Nail H. Ibragimov

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Preface

Volume II contains thirteen papers. Six of them, namely Papers 1, 2, 3, 9, 10 and 13, were not published previously in English. Moreover, Papers 3, 9 and 10 were sketched some 20 years ago but not published (though partially used in my short publications) since I hoped to work out more applications. Unfortunately, I could not find time to accomplish the whole project due to many academic and other engagements. Therefore, I decided to translate into English the original versions of these manuscripts as they were 20 years ago and publish them in this volume.

Papers 5 and 12 are presented here in their original unabridged versions.

Elena Avdonina translated Papers 1 and 4 into English. She also made the layout of Papers 5, 6, 7 and 8 in LATEX. Paper 2 was printed in LATEX by Elena Avdonina and Roza Yakushina. My wife Raisa carefully checked the formulae in all papers of this volume. I am cordially grateful to them.

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Nail H. Ibragimov
Karlskrona,
23 November 2006.

Contents

Paper 1

Optimal systems of subgroups and classification of invariant solutions of equations for planar non-stationary gas flows

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§ 1 Introduction

The purpose of this work is to classify the invariant solutions of the equations describing two-dimensional adiabatic gas motions. The similar problem for the one-dimensional gasdynamic equations has been considered by L.V. Ovsyannikov [103].

I begin with some explanations of the terminology used in what follows. The definitions and proofs of the statements mentioned below can be found in [103].

Let S be a given system of differential equations admitting a group G. The basic property of solutions for S is that any solution of the system S carried over by any transformation of the group G to a certain solution of the same system S. Therefore two solutions of the system S are said to be essentially different with respect to G if they are not transformed to each other by any transformation of the group G; otherwise they are termed unessentially different with respect to G.

If H is a subgroup of the group G, then it is obvious that solutions essentially different with respect to G are also essentially different with respect to H. However, solutions essentially different with respect to H do not necessarily have the same property with respect to G.

Let us assume that Φ_1 is a solution invariant with respect to H_1, (invariant H_1-solution), Φ_2 is a solution invariant with respect to H_2 (invariant H_2-solution), where H_1, H_2 are two subgroups of G. If there exists a transformation $T \in G$ such that

$$H_2 = TH_1T^{-1}, \tag{1}$$

then there is a one-to-one correspondence between H_1-solutions and H_2-solutions given by the formula

$$\Phi_2 = T\Phi_1. \tag{2}$$

Let S be a system of equations with respect to the unknown functions and admit the group H with τ functionally independent invariants. Then, provided that certain conditions are met, the problem of obtaining invariant solutions of the system S is reduced to solving the system of equations S/H, where the unknown functions (invariants of the group H in the given case) depend only on $\varrho = \tau - m$ independent variables. The number ϱ is referred to as the rank of invariant solutions. In search of all invariant solutions of the rank ϱ one has to find all subgroups of the admitted group G having the same number of invariants $\tau = m + \varrho$. Certainly, it is sufficient to find only solutions essentially different with respect to G. To this end one has to choose only subgroups which are not connected by the relation (1), where T is any transformation of the group G. Two subgroups H_1 and H_2 connected by the relation (1) are called similar. The set of all subgroups is classified into similar subgroups. A set of classes of similar subgroups of order h is called an optimal system of order h and is designated by Θ_h. The problem of classification of invariant solutions of the given system S consists actually in constructing systems Θ_h.

Due to one-to-one correspondence between subgroups and subalgebras we will identify optimal systems of subgroups with optimal systems of subalgebras and use for the latter the same notation Θ_h. Then one can compose systems Θ_h as follows. Transformation of H_1 into H_2 is inner automorphism of G. These automorphisms can be substituted by linear transformations of the corresponding Lie algebra. The linear transformations are carried out by the adjoint group of Lie algebra with operators in the form

$$E_\alpha = (X_\alpha, X_\beta)\frac{\partial}{\partial X_\beta} \tag{3}$$

calculated according to the table of commutators for operators of the group G. Selecting the appropriate transformation of the adjoint group one can find the simplest representative of the subgroup H in the system Θ_h.

Consider the system of equations for a two-dimensional adiabatic gas motion written in the form

$$\left.\begin{aligned} &u_t + uu_x + vu_y + \frac{1}{\rho}\,p_x = 0,\\ &v_t + uv_x + vv_y + \frac{1}{\rho}\,p_y = 0,\\ &\rho_t + u\rho_x + v\rho_y + \rho(u_x + v_y) = 0,\\ &p_t + up_x + vp_y + A(p,\rho)(u_x + v_y) = 0, \end{aligned}\right\} \tag{4}$$

where

$$A(p,\rho) \equiv -\rho\,\frac{\partial S/\partial\rho}{\partial S/\partial p},$$

and p, ρ, S are pressure, density and entropy, respectively. It is assumed that $\partial S/\partial p \neq 0$.

We consider two cases:

1) $A(p,\rho)$ is an arbitrary function of its arguments,

2) $A = \gamma^* p$, i.e. a polytropic gas.

The group G admitted by the system (4) is calculated in [103]. In the first case the Lie algebra of the group G is spanned by the operators

$$\begin{aligned} &X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial y},\\ &X_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \quad X_5 = t\frac{\partial}{\partial y} + \frac{\partial}{\partial v},\\ &X_6 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},\\ &X_7 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}\,\cdot \end{aligned} \tag{5}$$

In the second case the following operators are added to the above:

$$X_8 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} + 2\rho\,\frac{\partial}{\partial \rho}, \quad X_9 = \rho\,\frac{\partial}{\partial \rho} + p\,\frac{\partial}{\partial p}\,\cdot \tag{6}$$

Table 1: Table of commutators for the operators (5), (6), (7)

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}
X_1	0	0	0	X_2	X_3	X_1	0	X_1	0	X_6+X_8 $-4X_9$
X_2	0	0	0	0	0	X_2	$-X_3$	0	0	X_4
X_3	0	0	0	0	0	X_3	X_2	0	0	X_5
X_4	$-X_2$	0	0	0	0	0	$-X_5$	$-X_4$	0	0
X_5	$-X_3$	0	0	0	0	0	X_4	$-X_5$	0	0
X_6	$-X_1$	$-X_2$	$-X_3$	0	0	0	0	0	0	X_{10}
X_7	0	X_3	$-X_2$	X_5	$-X_4$	0	0	0	0	0
X_8	$-X_1$	0	0	X_4	X_5	0	0	0	0	X_{10}
X_9	0	0	0	0	0	0	0	0	0	0
X_{10}	$-X_6-X_8$ $+4X_9$	$-X_4$	$-X_5$	0	0	$-X_{10}$	0	$-X_{10}$	0	0

When $\gamma^*=2$ one more operator is added to (5), (6):

$$X_{10}=t^2\frac{\partial}{\partial t}+tx\frac{\partial}{\partial x}+ty\frac{\partial}{\partial y}+(x-tu)\frac{\partial}{\partial u}+(y-tv)\frac{\partial}{\partial v}-4tp\,\frac{\partial}{\partial p}-2t\rho\,\frac{\partial}{\partial \rho}\cdot \quad (7)$$

Let us denote in the first case the group admitted by the system (4) by G_7, and the corresponding Lie algebra by L_7. Likewise, in the second case G_9, L_9 and G_{10}, L_{10} when γ^* is arbitrary and $\gamma^*=2$, respectively.

It should be noted that according to the Cartan criterion, Lie algebras L_7 and L_9 are solvable and L_{10} is nonsolvable. L_{10} is also not a semisimple algebra since it contains a solvable ideal spanned by the operators X_2, X_3, X_4, X_5.

§ 2 Optimal system of one-parameter subgroups

1° Let us consider the group G_7. The matrix of the general inner automorphism of the algebra L_7 is a superposition of the following basic matrices:

$$A_1(a_1)=\begin{Vmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&a_1&0&1&0&0&0\\0&0&a_1&0&1&0&0\\a_1&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{Vmatrix};\ A_2(a_2)=\begin{Vmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&a_2&0&0&0&1&0\\0&0&-a_2&0&0&0&1\end{Vmatrix};$$

$$A_3(a_3) = \begin{Vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 & 1 & 0 \\ 0 & a_3 & 0 & 0 & 0 & 0 & 1 \end{Vmatrix}; \; A_4(a_4) = \begin{Vmatrix} 1 & a_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & a_4 & 0 & 1 \end{Vmatrix};$$

$$A_5(a_5) = \begin{Vmatrix} 1 & 0 & a_5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -a_5 & 0 & 0 & 1 \end{Vmatrix}; \; A_6(a_6) = \begin{Vmatrix} a_6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{Vmatrix};$$

$$A_7(a'_7, a''_7) = \begin{Vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a'_7 & a''_7 & 0 & 0 & 0 & 0 \\ 0 & -a''_7 & a'_7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a'_7 & a''_7 & 0 & 0 \\ 0 & 0 & 0 & -a''_7 & a'_7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{Vmatrix}. \tag{8}$$

Here a_i $(i = 1, \ldots, 7)$ are the parameters of transformations of the adjoint group, and $a'_7 = \cos a_7, a''_7 = \sin a_7$. The identical automorphism corresponds to $a_1 = a_2 = a_3 = a_4 = a_5 = a_7$ and $a_6 = 1$.

Any operator of the one-parameter subgroup of the group G_7 can be written in the form

$$X = e^\alpha X_\alpha \tag{9}$$

where e^α $(\alpha = 1, \ldots, 7)$ are some coefficients. Transformation of the vector $e\{e^1, \ldots, e^7\}$ by means of the matrix A^* transposed to A corresponds to transformation of the operator X by matrix A. The coordinates e^6 and e^7 do not change under the action of matrices A_i^*. Therefore it is convenient to consider different cases separately depending on whether e^6 and e^7 are equal to zero.

Let $e^6 = e^7 = 0$. If $e^4 = e^5 = 0$, then having $e^1 = 0$ and acting by the matrix

$$A_7^*(e^2/t, \; -e^3/t), \quad t^2 = (e^2)^2 + (e^3)^2,$$

one obtains the vector $\{0, 1, 0, 0, 0, 0, 0\}$. Whereas having $e^1 \neq 0$ and acting by the matrices

$$A_4^*(-e^2/e^1), \quad A_5^*(-e^3/e^1)$$

one obtains $\{1,0,0,0,0,0,0\}$. If $(e^4)^2+(e^5)^2=a^2\neq 0$, then having $e^1=0$ we act by the matrices

$$A_7^*(e^4/a,\ -e^5/a),\quad A_1^*\left(-\frac{e^2e^4+e^3e^5}{a^2}\right)$$

and in case $e^3e^4-e^2e^5=0$ we obtain $\{0,0,0,1,0,0,0\}$. In case $e^3e^4-e^2e^5\neq 0$ we act by one more matrix

$$A_6^*\left(\frac{a^2}{e^3e^4-e^2e^5}\right)$$

and obtain $\{0,0,1,1,0,0,0\}$. When $e^1\neq 0$ we act by the matrices

$$A_4^*(-e^2/e^1),\quad A_5^*(-e^3/e^1),\quad A_7^*(e^4/a,\ -e^5/a)$$

and obtain $\{1,0,0,1,0,0,0\}$.

Let $e^6=1,\ e^7=0$. Acting by the matrices

$$A_1^*(-e^1),\quad A_2^*(e^1e^4-e^2),\quad A_3^*(e^1e^5-e^3),$$

and in case $a^2=(e^4)^2+(e^5)^2\neq 0$ by one more matrix

$$A_7^*(e^4/a,\ -e^5/a),$$

we obtain the vector $\{0,0,0,\alpha,0,1,0\}$, where α is an arbitrary real number.

Let $e^6=0,\ e^7=1$. Under the action of the matrices

$$A_2^*(e^1e^4+e^3),\quad A_3^*(e^1e^5-e^2),\quad A_4^*(-e^5),\quad A_5^*(e^4)$$

we obtain the vectors $\{0,0,0,0,0,0,1\},\ \{1,0,0,0,0,0,1\}$.

Let $e^6=1,\ e^7=b\neq 0$. Acting by the matrices

$$A_1^*(-e^1),\quad A_2^*\left(\frac{e^1e^4-e^2+be^3-be^1e^5}{1+b^2}\right),$$

$$A_3^*\left(\frac{e^1e^5-e^3+be^1e^4-be^2}{1+b^2}\right),\quad A_4^*(-e^5/b),\quad A_5^*(e^4/b)$$

we obtain the vector $\{0,0,0,0,0,1,b\}$.

By means of the formula (9) we arrive to the conclusion that representatives of classes of operators of similar one-parameter subgroups of the group G_7 are the following operators

$$X_1,\ X_2,\ X_5,\ \alpha X_6+X_7,\ X_1+X_7,\ X_1+X_4,\ X_2+X_5,\ \alpha X_4+X_6. \quad (10)$$

2°. Let us consider the group G_9. Table (1) shows that X_9 generates the center of Lie algebra L_{10}. Therefore it is sufficient to consider the residue algebra L_{10}/X_9. The transposed basic matrices of inner automorphisms of the algebra L_{10} are as follows.

$$C_1(a_1) = \left\| \begin{matrix} 1 & 0 & 0 & 0 & 0 & a_1 & 0 & a_1 & 0 & a_1^2 \\ 0 & 1 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & a_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & a_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -4a_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{matrix} \right\|$$

$$C_2(a_2) = \left\| \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & a_2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{matrix} \right\|$$

$$C_3(a_3) = \left\| \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & a_3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & a_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & a_3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{matrix} \right\|$$

$$C_4(a_4) = \begin{Vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & a_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & a_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{Vmatrix}$$

$$C_5(a_5) = \begin{Vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_5 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -a_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & a_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{Vmatrix} \qquad (11)$$

$$C_6(a_6) = \begin{Vmatrix} a_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{a_6} \end{Vmatrix}$$

$$C_7(a_7', a_7'') = \begin{Vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_7' & -a_7'' & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_7'' & a_7' & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_7' & -a_7'' & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_7'' & a_7' & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{Vmatrix}$$

$$C_8(a_8) = \left\| \begin{array}{cccccccccc} \frac{1}{a_8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_8 \end{array} \right\|$$

$$C_9(a_9) = \left\| \begin{array}{cccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_9 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_9 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ a_9 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ a_9 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -4a_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ a_9^2 & 0 & 0 & 0 & 0 & a_9 & 0 & a_9 & 0 & 1 \end{array} \right\| \qquad \begin{array}{l} a_7' = \cos a_7, \\ a_7'' = \sin a_7. \end{array}$$

The identical automorphism occurs when

$$a_1 = a_2 = a_3 = a_4 = a_5 = a_7 = a_9 = 0, \quad a_6 = a_8 = 1.$$

Basic automorphisms of the algebra L_9/X_9 consist of the matrices B_i ($i = 1, \ldots, 8$) that are obtained from the matrices C_i ($i = 1, \ldots, 8$) by eliminating two last columns and rows. Matrices $B_i^*(a_i)$ act on operators of the form:

$$X = e^\alpha X_\alpha, \quad \alpha = 1, \ldots, \infty. \tag{12}$$

Let $e^6 = e^7 = e^8 = 0$. If $e^1 = 0$, then in the case of $e^4 = e^5 = 0$ acting by the matrix

$$B_7(e^2/t, \ -e^3/t), \quad t^2 = (e^2)^2 + (e^3)^2$$

we obtain $\{0, 1, 0, 0, 0, 0, 0, 0\}$, and in the case of $a^2 = (e^4)^2 + (e^5)^2 \neq 0$ acting by the matrices

$$B_1\left(-\frac{e^2 e^4 + e^3 e^5}{a^2}\right), \quad B_7(e^4/a, \ e^5/a)$$

we obtain $\{0, 1, 0, 0, 1, 0, 0, 0\}$, $\{0, 0, 0, 0, 1, 0, 0, 0\}$. When $e^1 \neq 0$, and in case $e^4 = e^5 = 0$ we act by the matrices

$$B_4(-e^2/e^1), \quad B_5(-e^3/e^1)$$

and obtain $\{1,0,0,0,0,0,0,0\}$ and in case $a^2=(e^4)^2+(e^5)^2\neq 0$ we act by the matrices

$$B_4(-e^2/e^1),\quad B_5(-e^3/e^1),\quad B_7(e^4/a,\ -e^5/a)$$

and obtain $\{1,0,0,1,0,0,0,0\}$.

Let $e^6=1,\ e^7=e^8=0$. When $e^4=e^5=0$ we act by

$$B_1(-e^1),\quad B_2(-e^2),\quad B_3(-e^3)$$

and obtain $\{0,0,0,0,0,1,0,0\}$. When $a^2=(e^4)^2+(e^5)^2\neq 0$ we act by

$$B_1(-e^1),\quad B_2(-e^2+e^1e^4),\quad B_3(-e^3+e^1e^5),\quad B_7(e^4/a,\ -e^5/a)$$

and obtain $\{0,0,0,1,0,1,0,0\}$.

Let $e^6=e^8=0,\ e^7=1$. Matrices

$$B_2(e^1e^4+e^3),\quad B_3(e^1e^5-e^2),\quad B_4(-e^5),\quad B_5(-e^4)$$

lead to the vectors $\{0,0,0,0,0,0,1,0\},\ \{1,0,0,0,0,0,1,0\}$.

Let $e^6=1,\ e^7=a,\ e^8=0$. We act by the matrices

$$B_1(-e^1),\quad B_2(a_2),\quad B_3(a_3),\quad B_4(-e^5/a),\quad B_5(e^4/a),$$

where a_2,a_3 are found from the system of equations

$$\left.\begin{aligned} a_2+aa_3&=e^1e^4-e^2\\ aa_2-a_3&=e^3-e^1e^5\end{aligned}\right\}$$

and obtain the vector $\{0,0,0,0,0,1,a,0\}$.

Let $e^6=a,\ e^7=0,\ e^8=1$. If $a\neq -1$ we act by the matrices

$$B_1\left(-\frac{e^1}{1+a}\right),\ B_2\left(\frac{e^1e^4-(1+a)e^2}{a+a^2}\right),\ B_3\left(\frac{e^1e^5-(1+a)e^3}{a+a^2}\right),$$

$$B_4(-e^4),\quad B_5(-e^5)$$

to obtain $\{0,0,0,0,0,a_6,0,1\}$. If $a=-1$, acting by

$$B_1(e^1),\quad B_2(e^2),\quad B_3(e^3),\quad B_4(-e^4),\quad B_5(-e^5)$$

we obtain the vectors $\{1,0,0,0,0,-1,0,1\},\ \{0,0,0,0,0,-1,0,1\}$.

Let $e^6=0,\ e^7=a,\ e^8=1$. Matrices

$$B_1(-e^1),\quad B_2\left(\frac{e^3-e^1e^5}{a}\right),\quad B_3\left(\frac{e^1e^4-e^2}{a}\right),$$

$$B_4\left(-\frac{e^4+ae^5}{1+a^2}\right), \quad B_5\left(\frac{ae^4-e^5}{1+a^2}\right)$$

yield the vector $\{0,0,0,0,0,0,a,1\}$.

Let $e^6=a,\ e^7=b,\ e^8=1$. If $a\neq -1$ we act by the matrices

$$B_1\left(-\frac{e^1}{1+a}\right), \quad B_2(a_2), \quad B_3(a_3), \quad B_4(a_4), \quad B_5(a_5),$$

where a_2, a_3, a_4, a_5 are determined from the system of equations

$$\left.\begin{aligned} aa_2+ba_3&=-e^2+\frac{e^1e^4}{1+a}\\ -ba_2+aa_3&=-e^3+\frac{e^1e^5}{1+a}\\ a_4-ba_5&=-e^4\\ ba_4+a_5&=-e^5 \end{aligned}\right\}$$

and obtain the vector $\{0,0,0,0,0,a,b,1\}$. If $a=-1$, we act by

$$B_2(a_2), \quad B_3(a_3), \quad B_4(a_4), \quad B_5(a_5),$$

where a_2, a_3, a_4, a_5 come from the system of equations

$$\left.\begin{aligned} -a_2+ba_3+e^1a_4&=-e^2\\ -ba_2-a_3+e^1a_5&=-e^3\\ a_4-ba_5&=-e^4\\ ba_4+a_5&=-e^5 \end{aligned}\right\},$$

and arrive at two vectors $\{1,0,0,0,0,-1,b,1\},\ \{0,0,0,0,0,-1,b,1\}$.

Let $e^6=e^7=0,\ e^8=1$. Matrices

$$B_1(-e^1), \quad B_4(-e^4), \quad B_5(-e^5)$$

yield the vectors $\{0,0,0,0,0,0,0,1\},\ \{0,1,0,0,0,0,0,1\}$.

Substituting the resulting vectors into Formula (12) and adding to every operator the term γX_9 with an arbitrary real γ and then adding to this system operator X_9, we obtain representatives of the system Θ_1 for the group G_9 in the form

$$\begin{aligned} &X_1+\gamma X_9, \quad X_2+\gamma X_9, \quad X_5+\gamma X_9, \quad X_1+X_4+\gamma X_9,\\ &X_2+X_5+\gamma X_9, \quad X_6+\alpha X_7+\gamma X_9, \quad X_7+\gamma X_9,\\ &X_1+X_7+\gamma X_9, \quad X_2+X_8+\gamma X_9, \quad X_4+X_6+\gamma X_9,\\ &\alpha X_6+\beta X_7+X_8+\gamma X_9, \quad X_1-X_6+\alpha X_7+X_8+\gamma X_9, \quad X_9. \end{aligned} \tag{13}$$

In the first five operators we can assume that $\gamma = 0; 1$, since when $\gamma \neq 0$ we can make $\gamma = 1$ by means of the automorphisms B_6, B_8.

3°. Let us consider the group G_{10}. The general inner automorphism of the group G_{10} transforms the coordinate e^{10} to the form

$$e'^{10} = \frac{1}{a_6 a_8}\left[z^2 e^1 + zw(e^6 + e^8) + w^2 e^{10}\right], \tag{14}$$

where the following notation is introduced $z = a_6 a_9,\ w = a_8 + a_1 a_6 a_9$. The brackets contain a quadratic form of z and w, whose discriminant is

$$\Delta = e^1 e^{10} - \frac{1}{4}(e^6 + e^8)^2 = q - \frac{1}{4}\,p^2,$$

where $p = e^6 - e^8,\ q = -e^1 e^{10} - e^6 e^8$ are invariants of any automorphism C_i. When $\Delta \leq 0$ there will be real z, w such that $e'^{10} = 0$, therefore, the operator

$$X = e^\alpha X_\alpha, \quad \alpha = 1, \ldots, 10 \tag{15}$$

in this case is similar to one of the operators (13). When $\Delta > 0$, i.e. $4e^1 e^{10} > (e^6 + e^8)^2$, we can consider that $e^1 = e^{10} = 1$. Indeed, let $e^{10} = 1,\ e^1 = t,\ 4t > (e^6 + e^8)^2$. Then under the action of the matrix $C_8(\sqrt{t})$ we obtain $e^1 = e^{10} = 1$. Let us investigate vectors provided by the case $\Delta > 0$.

Let $e^8 = e^6 = 0,\ e^7 = \alpha$. If $|\alpha| \neq 1$, acting by

$$C_2(a_2), \quad C_3(a_3), \quad C_4(a_4), \quad C_5(a_5),$$

where a_2, a_3, a_4, a_5 are found from the system of equations

$$\left.\begin{aligned} a_2 - \alpha a_5 &= -e^4 \\ a_3 + \alpha a_4 &= -e^5 \\ \alpha a_3 + a_4 &= -e^2 \\ -\alpha a_2 + a_5 &= -e^3 \end{aligned}\right\}$$

we arrive to the vector $\{1, 0, 0, 0, 0, 0, \alpha, 0, \gamma, 1\}$. If $|\alpha| = 1$, then by means of the matrices

$$C_4(-e^5), \quad C_5(-e^4)$$

we arrive either to the previous vector or to the vector $\{1, 1, 0, 0, 0, 0, 1, 0, \gamma, 1\}$.

Let $e^6 = a,\ e^7 = \alpha,\ e^8 = 0$. Acting by the matrices

$$C_2(a_2), \quad C_3(a_3), \quad C_4(a_4), \quad C_5(a_5),$$

where a_2, a_3, a_4, a_5 are derived from the equations

$$\left.\begin{aligned} &aa_2+\alpha a_3+a_4=-e^2\\ &-\alpha a_2+aa_3+a_5=-e^3\\ &a_2-\alpha a_5=-e^4\\ &a_3+\alpha a_4=-e^5 \end{aligned}\right\}$$

we arrive to the vector $\{1,0,0,0,0,a,\alpha,0,\gamma,1\}$.

The case $e^8\neq 0$ reduces to the two previous cases.

Operator X_5 of the system (13) transforms into X_2 under the action of automorphisms

$$C_1(-1),\quad C_7(0,\,1),\quad C_9(1).$$

The remaining operators of the system (13) transform neither to each other nor to operators obtained for the case $\Delta>0$ (due to invariance of Δ).

Finally the following representatives of the system Θ_1 for the group G_{10} are obtained:

$$\begin{aligned} &X_1+\gamma X_9,\quad X_2+\gamma X_9,\quad X_1+X_4+\gamma X_9,\quad X_2+X_5+\gamma X_9,\\ &X_6+\alpha X_7+\gamma X_9,\quad X_7+\gamma X_9,\quad X_1+X_7+\gamma X_9,\\ &X_2+X_8+\gamma X_9,\quad X_4+X_6+\gamma X_9,\quad \alpha X_6+\beta X_7+X_8+\gamma X_9,\\ &X_1-X_6+\alpha X_7+X_8+\gamma X_9,\quad X_9,\quad X_1+X_2+X_7+\gamma X_9+X_{10},\\ &X_1+\alpha X_6+\beta X_7+\gamma X_9+X_{10},\quad 0\le\alpha\le 2,\quad \text{when}\quad \alpha=0,\ \beta\ge 0. \end{aligned} \tag{16}$$

In the first four operators one can assume that $\gamma=0;1$.

§ 3 Optimal system of two-parameter subgroups

1°. Any operator of the Lie algebra of a two-parameter subgroup of a group can be written in the form

$$\lambda X+\mu Y, \tag{17}$$

where the basic operators X, Y have the form (9) and satisfy the condition

$$(X,Y)=aX+bY, \tag{18}$$

and λ, μ are two arbitrary real valued coefficients. For the sake of brevity the operator (17), or more accurately speaking the two-dimensional Lie algebra spanned by X and Y, will be often denoted by

$$<X,Y>. \tag{19}$$

The system Θ_2 can be constructed as follows. One of the operators (19) can be transformed to one of the operators (10) by the corresponding automorphism. Therefore we assume that the operator X of the pair (19) runs the system (10) and choose Y to satisfy the condition (18). Then we find an automorphism that keeps operator X unaltered and apply it to Y. Operators in the resulting set of pairs have a less number of arbitrary coefficients e^α. We fix one of the resulting pairs, let it be the "simplest" $< X^0, Y^0 >$. Then we take the next pair $< X^1, Y^1 >$ from the set of the resulting pars and check whether there is such an automorphism that transforms $< X^1, Y^1 >$ to $< X^0, Y^0 >$. To this end we follow the below procedure. Let for the sake of simplicity $< X^0, Y^0 >=< X_1, X_2 >$. We act on the operator $\lambda X^1 + \mu Y^1$ by the matrix A of the general automorphism and equate the result to the operator $\xi X_1 + \eta X_2$:

$$\lambda AX^1 + \mu AY^1 = \sum_i^7 = 1[\lambda p_i(A) + \mu q_i(A)]X_i = \xi X_1 + \eta X_2.$$

We arrive to the following system of equations determining ξ, η and parameters of the automorphism:

$$\left.\begin{aligned} \lambda p_1(A) + \mu q_1(A) &= \xi, \\ \lambda p_2(A) + \mu q_2(A) &= \eta \end{aligned}\right\} \qquad (20)$$

$$\left.\begin{aligned} \lambda p_3(A) + \mu q_3(A) &= 0, \\ &\cdots\cdots\cdots \\ \lambda p_7(A) + \mu q_7(A) &= 0. \end{aligned}\right\} \qquad (21)$$

If one of $p_i(A),\ i \geq 3$ is not equal to zero, then

$$\lambda = \frac{q_i(A)}{p_i(A)}\mu,$$

which contradicts the condition of independence of λ and μ. The same is the case with $q_i(A),\ i \geq 3$. Therefore, System (21) is equivalent to the system

$$\left.\begin{aligned} p_3(A) &= 0 \\ &\cdots \\ p_7(A) &= 0 \\ q_3(A) &= 0 \\ &\cdots \\ q_7(A) &= 0 \end{aligned}\right\} \qquad (22)$$

that helps to find the necessary automorphism. If the system (20)-(21) is not compatible, $< X', Y' >$ is not transformed into $< X^0, Y^0 >$. If it is compatible $< X', Y' >$ is transformed into $< X^0, Y^0 >$, and equations (20) are used to determine ξ, η. If $< X', Y' >$ is not transformed to $< X^0, Y^0 >$, we take the next pair and likewise check whether it can be transformed into one of the above pairs, etc. Following the same procedure with every operator of the system (10) one obtains the system Θ_2 for G_7.

Let us apply these considerations to arbitrary finite-dimensional Lie groups. Thus the same method is applied for constructing the system Θ_2 for the groups G_9 and G_{10}.

Let us take the first operator X_1 of the system (10). Condition (18) shows that a subgroup with X_1 is formed only by those operators in the form (9) which have $e^4 = e^5 = 0$. Automorphism that leaves X_1 unaltered occurs only when A_4, A_5, A_6 in (8) are identical. If $e^6 = e^7 = 0$, the matrix

$$A_7(e^2/t, \, -e^3/t), \quad t^2 = (e^2)^2 + (e^3)^2$$

yields the vector $\{0, 1, 0, 0, 0, 0, 0\}$. If $(e^6)^2 + (e^7)^2 \neq 0$, then the matrices

$$A_2(a_2), \quad A_3(a_3),$$

where a_2, a_3 are obtained from the system

$$\left.\begin{aligned} a_2 e^6 + a_3 e^7 &= -e^2 \\ -a_2 e^7 + a_3 e^6 &= -e^3 \end{aligned}\right\},$$

provide the vectors $\{0, 0, 0, 0, 0, \alpha, 1\}$, $\{0, 0, 0, 0, 0, 1, 0\}$.

Let us consider X_2. Condition (18) yields $e^7 = 0$. Matrices (8) do not change X_2 when A_6, A_7 are identical. If $e^6 = 1$, matrices

$$A_1(-e^1), \quad A_3(e^1 e^5 - e^3)$$

provide the vector $\{0, 0, 0, \alpha, \beta, 1, 0\}$. If $e^6 = 0$, in case $e^1 = 1$ matrix $A_5(-e^3)$ leads to the vector $\{1, 0, 0, \alpha, \beta, 0, 0\}$. When $e^1 = 0, e^5 = 1$, the matrix $A_1(-e^3)$ provides $\{0, 0, 0, \alpha, 1, 0, 0\}$. When $e^1 = 0$, $e^5 = 0$ one has

$$\{0, 0, \alpha, 1, 0, 0, 0\}, \quad \{0, 0, 1, 0, 0, 0, 0\}.$$

Let us take X_4. Condition (18) and the requirement for invariance of X_4 provide $e^1 = e^7 = 0$; A_1, A_7 are identical. If $e^6 = 1$, the matrices

$$A_2(-e^2), \quad A_3(-e^3)$$

yield the vector $\{0,0,0,0,\alpha,1,0\}$. If $e^6=0$, then depending on the values of the coefficients e^2, e^3, e^5 we obtain the following vectors:

$$\{0,1,\alpha,0,1,0,0\},\quad \{0,1,\alpha,0,0,0,0\},\quad \{0,0,1,0,1,0,0\},$$

$$\{0,0,1,0,0,0,0\},\quad \{0,0,0,0,1,0,0\}$$

acting in case $e^2\neq 0$ by the matrix

$$A_6\left(\frac{1}{e^2}\right),$$

and in case $e^2=0$, $e^3\neq 0$ by the matrix

$$A_6\left(\frac{1}{e^3}\right).$$

Let us consider αX_6+X_7. First assume that $\alpha=0$. Then our conditions provide $e^2=e^3=e^4=e^5=0$. If $e^6=0$, the matrix

$$A_6\left(\frac{1}{e^1}\right)$$

yields the vector $\{1,0,0,0,0,0,0\}$. If $e^6=1$, the matrix

$$A_1\left(-e^1\right)$$

yields the vector $\{0,0,0,0,0,1,0\}$. Now let $\alpha\neq 0$. Then condition (18) shows that either $e^2=e^3=e^4=e^5=e^6=e^7=0$ which provides the vector $\{1,0,0,0,0,0,0\}$, or $e^1=e^2=e^3=e^4=e^5=0$ which yields the pair $< X_6, X_7 >$.

Let us take X_1+X_4. Condition (18) provides $e^1=e^4,\ e^5=e^6=e^7=0$, therefore one obtains $\{0,0,1,0,0,0,0\}$, $\{0,1,\alpha,0,0,0,0\}$.

Operator X_1+X_7 yields the pair $< X_1, X_7 >$.

