, $\tilde x = -x$, $\tilde\psi = \psi$, $\tilde S = S$) and the Wigner time reflection I_t ($\tilde t = -t$, $\tilde x = x$, $\tilde\psi = \psi^*$, $\tilde S = S^*$).

Corollary 1. *For any equation of form* (2) *the value* $\rho S_{\rho\rho}/S_\rho$ *is preserved under any transformation which transforms this equation to an equation from the same class, excluding* I_t.

In particular, Theorem 1 results in the following statement on Lie symmetry operators of equations from class (2).

Theorem 2. *Any operator* Q *from the maximal Lie invariance algebra* $A^{\max}(S)$ *of equation* (2) *with an arbitrary function* S *can be presented in the form* $Q = D(\xi) + G(\chi) + \lambda M + \zeta I$, *where*

$$D(\xi) = \xi\partial_t + \frac{1}{2}\,\xi_t x\partial_x + \frac{1}{8}\,\xi_{tt}x^2 M,\quad G(\chi) = \chi\partial_x + \frac{1}{2}\chi_t x M, \qquad (5)$$

$$M = i(\psi\partial_\psi - \psi^*\partial_{\psi^*}),\quad I = \psi\partial_\psi + \psi^*\partial_{\psi^*}, \qquad (6)$$

where $\chi = \chi(t)$, $\xi = \xi(t)$, $\lambda = \lambda(t)$ *and* $\zeta = \zeta(t)$ *are arbitrary smooth real-valued functions of* t. *Moreover, the coefficients of* Q *should satisfy the classifying condition*

$$\xi S_t + \left(\frac{1}{2}\xi_t x + \chi\right) S_x + \nu\rho S_\rho + \xi_t S = \frac{1}{8}\,\xi_{ttt}x^2 + \frac{1}{2}\chi_{tt}x + \lambda_t - i\zeta_t + i\frac{1}{4}\,\xi_{tt}. \quad (7)$$

Assuming for S to be arbitrary and splitting (7) with respect to S, S_t, S_x and S_ρ, we obtain the following theorem.

Theorem 3. *The Lie algebra of the kernel of maximal Lie invariance groups of equations from class* (2) *is* $A^{\ker}_{\{S\}} = \langle M\rangle$.

5 General case of modular nonlinearity with potential

Let us pass to the subclass (1) of class (2) (i.e. $S = f(\rho) + V(t,x)$ where $f' \neq 0$). This subclass is separated from class (2) with the condition $S_{\rho t} = S_{\rho x} = 0$, i.e.

$$\psi S_{\psi t} + \psi^* S_{\psi^* t} = \psi S_{\psi x} + \psi^* S_{\psi^* x} = 0. \tag{8}$$

To find the equivalence group $G^{\sim}_{\{(f,V)\}}$ of class (1) in the framework of the direct method, we look for all local transformations in the space of the variables t, x, ψ, ψ^*, S and S^*, which preserve the system formed by equations (3) and (8). Moreover, in the same way we can classify all possible local transformations in class (1).

Theorem 4. *$G^{\sim}_{\{(f,V)\}}$ is formed by the transformations* (4) *where $T_{tt} = 0$ and $R_t = 0$. The subclass of* (1) *under the additional condition that $\rho S_{\rho\rho}/S_\rho$ is not a real constant have the same equivalence group and is normalized. There exist two different cases for additional (conditional) equivalence transformations in class* (1) *which are strongly normalized with respect to their equivalence groups in usual and extended sense correspondingly (below σ is a complex constant):*

1. $\rho S_{\rho\rho}/S_\rho = -1$, i.e. $f = \sigma \ln \rho$.

2. $\rho S_{\rho\rho}/S_\rho = \gamma - 1 \in \mathbb{R}$ and $\gamma \neq 0$, i.e. $f = \sigma \rho^\gamma$.

Note 5. It is possible to find equivalence transformations in another way, considering f and V as arbitrary elements instead of S. Then we have to look for all local transformations in the space of the variables t, x, ψ, ψ^*, f, f^*, V and V^*, which preserve the system formed by equations

$$i\psi_t + \psi_{xx} + S\psi = 0, \quad f_t = f_x = 0, \quad \psi f_\psi - \psi^* f_{\psi^*} = 0, \quad V_\psi = V_{\psi^*} = 0.$$

Due to the representation $S = f + V$ we additionally obtain only gauge equivalence transformations of the form $\tilde{f} = f + \beta$, $\tilde{V} = V - \beta$, where β is an arbitrary complex number and t, x and ψ are not changed. We neglect these transformations, choosing f in the most suitable form.

Below we adduce results only for the general case $\rho S_{\rho\rho}/S_\rho = \rho f_{\rho\rho}/f_\rho \neq \text{const} \in \mathbb{R}$. The cases $f = \sigma \ln \rho$ and $f = \sigma \rho^\gamma$ which admit extensions of admissible transformations and Lie symmetries have been also investigated completely and will be subjects of our future papers.

Corollary 2. *A potential V can be made to vanish in* (1) *by means of local transformations iff it is, up to trivial equivalence transformations, a real-valued function linear with respect to x.*

Theorem 5. *Any operator Q from the maximal Lie invariance algebra $A^{\max}(f,V)$ of equation* (1) *with an arbitrary nonlinearity f and an arbitrary potential V can be presented in the form $Q = c_0\partial_t + G(\chi) + \lambda M$, $c_0 = \text{const}$. Moreover, the coefficients of Q should satisfy the classifying condition*

$$c_0 V_t + \chi V_x = \frac{1}{2}\chi_{tt} x + \lambda_t. \tag{9}$$

Theorem 6. *The Lie algebra of the kernel of maximal Lie invariance groups of equations from class* (1) *is* $A^{\ker}_{\{(f,V)\}} = \langle M \rangle$.

Let us sketch shortly a chain of statements which results in complete group classification of class (1) in the general case.

Action of $G^{\sim}_{\{(f,V)\}}$ on f is only multiplication with non-zero real constants and/or complex conjugation. That is why we can fix an arbitrary function $f(\rho)$ and restrict our consideration on the set of nonlinearities which are proportional to f or f^* with real constant coefficients.

The set $A^{\cup}_{\{(f,V)\}} = \{Q = c_0\partial_t + G(\chi) + \lambda M\}$ is an (infinite-dimensional) Lie algebra under the usual Lie bracket of vector fields. For any $Q \in A^{\cup}_{\{(f,V)\}}$ where $(c_0, \chi) \neq (0,0)$ we can find V satisfying condition (9). Therefore, $A^{\cup}_{\{(f,V)\}} = \langle \bigcup_{(f,V)} A^{\max}(f,V) \rangle$.

$G^{\sim}_{\{(f,V)\}}$ generates an automorphism group on $A^{\cup}_{\{f,V\}}$ and the equivalence group on the set of equations of the form (9). Since these groups are isomorphic to $G^{\sim}_{\{(f,V)\}}$ we use the same notation for them. $A^{\ker}_{\{(f,V)\}}$ coincides with the center of the algebra $A^{\cup}_{\{(f,V)\}}$ and is invariant with respect to $G^{\sim}_{\{(f,V)\}}$.

Let A^1 and A^2 be the maximal Lie invariance algebras of some equations from class (1), and $\mathcal{S}^i = \{(f,V) \,|\, A^{\max}(f,V) = A^i\}$, $i = 1,2$. Then $\mathcal{S}^1 \sim \mathcal{S}^2 \bmod G^{\sim}_{\{(f,V)\}}$ iff $A^1 \sim A^2 \bmod G^{\sim}_{\{(f,V)\}}$.

A complete list of $G^{\sim}_{\{(f,V)\}}$-inequivalent one-dimensional subalgebras of $A^{\cup}_{\{f,V\}}$ is exhausted by the algebras $\langle \partial_t \rangle$, $\langle G(\chi) \rangle$, $\langle \lambda M \rangle$. (There exist additional equivalences into $\{\langle G(\chi)\rangle\}$ and $\{\langle \lambda M \rangle\}$, which are generated by equivalence transformations of t.)

Theorem 7. *A complete set of inequivalent cases of V admitting extensions of the maximal Lie invariance algebra of equations* (1) *in the case of arbitrary nonlinearity is exhausted by the potentials given in Table 1.*

Table 1. Results of classification in the case of arbitrary nonlinearity.

N	V	Basis of $A^{\max}$
0	$V(t,x)$	M
1	$V(x)$	$M,\ \partial_t$
2	$v(t)x^2 + iw(t)$	$M,\ G(\chi^1),\ G(\chi^2)$
3	0 or i	$M,\ \partial_t,\ G(1),\ G(t)$
4	$x^2 + i\nu$	$M,\ \partial_t,\ G(e^{-2t}),\ G(e^{2t})$
5	$-x^2 + i\nu$	$M,\ \partial_t,\ G(\cos 2t),\ G(\sin 2t)$

Here $v(t), w(t), \nu \in \mathbb{R}$, $(v_t, w_t) \neq (0,0)$. The functions $\chi^1 = \chi^1(t)$ and $\chi^2 = \chi^2(t)$ form a fundamental system of solutions for the ordinary differential equation $\chi_{tt} = 4v\chi$.

Proof. Suppose that equation (1) has extension of Lie symmetry for a parameter value (f,V), i.e. $A^{\max}(f,V) \neq A^{\ker}_{\{(f,V)\}}$. Then there exists an operator $Q = c_0\partial_t + G(\chi) + \lambda M \in A^{\max}(f,V)$ such that $(c_0, \chi) \neq (0,0)$.

If $c_0 \neq 0$ then $\langle Q\rangle \sim \langle \partial_t\rangle \bmod G^{\sim}_{\{(f,V)\}}$, i.e. we obtain Case 1.1. Investigation of additional extensions are reduced to the next case.

If $c_0 = 0$ then $\langle Q\rangle \sim \langle G(\chi)\rangle \bmod G^{\sim}_{\{(f,V)\}}$. It follows from (9) that the potential V have the form $V = v(t)x^2 + iw(t) + w^0(t)$, and $w^0 = 0 \bmod G^{\sim}_{\{(f,V)\}}$. For $(v_t, w_t) \neq (0,0)$ we have Case 1.2. The condition $v, w = \mathrm{const}$ results in Cases 1.3, 1.4 and 1.5 depending on the sign of v. If $v = 0$ and $w = \mathrm{const}$ we can reduce w by means of equivalence transformations to either 0 or 1. ■

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Group Analysis of a Nonlinear Model Describing Dissipative Media

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Group classifications of three equivalent mathematical models describing one-dimensional motion in nonlinear dissipative media are performed. The relationships between the symmetries of those models are explored.

1 Introduction

We consider the third order partial differential equation

$$w_{tt} = f(w_x)\,w_{xx} + \lambda_0\,w_{xxt}, \quad f,\,\lambda_0 > 0, \quad f' \neq 0, \tag{1}$$

where f is an arbitrary function of its argument, λ_0 is a positive real parameter, $w(t,x)$ is the dependent variable and subscripts denote partial derivative with respect to the independent variables t and x. Primes, here and in what follows, denote derivative of a function with respect to the only variable upon which it depends.

Some mathematical questions related to (1), as the existence, uniqueness and stability of weak solutions can be found in [1], while a study related to a generalized "shock structure" is shown in [2].

A physical prototype of the problem studied here arises when we consider purely longitudinal motions of a homogeneous viscoelastic bar of uniform cross-section and we assume that the material is a nonlinear Kelvin solid. That is we consider the equation of motion (the constant density is normalized to one)

$$w_{tt} = \tau_x \tag{2}$$

and assume a stress-strain relation of the following form:

$$\tau = \sigma(w_x) + \lambda_0\,w_{xt}, \tag{3}$$

where τ is the stress, x the position of a cross-section (which is assumed to move as a vertical plane section) in the homogeneous rest configuration of the bar, $w(t,x)$ the displacement at time t of the section from its rest position, $\sigma(w_x)$ the elastic tension (w_x is the strain), λ_0 the viscosity positive coefficient. Taking (3) into account and setting $\sigma' = f$ the equation (2) reduces to (1).

Moreover, the equation (1) occurs in the more well-known setting of one-dimensional motion of a viscous isentropic gas, treated from the lagrangian point of view. By putting $w_x = u$ and $w_t = v$, the equation (1) can be written as a 2×2 system of partial differential equations of the form

$$u_t = v_x, \tag{4}$$

$$v_t = f(u)\, u_x + \lambda_0\, v_{xx}, \tag{5}$$

where, u corresponds to the specific volume, $p\,(u) = \int^u f(s)\, ds$ is the pressure and v is the velocity.

The system (4)–(5), as it is well known, is equivalent to the equation (1), consequently, a symmetry of any one of the equation (1) and the system (4)–(5) defines a symmetry of the other. More specifically, because of the nonlocal transformation connecting (1) and (4)–(5), it is possible for a point symmetry of (4)–(5) to yield a contact symmetry of (1) (for details see [3]).

Moreover, it is worthwhile noticing that the system (4)–(5) can be regarded as the potential system associated to the following equation:

$$u_{tt} = \left[f(u)\, u_x + \lambda_0\, u_{xt}\right]_x . \tag{6}$$

In particular, point symmetries of the potential system (4)–(5) allows to obtain, if they exist, nonlocal symmetries (potential symmetries [4]) of the equation (6).

When $\lambda_0 = 0$, the equation (6) reduces to the nonlinear wave equation

$$u_{tt} = \left[f(u)\, u_x\right]_x ,$$

which was classified by Ames et al. [5] and give rise to numerous publications on symmetry analysis of nonlinear wave phenomena (see [6] and references therein for a review). While, for $\lambda_0 = \varepsilon \ll 1$, a study performed by means of the approximate symmetries can be found in [12].

In this paper we perform the complete group classification of the equation (1), the system (4)–(5) and the equation (6). After observing that the point symmetries of the system (4)–(5) do not produce any contact symmetry of the equation (1) and any nonlocal symmetry (potential symmetry) of the equation (6), we are able to demonstrate that the group classifications of (1), (4)–(5) and (6) are identical in the sense that, for any f, a point symmetry admitted by any one of (1), (4)–(5) and (6) induces a point symmetry admitted by the remaining two ones.

2 Group Classification of the Equation (1)

In order to discuss the group classification of the equation (1), we apply the classical Lie method and look for the one-parameter Lie group of infinitesimal transformations in (t, x, w)-space given by

$$\hat{t} = t + a\, \xi^1(t, x, w) + \mathcal{O}(a^2), \tag{7}$$

$$\hat{x} = x + a\, \xi^2(t, x, w) + \mathcal{O}(a^2), \tag{8}$$

$$\hat{w} = w + a\, \eta(t, x, w) + \mathcal{O}(a^2), \tag{9}$$

where a is the group parameter and the associated Lie algebra $\mathcal{L}$ is the set of vector fields of the form

$$X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial w}.$$

Then we require that the transformation (7)–(9) leaves invariant the set of solutions of the equation (1), in others words, we require that the transformed equation has the same form as the original one.

Following the well known monographs on this argument (see e.g. [6–9]), we introduce the third prolongation of the operator X in the form

$$X^{(3)} = X + \zeta_1 \frac{\partial}{\partial w_t} + \zeta_2 \frac{\partial}{\partial w_x} + \zeta_{11} \frac{\partial}{\partial w_{tt}} + \zeta_{22} \frac{\partial}{\partial w_{xx}} + \zeta_{221} \frac{\partial}{\partial w_{xxt}},$$

where we have set

$$\zeta_1 = D_t(\eta) - w_t\, D_t(\xi^1) - w_x\, D_t(\xi^2), \tag{10}$$

$$\zeta_2 = D_x(\eta) - w_t\, D_x(\xi^1) - w_x\, D_x(\xi^2), \tag{11}$$

$$\zeta_{11} = D_t(\zeta_1) - w_{tt}\, D_t(\xi^1) - w_{tx}\, D_t(\xi^2),$$
$$\zeta_{22} = D_x(\zeta_2) - w_{tx}\, D_x(\xi^1) - w_{xx}\, D_x(\xi^2),$$
$$\zeta_{221} = D_t(\zeta_{22}) - w_{xxt} D_t(\xi^1) - w_{xxx} D_t(\xi^2),$$

with operators D_t and D_x denoting total derivatives with respect to t and x.

The *determining system* of (1) arises from the following invariance condition:

$$X^{(3)}\left(w_{tt} - f(w_x)\, w_{xx} - \lambda_0\, w_{txx}\right) = 0, \tag{12}$$

under the constraints that the variable w_{tt} has to satisfy the equation (1). This latter allows us to find the infinitesimal generator of the symmetry transformations and, at the same time, gives the functional dependence of the constitutive function $f(w_x)$ for which the equation admits symmetries.

From (12) we obtain the following relations:

$$\begin{aligned} \xi^1 &= 2\, a_5\, t + a_1, \\ \xi^2 &= a_5\, x + a_2, \\ \eta &= a_6\, w + a_7\, x + a_4\, t + a_3, \\ &[(a_6 - a_5)\, w_x + a_7]\, f' + 2\, a_5\, f = 0, \end{aligned} \tag{13}$$

where a_i ($i = 1, 2, \ldots, 7$), are constants.

For arbitrary f we have that the *Principal Lie Algebra* $\mathcal{L}_{\mathcal{P}}$ of the equation (1) is four-dimensional and it is spanned by the operators

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial w}, \quad X_4 = t\frac{\partial}{\partial w},$$

otherwise we obtain the results summarized in Table 1.

Table 1. Group classification of the equation (1). f_0, p and q are constitutive constants with $f_0 > 0$, $p \neq 0$.

Case	Forms of $f(w_x)$	Extensions of $\mathcal{L}_{\mathcal{P}}$
I	$f(w_x) = f_0\, e^{\frac{w_x}{p}}$	$X_5 = 2\,t\,\dfrac{\partial}{\partial t} + x\,\dfrac{\partial}{\partial x} + (w - 2\,p\,x)\,\dfrac{\partial}{\partial w}$
II	$f(w_x) = f_0\,(w_x + q)^{\frac{1}{p}}$	$X_5 = 2\,t\,\dfrac{\partial}{\partial t} + x\,\dfrac{\partial}{\partial x} + [(1 - 2\,p)\,w - 2\,p\,q\,x]\,\dfrac{\partial}{\partial w}$

3 Group Classification of the System (4)–(5)

When we look for the one-parameter Lie group of infinitesimal transformations of the system (4)–(5) in the (t, x, u, v)-space, the associated Lie algebra $\bar{\mathcal{L}}$ is the set of vector fields of the form

$$\bar{X} = \bar{\xi}^1 \frac{\partial}{\partial t} + \bar{\xi}^2 \frac{\partial}{\partial x} + \eta^1 \frac{\partial}{\partial u} + \eta^2 \frac{\partial}{\partial v},$$

where the coordinates $\bar{\xi}^1$, $\bar{\xi}^2$, η^1, η^2 are functions of t, x, u and v.

Making use of the classical Lie method, from the invariance conditions which follow by applying the second prolongation of the operator $\bar{X}$ to (4)–(5), we give rise to the following result:

$$\begin{aligned}
&\bar{\xi}^1 = 2\,a_5\,t + a_1,\\
&\bar{\xi}^1 = a_5\,x + a_2,\\
&\eta^1 = (a_6 - a_5)\,u + a_7,\\
&\eta^2 = (a_6 - 2\,a_5)\,v + a_4,\\
&[(a_6 - a_5)\,u + a_7]\,f' + 2\,a_5\,f = 0.
\end{aligned}$$

For f arbitrary the *Principal Lie Algebra* $\bar{\mathcal{L}_{\mathcal{P}}}$ of the system (4)–(5) is three-dimensional and it is spanned by the operators

$$\bar{X}_1 = \frac{\partial}{\partial t}, \quad \bar{X}_2 = \frac{\partial}{\partial x}, \quad \bar{X}_3 = \frac{\partial}{\partial v}, \tag{14}$$

otherwise we obtain the results summarized in Table 2.

Table 2. Group classification of the system (4)–(5). f_0, p and q are constitutive constants with $f_0 > 0$, $p \neq 0$.

Case	Forms of $f(u)$	Extensions of $\bar{\mathcal{L}_{\mathcal{P}}}$
I	$f(u) = f_0\, e^{\frac{u}{p}}$	$\bar{X}_4 = 2t\dfrac{\partial}{\partial t} + x\dfrac{\partial}{\partial x} - 2p\dfrac{\partial}{\partial u} - v\dfrac{\partial}{\partial v}$
II	$f(u) = f_0\,(u + q)^{\frac{1}{p}}$,	$\bar{X}_4 = 2t\dfrac{\partial}{\partial t} + x\dfrac{\partial}{\partial x} - 2p(u+q)\dfrac{\partial}{\partial u} - (1+2p)v\dfrac{\partial}{\partial v}$

4 Group Classification of the Equation (6)

In order to obtain the complete point symmetry classification of the equation (6) we write the infinitesimal operator in the form

$$\tilde{X} = \tilde{\xi}^1 \frac{\partial}{\partial t} + \tilde{\xi}^2 \frac{\partial}{\partial x} + \tilde{\eta} \frac{\partial}{\partial u},$$

where the coordinates $\tilde{\xi}^1$, $\tilde{\xi}^2$, $\tilde{\eta}$ are functions of t, x and u.

Making use of the classical Lie method, from the invariance conditions which follow by applying the third prolongation of the operator $\tilde{X}$ to (6), we give rise to the following result:

$$\tilde{\xi}^1 = 2\, a_5\, t + a_1, \quad \tilde{\xi}^1 = a_5\, x + a_2, \quad \tilde{\eta} = (a_6 - a_5)\, u + a_7,$$
$$[(a_6 - a_5)\, u + a_7]\, f' + 2\, a_5\, f = 0.$$

For f arbitrary the *Principal Lie Algebra* $\tilde{\mathcal{L}_{\mathcal{P}}}$ of the equation (6) is two-dimensional and it is spanned by the operators

$$\tilde{X}_1 = \frac{\partial}{\partial t}, \quad \tilde{X}_2 = \frac{\partial}{\partial x},$$

otherwise the group classification is summarized in Table 3.

Table 3. Group classification of the equation (6). f_0, p and q are constitutive constants with $f_0 > 0$, $p \neq 0$.

Case	Forms of $f(u)$	Extensions of $\tilde{\mathcal{L}_{\mathcal{P}}}$
I	$f(u) = f_0\, e^{\frac{u}{p}}$	$\tilde{X}_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2p\frac{\partial}{\partial u}$
II	$f(u) = f_0\ (u+q)^{\frac{1}{p}}$,	$\tilde{X}_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2p(u+q)\frac{\partial}{\partial u}$

5 Discussions of the Group Classifications

By inspecting relations (14) and Table 2 we deduce easily that point symmetries of the system (4)–(5) do not produce any contact symmetry of the equation (1) and any nonlocal symmetry (potential symmetry) of the equation (6). Moreover, after observing that the following relations

$$\bar{\xi}^1 = \tilde{\xi}^1 = \xi^1 = 2\, a_5\, t + a_1,$$
$$\bar{\xi}^2 = \tilde{\xi}^2 = \xi^2 = a_5\, x + a_2,$$
$$\eta^1 = \tilde{\eta} = (a_6 - a_5)\, u + a_7$$

hold, it is worthwhile noticing that we can obtain the complete group classification of the equation (6) simply projecting in the (t, x, u)-space the operators appearing in (14) and in Table 2. Consequently, a symmetry of any one of the system (4)–(5) and the PDE (6) induces a symmetry of the other. After that, we will demonstrate the following statement.

Theorem 1. *For any f, a point symmetry admitted by* (1) *defines a point symmetry admitted by* (4)–(5) *and viceversa.*

Proof. Starting from the classification of the equation (1), in order to link the coordinates of the operators $\bar{X}$ to that ones of X, following the procedure concerning with the change of variables showed in [10, 11], we require the invariance of the transformation $u = w_x$, $v = w_t$, with respect to the operator

$$X^* = \eta\frac{\partial}{\partial w} + \zeta_1\frac{\partial}{\partial w_t} + \zeta_2\frac{\partial}{\partial w_x} + \eta^1\frac{\partial}{\partial u} + \eta^2\frac{\partial}{\partial v}.$$

That is, we perform the invariance conditions

$$X^*(u - w_x)|_{w_x=u,\, w_t=v} = 0, \qquad X^*(v - w_t)|_{w_x=u,\, w_t=v} = 0. \tag{15}$$

From (15) it follows

$$\eta^1 = \zeta_2|_{w_x=u,\, w_t=v} = (a_6 - a_5)u + a_7,$$
$$\eta^2 = \zeta_1|_{w_x=u,\, w_t=v} = (a_6 - 2\,a_5)v + a_4,$$

which give the remaining coordinates of the operator $\bar{X}$.

Conversely, starting from the classification of the system (4)–(5), taking (10) and (11) into account, from (15) we obtain

$$\zeta_2 = \eta_x + \eta_w\, w_x - a_5\, w_x = \eta^1\big|_{u=w_x,\, v=w_t} = (a_6 - a_5)\, w_x + a_7, \tag{16}$$
$$\zeta_1 = \eta_t + \eta_w\, w_t - 2\, a_5\, w_t = \eta^2\big|_{u=w_x,\, v=w_t} = (a_6 - 2\, a_5)\, w_t + a_4. \tag{17}$$

From (16) and (17) the relation (13) of η follows in a simple way.

■

So, we can conclude by affirming that:

Theorem 2. *The classifications of* (1), (4)–(5) *and* (6) *are identical in the sense that, for any f, a point symmetry admitted by any one of them induces a point symmetry admitted by the remaining two ones.*

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Symmetry Analysis of the Thermal Diffusion Equations in the Planar Case

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The model for convective motion of binary mixture with thermal diffusion effect is considered. It is shown that the equations admit infinite Lie symmetry algebra L that can be represented as the semi-direct sum of four-dimensional subalgebra and the infinite ideal spanned by three infinite-dimensional generators. The first and second order optimal systems of subalgebras for the algebra L are constructed.

1 Introduction

Thermal diffusion is a transport of matter associated with a thermal gradient. It may occur in mixtures of liquids and gases. As a result of thermal gradient, concentration gradients appear in the mixture. These gradients produce ordinary diffusion. A steady state is reached when the separating effect of thermal diffusion is balanced by the remixing effect of ordinary diffusion. As a result, partial separation is observed. Experimental results have shown in most cases a 'normal' behavior, i.e., the heavier components in the cold region, and the lighter components in the hot region. Also, there are systems with 'abnormal' behavior, where the situation is opposite. Thermal diffusion has various applications, such as separation of different mixtures, crystal growth, flows in oceans, and so on.