Let us consider X_3+X_4. We have $e^1=e^6=e^7=0$, and depending on the the values e^2, e^3, e^5 obtain the vectors

$$\{0,\alpha,\beta,0,1,0,0,\},\quad \{0,1,\alpha,0,0,0,0\},\quad \{0,0,1,0,0,0,0\}.$$

Let us take αX_4+X_6. Let $\alpha=0$. Then we have either $e^4=e^5=e^7=0$, which provides two pairs $< X_1, X_6 >,\ < X_2, X_6 >$, or $e^1=e^2=e^3=0$, which provides the pairs $< X_4, X_6 >,\ < X_6, X_7 >$. Let $\alpha\neq 0$. Then either $e^1=e^2=e^3=e^7=0$, which provides

$$\{0,0,0,\alpha,1,0,0\},\quad \{0,0,0,1,0,0,0\},$$

Table 2: Optimal system of two-parameter subgroups of G_7

1	$X_1,$	X_2	8	$X_2,$	X_4
2	$X_1,$	X_6	9	$X_2,$	X_1+X_5
3	$X_1,$	αX_6+X_7	10	$X_5,$	X_4
4	$X_2,$	X_3	11	$X_5,$	$X_2+\alpha X_3$
5	$X_2,$	X_3+X_4	12	$X_5,$	αX_4+X_6
6	$X_2,$	$X_1+X_4+\alpha X_5$	13	$X_2+X_5,$	$\alpha X_2+\beta X_3+X_4$
7	$X_2,$	$\alpha X_4+\beta X_5+X_6$	14	$X_7,$	X_6

or $e^1=e^4=e^5=e^6=e^7=0$ which yields

$$\{0,1,\alpha,0,0,0,0\},\quad \{0,0,1,0,0,0,0\}.$$

Now let us take $<X_1,X_2>$ as the first pair and check whether there is an automorphism that transforms for example the pair $<X_1,X_6>$ into $<X_1,X_2>$. Since the coordinate e^6 is an invariant of any automorphism we have $q_6(A)=1$ and these pairs cannot be transformed to each other. Upon the above checking procedure one obtains the following optimal system of two-parameter subgroups of G_7 :

2°. Let us consider the group G_9. In this case we consider the pairs (19) where X runs the system (13) and Y has the form

$$Y=e^{\alpha}X_{\alpha},\quad \alpha=1,\dots,9. \tag{23}$$

Let us take the operator $X_1+\gamma X_9$. Let $\gamma=0$. Then condition 18 yields $e^4=e^5=0$. If $e^6=e^7=0$, then in case $t^2=(e^2)^2+(e^3)^2\neq 0$ the matrix

$$B_7(e^2/t,\ -e^3/t)$$

provides the vector $\{0,1,0,0,0,0,0,\alpha,\beta\},\ \alpha=0;1$, and in case $e^2=e^3=0$ we have $\{0,0,0,0,0,0,0,1,\alpha\}$. When $(e^6)^2+(e^7)^2\neq 0$ then acting by the matrices

$$B_2(a_2),\quad B_3(a_3),$$

where a_2,a_3 are determined by the system

$$\left.\begin{aligned} e^6a_2+e^7a_3&=-e^2\\ -e^7a_2+e^6a_3&=-e^3\end{aligned}\right\},$$

we arrive to the vectors $\{0,0,0,0,0,1,\alpha,\beta,\gamma\},\ \{0,0,0,0,0,0,1,\alpha,\beta\}$. When $\gamma=1$ we have $e^4=e^5=0,\ e^6+e^8=0$, i.e., in the vectors obtained above it is considered that $e^6=-e^8$.

Let us consider the operator $X_2 + \gamma X_9$. When $\gamma = 0$ it is necessary that $e^7 = 0$, and when $\gamma = 1$ that $e^6 = e^7 = 0$. If $e^8 = 0$, then in case $e^6 = 0$, $e^1 = 1$ we act by the matrix $B_5(-e^3)$ and obtain $\{1, 0, 0, \alpha, \beta, 0, 0, 0, \gamma\}$, and in case $e^6 = e^1 = 0$ we obtain the following vectors:

$$\{0, 0, 0, \alpha, 1, 0, 0, 0, \beta\}, \quad \{0, 0, 1, 1, 0, 0, 0, 0, \alpha\},$$

$$\{0, 0, 1, 0, 0, 0, 0, 0, \alpha\}, \quad \{0, 0, 0, 1, 0, 0, 0, 0, \alpha\},$$

acting when $e^5 = 1$ by the matrix $B_1(-e^3)$, and when $e^6 = 1$ under the action of the matrices

$$B_1(-e^1), \quad B_3(-e^3 + e^1e^5)$$

we obtain $\{0, 0, 0, 1, \alpha, 1, 0, 0, \alpha\}$, $\{0, 0, 0, 0, 1, 1, 0, 0, \alpha\}$, $\{0, 0, 0, 0, 0, 1, 0, 0, \alpha\}$. If $e^8 = 1$, then in case $e^6 = 0$ matrices

$$B_1(-e^1), \quad B_4(-e^4), \quad B_5(-e^5)$$

yield $\{0, 0, \alpha, 0, 0, 0, 0, 1, \beta\}$ and in case $e^6 = -1$ matrices

$$B_3(e^3 - e^1e^5), \quad B_4(-e^4), \quad B_5(-e^5)$$

provide $\{0, 0, 0, 0, 0, -1, 0, 1, \alpha\}$, $\{1, 0, 0, 0, 0, -1, 0, 1, \alpha\}$, and in case $e^6 \neq 0, -1$ matrices

$$B_1\left(-\frac{e^1}{1+e^6}\right), \quad B_3\left(\frac{e^1 - e^3 - e^3e^6}{e^6 + (e^6)^2}\right), \quad B_4(-e^4), \quad B_5(-e^5)$$

provide $\{0, 0, 0, 0, 0, \alpha, 0, 1, \beta\}$, $\alpha \neq 0, -1$.

Consider $X_5 + \gamma X_9$. If $\gamma = 0$ it is necessary that $e^1 = e^7 = 0$, and if $\gamma = 1$ that $e^1 = e^7 = e^8 = 0$. If $e^8 = 0$, then in case $e^6 = 0$ we have the vectors

$$\{0, 0, 1, 1, 0, 0, 0, 0, \alpha\}, \quad \{0, 1, \alpha, 1, 0, 0, 0, 0, \beta\}, \quad \{0, 0, 0, 1, 0, 0, 0, 0, \alpha\},$$

$$\{0, 1, \alpha, 0, 0, 0, 0, 0, \beta\}, \quad \{0, 0, 1, 0, 0, 0, 0, 0, \alpha\},$$

and in case $e^6 = 1$ under the action of the matrices

$$B_2(-e^2), \quad B_3(-e^3)$$

we arrive to the vector $\{0, 0, 0, \alpha, 0, 1, 0, 0, \beta\}$. If $e^8 = 1$, then in case $e^6 = 0$ we act by the matrix $B_4(-e^4)$ and arrive to the vectors

$$\{0, 1, \alpha, 0, 0, 0, 0, 1, \beta\}, \quad \{0, 0, 1, 0, 0, 0, 0, 1, \alpha\}, \quad \{0, 0, 0, 0, 0, 0, 0, 1, \alpha\},$$

and in case $e^6 \neq 0$ we act by the matrices

$$B_2\left(-\frac{e^2}{e^6}\right), \quad B_3\left(-\frac{e^3}{e^6}\right), \quad B_4\left(-e^4\right)$$

and obtain $\{0,0,0,0,0,\alpha,0,1,\beta\}$, $\alpha \neq 0$.

Consider $X_1 + X_4 + \gamma X_9$. When $\gamma = 0$ we have $e^1 = e^4$, $e^5 = e^7 = e^6 + 2e^8 = 0$, and when $\gamma = 1$ we have $e^1 = e^4$, $e^5 = e^6 = e^7 = e^8 = 0$. If $e^6 = e^8 = 0$, we have two vectors $\{0,1,\alpha,0,0,0,0,0,\beta\}$, $\{0,0,1,0,0,0,0,0,\alpha\}$. If $e^8 = 1$, $e^6 = 2$, then matrices

$$B_2\left(\frac{e^2}{2}\right), \quad B_3\left(\frac{e^3}{2}\right)$$

provide $\{0,0,0,0,0,-2,0,1,\alpha\}$.

Let us take $X_2 + X_5 + \gamma X_9$. When $\gamma = 0$ we have $e^1 = e^7 = e^6 + e^8 = 0$, and when $\gamma = 1$ we have $e^1 = e^6 = e^7 = e^8 = 0$. If $e^8 = 0$ one has $\{0,\alpha,\beta,1,0,0,0,0,\gamma\}$, $\{0,\alpha,1,0,0,0,0,0,\beta\}$, $\{0,1,0,0,0,0,0,0,\alpha\}$. If $e^8 = 1$, we act by the matrices

$$B_2(e^2), \quad B_3(e^3), \quad B_4(-e^4)$$

and arrive to the vector $\{0,0,0,0,0,-1,0,1,\alpha\}$.

Consider $X_6 + \alpha X_7 + \gamma X_9$. If $\alpha = 0$ we have either $e^4 = e^5 = e^6 = e^7 = e^8 = e^9 = 0$, or $e^1 = e^2 = e^3 = 0$. If $e^1 = 0$ the first case is reduced to the vector $\{0,1,0,0,0,0,0,0,0\}$ by the matrix

$$B_7(e^2/\sqrt{(e^2)^2 + (e^3)^2}, -e^3/\sqrt{(e^2)^2 + (e^3)^2})$$

and if $e^1 = 1$ the case is reduced to $\{1,0,0,0,0,0,0,0,0\}$ under the action of the matrices

$$B_4(-e^2), \quad B_5(-e^3).$$

In the second case when $e^7 = e^8 = 0$ under the action of

$$B_7(e^4/\sqrt{(e^4)^2 + (e^5)^2}, -e^5/\sqrt{(e^4)^2 + (e^5)^2})$$

we arrive to the vector $\{0,0,0,1,0,0,0,0,\alpha\}$, and when $(e^7)^2 + (e^8)^2 \neq 0$ we obtain the vectors $\{0,0,0,0,0,0,\alpha,1,\beta\}$, $\{0,0,0,0,0,0,1,0,\alpha\}$ acting by the matrices

$$B_4(a_4), \quad B_5(a_5),$$

where a_4, a_5 are determined by the system

$$\left.\begin{aligned} e^8 a_4 - e^7 a_5 &= -e^4 \\ e^7 a_4 + e^8 a_5 &= -e^5 \end{aligned}\right\}.$$

When $\alpha \neq 0$ we have either the vector $\{1,0,0,0,0,0,0,0,0\}$, or $e^1 = e^2 = e^3 = e^4 = e^5 = 0$ which yields two vectors

$$\{0,0,0,0,0,\alpha,0,1,\beta\}, \quad \{0,0,0,0,0,1,0,0,\alpha\}.$$

Consider $X_7 + \gamma X_9$. We have $e^2 = e^3 = e^4 = e^5 = 0$. In case $e^6 = e^8 \neq 0$ we act by the matrix

$$B_1\left(-\frac{e^1}{e^6+e^8}\right)$$

and obtain the vectors $\{0,0,0,0,0,\alpha,0,1,\beta\},\ \alpha \neq -1,\ \{0,0,0,0,0,1,0,0,\alpha\}$, in case $e^6 + e^8 = 0,\ e^1 \neq 0$ we act by the matrix

$$B_8(e^1)$$

and obtain $\{1,0,0,0,0,-\alpha,0,\alpha,\beta\}$, $\alpha = 0; 1$, and in case $e^6+e^8 = 0,\ e^1 = 0$ we have the vector $\{0,0,0,0,0,-1,0,1,\alpha\}$.

Consider $X_1 + X_7 + \gamma X_9$. In this case $e^2 = e^3 = e^4 = e^5 = e^6 + e^8 = 0$ and we have two vectors $\{\alpha,0,0,0,0,-1,0,1,\beta\},\ \{1,0,0,0,0,0,0,0,\alpha\}$.

Let us take $X_2 + X_8 + \gamma X_9$. We have either the vectors

$$\{1,0,0,0,0,0,0,0,0,\}, \quad \{0,0,0,1,0,0,0,0,0\}, \quad \{0,0,0,\alpha,1,0,0,0,0\},$$

or $e^1 = e^4 = e^5 = e^6 = e^7 = 0$ and therefore the vectors

$$\{0,\alpha,1,0,0,0,0,0,\beta\}, \quad \{0,1,0,0,0,0,0,0,\alpha\}.$$

Consider $X_4 + X_6 + \gamma X_9$. Then we have either the vectors

$$\{0,\alpha,1,0,0,0,0,0,0\}, \quad \{0,1,0,0,0,0,0,0,0\},$$

or

$$\{0,0,0,\alpha,1,0,0,0,\beta\}, \quad \{0,0,0,1,0,0,0,0,\alpha\}.$$

Let us take $\alpha X_6 + \beta X_7 + X_8 + \gamma X_9$. First we consider the case $\alpha \neq -1$. Let $\alpha = \beta = 0$. Then if $t^2 = (e^4)^2 + (e^5)^2 \neq 0$ we act by the matrix

$$B_7(e^4/t,\ -e^5/t),$$

and obtain the vectors $\{1,0,0,0,0,0,0,0,0\},\ \{0,0,0,1,0,0,0,0,0\}$. If $e^1 = e^4 = e^5 = e^8 = 0$ provided that $e^6 = e^7 = 0$ under the action of the matrices

$$B_7(e^2/t,\ -e^3/t), \quad t^2 = (e^2)^2 + (e^3)^2$$

we obtain $\{0,1,0,0,0,0,0,0,\alpha\}$, and provided that $(e^6)^2 + (e^7)^2 \neq 0$ we act by the matrices

$$B_2(a_2), \quad B_3(a_3),$$

where a_2, a_3 are determined by the system

$$\left.\begin{array}{l} e^6 a_2 + e^3 a_3 = -e^2 \\ -e^7 a_2 + e^6 a_3 = -e^3 \end{array}\right\},$$

and obtain $\{0,0,0,0,0,\alpha,1,0,\beta\}$, $\{0,0,0,0,0,1,0,0,\alpha\}$. Let $\beta \neq 0$. Then we either have the vectors

$$\{0,0,0,0,0,\alpha,1,0,\beta\}, \quad \{0,0,0,0,0,1,0,0,\alpha\},$$

or $\{1,0,0,0,0,0,0,0,0\}$. Let $\alpha \neq 0$, $\beta = 0$. Then we have either the vectors

$$\{1,0,0,0,0,0,0,0,0\}, \quad \{0,1,0,0,0,0,0,0,0\}, \quad \{0,0,0,1,0,0,0,0,0\}$$

(and if $\alpha = -2$ there is one more vector $\{1,0,0,1,0,0,0,0,0\}$) or

$$\{0,0,0,0,0,\alpha,1,0,\beta\}, \quad \{0,0,0,0,0,1,0,0,\alpha\}.$$

Now let us turn to the case $\alpha = -1$. If $\beta = 0$, there are two possibilities:

1) $e^1 = e^6 = e^7 = e^8 = e^9 = 0$. Then under the action of the matrices

$$B_1(a_1), \quad B_7(a_7', a_7'')$$

where a_1, a_7 are obviously selected depending on the values e^2, e^3, e^4, e^5, we obtain the vectors

$$\{0,1,0,0,0,0,0,0,0\}, \quad \{0,0,0,1,0,0,0,0,0\}, \quad \{0,0,1,1,0,0,0,0,0\}.$$

2) $e^2 = e^3 = e^4 = e^5 = e^8 = 0$. Here if $e^6 = 1$ we act by the matrix

$$B_1(-e^1)$$

and obtain $\{0,0,0,0,0,1,\alpha,0,\beta\}$, and if $e^6 = 0$ we have

$$\{0,0,0,0,0,0,1,0,\alpha\}, \quad \{1,0,0,0,0,0,1,0,\alpha\}, \quad \{1,0,0,0,0,0,0,0,\alpha\}.$$

If $\beta \neq 0$ then $e^2 = e^3 = e^4 = e^5 = e^8 = 0$ and this case is contained in the latter case.

Let us take $X_1 - X_6 + \alpha X_7 + X_8 + \gamma X_9$. If $\alpha = 0$, then we obtain either the vector $\{0,1,0,0,0,0,0,0,0\}$, under the action of the matrix

$$B_7(e^2/t, -e^3/t), \quad t^2 = (e^2)^2 + (e^3)^2 \neq 0,$$

or $e^2 = e^3 = e^4 = e^5 = e^6 = e^8 = 0$ and consequently the vectors

$$\{1,0,0,0,0,0,\alpha,0,\beta\}, \quad \{0,0,0,0,0,0,1,0,\alpha\}.$$

Table 3: Representatives for the optimal system of two-parameter subgroups of the group G_9

1	$X_1+X_9,\quad \alpha X_1+X_7$	26	$X_1,\quad X_7+\alpha X_8+\beta X_9$
2	$X_1+X_9,\ \alpha X_1+X_6+\beta X_7-X_8$	27	$X_1,\quad X_6+\alpha X_7+\beta X_8+\gamma X_9$
3	$X_2+X_9,\quad X_3+X_4$	28	$X_1,\quad X_9$
4	$X_2+X_9,\ X_1+\alpha X_4+\beta X_5$	29	$X_2,\quad X_1+\alpha X_4+\beta X_5+\gamma X_9$ ($\alpha,\gamma=0;1$. When $\alpha=0,\ \beta=0;1$)
5	$X_2+X_9,\quad \alpha X_2+\beta X_4+X_5\ (\alpha=0;1)$		
6	$X_2+X_9,\ \alpha X_2+\beta X_3+X_8$	30	$X_2,\quad \alpha X_4+X_5+\beta X_9\quad (\beta=0;1)$
7	$X_1+X_4,\quad -2X_6+X_8+\alpha X_9$	31	$X_2,\quad X_3+X_4+\alpha X_9\quad (\alpha=0;1)$
8	$X_2+X_5,\ \alpha X_2+\beta X_3+X_4+\gamma X_9$ $(\gamma=0;1)$	32	$X_2,\quad X_3+\alpha X_9\quad (\alpha=0;1)$
		33	$X_2,\quad X_4+\alpha X_9\quad (\alpha=0;1)$
9	$X_2+X_5,\quad -X_6+X_8+\alpha X_9$	34	$X_2,\quad X_4+\alpha X_5+X_6+\beta X_9$
10	$X_7+\alpha X_9,\quad \beta X_6+X_8+\gamma X_9$	35	$X_2,\quad X_5+X_6+\alpha X_9$
11	$X_7+\alpha X_9,\quad X_6+\beta X_9$	36	$X_2,\quad X_6+\alpha X_9$
12	$X_7+\alpha X_9,\quad X_1-X_6+X_8+\beta X_9$	37	$X_2,\quad X_3+X_8+\alpha X_9$
13	$X_1+X_7+\alpha X_9, \beta X_1-X_6+X_8+\gamma X_9$	38	$X_2,\quad \alpha X_6+X_8+\beta X_9$
14	$X_6+\alpha X_7+\beta X_9,\quad \gamma X_7+X_8+\delta X_9$	39	$X_2,\quad X_1-X_6+X_8+\alpha X_9$
15	$X_9,\quad X_1+X_4$	40	$X_2,\quad X_9$
16	$X_9,\quad X_2+X_5$	41	$X_5,\quad X_3+X_9$
17	$X_9,\quad X_6+\alpha X_7$	42	$X_5,\ X_2+\alpha\beta X_3+X_4+\beta X_9\ (\beta=0;1)$
18	$X_9,\quad X_7$	43	$X_5,\quad X_3+X_4+X_9$
19	$X_9,\quad X_1+X_7$	44	$X_5,\quad X_4+\alpha X_9\quad (\alpha=0;1)$
20	$X_9,\quad X_2+X_8$	45	$X_5,\quad X_3+X_8+\alpha X_9$
21	$X_9,\quad X_4+X_6$	46	$X_5,\quad \alpha X_4+X_6+\beta X_9\quad (\alpha=0;1)$
22	$X_9,\quad \alpha X_6+\beta X_7+X_8$	47	$X_5,\quad X_2+\alpha X_3+X_8+\beta X_9$
23	$X_9,\quad X_1-X_6+\alpha X_7+X_8$	48	$X_5,\quad \alpha X_6+X_8+\beta X_9$
24	$X_1,\quad X_8+\alpha X_9$	49	$X_5,\quad X_9$
25	$X_1,\quad X_2+X_8+\alpha X_9$	50	$X_5+X_9,\quad \alpha X_4+\beta X_5+X_6$

If $\alpha \neq 0$, then $e^2 = e^3 = e^4 = e^5 = e^6 = e^8 = 0$ and the case is described in the latter case.

Operator X_9 composes pairs with all operators of the system (13) since X_9 is the invariant of any automorphism.

Finally, Table 3 of representatives for the optimal system of two-parameter subgroups of the group G_9 is obtained.

3°. Let us consider the group G_{10}. It follows from (13) and (18) that subgroups composed by means of the operators

$$X_1+\gamma X_9, \quad X_2+\gamma X_9, \quad X_1+X_4+\gamma X_9, \quad X_6+\alpha X_7+\gamma X_9, \quad X_1+X_7+\gamma X_9,$$

$$X_2 + X_8 + \gamma X_9, \quad X_4 + X_6 + \gamma X_9, \quad X_1 - X_6 + \alpha X_7 + X_8 + \gamma X_9$$

are contained in Table 3 and the corresponding subgroups are not simplified by automorphisms (11).

When $\alpha \neq -1$ operator $\alpha X_6 + \beta X_7 + X_8 + \gamma X_9$ also builds subgroups contained in Table 3. When $\alpha = -1$ it is sufficient to consider the case

$$e^1 e^{10} > \frac{1}{4}\,(e^6)^2,$$

since in the case

$$e^1 e^{10} \leq \frac{1}{4}\,(e^6)^2$$

there will be such real z, w that $e'^{10} = 0$ and in the operator $-X_6 + \beta X_7 + X_8+\gamma X_9$ the coordinate $e^{10} = 0$ with any z, w as follows from (14), therefore the resulting pairs in the given case will be contained in Table 3. In the case

$$e' e^{10} > \frac{1}{4}\,(e^6)^2$$

we have $e^2 = e^3 = e^4 = e^5 = 0,\ e^{10} = 1,\ e^1 > 0$ and the matrix

$$C_8(\sqrt{e^1})$$

provides the vector $\{1, 0, 0, 0, 0, \alpha, \beta, 0, \gamma, 1\}$.

Let us consider the operator $X_2+X_5+\gamma X_9$. The condition (18) provides that $e^1 = e^{10} = -e^7$, $e^6 = -e^8$. The case $e^1 = 0$ is contained in Table 3, and operator Y in case $e^1 = 1$ can be reduced to one of the last two operators of the system (16), therefore we shall not consider this case separately.

For the operator $X_7 + \gamma X_9$ we have $e^2 = e^3 = e^4 = e^5 = 0$ and arrive either to the pairs contained in Table 3 or to the vector $\{1, 0, 0, 0, 0, \alpha, 0, 0, \beta, 1\}$.

For the operator X_9 we take into consideration only the vectors

$$\{1, 1, 0, 0, 0, 0, 1, 0, 0, 1\}, \quad \{1, 0, 0, 0, 0, \alpha, \beta, 0, 0, 1\},$$

since all the rest pairs are contained in Table 3.

Consider operator $X_1 + \alpha X_6 + \beta X_7 + \gamma X_9 + X_{10}$. If $\alpha = 0$ then in case $\beta \neq 1$ we have $e^6 + e^8 = 0,\ e^1 = e^2 = e^3 = e^4 = e^5 = e^{10} = 0$, and in case $\beta = 1$ $e^1 = e^{10} = 0$, $e^4 = e^3$, $e^5 = -e^2$, $e^8 = -e^6$. The first case provides the vectors

$$\{0,0,0,0,0,1,\alpha,-1,\beta,0\}, \quad \{0,0,0,0,0,0,1,0,\alpha,0\},$$

and the second one yields

$$\{0,1,0,0,-1,0,0,0,\alpha,0\}, \quad \alpha = 0; 1,$$

$$\{0,0,0,0,0,1,\alpha,-1,\beta,0\}, \quad \{0,0,0,0,0,0,1,0,\alpha,0\},$$

provided that we act by the matrix

$$C_7(e^2/t,\ -e^3/t), \quad t^2 = (e^2)^2 + (e^3)^2$$

when $e^6 = e^7 = 0$, and when $(e^6)^2 + (e^7)^2 \neq 0$ by the matrices

$$C_2(a_5), \quad C_3(-a_4), \quad C_4(a_4), \quad C_5(a_5),$$

where a_4, a_5 are obtained from the equations

$$\left.\begin{array}{l} e^6 a_4 + e^7 a_5 = e^3 \\ -e^7 a_4 + e^6 a_5 = -e^2 \end{array}\right\}.$$

If $\alpha \neq 0$ we either have the vectors

$$\{0,0,0,0,0,1,\alpha,-1,\beta,0\}, \quad \{0,0,0,0,0,0,1,0,\alpha,0\},$$

or the pair

$$< X_1 + \alpha X_6 + \sqrt{1-\frac{\alpha^2}{4}}\, X_7 + \gamma X_9 + X_{10},\ \frac{\alpha}{2} X_9 + \sqrt{1-\frac{\alpha^2}{4}}\, X_3 + X_4 >, \quad \alpha \neq 0$$

that transforms into the pair

$$< X_2 + X_5,\ X_1 + \alpha X_6 - X_7 - \alpha X_8 + \beta X_9 + X_{10} >, \quad \alpha \neq 0$$

under the action of the matrices

$$C_1\left(-\frac{\alpha}{2}\right), \quad C_6\left(-\frac{1}{\sqrt{1-\frac{\alpha^2}{4}}}\right), \quad C_7(0,\ 1).$$

Table 4: Additional pairs for the group G_{10}

1	$X_9,$	$X_1 + X_2 + X_7 + X_{10}$
2	$X_9,$	$X_1 + \alpha X_6 + \beta X_7 + X_{10}, \quad 0 \le \alpha < 2$
3	$X_2 + X_5,$	$X_1 + \alpha X_6 - X_7 - \alpha X_8 + \beta X_9 + X_{10}, \quad \alpha \ne 0$
4	$X_2 + X_5,$	$X_1 + \alpha X_2 + \beta X_3 - X_7 + \gamma X_9 + X_{10}$
5	$X_2 + X_5 + X_9,$	$X_1 + \alpha X_2 + \beta X_3 - X_7 + X_{10}$
6	$X_7 + \gamma X_9,$	$X_1 + \alpha X_6 + \beta X_9 + X_{10}, \quad 0 \le \alpha < 2$
7	$X_6 + \delta X_7 - X_8 + \gamma X_9,$	$X_1 + \alpha X_6 + \beta X_7 + \kappa X_9 + X_{10}, \quad 0 \le \alpha < 2$

Consider the operator $X_1 + X_2 + X_7 + \gamma X_9 + X_{10}$. Condition (18) provides that $e^1 = e^6 = e^7 = e^8 = e^{10} = 0$. Therefore we obtain the vectors

$$\{0, 1, \alpha, \alpha, -1, 0, 0, 0, \beta, 0\}, \quad \{0, 0, 1, 1, 0, 0, 0, 0, \alpha, 0\},$$

that by means of the matrix

$$C_7(e^2/t, \, -e^3/t), \quad t^2 = (e^2)^2 + (e^3)^2$$

provides the pair

$$< X_2 + X_5 + \alpha X_9, \quad X_1 + \beta X_2 + \gamma X_3 - X_7 + \delta X_9 + X_{10} >, \quad \alpha = 0; 1.$$

Representatives of the optimal system of two-parameter subgroups of the group G_{10} are the pairs 1-40 of Table 3 and the following pairs.

§ 4 Invariant solutions of the rank 2.

Upon composing the optimal systems of one-parameter and two-parameter subgroups we can start to construct invariant solutions. This section describes some invariant solutions of the rank two obtained from one-parameter subgroups. Let us introduce the notation H_1 for one-parameter subgroups and H_2 for two-parameter subgroups. The variables U, V, P, R are functions of two arguments λ, μ with values being defined for different subgroups in every separate case.

1. Let us consider the system (10). The invariant H_1-solution in the form

$$u = U, \quad v = V, \quad p = P, \quad \rho = R; \quad \lambda = x, \quad \mu = y,$$

corresponds to the operator X_1. It is a stationary well investigated case, therefore we shall not take it into consideration.

2. The invariant H_1-solution

$$u = U, \quad v = V, \quad p = P, \quad \rho = R; \quad \lambda = t, \quad \mu = y$$

for the operator X_2 describes a one-dimensional gas flow analyzed by L.V. Ovsyannikov in [103].

3. Invariant H_1-solution corresponding to the operator X_5 has the form

$$u = U, \quad v = \frac{y}{t} + V, \quad p = \frac{1}{t}P, \quad \rho = \frac{1}{t}R; \quad \lambda = t, \quad \mu = x.$$

The system S/H_1 is

$$\left.\begin{aligned} &V_t + UV_x + \frac{1}{t}V = 0\\ &U_t + UU_x + \frac{1}{R}P_x = 0\\ &R_t + UR_x + RU_x = 0\\ &P_t + UP_x + A'U_x = 0 \end{aligned}\right\}$$

where the following notation is introduced

$$A' = A'(P, R) \equiv -R\,\frac{\partial S/\partial R}{\partial S/\partial P}.$$

For the values U, P, R the system of equations for a one-dimensional gas flow is obtained, and V is derived from the equation

$$V_t + UV_x + \frac{1}{t}V = 0$$

when the function U is known. Therefore this case can also be related to the case of a one-dimensional gas flow and omitted. We shall not consider one-parameter subgroups corresponding to the operators X_1, X_2, X_3, X_4, X_5, and two-parameter subgroups that have these operators as one of the generators.

4. Let us consider the subgroup H_1 corresponding to $\alpha X_6 + X_7$. If we introduce polar coordinates according to the formulae

$$r = \sqrt{x^2 + y^2}, \quad \varphi = \operatorname{arctg}\frac{y}{x}, \tag{24}$$

then

$$\alpha X_6 + X_7 = \alpha t\frac{\partial}{\partial t} + \alpha r\frac{\partial}{\partial r} - \frac{\partial}{\partial \varphi} + v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v},$$

and the invariant H_1-solution has the form

$$u = V\sin(U-\varphi),\quad v = V\cos(U-\varphi),\quad p = P,\quad \rho = R;\quad \lambda = te^{\alpha\varphi},\ \mu = re^{\alpha\varphi}.$$

5. Likewise the invariant H_1-solution for the operator X_1+X_7 is obtained

$$u = V\sin(U-\varphi),\quad v = V\cos(U-\varphi),\quad p = P,\quad \rho = R;\quad \lambda = t+\varphi,\quad \mu = r.$$

6. The invariant H_1-solution for the operator $X_1 + X_4$ has the form

$$u = t + U,\quad v = V,\quad p = P,\quad \rho = R;\quad \lambda = \frac{1}{2}t^2 - x,\quad \mu = y.$$

7. The invariant H_1-solution for the operator $X_2 + X_5$ has the form

$$u = U,\quad v = x + V,\quad p = P,\quad \rho = R;\quad \lambda = t,\quad \mu = tx - y.$$

8. The invariant H_1-solution for the operator $\alpha X_4 + X_6$ is

$$u = \frac{x}{t} + U,\quad v = V,\quad p = P,\quad \rho = R;\quad \lambda = \frac{y}{t},\quad \mu = \frac{x}{t} - \alpha\ln t.$$

2°. Let us consider the system (13). We will not take into account the operators of the system that differ from operators of the system (10) only by the term γX_9 since this term affects only p and ρ and the changed form of p and ρ can be easily found in every separate case.

1. For the operator X_9 the following complete set of functionally independent invariants is available:

$$J^1 = t,\quad J^2 = x,\quad J^3 = y,\quad J^4 = u,\quad J^5 = v,\quad J^6 = \frac{p}{\rho}\cdot$$

This set does not satisfy the necessary condition of existence of invariant H_1-solutions (see [103]), then there are no invariant H_1-solutions in this case. In case of two-parameter subgroups with the operator X_9 being one of the generators, the necessary condition is not satisfied either and that is why we will not consider this subgroups.

2. The invariant H_1-solution for the operator $X_2 + X_8 + \gamma X_9$ has the form

$$u = \frac{1}{t}U,\quad v = \frac{1}{t}V,\quad \rho = t^{2+\gamma}R,\quad p = t^{\gamma}P;\quad \lambda = y,\quad \mu = te^{-x}.$$

3. The operator $\alpha X_6 + \beta X_7 + X_8 + \gamma X_9$ is written in polar coordinates as

$$(\alpha+1)t\frac{\partial}{\partial t} + \alpha r\frac{\partial}{\partial r} - \beta\frac{\partial}{\partial\varphi} + (\beta v - u)\frac{\partial}{\partial u} - (\beta u + v)\frac{\partial}{\partial v} + (2+\gamma)\rho\frac{\partial}{\partial\rho} + \gamma p\frac{\partial}{\partial p}\cdot$$

If $\alpha \neq 0$, $\beta \neq 0$ the invariant H_1-solution is

$$u = r^{-\frac{1}{\alpha}} U \sin(V - \varphi), \quad v = r^{-\frac{1}{\alpha}} U \cos(V - \varphi), \quad p = r^{\frac{\gamma}{\alpha}} P, \quad \rho = r^{\frac{2+\gamma}{\alpha}} R;$$

$$\lambda = t^{-\alpha} r^{\alpha+1}, \quad \mu = r^{\beta} e^{\alpha\varphi}.$$

If $\alpha = 0$, $\beta \neq 0$ then the invariant H_1-solution has the form

$$u = e^{\frac{\varphi}{\beta}} U \sin(V - \varphi), \quad v = e^{\frac{\varphi}{\beta}} U \cos(V - \varphi), \quad p = t^{\gamma} P, \quad \rho = t^{2+\gamma} R;$$

$$\lambda = r, \quad \mu = t^{\beta} e^{\varphi}.$$

If $\alpha \neq 0$, $\beta = 0$, then

$$u = r^{-\frac{1}{\alpha}} U, \quad v = r^{-\frac{1}{\alpha}} V, \quad p = r^{\frac{\gamma}{\alpha}} P, \quad \rho = r^{\frac{\gamma+2}{\alpha}} R; \quad \lambda = \varphi, \quad \mu = t^{-\alpha} r^{\alpha+1}.$$

If $\alpha = \beta = 0$, then

$$u = \frac{1}{t} U, \quad v = \frac{1}{t} V, \quad p = t^{\gamma} P, \quad \rho = t^{\gamma+2} R; \quad \lambda = r, \quad \mu = \varphi.$$

4. The operator $X_1 - X_6 + \alpha X_7 + X_8 + \gamma X_9$ in polar coordinates has the form

$$\frac{\partial}{\partial t} - r\frac{\partial}{\partial r} - \alpha\frac{\partial}{\partial \varphi} + (\alpha v - u)\frac{\partial}{\partial u} - (\alpha u + v)\frac{\partial}{\partial v} + (2+\gamma)\rho\,\frac{\partial}{\partial \rho} + \gamma p\,\frac{\partial}{\partial p}\,\cdot$$

If $\alpha = 0$, then the invariant H_1-solution has the form

$$u = rU, \quad v = rV, \quad p = r^{-\gamma} P, \quad \rho = r^{-(\gamma+2)} R; \quad \lambda = \varphi, \quad \mu = re^{t}.$$

If $\alpha \neq 0$, then

$$u = rU \sin(V - \varphi), \quad v = rU \cos(v - \varphi), \quad p = r^{-\gamma} P, \quad \rho = r^{-(\gamma+2)} R;$$

$$\lambda = \varphi + \alpha t, \quad \mu = re^{t}.$$

3°. Let us consider the operator $X_1 + X_{10}$ of the system (16). The corresponding invariant H_1-solution has the form

$$u = \frac{tx}{1+t^2} + \frac{1}{x} U, \quad v = \frac{ty}{1+t^2} + \frac{1}{y} V, \quad p = \frac{1}{(1+t^2)^2} P, \quad \rho = \frac{1}{1+t^2} R;$$

$$\lambda = \frac{x}{\sqrt{1+t^2}}, \quad \mu = \frac{y}{\sqrt{1+t^2}}\,\cdot$$

§ 5 Invariant solutions of the rank 1.

These solutions are obtained from two-parameter subgroups. The variables U, V, P, R depend on one argument λ here. The values of λ will be indicated.

1°. Consider Table 3. If one omits the stationary and one-dimensional cases there should be only the pairs 13 and 14 to be considered.

1. For the pair $< X_2 + X_5,\ \alpha X_2 + \beta X_3 + X_4 >$ the following invariant H_2-solution is obtained

$$u = \frac{tx - y}{t^2 + \alpha t - \beta} + U, \quad v = \frac{(\alpha + t)y - \beta x}{t^2 + \alpha t - \beta} + V, \quad p = P, \quad \rho = R; \quad \lambda = t.$$

2. For the pair $< X_6,\ X_7 >$ the invariant H_2-solution in polar coordinates (24) is

$$u = U\cos\varphi + V\sin\varphi, \quad v = U\sin\varphi - V\cos\varphi, \quad p = P, \quad \rho = R; \quad \lambda = \frac{r}{t}\cdot$$

2°. According to the remarks made in Section § 4 on the operators $X_1,\ X_2,\ X_3,\ X_4,\ X_5,\ X_9$ it is sufficient to consider only the pairs 1-14, 50 of Table 3. Let us find the form of the invariant H_2-solution for some of these pairs.

1. Consider the pair $< X_7 + \alpha X_9,\ X_1 - X_6 + X_8 + \beta X_9 >$. The corresponding invariant H_2-solution in polar coordinates has the form

$$u = rV\sin(U - \varphi), \quad v = rV\cos(U - \varphi), \quad p = r^{-\beta}e^{-\alpha\varphi}P,$$
$$\rho = r^{-(2+\beta)}e^{-\alpha\varphi}R; \quad \lambda = re^t.$$

2. Let us take the second pair $< X_6 + \alpha X_7 + \beta X_9,\ \gamma X_7 + X_8 + \delta X_9 >$. Here different cases are possible depending on the values of the parameters α and γ. If $\gamma \neq 0$, the invariant H_2-solution has the form

$$u = r^{\frac{\alpha}{\gamma}}e^{\frac{\varphi}{\gamma}}V\sin(U - \varphi), \quad v = r^{\frac{\alpha}{\gamma}}e^{\frac{\varphi}{\gamma}}V\cos(U - \varphi), \quad p = r^{\beta-\delta}t^{\delta}P,$$
$$\rho = r^{\beta-\delta-2}t^{2+\delta}R; \quad \lambda = r^{\alpha-\gamma}t^{\gamma}e^{\varphi}.$$

If $\gamma = 0$, in case $\alpha \neq 0$ we have

$$u = \frac{r}{t}V\sin(U - \varphi), \quad v = \frac{r}{t}V\cos(U - \varphi), \quad p = r^{\beta-\delta}t^{\delta}P,$$
$$\rho = r^{\beta-\delta-2}t^{2+\delta}R; \quad \lambda = r^{-\alpha}e^{-\varphi},$$

and in case $\alpha = 0$ we have

$$u = \frac{r}{t}U, \quad v = \frac{r}{t}V, \quad p = r^{\beta-\delta}t^{\delta}P, \quad \rho = r^{\beta-\delta-2}t^{2+\delta}R; \quad \lambda = \varphi.$$

3°. Consider the pair $< X_7+\gamma X_9,\ X_1+X_{10} >$. The invariant H_2-solution in polar coordinates has the form

$$u = \frac{xt}{1+t^2} + \frac{1}{r} V \sin(U-\varphi), \quad v = \frac{yt}{1+t^2} + \frac{1}{r} V \cos(U-\varphi),$$

$$p = \frac{e^{-\gamma\varphi}}{(1+t^2)^2} P, \quad \rho = \frac{e^{-\gamma\varphi}}{1+t^2} R; \quad \lambda = \frac{r}{\sqrt{1+t^2}} \cdot$$

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Paper 2

Group properties of some differential equations

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Preface

The present work is dedicated to investigating group properties of several differential equations of physical significance. I give in Chapter 1 a brief introduction to the theory of group properties of differential equations using the terminology of the books [103], [104]. A detailed presentation of the theory, main definitions and profs of theorems can be found in these books.

Group analysis of the Einstein equations of the general relativity has lead me to a new geometrical notion - a group of generalized motions in Riemannian spaces. In Chapter 2 of the present work the generalized motions are introduced and investigated for arbitrary Riemannian spaces, independently on the Einstein equations. Applications of generalized motions to the Einstein equations are discussed in Chapter 3, § 6.

Note that § 6 and § 8 contain the solution of Problem 4 ("Find the group admitted by Einstein's equation of the general relativity") and a partial solution of Problem 5 ("Carry out classification of partially invariant solutions of gasdynamic equation in two and three dimensions") formulated by L.V. Ovsyannikov in [104], §19.

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Chapter 1
Transformation groups and symmetries of differential equations

This chapter contains a brief introduction to basic concepts of the theory of local Lie groups of transformations and is used in the subsequent chapters. I call local Lie groups for brevity simply Lie groups. The comprehensive presentation of § 1 and in part of § 2 can be found in [103], [104], [33].

§ 1 Point transformation groups

1.1 Groups, invariance and partial invariance

Let x^i $(i = 1, \dots, n)$ be independent variables and u^k $(k = 1, \dots, m)$ the dependent variables. We denote by $E(x, u)$ the $(n+m)$-dimensional Euclidean space of the variables $x = (x^1, \dots, x^n),\ u = (u^1, \dots, u^m)$ and consider an r-parameter group G_r of point transformations of the space $E(x, u)$ into itself given by the equations

$$\begin{aligned} x'^i &= f^i(x, u, a) \quad (i = 1, \dots, n), \\ u'^k &= \varphi^k(x, u, a) \quad (k = 1, \dots, m). \end{aligned} \tag{1.1}$$

It is assumed that the zero value of the group parameter $a = (a^1, \dots, a^r)$ corresponds to the identity transformation. To the group G_r there corresponds the n-dimensional Lie algebra L_r of operators

$$X = e^\alpha X_\alpha = \xi^i \frac{\partial}{\partial x^i} + \eta^k \frac{\partial}{\partial u^k}, \tag{1.2}$$

where $e^\alpha = \text{const}$. A basis of L_r is provided by operators

$$X_\alpha = \xi^i_\alpha \frac{\partial}{\partial x^i} + \eta^k_\alpha \frac{\partial}{\partial u^k}, \quad \alpha = 1, \dots, r,$$

where

$$\xi^i_\alpha = \frac{\partial f^i(x, u, a)}{\partial a^\alpha}\bigg|_{a=0}, \quad \eta^k_\alpha = \frac{\partial \varphi^k(x, u, a)}{\partial a^\alpha}\bigg|_{a=0}.$$

The elements of the algebra L_r are called the generators (or operators) of the group G_r. The transformations of the group G_r with known generators (1.2) can be found by solving the Lie equations with initial conditions:

$$\begin{gathered} \frac{\partial x'^i}{\partial a^\alpha} = \xi^i_\alpha(x', u'), \quad x'^i|_{a=0} = x^i, \\ \frac{\partial u'^k}{\partial a^\alpha} = \eta^k_\alpha(x', u'), \quad u'^k|_{a=0} = u^k, \\ (i = 1, \ldots, n; \quad k = 1, \ldots, m; \quad \alpha = 1, \ldots, r). \end{gathered} \tag{1.3}$$

Accordingly, the group G_r will further often be given by its generators. In cases when the quantities ξ^i $(i = 1, \ldots, n)$ in (1.2) are independent of the variables u^k $(k = 1, \ldots, m)$ we will write the operators (1.2) in the form

$$\overline{X} = X + \eta^k(x, u)\frac{\partial}{\partial u^k}, \tag{1.4}$$

where

$$X = \xi^i(x)\frac{\partial}{\partial x^i}\cdot$$

Most frequently we deal with operators (1.4) of the specific form:

$$\overline{X} = X + s^k_l(x)u^l\frac{\partial}{\partial u^k}\cdot \tag{1.5}$$

In this case we have

$$\overline{X}u^k = s^k_l u^l = (Su)^k,$$

where $S = \|s^k_l\|$ is an $m \times m$ matrix. Then the operator (1.5) is also written

$$\overline{X} = X + S. \tag{1.6}$$

Both of these forms will be used in what follows.

A function $J(x, u)$ is called an invariant of the group G_r if $J(x', u') = J(x, u)$. A function $J(x, u)$ is an invariant of G_r if and only if it solves the following equations for all generators X of the group G_r :

$$XJ = 0. \tag{1.7}$$

A manifold (surface) $M \subset E(x, u)$ is called an *invariant manifold* of the group G_r if all transformations of G_r transform every point of M into a point of the same manifold. Let M be given regularly by the equations

$$M : \quad \psi^\nu(x, u) = 0 \qquad (\nu = 1, \ldots, \mu).$$

The manifold M (or the system $\psi^\nu(x,u)=0$) is invariant if and only if the equations

$$(X\psi^\nu)|_M = 0 \qquad (\nu = 1,\ldots,\mu) \tag{1.8}$$

hold for all generators of the group G_r. The invariant manifold M is called *nonsingular* if $\operatorname{rank}\|\xi^i_\alpha,\ \eta^k_\alpha\|_M = r_*$, where

$$r_* = \operatorname{rank}\|\xi^i_\alpha,\ \eta^k_\alpha\| \tag{1.9}$$

is the general rank of the matrix $\|\xi^i_\alpha,\ \eta^k_\alpha\|$. Nonsingular invariant manifolds can be represented by equations of the form

$$M: \quad \Psi^\nu(J^1,\ldots,J^t) = 0 \qquad (\nu = 1,\ldots,\mu), \tag{1.10}$$

where

$$\operatorname{rank}\left|\left|\frac{\partial \Psi^\nu}{\partial \mathrm{u}^\mathrm{k}}\right|\right| = \mu$$

and $J^1(x,u),\ldots,J^\tau(x,u)$ is a complete set of functionally independent invariants of the group G_r. Thereafter we will consider only nonsingular invariant manifolds. The *rank* of the invariant manifold M is number

$$\varrho = \dim M - r_*\,.$$

A manifold $N \subset E(x,u)$ is called a *partially invariant manifold* of the group G_r if $N \subset M$, where M is an invariant manifold of the group. This manifold, in general, is distinct from $E(x,u)$. The least invariant manifold M containing the partially invariant manifold N will be called here the *determining manifold of the partially invariant manifold* N. The rank ϱ of the determining manifold M is known as the *rank of the partially invariant manifold* N while the number

$$\delta = \dim M - \dim N$$

is referred to as the *defect of invariance* of the partially invariant manifold N. Let the determining manifold M be given in the form (1.10). Then the following conditions hold:

$$\begin{gathered} \tau = n + m - r_*, \quad \varrho = \delta + n - r_*, \quad \mu = m - \delta, \\ \max\{r_* - n,\, 0\} \le \delta \le \min\{r_* - 1,\, m - 1\}\,. \end{gathered} \tag{1.11}$$

1.2 Groups admitted by differential equations

Consider a system of first-order differential equations

$$F_\alpha(x, u, p) = 0, \quad \alpha = 1, \ldots, s, \tag{1.12}$$

where $p = \{p_i^k\}$ is the set of the first-order partial derivatives $p_i^k = \frac{\partial u^k}{\partial x^i}$. Equations (1.12) define a manifold in the space $E(x, u, p)$. Let G_r be a group of transformations in $E(x, u)$ and $\widetilde{G}_r$ the prolongation of G_r to the space $E(x, u, p)$. The generators of $\widetilde{G}_r$ have the form:

$$\widetilde{X} = X + \zeta_i^k(x, u, p)\frac{\partial}{\partial p_i^k}, \tag{1.13}$$

where $\zeta_i^k(x, u, p)$ are defined by the *prolongation formula*

$$\zeta_i^k = D_i(\eta^k) - p_j^k D_i(\xi^j), \quad D_i = \frac{\partial}{\partial x^i} + p_i^k \frac{\partial}{\partial u^k} \cdot \tag{1.14}$$

The system of differential equations (1.12) is said to admit the group G_r if these equations define an invariant manifold of the prolonged group $\widetilde{G}_r$.

Let H be a group admitted by the system (1.12) and $u^k = \varphi^k(x)$ ($k = 1, \ldots, m$) be a solution of of the system. We will consider this solution as a manifold $\Phi \subset E(x, u)$. If Φ is an invariant manifold of rank ϱ for the group H then Φ is called an H*-invariant solution* of rank ϱ. If Φ is a partially invariant manifold of rank ϱ and of defect of invariance δ for the group H then Φ is called a *partially* H*-invariant solution* of rank ϱ and of defect of invariance δ. Using (1.11) one can show that if the general rank of a group admitted by Eqs. (1.12) is no less than $n + m - 1$, then the number of possible types of partially invariant solutions is $n \cdot m$.

The group admitted by Eqs. (1.12) is found from the invariance test of the manifold $F = 0$, where $F = (F_1, \ldots, F_s)$, namely, from the equations

$$\widetilde{X}(F_\alpha)\Big|_{F=0} = 0, \quad \alpha = 1, \ldots, s. \tag{1.15}$$

Eqs. (1.15) are called the *determining equations* of the group admitted by the system (1.12). Due to (1.14) Eqs. (1.15) are differential equations with respect to the functions $\xi^i(x, u)$, $\eta^k(x, u)$. The general solution to the determining equations provides the broadest Lie group of point transformations admitted by system (1.12).

In case of systems of higher-order equations the operator X should be prolonged to all derivatives involved in the system.

§ 2 Contact transformations

Consider a group G of point transformations

$$\begin{aligned} x'^i &= f^i(x,u,p,a), \quad u'^k = \varphi^k(x,u,p,a), \\ p_i'^k &= \psi_i^k(x,u,p,a), \quad (i=1,\dots,n; \quad k=1,\dots,m) \end{aligned} \tag{2.1}$$

of the space $E(x,u,p)$ into itself, where a is a group parameter. The generator of the group G is written in the form

$$X = \xi^i(x,u,p)\frac{\partial}{\partial x^i} + \eta^k(x,u,p)\frac{\partial}{\partial u^k} + \zeta_i^k(x,u,p)\frac{\partial}{\partial p_i^k}\,\cdot \tag{2.2}$$

Let

$$J^k \equiv du^k - p_i^k dx^i = 0 \quad (k=1,\dots,m). \tag{2.3}$$

If Eqs. (2.3) are invariant with respect to the group $\widetilde{G}$ obtained by prolongation of the transformations (2.1) of the group G to the differentials dx^i, du^k, dp_i^k, then the group G is called a group of *contact transformations* of the space $E(x,u)$. We will write the operator of the group $\widetilde{G}$ in the form

$$\widetilde{X} = X + \tilde{\xi}^i\frac{\partial}{\partial dx^i} + \tilde{\eta}^k\frac{\partial}{\partial du^k} + \cdots$$

(the terms with $\frac{\partial}{\partial dp_i^k}$ are omitted since they will be of no use). The quantities $\tilde{\xi}^i$ and $\tilde{\eta}^k$ are obtained by applying the operator $\frac{\partial}{\partial a}\big|_{a=0}$ to the equations

$$dx'^i = \frac{\partial f^i}{\partial x^j}\,dx^j + \frac{\partial f^i}{\partial u^l}\,du^l + \frac{\partial f^i}{\partial p_j^l}\,dp_j^l\,,$$

$$du'^k = \frac{\partial \varphi^k}{\partial x^j}\,dx^j + \frac{\partial \varphi^k}{\partial u^l}\,du^l + \frac{\partial \varphi^k}{\partial p_j^l}\,dp_j^l$$

and have the form:

$$\tilde{\xi}^i = \frac{\partial \xi^i}{\partial x^j}\,dx^j + \frac{\partial \xi^i}{\partial u^l}\,du^l + \frac{\partial \xi^i}{\partial p_j^l}\,dp_j^l \quad (i=1,\dots,n),$$

$$\tilde{\eta}^k = \frac{\partial \eta^k}{\partial x^j}\,dx^j + \frac{\partial \eta^k}{\partial u^l}\,du^l + \frac{\partial \eta^k}{\partial p_j^l}\,dp_j^l \quad (k=1,\dots,m).$$

Writing the invariance test (1.15) for Eqs. (2.3) and substituting the expressions of $\tilde{\xi}^i$ and $\tilde{\eta}^k$ we obtain:

$$\begin{aligned} &\frac{\partial \eta^k}{\partial x^j}\,dx^j + \frac{\partial \eta^k}{\partial u^l}\,p_j^l\,dx^j + \frac{\partial \eta^k}{\partial p_j^l}\,dp_j^l - \zeta_j^k dx^j \\ &-p_i^k\left(\frac{\partial \xi^i}{\partial x^j}\,dx^j + \frac{\partial \xi^i}{\partial u^l}\,p_j^l\,dx^j + \frac{\partial \xi^i}{\partial p_j^l}\,dp_j^l\right) = 0 \quad (k=1,\dots,m). \end{aligned} \tag{2.4}$$

Annulling the coefficients of the differentials dx^j and dp^l_j in (2.4) we get

$$\xi^i = \frac{\partial W}{\partial p_i}, \quad \eta = p_i \frac{\partial W}{\partial p_i} - W, \quad \zeta_i = -\frac{\partial W}{\partial x^i} - p_i \frac{\partial W}{\partial u}, \quad (i = 1, \ldots, n), \tag{2.5}$$

if $m = 1$. Here $W(x, u, p) = \xi^i p_i - \eta$. In the case of several dependent variables, i.e. $m > 1$, the similar procedure leads to the equations

$$\xi^i = \frac{\partial W^1}{\partial p^1_i}, \quad \eta^k = p^k_i \frac{\partial W^1}{\partial p^1_i} - W^k, \quad \zeta^k_i = -\frac{\partial W^k}{\partial x^i} - p^l_i \frac{\partial W^k}{\partial u^l}, \tag{2.6}$$

$$\frac{\partial W^k}{\partial p^l_i} = 0 \ (k \neq l), \quad \frac{\partial W^1}{\partial p^1_i} = \cdots = \frac{\partial W^m}{\partial p^m_i}, \quad W^k = \xi^i p^k_i - \eta^k, \tag{2.7}$$

where $W^k = \xi^i p^k_i - \eta^k, \ i = 1, \ldots, n; \ k = 1, \ldots, m$. Eqs. (2.7) yield that

$$W^k(x, u, p) = Q^i(x, u) p^k_i + P^k(x, u).$$

Substituting these expressions of W^k in Eqs. (2.6) we obtain:

$$\xi^i = \xi^i(x, u), \quad \eta^k = \eta^k(x, u), \quad \zeta^k_i = D_i(\eta^k) - p^k_j D_i(\xi^j). \tag{2.8}$$

Eqs. (2.8) are identical with the prolongation formulae (1.13). Hence, we have arrived at the following statement.

Theorem 2.1. In the case of several dependent variables ($m > 1$) all local groups of contact transformations are obtained by prolongation of local groups of point transformations. Hence, nontrivial contact transformation groups exist only in the case of one dependent variable ($m = 1$), and their generators (2.2) have the coefficients of the form (2.5).

§ 3 Higher-order tangent transformations

One can consider groups of higher-order tangent transformations as well. However, it turns out that they can be obtained by prolongation of contact (first-order tangent) transformation groups. This fact can be deduced from Theorem 2.1, but I will give an independent proof of the following theorem by considering, for the sake of brevity, second-order tangent transformations in the case of one independent variable x and one dependent variable u.

Theorem 2.2. Any group of second-order tangent transformation is the prolongation of a contact (first-order tangent) transformation group.

Proof. Denote by $p = u'(x)$ and $q = u''(x)$ the first and second derivatives of $u = u(x)$, respectively, and consider a group G of point transformations

$$x' = f(x,u,p,q,a), \quad u' = \varphi(x,u,p,q,a),$$

$$p' = \psi(x,u,p,q,a), \quad q' = \sigma(x,u,p,q,a).$$

in the space $E(x,u,p,q)$. If the equations

$$du = p\,dx, \quad dp = qdx \tag{3.1}$$

are invariant under the prolongation $\widetilde{G}$ of G to dx, du, dp, dq, then G is called a group of *second-order tangent transformations.* Let

$$X = \xi\frac{\partial}{\partial x} + \mu\frac{\partial}{\partial u} + \eta\frac{\partial}{\partial p} + \zeta\frac{\partial}{\partial q}$$

be the generator of the group G. If we prolong it to the differentials as in § 2 we will see that the invariance condition of Eqs. (3.1) yields:

$$\frac{\partial V}{\partial q} = 0, \quad \widetilde{D}(V) = W - 2q\,\frac{\partial W}{\partial q}, \tag{3.2}$$

$$\xi = -\frac{\partial W}{\partial q}, \quad \mu = V - p\,\frac{\partial W}{\partial q}, \quad \eta = W - q\,\frac{\partial W}{\partial q}, \quad \zeta = \widetilde{D}(W), \tag{3.3}$$

where $V = \mu - p\xi,\ W = \eta - q\xi$. Eqs. (3.2) yield:

$$W = \frac{\partial V}{\partial x} + p\,\frac{\partial V}{\partial u} - q\,\frac{\partial V}{\partial p}\,\cdot$$

Whereupon, using Eqs. (3.3) we obtain the following equation:

$$\xi = \xi(x,u,p), \quad \mu = \mu(x,u,p) \quad \eta = \eta(x,u,p),$$
$$\zeta = \widetilde{D}(\eta) - q\widetilde{D}(\xi), \quad \widetilde{D} = \frac{\partial}{\partial x} + p\,\frac{\partial}{\partial u} + q\,\frac{\partial}{\partial p}\,,$$

thus proving the theorem.

Hereafter, we will refer to a group of tangent transformations of any order as a local transformation group, considering point transformations as tangent transformations of the zero-order. We see from § 2 and § 3 that in the case of $m > 1$ dependent variables local transformation groups are restricted to Lie point transformation groups. In the case of one dependent variable they are restricted to contact (first-order tangent) transformations.

Chapter 2
Generalized motions in Riemannian spaces

Isometric motions (often called briefly *motions*) and *conformal transformations* have long been used in Riemannian geometry (see, e.g. [34]). They are also used for obtaining particular solutions of the Einstein equation in the general theory of relativity (see [108] and the references therein).

In the present chapter I introduce the concept of groups of generalized motions in Riemannian spaces and investigate their properties. In § 6 we will discuss application of generalized motions to Einstein's equations.

§ 4 Groups of isometric motions

Let V_n be an n-dimensional Riemannian space with a metric tensor

$$g_{ij} = g_{ij}(x) \quad (i, j = 1, \ldots, n) \tag{4.1}$$

and let H be a group of point transformations in V_n with a generator

$$X = \xi^i(x)\frac{\partial}{\partial x^i} \cdot \tag{4.2}$$

Let us associate with H the group $\overline{H}$ of transformations in the space of the variables x^i and g_{ij} $(i, j = 1, \ldots, n;\ i \leq j)$ with the generator

$$\overline{X} = \xi^i(x)\frac{\partial}{\partial x^i} - \left(g_{ik}\frac{\partial \xi^k}{\partial x^j} + g_{jk}\frac{\partial \xi^k}{\partial x^i}\right)\frac{\partial}{\partial g_{ij}} \cdot \tag{4.3}$$

The operator (4.3) is obtained via transition to infinitesimal transformations of coordinates in the transformation formula of the tensor g_{ij}.

Before defining generalized motions Riemannian spaces let us dwell on isometric motions. There exist several definitions of motions in V_n. We will consider three definitions and prove their equivalence by using the facts from § 1.

Definition 2.1. A group H is called a group of *isometric motions* (briefly *motions*) in V_n if the element of length

$$ds^2 = g_{ij}(x)dx^i dx^j$$

is an invariant of the group $\widetilde{H}$ obtained by extension of the group H to the differentials dx^i $(i = 1, \ldots, n)$.

Definition 2.2. A group H is called a group of isometric motions in V_n if the equations (4.1) specifying the metric tensor of the space are invariant withs respect to the group $\overline{H}$.

Definition 2.3. A group H is called a group of motions in V_n if the Killing equations are satisfied:

$$\xi^k \frac{\partial g_{ij}}{\partial x^k} + g_{ik} \frac{\partial \xi^k}{\partial x^j} + g_{jk} \frac{\partial \xi^k}{\partial x^i} = 0 \quad (i, j = 1, \ldots, n). \tag{4.4}$$

Let us prove the equivalence of these definitions.

a) 2.1 $\Longleftrightarrow$ 2.3. We will extend the operator (4.2) to the differentials regarding that transformations of the group H have the form:

$$x'^i = f^i(x, a) \quad (i = 1, \ldots, n).$$

Then if the equalities $dx'^i = \frac{\partial f^i}{\partial x^j} dx^j$ $(i = 1, \ldots, n)$ are acted on by the operator $\frac{\partial}{\partial a}\big|_{a=0}$ it yields an operator of the group extended to the differentials. The operator has the form:

$$\widetilde{X} = \xi^k \frac{\partial}{\partial x^k} + \frac{\partial \xi^k}{\partial x^l} dx^l \frac{\partial}{\partial dx^k}.$$

The necessary and sufficient condition for invariance of ds^2 will have the form:

$$\widetilde{X} ds^2 \equiv \left(\xi^k \frac{\partial g_{ij}}{\partial x^k} + g_{ik} \frac{\partial \xi^k}{\partial x^j} + g_{jk} \frac{\partial \xi^k}{\partial x^i} \right) dx^i dx^j = 0.$$

By setting the coefficients of all differentials in the equation equal to zero we obtain equations (4.4) thus proving the required equivalence.

b) 2.2 $\Longleftrightarrow$ 2.3. The proof immediately follows from the necessary and sufficient condition for invariance of the manifold (4.1) with respect to the group $\overline{H}$:

$$\overline{X}(g_{ij} - g_{ij}(x))\big|_{g_{ij}=g_{ij}(x)} \equiv -\left(\xi^k \frac{\partial g_{ij}(x)}{\partial x^k} + g_{ik} \frac{\partial \xi^k}{\partial x^j} + g_{jk} \frac{\partial \xi^k}{\partial x^i} \right)\Bigg|_{g_{ij}=g_{ij}(x)} = 0.$$

§ 5 Groups of generalized motions

5.1 Definition and basic properties

Definition 2.4. A group H is called a group of generalized motions in a Riemannian space V_n if the metric tensor (4.1) of V_n defines a partially invariant manifold of the group $\overline{H}$. The numbers ϱ and δ are called a rank and a defect, respectively, of the space V_n with respect to the group H.

For groups of generalized motions we have the following analogue of Killing's equations for isometric motions.

Theorem 2.3. Let H be a group of generalized motions in a Riemannian space V_n with a metric tensor (4.1) and let the determining manifold of the space V_n have the form:

$$\mathcal{M}: \quad \varphi^\nu(x,g) = 0, \quad \nu = 1,\ldots,\mu;\ \mu \le \frac{n(n+1)}{2}\cdot \tag{5.1}$$

Then the equations

$$\left[\left(\xi^k \frac{\partial g_{ij}}{\partial x^k} + g_{ik}\frac{\partial \xi^k}{\partial x^j} + g_{jk}\frac{\partial \xi^k}{\partial x^i}\right)\frac{\partial \varphi^\nu}{\partial g_{ij}}\right]_{g_{ij}=g_{ij}(x)} = 0 \quad (\nu = 1,\ldots,\mu) \tag{5.2}$$

are satisfied.

Remark 2.1. For brevity, we term as the *determining manifold of the space* V_n the determining manifold of the manifold given by Eqs. (4.1), i.e. the least invariant manifold containing the partially invariant manifold (4.1).

The proof of Theorem 2.3 is based on the following lemma.

Lemma 2.1. The manifold $\mathcal{M}$ defined by (5.1) is invariant under the group $\overline{H}$ if and only if

$$\left[\left(\xi^k \frac{\partial g_{ij}}{\partial x^k} + g_{ik}\frac{\partial \xi^k}{\partial x^j} + g_{jk}\frac{\partial \xi^k}{\partial x^i}\right)\frac{\partial \varphi^\nu}{\partial g_{ij}}\right]_{\mathcal{M}} = 0 \quad (\nu = 1,\ldots,\mu). \tag{5.3}$$

Proof. Since $\mu \le \frac{1}{2}n(n+1)$, all quantities x^k $(k = 1,\ldots,n)$ on the manifold (5.1) can be considered as independent parameters on which depend the quantities g_{ij} $(i,j = 1,\ldots,n)$. Hence,

$$\frac{d\varphi^\nu|_{\mathcal{M}}}{dx^k} \equiv \left(\frac{\partial \varphi^\nu}{\partial x^k} + \frac{\partial \varphi^\nu}{\partial g_{ij}}\frac{\partial g_{ij}}{\partial x^k}\right)\bigg|_{\mathcal{M}} = 0 \quad (k = 1,\ldots,n; \quad \nu = 1,\ldots,\mu).$$

Multiplying by ξ^k and summing over k from 1 to n we obtain:

$$\left(\xi^k\frac{\partial \varphi^\nu}{\partial x^k} + \xi^k\frac{\partial \varphi^\nu}{\partial g_{ij}}\frac{\partial g_{ij}}{\partial x^k}\right)\bigg|_{\mathcal{M}} = 0 \quad (\nu = 1,\ldots,\mu). \tag{5.4}$$

The invariance test $\overline{X}\varphi^\nu\Big|_{\mathcal{M}} = 0$ for the manifold (5.1) is written now:

$$\left[\xi^k\frac{\partial \varphi^\nu}{\partial x^k} - \left(g_{ik}\frac{\partial \xi^k}{\partial x^j} + g_{jk}\frac{\partial \xi^k}{\partial x^i}\right)\frac{\partial \varphi^\nu}{\partial g_{ij}}\right]_{\mathcal{M}} = 0 \quad (\nu = 1,\ldots,\mu). \tag{5.5}$$

Subtracting (5.5) from (5.4) we obtain (5.3). Conversely, Eqs. (5.3) and (5.4) yield (5.5). The lemma is proved.

Proof of Theorem 2.3. Equations (5.2) clearly result from (5.3) since under the conditions of the theorem the manifold (4.1) is contained in the manifold (5.1). This proves Theorem 2.3.

Let us introduce the notation:

$$h_{ij} = \xi^k \frac{\partial g_{ij}}{\partial x^k} + g_{ik} \frac{\partial \xi^k}{\partial x^j} + g_{jk} \frac{\partial \xi^k}{\partial x^i} \quad (i, j = 1, \ldots, n).$$

Then the metric tensor of the space obtained from V_n via transformations of the group H can be written in the form (see [34]):

$$\tilde{g}_{ij} = g_{ij} + h_{ij} a \quad (i, j = 1, \ldots, n). \tag{5.6}$$

5.2 Geometry of generalized motions

Let us use Theorem 2.3 for better understanding a geometric meaning of generalized motions. Let H be a group of generalized motions in a space V_n with the metric tensor (4.1) and the determining manifold (5.1). Equations (5.2) are linear equations (generally speaking, under-determined) with respect to the quantities h_{ij}. From these equations we can express μ quantities h_{ij} via the remaining δ quantities. Here $\delta = \frac{1}{2}n(n+1) - \mu$ is the defect of the space V_n with respect to the group H. These δ quantities h_{ij} can change arbitrarily thus causing the corresponding "distortion" of the space V_n due to formula (5.6). Thus, the defect δ shows the number of arbitrarily changing quantities h_{ij} by means of which all the remaining quantities are expressed linearly. For instance, for groups of motions we have $\delta = 0$, and Eqs. (5.2) yield that all h_{ij} vanish, hence the space is left undistorted. For conformal transformations equations we have [34]

$$h_{ij} = \sigma(x) g_{ij},$$

whence, assuming without loss of generality that $g_{11} \neq 0$ and eliminating the arbitrary function $\sigma(x)$ we obtain:

$$h_{ij} = \frac{g_{ij}}{g_{11}} h_{11}.$$

This means that the defect of the space V_n with respect to the group of conformal transformations equals to one as it was expected. We change here the arbitrary element $\sigma(x)$ by h_{11} in order to show clearly the meaning of the defect of invariance of a conformal group.

A more vivid geometric interpretation of generalized motions can be obtained by comparing them with motions and conformal transformations.

Recall that the metric at a point x_0 "carried by a group of motions" coincides at a point of destination x_1 with the local metric defined at the point x_1. Hence, it suffices to "measure" the metric properties of a space element at a starting point x_0 in order to know these properties (without additional measurements) at any point where we can arrive from x_0 by means of motions. Therefore, we can talk about equality of geometric figures having different positions, e.g. we call two triangles to be equal if they coincide after an appropriate translation and rotation (isometric motions).

Conformal transformations do not preserve all metric properties but only those that depend solely on ratios of the components of a metric tensor (angles), and hence they map geometric figures into similar figures. An additional measurement is to be taken in order to know all metric properties at a point attained by a conformal transformation. However, the advantage is that we can arrive by conformal transformations into a larger number of points from a given point as compared to motions. In this respect a conformal group has a greater capacity. Of course, arbitrary transformations are the most prolific in this respect since the allow us to reach from a given point into any other point. However, in general, there are no conserved quantities (angles, etc.) for such transformations, and hence all characteristics of space elements are to be measured at every attained point.

Groups of generalized motions fill the "gap" between motions ($\delta = 0$) and arbitrary transformations ($\delta = \frac{1}{2}n(n+1)$). Namely, for transformations with the defect δ there are $\mu = \frac{1}{2}n(n+1)) - \delta$ conserved quantities. Thus only δ metric characteristics have to be measured at a new point.