In this paper we consider convective motion of binary mixture supposing that its density linearly depends on concentration of the lighter component and temperature, $\rho = \rho_0(1-\beta_1 T-\beta_2 C)$. Here ρ_0 is the mixture density at mean values of temperature and concentration, T and C are the temperature and concentration variations that are supposed to be small, β_1 is the thermal expansion coefficient of the mixture, and β_2 is the concentration coefficient of density ($\beta_2 > 0$). The equations of motion in the Oberbeck–Boussinesq approximation have the form [1]

$$\begin{aligned}
&\boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\frac{1}{\rho_0}\nabla p + \nu\Delta\boldsymbol{u} - \mathbf{g}(\beta_1 T + \beta_2 C),\\
&T_t + \boldsymbol{u}\cdot\nabla T = \chi\Delta T,\\
&C_t + \boldsymbol{u}\cdot\nabla C = d\Delta C + \alpha d\Delta T,\\
&\mathrm{div}\boldsymbol{u} = 0,
\end{aligned} \tag{1}$$

where $\boldsymbol{u}$ is the velocity vector, p is the difference between actual and hydrostatic pressure, ν is the kinematic viscosity, χ is the thermal diffusivity, d is the diffusion coefficient, α is the thermal diffusion parameter, and $\mathbf{g}$ is the gravitational acceleration vector. We suppose that all characteristics of the medium are constant and correspond to the mean values of temperature and concentration.

Further, we consider the case of plane motion. The following notation is used: $\boldsymbol{x} = (x^1, x^2)$, $\boldsymbol{u} = (u^1, u^2)$, $\mathbf{g} = (0, -\mathrm{g})$, where g is the gravitational acceleration. In what follows we find symmetries for equations (1) and construct the first and second order optimal systems of subalgebras for the admissible Lie symmetry algebra.

2 The Admissible Lie Symmetry Algebra

As calculations show, equations (1) admit infinite Lie symmetry algebra that can be represented as the semi-direct sum $L = L_4 \oplus L_\infty$. The finite subalgebra L_4 is spanned by the generators

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{1}{\beta_1}\frac{\partial}{\partial T} - \frac{1}{\beta_2}\frac{\partial}{\partial C}, \qquad X_3 = \rho_0 \mathrm{g} x^2 \frac{\partial}{\partial p} + \frac{1}{\beta_2}\frac{\partial}{\partial C},$$

$$X_4 = 2t\frac{\partial}{\partial t} + x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2} - u^1\frac{\partial}{\partial u^1} - u^2\frac{\partial}{\partial u^2} - 2p\frac{\partial}{\partial p} - 3T\frac{\partial}{\partial T} - 3C\frac{\partial}{\partial C},$$

while the infinite ideal L_∞ has the basis

$$H_1(f^1(t)) = f^1(t)\frac{\partial}{\partial x^1} + f^1_t(t)\frac{\partial}{\partial u^1} - \rho_0 x^1 f^1_{tt}(t)\frac{\partial}{\partial p},$$
$$H_2(f^2(t)) = f^2(t)\frac{\partial}{\partial x^2} + f^2_t(t)\frac{\partial}{\partial u^2} - \rho_0 x^2 f^2_{tt}(t)\frac{\partial}{\partial p},$$
$$H_0(f^0(t)) = f^0(t)\frac{\partial}{\partial p},$$

where $f^i(t)$, $f^0(t)$ are smooth arbitrary functions. If the parameters entering the system satisfy the condition $\alpha = \beta_1(d-\chi)/\beta_2 d$, $d \neq \chi$, then the equations also admit the generator

$$X_5 = T\frac{\partial}{\partial T} - \frac{\beta_1}{\beta_2}T\frac{\partial}{\partial C}.$$

Further, it is assumed that the above relation does not hold and the generator X_5 is not admitted. The system (1) is also invariant under the discrete symmetries

$$d_1: \quad \widetilde{x^1} = -x^1, \quad \widetilde{u^1} = -u^1; \tag{2}$$

$$d_2: \quad \widetilde{x^2} = -x^2, \quad \widetilde{u^2} = -u^2, \quad \widetilde{T} = -T, \quad \widetilde{C} = -C.$$

3 The Optimal Systems of Subalgebras

To find all essentially different invariant solutions (i.e. the solutions that cannot be carried over into each other by the admissible transformations), we need to construct the optimal system of subalgebras for the Lie algebra L [2–4] (see also [5,6] for the examples of constructing optimal systems for infinite-dimensional algebras).

Let us introduce the following notation: $f^0(t)$, $\boldsymbol{f}(t) = (f^1(t), f^2(t))$ are smooth arbitrary functions, $H(\boldsymbol{f}) = H_1(f^1) + H_2(f^2)$ is the operator of the ideal L_∞. The functions $g^0(t)$, $h^0(t)$, $\boldsymbol{g}(t)$, $\boldsymbol{h}(t)$ are introduced in the same way. Using this notation, we calculate the commutators for the basic generators (see Table 1).

To construct the optimal system, it is necessary to find the group of inner automorphisms Int$L$. This group is generated by the automorphisms $A_i(a_i)$, $A^H(\boldsymbol{h})$, $A_0^H(h^0)$ corresponding to the basic generators X_i, $H(\boldsymbol{f})$, $H_0(f^0)$. Here $i = 1, 2, 3, 4$ and a_i, $\boldsymbol{h}(t)$, $h^0(t)$ are parameters. The group Int$L$ transforms the coordinates of general operator $X = k_1X_1 + \cdots + k_4X_4 + H(\boldsymbol{f}) + H_0(f^0)$ by the formula

$$\mathrm{Int}L: \quad (k_1, \ldots, k_4, \boldsymbol{f}(t), f^0(t)) \longrightarrow (\widetilde{k_1}, \ldots, \widetilde{k_4}, \widetilde{\boldsymbol{f}}(t), \widetilde{f^0}(t)).$$

The group action of Int$L$ is presented in Table 2, where the following notation is used:

$$p^0(t) = k_1h_t^0 + k_4(2th_t^0 + 2h^0), \qquad \boldsymbol{q}(t) = k_1\boldsymbol{h}_t + k_4(2t\boldsymbol{h}_t - \boldsymbol{h}),$$

$$q^0(t) = \rho_0\Big[\frac{k_1}{2}(\boldsymbol{h}_{ttt}\boldsymbol{h} - \boldsymbol{h}_t\boldsymbol{h}_{tt}) - k_3 \mathrm{g}h^2 + k_4(2\boldsymbol{h}_{tt}\boldsymbol{h} + t\boldsymbol{h}_{ttt}\boldsymbol{h} - t\boldsymbol{h}_{tt}\boldsymbol{h}_t) + \boldsymbol{f}_{tt}\boldsymbol{h} - \boldsymbol{f}\boldsymbol{h}_{tt}\Big].$$

Discrete automorphisms $A_i^d(\delta_i)$ generated by the discrete symmetries d_i, $i = 1, 2$ (see (2)) are also given in that table. Parameters δ_i have values $\{0, 1\}$. The automorphism $A_i^d(0)$ is an identity transformation.

At the first step, the optimal system for the finite Lie algebra L_4 is constructed. This algebra is decomposed as the semi-direct sum $L_4 = J \oplus N$ of its proper ideal $J = \{X_1, X_2\}$ and the subalgebra $N = \{X_3, X_4\}$. First of all, the optimal system ΘN is obtained. Then we find the optimal system ΘL_4, which is presented in Table 3. The numbers of subalgebras are specified in the first column, while the basic generators are given in the second column and denoted by their numbers. For example, the symbol $\lambda 2 + 3$ stands for $\lambda X_2 + X_3$, where λ is a real parameter. The numbers of normalizers are given in the third column. The equality sign marks self-normalized subalgebras. When constructing the optimal system, we used the discrete automorphisms $A_i^d(\delta_i)$.

At the second step, the first and second order optimal systems for the Lie algebra L are constructed. This algebra contains the infinite ideal L_∞ with the general operator

$$H(\bar{f}) = H_1(f^1) + H_2(f^2) + H_0(f^0), \qquad \bar{f}(t) = (f^1(t), f^2(t), f^0(t)).$$

Table 1. Commutators for the basic generators of Lie algebra L

$[\downarrow, \rightarrow]$	X_1	X_2	X_3	X_4	$H(\boldsymbol{g})$	$H_0(g^0)$
X_1	0	0	0	$2X_1$	$H(\boldsymbol{g}_t)$	$H_0(g_t^0)$
X_2	0	0	0	$-3X_2$	0	0
X_3	0	0	0	$-3X_3$	$H_0(-\rho_0 \mathrm{g} g^2)$	0
X_4	$-2X_1$	$3X_2$	$3X_3$	0	$H(2t\boldsymbol{g}_t - \boldsymbol{g})$	$H_0(2tg_t^0 + 2g^0)$
$H(\boldsymbol{f})$	$H(-\boldsymbol{f}_t)$	0	$H_0(\rho_0 \mathrm{g} f^2)$	$H(-2t\boldsymbol{f}_t + \boldsymbol{f})$	$H_0(\rho_0 \boldsymbol{f}_{tt}\boldsymbol{g} - \rho_0 \boldsymbol{f}\boldsymbol{g}_{tt})$	0
$H_0(f^0)$	$H_0(-f_t^0)$	0	0	$H_0(-2tf_t^0 - 2f^0)$	0	0

Table 2. Inner automorphisms of Lie algebra L

	$\widetilde{k}_1$	$\widetilde{k}_2$	$\widetilde{k}_3$	$\widetilde{k}_4$	$\widetilde{\boldsymbol{f}}(t)$	$\widetilde{f^0}(t)$
$A_1(a_1)$	$k_1 - 2a_1k_4$	k_2	k_3	k_4	$\boldsymbol{f}(t - a_1)$	$f^0(t - a_1)$
$A_2(a_2)$	k_1	$k_2 + 3a_2k_4$	k_3	k_4	$\boldsymbol{f}(t)$	$f^0(t)$
$A_3(a_3)$	k_1	k_2	$k_3 + 3a_3k_4$	k_4	$\boldsymbol{f}(t)$	$f^0(t) + a_3\rho_0 \mathrm{g} f^2(t)$
$A_4(a_4)$	$e^{2a_4}k_1$	$e^{-3a_4}k_2$	$e^{-3a_4}k_3$	k_4	$e^{a_4}\boldsymbol{f}(e^{-2a_4}t)$	$e^{-2a_4}f^0(e^{-2a_4}t)$
$A^H(\boldsymbol{h})$	k_1	k_2	k_3	k_4	$\boldsymbol{f}(t) + \boldsymbol{q}(t)$	$f^0(t) + q^0(t)$
$A_0^H(h^0)$	k_1	k_2	k_3	k_4	$\boldsymbol{f}(t)$	$f^0(t) + p^0(t)$
$A_1^d(\delta_1)$	k_1	k_2	k_3	k_4	$(-1)^{\delta_1}f^1(t), \quad f^2(t)$	$f^0(t)$
$A_2^d(\delta_2)$	k_1	$(-1)^{\delta_2}k_2$	$(-1)^{\delta_2}k_3$	k_4	$f^1(t), \quad (-1)^{\delta_2}f^2(t)$	$f^0(t)$

To find the first order optimal system, we need to classify two classes of subalgebras:

1) $\{H(\bar{f})\}$,

2) $\{P + H(\bar{f})\}$, $\{P\} \in \Theta_1 L_4$.

The first class belongs to the infinite ideal L_∞, while the second class has one-dimensional intersection with L_4. The subalgebras with basic operator P are taken from the first order optimal system $\Theta_1 L_4$ (see Table 3). The optimal system $\Theta_1 L$ is presented in Table 6.

To obtain the second order optimal system, it is necessary to classify three classes of subalgebras:

1) $\{H(\bar{f}),\, H(\bar{g})\}$,

2) $\{P + H(\bar{f}),\, H(\bar{g})\}$, $\{P\} \in \Theta_1 L_4$,

3) $\{P + H(\bar{f}),\, Q + H(\bar{g})\}$, $\{P,\, Q\} \in \Theta_2 L_4$.

The first class belongs to L_∞, while the second and third classes have one and two-dimensional intersections with L_4 respectively. The subalgebras $\{P, Q\}$ are taken from the second order optimal system $\Theta_2 L_4$ (see Table 3).

The infinite subalgebras from the optimal system $\Theta_2 L$ are presented in Table 4. Their basic generators depend on arbitrary functions that satisfy the equations given in the third column. In most cases these equations can be solved for the desired functions. Nevertheless, such form of presenting the results is preferable since the solution may be obtained for different functions entering into the equations.

The finite subalgebras from $\Theta_2 L$ are given in Table 5. The constants λ, μ, γ, σ, c in Tables 4, 5 have any real values unless otherwise indicated.

Table 3. The optimal system of subalgebras ΘL_4

i	Basis	NorF_i	i	Basis	NorF_i
1	1, 2, 3, 4	=1	11	1, 4	=11
2	1, 2, 3	1	12	2, 4	=12
3	1, 2, 4	=3	13	$\lambda 2 + 3$, 4	=13
4	2, 3, 4	=4	14	1	1
5	1, $\lambda 2 + 3$, 4	=5	15	2	1
6	1, 2	1	16	$\lambda 2 + 3$	1
7	2, 3	1	17	$1 + 2$	2
8	1, $\lambda 2 + 3$	1	18	$1 + \lambda 2 + 3$	2
9	$1 + 2$, $\lambda 1 + 3$	2	19	4	=19
10	$1 + 3$, 2	2	20	0	1

Table 4. Infinite subalgebras from the second order optimal system $\Theta_2 L$

i	Basis	Remark
1	$H_0(f^0),\ H_0(g^0)$	
2	$H_1(f^1)+H_2(f^2),\ H_0(g^0)$	
3	$H_2(f^2),\ H_2(g^2)+H_0(g^0)$	$f^2_{tt}g^2-f^2g^2_{tt}=0,\quad g^2\not\equiv cf^2,\quad g^0\not\equiv 0$
4	$H_1(f^1)+H_2(\lambda f^1),\ H_1(g^1)+H_2(\lambda g^1)+H_0(g^0)$	$f^1_{tt}g^1-f^1g^1_{tt}=0,\quad g^1\not\equiv cf^1,\quad g^0\not\equiv 0,\quad \lambda\geq 0$
5	$H_1(f^1)+H_2(f^2),\ H_1(g^1)+H_2(g^2)$	$f^1_{tt}g^1-f^1g^1_{tt}+f^2_{tt}g^2-f^2g^2_{tt}=0$
6	$X_2+H_0(f^0),\ H_0(g^0)$	$f^0\not\equiv 0$
7	$X_2+H_0(f^0),\ H_1(g^1)+H_2(g^2)$	$f^0\not\equiv 0$
8	$X_2+H_1(f^1)+H_2(f^2),\ H_0(g^0)$	
9	$X_2+H_2(f^2),\ H_2(g^2)+H_0(g^0)$	$f^2_{tt}g^2-f^2g^2_{tt}=0,\quad g^2\not\equiv cf^2,\quad f^2,\, g^0\not\equiv 0$
10	$X_2+H_1(f^1)+H_2(\lambda f^1),\ H_1(g^1)+H_2(\lambda g^1)+H_0(g^0)$	$f^1_{tt}g^1-f^1g^1_{tt}=0,\quad g^1\not\equiv cf^1,\quad f^1,\, g^0\not\equiv 0$
11	$X_2+H_1(f^1)+H_2(f^2),\ H_1(g^1)+H_2(g^2)$	$f^1_{tt}g^1-f^1g^1_{tt}+f^2_{tt}g^2-f^2g^2_{tt}=0$
12	$\lambda X_2+X_3,\ H_0(g^0)$	
13	$\lambda X_2+X_3+H_0(f^0),\ H_1(g^1)$	$f^0_{tt}g^1-f^0g^1_{tt}\not\equiv 0$
14	$\lambda X_2+X_3+H_1(f^1)+H_2(f^2),\ H_0(g^0)$	
15	$\lambda X_2+X_3+H_1(f^1)+H_2(f^2),$ $H_1(g^1)+H_2(g^2)+H_0(g^0)$	$f^1_{tt}g^1-f^1g^1_{tt}=0,\quad (f^2_{tt}-\mathrm{g})g^2-f^2g^2_{tt}=0,$ $f^1g^2-f^2g^1=0,\quad g^2,\, g^0\not\equiv 0$
16	$\lambda X_2+X_3+H_1(f^1)+H_2(f^2),\ H_1(g^1)+H_2(g^2)$	$f^1_{tt}g^1-f^1g^1_{tt}+f^2_{tt}g^2-f^2g^2_{tt}-\mathrm{g}g^2=0$
17	$X_2+H_0(f^0),\ X_3+H_1(g^1)+H_2(g^2)$	$f^0\not\equiv 0$
18	$X_2+H_1(f^1),\ X_3+H_0(g^0)$	$f^1_{tt}g^0-f^1g^0_{tt}\not\equiv 0$
19	$X_2+H_1(f^1)+H_2(f^2),$ $X_3+H_1(g^1)+H_2(g^2)+H_0(g^0)$	$f^1_{tt}g^1-f^1g^1_{tt}=0,\quad f^2_{tt}g^2+f^2(\mathrm{g}-g^2_{tt})=0,$ $f^1g^2-f^2g^1=0,\quad f^2,\, g^0\not\equiv 0$
20	$X_2+H_1(f^1)+H_2(f^2),\ X_3+H_1(g^1)+H_2(g^2)$	$f^1_{tt}g^1-f^1g^1_{tt}+f^2_{tt}g^2-f^2g^2_{tt}+\mathrm{g}f^2=0$

Table 5. Finite subalgebras from the second order optimal system $\Theta_2 L$

i	Basis	Remark
1	$X_1,\ H_0(1)$	
2	$X_1,\ H_2(1)$	
3	$X_1,\ H_1(1)+H_0(1)$	
4	$X_1,\ H_1(1)+H_2(\lambda)$	$\lambda \geq 0$
5	$X_1,\ H_0(e^{\pm t})$	
6	$X_1,\ H_2(e^{\pm t})$	
7	$X_1,\ H_1(e^{\pm t})+H_2(\lambda e^{\pm t})$	$\lambda \geq 0$
8	$X_1+X_2,\ H_0(1)$	
9	$X_1+X_2,\ H_1(1)+H_0(\lambda)$	$\lambda \geq 0$
10	$X_1+X_2,\ H_1(\lambda)+H_2(1)$	$\lambda \geq 0$
11	$X_1+\lambda X_2,\ H_0(e^{\pm t})$	$\lambda > 0$
12	$X_1+\lambda X_2,\ H_2(e^{\pm t})$	$\lambda > 0$
13	$X_1+\lambda X_2,\ H_1(e^{\pm t})+H_2(\mu e^{\pm t})$	$\lambda > 0,\ \mu \geq 0$
14	$X_1+\lambda X_2+X_3,\ H_1(1)+H_0(\mu)$	$\mu \geq 0$
15	$X_1+\lambda X_2+X_3,\ H_1(\mu)+H_2(1)+H_0(\rho_0 \mathrm{g} t)$	$\mu \geq 0$
16	$X_1+\lambda X_2+\mu X_3,\ H_0(e^{\pm t})$	$\mu > 0$
17	$X_1+\lambda X_2+\mu X_3,\ H_1(e^{\pm t})$	$\mu > 0$
18	$X_1+\lambda X_2+\mu X_3,$ $H_1(\sigma e^{\pm t})+H_2(e^{\pm t})+H_0(\mu\rho_0 \mathrm{g} t e^{\pm t})$	$\mu > 0,\ \sigma \geq 0$
19	$X_4,\ H_0(t^\gamma)$	$\gamma \neq -1$
20	$X_4,\ H_2(t^\gamma)$	$\gamma \neq 1/2$
21	$X_4,\ H_1(t^\gamma)+H_2(\lambda t^\gamma)$	$\gamma \neq 1/2,\ \lambda \geq 0$
22	$X_4,\ H_0(1/t)$	
23	$X_4,\ H_2(\sqrt{t})+H_0(\lambda/t)$	$\lambda \geq 0$
24	$X_4,\ H_1(\sqrt{t})+H_2(\mu\sqrt{t})+H_0(\lambda/t)$	$\lambda \geq 0,\ \mu \geq 0$
25	$X_1,\ X_2$	
26	$X_1,\ X_2+H_0(1)$	
27	$X_1,\ X_2+H_1(1)+H_0(\lambda)$	$\lambda \geq 0$
28	$X_1,\ X_2+H_1(\lambda)\pm H_2(1)$	$\lambda \geq 0$
29	$X_1,\ \lambda X_2+X_3$	
30	$X_1,\ \lambda X_2+X_3 \pm H_2(1)$	

Table 5. Continue

i	Basis	Remark
31	$X_1,\ \lambda X_2 + X_3 + H_1(1) + H_2(\mu)$	
32	$X_1 + X_2,\ \lambda X_1 + X_3 + H_1(\mu) + H_2(\sigma)$	$\mu \geq 0$
33	$X_1 + X_3,\ X_2 + H_1(\lambda) + H_0(\mu)$	$\lambda \geq 0$
34	$X_1 + X_3,\ X_2 + H_1(\lambda) + H_2(\mu) + H_0(\mu\rho_0 \mathrm{g}t)$	$\lambda \geq 0,\ \mu \neq 0$
35	$X_1,\ X_4$	
36	$X_2 + H_0(\lambda\sqrt{t}),\ X_4$	$\lambda > 0$
37	$X_2 + H_1(\lambda t^2) + H_2(\mu t^2),\ X_4$	$\lambda \geq 0$
38	$\lambda X_2 + X_3 + H_2(\frac{4}{9}\mathrm{g}t^2) + H_0(\mu\sqrt{t}),\ X_4$	$\mu > 0$
39	$\lambda X_2 + X_3 + H_1(\mu t^2) + H_2(\sigma t^2),\ X_4$	$\mu \geq 0$

Table 6. Optimal system of subalgebras $\Theta_1 L$

i	Basis	Remark
1	X_1	
2	$X_1 + X_2$	
3	$X_1 + \lambda X_2 + X_3$	
4	X_4	
5	$H_0(f^0)$	$f^0 \not\equiv 0$
6	$X_2 + H_0(f^0)$	$f^0 \not\equiv 0$
7	$H_1(f^1) + H_2(f^2)$	
8	$X_2 + H_1(f^1) + H_2(f^2)$	
9	$\lambda X_2 + X_3 + H_1(f^1) + H_2(f^2)$	

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Discrete Symmetries of the Black–Scholes Equation

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The full automorphism group of the Lie algebra associated to the Black–Scholes equation is computed and all the discrete symmetries of the equation are determined.

1 Introduction

Discrete symmetries of (partial) differential equations can be used in many ways. They map solutions to (possibly new) solutions. They may be used to create efficient numerical methods for the computation of solutions to boundary-value problems. Discrete and continuous groups of symmetries determine the nature of bifurcations in nonlinear dynamical systems. Equivariant bifurcation theory describes the effects of symmetries, but it may yield misleading results unless *all* the generators of the point symmetries, discrete and continuous, of the dynamical systems are known [2].

In general it is straightforward to find all one-parameter Lie groups of symmetries of a given system using techniques developed by Sophus Lie more than a century ago [5]. Yet until recently no method for finding all discrete symmetries was known. The main difficulty is that the determining equations for discrete symmetries typically form a highly-coupled nonlinear system.

A new approach to the problem of finding discrete point symmetries of a partial differential equation has recently been described by Hydon ([3,4]). The technique is based on the observation that every point symmetry yields an automorphism of the Lie algebra of Lie point symmetry generators. This results in a set of auxiliary equations that are satisfied by all point symmetries. These equations can be considerably simplified by factoring out the inner automorphisms of the Lie algebra. After that they can be solved by standard methods and their solutions are precisely the discrete symmetries we are looking for.

The present paper applies the whole procedure to the famous Black–Scholes partial differential equation (1). Unlike in Hydon [4] the full automorphism group of the Lie algebra is determined using purely algebraic techniques, such as construction of generators' centralizers and Lie algebra's radical. The final results are the description of the outer automorphism group, respectively, of the discrete symmetry group associated to Black–Scholes PDE.

2 Lie Symmetries of the Black–Scholes Equation

Consider the partial differential equation

$$u_t + \frac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu = 0, \tag{1}$$

where A, B, C are constant parameters of the model[1], $A \neq 0$, and assume that $\mathcal{D} \equiv B - A^2/2 \neq 0$.

The Lie (point) symmetries of equation (1) were computed by Gazizov and Ibragimov in [1]. They found an infinite-dimensional Lie algebra of infinitesimal symmetries generated by the following operators:

$$Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = x\frac{\partial}{\partial x}, \quad Y_3 = 2t\frac{\partial}{\partial t} + (\ln x + \mathcal{D}t)x\frac{\partial}{\partial x} + 2Ctu\frac{\partial}{\partial u},$$
$$Y_4 = A^2tx\frac{\partial}{\partial x} + (\ln x - \mathcal{D}t)u\frac{\partial}{\partial u},$$
$$Y_5 = 2A^2t^2\frac{\partial}{\partial t} + 2A^2tx\ln x\frac{\partial}{\partial x} + [(\ln x - \mathcal{D}t)^2 + 2A^2Ct^2 - A^2t]u\frac{\partial}{\partial u},$$
$$Y_6 = u\frac{\partial}{\partial u}, \quad Y_\alpha = \alpha(t,x)\frac{\partial}{\partial u},$$

where $\alpha(\cdot,\cdot)$ is an arbitrary solution of equation (1).

Consider the finite dimensional Lie algebra $\mathcal{L}$ generated by the first six operators. In order to simplify the computations, we change the set of generators of the Lie algebra as follows:

$$X_1 = \frac{1}{A^2}(Y_1 + \mathcal{D}Y_2 + CY_6), \quad X_2 = Y_2, \quad X_3 = Y_3 - \frac{1}{2}Y_6, \quad X_4 = Y_4,$$
$$X_5 = \frac{1}{2}Y_5, \quad X_6 = Y_6.$$

Their commutator table becomes

	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	0	$2X_1$	X_2	X_3	0
X_2	0	0	X_2	X_6	X_4	0
X_3	$-2X_1$	$-X_2$	0	X_4	$2X_5$	0
X_4	$-X_2$	$-X_6$	$-X_4$	0	0	0
X_5	$-X_3$	$-X_4$	$-2X_5$	0	0	0
X_6	0	0	0	0	0	0

3 Structure of the Lie Algebra

3.1 Centralizer Structure

It is known that for every operator $X \in \mathcal{L}$ its associated adjoint action

$$\mathcal{L} \ni Y \mapsto \mathrm{ad}_X(Y) = [Y, X] \in \mathcal{L}$$

[1]In the classical Black–Scholes model, $A = \sigma$, $B = r - q$ and $C = r$.

is a linear space endomorphism, its kernel being the centralizer $C_{\mathcal{L}}(X)$ of X and its image being the subspace $[X, \mathcal{L}]$. As a consequence of the Fundamental Isomorphism Theorem we have

$$\dim(C_{\mathcal{L}}(X)) = \operatorname{codim}[X, \mathcal{L}] \quad \forall X \in \mathcal{L}$$

Using the commutator table we get

$$[X_1, \mathcal{L}] = \langle X_1, X_2, X_3 \rangle, \qquad [X_2, \mathcal{L}] = [X_4, \mathcal{L}] = \langle X_2, X_4, X_6 \rangle,$$
$$[X_3, \mathcal{L}] = \langle X_1, X_2, X_4, X_5 \rangle, \quad [X_5, \mathcal{L}] = \langle X_4, X_5, X_6 \rangle, \quad [X_6, \mathcal{L}] = 0.$$

It becomes trivial to list all the centralizers of the generators of the Lie algebra

$$C_{\mathcal{L}}(X_1) = C_{\mathcal{L}}(X_2) = \langle X_1, X_2, X_6 \rangle, \quad C_{\mathcal{L}}(X_3) = \langle X_3, X_6 \rangle,$$
$$C_{\mathcal{L}}(X_4) = C_{\mathcal{L}}(X_5) = \langle X_4, X_5, X_6 \rangle, \quad C_{\mathcal{L}}(X_6) = \mathcal{L}$$

and its center

$$Z(\mathcal{L}) = \bigcap_{i=1}^{6} C_{\mathcal{L}}(X_i) = \langle X_6 \rangle.$$

3.2 Radical

All results and notation in this section are based on Ovsiannikov [6].