Let H be a group of generalized motions with a defect δ in a space V_n. Equations (5.2) allow one to obtain a group H' of motions in V_n without solving the complete system of Killing's equations (4.4) by taking H' as a subgroup of the group H. Indeed, it suffices to solve equations (5.2) for μ quantities h_{ij} (due to linearity of (5.2) they will be linear forms of the rest of h_{ij}) and to annul the remaining δ quantities h_{ij} for the coordinates $\xi'^i(x)$ of the desired subgroup H'. This approach is illustrated in the next section.

Given a group H, the space V_n having H as a group of generalized motions is sought as follows. First we construct the prolongation $\overline{H}$ of the group H in accordance with (4.3). Then we find a complete set of functionally independent invariants $J^1(x, y), \ldots, J^t(x, y)$ of the group $\overline{H}$ and express the determining manifold in the form (1.10). Any space with the metric tensor satisfying this system of equations will admit H as a group of generalized motions and have the defect δ.

Chapter 3
Symmetry analysis of some equations

This chapter is devoted to investigation of group properties of Einstein's empty space field equations, wave equations with zero mass and two-dimensional gasdynamic equations. The maximal group admitted by gasdynamic equations is already known [103]. Here the non-similar subgroups of this group are listed and used for investigating invariant and partially invariant solutions in gas dynamics. To the best of my knowledge, the problem of the maximal Lie groups of local transformations admitted by the other equations considered in the chapter has not been solved yet. Groups with generators (1.5) were considered for the Einstein and Maxwell equations while for the Dirac equations operators of the same form were analyzed but the operator X was assumed a priori to belong to a group of conformal transformations (about the Dirac and Maxwell equations see [20], [32], [95]). It is not obvious a priori that these groups are the broadest Lie groups admitted by the above equations, in particular for the Einstein equations due to their nonlinearity.The present chapter is dedicated to the general investigation of this problem. All cumbersome calculations are omitted and only the final results are presented.

§ 6 Einstein's empty space field equations

6.1 The maximal symmetry group

The notion of a group of generalized motions cited in the above section can be used to obtain particular solutions of the Einstein equations:

$$R_{ik} = T_{ik} - \frac{1}{2} T g_{ik} \quad (i, k = 1, \ldots, 4). \tag{6.1}$$

I will consider here Einstein's empty space field equations:

$$R_{ik} = 0 \quad (i, k = 1, \ldots, 4). \tag{6.2}$$

Let us begin with computing the maximal group of point transformations admitted by Eqs. (6.2).

It is well known that Eqs. (6.1) admit all transformations of coordinates, i.e. the infinite-dimensional Lie algebra of operators (4.3). However, the operators (4.3) do not provide the maximal admitted algebra for Eqs. (6.2). Indeed, Eqs. (6.2) admit, e.g. the dilations of the tensor g_{ij} :

$$x'^i = x^i, \quad g'_{ij} = a\, g_{ij}, \quad (i, j = 1, \dots, 4)$$

with the generator

$$X_0 = g_{ij} \frac{\partial}{\partial g_{ij}} \tag{6.3}$$

which is not of the form (4.3). Therefore, we will solve here the problem of computing the maximal Lie algebra admitted by equations (6.2) with generators of the form (1.2). For the sake of brevity, I will use the notation

$$g_{kl,i} \equiv \frac{\partial g_{kl}}{\partial x^i}; \quad g_{kl,ij} \equiv \frac{\partial^2 g_{kl}}{\partial x^i \partial x^j}\,.$$

Then

$$2R_{ik} = g^{lm}(g_{lk,im} + g_{im,lk} - g_{lm,ik} - g_{ik,lm}) + P_{ik}$$

where the terms P_{ik} are defined by

$$P_{ik} = 2g^{lm}(\Gamma^n_{im}\Gamma_{nlk} - \Gamma^n_{ik}\Gamma_{nlm})$$

and do not contain second derivatives of functions g_{ij} $(i, j = 1, \dots, 4)$.

We seek operators of the form:

$$X = \xi^i(x, g)\frac{\partial}{\partial x^i} + \eta_{ij}(x, g)\frac{\partial}{\partial g_{ij}}\,. \tag{6.4}$$

Equations (6.2) are of the second order, hence the operator X is to be prolonged by formulae (1.13), (1.14) twice. The prolonged operators are:

$$\widetilde{X} = X + \lambda_{ijk}\frac{\partial}{\partial g_{ij,\,k}}, \quad \lambda_{ijk} = D_k(\eta_{ij}) - g_{ij,\,s}D_k(\xi^s),$$

$$\widetilde{\widetilde{X}} = \widetilde{X} + \omega_{ijkl}\frac{\partial}{\partial g_{ij,\,kl}}, \quad \omega_{ijkl} = \widetilde{D}_l(\lambda_{ijk}) - g_{ij,\,ks}D_l(\xi^s),$$

where

$$D_k = \frac{\partial}{\partial x^k} + g_{ij,k}\frac{\partial}{\partial g_{ij}}, \qquad \widetilde{D}_k = D_k + g_{ij,kl}\frac{\partial}{\partial g_{ij,l}}\,.$$

We will write the invariance condition (6.2) in the form:

$$\widetilde{\widetilde{X}} R_{ik} = \Omega^{jl}_{ik}\, R_{jl} \qquad (i, k = 1, \dots, 4) \tag{6.5}$$

where Ω^{jl}_{ik} are undetermined coefficients depending, in general, on x^i as well as on g_{ij} and their first derivatives. Separating in (6.5) the terms with second derivatives of g_{ij} we obtain the determining equations given by

$$\begin{aligned}&[(g^{pj}g^{qr}\delta^s_i\delta^t_k + g^{ps}g^{qt}\delta^j_i\delta^r_k - g^{pj}g^{qt}\delta^s_i\delta^r_k - g^{ps}g^{qr}\delta^j_i\delta^t_k)\eta_{pq}\\ &+ g^{lm}(K^{jrst}_{lkim} + K^{jrst}_{imlk} - K^{jrst}_{lmik} - K^{jrst}_{iklm}) - g^{jt}\Omega^{sr}_{ik} - g^{sr}\Omega^{jt}_{ik}\\ &+ g^{jr}\Omega^{st}_{ik} + g^{st}\Omega^{jr}_{ik}]g_{jr,\,st} = 0, \quad (i,j,k,r,s,t = 1,\ldots,4),\end{aligned} \tag{6.6}$$

and

$$\begin{gathered}g^{lm}(M_{lkim} + M_{imlk} - M_{lmik} - M_{iklm}) + \widetilde{X}P_{ik} - \Omega^{jl}_{ik}\,P_{jl} = 0\\ (i,k = 1,\ldots,4).\end{gathered} \tag{6.7}$$

Here $M_{ijkl} = D_kD_l(\eta_{ij} - g_{ij,\,s}\,\xi^s)$ and

$$K^{pqrs}_{ijkl} = \left(\frac{\partial\eta_{ij}}{\partial g_{pq}} - g_{ij,\,t}\frac{\partial\xi^t}{\partial g_{pq}}\right)\delta^r_k\delta^s_l - \big(D_k(\xi^r)\,\delta^s_l + D_l(\xi^s)\,\delta^r_k\big)\,\delta^p_i\delta^q_j\,.$$

Equations (6.6), after setting the coefficients of $g_{ij,\,kl}$ equal to zero (symmetry of $g_{ij,\,kl}$ in both couples of indices should be taken into account), yield $\xi^i = \xi^i(x)$. Therefore we take $X - \overline{X}$ instead of X, where X is given by (6.4) and $\overline{X}$ is given by (4.3). In consequence we obtain $\xi^i = 0$ $(i = 1,\ldots,4)$ and simplify equations (6.2) and (6.7) considerably. By solving the resulting equations we finally arrive at the following statement.

Theorem 2.4. The operators (4.3) and (6.3) span the maximal Lie algebra admitted by Einstein's empty space field equations (6.2).

6.2 Spaces with a given group of generalized motions

Consider now the problem of finding exact particular solutions of the Einstein equations using the symmetries (4.3). According to Definition 2.2 in § 4, classification of invariant solutions of equations (6.1) geometrically means classification of space-times by groups of motions. A detailed discussion of this problem for equations (6.2) can be found in the book [108].

A similar classification can be done by using groups of generalized motions. In the case of general relativity, the classification by groups of generalized motions is equivalent to enumeration of all partially invariant solutions of the Einstein equations (6.1). I will discuss here the classification of the Einstein spaces only, i. e. partially invariant solutions of Eqs. (6.2).

According to the general theory [103], a search for Einstein's spaces admitting a given group H of generalized motions and having a rank ϱ and defect δ with respect to H leads to decomposition of Eqs. (6.2) into two sub-systems of equations. All functions of one sub-system depend on ϱ arguments, whereas the second sub-system (it is overdetermined) comprises equations with respect to δ functions g_{ij} depending on all variables $x^1, \ldots, x^4$. In particular, if $\varrho = 0$ then the first sub-system turns into relations among constants, and hence it remains to solve only the second sub-system. This is the case in Example 2.1 given below. It is obvious that the less the defect δ the narrower the class of solutions with a given rank ϱ. Thus in this regard the narrowest class is the class of the Einstein spaces admitting groups of motions.

Possible types of spaces V_4 admitting groups of generalized motions are listed in Table 1 obtained by applying the conditions (1.11) to the case $n = 4,\ m = 10$. The table presents all necessary characteristics (defect, rank etc.). The last column in the table indicates the form of the determining manifolds (1.10) written by using the convention that all invariants outside the parentheses are functions of the invariants in the parentheses. For instance, the notation $J^1, J^2, J^3(J^4, J^5)$ means that the determining manifold is to be taken in the form $J^1 = \Psi^1(J^4, J^5),\ \ldots,\ J^3 = \Psi^3(J^4, J^5)$. This kind of notation will be used in other tables as well. Note that types $1, 2, 4, 7$ of Table 1 correspond to groups of motions.

Example 2.1. We will find a partially invariant solution of Eqs. (6.2) of type 11 from Table 1. Let us take the group H_5 with the generators

$$X_i = \frac{\partial}{\partial x^i} \quad (i = 1, \ldots, 4), \qquad X_5 = x^1 \frac{\partial}{\partial x^2}\,,$$

with respect to which the desired space will have the rank $\varrho = 0$ and the defect of invariance $\delta = 1$. Calculations will be carried out in the synchronous coordinate system [81] which is invariant under the group H_5. The generators of the extended group $\overline{H}_5$ are given by (1.10) and have the form:

$$\overline{X}_i = \frac{\partial}{\partial x^i} \qquad \overline{X}_5 = x1\frac{\partial}{\partial x^2} - 2g_{12}\,\frac{\partial}{\partial g_{11}} - g_{22}\,\frac{\partial}{\partial g_{12}} - g_{23}\,\frac{\partial}{\partial g_{13}}\,\cdot$$

Let us find the invariants of the group $\overline{H}_5$ by solving the equations:

$$\overline{X}_i J(x, g) = 0 \quad (i = 1, \ldots, 4), \qquad \overline{X}_5 J(x, g) = 0.$$

The first four equations yield $J = J(g)$, Then the last equation becomes:

$$2g_{12}\,\frac{\partial J}{\partial g_{11}} + g_{22}\,\frac{\partial J}{\partial g_{12}} + g_{23}\,\frac{\partial J}{\partial g_{13}} = 0.$$

Solving the characteristic system

$$\frac{dg_{11}}{2g_{12}} = \frac{dg_{12}}{g_{22}} = \frac{dg_{13}}{g_{23}}$$

we obtain the following functionally independent invariants:

$$\begin{aligned} &J^1 = g_{11}g_{12} - g_{12}^2, \quad J^2 = g_{12}g_{23} - g_{13}g_{22},\\ &J^3 = g_{22}, \quad J^4 = g_{23}, \quad J^5 = g_{33}. \end{aligned} \tag{6.8}$$

We have here only five invariants instead of nine predicted by Table 1. The reason is that we have pre-fixed the values of four components of the metric tensor, and they are precisely the missing four invariants. To be more specific, we used the equations

$$g_{14} = g_{24} = g_{34} = 0, \quad g_{44} = -1$$

which define a synchronous coordinate system and which determine an invariant manifold for the group $\overline{H}_5$. Then we used the group induced by $\overline{H}_5$ on this manifold instead of the original group $\overline{H}_5$, keeping the same notation $\overline{H}_5$ for the induced group.

According to Table 1, we take the determining manifold in the form $J^k = c_k \quad (k = 1, \ldots, 5)$, express from these equations five components of the tensor g_{ij} via one of its components (since the defect $\delta = 1$) and five arbitrary constants c_k, and obtain:

$$g_{11} = \frac{1}{c_3}(c_1 + f^2), \quad g_{13} = \frac{1}{c_3}(c_4 f - c_2), \quad g_{22} = c_3, \quad g_{23} = c_4, \quad g_{33} = c_5,$$

where $f = g_{12}$ is an arbitrary function of four variables x^1, x^2, x^3, x^4. We can set here $c_1 = c_4 = 0,\ c_2 = 1,\ c_3 = -1$ by means of a suitable change of variables preserving synchronous coordinate systems. Then

$$g_{ij} = \begin{pmatrix} -f^2 & f & 1 & 0\\ f & -1 & 0 & 0\\ 1 & 0 & c_5 & 0\\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad g = \det|g_{ij}| = -1. \tag{6.9}$$

For (6.9) we obtain the following Christoffel symbols Γ^i_{kl} (all other components Γ^i_{kl} equal to zero except those that differ from the below only by permutation of subscripts):

$$\Gamma^1_{11} = f\frac{\partial f}{\partial x^3}, \quad \Gamma^1_{12} = -\frac{1}{2}\frac{\partial f}{\partial x^3}, \quad \Gamma^2_{11} = f^2\frac{\partial f}{\partial x^3} - f\frac{\partial f}{\partial x^2} - \frac{\partial f}{\partial x^1},$$

$$\Gamma^2_{12} = -\frac{1}{2}f\frac{\partial f}{\partial x^3}, \quad \Gamma^2_{13} = -\frac{1}{2}\frac{\partial f}{\partial x^3}, \quad \Gamma^2_{14} = -\frac{1}{2}\frac{\partial f}{\partial x^4}, \quad \Gamma^3_{11} = f^2\frac{\partial f}{\partial x^2},$$

$$\Gamma^3_{12} = -f\frac{\partial f}{\partial x^2}, \quad \Gamma^3_{13} = -\frac{1}{2}f\frac{\partial f}{\partial x^3}, \quad \Gamma^3_{22} = \frac{\partial f}{\partial x^2}, \quad \Gamma^3_{23} = \frac{1}{2}\frac{\partial f}{\partial x^3},$$

$$\Gamma^3_{14} = -\frac{1}{2}f\frac{\partial f}{\partial x^4}, \quad \Gamma^3_{24} = \frac{1}{2}\frac{\partial f}{\partial x^4}, \quad \Gamma^4_{11} = -f\frac{\partial f}{\partial x^4}, \quad \Gamma^4_{12} = \frac{1}{2}\frac{\partial f}{\partial x^4}\,.$$

By calculating the components of R_{ik} $(i,k = 1,\ldots,4)$ and equating them to zero we obtain 10 second-order differential equations for one function f. Solutions of these equations can be obtained in an explicit form. I will not carry out the calculations here but present the final result only. Neglecting the trivial case of the flat space and making some simplifications we can write the solution in the following form:

$$c^5 = 0, \quad f = x^2A(x^1) + x^4\sqrt{-2\left(A(x^1)^2 + \frac{dA(x^1)}{dx^1}\right)}\,, \tag{6.10}$$

where $A(x^1)$ is an arbitrary function. The metric of the obtained space is

$$ds^2 = -(f dx^1 - dx^2)^2 + 2dx^1dx^3 - (dx^4)^2, \tag{6.11}$$

where the function f is given in (6.10). If $A(x^1)$ is an arbitrary function, the space with the metric (6.11) is not flat. For instance, one of the non-zero components of Riemann's tensor is

$$R^4_{141} = \frac{3}{2}\left(A(x^1)^2 + \frac{dA(x^1)}{dx^1}\right).$$

Let us find out which subgroup H' of the group H_5 is a group of isometric motions of the metric (6.11). According to § 5, to answer this question it suffices to solve one equation $h_{ij} = 0$ with a certain pair (i,j). In order to identify the (i,j) we should find out a particular quantity h_{ij} via which all other h_{ij} can be expressed upon solving the equations (5.2):

$$h_{ij}\,\frac{\partial\varphi^\nu}{\partial g_{ij}}\bigg|_{g_{ij}=g_{ij}(x)} = 0, \quad \nu = 1,\ldots,5,$$

with $\varphi^\nu = J^\nu - c_\nu$ $(\nu = 1,\ldots,5)$, where J^ν are the invariants (6.8) of the group $\overline{H}_5$ and $g_{ij}(x)$ is the metric tensor (6.9). The reckoning shows that all quantities h_{ij} can be expressed via h_{12}. Hence, it is sufficient to annul the quantity h_{12} calculated for an arbitrary subgroup H' of the group H_5. Writing the generators of the subgroup H' in the form $X = \alpha^i X_i + \beta X_5$ with undetermined constant coefficients $\alpha^1, \ldots, \alpha^4, \beta$ and solving the equation $h_{12} = 0$ we find that $\alpha^1 = \alpha^2 = \alpha^4 = \beta = 0$ if $A(x^1)$ is an arbitrary function. Hence, $X = X_3$, i.e. H' is the one-parameter group of translations in x^3. It is obvious that the translations in x^3 do not change the metric (6.11) because the function f in (6.10) does not depend on x^3.

With some special types of the function $A(x^1)$ there are cases when not all of the $\alpha^1, \alpha^2, \alpha^4, \beta$ equal to zero. We will skip here this problem as well as the question about the maximal group of isometric motions (which is not necessarily a subgroup of H_5) of the metric (6.11). However, it is worth noting that in investigating Einstein spaces having a group of generalized motions H of special interest is the following problem.

Problem of symmetry augmentation. Let an Einstein space be sought as a space of the rank ϱ and the defect δ with respect to a group H. Is there a group H' (if yes, find this group) such that the space in question has a rank $\varrho' \leq \varrho$ and a defect $\delta' \leq \delta$ with respect to H'?

Hence the problem is to clarify the symmetry properties of the Einstein space obtained from the Riemannian space having a priori some partial symmetry. If this question can be settled before solving equations (6.2) then as an alternative to finding the Einstein space via the group H, with respect to which the desired space has the defect δ and the rank ϱ, the more facile problem of finding a space with a less rank and defect could be solved.

A good example to the problem on symmetry augmentation is provided by Birkhoff's theorem. It states that the spherically symmetric Einstein space is static. In this case H is the three-parameter group of rotations in the space of spatial coordinates (x^1, x^2, x^3), and the solution in question has the rank $\varrho = 2$ and the defect $\delta = 0$ with respect to H. The problem of symmetry augmentation has a positive solution. Namely, the spherically symmetric Einstein space has the rank $\varrho' = 1$ and the defect $\delta' = 0$ with respect to the four-parameter group H' obtained by adding to H the translations of the time variable x^4.

Remark 2.2. The problem on symmetry augmentation is closely related to possibility of *reductions* of partially invariant solutions ([104], §18). Namely, according to Ovsyannikov, a partially invariant solution having the rank ϱ and the defect δ with respect to a group H is said to be *reducible* if there exists a subgroup $H' \subset H$ such that the partially invariant solution in question has the rank $\varrho' = \varrho$ and a defect $\delta' \leq \delta$ with respect to H'.

Table 1: Einstein equations (n=4, m=10)

No	r_*	τ	δ	ϱ	μ	M
1	1	13	0	3	10	$J^1,\dots,J^{10}(J^{11},J^{12},J^{13})$
2	2	12	0	2	10	$J^1,\dots,J^{10}(J^{11},J^{12})$
3	2	12	1	3	9	$J^1,\dots,J^{9}(J^{10},J^{11},J^{12})$
4	3	11	0	1	10	$J^1,\dots,J^{10}(J^{11})$
5	3	11	1	2	9	$J^1,\dots,J^{9}(J^{10},J^{11})$
6	3	11	2	3	8	$J^1,\dots,J^{8}(J^{9},J^{10},J^{11})$
7	4	10	0	0	10	$J^k=C^k \quad (k=1,\dots,10)$
8	4	10	1	1	9	$J^1,\dots,J^{9}(J^{10})$
9	4	10	2	2	8	$J^1,\dots,J^{8}(J^{9},J^{10})$
10	4	10	3	3	7	$J^1,\dots,J^{7}(J^{8},J^{9},J^{10})$
11	5	9	1	0	9	$J^k=C^k \quad (k=1,\dots,9)$
12	5	9	2	1	8	$J^1,\dots,J^{8}(J^{9})$
13	5	9	3	2	7	$J^1,\dots,J^{7}(J^{8},J^{9})$
14	5	9	4	3	6	$J^1,\dots,J^{6}(J^{7},J^{8},J^{9})$
15	6	8	2	0	8	$J^k=C^k \quad (k=1,\dots,8)$
16	6	8	3	1	7	$J^1,\dots,J^{7}(J^{8})$
17	6	8	4	2	6	$J^1,\dots,J^{6}(J^{7},J^{8})$
18	6	8	5	3	5	$J^1,\dots,J^{5}(J^{6},J^{7},J^{8})$
19	7	7	3	0	7	$J^k=C^k \quad (k=1,\dots,7)$
20	7	7	4	1	6	$J^1,\dots,J^{6}(J^{7})$
21	7	7	5	2	5	$J^1,\dots,J^{5}(J^{6},J^{7})$
22	7	7	6	3	4	$J^1,\dots,J^{4}(J^{5},J^{6},J^{7})$
23	8	6	4	0	6	$J^k=C^k \quad (k=1,\dots,6)$
24	8	6	5	1	5	$J^1,\dots,J^{5}(J^{6})$
25	8	6	6	2	4	$J^1,\dots,J^{4}(J^{5},J^{6})$
26	8	6	7	3	3	$J^1,J^2,J^{3}(J^{4},J^{5},J^{6})$
27	9	5	5	0	5	$J^k=C^k \quad (k=1,\dots,5)$
28	9	5	6	1	4	$J^1,\dots,J^{4}(J^{5})$
29	9	5	7	2	3	$J^1,J^2,J^{3}(J^{4},J^{5})$
30	9	5	8	3	2	$J^1,J^{2}(J^{3},J^{4},J^{5})$
31	10	4	6	0	4	$J^k=C^k \quad (k=1,\dots,4)$
32	10	4	7	1	3	$J^1,J^2,J^{3}(J^{4})$
33	10	4	8	2	2	$J^1,J^{2}(J^{3},J^{4})$
34	10	4	9	3	1	$J^{1}(J^{2},J^{3},J^{4})$
35	11	3	7	0	3	$J^k=C^k \quad (k=1,2,3)$
36	11	3	8	1	2	$J^1,J^{2}(J^{3})$
37	11	3	9	2	1	$J^{1}(J^{2},J^{3})$
38	12	2	8	0	2	$J^k=C^k \quad (k=1,2)$
39	12	2	9	1	1	$J^{1}(J^{2})$
40	13	1	9	0	1	$J^1=C^1$

§ 7 Wave equations with zero mass

In this section we find the maximal Lie groups of local transformations admitted by the homogeneous Dirac, Maxwell and the scalar wave equations. Inspection of the determining equations shows that the generators of the point transformation groups admitted by these equations have the form (1.5). Moreover, the coordinates ξ^k $(k = 1, \dots, 4)$ of the admitted operators

$$X = \xi^k(x)\frac{\partial}{\partial x^k}$$

satisfy the equations:

$$\frac{\partial \xi^k}{\partial x^l} + \frac{\partial \xi^l}{\partial x^k} = \mu(x)\delta_{kl} \qquad (k, l = 1, \dots, 4). \tag{7.1}$$

I omit here calculations (rather time consuming) since they are performed in the standard way [103]. Equations (7.1) determine a group of conformal transformations of the Euclidean space. Finding the general solution of these equations we obtain the basis operators:

$$\begin{aligned} X_k &= \frac{\partial}{\partial x^k}\,, \qquad X_{kl} = x^l\frac{\partial}{\partial x^k} - x^k\frac{\partial}{\partial x^l} \quad (k < l), \\ T &= x^k\frac{\partial}{\partial x^k}\,, \quad Y_k = (2x^k x^l - |x|^2\delta^{kl})\frac{\partial}{\partial x^l}\,, \quad (k, l = 1, \dots, 4), \end{aligned} \tag{7.2}$$

where $|x|^2 = \sum_{k=1}^4 (x^k)^2$.

7.1 The Dirac equations

The Dirac equations with zero mass have the form:

$$\gamma^k\frac{\partial \psi}{\partial x^k} = 0 \tag{7.3}$$

The maximal group of point transformations admitted by Eqs. (7.3) is the 17-parameter group* with with the following generators of the form (1.6):

$$\begin{aligned} \overline{X} &= X + \frac{1}{8}\sum_{k,l=1}^{4}\frac{\partial \xi^k}{\partial x^l}(\gamma^k\gamma^l - \gamma^l\gamma^k - 3\delta^{kl}), \\ \overline{A} &= \gamma^1\gamma^2\gamma^3\gamma^4, \quad \overline{B} = I. \end{aligned} \tag{7.4}$$

*To be more precise we deal with an infinite group due to linearity of equations (7.3) Namely, all transformations of the form $\psi' = \psi + \varphi(x)$, where $\varphi(x)$ is an arbitrary solution of equations (7.3), are admitted. I excluded such transformations from consideration. The comment is valid for the other equations of this section as well.

Table 2: Dirac equations (n=4, m=4)

No	r_*	τ	δ	ϱ	μ	M
1	1	7	0	3	4	$J^1,\dots,J^4(J^5,J^6,J^7)$
2	2	6	0	2	4	$J^1,\dots,J^4(J^5,J^6)$
3	2	6	1	3	3	$J^1,J^2,J^3(J^4,J^5,J^6)$
4	3	5	0	1	4	$J^1,\dots,J^4(J^5)$
5	3	5	1	2	3	$J^1,J^2,J^3(J^4,J^5)$
6	3	5	2	3	2	$J^1,J^2(J^3,J^4,J^5)$
7	4	4	0	0	4	$J^k=C^k \quad (k=1,\dots,4)$
8	4	4	1	1	3	$J^1,J^2,J^3(J^4)$
9	4	4	2	2	2	$J^1,J^2(J^3,J^4)$
10	4	4	3	3	1	$J^1(J^2,J^3,J^4)$
11	5	3	1	0	3	$J^k=C^k \quad (k=1,2,3)$
12	5	3	2	1	2	$J^1,J^2(J^3)$
13	5	3	3	2	1	$J^1(J^2,J^3)$
14	6	2	2	0	2	$J^k=C^k \quad (k=1,2)$
15	6	2	3	1	1	$J^1(J^2)$
16	7	1	3	0	1	$J^1=C^1$

In (7.4) X runs trough the system of operators (7.2) and I stands for the unit 4×4 matrix corresponding to the dilation group

$$\psi' = a\psi.$$

Solving the Lie equations (1.3) for the operator $\overline{A}$ we get the following finite transformations of the corresponding one-parameter group: In (7.4) X runs trough the system of operators (7.2) and I stands for the unit 4×4 matrix corresponding to the dilation group

$$\psi' = a\psi.$$

Solving the Lie equations (1.3) for the operator $\bar{A}$ we get the following finite transformations of the corresponding one-parameter group:

$$\begin{aligned}
\psi'^1 &= \psi^1\cosh a + \psi^3\sinh a,\\
\psi'^2 &= \psi^2\cosh a + \psi^4\sinh a,\\
\psi'^3 &= \psi^1\sinh a + \psi^3\cosh a,\\
\psi'^4 &= \psi^2\sinh a + \psi^4\cosh a.
\end{aligned}$$

Note that the formula for $\overline{X}$ in (7.4) is a generalization to the conformal group of the following well-known spinor representation for the Lorenz group (see, e.g. [120], [115]):

$$\overline{X} = X + \frac{1}{8}\sum_{k,l=1}^{4} \frac{\partial \xi^k}{\partial x^l}(\gamma^k\gamma^l - \gamma^l\gamma^k).$$

The possible types of partially invariant solutions of the Dirac equations are given in Table 2.

7.2 The Maxwell equations

We will use the four-dimensional presentation of the Maxwell equations:

$$\frac{\partial F_{kl}}{\partial x^m} + \frac{\partial F_{lm}}{\partial x^k} + \frac{\partial F_{mk}}{\partial x^l} = 0 \quad (k, l, m = 1, \ldots, 4),$$
$$\sum_{l=1}^{4} \frac{\partial F_{kl}}{\partial x^l} = 0 \quad (k = 1, \ldots, 4). \tag{7.5}$$

The reckoning shows that the operators admitted by the Maxwell equations have the form (1.5). Namely, solution of the determining equations yields that the maximal Lie algebra admitted by Eqs. (7.5) is the 17-dimensional algebra spanned by

$$\overline{X} = X - \sum_{k<l}\left(F_{km}\frac{\partial \xi^m}{\partial x^l} + F_{ml}\frac{\partial \xi^m}{\partial x^k}\right)\frac{\partial}{\partial F_{kl}},$$
$$\bar{A} = \sum_{k<l}\widetilde{F}_{kl}\frac{\partial}{\partial F_{kl}}, \quad \overline{B} = \sum_{k<l} F_{kl}\frac{\partial}{\partial F_{kl}}, \tag{7.6}$$

where X runs through the system of operators (7.2) and $\widetilde{F}_{kl}$ is the dual tensor of the electromagnetic field defined by

$$\widetilde{F}_{kl} = \frac{\sqrt{-1}}{2}\sum_{m,n=1}^{4} e_{klmn}F_{mn} \quad (k, l = 1, \ldots, 4).$$

Note that $\overline{B}$ in (7.6) is the generator of the dilation group $F'_{kl} = aF_{kl}$, and $\bar{A}$ is the generator of the one-parameter group of dual transformation which is written in terms of the electric and magnetic field vectors as follows:

$$\boldsymbol{H}' = \boldsymbol{H}\cos a + \boldsymbol{E}\sin a,$$
$$\boldsymbol{E}' = \boldsymbol{E}\cos a - \boldsymbol{H}\sin a.$$

Table 3: Maxwell equations (n=4, m=6)

No	r_*	τ	δ	ϱ	μ	M
1	1	9	0	3	6	$J^1,\dots,J^6(J^7,J^8,J^9)$
2	2	8	0	2	6	$J^1,\dots,J^6(J^7,J^8)$
3	2	8	1	3	5	$J^1,\dots,J^5(J^6,J^7,J^8)$
4	3	7	0	1	6	$J^1,\dots,J^6(J^7)$
5	3	7	1	2	5	$J^1,\dots,J^5(J^6,J^7)$
6	3	7	2	3	4	$J^1,\dots,J^4(J^5,J^6,J^7)$
7	4	6	0	0	6	$J^k=C^k \quad (k=1,\dots,6)$
8	4	6	1	1	5	$J^1,\dots,J^5(J^6)$
9	4	6	2	2	4	$J^1,\dots,J^4(J^5,J^6)$
10	4	6	3	3	3	$J^1,J^2,J^3(J^4,J^5,J^6)$
11	5	5	1	0	5	$J^k=C^k \quad (k=1,\dots,5)$
12	5	5	2	1	4	$J^1,\dots,J^4(J^5)$
13	5	5	3	2	3	$J^1,J^2,J^3(J^4,J^5)$
14	5	5	4	3	2	$J^1,J^2(J^3,J^4,J^5)$
15	6	4	2	0	4	$J^k=C^k \quad (k=1,\dots,4)$
16	6	4	3	1	3	$J^1,J^2,J^3(J^4)$
17	6	4	4	2	2	$J^1,J^2(J^3,J^4)$
18	6	4	5	3	1	$J^1(J^2,J^3,J^4)$
19	7	3	3	0	3	$J^k=C^k \quad (k=1,2,3)$
20	7	3	4	1	2	$J^1,J^2(J^3)$
21	7	3	5	2	1	$J^1(J^2,J^3)$
22	8	2	4	0	2	$J^k=C^k \quad (k=1,2)$
23	8	2	5	1	1	$J^1(J^2)$
24	9	1	5	0	1	$J^1=C^1$

The operator $\overline{X}$ in (7.6) shows that for conformal transformations of coordinates x^k $(k = 1, \dots, 4)$ the quantities F_{kl} $(k, l = 1, \dots, 4)$ are transformed as components of a covariant tensor.

It is well known that the non-homogeneous Lorentz group (when X in the operators $\overline{X}$ runs through the operators X_k, X_{kl} from (7.2)) has the following two functionally independent invariants (see, e.g. [81]):

$$J^1 = \sum_{k<l} F_{kl}^2, \quad J^2 = \sum_{k<l} F_{kl}\widetilde{F}_{kl}.$$

In other words, the non-homogeneous Lorentz group is intransitive. The complete 17-parameter group has no invariants, i.e. it is transitive.

Table 3 lists the possible types of invariant ($\delta = 0$) and partially invariant solutions of the Maxwell equations.

7.3 The wave equation

The maximal Lie group of point transformations admitted by the wave equation with three spatial variables:

$$\left(\frac{\partial^2}{\partial t^2} - \triangle\right) u = 0 \tag{7.7}$$

is the group of conformal transformations [103] with the generators (7.2). It can be shown that the group of contact transformations admitted by equation (7.7) coincides with the extended group of point transformations.

§ 8 Two-dimensional gasdynamic equations

The system of equations of two-dimensional gas dynamics has the form:

$$\begin{aligned} &\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + \frac{1}{\rho}\operatorname{grad} p = 0, \\ &\frac{\partial \rho}{\partial t} + (\boldsymbol{v} \cdot \operatorname{grad} \rho) + \rho \operatorname{div} \boldsymbol{v} = 0, \\ &\frac{\partial p}{\partial t} + (\boldsymbol{v} \cdot \operatorname{grad} p) + A(p, \rho) \operatorname{div} \boldsymbol{v} = 0. \end{aligned} \tag{8.1}$$

Here p is pressure, ρ is density, $\boldsymbol{v} = \boldsymbol{v}(x, y, t)$ is the velocity vector with components u, v, and $A = -\rho\, S_\rho \big/ S_p$, where S is entropy. It is assumed that $S_p \equiv \frac{\partial S}{\partial p} \neq 0$. We will consider the case when $A(p, \rho)$ is an arbitrary

Table 4: Gasdynamic equations (n=3, m=4)

No	r_*	τ	δ	ϱ	μ	M
1	1	6	0	2	4	$J^1,\ldots,J^4(J^5,J^6)$
2	2	5	0	1	4	$J^1,\ldots,J^4(J^5)$
3	2	5	1	2	3	$J^1,J^2,J^3(J^4,J^5)$
4	3	4	0	0	4	$J^k=C^k \quad (k=1,\ldots,4)$
5	3	4	1	1	3	$J^1,J^2,J^3(J^4)$
6	3	4	2	2	2	$J^1,J^2(J^3,J^4)$
7	4	3	1	0	3	$J^k=C^k \quad (k=1,2,3)$
8	4	3	2	1	2	$J^1,J^2(J^3)$
9	4	3	3	2	1	$J^1(J^2,J^3)$
10	5	2	2	0	2	$J^k=C^k \quad (k=1,2)$
11	5	2	3	1	1	$J^1(J^2)$

function*. In this case, the maximal Lie algebra admitted by Eqs. (8.1) is the seven-dimensional algebra L_7 is spanned by the operators [103]:

$$
\begin{aligned}
&X_1=\frac{\partial}{\partial t}, \quad X_2=\frac{\partial}{\partial x}, \quad X_3=\frac{\partial}{\partial y},\\
&X_4=t\frac{\partial}{\partial x}+\frac{\partial}{\partial u}, \quad X_5=t\frac{\partial}{\partial y}+\frac{\partial}{\partial v},\\
&X_6=t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y},\\
&X_7=y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}+v\frac{\partial}{\partial u}-u\frac{\partial}{\partial v}\cdot
\end{aligned}
\tag{8.2}
$$

The usual test for solvability of Lie algebras [28] shows that L_7 is solvable. This fact will be useful in constructing optimal systems of subalgebr. One can see from the formula (1.9) that the the Lie algebra L_7 has general rank $r_*=5$. Using this equation we obtain Table 4 of the possible types of partially invariant solutions of Eqs. (8.1).

8.1 Optimal systems of subalgebras

In order to investigate all types of solutions from Table 4 we construct optimal systems of subgroups of all orders by employing the method of enumerating non-similar subgroups, or rather subalgebras of Lie algebras used

*For discussion of particular forms of $A(p,\rho)$ leading to an augmentation of the admitted group, see [103] and [50].

in [103]. In our case the solvability of the algebra L_7 is used as follows. Let us seek an optimal system Θ_s of s-dimensional subalgebras of the algebra L_7. Any subalgebra of L_7 is solvable by virtue of its own solvability. Hence, any s-dimensional subalgebra has at least one $(s-1)$-dimensional subalgebra. Thus if we know an optimal system Θ_{s-1} then we can obtain an optimal system Θ_s of s-dimensional subalgebras by supplementing every $(s-1)$-dimensional subalgebra from Θ_{s-1} till an s-dimensional subalgebra and eliminating the similar subalgebras of the resulting set of s-dimensional subalgebras. Choosing the values of s consecutively from 1 till 6 we will enumerate all non-similar subalgebras of L_7. Tables (8.3)-(8.8) contain the final results of the construction of the optimal systems $\Theta_1, \ldots, \Theta_6$.

Optimal system Θ_1

1	X_1	5	$X_1 + X_7$
2	X_2	6	$X_1 + X_4$
3	X_5	7	$X_2 + X_5$
4	$X_7 + \alpha X_6$	8	$X_6 + \alpha X_4$

(8.3)

Optimal system Θ_2

1	X_1	X_2
2		X_6
3		$X_7 + \alpha X_6$
4	X_2	X_3
5		$X_3 + X_4$
6		$X_1 + X_4 + \alpha X_5$
7		$X_6 + \alpha X_4 + \beta X_5$
8		X_4
9		$X_1 + X_5$
10	X_5	X_4
11		$X_2 + \alpha X_3$
12		$X_6 + \alpha X_4$
13	$X_2 + X_5$	$X_4 + \alpha X_2 + \beta X_3$
14	X_7	X_6

(8.4)

Optimal system Θ_3

1	X_1	X_2	X_3
2			X_4
3			$X_3 + X_4$
4			$X_6 + \alpha X_4$
5		X_6	X_7
6	X_2	X_3	$X_7 + \alpha X_6$
7			$X_1 + X_7$
8			$X_1 + X_5$
9			X_4
10			$X_6 + \alpha X_4$
11		X_4	$X_6 + \alpha X_5$
12			$X_1 + X_5$
13			X_5
14		$X_3 + X_4$	$X_1 + X_5$
15	X_4	X_5	$X_7 + \alpha X_6$
16			$X_2 + X_7 + \alpha X_6, \quad \alpha \neq 0$
17			X_6
18		$X_3 + \alpha X_2$	$X_6 + \alpha X_5$
19	$X_2 + X_5$	X_7	$X_3 - X_4$
20		X_3	$X_4 + \alpha X_2$

(8.5)

Optimal system Θ_4

1	X_1, X_2, X_3	X_6
2		$X_4 + \alpha X_6$
3		$X_7 + \alpha X_6$
4	X_1, X_2, X_4	X_6
5	X_2, X_3, X_7	X_6
6	X_2, X_3, X_4	$X_1 + X_5$
7		$X_6 + \alpha X_5$
8		X_5
9	X_2, X_4, X_5	X_6
10	X_4, X_5, X_6	X_7

(8.6)

Optimal system Θ_5

1	X_1, X_2, X_3, X_6	X_7
2	X_1, X_2, X_3, X_4	X_5
3		$X_6 + \alpha X_5$
4	X_2, X_3, X_4, X_5	X_6
5		$X_1 + X_7$
6		$X_7 + \alpha X_6$

(8.7)

Optimal system Θ_6

1	X_1, X_2, X_3, X_4, X_5	X_6
2		$X_7 + \alpha X_6$
3	X_2, X_3, X_4, X_5, X_6	X_7

(8.8)

In these tables α, β are arbitrary real constants. We will illustrate by an example the arrangement of the tables. For instance, the eights line of Table (8.6) represents the four-dimensional subalgebra spanned by X_2, X_3, X_4, X_5. The other cases are similar. Let us proceed to an analysis of partially invariant and invariant solutions.

8.2 Partially invariant solutions

In this section, we will examine partially invariant solutions of types 7, 10 and 11 (in classification of Table 4) in the listed order. Note that all partially invariant solutions constructed on the subalgebras of Table (8.8) are contained among the solutions constructed by using Table (8.7). Therefore we can exclude (8.8) from consideration.

Type 11. For this type we need subalgebras of the rank $r_* = 5$. They can be only from Θ_5. Inspecting Table (8.7), one can verify that the condition $r_* = 5$ is satisfied only for the subalgebra 2 from (8.7). It is spanned by $X_1,\ X_2,\ X_3,\ X_4,\ X_5$ and has two independent invariants ($\tau = 2$):

$$J^1 = \rho, \quad J^2 = p.$$

A solution of type 11 has the form:

$$p = p\,(\rho). \tag{8.9}$$

Substitution of (8.9) in Eqs. (8.1) yields:

$$\begin{gathered}
\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \frac{1}{\rho}\frac{dp}{d\rho}\,\mathrm{grad}\,\rho = 0,\\
\frac{\partial \rho}{\partial t} + (\boldsymbol{v}\cdot\ \mathrm{grad}\,\rho) + \rho\,\mathrm{div}\,\boldsymbol{v} = 0,\\
\Big(A - \rho\,\frac{dp}{d\rho}\Big)\,\mathrm{div}\,\boldsymbol{v} = 0.
\end{gathered} \tag{8.10}$$

The last equation of this system shows that two cases can be singled out:

$$a)\quad \mathrm{div}\,\boldsymbol{v} = 0,$$

$$b)\quad A(p,\rho) - \rho\,\frac{dp}{d\rho} = 0.$$

In the case a) Eqs. (8.10) are written:

$$\begin{aligned} \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \frac{1}{\rho}\frac{dp}{d\rho}\operatorname{grad}\rho = 0,\\ \frac{d\rho}{dt} \equiv \frac{\partial \rho}{\partial t} + (\boldsymbol{v}\cdot\operatorname{grad}\rho) = 0,\\ \operatorname{div}\boldsymbol{v} = 0. \end{aligned} \tag{8.11}$$

In the case b) Eqs. (8.10) have the form:

$$\begin{aligned} \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \frac{1}{\rho^2}A\operatorname{grad}\rho = 0,\\ \frac{\partial \rho}{\partial t} + (\boldsymbol{v}\cdot\ \operatorname{grad}\rho) + \rho\operatorname{div}\boldsymbol{v} = 0,\\ \rho\frac{dp}{d\rho} = A(p,\rho). \end{aligned} \tag{8.12}$$

Type 10. This type of solutions is based on the same subalgebra as type 11 and have the form $p = C_1,\ \rho = C_2$. Now Eqs. (8.1) yield:

$$\begin{aligned} \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = 0,\\ \operatorname{div}\boldsymbol{v} = 0. \end{aligned} \tag{8.13}$$

Type 7. All solutions of this type are based on the subalgebras 2, 3, 6-10 of the optimal system (8.6). Let us discuss these solutions in detail. Note that the solutions of type 7 are particular cases of the solutions of type 10 because p and ρ are invariants of all subgroups. Therefore, we can consider Eqs. (8.13) instead of (8.1).

Subalgebra 2. It is spanned by $X_1,\ X_2,\ X_3,\ X_4 + \alpha X_6$ and has the following functionally independent invariants:

$$J^1 = p, \quad J^2 = \rho, \quad J^3 = v.$$

Hence, the corresponding partially invariant solution is determined by

$$p = a, \quad \rho = b, \quad v = c,$$

where $a, b, c =$ const. By translation in v we can get $c = 0$, thus we will assume $v = 0$. Then Eqs. (8.13) yield $u_t = u_x = 0$. The desired solution is

$$p,\ \rho = \text{const.}, \quad v = 0, \quad u = f(y), \tag{8.14}$$

where $f(y)$ is an arbitrary function.

SUBALGEBRA 3. This subalgebra is spanned by $X_1,\ X_2,\ X_3,\ X_7+\alpha X_6$ and furnishes the following form of partially invariant solutions:

$$p = a, \quad \rho = b, \quad u^2 + v^2 = c. \tag{8.15}$$

Let us write Eqs. (8.13) in the form

$$u_t + uu_x + vu_y = 0, \quad v_t + uv_x + vv_y = 0, \quad u_x + v_y = 0. \tag{8.16}$$

Substitution of (8.15) in the second and third equation (8.16) yields:

$$v_t = -\frac{c}{\sqrt{c - v^2}}\, v_x, \quad v_y = -\frac{v}{\sqrt{c - v^2}}\, v_x. \tag{8.17}$$

The first equation (8.16) is satisfied due to (8.17). Hence, the result.

Theorem 2.5. The partially invariant solutions of type 7 based on the subalgebra spanned by $X_1,\ X_2,\ X_3,\ X_7+\alpha X_6$ is given by the equations

$$\begin{gathered} xu + yv - ct = F(v), \\ u^2 + v^2 = c \end{gathered} \tag{8.18}$$

with an arbitrary function $F(v)$.

Proof. Indeed, writing Eqs. (8.17) in the form

$$v_y = A(v)v_x, \quad v_t = B(v)v_x, \tag{8.19}$$

or

$$\frac{v_t}{B(v)} = \frac{v_x}{1} = \frac{v_y}{A(v)} = k(x, y, t),$$

we obtain along the surface $v =$ const. the following equation:

$$dv \equiv v_t\, dt + v_x\, dx + v_y\, dy = kB\, dt + k\, dx + kA\, dy = 0.$$

Since the functions A and B depend only on v, they keep constant values along $v =$ const. Ignoring the trivial solution $\boldsymbol{v} =$ const. (the constant flow), we can assume that $k \neq 0$. Then the above equation yields:

$$Bt + x + Ay = C.$$

Since this equation holds for any v, we have:

$$B(v)t + x + A(v)y = C(v).$$

Substituting the expressions of the functions $A(v)$ and $B(v)$ given by Eqs. (8.17) and (8.19), we finally arrive at (8.18).

SUBALGEBRA 6. This subalgebra yields the following form of solutions:

$$p = a, \quad \rho = b, \quad v = t + c.$$

Substitution in the second equation of the system (8.16) gives the contradiction $1 = 0$. Hence, the subalgebra 6 does not provide solutions of type 7. The reckoning shows that this statement is valid for the rest of subalgebras of the optimal system (8.6) as well. This proves the following theorem.

Theorem 2.6. All solutions of type 7 from Table 4 are similar to one of the solutions (8.14) or (8.18).

8.3 Invariant solutions

Let us examine the invariant solutions, i.e $\delta = 0$. They can be of type 1, 2 or 4 from Table 4. We will begin with the simplest of them, namely type 4. For this type, we can use (8.13) instead of (8.1) (cf. type 7 in Section 8.2).

Type 4. Solutions of this type are invariant solutions of the rank $\varrho = 0$. Consequently, construction of this type of solutions reduces the differential equations (8.1) to certain relations connecting four constants (by the number of invariants). All solutions of type 4 are obtained on subalgebras of the optimal system Θ_3 given in (8.5). Note that the subalgebras 2, 10-13, 19 and 20 from (8.5) are to be excluded since they do not satisfy the necessary condition for existence of invariant solutions given in [103], §15.

SUBALGEBRA 1. The first subalgebra from (8.5) is spanned by the operators

$$X_1, \quad X_2, \quad X_3$$

and has the functionally independent invariants

$$J^1 = p, \quad J^2 = \rho \quad J^3 = u, \quad J^4 = v$$

According to Table 4 we set the invariants to be equal to arbitrary constants and obtain the constant solution $p =$ const., $\rho =$ const., $\boldsymbol{v} =$ const. This trivial solution can be considered as a particular case of the solution (8.14).

SUBALGEBRA 3. The invariant solution based on the subalgebra spanned by the operators

$$X_1, \quad X_2, \quad X_3 + X_4$$

has the form

$$p = a, \quad \rho = b, \quad v = c, \quad u = y + d$$

with arbitrary constants a, b, c, d. Substitution in Eqs. (8.13) yields $c = 0$. Thus the invariant solution is

$$p = a, \quad \rho = b, \quad v = 0, \quad u = y + d.$$

It is a particular case of the solution (8.14).

SUBALGEBRA 4. This subalgebra with $\alpha \neq 0$ (if $\alpha = 0$ the flow is constant, cf. subalgebra 1) provides the solution

$$p = a, \quad \rho = b, \quad v = 0, \quad u = \alpha \ln y + d$$

with arbitrary constants a, b, d. It is also contained in the solution (8.14).

Simple calculations show that the other subalgebras from (8.5) do not provide invariant solutions, i.e. the expressions for the general form of the corresponding invariant solutions do not satisfy Eqs. (8.13). Hence the following statement holds.

Theorem 2.7. All solutions of type 4 from Table 4 are similar to the solutions obtained on the subalgebras 1, 3, 4. Moreover, the latter solutions are particular cases of the solution (8.14).

Type 2. These solutions have the rank $\varrho = 1$ and are based on the subalgebras of the optimal system Θ_2 listed in (8.4). We will not consider the subalgebras having in their bases one of the operators X_2, X_3, X_4, X_5 since the construction of the corresponding invariant solution reduces to integration of one-dimensional gasdynamic equations. The latter problem has already been investigated from group point of view [103] as is shown further while considering solutions of type 1. Subalgebras having X_1 as one of basic operators yield the stationary solutions and will not be considered here as well. Thus, we restrict the consideration to subalgebras 13 and 14 from (8.4). These subalgebras are Abelian and the corresponding invariant solutions are *similarity solutions* in accordance with [104], §15, Theorem 34.

SUBALGEBRA 13. The subalgebra is spanned by the operators

$$X_2 + X_5, \quad \alpha X_2 + \beta X_3 + X_4$$

and leads to solutions which are linear functions of the spatial variables x, y and have the form:

$$u = \frac{tx - y}{t^2 + \alpha t - \beta} + U(t), \qquad p = P(t),$$

$$v = \frac{(\alpha + t)y - \beta x}{t^2 + \alpha t - \beta} + V(t), \qquad \rho = R(t).$$

Substituting in Eqs. (8.1) and letting $\alpha = \beta = 0$ for simplicity we obtain

$$u = \frac{tx - y - a}{t^2} + \frac{b}{t}, \quad v = \frac{y + a}{t}, \quad \rho = \frac{c}{t^2}, \quad p = P(t),$$

where a, b, c are arbitrary constants, and the function $P(t)$ satisfies the first-order ordinary differential equation

$$\frac{dP}{dt} + \frac{2}{t} A\left(P, \frac{c}{t^2}\right) = 0.$$

Note that the equation for $P(t)$ can be easily solved for the polytropic gas.

SUBALGEBRA 14. The solutions based on the subalgebra spanned by the operators X_6, X_7 has the form:

$$\begin{aligned} u &= U(\lambda)\cos\varphi + V(\lambda)\sin\varphi, \qquad & p &= P(\lambda) \\ v &= U(\lambda)\sin\varphi - V(\lambda)\cos\varphi, \qquad & \rho &= R(\lambda), \end{aligned}$$

where

$$\lambda = \frac{r}{t}, \ \varphi = \arctan\frac{y}{x}, \ r = \sqrt{x^2 + y^2}.$$

Substitution in (8.1) yields the following system of ordinary differential equations for the functions $U(\lambda)$, $V(\lambda)$, $P(\lambda)$ and $R(\lambda)$:

$$\begin{aligned} (U - \lambda)V' + \frac{1}{\lambda} UV &= 0, \\ (U - \lambda)U' - \frac{1}{\lambda} V^2 + \frac{1}{R} P' &= 0, \\ (U - \lambda)R' + \left(U' + \frac{1}{\lambda} U\right) R &= 0, \\ (U - \lambda)P' + \left(U' + \frac{1}{\lambda} U\right) A &= 0, \end{aligned}$$

where the prime denotes here the differentiation with respect to λ.

Type 1. Solutions of this type have the rank $\varrho = 2$ and are based on the subalgebras of the optimal system Θ_1 listed in (8.3) and described by partial differential equations with two independent variables. Therefore we will not consider this case in detail. I will only show that in the case of the operators X_2 and X_5 the invariant solutions are described by one-dimensional gasdynamic equations. It is obvious that one can take the operators X_3 and X_4 instead of X_2 and X_5, respectively.

For the operator X_2 the invariant solution has the form

$$u = U(t, y), \quad v = V(t, y), \quad p = P(t, y), \quad \rho = R(t, y),$$

and Eqs. (8.1) become:

$$U_t + VU_y = 0, \quad V_t + VV_y + \frac{1}{R}P_y = 0,$$
$$R_t + VR_y + RV_y = 0, \quad P_t + VP_y + AV_y = 0.$$

The last three equations are precisely the one-dimensional gasdynamic equations while the first equation determines the function $U(t, y)$.

In the case of the operator X_5 we arrive at a similar result if we write the corresponding invariant solution in the form:

$$u = U(t, x), \quad v = \frac{y}{t} + V(t, x), \quad \rho = \frac{1}{t}R(t, x), \quad p = \frac{1}{t}P(t, x).$$

Then after introducing the function

$$\bar{A}(P, R) = -R\frac{\partial S}{\partial R}\Big/\frac{\partial S}{\partial P}$$

Eqs. (8.1) yield:

$$U_t + UU_x + \frac{1}{R}P_x = 0,$$
$$R_t + UR_x + RU_x = 0,$$
$$P_t + UP_x + \bar{A}U_x = 0,$$
$$V_t + UV_x + \frac{1}{t}V = 0.$$

Here the first three equations are the one-dimensional gas dynamics equations for the functions $U(t, x)$, $R(t, x)$, $P(t, x)$. The last equation of the system determines the function $V(t, x)$, provided that $U(t, x)$ is known.

We have investigated the general case when the form of the function $A(p, \rho)$ is not specified. In this case the class of invariant and partially invariant solutions is rather limited. For certain specific functions $A(p, \rho)$, i.e. if $A = \gamma p$ (polytropic gas) the admissible group extends [103] and augments the class of group invariant solutions. If $\gamma = 2$ the group extends even further [103] (in this case equations (8.1) describe the motion of a "shallow water"). Classification of invariant solutions of (8.1) for the above specific cases is given in [50]*.

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Paper 3

Theorem on projections of equivalence Lie algebras

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§ 1 Introduction

Mathematical models are often described in terms of differential equations containing unknown parameters or functions called "arbitrary elements". The group classification allows one to find specific values of these arbitrary elements when the symmetry group of the equations under consideration is wider compare to the group admitted by the equations for any values of the parameters or functions. Moreover, numerous applications manifest that equations provided by the group classification have physically interesting solutions. Therefore, the group classification has not only a theoretical significance, but it is important for applications as well.

Unfortunately, the group classification is a complicated problem since it requires integration of determining equations containing not only unknown coefficients of symmetry generators, but also unknown "arbitrary elements". In consequence, group classification is not widely used in mathematical modelling.

In this paper I suggest a simplified approach to the problem. It is hinted by the observation that most of extended symmetries are contained in the Lie algebra of the equivalence group (I call it briefly *equivalence Lie algebra*) of the equations in question. The method suggested here is based on the theorem of projections of the equivalence Lie algebra given in § 2. This theorem allows one to partially solve the problem of group classification, namely, to identify those values of arbitrary elements when the equations have *extended symmetries contained in the equivalence Lie algebra.*

I illustrate the method in § 3 by providing a partial solution to the problem of group classification of gasdynamic equations. The compete solution of this problem is given in [103] and requires rather complicated calculations.

§ 2 Description of the method

2.1 Projections of equivalence Lie algebras

Consider differential equations

$$F_\nu(x, u, u_{(1)}, \ldots, u_{(N)}, f(z)) = 0, \quad \nu = 1, \ldots, M, \tag{1}$$

where $x = (x^1, \ldots, x^n)$ and $u = (u^1, \ldots, u^m)$ denote independent and dependent variables, respectively. In most of cases the number M of equations in (1) is equal to the number of the dependent variables, $M = m$, but this requirement is not necessary for further considerations. The set of the first-order partial derivatives $u_i^\alpha = \partial u^\alpha / \partial x^i$ are denoted by $u_{(1)}$. Likewise, $u_{(2)}$ stands for the second-order derivatives $u_{ij}^\alpha = \partial^2 u^\alpha / \partial x^i \partial x^j$, etc. Furthermore, $f(z) = \big(f^1(z), \ldots f^l(z)\big)$ are unknown functions, where $z = (z^1, \ldots z^s)$ is a certain subset of the variables $x, u, u_{(1)}, u_{(2)}, \ldots$. The functions $f^1(z), \ldots f^l(z)$ are termed *arbitrary elements* [106] of Eqs. (1).

Definition 3.1. An equivalence transformation of Eqs. (1) is an invertible change of variables $\bar{x} = \bar{x}(x, u), \bar{u} = \bar{u}(x, u)$ taking the system (1) into a system of the same form,

$$F_\nu(\bar{x}, \bar{u}, \bar{u}_{(1)}, \ldots, \bar{u}_{(N)}, \bar{f}(\bar{z})) = 0, \quad \nu = 1, \ldots, M,$$

where, generally speaking, the functions $\bar{f}$ of $\bar{z} = (\bar{z}^1, \ldots \bar{z}^s)$ may be different from the functions f of $z = (z^1, \ldots z^s)$.

The set of all equivalence transformations form a group. According to Definition 3.1, the generators of continuous groups of equivalence transformations (in brief, *equivalence generators*) are written

$$Y = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \cdots + \mu^k \frac{\partial}{\partial f^k}\,, \tag{2}$$

where

$$\xi^i = \xi^i(x, u), \quad \eta^\alpha = \eta^\alpha(x, u), \quad \mu^k = \mu^k(x, u, u_{(1)}, \ldots, f).$$

The dots in (2) indicate that if z includes some of the derivatives $u_{(1)}, \ldots$, then the action of the differential operator Y is extended these derivatives by means of the usual prolongation formulae.

I will denote by X and Z the projections of the operator (2) to the (x, u)-space and to the (z, f)-space, respectively. Namely,

$$\begin{aligned} X &= \mathrm{pr}_{(x,u)}(Y) = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}\,, \\ Z &= \mathrm{pr}_{(z,f)}(Y) = \lambda^\sigma \frac{\partial}{\partial z^\sigma} + \mu^k \frac{\partial}{\partial f^k}\,\cdot \end{aligned} \tag{3}$$

Remark 3.1. In what follows, we will assume that the projection Z in (3) is well defined, specifically, that its coefficients λ^σ, μ^k depend only on the variables z, f. Note also that, by definition of the operator Y, the coefficients λ^σ in (3) comprise some of the coefficients ξ^i, η^α and some of the coefficients given in (2) by dots if z includes certain derivatives $u_{(1)}, \ldots$.

Example 3.1. Consider an equation (1) of the following the form:

$$u_t = f(x, u)u_{xx} + g(x, u_x).$$

Then $z = (x, u, u_x)$ and the operator (2) has the form

$$Y = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u} + \zeta_1 \frac{\partial}{\partial u_x} + \mu^1 \frac{\partial}{\partial f} + \mu^2 \frac{\partial}{\partial g}\,,$$

where ξ^1, ξ^2, η depend on x, t, u, the coordinate ζ_1 is given by the prolongation formula, $\zeta_1 = D_x(\eta) - u_x D_x(\xi^1) - u_t D_x(\xi^2)$, and μ^1, μ^2 depend, in general, on x, t, u, u_x, u_t, f, g. The projections (3) are written:

$$\begin{aligned} X &= \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u}\,, \\ Z &= \xi^1 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \zeta_1 \frac{\partial}{\partial u_x} + \mu^1 \frac{\partial}{\partial f} + \mu^2 \frac{\partial}{\partial g}\,\cdot \end{aligned}$$

According to Remark 3.1, the projection Z is well defined if ζ_1, μ^1, μ^2 do not depend on t and u_t.

2.2 Main theorem

Theorem 3.1. Let Y be a generator of a group of equivalence transformations for Eqs. (1). Its projection

$$X = \mathrm{pr}_{(x,u)}(Y)$$

given in (3) is admitted by the differential equations (1) with specific values

$$f^k = f^k(z), \quad k = 1, \ldots, l, \tag{4}$$

of the arbitrary elements $f^1, \ldots, f^l$ if and only if the projection

$$Z = \mathrm{pr}_{(z,f)}(Y)$$

of Y given in (3) is admitted by Eqs. (4).

Proof. According to Definition 3.1, an equivalence transformation does not alter Eqs. (1) with specific values (4) of the arbitrary elements $f^1, \ldots, f^l$ if and only if $\bar{f}(\bar{z}) = f(\bar{z})$. The latter equation guarantees that $X = \mathrm{pr}_{(x,u)}(Y)$ is admitted by the equations (1) with the given $f(z)$. On the other hand, the infinitesimal test for the validity of the equation $\bar{f}(\bar{z}) = f(\bar{z})$ is given by

$$Z\big[f^k - f^k(z)\big]\Big|_{(4)} = 0 \tag{5}$$

since the coordinates of the operator Z depend on z and f only. Hence, we conclude that Z is admitted by Eqs. (4) thus proving the theorem.

The following corollary of Theorem 3.1 allows one to easily find the *principal Lie algebra* (denoted by $L_{\mathcal{P}}$) for Eqs. (1), i.e. the maximal Lie algebra admitted by Eqs. (1) with arbitrary functions $f(z) = (f^1(z), \ldots, f^l(z))$.

Corollary 3.1. An operator

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}$$

is admitted by Eqs. (1) for any values of the functions $f^1(z), \ldots, f^l(z)$ if and only if it has the form

$$X = \mathrm{pr}_{(x,u)}(Y), \tag{6}$$

where Y is a generator of a group of equivalence transformations of Eqs. (1) such that its projection Z given in (3) vanishes:

$$Z = \mathrm{pr}_{(z,f)}(Y) = 0. \tag{7}$$

Proof. Let Y be a generator of a group of equivalence transformations such that $Z = \mathrm{pr}_{(z,f)}(Y) = 0$. Then Eq. (5) is satisfied by any functions $f^k(z)$, and hence $X = \mathrm{pr}_{(x,u)}(Y) \in L_{\mathcal{P}}$. Let us take now any $X \in L_{\mathcal{P}}$. Since X is admitted by Eqs. (1) with arbitrary $f(z)$, it is an equivalence generator, $Y = X$, such that Eqs. (5) hold for any $f(z)$, i.e.

$$\mu^k - \lambda^\sigma \frac{\partial f^k(z)}{\partial z^\sigma} = 0$$

for any $f^k(z)$. It follows that $\mu^k = 0,\ \lambda^\sigma = 0$. Hence, $Z = \mathrm{pr}_{(z,f)}(Y) = 0$.

§ 3 Application to gasdynamic equations

3.1 Equivalence algebra

Consider the gasdynamic equations:

$$\begin{aligned} \rho_t + \boldsymbol{v}\cdot\nabla\rho + \rho\,\mathrm{div}\,\boldsymbol{v} &= 0,\\ \rho\,[\boldsymbol{v}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}] + \nabla p &= 0,\\ p_t + \boldsymbol{v}\cdot\nabla p + A(p,\rho)\,\mathrm{div}\,\boldsymbol{v} &= 0, \end{aligned} \tag{8}$$

where $A(p,\rho)$ is an arbitrary function connected with the entropy $S(p,\rho)$ by the equation

$$A = -\rho\,\frac{\partial S/\partial\rho}{\partial S/\partial p}\,\cdot \tag{9}$$

The dependent variables are the velocity $\boldsymbol{v}$, the pressure p and the density ρ of the fluid. The independent variables are the time t and the position vector $\boldsymbol{x} = (x^1,\ldots,x^n)$, where n assumes the values 1, 2 or 3. Accordingly, the flow is termed one-dimensional, plane, or three-dimensional, respectively.

In order to find the generators

$$Y = \tau\frac{\partial}{\partial t} + \xi^i\frac{\partial}{\partial x^i} + \eta^i\frac{\partial}{\partial v^i} + \lambda^1\frac{\partial}{\partial p} + \lambda^2\frac{\partial}{\partial\rho} + \mu\frac{\partial}{\partial A} \tag{10}$$

of the continuous group of equivalence transformations, we rewrite Eqs. (8) as an extended system

$$\begin{aligned} \rho_t + \boldsymbol{v}\cdot\nabla\rho + \rho\,\mathrm{div}\,\boldsymbol{v} &= 0,\\ \rho\,[\boldsymbol{v}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}] + \nabla p &= 0,\\ p_t + \boldsymbol{v}\cdot\nabla p + A\,\mathrm{div}\,\boldsymbol{v} &= 0,\\ A_t = 0,\quad A_{x^i} = 0,\quad A_{v^i} &= 0, \end{aligned} \tag{11}$$

where A is treated as a new unknown variable depending on $t,\boldsymbol{x},\boldsymbol{v},p,\rho$. Accordingly, the coefficient μ of the operator (10) is, generally speaking, a function of $t,\boldsymbol{x},\boldsymbol{v},p,\rho,A$. All other coefficients of Y are functions of $t,\boldsymbol{x},\boldsymbol{v},p,\rho$. The last line in Eqs. (11) indicates that A does not depend on $t,\boldsymbol{x},\boldsymbol{v}$.