The Lie algebra $\mathcal{L}$ can be written as a direct sum

$$\mathcal{L} = \mathcal{R} \bigoplus \mathcal{N}, \quad \mathcal{R} = \langle X_2, X_4, X_6 \rangle, \quad \mathcal{N} = \langle X_1, X_3, X_5 \rangle.$$

$\mathcal{R}$ is a solvable ideal of the Lie algebra, its derived series being

$$\mathcal{R} = \langle X_2, X_4, X_6 \rangle \supset \mathcal{R}^{(1)} = \langle X_6 \rangle \supset \mathcal{R}^{(2)} = \{0\}.$$

On the other hand $\mathcal{N}$ is a nonsolvable subalgebra, its derived series being stationary. By Lemma 1 p. 177 in [6] $\mathcal{N}$ is semisimple. By the Structural Theorem p. 186 in [6] it is a simple Lie algebra. We conclude that $\mathcal{R}$ is the radical of $\mathcal{L}$ and $\mathcal{N}$ is a corresponding Levi factor ([6], p. 178).

3.3 Adjoint Action and Inner Automorphisms

The adjoint action of the one-parameter group generated by X_i is defined as follows:

$$\operatorname{ad}(X_i) : \mathcal{L} \to \mathcal{L}, \quad \operatorname{ad}(X_i)X = [X, X_i] \quad \forall X \in \mathcal{L}, \ i = 1, 2, \ldots, 6.$$

For every $\lambda \in \mathbf{R}$ and $i \in \{1, 2, \ldots, 6\}$ the exponential map

$$\exp(\lambda \operatorname{ad}(X_i)) : \mathcal{L} \to \mathcal{L}$$

is an inner automorphism of the Lie algebra and all inner automorphisms induced by $X_1, X_2, \ldots, X_6$ generate the inner automorphism group $\mathrm{Inn}(\mathcal{L})$. Their action $X_{ij}(\lambda) = \exp(\lambda \mathrm{ad}(X_i))X_j$ on the Lie algebra generators can be described as follows: $X_{ij}(\lambda) = X_j$ for all $i, j \in \{1, 2, \ldots, 6\}$, with the following exceptions:

$$X_{13}(\lambda) = X_3 - 2\lambda X_1, \quad X_{14}(\lambda) = X_4 - \lambda X_2, \quad X_{15}(\lambda) = X_5 - \lambda X_3 + \lambda^2 X_1,$$
$$X_{23}(\lambda) = X_3 - \lambda X_2, \quad X_{24}(\lambda) = X_4 - \lambda X_6, \quad X_{25}(\lambda) = X_5 - \lambda X_4 + \frac{\lambda^2}{2} X_6,$$
$$X_{31}(\lambda) = e^{2\lambda} X_1, \quad X_{32}(\lambda) = e^{\lambda} X_2, \quad X_{34}(\lambda) = e^{-\lambda} X_4, \quad X_{35}(\lambda) = e^{-2\lambda} X_5,$$
$$X_{41}(\lambda) = X_1 + \lambda X_2 + \frac{\lambda^2}{2} X_6, \quad X_{42}(\lambda) = X_2 + \lambda X_6, \quad X_{43}(\lambda) = X_3 + \lambda X_4,$$
$$X_{51}(\lambda) = X_1 + \lambda X_3 + \lambda^2 X_5, \quad X_{52}(\lambda) = X_2 + \lambda X_4, \quad X_{53}(\lambda) = X_3 + 2\lambda X_5.$$

Remark 1. Let α be an inner automorphism of $\mathcal{L}$. Then $\alpha(X_6) = X_6$. Moreover the i-th coordinate of $\alpha(X_i)$ is positive for any $i \in \{1, 2, 3, 4, 5\}$.

3.4 The Full Automorphism Group

We find all the automorphisms of the Lie algebra $\mathcal{L}$ that are pairwise nonequivalent with respect to the inner automorphism group.

Let $\theta : \mathcal{L} \to \mathcal{L}$ such an automorphism. Then it has to preserve the center $Z(\mathcal{L})$. Hence there exists a nonzero number Δ such that $\theta(X_6) = \Delta X_6$. Denote $\delta = \sqrt{|\Delta|}$ and $\varepsilon = \mathrm{sgn}(\Delta)$ so that $\theta(X_6) = \varepsilon\delta^2$. Any automorphism must preserve the radical $\mathcal{R}$. Hence

$$\theta(X_2) = b_{22}X_2 + b_{24}X_4 + b_{26}X_6, \quad \theta(X_4) = b_{42}X_2 + b_{44}X_4 + b_{46}X_6,$$

where the coefficients are such that

$$b_{22}b_{44} - b_{24}b_{42} = \varepsilon\delta^2.$$

If we premultiply η by $\exp(\lambda \mathrm{ad}(X_1))$, the resulting automorphism $\bar{\theta}$ satisfies

$$\bar{\theta}(X_2) = (b_{22} - \lambda b_{24})X_2 + b_{24}X_4 + b_{26}X_6,$$
$$\bar{\theta}(X_4) = (b_{42} - \lambda b_{44})X_2 + b_{44}X_4 + b_{46}X_6.$$

Case 1. $b_{44} \neq 0$. We choose $\lambda = b_{42}/b_{44}$. Then we premultiply $\bar{\theta}$ by $\exp(\lambda\, \mathrm{ad}(X_3))$, where $\lambda = \log(|b_{44}|/\delta)$. The resulting automorphism satisfies

$$\tilde{\theta}(X_2) = \varepsilon\varepsilon'\delta X_2 + \frac{b_{24}\delta}{|b_{44}|} X_4 + b_{26}X_6,$$
$$\tilde{\theta}(X_4) = \varepsilon'\delta X_4 + b_{46}X_6,$$

where $\varepsilon' = \mathrm{sgn}(b_{44})$. Next we can remove the coefficients b_{24}, b_{26} and b_{46} by premultiplying $\tilde{\theta}$ by appropriate inner automorphisms induced by X_5, X_4, X_2.

Finally we get that when $b_{44} \neq 0$, the automorphism θ is equivalent to an automorphism φ that satisfies:

$$\varphi(X_2) = \varepsilon\varepsilon'\delta X_2, \quad \varphi(X_4) = \varepsilon'\delta X_4.$$

Automorphism φ has to preserve the centralizer structure, that is,

$$\langle \varphi(X_1), \varphi(X_2), \varphi(X_6)\rangle = \varphi(C_{\mathcal{L}}(X_2)) = C_{\mathcal{L}}(X_2) = C_{\mathcal{L}}(\varphi(X_2)) = \langle X_1, X_2, X_6\rangle,$$
$$\langle \varphi(X_4), \varphi(X_5), \varphi(X_6)\rangle = \varphi(C_{\mathcal{L}}(X_4)) = C_{\mathcal{L}}(\varphi(X_4)) = C_{\mathcal{L}}(X_4) = \langle X_4, X_5, X_6\rangle.$$

Hence it satisfies

$$\varphi(X_1) = b_{11}X_1 + b_{12}X_2 + b_{16}X_6,$$
$$\varphi(X_5) = b_{54}X_4 + b_{55}X_5 + b_{56}X_6,$$
$$\varphi(X_3) = \varphi([X_1, X_5]) = b_{11}b_{54}X_2 + b_{11}b_{55}X_3 + b_{12}b_{55}X_4 + b_{12}b_{54}X_6.$$

Equations $[X_1, X_4] = X_2$, $[X_2, X_5] = X_4$, $[X_1, X_3] = 2X_1$, $[X_3, X_5] = 2X_5$ give $b_{12} = b_{54} = b_{16} = b_{56} = 0$ and $b_{11} = b_{55} = \varepsilon$. Automorphism φ gets a simple form:

X	X_1	X_2	X_3	X_4	X_5	X_6
$\varphi(X)$	εX_1	$\varepsilon\varepsilon'\delta X_2$	X_3	$\varepsilon'\delta X_4$	εX_5	$\varepsilon\delta^2 X_6$

Case 2. $b_{44} = 0$. As in the previous case we can prove that automorphism θ is equivalent to an automorphism ψ that satisfies:

X	X_1	X_2	X_3	X_4	X_5	X_6
$\psi(X)$	εX_5	$-\varepsilon\varepsilon'\delta X_4$	$-X_3$	$\varepsilon'\delta X_2$	εX_1	$\varepsilon\delta^2 X_6$

where $\varepsilon' = \mathrm{sgn}(b_{42})$. The set $\mathcal{G} = \{\varphi_{\varepsilon\varepsilon'}(\delta), \psi_{\varepsilon\varepsilon'}(\delta) |\ \varepsilon, \varepsilon' \in \{-1, 1\}, \delta > 0\}$ of the automorphisms constructed above possesses the property that any automorphism of the Lie algebra $\mathcal{L}$ is equivalent, modulo $\mathrm{Inn}(\mathcal{L})$, to one element of $\mathcal{G}$. Moreover Remark 1 shows that the automorphisms in $\mathcal{G}$ are pairwise not equivalent. Therefore $\mathcal{G}$ represents a complete set of representatives of $\mathrm{Aut}(\mathcal{L})$ modulo $\mathrm{Inn}(\mathcal{L})$.

We observe that $(\mathcal{G}, \circ)$ is a group itself, its multiplication table being:

$\circ$	$\varphi_{++}(\rho)$	$\varphi_{+-}(\rho)$	$\varphi_{-+}(\rho)$	$\varphi_{--}(\rho)$	$\psi_{++}(\rho)$	$\psi_{+-}(\rho)$	$\psi_{-+}(\rho)$	$\psi_{--}(\rho)$
$\varphi_{++}(\sigma)$	$\varphi_{++}(\delta)$	$\varphi_{+-}(\delta)$	$\varphi_{-+}(\delta)$	$\varphi_{--}(\delta)$	$\psi_{++}(\delta)$	$\psi_{+-}(\delta)$	$\psi_{-+}(\delta)$	$\psi_{--}(\delta)$
$\varphi_{+-}(\sigma)$	$\varphi_{+-}(\delta)$	$\varphi_{++}(\delta)$	$\varphi_{--}(\delta)$	$\varphi_{-+}(\delta)$	$\psi_{+-}(\delta)$	$\psi_{++}(\delta)$	$\psi_{--}(\delta)$	$\psi_{-+}(\delta)$
$\varphi_{-+}(\sigma)$	$\varphi_{-+}(\delta)$	$\varphi_{--}(\delta)$	$\varphi_{++}(\delta)$	$\varphi_{+-}(\delta)$	$\psi_{--}(\delta)$	$\psi_{-+}(\delta)$	$\psi_{+-}(\delta)$	$\psi_{++}(\delta)$
$\varphi_{--}(\sigma)$	$\varphi_{--}(\delta)$	$\varphi_{-+}(\delta)$	$\varphi_{+-}(\delta)$	$\varphi_{++}(\delta)$	$\psi_{-+}(\delta)$	$\psi_{--}(\delta)$	$\psi_{++}(\delta)$	$\psi_{+-}(\delta)$
$\psi_{++}(\sigma)$	$\psi_{++}(\delta)$	$\psi_{+-}(\delta)$	$\psi_{-+}(\delta)$	$\psi_{--}(\delta)$	$\varphi_{+-}(\delta)$	$\varphi_{++}(\delta)$	$\varphi_{--}(\delta)$	$\varphi_{-+}(\delta)$
$\psi_{+-}(\sigma)$	$\psi_{+-}(\delta)$	$\psi_{++}(\delta)$	$\psi_{--}(\delta)$	$\psi_{-+}(\delta)$	$\varphi_{++}(\delta)$	$\varphi_{+-}(\delta)$	$\varphi_{-+}(\delta)$	$\varphi_{--}(\delta)$
$\psi_{-+}(\sigma)$	$\psi_{-+}(\delta)$	$\psi_{--}(\delta)$	$\psi_{++}(\delta)$	$\psi_{+-}(\delta)$	$\varphi_{-+}(\delta)$	$\varphi_{--}(\delta)$	$\varphi_{++}(\delta)$	$\varphi_{+-}(\delta)$
$\psi_{--}(\sigma)$	$\psi_{--}(\delta)$	$\psi_{-+}(\delta)$	$\psi_{+-}(\delta)$	$\psi_{++}(\delta)$	$\varphi_{--}(\delta)$	$\varphi_{-+}(\delta)$	$\varphi_{+-}(\delta)$	$\varphi_{++}(\delta)$

where $\delta = \sigma\rho$. Hence $\mathcal{G}$ is isomorphic to the *outer automorphism group* of the Lie algebra $\mathcal{L}$. Its subgroups $\mathcal{H} = \{\varphi_{\varepsilon\varepsilon'}(1), \psi_{\varepsilon\varepsilon'}(1) |\ \varepsilon, \varepsilon' \in \{-1, 1\}\}$ and $\mathcal{K} = \{\varphi_{++}(\delta) |\ \delta > 0\}$ are respectively isomorphic to the dihedral group D_8 and to the multiplicative group of the positive numbers. Moreover $\mathcal{G}$ is the direct product of its subgroups $\mathcal{H}$ and $\mathcal{K}$. All these remarks are helping us to establish the structure of the full automorphism group $\mathrm{Aut}\,(\mathcal{L})$.

Proposition 1. *The outer automorphism group* $\mathrm{Out}\,(\mathcal{L})$ *is isomorphic to the direct product* $D_8 \times (0, \infty)$*.*

Corollary 1. *The full automorphism group* $\mathrm{Aut}\,(\mathcal{L})$ *is an extension of the inner automorphism group* $\mathrm{Inn}\,(\mathcal{L})$ *by a direct product* $D_8 \times (0, \infty)$*.*

4 Discrete Symmetries

Let Γ be a discrete symmetry that maps the independent and dependent variables (t, x, u) into $(\hat{t}, \hat{x}, \hat{u})$. Then for any generator of the Lie algebra $\mathcal{L}$

$$X_i = \xi_i^0(t, x, u)\frac{\partial}{\partial t} + \xi_i^1(t, x, u)\frac{\partial}{\partial x} + \eta_i(t, x, u)\frac{\partial}{\partial u}$$

we have according to [4]

$$\begin{cases} X_i\hat{t} = \sum_{j=1}^{6} b_i^j \xi_j^0(\hat{t}, \hat{x}, \hat{u}), \\ X_i\hat{x} = \sum_{j=1}^{6} b_i^j \xi_j^1(\hat{t}, \hat{x}, \hat{u}), \\ X_i\hat{u} = \sum_{j=1}^{6} b_i^j \eta_j(\hat{t}, \hat{x}, \hat{u}), \end{cases}$$

for every $i \in \{1, 2, \dots, 6\}$. The coefficients b_i^j are elements of the matrix $B = (b_i^j)_{i,j=1,6}$ associated to the automorphism that maps the generating system $\{\hat{X}_i = \Gamma X_i \Gamma^{-1} |\ i = 1, 6\}$ into $\{X_i |\ i = 1, 6\}$. If we can solve this system of 18 equations, we get the components of the symmetry Γ, (i. e., $\hat{t}, \hat{x}, \hat{u}$) as functions of t, x, u. The usual way to do it is to solve the first subsystem of six equations, use its solution $\hat{t}(x, t, u)$ to solve the second subsystem and so on.

Consider firstly that the symmetry Γ is associated to an automorphism of the type $\varphi_{\varepsilon\varepsilon'}(\delta)$. Its corresponding matrix is $B = \mathrm{diag}(\varepsilon, \varepsilon\varepsilon'\delta, 1, \varepsilon'\delta, \varepsilon, \varepsilon\delta^2)$. Equations 2 and 6 of the first subsystem show that $\hat{t}$ depends only on t. The rest of the subsystem gives

$$\hat{t} = \varepsilon t.$$

Solving the second subsystem we get

$$\hat{x} = \exp[\mathcal{D}\varepsilon(1 - \varepsilon'\delta)t + \varepsilon\varepsilon'\delta \log x].$$

Equations 1, 3 and 6 of the third subsystem are compatible only if $\varepsilon\delta^2 = 1$, which implies $\varepsilon = \delta = 1$. Finally we get the solution

$$\hat{u} = \mu u, \quad \mu \neq 0.$$

Since we are interested in finding discrete symmetries that are pairwise not equivalent modulo the continuous ones, one may consider $\mu = 1$. The reason for that is the existence of continuous symmetries that preserve the independent variables and multiply the dependent one by any nonzero constant (see [1], p. 394). Therefore we have found two discrete symmetries, Γ_0 and Γ, the first one being the identity and the second one having the following description:

$$\Gamma \begin{cases} \hat{t} = t, \\ \hat{x} = \exp(2\mathcal{D}t - \log x), \\ \hat{u} = u. \end{cases}$$

We apply the same procedure to an automorphism $\psi_{\varepsilon\varepsilon'}(\delta)$. The corresponding matrix is

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & \varepsilon'\delta & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -\varepsilon\varepsilon'\delta & 0 & 0 & 0 & 0 \\ \varepsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \varepsilon\delta^2 \end{pmatrix}.$$

The first subsystem gives

$$\hat{t} = -\varepsilon A^{-4} t^{-1}$$

and the second one

$$\hat{x} = \exp(A^{-2}\mathcal{D}\varepsilon\varepsilon'\delta - A^{-4}\mathcal{D}\varepsilon t^{-1} - A^{-2}\varepsilon\varepsilon'\delta t^{-1}\log x).$$

The last subsystem is compatible iff $\varepsilon = \delta = 1$. In this case we get

$$\hat{u} = \nu \cdot \sqrt{|t|} \cdot \exp\left\{-\frac{1}{2A^2 t}\left[(\log x - \mathcal{D}t)^2 + 2A^2 C t^2 + \frac{2C}{A^2}\right]\right\} \cdot u,$$

where ν is a nonzero constant. The same argument that was used above allows us to choose $\nu = A$. Therefore we have found two new discrete symmetries, namely

$$\Gamma_+ \begin{cases} \hat{t} = -A^{-4}t^{-1}, \\ \hat{x} = \exp(A^{-2}\mathcal{D} - A^{-4}\mathcal{D}t^{-1} - A^{-2}t^{-1}\log x), \\ \hat{u} = A\cdot\sqrt{|t|}\cdot\exp\left\{-\frac{1}{2}A^{-2}t^{-1}\left[(\log x - \mathcal{D}t)^2 + 2A^2Ct^2 + 2A^{-2}C\right]\right\}\cdot u, \end{cases}$$

$$\Gamma_- \begin{cases} \hat{t} = -A^{-4}t^{-1}, \\ \hat{x} = \exp(-A^{-2}\mathcal{D} - A^{-4}\mathcal{D}t^{-1} + A^{-2}t^{-1}\log x), \\ \hat{u} = A\cdot\sqrt{|t|}\cdot\exp\left\{-\frac{1}{2}A^{-2}t^{-1}\left[(\log x - \mathcal{D}t)^2 + 2A^2Ct^2 - 2A^{-2}C\right]\right\}\cdot u. \end{cases}$$

We observe that $\Gamma_+^{-1} = \Gamma_-$ and that $\Gamma_+^2 = \Gamma_-^2 = \Gamma = \Gamma^{-1}$, which means that the four discrete symmetries listed above form a cyclic group. In fact we have just proved the following results:

Theorem 1. *Black–Scholes equation's discrete symmetry group is cyclic of order 4, generated by* Γ_+.

Corollary 2. *Any symmetry of the Black–Scholes equation is a product of the type* $\Upsilon\Gamma_+^k$, *where* Υ *is a continuous symmetry and* $k \in \{0, 1, 2, 3\}$.

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Differential Invariants of Semilinear Wave Equations

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We employ the infinitesimal method for calculating invariants of families of differential equations using equivalence groups. We apply the method to the class of semilinear wave equations $u_{tt}-u_{xx}=f(x,u,u_t,u_x)$. We show that this class of equations admits four functionally independent differential invariants of second order. We employ these invariants to derive necessary and sufficient conditions such that this class of wave equations can be mapped into the linear wave equation $u_{tt}-u_{xx}=0$.

1 Introduction

In this paper we consider a class of nonlinear one-dimensional wave equations of the form

$$u_{tt}-u_{xx}=f(x,u,u_t,u_x) \tag{1}$$

Wave equations have a variety of applications in the physical and biological sciences [1, 2, 16]. For example, population dynamics, tides and waves, chemical reactors, flame and combustion problems, theory of transonic aerodynamics etc.

The differential invariants of the Lie groups of continuous transformations play important role in mathematical modelling, non-linear science and differential geometry. First it was noted by S. Lie [8], who showed that every invariant system of differential equations [9], and every variational problem [10], could be directly expressed in terms of the differential invariants. Lie also demonstrated [9], how differential invariants can be used to integrate ordinary differential equations, and succeeded in completely classifying all the differential invariants for all possible finite-dimensional Lie groups of point transformations in the case of one independent and one dependent variable. These results were generalized by Tresse [15] and Ovsiannikov [12]. The general theory of differential invariants of Lie groups including the notion of differential invariant of a transformation group and algorithms of construction of differential invariants can be found in [11, 12].

Following the method proposed by Ibragimov [3–5] we calculate invariants of equivalence transformations of the class (1). In [14] the invariants of equivalence transformations of the class $u_{tt} - u_{xx} = f(u, u_t, u_x)$ were found and in [13] these invariants were used to derive linearising mappings for this class of semilinear wave equations.

Our paper is organized as follows. In the next section we adduce some general results on form-preserving transformations for a generalised class of wave equations. Equivalence transformations of the class (1) were found in Section 3. In Section 4 we prove nonexistence of differential invariants of order zero and one and find the basis of the second order differential invariants. An example of application of these results to linearisation of wave equations is given in Section 5.

2 Form-Preserving Transformations for Generalised Wave Equations

In this section we present some general results on a generalised class of wave equations. We examine the nature of point transformations that connect equations belonging in the class

$$u_{tt} = H(x, t, u, u_t, u_1, u_2, \ldots, u_k, \ldots, u_n), \tag{2}$$

where $u_k = \partial^k u/\partial x^k$, $k = 1, 2, \ldots n$. This class of equations include many models of physical phenomena, especially wave-type motions. For example, the axially symmetric wave equation $u_{tt} = u_{xx} + x^{-1}u_x$, the family of equations $u_{tt} = (f(u)u_x)_x$, certain Boussinesq-type equations and many others.

The work here is similar to the analysis that is presented in [6], where the class (2) with $H_{u_t} = 0$ was considered. For the proofs of the results that follow, the reader may refer to [6].

Theorem 1. *The point transformation $x' = P(x, t, u)$, $t' = Q(x, t, u)$, $u' = R(x, t, u)$ transforms*

$$u'_{t't'} = H'(x', t', u', u'_{t'}, u'_1, u'_2, \ldots, u'_k, \ldots, u'_n) \quad \textit{into}$$

$$u_{tt} = H(x, t, u, u_t, u_1, u_2, \ldots, u_k, \ldots, u_n)$$

where $n \geq 3$ if and only if $P = P(x)$ and $Q = Q(t)$. Also the following identity holds:

$$H' = Q_t^{-3}\left[Q_t R_u H + Q_t R_{uu} u_t^2 + (2Q_t R_{tu} - Q_{tt} R_u) u_t + Q_t R_{tt} - Q_{tt} R_t\right]. \tag{3}$$

We note that if H' and H are linear in $u'_{t'}$ and u_t, respectively, then R is linear in u. Furthermore if $H'_{u'_{t'}} = H_{u_t} = 0$, then $R = A(x)Q_t^{1/2}u + B(x, t)$ [6].

Now we consider equations (2) with $n = 2$ and we present the following theorem:

Theorem 2. *Point transformations* $x' = P(x,t,u)$, $t' = Q(x,t,u)$, $u' = R(x,t,u)$ *which transforms*

$$u'_{t't'} = H'(x',t',u',u'_{t'},u'_{x'},u'_{x'x'}) \quad \textit{into} \quad u_{tt} = H(x,t,u,u_t,u_x,u_{xx})$$

where $H'_{u'_{x'x'}} \neq 0$, *belongs to one of the three categories:*

(a) P, Q *and* H' *restricted as in the conditions of Theorem 1;*

(b) $P = P(t)$, $Q = Q(x)$ *and the following identity holds:*

$$H' = Q_x^{-3}\left[Q_xR_uu_{xx} + Q_xR_{uu}u_x^2 + (2Q_xR_{ux} - Q_{xx}R_u)u_x + Q_xR_{xx} - Q_{xx}R_x\right],$$

where

$$u'_{t'} = Q_x^{-1}(R_x + R_uu_x), \;\; u'_{x'} = P_t^{-1}(P_t + R_uu_t),$$
$$u'_{x'x'} = P_t^{-3}\left[P_tR_uH + P_tR_{uu}u_t^2 + (2P_tR_{tu} - P_{tt}R_u)u_t + P_tR_{tt} - P_{tt}R_t\right];$$

(c) no restrictions on the forms of P, Q, R *and*

$$H' = \frac{(P_x + P_uu_x)(P_t + P_uu_t)}{(Q_x + Q_uu_x)(Q_t + Q_uu_t)}u'_{x'x'} + G'(x',t',u',u'_{t'},u'_{x'}),$$
$$H = \frac{(P_t + P_uu_t)(Q_t + Q_uu_t)}{(P_x + P_uu_x)(Q_x + Q_uu_x)}u_{xx} + G(x,t,u,u_x,u_t).$$

We note that in (c) the most general point transformation applies. If there is no dependence on u_t (see [6]), then $P_u = Q_u = 0$. Hence, the appearance of u_t in the wave equation leads to hodograph-type transformations (transformations where one of the old independent variables depends on the new dependent variable). For example, the hodograph-type transformation

$$x' = u, \quad t' = x + t, \quad u' = x$$

maps

$$u'_{t't'} = \frac{u'_{t'}(u'_{t'} - 1)}{(u'_{x'})^2}u'_{x'x'} \quad \text{into} \quad u_{tt} = \frac{u_t}{u_x}u_{xx}.$$

The similar results of Theorems 1 and 2 corresponding to infinitesimal transformations can be found in [7].