According to the general theory [106], the operator (10) is a generator of an equivalence group for Eqs. (8) if it is admitted by the extended system (11). The latter condition is guaranteed by the *determining equations*

$$\begin{aligned} \widetilde{Y}\big(\rho_t + \boldsymbol{v}\cdot\nabla\rho + \rho\,\mathrm{div}\,\boldsymbol{v}\big) &= 0,\\ \widetilde{Y}\big(\rho\,[\boldsymbol{v}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}] + \nabla p\big) &= 0,\\ \widetilde{Y}\big(p_t + \boldsymbol{v}\cdot\nabla p + A\,\mathrm{div}\,\boldsymbol{v}\big) &= 0,\\ \widetilde{Y}(A_t) = 0, \widetilde{Y}(A_{x^i}) = 0, \widetilde{Y}(A_{v^i}) &= 0, \end{aligned} \tag{12}$$

which should be satisfied, as usual, on solutions of Eqs. (11). The *prolonged operator* $\widetilde{Y}$ is obtained from Y by the usual prolongation procedure with the exception of prolongation to the derivatives of A. For example, the prolongation to A_t is written $\tilde{\mu}_t \frac{\partial}{\partial A_t}$ with

$$\tilde{\mu}_t = \widetilde{D}_t(\mu) - A_t\widetilde{D}_t(\tau) - A_{x^i}\widetilde{D}_t(\xi^i) - A_{v^i}\widetilde{D}_t(\eta^i) - A_p\widetilde{D}_t(\lambda^1) - A_\rho\widetilde{D}_t(\lambda^2),$$

where

$$\widetilde{D}_t = \frac{\partial}{\partial t} + A_t\frac{\partial}{\partial A}\cdot$$

Taking into account the last line in Eqs. (11) we write $\widetilde{D}_t = \frac{\partial}{\partial t}$ and obtain

$$\tilde{\mu}_t = \mu_t - A_p\lambda^1_t - A_\rho\lambda^2_t.$$

Now the equation $\widetilde{Y}(A_t) = 0$ from (12) is written $\tilde{\mu}_t = 0$, i.e.

$$\mu_t - A_p\lambda^1_t - A_\rho\lambda^2_t = 0,$$

and yields $\mu_t = \lambda^1_t = \lambda^2_t = 0$, since $\mu, \lambda^1, \lambda^2$ do not depend on A_p and A_ρ. Using the equations $\widetilde{Y}(A_{x^i}) = 0, \widetilde{Y}(A_{v^i}) = 0$ in a similar way, we conclude that λ^1, λ^2 can depend on p and ρ only, whereas μ_t can depend on p, ρ, A. Using this information and solving the remaining equations (12), one can show that the equivalence Lie algebra for Eqs. (8) is spanned by

$$\begin{gathered} Y_1 = \frac{\partial}{\partial t}, \quad Y_{2i} = \frac{\partial}{\partial x^i}, \quad Y_3 = t\frac{\partial}{\partial t} + x^i\frac{\partial}{\partial x^i}, \quad Y_{4i} = t\frac{\partial}{\partial x^i} + \frac{\partial}{\partial v^i}, \\ Y_{ik} = x^k\frac{\partial}{\partial x^i} - x^i\frac{\partial}{\partial x^k} + v^k\frac{\partial}{\partial v^i} - v^i\frac{\partial}{\partial v^k}, \quad Y_5 = \frac{\partial}{\partial p}, \\ Y_6 = x^i\frac{\partial}{\partial x^i} + v^i\frac{\partial}{\partial v^i} - 2\rho\frac{\partial}{\partial \rho}, \quad Y_7 = \rho\frac{\partial}{\partial \rho} + p\frac{\partial}{\partial p} + A\frac{\partial}{\partial A}\cdot \end{gathered} \tag{13}$$

3.2 Projections of the equivalence algebra

Using the notation of § 2, we have:

$$x = (t, \boldsymbol{x}), \quad u = (p, \rho, \boldsymbol{v}), \quad z = (p, \rho), \quad f = A.$$

The projections (3) of the operator (10) are given by

$$X = \mathrm{pr}_{(x,u)}(Y) = \tau\frac{\partial}{\partial t} + \xi^i\frac{\partial}{\partial x^i} + \eta^i\frac{\partial}{\partial v^i} + \lambda^1\frac{\partial}{\partial p} + \lambda^2\frac{\partial}{\partial \rho},$$

$$Z = \mathrm{pr}_{(z,f)}(Y) = \lambda^1\frac{\partial}{\partial p} + \lambda^2\frac{\partial}{\partial \rho} + \mu\frac{\partial}{\partial A}\cdot \tag{14}$$

Accordingly, among the basic equivalence generators (13), only Y_5, Y_6 and Y_7 have non-vanishing (z, f)-projections:

$$\mathrm{pr}_{(z,f)}(Y_5) = \frac{\partial}{\partial p}\,, \quad \mathrm{pr}_{(z,f)}(Y_6) = -2\rho\frac{\partial}{\partial \rho}\,, \quad \mathrm{pr}_{(z,f)}(Y_7) = \rho\frac{\partial}{\partial \rho}+p\frac{\partial}{\partial p}+A\frac{\partial}{\partial A}\,\cdot$$

It is convenient to take their linear combinations and deal with the operators

$$Z_1 = \frac{\partial}{\partial p}\,, \quad Z_2 = \rho\frac{\partial}{\partial \rho}\,, \quad Z_3 = p\frac{\partial}{\partial p} + A\frac{\partial}{\partial A}\,\cdot \tag{15}$$

Thus,

$$Z_1 = \mathrm{pr}_{(z,f)}(Y_5), \quad Z_2 = \mathrm{pr}_{(z,f)}\Big(-\frac{1}{2}\,Y_6\Big), \quad Z_3 = \mathrm{pr}_{(z,f)}\Big(Y_7 + \frac{1}{2}\,Y_6\Big). \tag{16}$$

3.3 The principal Lie algebra

The general equivalence generator (10) is the linear combination

$$Y = C_1Y_1 + C_{2i}Y_{2i} + C_3Y_3 + C_{4i}Y_{4i} + C_{ij}Y_{ij} + C_5Y_5 + C_6Y_6 + C_7Y_7 \tag{17}$$

of the operators (13) with arbitrary constant coefficients $C_1, C_{2i}, \ldots, C_7$. The projection $Z = \mathrm{pr}_{(z,f)}(Y)$ of the operator (17) vanishes if $C_5 = C_6 = C_7 = 0$. Hence, according to Corollary 3.1, the principal Lie algebra for gasdynamic equations (8), i.e. the maximal Lie algebra admitted by Eqs. (8) with arbitrary $A(p, \rho)$, is spanned by

$$\begin{aligned} &X_1 = \frac{\partial}{\partial t}\,, \quad X_{2i} = \frac{\partial}{\partial x^i}\,, \quad X_3 = t\frac{\partial}{\partial t} + x^i\frac{\partial}{\partial x^i}\,, \\ &X_{4i} = t\frac{\partial}{\partial x^i} + \frac{\partial}{\partial v^i}\,, \quad X_{ik} = x^k\frac{\partial}{\partial x^i} - x^i\frac{\partial}{\partial x^k} + v^k\frac{\partial}{\partial v^i} - v^i\frac{\partial}{\partial v^k}\,\cdot \end{aligned} \tag{18}$$

3.4 Optimal systems of subalgebras spanned by (15)

3.4.1 One-dimensional subalgebras

Elements Z of the three-dimensional Lie algebra L_3 spanned by the operators (15) are written

$$Z = e^1Z_1 + e^2Z_2 + e^3Z_3. \tag{19}$$

Taking the commutators $[Z_\alpha, Z_\beta] = c^\gamma_{\alpha\beta} Z_\gamma$ of the operators (15) one obtains the following commutator table:

	Z_1	Z_2	Z_3
Z_1	0	0	Z_1
Z_2	0	0	0
Z_3	$-Z_1$	0	0

(20)

Thus, the only non-vanishing structure constants are $c^1_{13} = 1$ and $c^1_{31} = -1$. According to the formula

$$E_\alpha = c^\gamma_{\alpha\beta} e^\beta \frac{\partial}{\partial e^\gamma}$$

we find non-vanishing generators of the inner automorphisms:

$$E_1 = c^1_{13} e^3 \frac{\partial}{\partial e^1} = e^3 \frac{\partial}{\partial e^1}\,, \quad E_3 = c^1_{31} e^1 \frac{\partial}{\partial e^1} - e^1 \frac{\partial}{\partial e^1}\,\cdot$$

These generators provide the transformations

$$\bar{e}^1 = e^1 + a_1 e^3, \quad \bar{e}^2 = e^2, \quad \bar{e}^3 = e^3 \tag{21}$$

and

$$\bar{e}^1 = a_3 e^1 \ (a_3 > 0), \quad \bar{e}^2 = e^2, \quad \bar{e}^3 = e^3, \tag{22}$$

respectively. To construct the optimal system Θ_1 of one-dimensional subalgebras of L_3, we have to partition the coordinate vectors $e = (e^1, e^2, e^3)$ of (19) into similarity classes with respect to the transformations (21)-(22) and the reflection $p \mapsto -p$ which is a discrete equivalence transformation and maps Z_1 into $-Z_1$, and hence $e^1 \mapsto -e^1$. It is clear from (21) that we have to distinguish the cases $e^3 = 0$ and $e^3 \neq 0$.

If $e^3 = 0$, we have three possibilities:

$$\text{(i) } e^1 \neq 0,\ e^2 = 0, \quad \text{(ii) } e^1 = 0,\ e^2 \neq 0, \quad \text{(iii) } e^1 \neq 0,\ e^2 \neq 0.$$

Since the vector e is determined up to a constant factor, we we can assume: (i) $e^1 = 1$, i.e. $e = (1, 0, 0)$, (ii) $e^2 = 1$, i.e. $e = (0, 1, 0)$. In the case (iii) we use (22) and the reflection $p \mapsto -p$ to obtain $e = (1, 1, 0)$.

If $e^3 \neq 0$, we let $e^3 = 1$, apply the transformation (21) with $a_1 = -e^1$ and map the vector $e = (e^1, e^2, 1)$ to $\bar{e} = (0, c, 1)$ with an arbitrary c.

Thus, the optimal system Θ_1 of one-dimensional subalgebras comprises

$$Z_1, \quad Z_2, \quad Z_1 + Z_2, \quad Z_3 + cZ_2. \tag{23}$$

3.4.2 Two-dimensional subalgebras

The optimal system Θ_2 of two-dimensional subalgebras can be easily constructed by the procedure explained in §3 of my MSc Thesis*. Namely, one can construct two-dimensional subalgebras L_2 taking each member of the optimal system Θ_1 as one of basic operators of L_2. It remains to partition the set of resulting two-dimensional subalgebras L_2 into similarity classes with respect to the transformations (21)-(22) and the reflection $p \mapsto -p$.

Let us take the operator Z_1 from the optimal system (23) and consider the two-dimensional vector space L_2 spanned by Z_1 and $Z = e^2 Z_2 + e^3 Z_3$. The commutator table (20) shows that $[Z_1, Z] = e^3 Z_1$. Hence L_2 is a two-dimensional Lie algebra for arbitrary e^2 and e^3. We have two possibilities:

(i) $e^3 = 0, e^2 \neq 0$. Then we set $e^2 = 1$ and obtain the subalgebra L_2 spanned by Z_1, Z_2 and denoted by

$$< Z_1, Z_2 > .$$

(ii) $e^3 \neq 0$. Then we set $e^3 = 1, e^2 = c$ and obtain the subalgebra L_2 spanned by $Z_1, Z_3 + cZ_2$, i.e.

$$< Z_1, Z_3 + cZ_2 >$$

with an arbitrary constant c.

Let us take the operator Z_2 from (23) and consider the two-dimensional vector space L_2 spanned by Z_2 and $Z = e^1 Z_1 + e^3 Z_3$. The commutator table (20) shows that $[Z_1, Z] = 0$. Hence L_2 is a two-dimensional Lie algebra for arbitrary e^1 and e^3. If $e^3 = 0, e^1 \neq 0$ we set $e^1 = 1$ and arrive at the subalgebra $< Z_1, Z_2 >$ obtained above. If $e^3 \neq 0$ we make $e^1 = 0$ using the transformation (21) and obtain the subalgebra

$$< Z_2, Z_3 > .$$

Let us take the operator $Z_1 + Z_2$ from (23) and consider the two-dimensional vector space L_2 spanned by $Z_1 + Z_2$ and $Z = e^2 Z_2 + e^3 Z_3$. We assume that $e^3 \neq 0$ and set $e^3 = 1$, since if $e^3 = 0$ we return to the subalgebra $< Z_1, Z_2 >$ considered above. We have $[Z_1, Z] = Z_1$. Therefore the subalgebra condition is written $Z_1 = a(Z_1 + Z_2) + b(Z_3 + e^2 Z_2)$ or $(a-1)Z_1 + (a + be^2)Z_2 + bZ_3 = 0$. Since the operators Z_1, Z_2 and Z_3 are linearly independent, this equation yields $a - 1 = 0, a + be^2 = 0, b = 0$. This is impossible. Hence, $Z_1 + Z_2$ does not lead to new subalgebras.

Finally, we take the operator $Z_3 + cZ_2$ from (23) and consider the two-dimensional vector space L_2 spanned by $Z_3 + kZ_2$ and $Z = e^1 Z_1 + e^2 Z_2$. If $e^1 = 0$ or $e^2 = 0$ we arrive at the subalgebras $< Z_2, Z_3 >$ or $< Z_1, Z_3 + kZ_2 >$,

**Author's note to this 2006 edition:* Paper 1 of the present volume.

respectively. They have been obtained above. Let $e^1 \neq 0, e^2 \neq 0$. Then, using the transformation (22) and the reflection $p \mapsto -p$, we arrive to the subalgebra $< Z_1 + Z_2, Z_3 + cZ_2 >$ which also has been obtained above.

Thus, the optimal system Θ_2 of two-dimensional subalgebras comprises

$$< Z_1, Z_2 >, \quad < Z_2, Z_3 >, \quad < Z_1, Z_3 + cZ_2 > . \tag{24}$$

3.5 Equations with extended symmetry algebras

Let us find the invariant equations $A = F(p, \rho)$ for the operators (23) of the optimal system Θ_1. For each case, the corresponding extension of the principal Lie algebra (18) is obtained by means of Theorem 3.1.

For the first operator, $Z_1 = \frac{\partial}{\partial p}$, the invariant equation test is written $F_p = 0$ and yields $F(p, \rho) = f(\rho)$ with an arbitrary function $f(\rho)$. According to Theorem 3.1 and Eqs. (16), (13), (14), the gasdynamic equations (8) with

$$A = f(\rho) \tag{25}$$

have, along with (18), the additional symmetry

$$X_8 = \mathrm{pr}_{(x,u)}(Y_5) = \frac{\partial}{\partial p} \cdot \tag{26}$$

Likewise, using the second operator from (23), $Z_2 = \rho\frac{\partial}{\partial \rho}$, and Eqs. (16), (13) and (14), one can verify that the gasdynamic equations (8) with

$$A = f(p) \tag{27}$$

have the additional symmetry

$$X_8 = -\frac{1}{2}\mathrm{pr}_{(x,u)}(Y_6)$$

or, neglecting the immaterial coefficient $(-1/2)$,

$$X_8 = x^i \frac{\partial}{\partial x^i} + v^i \frac{\partial}{\partial v^i} - 2\rho\frac{\partial}{\partial \rho} \cdot \tag{28}$$

For the third operator from (23), $Z_1 + Z_2$, the invariant equation test is written

$$\frac{\partial F}{\partial p} + \rho\frac{\partial F}{\partial \rho} = 0$$

and yields $F(p, \rho) = f(p - \ln \rho)$. Hence, the gasdynamic equations (8) with

$$A = f(p - \ln \rho) \tag{29}$$

have the additional symmetry

$$X_8 = \mathrm{pr}_{(x,u)}\Big(Y_5 - \frac{1}{2}Y_6\Big)$$

or

$$X_8 = \frac{\partial}{\partial p} + \rho\frac{\partial}{\partial \rho} - \frac{1}{2}\Big(x^i\frac{\partial}{\partial x^i} + v^i\frac{\partial}{\partial v^i}\Big). \tag{30}$$

Finally, according to (15), the forth operator from (23) has the form

$$Z_3 + cZ_2 = A\,\frac{\partial}{\partial A} + p\frac{\partial}{\partial p} + c\rho\,\frac{\partial F}{\partial \rho}$$

and the invariance test for the equation $A - F(p,\rho) = 0$ is written

$$\Big[A - p\frac{\partial F}{\partial p} - c\rho\,\frac{\partial F}{\partial \rho}\Big]_{A=F} = 0,$$

or

$$p\frac{\partial F}{\partial p} + c\rho\,\frac{\partial F}{\partial \rho} = F.$$

The general solution of this equation has the form $F(p,\rho) = pf(p^c\rho^{-1})$. Hence, the gasdynamic equations (8) with

$$A = pf(p^c\rho^{-1}) \tag{31}$$

have the additional symmetry

$$X_8 = \mathrm{pr}_{(x,u)}\Big(Y_7 + \frac{1}{2}(1-c)Y_6\Big)$$

or

$$X_8 = \frac{1-c}{2}\Big(x^i\frac{\partial}{\partial x^i} + v^i\frac{\partial}{\partial v^i}\Big) + c\rho\frac{\partial}{\partial \rho} + p\frac{\partial}{\partial p}\,\cdot \tag{32}$$

Summarizing Eqs. (25)-(32) we have the following special functions $A(p,\rho)$ and the corresponding additional symmetry of Eqs. (8) for each specialization:

$$\begin{aligned}
&A = f(\rho): && X_8 = \frac{\partial}{\partial p}\,,\\
&A = f(p): && X_8 = x^i\frac{\partial}{\partial x^i} + v^i\frac{\partial}{\partial v^i} - 2\rho\frac{\partial}{\partial \rho}\,,\\
&A = f(p - \ln\rho): && X_8 = \frac{\partial}{\partial p} + \rho\frac{\partial}{\partial \rho} - \frac{1}{2}\Big(x^i\frac{\partial}{\partial x^i} + v^i\frac{\partial}{\partial v^i}\Big),\\
&A = pf(p^c\rho^{-1}): && X_8 = \frac{1-c}{2}\Big(x^i\frac{\partial}{\partial x^i} + v^i\frac{\partial}{\partial v^i}\Big) + c\rho\frac{\partial}{\partial \rho} + p\frac{\partial}{\partial p}\,\cdot
\end{aligned} \tag{33}$$

We apply now the similar procedure to the optimal system Θ_2. Let us find the invariant equations $A = F(p, \rho)$ for the subalgebra $< Z_1, Z_2 >$ from (24). We have to solve the equations

$$Z_1(A - F(p,\rho)\big|_{A=f} = 0, \quad Z_2(A - F(p,\rho)\big|_{A=f} = 0.$$

Substituting the expressions (15) for Z_1, Z_2 and dividing the second equation by ρ we obtain the system

$$\frac{\partial F}{\partial p} = 0, \quad \frac{\partial F}{\partial \rho} = 0$$

with the general solution $F = \text{const}$. Using an equivalence transformation we set $F = 1$. Thus, according to Theorem 3.1 and Eqs. (16), (13), (14), the gasdynamic equations (8) with $A = 1$ have two additional symmetries:

$$X_8 = \frac{\partial}{\partial p}, \quad X_9 = x^i\frac{\partial}{\partial x^i} + v^i\frac{\partial}{\partial v^i} - 2\rho\frac{\partial}{\partial \rho}\cdot$$

Likewise, the invariance under the subalgebra $< Z_2, Z_3 >$ from (24) yields the system (cf. derivation of Eq. (31))

$$\frac{\partial F}{\partial \rho} = 0, \quad p\frac{\partial F}{\partial p} = F,$$

whence $F(p, \rho) = \gamma p$ with an arbitrary constant γ. The corresponding two additional symmetries are $\text{pr}_{(x,u)}\big(-\frac{1}{2}Y_6\big)$ and $\text{pr}_{(x,u)}\big(Y_7 + \frac{1}{2}Y_6\big)$. Since we can take their linear combinations, we set $X_8 = \text{pr}_{(x,u)}(Y_6),\ X_9 = \text{pr}_{(x,u)}(Y_7)$:

$$X_8 = x^i\frac{\partial}{\partial x^i} + v^i\frac{\partial}{\partial v^i} - 2\rho\frac{\partial}{\partial \rho}, \quad X_9 = p\frac{\partial}{\partial p} + \rho\frac{\partial}{\partial \rho}\cdot$$

Finally, the subalgebra $< Z_1, Z_3 + cZ_2 >$ from (24) yields the system

$$\frac{\partial F}{\partial p} = 0, \quad p\frac{\partial F}{\partial p} + c\rho\frac{\partial F}{\partial \rho} = F,$$

whence $F(p, \rho) = A_0\rho^s,\ s = \frac{1}{c},\ A_0 = \text{const}$. The additional symmetries are

$$X_8 = \frac{\partial}{\partial p}, \quad X_9 = \frac{s-1}{2}\Big(x^i\frac{\partial}{\partial x^i} + v^i\frac{\partial}{\partial v^i}\Big) + sp\frac{\partial}{\partial p} + \rho\frac{\partial}{\partial \rho}\cdot$$

Thus, we have obtained the following inequivalent specifications of the function $A(p,\rho)$ when Eqs. (8) have two additional symmetries:

$$
\begin{aligned}
&A = 1: && X_8 = \frac{\partial}{\partial p}\,, \quad X_9 = x^i\frac{\partial}{\partial x^i} + v^i\frac{\partial}{\partial v^i} - 2\rho\frac{\partial}{\partial \rho}\,,\\
&A = \gamma p: && X_8 = x^i\frac{\partial}{\partial x^i} + v^i\frac{\partial}{\partial v^i} - 2\rho\frac{\partial}{\partial \rho}\,, \quad X_9 = p\frac{\partial}{\partial p} + \rho\frac{\partial}{\partial \rho}\,,\\
&A = A_0\rho^s: && X_8 = \frac{\partial}{\partial p}\,, \ X_9 = \frac{s-1}{2}\Big(x^i\frac{\partial}{\partial x^i} + v^i\frac{\partial}{\partial v^i}\Big) + sp\frac{\partial}{\partial p} + \rho\frac{\partial}{\partial \rho}\cdot
\end{aligned}
\tag{34}
$$

Remark 3.2. Equations (33) and (34) represent all cases of extensions of the principal Lie algebra (18) by one symmetry and by two symmetries obtained in [103], §22, by direct method. However, the simplified approach presented here does not reveal the case of extension by three symmetries found in [103], namely, $A = \gamma p$ with $\gamma = (n+2)/n$. In this case Eqs. (8) have, along with two additional symmetries given in (34) for $A = \gamma p$, a third additional symmetry, namely, the generator of the projective group:

$$
X_{10} = t^2\frac{\partial}{\partial t} + tx^i\frac{\partial}{\partial x^i} + (x^i - tu^i)\frac{\partial}{\partial u^i} - (n+2)tp\frac{\partial}{\partial p} - nt\rho\frac{\partial}{\partial \rho}\cdot \tag{35}
$$

The operator X_{10} is not projection $X = \mathrm{pr}_{(x,u)}(Y)$ of an equivalence operator Y of the form (17) and therefore cannot be obtained by our approach.

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Paper 4

Quasi-local symmetries of non-linear heat conduction type equations

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Doklady Akademii Nauk SSSR, Mat. Fiz., Tom 295, (1987), No. 1, pp. 75–78.

1. We have developed in [2] the concept of quasi-local symmetries and gave a constructive method for their computation. Our approach is based on the observation that Bäcklund transformations can augment invariance properties of differential equations ([58], § 17.1, Remark 3). The essence of the method is as follows. Let us consider evolutionary equations $u_t = F$ and $v_t = G$ connected by a Bäcklund transformation in the form $y = \varphi, v = \Phi$. Here $F, \varphi, \Phi \in \mathcal{A}[x, u]$ and $G \in \mathcal{A}[y, v]$, where $\mathcal{A}[x, u]$ and $\mathcal{A}[y, v]$ are the spaces of differential functions of local variables $x, u, u_1, \ldots$ and $y, v, v_1, \ldots$, respectively. Let the equation $u_t = F$ admit a canonical Lie-Bäcklund operator $X = f_u \frac{\partial}{\partial u} + \cdots$ with $f_u \in \mathcal{A}[x, u]$. The Bäcklund transformation connecting the equations in question maps f_u into $f_v \in \mathcal{A}[y, v]$, where f_u and f_v are connected by the following *transition formula* (see [58], Eq. (19.44d)):

$$D_x(\varphi) f_v = (D_x(\varphi)\Phi_* - D_x(\Phi)\varphi_*) f_u. \tag{1}$$

The asterisk indicates transition from differential functions to differential operators by the formula

$$\varphi_* = \sum_{i \geq 0} \frac{\partial \varphi}{\partial u_i} D_x^i = \frac{\partial \varphi}{\partial u} + \frac{\partial \varphi}{\partial u_1} D_x + \frac{\partial \varphi}{\partial u_2} D_x^2 + \cdots .$$

Extension of Equation (1) to the case when $f_v \notin \mathcal{A}[y, v]$ necessitates introduction of new, *nonlocal,* variables Q determined from (1) up to arbitrary

Table 1: Equivalence transformations ($\alpha \neq 0,\ \beta_1\beta_4 - \beta_2\beta_3 \neq 0$)

$w_t = H(w_{xx}) \quad \overset{w_x=v}{\longrightarrow}$	$v_t = h(v_x)v_{xx} \quad \overset{v_x=u}{\longrightarrow}$	$u_t = [h(u)u_x]_x$
$\tilde{t} = \alpha t + \gamma_1$	$\tilde{t} = \alpha t + \gamma_1$	$\tilde{t} = \alpha t + \gamma_1$
$\tilde{x} = \beta_1 x \boxed{+\beta_2 w_x} + \gamma_2$	$\tilde{x} = \beta_1 x + \beta_2 v + \gamma_2$	$\tilde{x} = \beta_1 x \boxed{+\beta_2 v} + \gamma_2$
$\tilde{w} = \beta_1\gamma_3 x + \beta_1\beta_4 w + \frac{1}{2}\beta_1\beta_3 x^2 + \gamma_4 t + \gamma_5$ $\boxed{+\beta_2\beta_3(xw_x - w) + \beta_2\gamma_3 w_x + \frac{1}{2}\beta_2\beta_4 w_x^2}$	$\tilde{v} = \beta_3 x + \beta_4 v + \gamma_3$	$\tilde{u} = \frac{\beta_3 + \beta_4 u}{\beta_1 \boxed{+\beta_2 u}}$
$\tilde{H} = \frac{\beta_1\beta_4 \boxed{-\beta_2\beta_3}}{\alpha} H + \frac{\gamma_4}{\alpha}$	$\tilde{h} = \frac{(\beta_1 + \beta_2 v_x)^2}{\alpha} h$	$\tilde{h} = \frac{\left(\beta_1 \boxed{+\beta_2 u}\right)^2}{\alpha}$

functions of time. Dependence of Q on time is specified from the invariance condition for the equation $v_t = G$ with respect to the operator

$$X = f_v(y, v, v_1, \ldots, Q)\frac{\partial}{\partial v} + \cdots .$$

The resulting function f_v is called a *quasi-local symmetry, associated with the local symmetry* f_u.

2. Here the method is applied to the sequence of equations

$$w_t = H(w_{xx}) \quad \overset{v=w_x}{\longrightarrow} \quad v_t = h(v_x)v_{xx} \quad \overset{u=v_x}{\longrightarrow} \quad u_t(h(u)u_x)_x, \quad h = H', \tag{2}$$

connected by the simplest Bäcklund transformations indicated over the arrows. We classify these equations according to their quasi-local symmetries, associated with point symmetry groups. The first equation in (2) is the equation of inertia-free string vibrations in a liquid with nonlinear resistance, the second equation is the equation for filtration of non-Newtonian liquid in a porous medium, and the third one is the nonlinear heat equation. First, group classification of every term of the sequence (2) is carried out with respect to local symmetries (the second and the third equations are classified in [2], [102], see also [106]). Then, the corresponding groups of quasi-local symmetries are constructed. Application of the groups of equivalence transformations extended by quasi-local equivalence transformations completes the classification. The result is given in Tables 1 and 2, where the operators and equivalence transformations included in boxes are non-point. The well-known case of linear heat conduction is omitted here.

Note that all non-point symmetries and equivalence transformations for the first equation from (2) are tangent, i.e. local. Furthermore, the case

$h(u) = u^{-4/3}$ of group extension for the nonlinear heat equation obtained in [102] is equivalent to the case $h(u) = u^{-2/3}$ and is obtained by the quasi-local equivalence transformation $\tilde{x} = v, \tilde{u} = u^{-1}$. The corresponding chain of operators for $h(u) = u^{-4/3}$ has the form

$$Z_7 = x^2\frac{\partial}{\partial x} + xw\frac{\partial}{\partial w} \rightarrow Y_6 = x^2\frac{\partial}{\partial x} + (w - xv)\frac{\partial}{\partial v} \rightarrow X_5 = x^2\frac{\partial}{\partial x} - 3xu\frac{\partial}{\partial u}\cdot$$

3. Quasi-local symmetries allow one to extend the class of exact solutions provided by the group method. Indeed, the number of invariant solutions increases. Moreover, the new approach allows us to *construct partially invariant solutions for scalar differential equations possessing a Bäcklund transformation.*

Consider, e.g. the nonlinear heat equation with $h(u) = (1 + u^2)^{-1}$. In this case the group is extended by adding the quasi-local symmetry

$$X_4 = -v\frac{\partial}{\partial x} + (1 + u^2)\frac{\partial}{\partial u}\cdot$$

We will be interested in partially invariant solutions with the defect $\delta = 1$ and the rank $\rho = 1$ (see [106]). Let us consider the over-determined system

$$u_t = \left(\frac{u_x}{1 + u^2}\right)_x, \quad v_x = u, \quad v_t = \frac{v_{xx}}{1 + v_x^2}\cdot \qquad (3)$$

In order to determine all essentially different invariant and partially invariant solutions of the rank $\rho = 1$ for the system (3), one has to compose the optimal systems θ_1 and θ_2 [106] of non-similar one-dimensional and two-dimensional subalgebras, respectively, for the five-dimensional Lie algebra with the basis

$$Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial x}, \quad Y_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v},$$

$$Y_4 = \frac{\partial}{\partial v}, \quad Y_5 = -v\frac{\partial}{\partial x} + x\frac{\partial}{\partial v} + (1 + u^2)\frac{\partial}{\partial u}\cdot$$

The optimal system θ_1 is provided by the operators

$$Y_1,\ Y_2,\ Y_1 + Y_2,\ Y_5,\ Y_1 + Y_5,\ Y_1 - Y_5,\ Y_3 + \alpha Y_5,$$

where α is an arbitrary parameter. Using θ_1, one can easily show that the optimal system θ_2 is given by the following pairs of operators:

$$< Y_1, Y_2 >,\ < Y_1, Y_5 >,\ < Y_1, Y_3 + \alpha Y_5 >,\ < Y_2, Y_3 >,\ < Y_3, Y_5 > .$$

Table 2: Group classification $(h(\xi) = H'(\xi))$

$H(w_{xx})$	$w_t = H(w_{xx}) \xrightarrow{w_x = v}$	$v_t = h(v_x)v_{xx} \xrightarrow{v_x = u}$	$u_t = [h(u)u_x]_x$
Arbitrary function	$Z_1 = \frac{\partial}{\partial t}$	$Y_1 = \frac{\partial}{\partial t}$	$X_1 = \frac{\partial}{\partial t}$
	$Z_2 = \frac{\partial}{\partial x}$	$Y_2 = \frac{\partial}{\partial x}$	$X_2 = \frac{\partial}{\partial x}$
	$Z_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2w\frac{\partial}{\partial w}$	$Y_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v}$	$X_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$
	$Z_4 = x\frac{\partial}{\partial w}$	$Y_4 = \frac{\partial}{\partial v}$	-
	$Z_5 = \frac{\partial}{\partial w}$	-	-
$e^{w_{xx}}$	$Z_6 = t\frac{\partial}{\partial t} - \frac{x^2}{2}\frac{\partial}{\partial w}$	$Y_5 = t\frac{\partial}{\partial t} - x\frac{\partial}{\partial v}$	$X_4 = t\frac{\partial}{\partial t} - \frac{\partial}{\partial u}$
$\frac{1}{\sigma}w_{xx}^{\sigma},\ \sigma \neq 1$	$Z_6 = (1-\sigma)t\frac{\partial}{\partial t} + w\frac{\partial}{\partial w}$	$Y_5 = (1-\sigma)t\frac{\partial}{\partial t} + v\frac{\partial}{\partial v}$	$X_4 = (1-\sigma)t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$
$\sigma = \frac{1}{3}$	$Z_7 = w\frac{\partial}{\partial x}$	$\boxed{Y_6 = w\frac{\partial}{\partial x} - v^2\frac{\partial}{\partial v}}$	$\boxed{X_5 = w\frac{\partial}{\partial x} - 3uv\frac{\partial}{\partial u}}$
$\ln w_{xx}$	$Z_6 = t\frac{\partial}{\partial t} + (w+t)\frac{\partial}{\partial w}$	$Y_5 = t\frac{\partial}{\partial t} + v\frac{\partial}{\partial v}$	$X_4 = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$
$\arctan w_{xx}$	$\boxed{Z_6 = -w_x\frac{\partial}{\partial x} + \left(\frac{x^2 - w_x^2}{2} + t\right)\frac{\partial}{\partial w}}$	$Y_5 = -v\frac{\partial}{\partial x} + x\frac{\partial}{\partial v}$	$\boxed{X_4 = -v\frac{\partial}{\partial x} + (1+u^2)\frac{\partial}{\partial u}}$
$\frac{1}{\lambda}e^{\lambda \arctan w_{xx}}$	$\boxed{Z_6 = -\lambda t\frac{\partial}{\partial t} - w_x\frac{\partial}{\partial x} + \frac{x^2 - w_x^2}{2}\frac{\partial}{\partial w}}$	$Y_5 = -\lambda t\frac{\partial}{\partial t} - v\frac{\partial}{\partial x} + x\frac{\partial}{\partial v}$	$\boxed{X_4 = -\lambda t\frac{\partial}{\partial t} - v\frac{\partial}{\partial x} + (1+u^2)\frac{\partial}{\partial u}}$

Let us illustrate the method by obtaining partially invariant solutions for the subalgebra $< Y_3, Y_5 >$. The independent invariants of this subalgebra are $I_1 = (x^2+v^2)/t, I_2 = (xu-v)/(uv+x)$. Therefore the partially invariant solutions with the defect $\delta = 1$ and the rank $\rho = 1$ are sought in the form $I_2 = \varphi(I_1)$, i.e.

$$u = \frac{v + x\varphi(z)}{x - v\varphi(z)}, \quad \text{where} \quad z = \frac{x^2 + v^2}{t} \cdot \tag{4}$$

Substituting (4) in the first equation of the system (3) and taking into account the two remaining equations one obtains an ordinary differential equation for $\varphi(z)$. Upon one integration the latter equation takes the form

$$\varphi' = \frac{1+\varphi^2}{2}\left[C - \left(\frac{1}{2} + \frac{1}{z}\right)\varphi\right], \quad C = \text{const.}$$

Now the problem reduces to integration of the compatible system

$$v_t = \left(\frac{2z\varphi'}{1+\varphi^2} + \varphi\right)(x - \varphi v)^{-1}, \quad v_x = \frac{v + x\varphi}{x - \varphi v} \cdot$$

Its solution has the form

$$v = \sqrt{v^2 + x^2}\,\sin(C\ln t + \lambda(z)), \quad \lambda'(z) = \pm\frac{\varphi(z)}{2z} \cdot$$

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Paper 5

Nonlocal symmetries

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Preface

We have suggested in [1] a method for computing special types of nonlocal symmetries (which we call quasi-local symmetries) for differential equations having Bäcklund transformations. The quasi-local symmetries are connected with local (Lie or Lie-Bäcklund) symmetries, and hence the Lie equations corresponding to quasi-local symmetries can be integrated. In the present paper we discuss key points of our approach and apply the method to nonlinear diffusion models and one-dimensional gasdynamic equations.

The first chapter contains the key ideas and methods for computing quasi-local symmetries. Nonlocal variables arising in this approach are used for constructing more general nonlocal symmetries.

The second and third chapters may be useful for those who want to use nonlocal symmetries in their own problems. In these chapters, we provide detailed calculations for obtaining local and nonlocal symmetries as well as group invariant solutions for nonlinear diffusion equations and the one-dimensional gasdynamic equations. Introducing nonlocal variables, we find, e.g. the gasdynamic equations that are invariant with respect to passage to uniformly accelerated coordinate systems (see Example 5.6 in § 16). Furthermore, this approach allows us to disclose a hidden symmetry of the so-called *Chaplygin gas* corresponding to the state equation $p(\rho) = p_0 - \frac{1}{\rho} a^2(S)$ (see Example 5.8 in § 16).

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* *Author's note to this 2006 edition:* I give here the translation of the original Russian manuscript a modified version of which has been published in [7].

Chapter 1
Preliminaries

§ 1 Introduction

Sophus Lie gave an efficient method for constructing groups of point and contact transformations admitted by differential equations. However, it is significant for applications to investigate more general symmetries (see, e.g. [84], p. 223, and [105], §9). The theory of Lie-Bäcklund transformation groups (see [56], [58]) provides a method for computing rather general types of non-point and non-contact symmetries defined by Lie-Bäcklund operators

$$X = \xi^i(x, u, u_{(1)}, u_{(2)}, \ldots)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u, u_{(1)}, u_{(2)}, \ldots)\frac{\partial}{\partial u^\alpha} \tag{1.1}$$

with coordinates $\xi^i, \eta^\alpha \in \mathcal{A}$. Here $\mathcal{A}$ denotes the space of *differential functions*, i.e. analytic functions of independent variables $x = (x^1, \ldots, x^n)$, dependent variables $x = (u^1, \ldots, u^m)$ and an arbitrary *finite* number of partial derivatives $u_{(1)} = \{u_i^\alpha\}, u_{(2)} = \{u_{ij}^\alpha\}, \ldots$, etc. Symmetries provided by point, contact and Lie-Bäcklund operators are called *local symmetries.* Most of known symmetries belong to this category. However, in practice differential equations are encountered that admit operators of the form (1.1) such that $\xi^i, \eta^\alpha \notin \mathcal{A}$. These kind of generalized symmetries are termed *nonlocal symmetries.* They have coordinates depending either on an infinite number of derivatives of u or on expressions involving integrals of u. Discussions of nonlocal symmetries and conservation laws can be found, e.g. in [24], [122], [123], [124], [127], [126], [58], [69], [73], [75], [79], [80], [40], [41], [42], [74], [8], [9], [21], [27], [29], [49], [57], [84], [76].

Let us consider the case of two independent variables, t and x, one dependent variable u, denote the derivatives by u_t, $u_1 = u_x, u_2 = u_{xx}, \ldots$ and use the *canonical representation* (see further Equation (2.8)) of the operator (1.1):

$$X = f(x, u, u_1, u_2, \ldots)\frac{\partial}{\partial u} + \cdots . \tag{1.2}$$

If the operator (1.2) is admitted by a differential equation, the term *symmetry* applies both to the operator X and its coordinate f.

In a number of cases nonlocal symmetries may be easily obtained by using recursion operators as is shown in the following example.

Example 5.1. The Korteweg-de Vries equation $u_t = u_3 + uu_1$ has the recursion operator

$$L = D^2 + \frac{2}{3}u + \frac{1}{3}u_1 D^{-1}.$$

Its action on the coordinate $f^{(1)} = 1 + tu_1$ of the canonical operator (1.2) of the Galilean group provides a new symmetry (1.2) with the coordinate

$$f^{(3)} = Lf^{(1)} = t(u_3 + uu_1) + \frac{1}{3}xu_1 + \frac{2}{3}u.$$

Here $f^{(1)}, f^{(3)} \in \mathcal{A}$, but further action of the recursion operator L leads to the nonlocal symmetry [69], [74]

$$f^{(5)} = t\Big(u_5 + \frac{5}{3}uu_3 + \frac{10}{3}u_1u_2 + \frac{5}{6}u^2u_1\Big) + \frac{1}{3}x(u_3 + uu_1) + \frac{4}{3}u_2 + \frac{4}{9}u^2 + \frac{1}{9}u^2 + \frac{1}{9}u_1\varphi,$$

where φ is a nonlocal variable defined by the over-determined system of differential equations

$$\varphi_x = u, \quad \varphi_t = u_2 + \frac{u^2}{2}\cdot$$

The integrability condition $\varphi_{xt} = \varphi_{tx}$ of this system coincides with the Korteweg-de Vries equation.

The symmetry $f^{(5)}$ can also be found by direct calculation if one permits dependence of f on a nonlocal variable $u_{-1} = D^{-1}(u)$, i.e. $D(u_{-1}) = u$. Developing this observation one could introduce instead of $\mathcal{A}$ the space $\overline{\mathcal{A}}$ of analytic functions of finite numbers of local $(x, u, u_1 ...)$ and "natural" nonlocal $(u_{-1}, u_{-2} ...)$ variables and seek operators with coordinates from $\overline{\mathcal{A}}$. However, the action of L on $f^{(5)}$ provides a nonlocal symmetry

$$f^{(7)} = t\big(u_7 + \frac{7}{3}uu_5 + 7u_1u_4 + \frac{35}{3}u_2u_3 + \frac{35}{18}u^2u_3 + \frac{70}{9}uu_1u_2 + \frac{35}{18}u_1^3 + \frac{35}{54}u^3u_1\big) + \frac{1}{3}x\big(u_5 + \frac{5}{3}uu_3 + \frac{10}{3}u_1u_2 + \frac{5}{6}u^2u_1\big) + 2u_4 + \frac{8}{3}uu_2 + \frac{1}{9}\big(u_3 + uu_1\big)\varphi + \frac{1}{18}u_1\psi$$

with a new nonlocal variable ψ determined by the following equations:

$$\psi_x = u^2, \quad \psi_t = 2uu_2 - u_1^2 + \frac{2}{3}u^3.$$

The nonlocal variable ψ does not belong to $\overline{\mathcal{A}}$ since it involves infinite number of "natural" nonlocal variables, namely it is given by an infinite sum:

$$\psi = D^{-1}(u^2) = uu_{-1} - u_1u_{-2} + u_2u_{-3} - \cdots . \tag{1.3}$$

Therefore, the use of the "natural" variables $u, u_1, u_{-1}, \ldots$ would be unsuccessful. Eq. (1.3) follows from the equations $u = D(u_{-1}),\ u_{-1} = D(u_{-2}), \ldots$ and from the rule of integration by parts written in the form

$$D^{-1}(uDv) = uv - D^{-1}(vDu).$$

Indeed, we have $D^{-1}(u^2) = D^{-1}\big(uD(u_{-1})\big) = uu_{-1} - D^{-1}(u_{-1}Du)$ or

$$D^{-1}(u^2) = uu_{-1} - D^{-1}(u_1u_{-1}).$$

Integrating by parts again, we have

$$D^{-1}(u_1u_{-1}) = D^{-1}\big(u_1D(u_{-2})\big) = u_1u_{-2} - D^{-1}(u_2u_{-2}),$$

and hence:

$$D^{-1}(u^2) = uu_{-1} - u_1u_{-2} + D^{-1}(u_2u_{-2}).$$

The iteration of the procedure leads to Eq. (1.3).

If one acts by the recursion operator L on $f^{(7)}$ one obtains the symmetry $f^{(9)} = Lf^{(7)}$ depending on φ, ψ and on a new nonlocal variable θ defined by

$$\theta_x = \frac{u^3}{3} - u_1^2, \quad \theta_t = u_2^2 - 2u_1u_2 + u^2u_2 - 2uu_1^2 + \frac{u^4}{4}\,\cdot$$

A formal theory of nonlocal symmetries can be developed by introducing functions of an infinite number of variables $u, u_1, u_{-1}, \ldots$ (see, e.g. [73]). However in this way one loses the possibility of a constructive computation of nonlocal symmetries.

A basic problem for computing nonlocal symmetries is the proper choice of nonlocal variables. They are defined by integrable systems of differential equations which relate the nonlocal variables to the original differential variable u. The choice of these differential equations requires additional considerations. In Example 5.1 such an additional consideration was the recurrent determination of symmetries.

In the present paper we develop the approach suggested in [58] and based on the observation (cf. below Examples 5.2 and 5.3) that the existence of Bäcklund transformations can lead to extension of the admissible group due to nonlocal symmetries. The method of obtaining non-point symmetries with the help of a nonlocal transformation (the Fourier transform) was earlier used by Fock [39] (cf. also [107]) for discovering additional symmetries of the Schrödinger equation for the hydrogen atom.

The nonlocal variables obtained in what follows are connected with local or nonlocal conservation laws. In Example 5.1 they are connected with the familiar series of conservation laws for the Korteweg-de Vries equation. In particular, the compatibility conditions of the system of equations for φ, ψ, and θ give, respectively, the first three conservation laws of this series:

$$D_t(u) + D_x\left(-u_2 - \frac{u^2}{2}\right) = 0,$$

$$D_t(u^2) + D_x\left(u_1^2 - 2uu_2 - \frac{2}{3}u^3\right) = 0,$$

$$D_t\left(\frac{u^3}{3} - u_1^2\right) + D_x\left(2u_1u_3 - u_2^2 - u^2u_2 + 2uu_1^2 - \frac{u^4}{4}\right) = 0.$$

Example 5.2. The wave equation $u_{tt} = u_{xx}$ is invariant with respect to the Bäcklund transformation ([58], Section 17.1) $v_t = u_x, v_x = u_t$ and therefore has the nonlocal symmetry $f = \eta(u + v)$ with an arbitrary function η.

Example 5.3. The linear heat equation $u_t - u_2 = 0$, in view of its invariance with respect to differentiation and integration, has the nonlocal symmetries $f_1 = v$ (where $v_x = u, v_t = u_x$) and $f_2 = 2tu + w$ (where $w_x = xu, w_t = xu_x - u$).

Example 5.4. The Burgers equation $u_t - uu_1 - u_2 = 0$ is connected with the heat equation $v_t - v_2 = 0$ by the transformation ([38], [49], [29]) $u = -2\frac{v_1}{v}$ and therefore has nonlocal symmetries, e.g. $f = (hu + 2h_x)\mathrm{e}^{\varphi/2}$, where φ is a nonlocal variable defined by the system $\varphi_x = u, \varphi_t = u_1 - \frac{1}{2}u^2$, and $h(t, x)$ is an arbitrary solution of the heat equation, $h_t - h_{xx} = 0$.

We will consider only equations having particular Bäcklund transformations, namely differential substitutions (or coverings in terms of [123], [124]). We will use these transformations for constructing quasi-local symmetries. The quasi-local symmetries augment the set of group-invariant solutions. In particular, using quasi-local symmetries one can find partially invariant solutions for scalar equations. Note that partially invariant solutions provided by local symmetries exist only for systems of equations [103].

§ 2 Lie-Bäcklund operators

We use the following notation of [58], [57]: $x = (x^1, ..., x^n)$ are independent variables and $u = (u^1, \ldots, u^m)$ are the dependent variables (called also differential variables) with successive derivatives $u_{(1)} = \{u_i^\alpha\},\ u_{(2)} = \{u_{ij}^\alpha\}, \ldots,$ where $u_i^\alpha = D_i(u^\alpha),\ u_{ij}^\alpha = D_j(u_i^\alpha), \ldots$. Here

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha u_i^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots$$

are the *total differentiations* with respect to x^i.

Locally analytic functions of a finite number of variables $x, u, u_{(1)}, \ldots$ are termed *differential functions* and the space of all differential functions is denoted by $\mathcal{A}[x, u]$.

Any function $F \in \mathcal{A}[x, u]$ determines a *differential manifold* $[F]$ defined by the infinite system of equations

$$[F]: \quad F = 0, \quad D_i F = 0, \quad D_i D_j F = 0, \ldots . \tag{2.1}$$

It is convenient to associate with any differential function $F \in \mathcal{A}[x, u]$ the following differential operators:

$$(F_*)_\alpha = \frac{\partial F}{\partial u^\alpha} + \frac{\partial F}{\partial u^\alpha_j} D_i + \frac{\partial F}{\partial u^\alpha_{ij}} D_i D_j + \ldots \tag{2.2}$$

Recall that continuous one-parameter groups of point transformations are determined by their generators (known also as infinitesimal operators)

$$X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha} \cdot \tag{2.3}$$

Likewise, Lie-Bäcklund transformation groups are determined by their generators of the form (see [56])

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta^\alpha_i \frac{\partial}{\partial u^\alpha_i} + \zeta^\alpha_{ij} \frac{\partial}{\partial u^\alpha_{ij}} + \ldots, \tag{2.4}$$

where $\xi^i, \eta^\alpha \in \mathcal{A}[x, u]$ and $\zeta^\alpha_i, \zeta^\alpha_{ij}, \ldots$ are given by the *prolongation formulae:*

$$\begin{aligned} \zeta^\alpha_i &= D_i(\eta^\alpha) - u^\alpha_k D_i(\xi^k), \\ \zeta^\alpha_{ij} &= D_i(\zeta^\alpha_j) - u^\alpha_{kj} D_i(\xi^k), \ldots . \end{aligned} \tag{2.5}$$

Operators of the form (2.4) are called Lie-Bäcklund operators. Bearing Eqs. (2.5) in mind, Lie-Bäcklund operators can be written for short as follows:

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \cdots . \tag{2.6}$$

The action of the operator (2.6) on any function from $\mathcal{A}[x, u]$ is well-defined. Hence the Lie bracket $[X, Y] = XY - YX$ of arbitrary operators (2.4) is an operator of the same form. Therefore one can consider the Lie algebra L of Lie-Bäcklund operators. It is called the Lie-Bäcklund algebra.

A differential equation $F = 0$ admits a Lie-Bäcklund operator (2.4) if the group generated by this operator leaves invariant the differential manifold

$[F]$ defined by Eqs. (2.1). A necessary and sufficient condition for the invariance is written in the form of the following *determining equation:*

$$XF|_{[F]} = 0. \tag{2.7}$$

Any operator $X_* = \xi^i D_i$ with arbitrary $\xi^i \in \mathcal{A}[x, u]$ is a Lie-Bäcklund operator. Moreover, $X_* = \xi^i D_i$ is admitted by any differential equation and the set L_* of all operators $X_* = \xi^i D_i$ is an ideal in the Lie-Bäcklund algebra L. Hence, instead of the algebra L we can deal with the quotient-algebra L/L_* by considering operators X and Y to be equivalent if $X - Y \in L_*$. In particular, any operator (2.6) is equivalent to the operator

$$Y = X - \xi^i D_i = (\eta^\alpha - \xi^i u_i^\alpha)\frac{\partial}{\partial u^\alpha} + \cdots .$$

Operators of this form, i.e.

$$X = f^\alpha \frac{\partial}{\partial u^\alpha} + \cdots \tag{2.8}$$

with any $f \in \mathcal{A}[x, u]$ are called canonical Lie-Bäcklund operators. For them the prolongation formulae (2.5) have the simple form

$$\zeta_i^\alpha = D_i(\eta^\alpha), \quad \zeta_{ij}^\alpha = D_i(\zeta_j^\alpha), \ldots . \tag{2.9}$$

Thus, the theory of Lie–Bäcklund operators lets us to restrict our consideration to the canonical operators. For these operators the independent variables x^i are invariants. It is worth noting that a canonical Lie-Bäcklund operator (2.8) is equivalent to an operator of the group of point transformations of the form (2.3) if and only if f^α can be represented in the form

$$f^\alpha = \eta^\alpha(x, u) - \xi^i(x, u)u_i^\alpha. \tag{2.10}$$

§ 3 Transition formula

In what follows we will need the formula for transforming coordinates of Lie-Bäcklund operators under changes of variables $(x, u) \to (y, v)$ given by differential substitutions:

$$y^j = \varphi^j, \quad v^\beta = \Phi^\beta, \tag{3.1}$$

where $\varphi^j, \Phi^\beta \in \mathcal{A}[x, u]$. The change of variables (3.1) maps a canonical Lie-Bäcklund operator (2.8), $X = f_u^\alpha \frac{\partial}{\partial u^\alpha} + \cdots$ into the operator

$$\widetilde{X} = X(\varphi^j)\frac{\partial}{\partial y^j} + X(\Phi^\beta)\frac{\partial}{\partial v^\beta} + \cdots$$

which is equivalent to the canonical operator $Y = f_v^\beta \frac{\partial}{\partial v^\beta} + ...$ with

$$f_v^\beta = X(\Phi^\beta) - v_j^\beta X(\varphi^j).$$

Using the notation (2.2), we arrive at the following formulae for transformation of coordinates of canonical Lie-Bäcklund operators:

$$f_v^\beta = [(\Phi_*^\beta)_\alpha - v_j^\beta (\varphi_*^j)_\alpha] f_u^\alpha. \tag{3.2}$$

The formulae (3.2) can be used both for passing from f_u to f_v and from f_v to f_u. In this procedure, one should the change of variables (3.1) together with the transformations of the total differentiation operators:

$$D_{x^i} = D_{x^i}(\varphi^j) D_{y^j}. \tag{3.3}$$

§ 4 Quasi-local symmetries

Let a differential equation $F = 0$, where $F \in \mathcal{A}[x, u]$, be connected with an equation $G = 0$, $G \in \mathcal{A}[y, v]$, by a Bäcklund transformation of the form (3.1). Let the equation $F = 0$ admit a canonical Lie-Bäcklund operator with a coordinate $f_u \in \mathcal{A}[x, u]$. If the the Bäcklund transformation connecting the equations $F = 0$ and $G = 0$ maps f_u into a differential function $f_v \in \mathcal{A}[y, v]$, then the equation $G = 0$ admits the canonical Lie-Bäcklund operator with the coordinate f_v, and the differential functions f_u and f_v are related by the *transition formulae* (3.2). Extension of (3.2) to the case $f_v \notin \mathcal{A}[u, v]$ leads to the necessity of introducing new, *nonlocal*, variables Q^j. They are defined from (3.2) and from the invariance condition of the differential manifold $[G]$ with respect to the operator with the coordinate $f_v(y, v, v_1, \ldots, Q)$. The function f_v calculated in this way is called in [1], [5] the *quasi-local symmetry** *associated with the local symmetry* f_u.

In what follows we will consider systems of evolution equations

$$u_t = F, \quad F \in \mathcal{A}[x, u], \tag{4.1}$$

$$v_s = G, \quad G \in \mathcal{A}[y, v], \tag{4.2}$$

connected by a Bäcklund transformation of the form

$$s = t, \quad y = \varphi(x, u, u_1, \ldots, u_k), \quad v^\alpha = \Phi^\alpha(x, u, u_1, \ldots, u_k). \tag{4.3}$$

**Author's note to this 2006 edition:* The closely related notion of a *potential symmetry* for PDEs written in a conserved form is introduced in [22].

Here t and s are time, x and y are scalar independent variables, u and v are vector-valued differential variables with the successive derivatives $u_1, u_2, \ldots$ and $v_1, v_2, \ldots$, respectively, where

$$u_0 = u, \quad u_{i+1} = D_x(u_i); \quad v_0 = v, \quad v_{i+1} = D_y(v_i).$$

Let the system (4.1) admit an algebra of local symmetries, i.e. a Lie-Bäcklund algebra

$$\mathcal{A}_F[x,u] = \left\{ f_u \in \mathcal{A}[x,u] : \frac{\partial f_u}{\partial t} - \{F, f_u\} = 0 \right\}.$$

Here $\frac{\partial f_u}{\partial t} - \{F, f_u\} = 0$ is the determining equation (2.7) written for the evolution equation $u_t = F$, and $\{F, f\}$ is the Lie bracket defined as the m-dimensional vector from $\mathcal{A}[x,u]$ with the components

$$\{F, f\}^\alpha = F^\alpha_* \cdot f - f^\alpha_* \cdot F,$$

where

$$F^\alpha_* \cdot f = \sum_{\beta=1}^{m} (F^\alpha_*)_\beta f^\beta, \quad (F^\alpha_*)_\beta = \sum_{i \geq 0} \frac{\partial F^\alpha}{\partial u^\beta_l} D^i_x, \quad \beta = 1, \ldots, m.$$

If the transformation (4.3) maps $f_u \in \mathcal{A}_F[x,u]$ into $f_v \in \mathcal{A}_G[y,v]$, then f_u and f_v are connected by the transition formulae (3.2) which in the present case can be rewritten, using (3.3), in the form

$$D_x(\varphi) f^\alpha_v = [D_x(\varphi)\Phi^\alpha_* - D_x(\Phi^\alpha)\varphi_*] f_u, \tag{4.4}$$

$$D_t = D_x + D_t(\varphi) D_y, \quad D_x = D_x(\varphi) D_y. \tag{4.5}$$

The extension of (4.4) to arbitrary f_v (in general $f_v \notin \mathcal{A}[y,v]$) leads to quasi-local symmetries calculated in accordance with the following definition.

Definition 5.1. A quasi-local symmetry of Eqs. (4.2) associated with a local symmetry $f_u \in \mathcal{A}_F[x,u]$ is a vector-function f_v which is connected with f_u by the transition formulae (4.4) and depends, along with $s, y, u, v_1, \ldots$, also on new variables $Q^1, ..., Q^l$ satisfying the following equation:

$$\frac{\partial f_v}{\partial s} - \{G, f_v\} + \sum_{i=1}^{l} \frac{\partial f_v}{\partial Q^i} Q^i_s = 0.$$

In calculating the Lie bracket $\{G, f_v\}$ one uses the total differentiation extended to nonlocal variables Q^i as to ordinary differential variables.

Inverting the sequence of arguments in this definition one can get quasi-local symmetries f_u for Eq. (4.1) associated with local symmetries f_v for Eq. 4.2). Furthermore, the transition formulae (4.4) are applicable also to the case when both symmetries f_u and f_v are quasi-local.

Chapter 2
Diffusion equations

This chapter is devoted to the classification of the sequence of equations

$$w_t = H(w_2) \quad \xrightarrow{v=w_1} \quad v_t = h(v_1)v_2 \quad \xrightarrow{u=v_1} \quad u_t = (h(u)u_1)_1 \tag{H}$$

related to nonlinear heat equations and connected with one another by the simplest Bäcklund transformations (introducing potentials). The classification is made with respect to quasi-local symmetries associated with point symmetries. The functions H and h, where $H(\xi) = \int h(\xi)d\xi$, are arbitrary elements with respect to which one performs the group classification. The first equation of (H) describes the inertia-free motion of a string in a medium with nonlinear resistance. The second is used in mechanics for the study of the shear currents of a nonlinear viscoplastic medium and processes of filtration of a non-Newtonian fluid [13], [18], and also in the physics of the ocean for describing the propagation of oscillations of temperature and salinity to depths [122]. The third equation of this sequence is the classical equation of nonlinear thermal conductivity. The Bäcklund transformations (in the present case differentiation and integration) let one transfer a symmetry of one term of the sequence with a certain function h to a symmetry of any other term of this sequence with the same function h. In this way, we naturally arrive at quasi-local symmetries [4], [5]. We note that the nonlocal symmetries of the sequence (H) were found independently and simultaneously with us in [75]. It is also shown in [75] that the nonlinear heat equation does not have other transformations not depending explicitly on t, x and leading to nonlocal symmetries, besides the Bäcklund transformations indicated in (H).

A possibility of increasing the number of exact solutions of equations of the sequence (H) at the expense of invariant solutions of other equations of this sequence is indicated in [100].

§ 5 Equivalence transformations

At the first stage of the group classification it is necessary to calculate the equivalence transformations for each term of the sequence (H). Let us begin with the nonlinear filtration equation

$$v_t = h(v_1)v_2 \tag{5.1}$$

and find all pointwise equivalence transformations, i.e. non-degenerate changes of variables t, x, v, carrying any equation of the form (5.1) into an equation of the same form, generally with a different filtration coefficient $h(v_1)$. Equations which are related by equivalence transformations admit similar groups and are considered indistinguishable in the group classification [106].

Direct search for the set $\mathcal{E}$ of all equivalence transformations is connected with considerable computational difficulties. On the other hand, since the set $\mathcal{E}$ forms a group we can apply the infinitesimal method and find the continuous subgroup $\mathcal{E}_c \subset \mathcal{E}$ relatively easily. Then we will complete $\mathcal{E}_c$ to the general equivalence group $\mathcal{E}$.

5.1 Filtration equation: continuous equivalence group

We shall seek the operator

$$E = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial v} + \mu \frac{\partial}{\partial h} \tag{5.2}$$

of the group $\mathcal{E}_c$ from the condition of invariance of (5.1) written as a system

$$v_t = hv_2, \quad h_t = 0, \quad h_x = 0, \quad h_v = 0, \quad h_{v_t} = 0. \tag{5.3}$$

Here v and h are considered as differential functions: v in the space (t, x) and h in the extended space (t, x, v, v_t, v_1). The coordinates ξ^1, ξ^2, η of the operator E are sought as functions of the variables t, x, v, and the coordinate μ as a function of t, x, v, v_t, v_1, h. If along with the usual total differentiations

$$\begin{aligned} D_1 \equiv D_t &= \frac{\partial}{\partial t} + v_t \frac{\partial}{\partial v} + v_{tt} \frac{\partial}{\partial v_t} + v_{1t} \frac{\partial}{\partial v_1}\,, \\ D_2 \equiv D_x &= \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial v} + v_{1t} \frac{\partial}{\partial v_t} + v_2 \frac{\partial}{\partial v_1}\,, \end{aligned} \tag{5.4}$$

we introduce the following differentiations in the extended space:

$$\widetilde{D}_1 \equiv \widetilde{D}_t = \frac{\partial}{\partial t} + h_t \frac{\partial}{\partial h} + h_{tt} \frac{\partial}{\partial h_t} + h_{tx} \frac{\partial}{\partial h_x} + h_{tv} \frac{\partial}{\partial h_v} + h_{tv_t} \frac{\partial}{\partial h_{v_t}}\,, \tag{5.5}$$

$$\widetilde{D}_2 \equiv \widetilde{D}_x = \frac{\partial}{\partial x} + h_x \frac{\partial}{\partial h} + h_{xt} \frac{\partial}{\partial h_t} + h_{xx} \frac{\partial}{\partial h_x} + h_{xv} \frac{\partial}{\partial h_v} + h_{xv_t} \frac{\partial}{\partial h_{v_t}}\,,$$

$$\widetilde{D}_3 \equiv \widetilde{D}_v = \frac{\partial}{\partial v} + h_v \frac{\partial}{\partial h} + h_{vt} \frac{\partial}{\partial h_t} + h_{vx} \frac{\partial}{\partial h_x} + h_{vv} \frac{\partial}{\partial h_v} + h_{vv_t} \frac{\partial}{\partial h_{v_t}}\,,$$

$$\widetilde{D}_4 \equiv \widetilde{D}_{v_t} = \frac{\partial}{\partial v_t} + h_{v_t} \frac{\partial}{\partial h} + h_{tv_t} \frac{\partial}{\partial h_t} + h_{xv_t} \frac{\partial}{\partial h_x} + h_{vv_t} \frac{\partial}{\partial h_v} + h_{v_tv_t} \frac{\partial}{\partial h_{v_t}}\,,$$

then the the prolongation of the operator (5.2) is written

$$\widetilde{E} = E + \zeta_1 \frac{\partial}{\partial v_t} + \zeta_2 \frac{\partial}{\partial v_1} + \zeta_{22} \frac{\partial}{\partial v_2} + \mu_1 \frac{\partial}{\partial h_t} + \mu_2 \frac{\partial}{\partial h_x} + \mu_3 \frac{\partial}{\partial h_v} + \mu_4 \frac{\partial}{\partial h_{v_t}}\,,$$

where

$$\begin{gathered}\zeta_i = D_i(\eta) - v_t D_i(\xi^1) - v_1 D_i(\xi^2),\\ \zeta_{22} = D_2(\zeta_2) - v_{1t} D_2(\xi^1) - v_2 D_2(\xi^2),\\ \mu_i = \widetilde{D}_i(\mu) - h_t \widetilde{D}_i(\xi^1) - h_x \widetilde{D}_i(\xi^2) - h_v \widetilde{D}_i(\eta) - h_{v_t} \widetilde{D}_i(\zeta_1) - h_{v_1} \widetilde{D}_i(\zeta_2).\end{gathered} \tag{5.6}$$

The infinitesimal test for invariance of the system (5.3) has the form

$$(\zeta_1 - h\zeta_{22} - \mu v_2)\Big|_{(5.3)} = 0, \tag{5.7}$$

$$\mu_i\Big|_{(5.3)} = 0, \quad i = 1, \ldots, 4. \tag{5.8}$$

Invoking that the functions ξ, η, ζ do not depend on h, we write Eqs. (5.8) in the form

$$\begin{gathered}\frac{\partial \mu}{\partial t} - h_{v_1} \frac{\partial \zeta_2}{\partial t} = 0, \quad \frac{\partial \mu}{\partial x} - h_{v_1} \frac{\partial \zeta_2}{\partial x} = 0,\\ \frac{\partial \mu}{\partial v} - h_{v_1} \frac{\partial \zeta_2}{\partial v} = 0, \quad \frac{\partial \mu}{\partial v_t} - h_{v_1} \frac{\partial \zeta_2}{\partial v_t} = 0.\end{gathered}$$

Since here h is a differential variable (so that h and h_{v_1} are algebraically independent), the above equations give

$$\mu = \mu(v_1, h), \quad \frac{\partial \zeta_2}{\partial t} = 0, \quad \frac{\partial \zeta_2}{\partial x} = 0, \quad \frac{\partial \zeta_2}{\partial v} = 0, \quad \frac{\partial \zeta_2}{\partial v_t} = 0. \tag{5.9}$$

Since $\zeta_2 = \eta_x + v_1\eta_v - v_t\xi^1_x - v_t v_1 \xi^1_v - v_1\xi^2_x - v_1^2\xi^2_v$, Eqs. (5.9) yield

$$\xi^1 = \xi^1(t), \quad \xi^2 = A_1(t)x + C_1 v + A_2(t), \quad \eta = A_3(t)v + C_2 x + A_4(t),$$

where $C_i =$ const. Substituting the expressions for ξ^i, η, μ in Eq. (5.7), we obtain the general solution of the determining equations (5.7), (5.8):

$$\xi^1 = C_1 t + C_2, \quad \xi^2 = C_3 x + C_4 v + C_5,$$

$$\eta = C_6 x + C_7 v + C_8, \quad \mu = (2C_4 v_1 + 2C_3 - C_1)h.$$

Substitution in (5.2) yields the eight-dimensional Lie algebra spanned by

$$E_1 = \frac{\partial}{\partial t}\,, \quad E_2 = \frac{\partial}{\partial x}\,, \quad E_3 = \frac{\partial}{\partial v}\,, \quad E_4 = t\frac{\partial}{\partial t} - h\frac{\partial}{\partial h}\,,$$

$$E_5 = x\frac{\partial}{\partial x} + 2h\frac{\partial}{\partial h}\,, \quad E_6 = v\frac{\partial}{\partial x} + 2v_1 h\frac{\partial}{\partial h}\,, \quad E_7 = x\frac{\partial}{\partial v}\,, \quad E_8 = v\frac{\partial}{\partial v}\,\cdot$$

Taking the prolongation of these operators to v_1 and solving the Lie equations, we obtain the 8-parameter group $\mathcal{E}_c$ of equivalence transformations

$$\begin{gathered} \tilde{t} = \alpha t + \gamma_1, \quad \tilde{x} = \beta_1 x + \beta_2 v + \gamma_2, \quad \tilde{v} = \beta_3 x + \beta_4 v + \gamma_3, \\ \tilde{v}_1 = \frac{\beta_3 + \beta_4 v_1}{\beta_1 + \beta_2 v_1}, \quad \tilde{h} = \frac{(\beta_1 + \beta_2 v_1)^2}{\alpha}\, h \end{gathered} \tag{5.10}$$

with the coefficients

$$\alpha = a_4, \quad \beta_1 = a_5, \quad \beta_2 = a_6, \quad \beta_3 = a_5 a_7 a_8, \quad \beta_4 = (1 + a_6 a_7) a_8,$$

$$\gamma_1 = a_1 a_4, \quad \gamma_2 = a_2 a_5 + a_3 a_6, \quad \gamma_3 = (a_3 + a_2 a_5 a_7 + a_3 a_6 a_7) a_8.$$

Here a_i is the parameter of the subgroup with operator E_i. Therefore a_4, a_5, a_8 are positive, and hence in (5.10) we have $\alpha > 0,\ \beta_i > 0,\ \alpha(\beta_1\beta_4 - \beta_2\beta_3) > 0$.

5.2 Filtration equation: complete equivalence group

Note that the reflections $t \to -t$ and $x \to -x$ are also equivalence transformations, i.e. they are contained in the group $\mathcal{E}$. Adding them to the continuous group $\mathcal{E}_c$ found in the previous section, we get transformations (5.10) with arbitrary coefficients $\alpha, \beta_i, \gamma_i$ satisfying merely the non-degeneracy condition $\alpha(\beta_1\beta_4 - \beta_2\beta_3) \neq 0$. We will prove now that in this way we obtain the most general equivalence group.

Theorem 5.1. The complete equivalence group $\mathcal{E}$ for the filtration equations (5.1) is provided by the transformations (5.10),

$$\begin{gathered} \tilde{t} = \alpha t + \gamma_1, \quad \tilde{x} = \beta_1 x + \beta_2 v + \gamma_2, \\ \tilde{v} = \beta_3 x + \beta_4 v + \gamma_3, \quad \tilde{h} = \frac{(\beta_1 + \beta_2 v_1)^2}{\alpha}\, h, \end{gathered}$$

with arbitrary coefficients $\alpha, \beta_i, \gamma_i$ satisfying the non-degeneracy condition

$$\alpha(\beta_1\beta_4 - \beta_2\beta_3) \neq 0.$$

Proof. Let the transformation

$$y = \varphi(t, x, v), \quad s = \psi(t, x, v), \quad \tilde{v} = \Phi(t, x, v)$$

be an equivalence transformation for Eq. (5.1). In other words, we assume that it maps the equation $v_t = h(v_x) v_{xx}$ into the equation

$$\tilde{v}_s = \tilde{h}(\tilde{v}_y)\tilde{v}_{yy}. \tag{5.11}$$

Using the transformation formulae for the differentiation operators D_t, D_x :

$$D_t = D_t(\varphi)D_y + D_t(\psi)D_s, \quad D_x = D_x(\varphi)D_y + D_x(\psi)D_s,$$

we obtain the following expression for $\tilde{v}_1 = \tilde{v}_y$:

$$\tilde{v}_1 = \frac{D_x(\Phi)D_t(\psi) - D_t(\Phi)D_x(\psi)}{D_x(\varphi)D_t(\psi) - D_t(\varphi)D_x(\psi)}\,.$$

It follows from the definition of equivalence transformations that the right-hand side of the above equation should depend only on v_1 so its derivatives with respect to t, x, v, v_t are equal to zero. This condition leads to a system of equations on the functions φ, ψ, Φ whose solution has the form

$$\varphi = A_1(t)(\beta_1 x + \beta_2 v) + A_2(t), \quad \psi = \psi(t),$$

$$\Phi = A_1(t)(\beta_3 x + \beta_4 v) + A_3(t),$$

where $\beta_1\beta_4 - \beta_2\beta_3 \neq 0, \psi'(t) \neq 0, A_1(t) \neq 0.$

The specification of the functions $A_i(t)$ and $\psi(t)$ is effected by substituting the expressions for φ, ψ, Φ in Eq. (5.11) and leads to Eqs. (5.10).