3 Equivalence Transformations

In order to construct equivalence transformations of the class (1) one can use direct or infinitesimal method. After restricting ourselves to studying of the connected component of unity in equivalence group we can use the Lie infinitesimal method. To find equivalence transformations of the class (1) in the framework of this approach, we have to investigate Lie symmetries of the system that consists of equation (1) and additional conditions $f_t = 0$.

Consider a one-parameter Lie group $\mathcal{E}$ of local transformations in (t, x, u, f) with an infinitesimal operator of the form

$$Y = \xi^1\partial_t + \xi^2\partial_x + \eta\partial_u + \zeta^1\partial_{u_t} + \zeta^2\partial_{u_x} + \mu\partial_f.$$

Using the classical Lie approach we find the invariance algebra of the above system that is the linear span of operators of the form

$$\begin{aligned} &Y_1 = \partial_t, \quad Y_2 = \partial_x, \quad Y_3 = t\partial_t + x\partial_x - 2f\partial_f - u_t\partial_{u_x} - u_x\partial_{u_x}, \\ &Y_\phi = \phi\partial_u + (\phi_u f + \phi_{uu}(u_t^2 - u_x^2) - 2\phi_{xu}u_x - \phi_{xx})\partial_f \\ &\quad + \phi_u u_t\partial_{u_t} + (\phi_x + \phi_u u_x)\partial_{u_x}. \end{aligned} \tag{4}$$

Here and below $\phi = \phi(x, u)$.

It can be proved by the direct method that equivalence group $\mathcal{E}$ of the class (1) coincides with the group generated by the following transformations

$$\begin{aligned} &t' = c_1 t + c_2, \quad x' = \varepsilon c_1 x + c_3, \quad u' = R(x, u), \\ &f' = c^{-1}(R_u f + (u_t^2 - u_x^2)R_{uu} - 2R_{xu}u_x - R_{xx}). \end{aligned} \tag{5}$$

Here $c_1 \neq 0$, c_2, c_3 are arbitrary constants, $\varepsilon = \pm 1$ and u_t, u_x are transformed as follows

$$u'_{t'} = c_1^{-1}R_u u_t, \quad u'_{x'} = c_1^{-1}\varepsilon(R_u u_x + R_x).$$

Differential invariants of order s of the class (1) are functions of the independent variables t, x, the dependent variable u and its derivatives u_t, u_x, as well as of the function f and its derivatives of maximal order s, that are invariants with respect to the equivalence group $\mathcal{E}$.

4 Differential Invariants

We search for invariants of order zero. That is, invariants of the form

$$J = J(t, x, u, u_t, u_x, f).$$

We apply the invariant test $Y(J) = 0$ to the operators Y_1, Y_2, Y_3 and Y_ϕ and using the fact that $\phi(x, u)$ is arbitrary function, we obtain $J = \text{const}$. Hence, equations (1) do not admit differential invariants of order zero.

In order to determine differential invariants of the nth order,

$$J = J(t, x, u, u_t, u_x, f_x, f_u, f_{u_{(1)}}, \ldots, f_{u_{(n)}})$$

we need to find the nth prolongation $Y^{(n)}$ of Y, considering t, x, u, $u_{(1)}, \ldots, u_{(n)}$ as independent variables, f and derivatives of f as dependent variables. Here and below $u_{(k)}$, is the sets of all kth order partial derivatives of u, $f_{(k)}$ denotes the sets of all kth order partial derivatives of f.

We note that $Y_1^{(n)} = Y_1$, $Y_2^{(n)} = Y_2$. Hence for any order of differential invariants $J_t = 0$, $J_x = 0$. Likewise considering $Y_\phi^{(n)}$ we obtain that $J_u = 0$. That is below we search for invariants of the form $J = J(u_t, u_x, f_x, f_u, f_{u_{(1)}}, \ldots, f_{u_{(n)}})$ only.

The first prolongation of operators (4) has the form

$$\begin{aligned}
Y_3^{(1)} &= Y_3 - 3f_x\partial_{f_x} - 2f_u\partial_{f_u} - f_{u_t}\partial_{f_{u_t}} - f_{u_x}\partial_{f_{u_x}}, \\
Y_\phi^{(1)} &= Y_\phi + [\phi_{xu}(f - f_{u_t}u_t - f_{u_x}u_x) - \phi_{xxx} - \phi_{xuu}(u_t^2 - u_x^2) \\
&- 2\phi_{xxu}u_x + f_x\phi_u - \phi_{xx}f_x - f_u\phi_x]\partial_{f_x} + [\phi_{uuu}(u_t^2 - u_x^2) \\
&+ \phi_{uu}(f - f_{u_t}u_t - f_{u_x}u_x) - 2\phi_{xuu}u_x - \phi_{xxu} - f_{u_x}\phi_{xu}]\partial_{f_u} \\
&+ 2\phi_{uu}u_t\partial_{f_{u_t}} - 2(\phi_{uu}u_x + \phi_{xu})\partial_{f_{u_x}}
\end{aligned} \tag{6}$$

Because all derivatives of $\phi(x, u)$ can be considered as independent functions, for searching of differential invariants of the first order

$$J = J(u_t, u_x, f, f_x, f_u, f_{u_t}, f_{u_x})$$

we obtain the system of 10 linear partial first order differential equations on function J depending of 7 variables. It is not difficult to show that rank r of the matrix of this system is equal to 7, so it admits only the trivial solution $J = \mathrm{const}$ and therefore equations from class (1) do not admit differential invariants of the first order.

The invariant test

$$E_3 = Y_3^{(2)}(J) = 0, \qquad E_\phi = Y_\phi^{(2)}(J) = 0 \tag{7}$$

for finding second order differential invariants produces the system of 13 linear partial first order differential equations of rank 13 on function J depending of 17 variables. Therefore, the set of all solutions of this system forms 4-dimensional vector space $\mathcal{J}^{(2)}$. Below we shall explain in detail how to construct a basis of $\mathcal{J}^{(2)}$ (the complete set of inequivalent second order differential invariants).

Coefficients of ϕ_{uuuu}, ϕ_{xxxx}, ϕ_{xxxu}, and ϕ_{xxx} in $E_\phi = 0$ give $J_{f_{uu}} = J_{f_{xu}} = J_{f_{xx}} = J_{f_x} = 0$. Hence,

$$J = J(u_t, u_x, f, f_u, f_{u_t}, f_{u_x}, f_{xu_t}, f_{xu_x}, f_{uu_t}, f_{uu_x}, f_{u_tu_t}, f_{u_tu_x}, f_{u_xu_x}).$$

Singling out the terms with different derivatives of ϕ, one can obtain from (7) the system of linear partial first order differential equations on function J:

$$
\begin{aligned}
&2fJ_f + 2f_uJ_{f_u} + f_{u_x}J_{f_{u_x}} + f_{u_t}J_{f_{u_t}} + f_{uu_x}J_{f_{uu_x}} + f_{uu_t}J_{f_{uu_t}}\\
&\quad +2f_{xu_x}J_{f_{xu_x}} + 2f_{xu_t}J_{f_{xu_t}} + u_xJ_{u_x} + u_tJ_{u_t} = 0,\\
&(u_t^2 - u_x^2)J_{f_u} - 2u_xJ_{f_{uu_x}} + 2u_tJ_{f_{uu_t}} = 0,\\
&-u_xJ_{f_u} - J_{f_{uu_x}} - u_xJ_{f_{xu_x}} + u_tJ_{f_{xu_t}} = 0,\\
&J_{f_u} + 2J_{f_{xu_x}} = 0,\\
&(u_t^2 - u_x^2)J_f + (f - u_xf_{u_x} - u_tf_{u_t})J_{f_u} - 2u_xJ_{f_{u_x}} - 2J_{f_{u_xu_x}} + 2u_tJ_{f_{u_t}}\\
&\quad +2J_{f_{u_tu_t}} - (u_xf_{u_xu_x} + u_tf_{u_tu_x})J_{f_{uu_x}} - (u_xf_{u_tu_x} + u_tf_{u_tu_t})J_{f_{uu_t}} = 0,\\
&-2u_xJ_f - f_{u_x}J_{f_u} - 2J_{f_{u_x}} - f_{u_xu_x}J_{f_{uu_x}} - f_{u_tu_x}J_{f_{uu_t}}\\
&\quad -(u_xf_{u_xu_x} + u_tf_{u_tu_x})J_{f_{xu_x}} - (u_xf_{u_tu_x} + u_tf_{u_tu_t})J_{f_{xu_t}} = 0,\\
&J_f + f_{u_xu_x}J_{f_{xu_x}} + f_{u_tu_x}J_{f_{xu_t}} = 0,\\
&fJ_f - f_{u_xu_x}J_{f_{u_xu_x}} - f_{u_tu_x}J_{f_{u_tu_x}} - f_{u_tu_t}J_{f_{u_tu_t}} - f_{uu_x}J_{f_{uu_x}}\\
&-f_{uu_t}J_{f_{uu_t}} + u_xJ_{u_x} + u_tJ_{u_t} = 0,\\
&-f_{uu_x}J_{f_{xu_x}} - f_{uu_t}J_{f_{xu_t}} + J_{u_x} = 0.
\end{aligned}
\tag{8}
$$

Now solving the system (8) we obtain the following second order differential invariants

$$J_1 = \frac{f_{u_xu_x} + f_{u_tu_t}}{f_{u_xu_t}}, \tag{9}$$

$$J_2 = \frac{u_tf_{u_tu_t} - f_{u_t}}{u_tf_{u_xu_t}}, \tag{10}$$

$$
\begin{aligned}
J_3 = [&2ff_{u_xu_x} + 4f_u - f_{u_x}^2 + 2u_tf_{u_x}f_{u_xu_t} - u_tf_{u_t}f_{u_xu_x}\\
&+ f_{u_t}^2 - u_tf_{u_t}f_{u_tu_t} - 2u_xf_{uu_x} - 2u_tf_{uu_t} - 2f_{xu_x}]u_t^{-2}f_{u_xu_t}^{-2},
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
J_4 = [&2ff_{u_tu_x} + u_tf_{u_x}f_{u_xu_x} + u_tf_{u_x}f_{u_tu_t} - 2u_tf_{u_t}f_{u_tu_x} - 2u_tf_{uu_x}\\
&- 2u_xf_{uu_t} - 2f_{xu_t}]u_t^{-2}f_{u_xu_t}^{-2}.
\end{aligned}
\tag{12}
$$

Equivalence transformations (4) have the following invariant equations:

$$f_{u_xu_t} = 0, \tag{13}$$

$$f_{u_xu_x} + f_{u_tu_t} = 0, \tag{14}$$

$$u_tf_{u_tu_t} - f_{u_t} = 0, \tag{15}$$

$$
\begin{aligned}
&2ff_{u_xu_x} + 4f_u - f_{u_x}^2 + 2u_tf_{u_x}f_{u_xu_t} - u_tf_{u_t}f_{u_xu_x} + f_{u_t}^2 - u_tf_{u_t}f_{u_tu_t},\\
&\quad -2u_xf_{uu_x} - 2u_tf_{uu_t} - 2f_{xu_x} = 0,
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
&2ff_{u_tu_x} + u_tf_{u_x}f_{u_xu_x} + u_tf_{u_x}f_{u_tu_t} - 2u_tf_{u_t}f_{u_tu_x} - 2u_tf_{uu_x}\\
&\quad -2u_xf_{uu_t} - 2f_{xu_t} = 0.
\end{aligned}
\tag{17}
$$

5 Linearisation Using Differential Invariants

The linear wave equation

$$u_{tt} - u_{xx} = 0, \tag{18}$$

that is $f = 0$, has no differential invariants. However the invariant equations (13)–(17) are all satisfied when $f = 0$. Hence any equation of the form (1) that is connected with the linear wave equation (18) satisfies the five invariant equations. Consequently, the solution of the system (13)–(17) will provide necessary conditions for an equation of the form (1) to be mapped into (18).

From equations (13)–(17) we deduce that

$$f(x, u, u_x, u_t) = \alpha(x, u)(u_t^2 - u_x^2) + \beta(x, u)u_x + \gamma(x, u) \tag{19}$$

where

$$2\alpha_x + \beta_u = 0, \tag{20}$$

$$-4\gamma_u + 2\beta_x + 4\alpha\gamma + \beta^2 = 0. \tag{21}$$

Now if we set

$$\alpha(x, u) = A_u(x, u), \tag{22}$$

where $A(x, u)$ is an arbitrary function, equations (20) and (21) give

$$\beta(x, u) = -2A_x + B^1(x), \tag{23}$$

$$\gamma(x, u) = e^A B^2(x) + e^A \int^u \left[\frac{1}{2}(B_x^1 - 2A_{xx}(x, u')) + \frac{1}{4}(B^1 - 2A_x(x, u'))^2\right] e^{-A(x,u')}\mathrm{d}u' \tag{24}$$

where $B^1(x)$ and $B^2(x)$ are also arbitrary function. Hence, any equation of the class (1) which is connected under the equivalence transformation (5) with the linear wave equation (18) is also a member of the restricted class

$$u_{tt} - u_{xx} = A_u(x, u)\left[u_t^2 - u_x^2\right] + [-2A_x + B^1(x)]u_x + e^A B^2(x) + e^A \int^u \left[\frac{1}{2}(B_x^1 - 2A_{xx}(x, u')) + \frac{1}{4}(B^1 - 2A_x(x, u'))^2\right] e^{-A(x,u')}\mathrm{d}u' \tag{25}$$

Now we apply Theorem 2 to find transformations that link members of (25) and (18). From the form of the equivalence transformation (5) we deduce that part (a) of Theorem 2 applies. We substitute into identity (3) $P = x$, $Q = t$ (we take without loss of generality $c_1 = \epsilon = 1$), $R = R(x, u)$, $H' = u'_{x'x'}$ and $H = u_{xx}+$ RHS of equation (25). The resulting equation leads to the form of R and may provide restrictions on the forms of $A(x, u)$, $B^1(x)$ and $B^2(x)$.

Straightforward calculations show that

$$R(x,u) = \phi(x) \int^u e^{-A(x,u')} \mathrm{d}u' + \psi(x),$$

where

$$\phi(x) = e^{\int^x \frac{1}{2} B^1(x')\mathrm{d}x'}, \quad \psi''(x) = B^2(x) e^{\int^x \frac{1}{2} B^1(x')\mathrm{d}x'} \tag{26}$$

The above form of R satisfies (3) with no restrictions imposed on the functional forms of $H(x,u)$, $B^1(x)$ and $B^2(x)$. This shows that equation (25) is the most general equation of the class (1) that can be transformed into the linear wave equation (18). The transformation that link these two latter equations is

$$u \mapsto \phi(x) \int^u e^{-A(x,u')} \mathrm{d}u' + \psi(x), \tag{27}$$

where $\phi(x)$ and $\psi(x)$ are defined by equation (26). Therefore we have shown that:

Theorem 3. *Invariant equations* (13)–(17) *provide necessary and sufficient conditions for linking equations of the class* (1) *and the linear wave equation* (18) *under the equivalence transformation admitted by* (1).

6 Remarks

We have shown that the class of equations (1) has no differential invariants of order zero and order one. We have determined four functionally independent differential invariants of second order. These invariants were employed to derive necessary and sufficient conditions such that class (1) can be mapped into the linear wave equation $u_{tt} - u_{xx} = 0$.

We plan to continue investigations of this subject. For the class under consideration we plan to find the basis of differential invariants and operators of invariant differentiation. Another direction for us to develop the above results to linearize equations of the class (1). For example, to construct necessary and sufficient conditions for linking equations of the class (1) and the linear wave equation $u_{tt} - u_{xx} = u$. In order to be able to derive such conditions, we need to consider third order differential invariants for (1).

To produce higher order invariants, we need to follow the procedure as above by considering higher order prolongations, or alternatively we can introduce the idea of invariant differentiation. Details about invariant differentiation can be found in [4, 12].

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Bäcklund Transformations of One Class of Partially Invariant Solutions of the Navier–Stokes Equations

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A Lie group of Bäcklund transformations is constructed in the manuscript. This Lie group is admitted by a system of partial differential equations which obtained as a reduced system for one class of partially invariant solutions of the Navier–Stokes equations.

1 Introduction

Group analysis is one of the methods for constructing particular exact solutions of partial differential equations. A survey of this method can be found in [1, 2]. The first step in application of group analysis to partial differential equations consists of finding an admitted Lie group of transformations. The transformations can be point, contact or finite order tangent transformations. The Bäcklund theorem states that there are no nontrivial tangent transformations of finite order except contact transformations. This theorem is proven under assumption that all derivatives involved in the transformations are free: they only satisfy the tangent conditions. On the other hand, if the derivatives appearing in a system of partial differential equations satisfy additional relations other than the tangent conditions, then there may exist nontrivial tangent transformations of finite order. These transformations are called Bäcklund transformations [3].

In this manuscript we construct Lie group of Bäcklund transformations for a system of partial differential equations which arises from the study of partially invariant solutions of the Navier–Stokes equations.

Recall that the construction of partially invariant solutions consists of choosing a subgroup of the admitted group, finding a representation of a solution, substituting the representation into the studied system of equations and studying the compatibility of the obtained (reduced) system of equations.

2 One Class of Partially Invariant Solutions of the Navier–Stokes Equations

Unsteady motion of incompressible viscous fluid is governed by the Navier–Stokes equations

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \Delta \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0, \tag{1}$$

where $\mathbf{u} = (u_1, u_2, u_3) = (u, v, w)$ is the velocity field, p is the fluid pressure, ∇ is the gradient operator in the three-dimensional space $\mathbf{x} = (x_1, x_2, x_3) = (x, y, z)$, and Δ is the Laplacian. The Lie group admitted by the Navier–Stokes equations is infinite [4]. Its Lie algebra L can be presented in the form of a direct sum $L = L^\infty \oplus L^5$, where the infinite-dimensional ideal L^∞ is generated by the operators

$$X_{\phi_i} = \phi_i(t)\partial_{x_i} + \phi_i'(t)\partial_{u_i} - \phi_i''(t)x_i\partial_p\,, \quad X_\psi = \psi(t)\partial_p$$

with arbitrary functions $\phi_i(t)$, $(i = 1, 2, 3)$ and $\psi(t)$. The subalgebra L^5 has the following basis:

$$Y = 2t\partial_t + x_i\partial_{x_i} - u_i\partial_{u_i} - 2p\partial_p, \quad Z_0 = \partial_t,$$
$$Z_{ik} = x_i\partial_{x_k} - x_k\partial_{x_i} + u_i\partial_{u_k} - u_k\partial u_i, \qquad i < k \leq 3.$$

There is no complete classification of all subalgebras of the algebra L yet. However, attempts for classification of infinite dimensional algebras were considered in [5,6]. In this manuscript we study solutions constructed on the basis of the Lie group corresponding the subalgebra

$$\{X_2,\ X_3,\ X_5,\ X_4 + \beta X_6 + X_{10}\}.$$

where β is an arbitrary constant and

$$X_2 = \partial_y, \quad X_3 = \partial_z, \quad X_4 = t\partial_x + \partial_u, \quad X_5 = t\partial_y + \partial_v,$$
$$X_6 = t\partial_z + \partial_w, \quad X_{10} = \partial_t.$$

This subgroup is taken from the optimal system of admitted subalgebras constructed for the gas dynamic equations [7]. Partially invariant solutions of the gas dynamic equations for these groups were considered in [8].

Invariants of the Lie group G_4 are

$$u - t,\ w - \beta t,\ p,\ x - t^2/2.$$

The rank of the Jacobi matrix of the invariants with respect to the dependent variables is equal to three. Since this rank is less than the number of the dependent variables, there are no nonsingular invariant solutions that are invariant with respect to this group. A minimally possible defect of a partially invariant solution with respect this group is equal to one. In this case a representation of a regular partially invariant solution is

$$u = U(s) + t, \quad w = W(s) + \beta t, \quad p = P(s), \quad s = x - t^2/2,$$

while the function $v = v(t, x, y, z)$ still depends on all independent variables. Substituting this representation of a solution into the Navier–Stokes equations, one obtains

$$U'' - UU' - P' - 1 = 0, \tag{2}$$
$$v_t + (U + t)v_x + vv_y + (W + \beta t)v_z - (v_{xx} + v_{yy} + v_{zz}) = 0, \tag{3}$$
$$W'' - UW' - \beta = 0, \tag{4}$$
$$U' + v_y = 0. \tag{5}$$

Integrating equations (2) and (5), one has

$$P = U' - \frac{1}{2}U^2 - s + C_1, \quad v = -U'y + V(t, s, z).$$

Substituting v into equation (3), one arrives at the equation

$$V_t + UV_s - VU' + (W + \beta t)V_z - V_{ss} + V_{zz} + y(U''' - UU'' + U'^2) = 0.$$

Since U, V and W do not depend on y, the last equation can be split with respect to y:

$$U''' - UU'' + U'^2 = 0,$$
$$V_t + UV_s - VU' + (W + \beta t)V_z - V_{ss} + V_{zz} = 0.$$

Thus the studied partially invariant solution of the Navier–Stokes equations is

$$u = U(s) + t, \quad v = -U' + V(t, s, z), \quad w = W(s) + \beta t,$$
$$p = U' - \frac{1}{2}U^2 - s + C_1,$$

where $s = x - t^2/2$ and the function $U(s)$, $W(s)$ and $V(t, s, z)$ satisfy the reduced system

$$\begin{aligned} &U''' - UU'' + U'^2 = 0,\\ &W'' - UW' - \beta = 0,\\ &V_t + UV_s - VU' + (W + \beta t)V_z - V_{ss} + V_{zz} = 0. \end{aligned} \tag{6}$$

3 Lie Groups of Bäcklund Transformations

The notion of a Lie group of point transformations has been generalized to involve derivatives in the transformation[1]. Assume that the transformations

$$\overline{x}_k = f_k(x, u, p; a), \quad \overline{u}^j = \phi^j(x, u, p; a), \quad \overline{p}^j_\alpha = \psi^j_\alpha(x, u, p; a), \tag{7}$$
$$k = 1, 2, \ldots, n; \ j = 1, 2, \ldots, m; \ |\alpha| \le q$$

[1]For details one can see [3, 9].

form a one parameter Lie group G^1, where the functions f_k, ϕ^j and ψ^j_α depend on the independent variables x, the dependent variables u, and the derivatives p^k_β, $(k = 1, 2, \dots, m;\ |\beta| \le q)$ of order up to q. The infinitesimal generator of the group G^1 is given by the vector field (ξ, η, ζ)

$$X = \xi^k \partial_{x_k} + \eta^j \partial_{u^j} + \zeta^j_\alpha \partial_{p^j_\alpha},$$

where

$$\xi^k = \frac{df_k}{da}\Big|_{a=0}, \quad \eta^j = \frac{d\phi^j}{da}\Big|_{a=0}, \quad \zeta^j_\alpha = \frac{d\psi^j_\alpha}{da}\Big|_{a=0},$$
$$(k = 1, 2, \dots, n;\ j = 1, 2, \dots, m;\ |\alpha| \le q).$$

The coefficients of the infinitesimal generator for the derivatives of any order higher than q are defined by the recurrent prolongation formulae

$$\zeta^j_{\alpha,i} = D_i \zeta^j_\alpha - p^j_{\alpha,k} D_i, \qquad |\alpha| = q, q+1, \dots.$$

Here D_i is the operator of the total derivative with respect to x_i.

The infinitesimal generator X is prolonged for the differentials du^j, dx_i and dp^j_α:

$$\widetilde{X} = X + \widetilde{\xi^i}\, \partial_{dx_i} + \widetilde{\eta^k} \partial_{du^k} + \widetilde{\zeta^j_\alpha}\, \partial_{dp^j_\alpha}$$

by the usual formulae for the differentials

$$\begin{aligned}
\widetilde{\xi^i} &= d\xi^i = \frac{\partial \xi^i}{\partial x_l} dx_l + \frac{\partial \xi^i}{\partial u^s} du^s + \frac{\partial \xi^i}{\partial p^s_\beta} dp^s_\beta, \\
\widetilde{\eta^k} &= d\eta^k = \frac{\partial \eta^k}{\partial x_l} dx_l + \frac{\partial \eta^k}{\partial u^s} du^s + \frac{\partial \eta^k}{\partial p^s_\beta} dp^s_\beta, \\
\widetilde{\zeta^j_\alpha} &= d\zeta^j_\alpha = \frac{\partial \zeta^j_\alpha}{\partial x_l} dx_l + \frac{\partial \zeta^j_\alpha}{\partial u^s} du^s + \frac{\partial \zeta^j_\alpha}{\partial p^s_\beta} dp^s_\beta.
\end{aligned} \tag{8}$$

The transformations (7) are called tangent transformations if the tangent conditions

$$du^j - p^j_i dx_i = 0, \quad dp^j_\alpha - p^j_{\alpha,k} dx_k = 0, \tag{9}$$

are invariant with respect to them. Tangent transformations of finite order are also called Bäcklund transformations. Hence, a Lie group of tangent transformations has to satisfy the determining equations

$$\left(\widetilde{\eta^j} - \widetilde{\zeta^j_i}\, dx_i - p^j_i \widetilde{\xi^i}\right)\Big|_{(9)} = 0, \quad \left(\widetilde{\zeta^j_\alpha} - \widetilde{\zeta^j_{\alpha,k}} dx_k - p^j_{\alpha,k} \widetilde{\xi^k}\right)\Big|_{(9)} = 0. \tag{10}$$

The Bäcklund theorem [3] states that any tangent transformation is a prolongation of a Lie group of either contact transformations or point transformations. The first case is only possible for $m = 1$. Notice that the Bäcklund theorem was proven under the assumption that all derivatives only satisfy the tangent conditions.

A Lie group of tangent transformations with generator X is called admitted by a system of partial differential equations

$$(S) \qquad F(x,u,p) = 0$$

if the coefficients of the infinitesimal generator satisfy the determining equations

$$XF|_{(S)} = 0. \tag{11}$$

Invariance of tangent conditions for the admitted Lie group of tangent transformations becomes

$$\left(\widetilde{\eta^j} - \widetilde{\zeta_i^j}\, dx_i - p_i^j \widetilde{\xi^i}\right)\Big|_{(9),\,(S)} = 0, \quad \left(\widetilde{\zeta_\alpha^j} - \widetilde{\zeta_{\alpha,k}^j} dx_k - p_{\alpha,k}^j \widetilde{\xi^k}\right)\Big|_{(9),\,(S)} = 0. \tag{12}$$

In contrast to the Bäcklund theorem, admitted tangent transformations involve additional relations for the derivatives occurring in (S). This allows for the existence of Bäcklund transformations, namely tangent transformations of finite order.

For example, direct calculations show that the Lie group of transformations corresponding to the generators

$$Y_1 = U'\partial_V, \quad Y_2 = (tU' + 1)\partial_V, \quad Y_3 = (\beta t + W + zU')\partial_V \tag{13}$$

is admitted by the system of equations (6). The corresponding transformations are:

$$\begin{aligned} Y_1: \quad & \bar{t} = t, \ \ \bar{s} = s, \ \ \bar{z} = z, \\ & \overline{U} = U, \ \ \overline{U}' = U'', \ \ \overline{U}'' = UU'' - (U')^2, \\ & \overline{W} = W, \ \ \overline{W}' = W', \ \ \overline{W}'' = W'', \\ & \overline{V} = V + aU', \ \ \overline{V}_t = V_t, \ \ \overline{V}_z = V_z, \ \ \overline{V}_{zz} = V_{zz}, \\ & \overline{V}_s = V_s + aU'', \ \ \overline{V}_{ss} = V_{ss} + a(UU'' - (U')^2), \end{aligned}$$

$$\begin{aligned} Y_2: \quad & \bar{t} = t, \ \ \bar{s} = s, \ \ \bar{z} = z, \\ & \overline{U} = U, \ \ \overline{U}' = U'', \ \ \overline{U}'' = UU'' - (U')^2, \\ & \overline{W} = W, \ \ \overline{W}' = W', \ \ \overline{W}'' = W'', \\ & \overline{V} = V + a(tU' + 1), \ \ \overline{V}_t = V_t + aU', \ \ \overline{V}_z = V_z, \ \ \overline{V}_{zz} = V_{zz}, \\ & \overline{V}_s = V_s + atU'', \ \ \overline{V}_{ss} = V_{ss} + at(UU'' - (U')^2), \end{aligned}$$

$$
\begin{aligned}
Y_3: \quad & \overline{t} = t, \ \ \overline{s} = s, \ \ \overline{z} = z, \\
& \overline{U} = U, \ \ \overline{U}' = U'', \ \ \overline{U}'' = UU'' - (U')^2, \\
& \overline{W} = W, \ \ \overline{W}' = W', \ \ \overline{W}'' = W'', \\
& \overline{V} = V + a(\beta t + W + zU'), \ \ \overline{V}_t = V_t + a\beta, \\
& \overline{V}_z = V_z + aU', \ \ \overline{V}_{zz} = V_{zz}, \\
& \overline{V}_s = V_s + aW' + azU'', \ \ \overline{V}_{ss} = V_{ss} + aW'' + az(UU'' - (U')^2).
\end{aligned}
$$

The Lie groups of transformations (13) were originally found by seeking an admitted Lie group of point transformations for the equivalent system

$$
\begin{aligned}
& \widetilde{U}'' - U\widetilde{U}' + \widetilde{U}^2 = 0, \\
& W'' - UW' - \beta = 0, \\
& V_t + UV_s - V\widetilde{U} + (W + \beta t)V_z - V_{ss} + V_{zz} = 0, \\
& \widetilde{U} = U'.
\end{aligned} \tag{14}
$$

For system (14) the dependent variables are

$$
u^1 = U, \quad u^2 = V, \quad u^3 = W, \quad u^4 = U'.
$$

The basis of the Lie algebra corresponding to the Lie group of point transformations admitted by system (14) is

$$
\begin{aligned}
& X_1 = \partial_s, \quad X_2 = \partial_z, \quad X_3 = V\partial_V, \quad X_4 = \partial_t + \beta t\partial_z, \quad X_5 = t\partial_z + \partial_W, \\
& X_6 = \widetilde{U}'\partial_V, \quad X_7 = (t\widetilde{U}' + 1)\partial_V, \quad X_8 = (\beta t + W + z\widetilde{U}')\partial_V.
\end{aligned}
$$

The generators X_6, X_7, X_8 correspond to the operators Y_1, Y_2, Y_3, respectively. Notice that if one looks for an admitted group by considering the dependent variables

$$
\begin{aligned}
& u^1 = U, \quad u^2 = V, \quad u^3 = W, \quad u^4 = U', \\
& u^5 = W', \quad u^6 = V_t, \quad u^7 = V_s, \quad u^8 = V_y,
\end{aligned}
$$

then one obtains the following admitted generators

$$
\begin{aligned}
& X_1 = \partial_s, \quad X_2 = \partial_z, \quad X_3 = V\partial_V + V_t\partial_{V_t} + V_s\partial_{V_s} + V_z\partial_{V_z}, \\
& X_4 = t\partial_z + \partial_W - V_z\partial_{V_t}, \quad X_5 = \partial_t - \beta t\partial_w.
\end{aligned}
$$

Note that Y_1, Y_2, and Y_3 are no longer among these generators.

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Integrable Nonlinear Partial Differential Equations with Variable Coefficients from the Painlevé Test

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In this paper we present new higher-dimensional equations with variable coefficients which are derived from the Burgers and KdV equations by applying the Painlevé test.

1 Introduction

Modern theories of nonlinear science have been highly developed over the last half century. Particularly, integrable nonlinear systems have attracted great interests of a number of mathematicians and physicists. One of such attentions is the algebraic solvability of integrable nonlinear partial differential equations. In addition to their theoretical importance, they have remarkable applications to many physical systems that are thought of as perturbations of integrable systems such as hydrodynamics, nonlinear optics, plasma and certain field theories and so on [1]. On the other hand the definition of the infinite dimensional integrable systems [2] is still not unified precisely but rather is characterized generally by various interrelated common features including the space-localized solutions (solitons), Lax pairs, Bäcklund transformations and some Painlevé properties [3,4]. Finding new integrable systems is very important, but it is a difficult problem because of their ambiguous properties of integrable systems and the undeveloped mathematical backgrounds.

For the discovery of new integrable systems, many researchers have mainly investigated autonomous and lower-dimensional nonlinear systems [5–12]. Thus many autonomous $(1 + 1)$-dimensional integrable nonlinear partial differential equations have been found. On the other hand, there are few studies to find integrable nonlinear partial differential equations with variable coefficients, since they are essentially complicated and their theory is still in its early stages. In physical systems, however, integrable nonlinear equations with variable coefficients are one of the exciting subjects in integrable systems [13–15]. Analysis of higher-dimensional equations is also an active topic in the theory of integrable systems. Since then, the study of higher dimensions has attracted much more attention. Higher-dimensional generalizations of integrable systems are not usually unique, in the sense that there exist several equations that reduce to an original one under dimensional reductions. So our goal in this paper is to extend integrable equations with variable coefficients to higher-dimensions by applying the Painlevé test.

2 Investigation of Equations with Variable Coefficients in (2 + 1) Dimensions

In this section, we first give a brief review of both of the Painlevé property and of the Painlevé test. Next we will present higher-dimensional equations with variable coefficients of the Burgers and KdV equations by using Painlevé test. It is widely known that the Painlevé test, in the sense of Weiss–Tabor–Carnevale (WTC) method [3, 4], is a powerful tool for investigating integrable equations.

2.1 Weiss–Tabor–Carnevale's Method of the Painlevé Test

Weiss *et. al.* [3] said that a partial differential equation (PDE) has the Painlevé property when the solutions of the PDE are single-valued about the movable singularity manifold. They have proposed a technique which determine whether or not a given system is integrable, that we call WTC's method.

When the singularity manifold is determined by

$$\phi(z_1, \cdots, z_n) = 0, \tag{1}$$

and $u = u(z_1, \cdots, z_n)$ is a solution of the given PDE, then we assume that

$$u = \phi^\alpha \sum_{j=0}^{\infty} u_j \phi^j, \tag{2}$$

where $\phi = \phi(z_1, \cdots, z_n),\ u_j = u_j(z_1, \cdots, z_n),\ j = 0, 1, 2, \cdots,\ u_0 \neq 0$ are analytic functions of z_j in a neighborhood of the manifold (1) and α is a negative integer (so-called the leading order). Substitution of (2) into the PDE determines the value of α and defines the recursion relations for u_j. When the ansatz (2) is correct, the PDE possesses the Painlevé property and it is conjectured to be integrable.

2.2 Burgers Equation with Variable Coefficients in (2 + 1) Dimensions

Consider the following equation:

$$\begin{aligned} &u_t + a(x,z,t)u + b(x,z,t)u_x + c(x,z,t)u_z + d(x,z,t)uu_z \\ &+e(x,z,t)u_x\partial_x^{-1}u_z + f(x,z,t)u_{xz} + g(x,z,t) = 0, \end{aligned} \tag{3}$$

where $d(x,z,t)+e(x,z,t) \neq 0$, $f(x,z,t) \neq 0$ and the subscripts with respect to the independent variables denote partial derivatives, for example, $u_x = \partial u/\partial x$, $u_{xz} = \partial^2 u/(\partial x \partial z)$ etc, and $\partial_x^{-1} u := \int^x u(s)ds$. Here $a(x,z,t)$, $b(x,z,t), \ldots, g(x,z,t)$ are functions of two spatial variables x, z and one temporal variable t. If we choose $a(x,z,t) = b(x,z,t) = c(x,z,t) = g(x,z,t) = 0$ and $d(x,z,t) = e(x,z,t) = f(x,z,t) = 1$, equation (3) is reduced to a $(2+1)$-dimensional Burgers equation:

$$u_t + uu_z + u_x \partial_x^{-1} u_z + u_{xz} = 0, \tag{4}$$

which, by setting $\partial/\partial z = \partial/\partial x$, reads the (ordinary) Burgers equation which is widely-known to be linearisable or integrable.

Here our main goal is to find new integrable equations. We apply the Painlevé test to equation (3) and determine the coefficients by conditions from the Painlevé test. The Painlevé test for equation (3) requires an elimination of the non-local term. Through operations of division and differentiation, equation (3) is transformed to

$$\begin{aligned}
&(ea_x - ae_x)uu_x + (ec_x - ce_x)u_x u_z + (ed_x - de_x)uu_x u_z \\
&+(eg_x - ge_x)u_x + e(d+e)u_x^2 u_z + (ae + eb_x - be_x)u_x^2 - e_x u_t u_x \\
&+eu_x u_{xt} + deuu_x u_{xz} + (ce + ef_x - fe_x)u_x u_{xz} - egu_{xx} - aeuu_{xx} \\
&-eu_t u_{xx} - ceu_{xx}u_z - deuu_{xx}u_z - efu_{xx}u_{xz} + efu_x u_{xxz} = 0,
\end{aligned} \tag{5}$$

where a, b, c, d, e, f and g denote $a(x,z,t)$, $b(x,z,t)$, $c(x,z,t)$, $d(x,z,t)$, $e(x,z,t)$, $f(x,z,t)$ and $g(x,z,t)$ respectively. We assume the following singularity manifold expansion with $\phi = \phi(x,z,t)$ for $u = u(x,z,t)$:

$$u = \phi^\alpha \sum_{j=0}^{\infty} u_j \phi^j, \tag{6}$$

where ϕ and the coefficients u_j are analytic functions of the independent variables x, z, t, and $\phi(x,z,t) = 0$ defines the singularity manifold. By a leading order analysis, substituting

$$u = \phi^\alpha u_0, \tag{7}$$

into equation (5), we obtain $\alpha = -1$ and $u_0 \neq 0$. By the substitution of expansion (7) with $\alpha = -1$ into equation (5), the recursion relations for the u_j are presented as follows

$$\begin{aligned}
&(j-1)(j-2)(j+1)e(x,z,t)f(x,z,t)^2 \phi_x^4 \phi_z u_j \\
&= F(u_{j-1}, \cdots, u_0, \phi_t, \phi_x, \phi_z, \ldots),
\end{aligned} \tag{8}$$

where the explicit dependence on t, x, z of the right-hand side comes from that of the coefficients. It is found that the resonances occur at

$$j = -1, 1, 2. \tag{9}$$

Let us note here that the leading order and resonances are the same result as for the $(2+1)$-dimensional Burgers equation (4). From recurrence relations, we find

$$j=0: \quad u_0 = \frac{2f}{d+e}\phi_x, \tag{10}$$

$$j=1: \quad \frac{8e^2f^2z}{\{d+e\}^4}\times\Big[\{f_z(d+e)-f(d_z+e_z)\}\phi_x^5 - \{f_x(d+e)-f(d_x+e_x)\}\phi_x^4\phi_x\Big]=0, \tag{11}$$

$$j=2: \quad \frac{1}{(d+e)^5}\Big[4ef^3(d-e)(d+e)^2\{\phi_x^2\phi_{xx}\phi_{xz}+\phi_x^2\phi_{xxx}\phi_z-\phi_{xx}^2\phi_x\phi_z - \phi_x^3\phi_{xxz}\} + \cdots + f(u_1(d_z+e_z)+3b_x-2f_{xz}))\}]\phi_x^4\Big]=0, \tag{12}$$

in lower orders. Now we look into cases to pass the Painlevé test. We take into account only the following cases

1. $e=0, \quad f\neq 0,$
2. $e\neq 0, \quad f\neq 0, \quad d=d(t), \quad e=e(t), \quad f=f(t),$
3. $e\neq 0, \quad f\neq 0, \quad f=(d+e)\exp h(t),$

where $h(t)$ is a constant of integration with respect to x and z. It is easily checked that Case 1 is not determined the leading order and the resonances. And it is easy to see that Case 2 is a special case of Case 3. Now we discuss the following form of equation in the Case 3:

$$\begin{aligned}&u_t + a(x,z,t)u + b(x,z,t)u_x + c(x,z,t)u_z + d(x,z,t)uu_z \\ &+e(x,z,t)u_x\partial_x^{-1}u_z + \exp\{h(t)\}\{d(x,z,t)+e(x,z,t)\}u_{xz} \\ &+g(x,z,t)=0.\end{aligned} \tag{13}$$

Substituting (6) into equation (13), we find

$$\begin{aligned}&j=0: \quad u_0 = 2\exp\{h(t)\}\phi_x,\\ &j=1: \quad u_1 : \text{arbitrary},\end{aligned}$$

from the recurrence relations (10) and (11). For $j=2$, we have

$$\begin{aligned}&4\exp\{2h(t)\}(eb_x-be_x-ae-eh'(t))\phi_x^4 + 4\exp\{2h(t)\}(ec_x-ce_x)\phi_x^3\phi_z \\ &+4\exp\{3h(t)\}e(d-e)(\phi_x^2\phi_{xx}\phi_{xz}-\phi_{xx}^2\phi_x\phi_z-\phi_x^3\phi_{xxz}+\phi_x^2\phi_{xxx}\phi_z) \\ &-4\exp\{2h(t)\}e_x\phi_x^3\phi_t + 4\exp\{3h(t)\}(ed_x-de_x)\phi_x^2\phi_{xx}\phi_z = 0.\end{aligned} \tag{14}$$

Only when setting

$$a(x,z,t)=b_x(x,z,t)-h'(t), \quad c=c(z,t), \quad d=d(z,t), \quad e=d(z,t), \tag{15}$$

the resonance at $j = 2$ occurs. Here the prime of $h(t)$ stands for differentiation with respect to the temporal variable t. This leads to a (2+1)-dimensional Burgers equation with variable coefficients:

$$u_t + \{b_x(x,z,t) - h'(t)\}u + b(x,z,t)u_x + c(z,t)u_z + d(z,t)uu_z$$
$$+d(z,t)u_x\partial_x^{-1}u_z + 2\exp\{h(t)\}d(z,t)u_{xz} + g(x,z,t) = 0. \qquad (16)$$

From the arbitrariness of resonance functions u_1 and u_2, we can set a generalized Cole–Hopf transformation for equation (16):

$$u = u_0\phi^{-1} = 2\exp\big\{h(t)\big\}\frac{\phi_x}{\phi}. \qquad (17)$$

In the case of $g(x,z,t) = 0$, by transformation (17), equation (16) is reduced to the linear equation:

$$\phi_t + b(x,z,t)\phi_x + c(z,t)\phi_z + 2\exp\big\{h(t)\big\}d(z,t)\phi_{xz} = 0. \qquad (18)$$

Setting $\partial/\partial z = \partial/\partial x$, equation (16) is dimensionally reduced to the $(1+1)$-dimensional Burgers equation with variable coefficients appeared in the reference [5], which demonstrates that lower dimensional Burgers equation with variable coefficients can be reduced to the autonomous Burgers equation.

2.3 KdV Equation with Variable Coefficients in (2 + 1) Dimensions

We discuss the following higher-dimensional KdV type equation for $u = u(x,z,t)$:

$$u_t + a(x,z,t)u + b(x,z,t)u_x + c(x,z,t)u_z + d(x,z,t)uu_z$$
$$+e(x,z,t)u_x\partial_x^{-1}u_z + f(x,z,t)u_{xxz} + g(x,z,t) = 0. \qquad (19)$$

Equation (19) includes the standard higher-dimensional KdV equation [16]:

$$u_t + uu_z + \frac{1}{2}u_x\partial_x^{-1}u_z + \frac{1}{4}u_{xxz} = 0, \qquad (20)$$

which, by setting $\partial/\partial z = \partial/\partial x$, reads the (ordinary) KdV equation well-known to be integrable. We determine the coefficients of equation (19) to pass the Painlevé test. Here a potential field $U = U(x,z,t)$ for the original u is defined as $u = U_x$, since the non-local term of equation (19) should eliminate to perform the Painlevé test. Then we are now looking for a solution of equation (19) in terms of U in the Laurent series expansion:

$$U = \phi^\alpha \sum_{j=0}^{\infty} U_j\phi^j, \qquad (21)$$

where U_j are analytic functions of the independent variables in a neighborhood of $\phi = 0$. In this case, the leading order -1 and

$$U_0 = 12f(x,z,t)\phi_x/(d(x,z,t) + e(x,z,t))$$

are given. Substituting expansion (21) with $\alpha = -1$, the recursion relations for the U_j are presented as follows:

$$
\begin{aligned}
&(j-1)(j-4)(j-6)(j+1)f(x,z,t)\phi_x^3\phi_z U_j \\
&= F(U_{j-1},\cdots,U_0,\phi_t,\phi_x,\phi_z,\cdots),
\end{aligned} \tag{22}
$$

where the explicit dependence on x, z, t of the right-hand side comes from that of the coefficients. Then, it is found that the resonances occur at $j = -1$, 1, 4 and 6, substituting expansion (21) with $\alpha = -1$ into equation (19) in terms of U. We are succeeded in finding two types of the higher-dimensional KdV equation with variable coefficients. One of them is

$$
\begin{aligned}
&u_t + \frac{2}{3}x\big\{\alpha(z,t) - \beta(t) + c_z(z,t)\big\}u_x + c(z,t)u_z \\
&+\left(\frac{d'(t)}{d(t)} - \frac{f'(t)}{f(t)} + \frac{4}{3}\big\{\alpha(z,t) - \beta(t) + c_z(z,t)\big\}\right)u \\
&+d(t)uu_z + \frac{d(t)}{2}u_x\partial_x^{-1}u_z + f(t)u_{xxz} + g(z,t) = 0,
\end{aligned} \tag{23}
$$

and another is

$$
\begin{aligned}
&u_t + \Big(2B_1(z,t) - \eta'(t)\Big)u + c(z,t)u_z + \big\{B_1(z,t)x + B_2(z,t)\big\}u_x \\
&+d(z,t)uu_z + \frac{d(z,t)}{2}u_x\partial_x^{-1}u_z + \frac{3}{2}\exp\{\eta(t)\}d(z,t)u_{xxz} + g(z,t) = 0,
\end{aligned} \tag{24}
$$

where $\alpha(z,t)$, $\beta(t)$, $B_1(z,t)$, $B_2(z,t)$ and $\eta(t)$ being arbitrary functions. Setting $\partial/\partial z = \partial/\partial x$, equations (23) and (24) are reduced to the (1+1)-dimensional KdV equations with variable coefficients [5, 8, 9].

3 Conclusions

In this paper, we have presented new $(1+1)$- and $(2+1)$- dimensional integrable nonlinear equations with variable coefficients. In section 2 we have reviewed the Painlevé test and constructed a higher-dimensional Burgers equation with variable coefficients (16). Via truncating the Laurent expansion, we have presented a generalized Cole-Hopf transformation. And then we have also obtained higher-dimensional KdV equations with variable coefficients (23) and (24). We presented exact solutions, hierarchies and families of equations (16), (23) and (24) in reference [17].

Finally let us mention a special and interesting case of equation (23). Setting the following condition of variable coefficients:

$$
\alpha(z,t) - \beta = \frac{3}{4}\frac{G'(t)}{G(t)}, \quad c = 0,\ d = 1,\ f = \frac{1}{4},\ g = 0, \tag{25}
$$

equation (23) becomes

$$u_t + uu_z + \frac{1}{2}u_x\partial_x^{-1}u_z + \frac{1}{4}u_{xxz} - \frac{G'(t)}{G(t)}u - \frac{xG'(t)}{2G(t)}u_x = 0. \tag{26}$$

Equation (26) is a higher-dimensional integrable version of the general KdV (GKdV) equation [18]:

$$u_t + \frac{3}{2}uu_x + \frac{1}{4}u_{xxx} - \frac{G'(t)}{G(t)}u - \frac{xG'(t)}{2G(t)}u_x = 0, \tag{27}$$

by the dimensional reduction $\partial/\partial z = \partial/\partial x$. And using the *Lax-pair Generating Technique* [19, 20], we obtain its Lax Pair given by

$$L = \frac{1}{G(t)}\left(\partial_x^2 + u\right) - \lambda \equiv \frac{1}{G(t)}L_{\mathrm{GKdV}} - \lambda, \tag{28}$$

$$T = \partial_z L_{\mathrm{GKdV}} + \widetilde{T} + \partial_t, \tag{29}$$

which

$$\widetilde{T} = \frac{1}{2}\left(\partial_x^{-1}u_z - \frac{xG'(t)}{G(t)}\right)\partial_x - \frac{1}{4}\left(u_z - \frac{G'(t)}{G(t)}\right). \tag{30}$$

Here $\lambda = \lambda(z,t)$ is the spectral parameter and satisfies the non-isospectral condition $\lambda_t = \lambda\lambda_z$ [21–23]. In reference [17] we have presented modified GKdV and general Calogero-Degasperis-Fokas equations [24, 25] from the Lax pair (28) and (29) using the Lax-pair Generating Technique.

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Symmetries and Group Invariant Reductions of Integrable Partial Difference Equations

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The interplay between the symmetries of compatible discrete and continuous integrable systems in two dimensions is investigated. Master and higher symmetries for certain quadrilateral lattice equations are found. The usage of symmetries in obtaining group invariant reductions on the lattice is also discussed.

1 Introduction

The investigations of Bäcklund, in the late nineteenth century, of possible extensions of Lie contact transformations led him to introduce an important class of surface transformations in ordinary space. The intimate connection of Bäcklund transformations with certain type nonlinear equations, which from a modern perspective are called integrable systems, has been the subject of intensive investigations over the past century. A detailed account on Bäcklund transformations can be found in the recent works [1, 2]. Integrable systems are also characterized by an extremely high degree of symmetry. As a result, Lie symmetries and their generalizations have proven to be invaluable tools for generating solutions and obtaining classification results for this kind of systems, cf [3] and contributions in this volume.

Due to a commutativity property, Bäcklund transformations possess the interesting feature that repeated applications can be performed in a purely algebraic fashion. This is known in classical geometry as the Bianchi permutability theorem and represents a nonlinear analogue of the superposition principle for linear homogeneous differential equations. The prototypical example is given by the equation

$$(p-q)\tan\left(\frac{u_{12}-u}{4}\right) = (p+q)\tan\left(\frac{u_2-u_1}{4}\right). \tag{1}$$

It relates a solution u_{12} of the sine-Gordon equation

$$u_{xy} = \sin u\,, \tag{2}$$

with an arbitrary seed solution u and two solutions u_1 and u_2 obtained from u via the Bäcklund transformations specified by the parameters values p and q, respectively.

On the other hand, equation (1) may be interpreted as a partial difference equation. This interpretation is obtained by simply identifying u_1 and u_2, respectively, with the values attained by the dependent variable u when the discrete independent variables n_1 and n_2 change by a unit step.

Recent advances in the theory of integrable systems show that discrete systems are equally important to their continuous analogues, and their study has led to new insights into the structures behind the more familiar continuous systems. Thus, standard symmetry techniques applied to integrable discrete equations have attracted the attention of many investigators, see e.g. [4–11]. More general symmetry approaches are being pursued starting from different philosophies, see e.g. [12–18] and references therein.

In the present work, symmetries and invariant reductions of certain partial difference equations on elementary quadrilaterals are investigated. The approach to this problem originates in the interplay between integrable quadrilateral equations and their compatible continuous PDEs, as this has been addressed recently in [8, 11].

2 Symmetries of Quadrilateral Equations

Central to our considerations on the discrete level are equations on quadrilaterals, i.e. equations of the form

$$\mathcal{H}(F_{(0,0)}, F_{(1,0)}, F_{(0,1)}, F_{(1,1)}; p, q) = 0\,. \tag{3}$$

They may be regarded as the discrete analogues of hyperbolic type partial differential equations (PDEs) involving two independent variables. The dependent variables (fields) are assigned on the vertices at sites (n_1, n_2) which vary by unit steps only, and the continuous lattice parameters $p, q \in \mathbb{C}$ are assigned on the edges of an elementary quadrilateral (Fig. 1). The updates of a lattice variable $F \in \mathbb{C}$, along a shift in the n_1 and n_2 direction of the lattice are denoted by $F_{(0,1)}, F_{(0,1)}$ respectively, i.e.

$$F_{(1,0)} = F(n_1+1, n_2), \quad F_{(0,1)} = F(n_1, n_2+1), \quad F_{(1,1)} = F(n_1+1, n_2+1)\,. \tag{4}$$

A specific equation of the type (3) is given by the Bianchi lattice (1). Its linearized version is the partial difference equation (PΔE)

$$(p-q)(f_{(1,1)} - f) = (p+q)(f_{(0,1)} - f_{(1,0)})\,. \tag{5}$$

The aim now is to find the symmetries of equation (5) and successively to find the corresponding group invariant solutions. An indirect approach in dealing with such a problem is to derive first a compatible set of differential-difference and partial differential equations, by interchanging the role of the discrete variables (n_1, n_2) with that of the continuous parameters (p, q). The reasoning behind this construction is that one could set up a natural framework for the description of

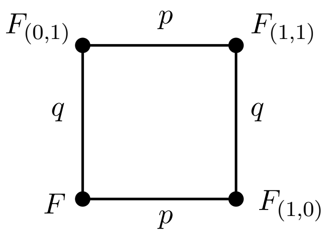

Figure 1. An elementary quadrilateral.

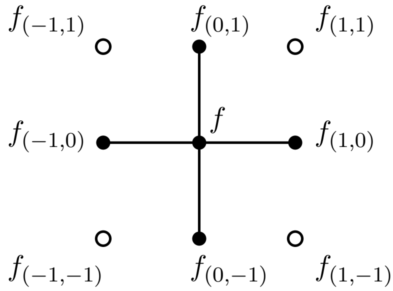

Figure 2. A cross configuration.

the symmetry and the symmetry reduction of discrete systems, by exploiting the notion of Lie-point symmetries and the infinitesimal methods for obtaining them, which are well known for the continuous PDEs. We next illustrate the relevant construction for the PΔE (5).

A particular solution of equation (5) is

$$f = \left(\frac{p-\lambda}{p+\lambda}\right)^{n_1}\left(\frac{q-\lambda}{q+\lambda}\right)^{n_2}, \tag{6}$$

$\lambda \in \mathbb{C}$. Differentiating f with respect to p, (respectively q) and rearranging terms, we easily find that f also satisfies the differential-difference equations (DΔEs)

$$f_p = \frac{n_1}{2p}(f_{(-1,0)} - f_{(1,0)}), \quad f_q = \frac{n_2}{2q}(f_{(0,-1)} - f_{(0,1)}), \tag{7}$$

where the minus sign denotes backward shift in the direction of the corresponding discrete variable.

By interchanging completely the role of the lattice variables n_1, n_2 with that of the continuous lattice parameters p, q, the aim now is to find a PDE which is compatible with equations (5) and (7). Using similar considerations as in [19], we find that such a PDE is the fourth order equation obtained from the Euler-Lagrange equation

$$\partial_{pq}\left(\frac{\partial L}{f_{pq}}\right) - \partial_p\left(\frac{\partial L}{f_p}\right) - \partial_q\left(\frac{\partial L}{f_q}\right) = 0, \tag{8}$$

for the variational problem associated with the Lagrangian density

$$L = \frac{1}{2}(p^2-q^2)(f_{pq})^2 + \frac{2}{p^2-q^2}(n_2 f_p - n_1 f_q)(n_2 p^2 f_p - n_1 q^2 f_q). \tag{9}$$

Two of the divergence symmetries of Lagrangian L are the scaling transformations

$$p \mapsto \alpha p, \quad q \mapsto \alpha q, \quad f \mapsto \beta f, \quad \alpha, \beta \in \mathbb{C}, \ \alpha, \beta \neq 0. \tag{10}$$

Since every divergence symmetry of a variational problem is inherited as a Lie-point symmetry by the associated Euler-Lagrange equations, the transformations (10) are Lie-point symmetries of equations (8). They correspond to the characteristic symmetry generator

$$X_Q = Q\,\partial_f, \quad \text{where} \quad Q = c_1(pf_p + qf_q) + c_2\,f, \quad c_1\,, c_2 \in \mathbb{C}. \tag{11}$$

In view of the compatible DΔEs (7), the characteristic Q takes the form

$$Q = \frac{c_1}{2}\left(n_1(f_{(-1,0)} - f_{(1,0)}) + n_2(f_{(0,-1)} - f_{(0,1)})\right) + c_2 f\,. \tag{12}$$

Equations (5), (7) and (8) form a compatible set of equations, in the sense that they share a common set of solutions. By virtue of this fact and since the symmetry generator X_Q given by (11) maps solutions to solutions of PDE (8), X_Q, with Q given by (12), should generate a symmetry of the discrete equation (5). In other words, Q given by (12) should satisfy

$$(p-q)(Q_{(1,1)} - Q) = (p+q)(Q_{(0,1)} - Q_{(1,0)})\,, \tag{13}$$

for all solutions f of (5). It should be noted that Q depends on the values of f and the four adjacent values on the lattice. Taking into account equation (5) and its backward discrete consequences, we easily find that equation (13) holds. Thus, Q is indeed a symmetry characteristic of equation (5).

The above considerations lead us naturally to assume that the symmetry characteristic Q of a general quadrilateral equation (3) initially depends on the values of f assigned on the points which form the *cross* configuration of Fig. 2. In other words, we are led to adopt the following definition.

Let Q be a scalar function which depends on the values of F and their shifts forming the cross configuration of Fig. 2. We denote the *first prolongation* of a vector field $X_Q = Q\,\partial_F$, by the vector field

$$X_Q^{(1)} = Q\,\partial_F + Q_{(-1,0)}\,\partial_{F_{(-1,0)}} + Q_{(0,1)}\,\partial_{F_{(0,1)}} + Q_{(0,-1)}\,\partial_{F_{(0,-1)}} + Q_{(0,1)}\,\partial_{F_{(0,1)}}. \tag{14}$$

Similarly, the second prolongation of X_Q is denoted by

$$X_Q^{(2)} = X_Q^{(1)} + Q_{(-1,-1)}\partial_{F_{(-1,-1)}} + Q_{(-1,1)}\partial_{F_{(-1,1)}} + Q_{(1,-1)}\partial_{F_{(1,-1)}} + Q_{(1,1)}\partial_{F_{(1,1)}}. \tag{15}$$

We say that $X_Q = Q\,\partial_F$ is a symmetry generator of the quadrilateral equation (3), if and only if

$$X_Q^{(2)}(\mathcal{H}) = 0\,, \tag{16}$$

holds, where equation (3) and its backward discrete consequences should be taken into account.

2.1 Symmetries of the Linearized Bianchi Lattice

The symmetries of equation (5) are determined from the functional equation (13). Two simple solutions of the latter give the symmetry generators

$$X_1 = (\,\mu + \lambda(-1)^{n_1+n_2}\,)\partial_f, \quad X_2 = f\partial_f\,. \tag{17}$$

Symmetry characteristics corresponding to the cross configuration of Fig. 2, and which can be found by exploiting the correspondence with the continuous PDE, are given by the vector fields

$$Y_1 = \big(f_{(1,0)} - f_{(-1,0)}\big)\,\partial_f, \quad Y_2 = \big(f_{(0,1)} - f_{(0,-1)}\big)\,\partial_f, \tag{18}$$

$$Z = \big(n_1(f_{(1,0)} - f_{(-1,0)}) + n_2(f_{(0,1)} - f_{(0,-1)})\big)\,\partial_f\,. \tag{19}$$

The latter serve to construct an infinite number of symmetries. This follows from the fact that the commutator of two symmetry generators is again a symmetry generator. Let

$$Q_{[i,0]} = f_{(i,0)} - f_{(-i,0)}\,, \quad Q_{[0,j]} = f_{(0,j)} - f_{(0,-j)} \quad i,j \in \mathbb{N}\,, \tag{20}$$

be the characteristics of the vector fields

$$Y_{Q_{[i,0]}} = Q_{[i,0]}\partial_f\,, \quad Y_{Q[0,j]} = Q_{[0,j]}\partial_f\,, \quad i,j \in \mathbb{N}\,. \tag{21}$$

By induction we find that

$$Y_{Q_{[i-1,0]}} + \frac{1}{i}\,[Z, Y_{Q_{[i,0]}}] = Y_{Q_{[i+1,0]}}, \;\; Y_{Q_{[0,j-1]}} + \frac{1}{j}\,[Z, Y_{Q_{[0,j]}}] = Y_{Q_{[0,j+1]}}, \tag{22}$$

holds $\forall\, i,j \in \mathbb{N} \setminus \{0\}$. Repeated applications of the commutation relations (22) produce new symmetries of equation (5), and thus the vector field Z represents a *master symmetry*. The generated new symmetries correspond to extended cross configurations.

2.2 Symmetries of the Discrete Korteweg–de Vries Equation

We next demonstrate how the above considerations can be applied equally well to a nonlinear discrete equation, namely the discrete Korteweg–de Vries (KdV) equation [20]

$$(f_{(1,1)} - f)(f_{(1,0)} - f_{(0,1)}) = p - q\,. \tag{23}$$

Recently in [22], the compatible differential-difference system

$$f_p = \frac{n_1}{f_{(1,0)} - f_{(-1,0)}}\,, \quad f_q = \frac{n_2}{f_{(0,1)} - f_{(0,-1)}}\,, \tag{24}$$

was derived, along with the compatible PDE which is the Euler-Lagrange equation for the variational problem associated with the Lagrangian density

$$L = (p-q)\frac{(f_{pq})^2}{f_p f_q} + \frac{1}{p-q}\left((n_2)^2 \frac{f_p}{f_q} + (n_1)^2 \frac{f_q}{f_p}\right) . \tag{25}$$

The importance of the above Lagrangian stems from the fact that the commuting generalized symmetries of the associated Euler-Lagrange equation *generate* the complete hierarchy of the KdV soliton equations, (cf [21] for generalizations of the above results). Moreover, the Euler–Lagrange equation acquires a certain physical significance, since it incorporates the hyperbolic Ernst equation for an Einstein–Weyl field [23]. Thus, it would be interesting to find symmetries and special solutions on the discrete level as well.

Exploiting the symmetries of the continuous PDE and the interplay between the compatible set of differential and difference equations, we find the following symmetries of the discrete KdV equation

$$X_1 = \partial_f , \quad X_2 = (-1)^{n_1+n_2} f\partial_f , \tag{26}$$

$$Y_1 = \frac{1}{f_{(1,0)} - f_{(-1,0)}}\,\partial_f, \quad Y_2 = \frac{1}{f_{(0,1)} - f_{(0,-1)}}\,\partial_f, \tag{27}$$

$$Z_1 = \left(\frac{n_1}{f_{(1,0)} - f_{(-1,0)}} + \frac{n_2}{f_{(0,1)} - f_{(0,-1)}}\right)\partial_f , \tag{28}$$

$$Z_2 = \left(\frac{n_1\, p}{f_{(1,0)} - f_{(-1,0)}} + \frac{n_2\, q}{f_{(0,1)} - f_{(0,-1)}} - \frac{1}{2}f\right)\partial_f , \tag{29}$$

Taking the commutator of Z_1 with Y_1, one finds the new symmetry generator

$$[Z_1, Y_1] = \frac{1}{(f_{(1,0)} - f_{(-1,0)})^2}\left(\frac{1}{f - f_{(2,0)}} + \frac{1}{f_{(-2,0)} - f}\right)\partial_f \tag{30}$$

and a similar relation can be found for the commutator $[Z_1, Y_2]$. Further new symmetries are obtained by taking the commutator of Z_1 with the resulting new symmetries, which are omitted here because of their lenght.

3 Symmetry Reduction on the Lattice

Let $\mathcal{H} = 0$ be a quadrilateral equation of the form (3) and X_Q a symmetry generator. In analogy with the continuous PDEs, we say that a solution of $\mathcal{H} = 0$ is invariant under X_Q, if it satisfies the compatible constraint $Q = 0$.

Let us now consider the linearized Bianchi lattice (5) and a linear combination of the symmetries Y_1 and Y_2 given by equation (18). The corresponding invariant solutions are obtained from the compatible system

$$(p-q)(f_{(1,1)}-f) = (p+q)(f_{(0,1)}-f_{(1,0)}) , \quad f_{(1,0)}-f_{(-1,0)} = c(f_{(0,1)}-f_{(0,-1)}). \tag{31}$$

The method for obtaining the invariant solutions on the lattice is similar to the direct substitution method for the invariant solutions of PDEs. The aim is to derive from the above discrete system, equations where the variables are given in terms of only one direction of the lattice, i.e. to derive an ordinary difference equation. To this end, we define auxiliary dependent variables

$$x = f_{(1,1)} - f\,, \qquad a = f_{(1,0)} - f_{(-1,0)}\,, \tag{32}$$

$$y = f_{(1,0)} - f_{(0,1)}\,, \quad b = f_{(0,1)} - f_{(0,-1)}\,. \tag{33}$$

It follows from equations (32)–(33) that

$$b_{(1,0)} = x - y_{(0,-1)}\,, \quad b = x_{(0,-1)} - y\,, \tag{34}$$

$$a_{(0,1)} = x + y_{(-1,0)}\,, \quad a = x_{(-1,0)} + y\,. \tag{35}$$

Using the above relations and the system (31), we arrive at the second order linear ordinary difference equation (OΔE) for the variable x

$$x_{(2,0)} - \big(c\,(r - r^{-1}) - (r + r^{-1})\big)\,x_{(1,0)} + x = 0\,, \tag{36}$$

where $r = (q+p)/(q-p)$. Equation (36) can be easily solved, giving

$$x = c_1(n_2)\,{\mu_1}^{n_1} + c_2(n_2)\,{\mu_2}^{n_2}\,, \tag{37}$$

where μ_1, μ_2 are the two roots of the characteristic polynomial of equation (36). In a similar manner, the arbitrary functions c_1, c_2 of n_2 are determined from (31), (32)–(33) and their consequences, leading finally to the invariant form of f.

We conclude this section by considering a specific symmetry reduction of the discrete KdV (23). For the compatible symmetry constraint we choose a linear combination of Y_1 and Y_2 given by (27), leading to the same symmetry constraint as in the previous case ($\tilde{c} = 1/c$). With the help of the same auxiliary variables (32)–(33), we arrive at the following OΔE

$$w_{(1,0)} = \frac{\alpha\,w + \beta}{\gamma\,w + \delta}\,, \tag{38}$$

where $w = x\,x_{(-1,0)}$ and the parameters are given by $\alpha = -\delta = r\,\tilde{c}$, $\beta = r^2(1+\tilde{c})$, $\gamma = 1 - \tilde{c}$ and $r = p - q$. Equation (38) is a discrete Riccati equation which can be solved explicitly, by using the symmetry generator

$$X = \big(\gamma w^2 + (\delta - \alpha)w - \beta\big)\partial_w\,. \tag{39}$$

It should be noted that, when $\tilde{c} = -1$, the invariant solutions obtained above correspond to the periodic reduction $f_{(-1,1)} = f_{(1,-1)}$.

4 Concluding Remarks

The main purpose of this work was to demonstrate that the notions of symmetry and invariance on the discrete level arise naturally from the interplay between PΔEs and PDEs that share a common set of solutions. Moreover, certain symmetry characteristics which admit the aforementioned cross configuration can be used to derive invariant solutions, in exact analogy with the invariant solutions of the continuous PDEs. In connection with the latter issue, recently in [11], a parameter family of discrete OΔEs which are compatible with the full Painlevé VI differential equations was derived. More recently in [19], the discrete multi-field Boussinesq system and the compatible PDEs were investigated. It was shown that scaling invariant solutions of the relevant PDEs are built from solutions of higher Painlevé equations, which potentially lead to solutions in terms of new transcendental functions. Thus, it is even more interesting to find the compatible discrete reduced system.

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Classes of Linearizable Wave Equations

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In some recent papers [1–3] we took in consideration a class of nonlinear wave equations. In those papers, we obtained its equivalence algebra $\mathcal{L}_{\mathcal{E}}$, first order differential invariant and invariant equations with respect to the equivalence group $G_{\mathcal{E}}$. Further, by means of the invariant equations and/or the differential invariants, we found subclasses of linearizable equations. Here, we continue this research and characterize a new subclass of equations which can be mapped by an equivalence transformation of $G_{\mathcal{E}}$ in a linear equation of the same family.

1 Introduction

In this short note, by using the results obtained in the papers [1–3], we characterize a new subclass of the family of nonlinear wave equations

$$u_{tt} - u_{xx} = f(u, u_t, u_x) \tag{1}$$

which can be transformed by an equivalence transformation of the group $G_{\mathcal{E}}$ in a linear equation of the same class.

The equations of this type arise in different fields of mathematical physics. They describe, for instance, the motion of vibrating strings, sound and electromagnetic waves. Moreover, they are used in gas dynamics, chemical technology, to cite only a few application fields.

In the next section we recall in short the results obtained in [1, 2] which we need in the following. In the section 3, we illustrate the method which allows us to obtain, by using the invariant equations and/or the differential invariants, the new subclass of family (1) which can be transformed in a linear equation.

2 Preliminary Results

In the paper [1] we considered the family of nonlinear wave equations (1) and, by using the infinitesimal method, we found that its equivalence algebra $\mathcal{L}_{\mathcal{E}}$ is infinite-dimensional and is spanned by the operators:

$$Y_0 = \partial_t, \quad Y_1 = \partial_x, \quad Y_2 = x\partial_t + t\partial_x - u_x\partial_{u_t} - u_t\partial_{u_x},$$
$$Y_3 = t\partial_t + x\partial_x - 2f\partial_f - u_t\partial_{u_t} - u_x\partial_{u_x},$$
$$Y_\varphi = \varphi\partial_u + [\varphi' f + \varphi''(u_t^2 - u_x^2)]\partial_f + \varphi' u_t\partial_{u_t} + \varphi' u_x\partial_{u_x},$$

where $\varphi = \varphi(u)$ is an arbitrary function of u and prime denotes derivative with respect to u.

Following the method proposed by Ibragimov [4, 5], by setting

$$\lambda_1 \equiv (u_t + u_x)(f_{u_t} + f_{u_x}) - 2f, \quad \lambda_2 \equiv (u_t - u_x)(f_{u_t} - f_{u_x}) - 2f,$$

we showed in [2] that the class of equations (1):

- at zero order possesses the invariant equation $u_t^2 - u_x^2 = 0$, but does not have differential invariants;
- at first order possesses two invariant equations, namely $\lambda_1 = 0$, $\lambda_2 = 0$ and the following differential invariant with respect to the equivalence group $G_{\mathcal{E}}$:

$$\lambda \equiv \frac{\lambda_2}{\lambda_1}.$$

3 Applications

Here we show an application in which the knowledge of the invariant equations and/or the differential invariants can be useful for the linearization of subclasses of equations (1).

We consider the case in which only the invariant equation

$$\lambda_1 \equiv (u_t + u_x)(f_{u_t} + f_{u_x}) - 2f = 0 \tag{2}$$

is satisfied.
In this case, we observe that the general form of f satisfying the invariant equation $\lambda_1 = 0$ is

$$f = (u_t + u_x)g(u, u_t - u_x),$$

with g an arbitrary function of its arguments.
So, we consider the subclass of the equations (1) of the form

$$u_{tt} - u_{xx} = (u_t + u_x)g(u, u_t - u_x). \tag{3}$$

Since the equation (2) is invariant with respect to $G_{\mathcal{E}}$, any equation (3) is transformed by an equivalence transformation of the group $G_{\mathcal{E}}$ into an equation on the same subclass.

We observe that the linear equation

$$u_{tt} - u_{xx} = (u_t + u_x)k \tag{4}$$

falls in this subclass, where k is an arbitrary constant.

In the following, we suppose $k \neq 0$ because if $k = 0$ we fall in the case $\lambda_1 = \lambda_2 = 0$ already studied in [3].

In order to get further information about the possibility to map an equation of the form (3) in the linear form (4), we look for the second order differential invariants of the subclass (3).

With this aim, we consider the following change of variables:

$$\bar{t} = t, \qquad \bar{x} = x, \qquad \bar{u} = u, \tag{5}$$

$$\sigma = u_t + u_x, \quad \tau = u_t - u_x, \quad \sigma g(\bar{u}, \tau) = f \tag{6}$$

and we write the subclass (3) in the form

$$\bar{u}_{\bar{t}\bar{t}} - \bar{u}_{\bar{x}\bar{x}} = \sigma g(\bar{u}, \tau). \tag{7}$$

So we search for the second order differential invariants of the class (7), that is, we search for functions of the form

$$J = J(\bar{t}, \bar{x}, \bar{u}, \tau, g, g_{\bar{u}}, g_\tau, g_{\bar{u}\bar{u}}, g_{\bar{u}\tau}, \ldots),$$

which are invariant with respect to the infinitesimal equivalence generator

$$\bar{Y} = \bar{\xi}^1 \partial_{\bar{t}} + \bar{\xi}^2 \partial_{\bar{x}} + \bar{\eta} \partial_{\bar{u}} + \Sigma \partial_\sigma + T \partial_\tau + \nu \partial_g.$$

In order to obtain the new coordinates $\bar{\xi}^1$, $\bar{\xi}^2$, $\bar{\eta}$, Σ, T, ν, taking into account the procedure concerning with the change of variables stated in [6], we require the invariance of the change of variables (5) and (6) with respect to the generator

$$\begin{aligned} Y + \bar{Y} \equiv c_i Y_i + \bar{Y} \equiv \xi^1 \partial_t + \xi^2 \partial_x + \eta \partial_u + \zeta^1 \partial_{u_t} + \zeta^2 \partial_{u_x} + \mu \partial_f + \\ + \bar{\xi}^1 \partial_{\bar{t}} + \bar{\xi}^2 \partial_{\bar{x}} + \bar{\eta} \partial_{\bar{u}} + \Sigma \partial_\sigma + T \partial_\tau + \nu \partial_g. \end{aligned}$$

This request leads

$$\bar{\xi}^1 = \xi^1, \quad \bar{\xi}^2 = \xi^2, \quad \bar{\eta} = \eta, \quad \Sigma = \zeta^1 + \zeta^2, \quad T = \zeta^1 - \zeta^2, \quad \Sigma g + \sigma\nu = \mu.$$

Then, we are able to write the equivalence generator $\bar{Y}$ in the form

$$\begin{aligned} \bar{Y} = (c_3 \bar{t} + c_0 \bar{x} + c_1) \partial_{\bar{t}} + (c_0 \bar{t} + c_3 \bar{x} + c_2) \partial_{\bar{x}} + \varphi(\bar{u}) \partial_{\bar{u}} + \\ + (c_0 - c_3 + \varphi') \sigma \partial_\sigma + (\varphi' - c_0 - c_3) \tau \partial_\tau + (\varphi'' \tau - c_0 g - c_3 g) \partial_g. \end{aligned}$$

After having got $\bar{Y}$, it is a simple matter to ascertain that, as expected, the equations (7) do not possess differential invariants of zero and first order, while admit the following invariant differential equation of first order:

$$\tau g_\tau - g = 0.$$

We observe that for the linear equation (4) is $g = k \neq 0$ and $\tau g_\tau - g \neq 0$.

So we look for second order differential invariants of equations (7).

After having performed the invariant tests

$$\bar{Y}^{(2)} \left[J(\bar{t}, \bar{x}, \bar{u}, \tau, g, g_{\bar{u}}, g_\tau, g_{\bar{u}\bar{u}}, g_{\bar{u}\tau}, g_{\tau\tau}) \right] = 0,$$

where $\bar{Y}^{(2)}$ is the second prolongation of generator $\bar{Y}$, we give rise to

$$J = J(r_1, r_2)$$

where r_1 and r_2 are defined by

$$r_1 = \frac{\tau g_{\bar{u}} - \tau^2 g_{\bar{u}\tau}}{(g - \tau g_\tau)^2}, \quad r_2 = \frac{\tau^2 g_{\tau\tau}}{g - \tau g_\tau}.$$

For the linear equations (4) it is $r_1 = r_2 = 0$.

So an equation of the subclass (7) can be equivalent to the linear equation (4) only if it has the same differential invariants with rwspect to $G_{\mathcal{E}}$, that is

$$r_1 = r_2 = 0. \tag{8}$$

Then, after looking for the general form of the functions g, which satisfy the conditions (8), we obtain

$$g = \tau h(\bar{u}) + l_0,$$

with h an arbitrary function of $\bar{u}$ and l_0 an arbitrary constant.

Since the conditions (8) are invariant with respect to the equivalence group, any equation of the subclass (7) of the form

$$\bar{u}_{\bar{t}\bar{t}} - \bar{u}_{\bar{x}\bar{x}} = \sigma \left[\tau h(\bar{u}) + l_0\right] \tag{9}$$

is transformed by the equivalence group into an equation of the same form.

Then by recalling the change of variables (5) and (6) we can rewrite equations (9) by using the old variables and we can affirm that the equations of the subclass (1) which could be transformed in the linear form (4) are

$$u_{tt} - u_{xx} = (u_t + u_x)[(u_t - u_x)h(u) + l_0]. \tag{10}$$

Conversely we will show that there exists at least an equivalence transformation of $G_{\mathcal{E}}$ mapping the equations (10) in (4). By applying to the equations (10) the transformation

$$u = \psi(v),$$

with $\psi'(v) \neq 0$, we get

$$v_{tt} - v_{xx} = (v_t + v_x)(v_t - v_x)\left(\psi' h(\psi) - \frac{\psi''}{\psi'}\right) + (v_t + v_x)l_0.$$

So, when ψ satisfies the ODE

$$\psi' h(\psi) - \frac{\psi''}{\psi'} = 0,$$

the equation (10) is transformed in the linear equation (4). As a result, we can affirm:

Theorem 1. *An equation belonging to the class* (1) *can be transformed in the linear form* (4) *by an equivalence transformation of* $G_{\mathcal{E}}$ *if and only if the function* f *is given by*

$$f = (u_t + u_x)[(u_t - u_x)h(u) + l_0].$$

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Approximate Symmetries for a Model Describing Dissipative Media

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Approximate symmetries of a mathematical model describing one-dimensional motion in a nonlinear medium with a small dissipation are studied. In a physical application, the approximate solution is calculated making use of the approximate generator of the first-order approximate symmetry.

1 Introduction

We consider the nonlinear wave equation with a small dissipation of the form

$$w_{tt} - f(w_x)\, w_{xx} = \varepsilon\, w_{xxt}, \tag{1}$$

where f is a smooth function, $w(t,x)$ is the dependent variable, $\varepsilon \ll 1$ is a small parameter and subscripts denote partial derivative with respect to the independent variables t and x.

The equation (1) can describe one-dimensional wave propagation in nonlinear dissipative media and some mathematical questions related to (1), as the existence, uniqueness and stability of weak solutions can be found in [1], moreover a study related to a generalized "shock structure" is showed in [2], while, for $\varepsilon = \lambda_0$ (λ_0 is the viscosity positive coefficient), a symmetry analysis is performed in [12].

As it is well known, a small dissipation is able to prevent the breaking of the wave profile allowing to study the so called "far field".

A technique widely used in studying nonlinear problems is the perturbation analysis performed by expanding the dependent variables in power series of a small parameter (may be a physical parameter or often artificially introduced).

Combination of the Lie group theory and the perturbation analysis give rise to the so-called approximate symmetry theories. The first paper on this subject is due to Baikov, Gazizov and Ibragimov [3]. Successively another method for finding approximate symmetries was proposed by Fushchich and Shtelen [4]. In the method proposed by Baikov, Gazizov and Ibragimov, the Lie operator is expanded in a perturbation series so that an approximate operator can be found. While in the method proposed by Fushchich and Shtelen the dependent variables are expanded in a perturbation series; equations are separated at each order of approximation and the approximate symmetries of the original equations are defined to be the exact symmetries of the system coming out from equating to zero the coefficients of the smallness parameter. Pakdemirli et al. in a recent paper [5] have made a comparison of those two methods. We summarize the main results of their analysis in the following two statements:

a) The expansion of the approximate operator assumed in the method proposed by Baikov, Gazizov and Ibragimov, does not reflect well an approximation in the perturbation sense; in fact, even if one uses a first order approximate operator, the corresponding approximate solution could contain higher order terms;

b) The method proposed by Fushchich and Shtelen is consistent with the perturbation theory and yields correct terms for the approximate solutions but it is impossible to work in hierarchy; in the searching of symmetries there is a coupled system between the equations at several order of approximation, therefore the algebra can increase enormously.

In this paper we follow the guide lines of the method proposed by Fushchich and Shtelen [4] and remove the "drawback" of the impossibility to work in hierarchy. We perform the group classification of the nonlinear function $f(w_x)$ through which equation (1) with the small parameter ε is approximately invariant and search for approximate solutions.

The plan of the paper is the following: the approximate symmetry method is introduced in the next section; the group classification via approximate symmetries is performed in Sec.3; in Sec.4, in a physical application, the approximate solution is calculated by means of the approximate generator of the first-order approximate group of transformations.

2 Approximate Symmetry Method

In general, any solution of (1) will be of the form $w = w(t,x,\varepsilon)$and the one-parameter Lie group of infinitesimal transformations in the (t,x,w)-space of the equation (1), can be considered in the following form:

$$\begin{aligned} \hat{t} &= t + a\,\xi^1(t,x,w(t,x,\varepsilon),\varepsilon) + \mathcal{O}(a^2), \\ \hat{x} &= x + a\,\xi^2(t,x,w(t,x,\varepsilon),\varepsilon) + \mathcal{O}(a^2), \\ \hat{w} &= w + a\,\eta(t,x,w(t,x,\varepsilon),\varepsilon) + \mathcal{O}(a^2), \end{aligned} \tag{2}$$

where a is the group parameter.

Let us suppose that $w(t,x,\varepsilon)$ and $\hat{w}(\hat{t},\hat{x},\varepsilon)$, analytic in ε, can be expanded in power series of ε, i.e.

$$w(t,x,\varepsilon) = w_0(t,x) + \varepsilon\,w_1(t,x) + \mathcal{O}(\varepsilon^2), \tag{3}$$

$$\hat{w}(\hat{t},\hat{x},\varepsilon) = \hat{w}_0(\hat{t},\hat{x}) + \varepsilon\,\hat{w}_1(\hat{t},\hat{x}) + \mathcal{O}(\varepsilon^2), \tag{4}$$

where: w_0 and w_1 are some smooth functions of t and x; $\hat{w}_0$ and $\hat{w}_1$, are some smooth functions of $\hat{t}$ and $\hat{x}$.

Upon formal substitution of (3) in (1), equating to zero the coefficients of zero and first degree powers of ε we arrive at the following system of PDEs

$$L_0 := w_{0tt} - f(w_{0x})\, w_{0xx} = 0, \tag{5}$$
$$L_1 := w_{1tt} - f(w_{0x})\, w_{1xx} = g(w_{0x})\, w_{0xx}\, w_{1x} + w_{0xxt}, \tag{6}$$

where we have set

$$f(w_{0x}) = f(w_x)\,|_{\varepsilon=0}, \quad g(w_{0x}) = \frac{d\, f(w_x)}{d\, w_x}\Big|_{\varepsilon=0}.$$

Hence, w_0 is a solution of the nonlinear wave equation (5) which we call *unperturbed equation*, while w_1 can be determined from the linear equation (6).

In order to have an one-parameter Lie group of infinitesimal transformations of the system (5)–(6), which is consistent with the expansions of the dependent variables (3) and (4), we introduce these expansions in the infinitesimal transformations (2). Upon formal substitution, equating to zero the coefficients of zero and first degree powers of ε, we get the following one-parameter Lie group of infinitesimal transformations in the (t, x, w_0, w_1)-space

$$\begin{aligned} \hat{t} &= t + a\, \xi_0^1(t, x, w_0) + \mathcal{O}(a^2), \\ \hat{x} &= x + a\, \xi_0^2(t, x, w_0) + \mathcal{O}(a^2), \\ \hat{w}_0 &= w_0 + a\, \eta_0(t, x, w_0) + \mathcal{O}(a^2), \\ \hat{w}_1 &= w_1 + a\, [\eta_{10}(t, x, w_0) + \eta_{11}(t, x, w_0)\, w_1] + \mathcal{O}(a^2), \end{aligned} \tag{7}$$

where we have set

$$\xi_0^i(t, x, w_0) = \xi^i(t, x, w(t, x, \varepsilon), \varepsilon)\,|_{\varepsilon=0}, \quad i = 1, 2$$

$$\eta_0(t, x, w_0) = \eta(t, x, w(t, x, \varepsilon), \varepsilon)\,|_{\varepsilon=0},$$

$$\eta_{10}(t, x, w_0) + \eta_{11}(t, x, w_0)\, w_1 = \frac{d\,\eta}{d\,\varepsilon}\Big|_{\varepsilon=0}.$$

Similarly to Fushchich and Shtelen [4], we give the following definition:

Definition 1. We call *approximate symmetries* of equation (1) the (exact) symmetries of the system (5)–(6) through the one-parameter Lie group of infinitesimal transformations (7).

Consequently, the one-parameter Lie group of infinitesimal transformations (7) the associated Lie algebra and the corresponding infinitesimal operator

$$X = \xi^1(t, x, w_0)\frac{\partial}{\partial t} + \xi^2(t, x, w_0)\frac{\partial}{\partial x} + \eta(t, x, w_0)\frac{\partial}{\partial w_0} + [\eta_{10}(t, x, w_0) + \eta_{11}(t, x, w_0)\, w_1]\frac{\partial}{\partial w_1}, \tag{8}$$

are called the approximate Lie group, the approximate Lie algebra and the approximate Lie operator of the equation (1), respectively.

Moreover, after putting

$$X_0 = \xi_0^1(t,x,w_0)\frac{\partial}{\partial t} + \xi_0^2(t,x,w_0)\frac{\partial}{\partial x} + \eta_0(t,x,w_0)\frac{\partial}{\partial w_0}, \tag{9}$$

the approximate Lie operator (8) can be rewritten as

$$X = X_0 + [\eta_{10}(t,x,w_0) + \eta_{11}(t,x,w_0)\, w_1]\frac{\partial}{\partial w_1} \tag{10}$$

and X_0 can be regarded as the infinitesimal operator of the unperturbed equation (5).

It is worthwhile noticing that, thanks to the functional dependencies of the coordinates of the approximate Lie operator (8) (or (10)), now we are able to work in hierarchy in finding the invariance conditions of the system (5)–(6): firstly, by classifying the unperturbed equation (5) through the operator (9) and after by determining η_{10} and η_{11} from the invariance condition that follows by applying the operator (10) to the linear equation (6). In fact the invariance condition of the system (5)–(6) reads:

$$X_0^{(2)}(L_0)\Big|_{L_0=0} = 0, \tag{11}$$

$$X^{(3)}(L_1)\Big|_{L_0=0,\ L_1=0} = 0, \tag{12}$$

where $X_0^{(2)}$ and $X^{(3)}$ are the second and third extensions of the operators X_0 and X, respectively.

Finally, the procedure outlined above is a variant of that developed by Donato and Palumbo [7,8] and successively by Wiltshire [9].

3 Group Classification via Approximate Symmetries

The classification of the equation (5) is well known (see for details Ibragimov [6] and bibliography therein). From (11), we arrive at the following result:

$$\begin{aligned}
&\xi_0^1 = a_5\, t^2 + a_3\, t + a_1, \quad \xi_0^2 = a_4\, x + a_2,\\
&\eta_0 = (a_5\, t + a_6)\, w_0 + a_7\, t\, x + a_8\, t + a_9\, x + a_{10},\\
&[(a_6 - a_4)\; w_{0x} + a_9]\; \frac{d\, f(w_{0x})}{d\, w_{0x}} - 2\,(a_4 - a_3)\, f(w_{0x}) = 0,\\
&(a_5\, w_{0x} + a_7)\, \frac{d\, f(w_{0x})}{d\, w_{0x}} + 4\, a_5\, f(w_{0x}) = 0,
\end{aligned} \tag{13}$$

where a_i, $i = 1, 2, \ldots, 10$ are constants.

Taking (13) into account, from (12) we obtain the following additional conditions:

$$a_5 = a_7 = 0, \tag{14}$$

$$\eta_{10} = a_{11}\, t + a_{12}, \quad \eta_{11} = a_3 - 2\, a_4 + a_6, \tag{15}$$

$$[(a_6 - a_4)\ w_{0x} + a_9]\ \frac{d\, g(w_{0x})}{d\, w_{0x}} + (2\, a_3 - 3\, a_4 + a_6)\, g(w_{0x}) = 0, \tag{16}$$

with a_{11} and a_{12} constants.

After observing that conditions (14) impose restrictions upon to X_0, summarizing we have to manage the following relations:

$$\xi_0^1 = a_3\, t + a_1, \quad \xi_0^2 = a_4\, x + a_2, \quad \eta_0 = a_6\, w_0 + a_8\, t + a_9\, x + a_{10}, \tag{17}$$

$$\eta_{10} = a_{11}\, t + a_{12}, \quad \eta_1 = a_3 - 2\, a_4 + a_6, \tag{18}$$

$$[(a_6 - a_4)\ w_{0x} + a_9]\ \frac{d\, f(w_{0x})}{d\, w_{0x}} - 2\, (a_4 - a_3)\, f(w_{0x}) = 0, \tag{19}$$

$$[(a_6 - a_4)\ w_{0x} + a_9]\ \frac{d\, g(w_{0x})}{d\, w_{0x}} + (2\, a_3 - 3\, a_4 + a_6)\, g(w_{0x}) = 0. \tag{20}$$

For f an arbitrary function we obtain $a_6 = a_4 = a_3$, $a_9 = 0$, from which it follows that g is also an arbitrary function.

We call the associate seven-dimensional Lie algebra the *Approximate Principal Lie Algebra* of equation (1). It is spanned by the seven operators

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial w_0}, \quad X_4 = t\, \frac{\partial}{\partial w_0},$$

$$X_5 = t\, \frac{\partial}{\partial t} + x\, \frac{\partial}{\partial x} + w_0\, \frac{\partial}{\partial w_0}, \quad X_6 = \frac{\partial}{\partial w_1}, \quad X_7 = t\, \frac{\partial}{\partial w_1}$$

and we denote it by $\mathcal{A}pprox\mathcal{L}_{\mathcal{P}}$.

Otherwise, from (19) and (20) we obtain that f and g are linked by the relation

$$g(w_{0x}) = \frac{d\, f(w_{0x})}{d\, w_{0x}},$$

as we hoped and expected in order to be consistent with the perturbation theory.

The classification of $f(w_{0x})$ and the corresponding extensions of $\mathcal{A}pprox\mathcal{L}_{\mathcal{P}}$ arising from (17)–(19), are reported in Table 1.

Table 1. Classification of $f(w_{0x})$ and corresponding extensions of $\mathcal{A}pprox\mathcal{L}_{\mathcal{P}}$. f_0, p and q are constitutive constants with $f_0 > 0$, $p \neq 0$.

Case	Forms of $f(w_{0x})$	Extensions of $\mathcal{A}pprox\mathcal{L}_{\mathcal{P}}$
I	$f(w_{0x}) = f_0\, e^{w_{0x}/p}$	$X_8 = x\, \frac{\partial}{\partial x} + (w_0 + 2\, p\, x)\, \frac{\partial}{\partial w_0} - w_1\, \frac{\partial}{\partial w_1}$
II	$f(w_{0x}) = f_0\ (w_{0x} + q)^{2/p}$	$X_8 = x\, \frac{\partial}{\partial x} + [(1+p)\, w_0 + p\, q\, x]\, \frac{\partial}{\partial w_0} + (p-1)\, w_1\, \frac{\partial}{\partial w_1}$

4 A Physical Application

Let us consider a homogeneous viscoelastic bar of uniform cross-section and assume the material to be a nonlinear Kelvin solid. This model is described by the classical equation of motion (the constant density is normalized to 1 and the mass forces are neglected)

$$w_{tt} = \tau_x \tag{21}$$

and by assuming a stress-strain relation of the following form:

$$\tau = \sigma(w_x) + \lambda_0\, w_{xt}, \tag{22}$$

where τ is the stress, x the position of a cross-section in the homogeneous rest configuration of the bar, $w(t,x)$ the displacement at time t of the section from its rest position, $\sigma(w_x)$ the elastic tension (w_x is the strain), λ_0 the viscosity positive coefficient. Taking (22) into account and setting

$$\frac{d\,\sigma(w_x)}{d\,w_x} = f, \quad \lambda_0 = \varepsilon,$$

the equation (21) reduces to (1).

Let us consider the following form of the tension $\sigma(w_x)$:

$$\sigma(w_x) = \sigma_0\, \log(1 + w_x), \tag{23}$$

which was suggested by G. Capriz [10, 11].

So, we fall in the Case II of Table 1 with the following identifications:

$$f_0 = \sigma_0, \quad p = -2, \quad q = 1.$$

In this case, the approximate Lie operator X_8 assumes the form

$$X_8 = x\,\frac{\partial}{\partial x} - (w_0 + 2\,x)\,\frac{\partial}{\partial w_0} - 3\;w_1\,\frac{\partial}{\partial w_1}$$

and from the corresponding invariant surface conditions we obtain the following representation for the different terms in the expansion of w:

$$w_0 = \frac{\psi(t)}{x} - x, \quad w_1 = \frac{\chi(t)}{x^3}, \tag{24}$$

which give the form of an invariant solution approximate at the first order in ε.

The functions ψ and χ must satisfy the following system of ODEs to which, after (23), the system (5) is reduced through (24):

$$\psi_{tt} + 2\,\sigma_0 = 0, \quad \chi_{tt} + \frac{6\,\sigma_0}{\psi}\,\chi - 2\,\psi_t = 0. \tag{25}$$

After solving (25) and taking (24) into account, we have

$$w_0 = -\,\sigma_0\,\frac{t^2}{x} - x, \quad w_1 = -\,\frac{(40\,\sigma_0\,\log t - 8\,\sigma_0 - 25)\,t^5 - 25}{50\,t^2\,x^3}\,.$$

Therefore, the invariant solution up to the first order in ε is

$$w(t,x,\varepsilon) = -\,\sigma_0\,\frac{t^2}{x} - x - \varepsilon\,\frac{(40\,\sigma_0\,\log t - 8\,\sigma_0 - 25)\,t^5 - 25}{50\,t^2\,x^3} + \mathcal{O}(\varepsilon^2).$$

We have an unperturbed state represented by a stretching modified by the viscosity effect. For large time this latter becomes dominant and the linear expansion is not longer valid. This can be probably ascribed to the stress-strain relation (22) which is linear in the viscosity. More sophisticated model with a non linear viscosity are currently under investigation by the author and will be the subject of a future paper.

5 Conclusions

In this paper we perform the group analysis of the nonlinear wave equation with a small dissipation (1) in the framework of the approximate symmetries.

We follow the guide lines of the method proposed by Fushchich and Shtelen [4], expanding in a perturbation series the dependent variables and removing the "drawback" of the impossibility to work in hierarchy in calculating symmetries.

In order to remove that "drawback", we introduce, according to the perturbation theory, the expansions of the dependent variables in the one-parameter Lie group of infinitesimal transformations of the equation (1). Equating to zero the coefficients of zero and first degree powers of ε, we obtain an approximate Lie operator which permits to solve in hierarchy the invariance condition of the system (5)–(6) starting from the classification of the unperturbed non linear wave equation (5).

The proposed strategy is consistent with the perturbation point of view and can be generalized in a simple way to the higher orders of approximation in ε.

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Symmetry Investigations in Modelling Surface Processes

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In our paper we show the results of methods of group analysis to equations describing a special kind of filtering processes. We consider two pipes separated by a porous wall. A loaded gas flux passes this wall and the solid ingredients are deposited on the surface of the wall. All calculations will be done by the *Mathematica*-tools *MathLie* [1] and *MathLieAlg* [2].

1 Introduction

In nature a two phase flow passing through a porous medium plays an important role. Especially those flows carring solid particles are significant in technical applications. Such kinds of processes can be the filter function of the lung avioli, the cleaning of a gas flux from particles or other ways of filtration actions. Our presentation is concerned with the second way.

Power production by combustion is one of the oldest techniques of mankind. Approximately 90% of our current power requirement are based on combustion processes. A large part of this amount is devoted to the personal transportation by cars driven by Otto or Diesel engines. Both kinds of engines generate a large amount of pollutants and consume large quantities of fossil fuel. In future Diesel engines will become a preferable and superiour driving concept. The reason is that their consumption of fuel is less compared with otto engines. This point plays an important rule due to the shortage of fossil fuel and the tendency of minimizing the output of carbone dioxid.

However, diesel engines have the disadvantage of emitting nitrogen oxide and poisonous particles. The NO_x $(x = 1, 2, 3, \ldots)$ exhaust is considered as polluter of photochemical smog and is responsible for the generation of ozone in the lower part of the atmosphere and especially for acid rain [3].

In view of these facts, the increasing number of diesel cars and the growing importance of nature, government decided to regulate the exhaust gas concentrations by laws, which become stricter each time. This leads to an improvement of the exhaust aftertreatment systems and so to a high effort in their design. The consequence is that the understanding of physical and chemical processes must be improved and the derivation of mathematical models must be adapted to this situation. This is necessary because particles formed during the combustion processes are classified to be carcinogenic by the World Health Organization.

In section two we derive model equations describing such a particulate trap incorporating the technical assumptions. We discuss ways to modify the term for the pressure difference between inlet and outlet channel in the third part. The fourth part is dedicated to symmetry investigations of a general model and at last its submodels in the fifth part. At the end of the work a short discussion is given.

2 Derivation of Model Equations Describing a Particulate Trap

It is a great challenge to describe consistently the controlled loading and regeneration of a ceramic particulate filter by a mathematical model. A review of different trap models was given by Opris [4], who extended also one-dimensional models to a two-dimensional spacial model incorporating the properties of Navier–Stokes equation [4]. A theoretical study of the performance of traps with respect to pressure drop and flow velocities was carried out by Konstandopoulos [5]. His model is based on phenomenological assumptions for fluid friction.

To improve Konstandopoulos's model we take into our consideration:

- a one-dimensional plug-flow model.
- The density along the x-axis is constant.
- We only consider the loading process. There is no regeneration.
- The changing in the geometry will be neglected. There is no changing of the cross section diameter.
- There is no interaction between the flow and the particles.
- The system is considered to be isothermal. There is no energy balance.

Based on these conditions, we consider a pipe consisting of two channels. As shown in Figure 1. the inlet channel with pressure P_1 and velocity v_1 is closed at the end. In the outlet part of the filter pressure and velocity are denoted by P_2 and v_2. Both channels are connected by a porous wall through which gas can flow with a velocity $4u_w$.

Taking into account the assumptions listed above we use balance equations to derive the governing equations. The major balances are mass balance:

$$\varrho\frac{\partial u_1}{\partial x} = -\frac{4u_w\varrho}{a}, \quad \varrho\frac{\partial u_2}{\partial x} = -\frac{4u_w\varrho}{a}, \tag{1}$$

momentum balance:

$$\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x} = -\frac{\partial P_1}{\partial x}\frac{1}{\varrho} + \frac{\mu}{\varrho}\frac{\partial^2 u_1}{\partial x^2}, \; \frac{\partial u_2}{\partial t} + u_2\frac{\partial u_2}{\partial x} = -\frac{\partial P_2}{\partial x}\frac{1}{\varrho} + \frac{\mu}{\varrho}\frac{\partial^2 u_2}{\partial x^2}, \tag{2}$$

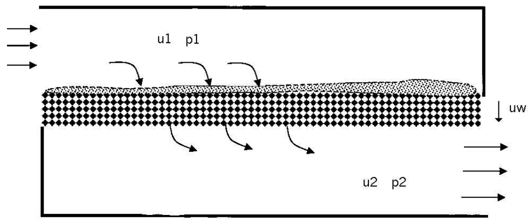

Figure 1. Scetch of a pfug flow trap (wall flow type)

where a is the cross section diameter, u_w the wall velocity, k the specific permeability in cm^2 and w_s the wall thickness.

To write out the equations (1), (2) in nondimensional form, we introduce scales for length, time, velocity and pressure:

$$\hat{x} = \frac{x}{L}, \quad \hat{t} = \frac{t}{T}, \quad \hat{u}_i = \frac{u_i}{U}, \quad \hat{u}_w = \frac{4L}{aU} u_w, \quad \hat{P}_i = \frac{P_i - P_{atm}}{P^*}, \quad P^* = \frac{\mu a U w_s}{4Lk},$$

where L is the characteristic length, T is the characteristic time, U is the characteristic velocity. Mass balance of the flow results in:

$$\frac{\partial u_1}{\partial x} = -u_w, \quad \frac{\partial u_2}{\partial x} = u_w, \tag{3}$$

Momentum balance gives:

$$\frac{\partial P_1}{\partial x} + B_1 \frac{\partial u_1}{\partial t} + B_2 u_1 \frac{\partial u_1}{\partial x} + B_3 \frac{\partial^2 u_1}{\partial x^2} = 0, \tag{4}$$

$$\frac{\partial P_2}{\partial x} + B_1 \frac{\partial u_2}{\partial t} + B_2 u_2 \frac{\partial u_2}{\partial x} + B_3 \frac{\partial^2 u_2}{\partial x^2} = 0. \tag{5}$$

The hated quantities are replaced for simplicity by non-hated ones. Abbreviations used are: $B_1 = 4L^2 k/(\mu a w_s T)$, $B_2 = 4kLRe/(a w_w a)$, $B_3 = 4k/a w_s$ with the Reynolds number Re.

To simplify system (3) we added both equations and integrated the result. After applying the initial conditions $u_1(0) = 1$, $u_2(0) = 0$ we get

$$u_1 + u_2 = 1. \tag{6}$$

Subtracting the second from the first of equation (4) and applying (6) we find:

$$\frac{\partial}{\partial x}(P_1 - P_2) - 2B_1 \frac{\partial u_2}{\partial t} - B_2 \frac{\partial u_2}{\partial x} - 2B_3 \frac{\partial u_2}{\partial x^2} = 0. \tag{7}$$

3 Modelling the Pressure Term

In the literature we find a large number of models describing the pressure difference in equation (7) (see [6] and the papers cited therein). The question here is which of these models are most appropriate to describe a real situation.

The first consideration is based on Darcy's law [6]:

$$\Delta P = U\frac{e}{k}, \tag{8}$$

where U is the velocity, e is the thickness of the substrate (sand) and ΔP is the pressure difference. Darcy's law assumes that:

- an incompressible fluid will be treated;
- an isothermic situation is considered;
- a Newtonian creeping flow will be discussed which means that the flow velocity is very small;
- a quite long uniform and isotropic medium with low hydraulic conductivity k will be treated.

At this point it should be mentioned that k depends on the fluid properties and is related with the porous medium.

Due to the fact that no viscosity effects were taken into account by Darcy expression (8) was changed by Hazen to

$$\Delta P = U\frac{\mu}{K}e, \tag{9}$$

where K is the specific permeability of the material which is assumed to be independent of fluid properties, and μ is the dynamic viscosity. Relations (9) is called the Hazen–Darcy equations.

The next step to improve Darcy's law was suggested done by Dupuit in 1863. He established the equation

$$0 = \frac{\Delta P}{\Delta x} - \alpha U - \beta U^2. \tag{10}$$

We note, that in the derivation of equation (10) weight effects are neglected and the polynomial form of the expression of the velocity represents only a fit of experimental data.

In 1947 Brinkman compared the multidimensional differential form of the Hazen–Darcy law [6]

$$0 = -\nabla P - \frac{\mu}{K}U \tag{11}$$

with Stokes flow for creeping motion of the fluid. This flow is related to the Navier–Stokes equation:

$$0 = -\nabla P + \mu\nabla^2 U. \tag{12}$$

It was suggested by Brinkman that the equation (11) should be modified by adding the Laplacian term equation to (12). Brinkman also recognized that a number of viscous terms are motivated by the physical properties such as viscous shear stress and solid viscous drag. In the limited case of low permeability the first effect is smaller compared with the second one and can be neglected here.

By using the argumentation of Brinkmanm, we have to replace the Hazen–Dupuit–Darcy equation by

$$0 = -\nabla(\Phi P) + \mu_e \nabla^2 \mathbf{U} - \frac{\mu}{K}\phi \mathbf{U} - C\varrho\phi^2 |U| \mathbf{U}, \tag{13}$$

with the effective fluid viscosity μ_e, a function of the fluid and the geometry of the permeable medium and the surface porosity $\Phi = A_f/A_m$. Here A_f and A_m represent the values of volumes occupied by the fluid and by the solid.

A more formal model was presented by Vafai/Tien and Hsu/Cheng [6], which dealt with a general equation for the flow through an isotropic rigid homogenous medium. Their final equation reads

$$\varrho\left[\frac{\partial \mathbf{U}}{\partial t} + (U\nabla)\mathbf{U}\right] = -\nabla(\Phi P) + \mu_e \nabla^2 \mathbf{U} - \frac{\mu}{K}\phi \mathbf{U} + C\varrho\phi^2 |U| \mathbf{U}. \tag{14}$$

There are six main physical parameters characterising the fluid: the fluid density ϱ, the fluid dynamic viscosity μ, the effective viscosity μ_e, permeablility K, porosity Φ and the form coefficient C. The values ϱ, μ, Φ can be measured independently but the other three, μ_e, K, C depend on the geometry of the permeable medium and can not be measured directly, nor calculated analytically because of the absence of a model in which they might be bounded with measurable data in porous media.

4 Group Analysis of Filtration Processes

In the previous part we have demonstrated several ways to model the pressure term. In accordance with these considerations, we deal with the following expression for the gradient of pressure by taking into account the approximation $\nabla P \approx P_1 - P_2 := \Delta P$: $\nabla P = f(u_w, \nabla^2 u_w)$.Using the second of equations (3) we find $\nabla P = g(\partial u_2/\partial x, \partial^3 u_2/\partial x^3)$. From equation (7) it follows

$$\frac{\partial u_2}{\partial t} + \frac{1}{2}\beta\frac{\partial u_2}{\partial x} + \left(\gamma - \frac{1}{2B_1}\frac{\partial g}{\partial u_{2x}}\right)\frac{\partial^2 u_2}{\partial x^2} - \frac{1}{2B_1}\frac{\partial g}{\partial u_{2xxx}}\frac{\partial^4 u_2}{\partial x^4} = 0.$$

By substituting

$$\left(\gamma - \frac{1}{2B_1}\frac{\partial g}{\partial u_{2x}}\right) = \frac{\partial K}{\partial u_{2x}}, \quad -\frac{1}{2B_1}\frac{\partial g}{\partial u_{2xxx}} = \frac{\partial K}{\partial u_{2xxx}},$$

we gain

$$\frac{\partial u_2}{\partial t} + \frac{1}{2}\beta\frac{\partial u_2}{\partial x} + \frac{\partial K}{\partial (u_2)_x}\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial K}{\partial (u_2)_{xxx}}\frac{\partial^4 u_2}{\partial x^4} = 0. \tag{15}$$

We applied modern group analysis methods to equation (15). The methods for calculating the equivalence algebra and the classification of the arbitrary function are described in [7–12]. By applying these methods to equation (15) and the additional equations $K_t = 0$, $K_x = 0$, $K_u = 0$, $K_{u_t} = 0$, $K_{u_{tt}} = 0$, $K_{u_{tx}} = 0$, $K_{u_{xx}} = 0$, $K_{u_{ttt}} = 0$, $K_{u_{ttx}} = 0$, $K_{u_{txx}} = 0$ coming from the extension of the considered space [10]. The result is an Abelian algebra of translations. The related infinitesimals are $\xi^1 = k_1$, $\xi^2 = k_2$, $\eta^1 = k_3$, $\mu = k_4$. Here ξ^1, ξ^2, η^1, μ are the coordinates of the generator

$$\begin{aligned} Y &= \xi^1\frac{\partial}{\partial t} + \xi^2\frac{\partial}{\partial x} + \eta^1\frac{\partial}{\partial u} + \zeta^1\frac{\partial}{\partial u_t} + \zeta^2\frac{\partial}{\partial u_x} + \zeta 11\frac{\partial}{\partial u_{tt}} + \zeta 12\frac{\partial}{\partial u_{tx}} \\ &+\zeta^{22}\frac{\partial}{\partial u_{xx}} + \zeta^{111}\frac{\partial}{\partial u_{ttt}} + \zeta^{112}\frac{\partial}{\partial u_{ttx}} + \zeta^{122}\frac{\partial}{\partial u_{txx}} + \zeta^{222}\frac{\partial}{\partial u_{xxx}} + \mu\frac{\partial}{\partial K}. \end{aligned}$$

From the largest symmetry group we get for the case $K = \text{const}$ Darcy's law. The related equation of motion can be transformed to the diffusion equation.

5 Discussion of the Special Cases

The first case is given by applying Darcy's law in the following form: $\Delta P = -(\mu/K)\partial u_2/\partial x$, which leads to the equation

$$\frac{\partial u_2}{\partial t} + \frac{1}{2}\beta\frac{\partial u_2}{\partial x} - \alpha\frac{\partial^2 u_2}{\partial x^2} = 0, \qquad (16)$$

with coefficients $\beta = B_2/B_1$, $\alpha = (1 - 2B_3)/(2B_1)$. This kind of equation was investigated in [13] and is directly related to the diffusion equation.

The law of Hazen–Dupuit–Darcy leads to the second case: $\Delta P = \alpha\partial u_2/\partial x + \beta(\partial u_2/\partial x)^2$, from which follows

$$\alpha\frac{\partial^2 u_2}{\partial x^2} + 2\beta\frac{\partial u_2}{\partial x}\frac{\partial^2 u_2}{\partial x^2} - 2B_1\frac{\partial u_2}{\partial t} - B_2\frac{\partial u_2}{\partial x} - 2B_3\frac{\partial^2 u_2}{\partial x^2} = 0. \qquad (17)$$

Applying group analysis methods to (17) a five-dimensional algebra is found:

$$\begin{aligned} &V_1 = \partial_{u_2}, \quad V_2 = \partial_t, \quad V_4 = \partial_x, \\ &V_3 = t\partial_t + \left(2u_2 + \frac{B_2B_3}{2B_1\beta} - \frac{B_3}{\beta} - \frac{B_2t\alpha}{4B_1\beta} + \frac{x\alpha}{2\beta}\right)\partial_{u_2} + x\partial x, \\ &V_5 = \frac{2B_1t}{B_2}\partial_t + \left(-\frac{2B_1u_2}{B_2} - \frac{B_3t}{\beta} + \frac{2B_1B_3x}{B_2\beta} + \frac{t\alpha}{2\beta} - \frac{B_1x\alpha}{B_2\beta}\right)\partial_{u_2} + t\partial_x, \end{aligned}$$

with the nontrivial commutator relations:

$$\begin{aligned} &[V_1, V_3] = 2V_1, \quad [V_1, V_5] = \frac{2B_1}{B_2}V_1, \quad [V_2, V_3] = -\frac{2B_2B_3 + B_2\alpha}{4B_1\beta}V_1 - V_2, \\ &[V_2, V_5] = -\frac{2B_3 - \alpha}{2\beta}V_1 + \frac{2B_1}{B_2}V_2 + V_4, \quad [V_3, V_4] = \frac{2B_3 - \alpha}{2\beta}V_1 - V_4, \\ &[V_4, V_5] = -\frac{-2B_1B_3 + B_1\alpha}{B_2\beta}V_1, \end{aligned}$$

representing a non semisimple, solvable and non nilpotent algebra. The investigation of this case can be found in [13].

The third case will appear by using Stoke's flow with pressure difference in the following form: $\Delta P = \mu \nabla^3 u_2$. The related equation is:

$$\mu \frac{\partial^4 u_2}{\partial x^4} - 2B_1 \frac{\partial u_2}{\partial t} - B_2 \frac{\partial u_2}{\partial x} - 2B_3 \frac{\partial^2 u_2}{\partial x^2} = 0.$$

In this case we found infinite dimensional algebra with the coordinates $\xi^1 = k_1$, $\xi^2 = k_2$, $\eta^1 = u_2 k_3 + \mathcal{F}(t,x)$, where $\mathcal{F}$ has to satisfy the equation

$$B_2 \frac{\partial \mathcal{F}(t,x)}{\partial x} + 2B_3 \frac{\partial \mathcal{F}(t,x)}{\partial x} - \frac{\partial^4 \mathcal{F}(t,x)}{\partial x^4} + 2B_1 \frac{\partial \mathcal{F}(t,x)}{\partial t} = 0.$$

The discrete part of the symmetry group contains the generators $V_1 = \partial_t$, $V_2 = \partial_x$ and $V_3 = u_2 \partial_{u_2}$ which are also Abelian.

The fourth case deals with Brinkman–Hazen–Dupuit–Darcy law: $\nabla(\phi P) = \mu_e \nabla^3 u_2 - \mu K^{-1} \phi \nabla u_2 + C \varrho \phi^2 (\nabla u_2)^2$, which gives the following equation:

$$-B_2 \frac{\partial u_2}{\partial x} - 2B_3 \frac{\partial^2 u_2}{\partial x^2} - B_5 \frac{\partial^2 u_2}{\partial x^2} + 2B_6 \frac{\partial u_2}{\partial x} \frac{\partial^2 u_2}{\partial x} + B_4 \frac{\partial^4 u_2}{\partial x^4} - 2B_1 \frac{\partial u_2}{\partial t} = 0.$$

The symmetry group of this equation contains generators belonging to translations in t, x, and u_2-direction and the generator

$$V_4 = \frac{8B_1}{3B_2} t \partial_t - \left(\frac{2B_3}{3B_6} t - \frac{B_5}{3B_6} t - \frac{2B_1}{3B_2} u_2 + \frac{4B_1 B_3}{3B_2 B_6} x + \frac{2B_1 B_5}{3B_2 B_6} x \right) \partial_{u_2}$$
$$+ t \partial_x + \frac{2B_1}{3B_2} x \partial_x .$$

This algebra is not semisimple, not solvable and not nilpotent.

Finally, we consider the general case with pressure difference ∇P of the following representation

$$\nabla P = \frac{\mu_e}{\phi} \nabla^2 \frac{\partial u_2}{\partial x} - \frac{\mu}{K} \frac{\partial u_2}{\partial x} + C \varrho \phi (\frac{\partial u_2}{\partial x})^2 - \frac{\varrho}{\phi} \left(\frac{\partial}{\partial t} \frac{\partial}{\partial x} u_2 + \frac{\partial u_2}{\partial x} \frac{\partial^2 u_2}{\partial x} \right),$$

resulting to the equation:

$$\frac{\mu_e}{\phi} \frac{\partial^4 u_2}{\partial x^4} - \frac{\mu}{K} \frac{\partial^2 u_2}{\partial x^2} + 2C \varrho \phi \frac{\partial u_2}{\partial x} \frac{\partial^2 u_2}{\partial x^2} - \frac{\rho}{\phi} \left(\frac{\partial^3 u_2}{\partial t \partial x^2} + \left(\frac{\partial^2 u_2}{\partial x^2} \right)^2 + \frac{\partial u_2}{\partial x} \frac{\partial^3 u_2}{\partial x^3} \right)$$
$$-2B_1 \frac{\partial u_2}{\partial t} - B_2 \frac{\partial u_2}{\partial x} - 2B_3 \frac{\partial^2 u_2}{\partial x^2} = 0.$$

The algebra of this equation is an Abelian one which is generated by translations in time, space and velocity.

6 Conclusions

In our presentation we have seen that there are several ways to model the pressure difference in a pipe. Most of these models are based on empirical considerations which are only valid in a small range of hydrodynamics. A general model and some submodels were investigated by the method of modern group analysis. From viewpoint of symmetries one can say that the general model allows only translations. The variety of transformations leaving the equations invariant is missing and connected with them the existence of conservation laws. That means that in this context an improvement of the model is necessary. Especially the empirical laws must be substituted by physical ones deriving from basic physical laws.

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The Application of the Weiss Algorithm of Painlevé Analysis to Create Solutions of Reaction–Diffusion Systems

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In this paper we discuss a system of reaction–diffusion equations which have application in mathematical biology. The Weiss algorithm used in Painlevé analysis is applied to systems of two reaction–diffusion equations whose sources terms may be expressed in terms of cubic polynomials. It is shown how Bäcklund transformations may be constructed and these give rise to new solutions of particular reaction–diffusion systems.

1 Introduction

In recent years there has been much interest in of systems reaction diffusion equations which occur in many applications for example mathematical biology including models for multi-spaces chemical reactions and predictor prey systems. In particular, Gierer and Meinhardt [1] present models for morphological pattern formation based upon coupled reaction diffusion models of the type

$$u_t = u_{xx} + a_{00} + a_{10}u + a_{01}v + c_1u^mF(v),$$
$$v_t = kv_{xx} + b_{00} + b_{10}u + b_{01}v + c_2u^mG(v),$$

where u, v are functions of x and t and a suffix means a derivative. Also $F(v)$ and $G(v)$ have numerous specifications, for example the case $F(v) = v$, $G(v) = v$ with $m = 2$, was used by Prigogene and Lefever [2] to describe the 'Brusselator' reaction mechanism. It is our aim to discuss coupled one dimensional reaction diffusion equations in the general form

$$u_t = u_{xx} + f(u, v), \quad v_t = kv_{xx} + g(u, v), \ k \neq 0 \tag{1}$$

and to consider some conditions under which these equations may become integrable. In a recent paper Archilla *et. al.* [3] demonstrated how a classical symmetry analysis of the '$\lambda - \omega$' reaction system gave rise to an example of integrable systems. However in general, [4–6], classical symmetry analysis gives little insight into conditions of integrability (1). For example the primitive predator prey model where $f = uv = -g$ does not produce an integrable system.

It is for this reason that we consider an alternative approach based upon the Painlevé property Ablowitz *et. al.* [7] for the integrability of partial differential equations and use the related Weiss algorithm [8–10] to construct solutions of (1). Such solutions have the form

$$u = \phi^{\lambda} \sum_{j=0}^{\infty} U_j \phi^j, \quad v = \phi^{\mu} \sum_{j=0}^{\infty} V_j \phi^j,$$

where λ, $\mu < 0$ and ϕ, U_j, V_j are functions of x and t and where the condition $\phi = 0$ determines a movable singularity manifold. Our analysis uses Bäcklund transformations to construct new solutions and extends the analysis of Vani *et. al.* [11] and Larsen [12] who focussed on a particular reaction diffusion system.

2 Equations with Cubic Sources

2.1 Introduction

We begin by considering the particular system

$$u_t = \Delta u + \sum_{i,j=0}^{3} a_{ij} u^i v^j, \quad v_t = k\Delta v + \sum_{i,j=0}^{3} b_{ij} u^i v^j, \tag{2}$$

where $i + j \leq 3$ and $i + j \neq 0$. The first step in the Weiss-algorithm is to look for the dominant behaviour about a movable singular manifold $\phi(x,t) = 0$. So we write

$$u = U\phi^{\lambda},\ \lambda < 0, \quad v = V\phi^{\mu},\ \mu < 0 \tag{3}$$

and substitute these relationships into the system (2). On equating highest order singular terms it may be shown that

$$\lambda = -1, \quad \mu = -1 \tag{4}$$

and further

$$\phi_x^2 = -\frac{a_{03}V^3 + a_{12}UV^2 + a_{21}U^2V + a_{30}U^3}{2U} = -\frac{b_{03}V^3 + b_{12}UV^2 + b_{21}U^2V + b_{30}U^3}{2kV}.$$

The latter equations may be solved simultaneously for U and V and we consider here the particularly straight forward case when

$$U = \alpha n \phi_x, \quad V = \alpha \phi_x, \tag{5}$$

where $\alpha^2 > 0$ satisfies

$$\alpha^2 = -\frac{2n}{a_{30}n^3 + a_{21}n^2 + a_{12}n + a_{03}} = -\frac{2k}{b_{30}n^3 + b_{21}n^2 + b_{12}n + b_{03}}. \tag{6}$$

Note that (6) not only defines α but also defines a relationships between polynomial coefficients of the source terms that are strictly cubic in nature. Those that are strictly quadratic are considered separately later. The results (3), (4) and (5) may be taken together to form the following Bäcklund transform (also known as Hopf–Cole transformation)

$$u = \frac{\alpha n \phi_x}{\phi}, \quad v = \frac{\alpha \phi_x}{\phi}$$

and substituted into the system (2). Equating coefficients of the respective singular terms gives the following set of over determined equations

$$\begin{aligned}
&\phi_t = 3\phi_{xx} - \frac{\alpha\phi_x}{n}\left(a_{20}n^2 + a_{11}n + a_{02}\right) = 3k\phi_{xx} - \alpha\phi_x\left(b_{20}n^2 + b_{11}n + b_{02}\right) \\
&\Rightarrow 3\left(1-k\right)\phi_{xx} + \alpha\left(b_{20}n^2 - a_{20}n + b_{11}n - \frac{a_{02}}{n} - a_{11} + b_{02}\right)\phi_x = 0, \\
&-2\phi_{xxx} + \alpha\left(a_{20}n + \frac{a_{02}}{n} + a_{11}\right)\phi_{xx} + \left(\frac{a_{01}}{n} + a_{10}\right)\phi_x = 0, \\
&-2k\phi_{xxx} + \alpha\left(b_{20}n^2 + b_{11}n + b_{02}\right)\phi_{xx} + \left(b_{10}n + b_{01}\right)\phi_x = 0.
\end{aligned}$$

These equations may be solved in three cases:

(i) $k \neq 1$, $B = \left(b_{20}n^2 - a_{20}n + b_{11}n - \frac{a_{02}}{n} - a_{11} + b_{02}\right) \neq 0$, $\phi = l_1(t)e^{\frac{Bx}{3(k-1)}} + l_2(t)$;
(ii) $k \neq 1$, $B = \left(b_{20}n^2 - a_{20}n + b_{11}n - \frac{a_{02}}{n} - a_{11} + b_{02}\right) = 0$, $\phi = l_1(t)x + l_2(t)$;
(iii) $k = 1$, $B = \left(b_{20}n^2 - a_{20}n + b_{11}n - \frac{a_{02}}{n} - a_{11} + b_{02}\right) = 0$, $b_{10}n^2 + \left(b_{01} - a_{10}\right)n - a_{01} = 0$ and $\phi(x,t)$ satisfies the equation

$$\phi_{xxx} - \alpha(b_{02} + b_{11}n + b_{20}n^2)\phi_{xx} - (b_{01} + b_{10}n)\phi_x = 0.$$

Here we consider two examples of the above cases. The complete solution of the problem will be consider elsewhere.

2.2 Example 1: $\phi = px + qt + c$

We find that the solution of the determining equations is

$$\begin{aligned}
&\phi = px + qt + c, \quad \frac{q}{p} = -\alpha\left(a_{20}n + \frac{a_{02}}{n} + a_{11}\right), \quad 0 = \frac{a_{01}}{n} + a_{10}, \\
&0 = b_{10}n + b_{01}, \quad 0 = b_{20}n^2 - a_{20}n + b_{11}n - \frac{a_{02}}{n} - a_{11} + b_{02},
\end{aligned}$$

where α is defined by (6). Thus

$$u = \frac{\alpha n p}{px + qt + c}, \quad v = \frac{\alpha p}{px + qt + c}.$$

Hence for the particular system

$$u_t = \Delta u - \frac{2u^2 v}{\alpha^2 n} + bnuv, \quad v_t = k\Delta v - \frac{2ku^2 v}{\alpha^2 n^2} + buv, \quad a > 0 \tag{7}$$

we find

$$\phi = p(x - b\alpha nt) + c, \quad u = nv, \quad v = \frac{\alpha}{x - b\alpha nt + \bar{c}}\,.$$

2.3 Example 2: $\phi = c_0 + e^{qt}(c_1 e^{p_1 x} + c_2 e^{p_2 x})$, $k = 1$

In this example, which falls into case (iii) we have further solutions. For example when $b_{10}n + b_{01} \neq 0$, $a_{10}n + a_{01} \neq 0$ then

$$\phi = c_0 + e^{qt}(c_1 e^{p_1 x} + c_2 e^{p_2 x}),$$

where p_1 and p_2 are solutions of the quadratic:

$$-2p^2 + \alpha p \left(a_{20}n + \frac{a_{02}}{n} + a_{11}\right) + \frac{a_{01}}{n} + a_{10} = 0$$

and q satisfies

$$q - 3p^2 + \alpha p \left(a_{20}n + \frac{a_{02}}{n} + a_{11}\right) = 0,$$

where

$$0 = b_{20}n^3 - (b_{11} - a_{20})\, n^2 + (b_{02} - a_{11})\, n - a_{02},$$
$$0 = b_{10}n^2 + (b_{01} - a_{10})\, n - a_{01}$$

and additionally (6) applies. The corresponding solution is thus

$$u = nv, \quad v = \frac{\alpha\left(c_1 p_1 e^{p_1 x + q_1 t} + c_2 p_2 e^{p_2 x + q_2 t}\right)}{c_0 + c_1 e^{p_1 x + q_1 t} + c_2 e^{p_2 x + q_2 t}}\,.$$

So when

$$u_t = \Delta u + \frac{2u^2 v}{\alpha^2} + bu, \quad v_t = \Delta v - \frac{2u^2 v}{\alpha^2} - bu, \quad a > 0$$

we have $n = -1$ and

$$\phi = e^{3p^2 t}\left(c_1 e^{px} + c_2 e^{-px}\right) + c_0, \quad b = -2p^2,$$
$$\phi = e^{-3p^2 t}\left(c_1 \sin(px) + c_2 \cos(px)\right) + c_0, \quad b = 2p^2.$$

However in example (7) we have $b_{10}n + b_{01} = 0 = a_{10}n + a_{01}$ and so if $k = 1$ then

$$\phi = c_0 + c_1 (x - 2pt) + c_2 e^{px + p^2 t}, \quad p = \frac{\alpha bn}{2}.$$

3 Determination of Resonances

3.1 Introduction

We may find additional solutions of the reaction diffusion equation by writing a Bäcklund transform of the type

$$u = \frac{\alpha n \phi_x}{\phi} + r\phi^N, \quad v = \frac{\alpha \phi_x}{\phi} + s\phi^N, \tag{8}$$

where $r = r(x,t)$ and $s = s(x,t)$. Now we substitute (8) into (2) and equate the singular terms of the highest order we see that (6) must again be imposed and additionally there is the matrix condition

$$\Omega \mathbf{r} = \mathbf{0}, \quad \Omega = \begin{bmatrix} Z + 3a_{30}n^2 + 2a_{21}n + a_{12} & a_{21}n^2 + 2a_{12}n + 3a_{03} \\ 3b_{30}n^2 + 2b_{21}n + b_{12} & kZ + b_{21}n^2 + 2b_{12}n + 3b_{03} \end{bmatrix},$$

where $\mathbf{r} = [r \quad s]^T$ and

$$\alpha^2 Z = N^2 - N. \tag{9}$$

In other words solving $\det(\Omega) = 0$ and substituting (9) we find four solutions for N. Two of these are $N = 3$, $N = -2$ whilst two further values may be found by solving:

$$N^2 - N + \frac{n\left(6b_{30}n^3 + (4b_{21} - 6a_{30}k)\,n^2 + (2b_{12} - 4a_{21}k)\,n - 2a_{12}k\right)}{k\left(a_{30}n^3 + a_{21}n^2 + a_{12}n + a_{03}\right)}. \tag{10}$$

3.2 A Particular Solution Employing Resonances

We consider the system

$$\begin{aligned} u_t &= u_{xx} + a_{12}uv^2 + a_{21}u^2v + (b_{10}n^2 + (b_{01} - a_{10})\,n)v + a_{10}u, \\ v_t &= v_{xx} + \frac{a_{12}}{n}uv^2 + \frac{a_{21}}{n}u^2v + b_{01}v + b_{10}u, \end{aligned} \tag{11}$$

so that (6) becomes $\alpha^2 = -2/(a_{21}n + a_{12}) > 0$ and the solutions of (10) are $N = 0$ and $N = 1$. Thus in the case $N = 0$ we write

$$u = r + \frac{\alpha n \phi_x}{\phi}, \quad v = s + \frac{\alpha \phi_x}{\phi}$$

and substitute into (11). Following the equating of singular terms find the four over-determined equations

$$\begin{aligned} &\phi_x\left(a_{21}\alpha\,(r - ns) - \frac{4s}{\alpha} - \frac{2r}{\alpha n}\right) - 3\phi_{xx} + \phi_t = 0, \\ &\phi_x\left(a_{21}\left(\frac{r^2}{n} - ns^2\right) - \frac{2s^2}{\alpha^2} - \frac{4rs}{\alpha^2 n} + b_{10}n + b_{01}\right) + \phi_{xxx} - \phi_{xt} = 0, \\ &r_{xx} + a_{12}rs^2 + a_{21}r^2s + (b_{10}n^2 + (b_{01} - a_{10})\,n)s + a_{10}r = r_t, \\ &s_{xx} + \frac{a_{12}}{n}rs^2 + \frac{a_{21}}{n}r^2s + b_{01}s + b_{10}r = s_t. \end{aligned} \tag{12}$$

The third and fourth of these are replicas of the original system (11).

When both r and s are constant then the solution of (12) is

$$r = \pm n\alpha\omega, \quad s = \pm\alpha\omega, \quad \omega = \sqrt{\frac{b_{10}n + b_{01}}{2}}.$$

The positive solutions give rise to

$$\phi = c_0 + c_1 e^{-2\omega x} + c_2 e^{-\omega x - 3\omega^2 t}$$

and so:

$$u = nv, \quad v = \alpha\omega\left[1 - \frac{2c_1 e^{-2\omega x} + c_2 e^{-\omega x - 3\omega^2 t}}{c_0 + c_1 e^{-2\omega x} + c_2 e^{-\omega x - 3\omega^2 t}}\right],$$

whilst the negative solutions give

$$\phi = c_0 + c_1 e^{2\omega x} + c_2 e^{\omega x - 3\omega^2 t}$$

which result in

$$u = nv, \quad v = \alpha\omega\left[-1 + \frac{2c_1 e^{2\omega x} + c_2 e^{\omega x - 3\omega^2 t}}{c_0 + c_1 e^{2\omega x} + c_2 e^{\omega x - 3\omega^2 t}}\right].$$

4 Equations with Quadratic Sources

Consider now the particular system

$$u_t = u_{xx} + \sum_{i,j=0}^{2} a_{ij} u^i v^j, \quad v_t = v_{xx} + \sum_{i,j=0}^{2} b_{ij} u^i v^j, \tag{13}$$

where $i+j \leq 2$ and $i+j \neq 0$. On this occasion applications of the substitutions (3) reveals that singular terms of the highest order may be equated by writing

$$\lambda = -2, \quad \mu = -2 \tag{14}$$

and further

$$\phi_x^2 = -\frac{a_{02}V^2 + a_{11}UV + a_{20}U^2}{6U} = -\frac{b_{02}V^2 + b_{11}UV + b_{20}U^2}{6V}. \tag{15}$$

Using equation (3) together with (14) and (15) implies that we may write

$$u = \alpha n \left(\frac{\phi_x}{\phi}\right)^2, \quad v = \alpha \left(\frac{\phi_x}{\phi}\right)^2, \tag{16}$$

where $\alpha \neq 0$ is

$$\alpha = -\frac{6n}{a_{20}n^2 + a_{11}n + a_{02}} = -\frac{6}{b_{20}n^2 + b_{11}n + b_{02}}. \tag{17}$$

Note that (17) not only defines α but also defines a relationship between the quadratic coefficients of the source terms. Substitution of (16) into (13) and equating coefficients of respective singular terms gives the following set of over determined equations

$$\phi_t = 5\phi_{xx}, \quad 0 = 2\frac{\phi_{xxx}}{\phi} + 2\left(\frac{\phi_{xx}}{\phi_x}\right)^2 - 2\frac{\phi_{xt}}{\phi_x} + b_{10}n + b_{01},$$
$$0 = b_{10}n + b_{01} - a_{10} - \frac{a_{01}}{n}.$$

The first two of these equations may be solved simultaneously to give

$$\phi = c_1 e^{mx+5m^2t} + c_0, \quad m^2 = \frac{b_{10}n + b_{01}}{6} > 0.$$

Hence finally

$$u = \alpha n m^2 \Psi, \quad v = \alpha\Psi, \quad \Psi = \left(\frac{c_1}{c_1 + c_0 e^{-mx-5m^2t}}\right)^2. \tag{18}$$

To determine any resonances we now consider solutions of the type

$$u = r + \alpha n\left(\frac{\phi_x}{\phi}\right)^2, \quad v = s + \alpha\left(\frac{\phi_x}{\phi}\right)^2$$

and on substitution into the reaction system (13) and equating singular terms of the highest order gives the matrix equation

$$\Omega\mathbf{r} = \mathbf{0}, \quad \Omega = \begin{bmatrix} \alpha Z + 2a_{20}n + a_{11} & a_{11}n + 2a_{02} \\ 2b_{20}n + b_{11} & \alpha Z + b_{11}n + 2b_{02} \end{bmatrix},$$

where $\mathbf{r} = [r \quad s]^T$ and $\alpha^2 Z = N^2 - N$. On solving $\det(\Omega) = 0$ and substituting (17) we find four solutions for N. Two of these are $N = -3, 4$ whilst two further values may be found by solving

$$N^2 - N + \frac{6\left(2b_{20}n^2 - 2a_{20}n + b_{11}n - a_{11}\right)}{b_{20}n^2 + b_{11}n + b_{02}} = 0.$$

4.1 Example

Consider the particular system

$$u_t = u_{xx} - \frac{6uv}{\alpha} + 6mu, \quad v_t = v_{xx} - \frac{6uv}{\alpha n} + 6mv, \quad \alpha > 0.$$

The solution is given by (18) with additional resonances $N = 0, 1$. With $m = -p^2$ a solution for $N = 0$ is

$$u = nv, \quad v = -\alpha p^2\left[1 - \left(\frac{c_1 e^{px+5p^2t}}{c_1 e^{px+5p^2t} + c_0}\right)^2\right].$$

5 Discussion

In this paper we have constructed new solutions of certain reaction diffusion equations by applying the Weiss algorithm with and without consideration of resonances. Our consideration of resonance solutions has been confined to the simple example $N = 0$ and clearly more work is necessary to evaluate other cases. In addition there is further need to a complete a full Painlevé analysis incorporating all possible resonances to determine the integrability of the cubic and quadratic sourced reaction diffusion equations considered here. Finally the analysis begun needs to be extended to source terms that may written as a Laurent series in either u or v. These matters will be considered elsewhere.

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