5.3 Equivalence group for the equation $w_t = H(w_2)$

the equivalence transformations for the equation

$$w_t = H(w_2) \tag{5.12}$$

are calculated as in Section 5.1. Namely, we rewrite Eq. (5.12) in the form of the system

$$\begin{gathered} w_t = H, \quad H_t = 0, \quad H_x = 0, \quad H_w = 0, \\ H_{w_t} = 0, \quad H_{w_1} = 0, \quad H_{w_{tt}} = 0, \quad H_{w_{1t}} = 0, \end{gathered} \tag{5.13}$$

where w depends on t, x and H depends on $t, x, w, w_t, w_1, w_{tt}, w_{1t}, w_2$. In other words, we consider w as a differential variable in the space (t, x) and H as a differential variable in the extended space $(t, x, w, w_t, w_1, w_{tt}, w_{1t}, w_2)$. The generators of the continuous group of equivalence transformations are sought as the operators

$$E = \xi^1\frac{\partial}{\partial t} + \xi^2\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial w} + \mu\frac{\partial}{\partial H} \tag{5.14}$$

admitted by the system (5.13), where ξ^1, ξ^2, η depend on t, x, w and μ depends on $t, \ldots, w_2, H$. Acting on Eqs. (5.13) by the prolonged operator

$$\widetilde{E} = E + \zeta_1\frac{\partial}{\partial w_t} + \zeta_2\frac{\partial}{\partial w - 1} + \mu_1\frac{\partial}{\partial H_t} + \mu_2\frac{\partial}{\partial H_r} +$$

$$+\mu_3\frac{\partial}{\partial H_w}+\mu_4\frac{\partial}{\partial H_{w_t}}+\mu_5\frac{\partial}{\partial H_{w_1}}+\mu_6\frac{\partial}{\partial H_{w_{tt}}}+\mu_7\frac{\partial}{\partial H_{w_{1t}}}$$

we obtain the determining equations

$$(\zeta_1-\mu)\big|_{(5.13)}=0 \tag{5.15}$$

and

$$\mu_i\big|_{(5.13)}=0,\quad i=1,\ldots,7. \tag{5.16}$$

Here

$$\mu_i=\widetilde{D}_i(\mu)-H_t\widetilde{D}_i(\xi^1)-H_x\widetilde{D}_i(\xi^2)-H_w\widetilde{D}_i(\eta)-H_{w_t}\widetilde{D}_i(\zeta_1)-$$
$$-H_{w_1}\widetilde{D}_i(\zeta_2)-H_{w_{tt}}\widetilde{D}_i(\zeta_{11})-H_{w_{1t}}\widetilde{D}_i(\zeta_{12})-H_{w_2}\widetilde{D}_i(\zeta_{22}),$$

where $D_i,\ \widetilde{D}_i,\ \zeta_i,\ \zeta_{jk}$ are obtained from Eqs. (5.4)-(5.6) by replacing v and h by w and H, respectively.

Rewriting Eqs. (5.16) in the form

$$\frac{\partial\mu}{\partial t}-H_{w_2}\frac{\partial\zeta_{22}}{\partial t}=0,\quad \frac{\partial\mu}{\partial x}-H_{w_2}\frac{\partial\zeta_{22}}{\partial x}=0,\quad \frac{\partial\mu}{\partial w}-H_{w_2}\frac{\partial\zeta_{22}}{\partial w}=0,$$
$$\frac{\partial\mu}{\partial w_t}-H_{w_2}\frac{\partial\zeta_{22}}{\partial w_t}=0,\quad \frac{\partial\mu}{\partial w_1}-H_{w_2}\frac{\partial\zeta_{22}}{\partial w_1}$$
$$\frac{\partial\mu}{\partial w_{tt}}-H_{w_2}\frac{\partial\zeta_{22}}{\partial w_{tt}}=0,\quad \frac{\partial\mu}{\partial w_{1t}}-H_{w_2}\frac{\partial\zeta_{22}}{\partial w_{1t}}=0$$

and splitting with respect to H_{w_2}, one obtains

$$\mu=\mu(w_2,H),\quad \xi^1=\xi^1(t),\quad \xi^2=A_1(t)x+A_2(t),$$
$$\eta=[2A_1(t)+C_1]w+C_2x^2+A_3(t)x+A_4(t).$$

Substitution of these expressions in Eq. (5.15) yields the following general solution of the determining equations (5.15), (5.16):

$$\xi^1=C_1t+C_2,\quad \xi^2=C_3x+C_4,$$
$$\eta=C_5w+C_6x^2+C_7x+C_8t+C_9,\quad \mu=C_8+H(C_5-C_1)$$

involving nine arbitrary constants $C_1,\ldots,C_9$. Substitution in (5.14) yields the nine-dimensional Lie algebra. Solving the Lie equations for the basic generators of this algebra and taking the composition of the resulting nine one-parameter groups, we obtain the nine-parameter continuous group $\mathcal{E}_c$. Finally, we add to $\mathcal{E}_c$ the reflections $t\to -t, x\to -x, w\to -w$ and arrive at the following complete equivalence transformation group $\mathcal{E}$ for Eq. (5.12):

$$\tilde{t}=\alpha t+\gamma_1,\quad \tilde{x}=\beta_1x+\gamma_2,$$
$$\tilde{w}=\delta_1w+\delta_2x^2+\delta_3x+\delta_4t+\delta_5,\quad H=\frac{\delta_1}{\alpha}H+\delta_4, \tag{5.17}$$

where $\alpha\beta_1\delta_1\neq 0$.

5.4 Equivalence group for the nonlinear heat equation

The equivalence transformations for the nonlinear heat equation

$$u_t = (h(u)u_1)_1$$

have the form [106]:

$$\begin{gathered}\tilde{t} = \alpha t + \gamma_1, \quad \tilde{x} = \beta_1 x + \gamma_2,\\ \tilde{u} = \delta_1 u + \delta_2, \quad \tilde{h} = \frac{\beta_1^2}{\alpha}\, h, \quad \alpha\beta_1\delta_1 \neq 0.\end{gathered} \tag{5.18}$$

§ 6 Classification of the equations $w_t = H(w_2)$

6.1 The determining equations

Let us classify the equations

$$w_t = H(w_2) \tag{6.1}$$

in accordance with the admitted point transformation groups. The classification is simplified by using the equivalence transformations. Eliminating from consideration the well-studied case of linear heat equation $w_t = kw_2$, we assume that $H'' \neq 0$.

An admissible operator is sought in the form

$$Z = \xi^1(t,x,w)\frac{\partial}{\partial t} + \xi^2(t,x,w)\frac{\partial}{\partial x} + \eta(t,x,w)\frac{\partial}{\partial w}$$

and is found from the determining equation

$$\zeta_1 - H'(w_2)\zeta_{22} = 0.$$

where ζ_1 and ζ_{22} are given by Eqs. (5.6) with v replaced by w. In solving the determining equation, we replace w_t by $H(w_2)$ and consider w_1, w_2 and w_{1t} as free variables. Splitting with respect to the variables w_{1t}, w_1, we have:

$$\xi^1_x = 0, \quad \xi^1_x = 0, \quad \xi^2_{ww} = 0, \quad \eta_{ww} = 2\xi^2_{xw}, \tag{6.2}$$

$$\xi^2_t + \xi^2_w\, H + (2\eta_{xw} - \xi^2_{xx} - 3w_2\xi^2_w)\, H' = 0, \tag{6.3}$$

$$\eta_t + (\eta_w - \xi^1_t)H - [\eta_{xx} + (\eta_w - 2\xi^2_x)w_2]H' = 0. \tag{6.4}$$

Eqs. (6.2) yield:

$$\xi^1 = \xi^1(t), \quad \xi^2 = \alpha(t,x)w + \beta(t,x), \quad \eta = \alpha_x(t,x)w^2 + \gamma(t,x)w + \delta(t,x).$$

Substituting these expressions in Eqs. (6.3) - (6.4) and splitting with respect to w, we obtain $\alpha_{xt} - H'\alpha_{xxx} = 0$ and

$$\alpha_t + 3\alpha_{xx}H' = 0, \tag{6.5}$$

$$\beta_t + \alpha H + (2\gamma_x - \beta_{xx} - 3\alpha w_2)H' = 0, \tag{6.6}$$

$$\gamma_t + 2\alpha_x H - \gamma_{xx}H' = 0, \tag{6.7}$$

$$\delta_t + (\gamma - \xi^1_t)H - [\delta_{xx} + (\gamma - 2\beta_x)w_2]H' = 0. \tag{6.8}$$

Since $H'' \neq 0$, Eq. (6.5) yields $\alpha_t = 0,\ \alpha_{xx} = 0$. Differentiating Eq. (6.6) with respect to w_2 and x, Eq. (6.7) with respect to w_2 and adding the results, we get

$$(\gamma_{xx} - \beta_{xxx} - 3\alpha_x w_2)H'' = 0,$$

whence $\alpha_x = 0,\ \beta_{xxx} = \gamma_{xx}$. Now Eq. (6.7) yields $\gamma_t = 0,\ \gamma_{xx} = 0$, and hence $\beta_{xxx} = 0$. Thus

$$\alpha = C_1 = \text{const.}, \quad \beta = \beta_1(t)x^2 + \beta_2(t)x + \beta_3(t), \quad \gamma = C_2 x + C_3.$$

Furthermore Eq. (6.6) reduces the expression for β to

$$\beta = C_4 x^2 + C_5 x + C_6 t + C_7.$$

Collecting the above results, we obtain the expressions

$$\xi^1 = \xi^1(t), \quad \xi^2 = C_1 w + C_4 x^2 + C_5 x + C_6 t + C_7,$$

$$\eta = C_1 w^2 + (C_2 x + C_3)w + \delta(t,x)$$

for the coordinates of the operator Z and conclude that the determining equations reduce to the following equations:

$$C_6 + C_1 H + 2(C_2 - C_4)H' - 3C_1 w_2 H' = 0, \tag{6.9}$$

$$\delta_t + (C_2 x + C_3 - \xi^1_t)H - \delta_{xx}H' - [(C_2 - 4C_4)x + C_3 - 2C_5]w_2 H' = 0. \tag{6.10}$$

In the case of an arbitrary function $H(w_2)$ all the coefficients in (6.9) and (6.10) should vanish, i.e.

$$C_6 = 0, \quad C_1 = 0, \quad C_2 - C_4 = 0, \quad \delta_t = 0,$$

$$C_2 x + C_3 - \xi^1_t = 0, \quad \delta_{xx} = 0, \quad (C_2 - 4C_4)x + C_3 - 2C_5 = 0.$$

It follows that

$$C_1 = C_2 = C_4 = C_6 = 0, \quad C_3 = 2C_5,$$

$$\xi^1 = 2C_5 t + K_1, \quad \delta = K_2 x + K_3,$$

whence finally

$$\xi^1 = 2C_5 t + K_1, \quad \xi^2 = C_5 x + C_7, \quad \eta = 2C_5 w + K_2 x + K_3.$$

Thus, Eq. (6.1) with an arbitrary $H(w_2)$ admits the five-dimensional Lie algebra spanned by

$$\begin{gathered} Z_1 = \frac{\partial}{\partial t}, \quad Z_2 = \frac{\partial}{\partial x}, \quad Z_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2w\frac{\partial}{\partial w}, \\ Z_4 = x\frac{\partial}{\partial w}, \quad Z_5 = \frac{\partial}{\partial w} \cdot \end{gathered} \tag{6.11}$$

6.2 The classifying relation

Extensions of the symmetry algebra are possible if Eqs. (6.9) and (6.10) do not imply that all coefficients in these equations vanish. To find all possible cases, note that Eq. (6.10), as well as (6.9), is equivalent to the equation

$$a + bH + (c + dw_2)H' = 0 \tag{6.12}$$

with constant coefficients a, b, c, and d. Indeed, since H depends only on w_2, Eq. (6.10) holds only when all its coefficients either vanish identically or are proportional (with constant coefficients) to a function $\lambda(t, x) \neq 0$, i.e.

$$\delta_t = a\lambda(t, x), \quad C_2 x + C_3 - \xi^1_t = b\lambda(t, x),$$

$$-\delta_{xx} = c\lambda(t, x), \quad (4C_4 - C_2)x - C_3 + 2C_5 = d\lambda(t, x).$$

If all coefficients in (6.9) and (6.10) are simultaneously equal to zero, then this corresponds to the case of arbitrary function H. Hence, extension of the group is possible only for the functions $H(w_2)$ satisfying an equation of the form (6.12) with constant coefficients a, b, c, and d not simultaneously equal to zero. Thus, the problem of group classification of Eq. (6.1) reduces to listing of all functions H obeying Eq. (6.12). Therefore we call Eq. (6.12) the *classifying relation* for Eq. (6.1).

Remark 5.1. It is significant in group classification that the classifying relation (6.12) inherits the equivalence transformations of Eqs. (6.1). Namely, after the equivalence transformation (5.17) Eq. (6.12) preserves its form,

$$\tilde{a} + \tilde{b}\,\widetilde{H} + (\tilde{c} + \tilde{d}\,\tilde{w}_2)\,\widetilde{H}' = 0$$

where

$$\tilde{a} = a - \frac{a\delta_4}{\delta_1}\,b, \quad \tilde{b} = \frac{\alpha}{\delta_1}\,b, \quad \tilde{c} = \frac{\alpha}{\beta_1^2}\,c - \frac{2\alpha\delta_4}{\delta_1\beta_1^2}\,d, \quad \tilde{d} = \frac{\alpha}{\delta_1}\,d. \tag{6.13}$$

Let in (6.12) $d = 0$. Then we obtain from Eq. (6.12), due to the condition $H'' \neq 0$, that $c \neq 0$ and $b \neq 0$. Therefore one can reduce (6.12) to the form $H' = H$ using (6.13) with suitable parameters α, β, δ. Hence, we have

$$H = \mathrm{e}^{w_2}. \tag{6.14}$$

If $d \neq 0$ and $b = 0$ then one can reduce (6.12) by the transformation (6.13) to the form $w_2 H' = 1$, whence

$$H = \ln w_2. \tag{6.15}$$

If $d \neq 0$ and $b \neq 0$ then one can reduce (6.12) to $w_2\, H' = \sigma H$, whence

$$H = w_2^\sigma, \quad \sigma = \text{const.} \tag{6.16}$$

The condition $H'' \neq 0$ requires that $\sigma \neq 0$ and $\sigma \neq 1$. Note that in the derivation of Eqs. (6.14)-(6.16) the constants of integration are eliminated by an appropriate dilation of t in Eq. (6.1).

6.3 Additional symmetries

Now we solve the determining equations (6.9) and (6.10) for all three cases (6.14)-(6.16). Substituting (6.14) in (6.9) and (6.10) and splitting with respect to w_2 we get

$$C_6 = 0, \quad C_2 - C_4 = 0, \quad C_1 = 0, \quad \delta_t = 0,$$

$$C_2 x + C_3 - \xi^1_t - \delta_{xx} = 0, \quad C_2 - 4C_4 = 0, \quad C_3 - 2C_6 = 0.$$

Hence,

$$\xi^1 = A_1 t + A_2, \quad \xi^2 = A_3 x + A_4, \eta = 2A_3 w + \left(A_3 - \frac{A_1}{2}\right) x^2 + A_5 x + A_6.$$

Consequently, in the case (6.14) Eq. (6.1) has, along with the operators (6.11), the following additional symmetry:

$$Z_6 = 2t\frac{\partial}{\partial t} - x^2\frac{\partial}{\partial w}\,\cdot \tag{6.17}$$

Similar calculations in the case (6.15) give the supplementary operator

$$Z_6 = x\frac{\partial}{\partial x} - 2t\frac{\partial}{\partial w}\,\cdot \tag{6.18}$$

For the function (6.16), Eqs. (6.9) and (6.10) give the system

$$C_6 = 0, \quad C_2 = C_4, \quad \delta_t = 0, \quad \delta_{xx} = 0,$$

$$\xi^1_t = C_3(1-\sigma) + 2C_5\sigma, \quad C_1(1-3\sigma) = 0, \quad C_2(1+3\sigma) = 0.$$

The last two equations of this system provide additional classifying relations. They show that if $\sigma \neq \pm 1/3$, one should add to (6.11) only one operator

$$Z_6 = (1-\sigma)t\frac{\partial}{\partial t} + w\frac{\partial}{\partial w}\cdot \tag{6.19}$$

If $\sigma = 1/3$ or $\sigma = -1/3$ then Eq. (6.19) has, along with (6.11), (6.19), the seventh symmetry, namely

$$Z_7 = w\frac{\partial}{\partial x} \tag{6.20}$$

for $\sigma = 1/3$ and

$$Z_7 = x^2\frac{\partial}{\partial x} + xw\frac{\partial}{\partial x} \tag{6.21}$$

for $\sigma = -1/3$.

§ 7 Equations of nonlinear filtration

7.1 Determining equations

Here we give detailed calculations in solving the problem of group classification of the filtration equations ([2], [3])

$$v_t = h(v_1)v_2 \tag{7.1}$$

with $h'(v_1) \neq 0$. The operator of the admitted group is sought in the form

$$Y = \xi^1(t,x,v)\frac{\partial}{\partial t} + \xi^2(t,x,v)\frac{\partial}{\partial x} + \eta(t,x,v)\frac{\partial}{\partial v}$$

and is found from the determining equation

$$\zeta_1 - h'(v_1)v_2\zeta_2 - h(v_1)\zeta_{22} = 0, \tag{7.2}$$

where $\zeta_i, \zeta_{22}, i = 1,2$ are given by the prolongation formulae (5.6). In solving Eq. (7.2) v_t is replaced by $h(v_1)v_2$ and v_1, v_2, v_{1t} are considered as free variables. The decomposition of (7.2) with respect to the free variables v_{1t}, v_2 leads to the equations

$$\xi^1_x = 0, \quad \xi^1_v = 0, \tag{7.3}$$

$$(2\xi^2_x - \xi^1_t + 2\xi^2_v v_1)h = [\eta_x + (\eta_v - \xi^2_x)v_1 - \xi^2_v v_1^2]h', \tag{7.4}$$

$$\eta_t - \xi^2_t v_1 = [\eta_{xx} + 2(\eta_{xv} - \xi^2_{xx})v_1 + (\eta_{vv} - 2\xi^2_{xv})v_1^2 - \xi^2_{vv}v_1^3]h. \tag{7.5}$$

In the case of an arbitrary function $h(v_1)$ all coefficients in (7.4) and (7.5) should vanish:

$$\begin{gathered}\xi^1_t - 2\xi^2_x = 0, \quad \xi^2_v = 0, \quad \eta_x = 0, \quad \eta_v - \xi^2_x = 0,\\ \eta_t = 0, \quad \xi^2_t = 0, \quad 2\eta_{xv} - \xi^2_{xx} = 0, \quad \eta_{vv} - 2\xi^2_{xv} = 0.\end{gathered} \tag{7.6}$$

Using Eqs. (7.3), (7.6) one can easily see that Eq. (7.1) with an arbitrary function $h(v_1)$ admits the four-dimensional Lie algebra spanned by

$$Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial x}, \quad Y_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v}, \quad Y_4 = \frac{\partial}{\partial v}\cdot \tag{7.7}$$

Let us find all cases of extension of the group. We note that Eq. (7.4) is equivalent to the following *classifying relation:*

$$h(a + 2bv_1) = h'(c + dv_1 - bv_1^2) \tag{7.8}$$

with constant coefficients a, b, c, and d. Indeed, since h depends only on v_1 Eq. (7.4) implies that its coefficients either vanish identically or are proportional (with constant coefficients) to a function $\lambda(t, x, v) \neq 0$, i.e.,

$$2\xi^2_x - \xi^1_t = a\lambda(t, x, v), \quad \xi^2_v = b\lambda(t, x, v),$$

$$\eta_x = c\lambda(t, x, v), \quad \eta_v - \xi^2_x = d\lambda(t, x, v).$$

One can readily verify that vanishing of all coefficients in (7.4) corresponds to the case of an arbitrary function h. Consequently, extension of the group is only possible for functions h satisfying an equation of the form (7.8) with constant coefficients a, b, c, and d such that the polynomials $a + 2bv_1$ and $c + dv_1 - bv_1^2$ do not vanish. Hence, one can rewrite the classifying relation (7.8) in the form

$$\frac{h'}{h} = \frac{a + 2bv_1}{c + dv_1 - bv_1^2}\,\cdot \tag{7.9}$$

7.2 Analysis of the classifying relation

The classifying relation (7.9) inherits the equivalence transformations of Eqs. (7.1) (cf. Section 6.2). Indeed, two last equations from (5.10) yield

$$\frac{\tilde{h}'}{\tilde{h}} = \frac{1}{B}\Big[2\beta_2(\beta_1 + \beta_2 v_1) + (\beta_1 + \beta_2 v_1)^2\frac{h'}{h}\Big], \tag{7.10}$$

where $B = \beta_1\beta_4 - \beta_2\beta_3 \neq 0$. We solve the equation for $\tilde{v}_1$ in (5.10) with respect to v_1, substitute the resulting expression $v_1 = (\beta_1\tilde{v}_1 - \beta_3)/(\beta_4 - \beta_2\tilde{v}_1)$ in (7.10) and obtain an equation of the form (7.9):

$$\frac{\tilde{h}'}{\tilde{h}} = \frac{\tilde{a} + 2\tilde{b}\tilde{v}_1}{\tilde{c} + \tilde{d}\tilde{v}_1 - \tilde{b}\tilde{v}_1^2} \tag{7.11}$$

with the coefficients $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ related to a, b, c, d by the equations

$$\begin{gathered} a = B\tilde{a} + 2\beta_3\beta_4\tilde{b} - 2\beta_1\beta_2\tilde{c} - 2\beta_2\beta_3\tilde{d}, \\ b = \beta_4^2\tilde{b} - \beta_2^2\tilde{c} - \beta_2\beta_4\tilde{d}, \quad c = -\beta_3^2\tilde{b} + \beta_1^2\tilde{c} + \beta_1\beta_3\tilde{d}. \\ d = -2\beta_3\beta_4\tilde{b} + 2\beta_1\beta_2\tilde{c} + (\beta_1\beta_4 + \beta_2\beta_3)\tilde{d}. \end{gathered} \tag{7.12}$$

Using Eqs. (7.12) one can prove the following.

Lemma 5.1. Under the action of the equivalence transformation (5.10) the discriminant $\Delta = d^2 + 4bc$ of the trinomial $c + dv_1 - bv_1^2$ undergoes the conformal transformation, namely $\tilde{\Delta} = B^{-2}\Delta$.

Theorem 5.2. Every equation of the form (7.9) belongs to one of the three distinctly different classes determined by the conditions $\Delta = 0,\ \Delta > 0$ and $\Delta < 0$. Furthermore:

(i) all equations (7.9) with $\Delta = 0$ are equivalent to the equation

$$\frac{\tilde{h}'}{\tilde{h}} = 1, \tag{7.13}$$

(ii) each equation (7.9) with $\Delta > 0$ can be reduced to the form

$$\frac{\tilde{h}'}{\tilde{h}} = \frac{\sigma - 1}{\tilde{v}_1}, \quad \sigma \geq 0, \tag{7.14}$$

(iii) each equation (7.9) with $\Delta < 0$ can be reduced to the form

$$\frac{\tilde{h}'}{\tilde{h}} = \frac{\nu - 2\tilde{v}_1}{1 + \tilde{v}_1^2}, \quad \nu \geq 0. \tag{7.15}$$

Proof. Lemma 5.1 shows that the conditions $\Delta = 0,\ \Delta > 0$ and $\Delta < 0$ are invariant under the equivalence transformations (5.10), and hence these conditions divide the equations (7.9) into three distinctly different classes.

Let us verify that in the case $\Delta = 0$ all equations (7.9) can be obtained from (7.13) by equivalence transformations. Substitution of (7.13) in (7.10) shows that the equations that are equivalent to (7.13) have the form

$$\frac{h'}{h} = \frac{(B - 2\beta_1\beta_2) - 2\beta_2^2 v_1}{(\beta_1 + \beta_2 v_1)^2} = \frac{(\beta_1\beta_4 - \beta_2\beta_3 - 2\beta_1\beta_2) - 2\beta_2^2 v_1}{(\beta_1 + \beta_2 v_1)^2}\,.$$

On the other hand, invoking that $c = -d^2/(4b)$ due to $\Delta = 0$, one can write an arbitrary equation (7.9) in the form

$$\frac{h'}{h} = \frac{-(a/b) - 2v_1}{[v_1 - d/(2b)]^2} \cdot$$

Therefore, to prove the statement (i) we have to verify that these two equations coincide upon an appropriate choice of the parameters $\beta_1, \ldots, \beta_4$. But this is quite obvious, since these four parameters are arbitrary. For example, it suffices to set $\beta_1 = -d/(2b), \beta_2 = 1, \beta_3 = a/b, \beta_4 = 2$.

Likewise, for (7.14) Eq. (7.10) yields

$$\frac{h'}{h} = \frac{[B\sigma - (\beta_1\beta_4 + \beta_2\beta_3)] - 2\beta_2\beta_4 v_1}{(\beta_1 + \beta_2 v_1)(\beta_1 + \beta_4 v_1)} \tag{7.16}$$

and the statement (ii) follows again from the arbitrariness of the parameters $\beta_1, \ldots, \beta_4$. Furthermore, Eq. (7.16) yields the following expression for calculating σ for given coefficients of Eq. (7.9):

$$\sigma = \begin{cases} |(a/b) + r_1 + r_2||r_1 - r_2|^{-1}, & b \neq 0, \\ |(a/d) + 1|, & b = 0, \end{cases} \tag{7.17}$$

where r_1 and r_2 are the zeros for the trinomial $c + dr - br^2$. The condition $\sigma \geq 0$ is achieved by the equivalence transformation

$$\tilde{x} = v, \quad \tilde{v} = x$$

mapping Eq. (7.14) with arbitrary σ into the equation

$$\frac{h'}{h} = -\frac{\sigma + 1}{v_1} \cdot$$

The statement (iii) is proved similarly. Here one gets the formula

$$v = 2|a + d|/\sqrt{-\delta}. \tag{7.18}$$

Thus, the set of Eqs. (7.9) splits into equivalence classes with representatives (7.13)- (7.15). Omitting in these equations the symbol " $\sim$ " over v and h and integrating each of these equations, we obtain:

$$h = \mathrm{e}^{v_1}, \tag{7.19}$$

$$h = v_1^{\sigma - 1}, \quad \sigma \geq 0, \tag{7.20}$$

$$h = (1 + v_1^2)^{-1} \exp(\nu \arctan v_1), \quad \nu \geq 0, \tag{7.21}$$

The constant of integration is eliminated by dilation in (7.1).

Remark 5.2. The problem of group classification of Eqs. (7.1) has been discussed in [83] in connection with the study of shear motions of nonlinear visco-elastic fluids. In [83] only the cases (7.19) and (7.20) are indicated but the complete solution of the problem of group classification is not given.

Remark 5.3. It is shown in [116] that (7.21) is carried into (7.20) by composition of the complex substitution $x \to ix, \nu i \to 2(n+1)$ and the equivalence transformation (5.10). Here one takes into account the relation $\arctan(ix) = \frac{i}{2} \ln \frac{1+x}{1-x}$.

Remark 5.4. Since Eqs. (7.20) with $h'(v_1) = 0$ have been eliminated from consideration, it is necessary to assume that $\sigma \neq 1$. As is clear from (7.16), the equation $v_t = h(v_1)v_2$ is equivalent to the linear heat equation if and only if $h(v_1) = (Av_1 + B)^{-2}$, where A and B are arbitrary constants.

7.3 Additional symmetries

Now we return to the solution of the determining equations (7.3)-(7.5). In the case (7.19) we obtain from Eqs. (7.4) and (7.5):

$$-\xi^1_t + 2\xi^2_x = \eta_x, \quad \eta_v - \xi^2_x = 0, \quad \xi^2_v = 0,$$

$$\eta_t = 0, \quad \xi^2_t = 0, \quad \eta_{xx} = 0, \quad 2\eta_{xv} - \xi^2_{xx} = 0,$$

whence, taking into account Eqs. (7.3) we have:

$$\xi^1 = (2A_1 - A_2)t + A_3, \quad \xi^2 = A_1 x + A_4, \quad \eta = A_2 x + A_1 v + A_5.$$

These equations furnish the previous four operators (7.7) and one additional symmetry

$$Y_5 = t\frac{\partial}{\partial t} - x\frac{\partial}{\partial v} \cdot$$

Proceeding likewise in the cases (7.20) and (7.21), we obtain the supplementary symmetry

$$Y_5 = (1-\sigma)t\frac{\partial}{\partial t} + v\frac{\partial}{\partial v}$$

for (7.20) with $\sigma \neq 1$ and

$$Y_5 = -\nu t\frac{\partial}{\partial t} - v\frac{\partial}{\partial x} + x\frac{\partial}{\partial v}$$

for (7.21).

§ 8 Quasi-local transformations

8.1 Lacunary table. Quasi-local symmetries

We have summarized in Table 1 the results of the group classification of the first two members of the sequence (H) obtained in § 5 and § 6 and of the nonlinear heat equations $u_t = (h(u)u_x)_x$ obtained in [102]. The table shows that extension of symmetry properties occurs not always simultaneously for all equations of the sequence (H). For example, if $H = w_2^\sigma$ with arbitrary σ, all members of the sequence (H) admit an extension of the symmetry algebra by one operator, but for $\sigma = 1/3$ a supplementary symmetry appears only for the equation $w_t = H(w_2)$. Therefore Table 1 contains empty blocs, "lacunae". The latter testify to the existence of hidden symmetries. To find them, we fill the lacunae using the transition formulae (4.4):

$$D_x(\varphi)f_v = (D_x(\varphi)\Phi_* - D_x(\Phi)\varphi_*)f_u,$$

$$\varphi_* = \frac{\partial\varphi}{\partial u} + \frac{\partial\varphi}{\partial u_1}D_x + \cdots, \quad \Phi_* = \frac{\partial\Phi}{\partial u} + \frac{\partial\Phi}{\partial u_1}D_x + \cdots,$$

connecting the symmetries f_u and f_v of scalar equations $u_t = F(x, u, u_1...)$ and $v_t = G(y, v, v_1...)$ related by (4.3), $y = \varphi(x, u, u_1...)$, $v = \Phi(x, u, u_1...)$.

We will denote the symmetries of the first, second, and third equation of the sequence (H) by f_w, f_v and f_u, respectively. Then, since $y = x$ and $v = w_1,\ u = v_1$, the above transition formulae are written as follows:

$$f_v = D_x(f_w), \tag{8.1}$$

$$f_u = D_x(f_v). \tag{8.2}$$

Example 5.5. Let us translate the operator $Y_5 = x\frac{\partial}{\partial v} - v\frac{\partial}{\partial x}$ admitted by the equation $v_t = v_2/(1 + v_1^2)$ into the corresponding operators admitted by the other two equations of the sequence (H). Using Eq. (2.10) written as

$$f_v = \eta - \xi^1 v_t - \xi^2 v_1$$

we obtain the coordinate $f_v = x + vv_1$ of the canonical representation of the operator Y_5. Now the transition formula (8.2) yields

$$f_u = D_x(f_v) = 1 + v_1^2 + vv_2 \equiv 1 + u^2 + vu_1.$$

Hence, the equation $u_t = \left(\frac{u_1}{1+u^2}\right)_1$ has a quasi-local symmetry $f_u = 1 + u^2 + vu_1$ with a nonlocal variable v defined by the equations $v_1 = u, v_t = u_1/(1 + u^2)$. The corresponding generator can be written in the usual form:

$$X_4 = -v\frac{\partial}{\partial x} + (1 + u^2)\frac{\partial}{\partial u}\cdot$$

Further we calculate the corresponding quasi-local symmetry f_w for the equation $w_t = \arctan w_2$. The use of (8.1) leads to the equation

$$D_x(f_w) = x + vv_1 \equiv x + w_1 w_2$$

for determining f_w. The solution of this equation can be written in the form

$$f_w = \frac{x^2 + w_1^2}{2} + C(t).$$

The function $C(t)$ is determined from the invariance condition of the equation $w_t = \arctan w_2$ with respect to the operator $f_w \frac{\partial}{\partial w}$. The invariance condition is $C(t) = t$. Consequently, the operator Y_5 becomes the generator

$$Z_6 = -w_1 \frac{\partial}{\partial x} + \left(\frac{x^2 - w_1^2}{2} + t\right) \frac{\partial}{\partial w}$$

of a group of contact transformations ([58], Sec. 16.2).

Proceedings as in this example, we fill all lacunae in Table 1. The result is given in Table 2. The new symmetries obtained in this way for the first equation of the sequence (H) are contact symmetries*, i.e. local, whereas for the second and third equations from (H) they are quasi-local. Quasi-local symmetries for the linear heat equation are given in § 1, Example 5.3.

8.2 Nonlocal equivalence transformations

Using Bäcklund transformations, we translate the equivalence transformations (5.10) of the filtration equation into equivalence transformations of the other two members of the sequence (H). This adds to the pointwise equivalence transformations (5.17) and (5.18) contact and nonlocal equivalence transformations (cf. Table 3). In particular, the equivalence group of the filtration equation contains the substitution $\tilde{x} = v, \tilde{v} = x$ which is carried into the nonlocal transformation $\tilde{x} = v, \tilde{u} = u^{-1}$ for the heat equation† and into the contact transformation $\tilde{x} = w_1, \tilde{w} = xw_1 - w$ for the first equation of (H).

The extended equivalence groups are used afterwards for completing the group classification. As a result one imposes the restriction $\sigma > 0$ in

*It is shown in [43] that the classification of the first equation of the sequence (H) with respect to groups of contact transformations does not lead to other cases of extension.

†A direct proof of the fact that the nonlocal transformation $\tilde{x} = v, \tilde{u} = u^{-1}$ is an equivalence transformation of the nonlinear heat equation is given in [27], [96]. This question is also discussed in [8], [9], [110], [111].

the case $H = \sigma^{-1} w_2^\sigma$. In particular, the known [102] case $h(u) = u^{-4/3}$ of extension of the symmetry group for the nonlinear heat equation turns out to be equivalent to the case $h(u) = u^{-2/3}$ and is obtained from the latter by the nonlocal transformation $\tilde{x} = v, \tilde{u} = u^{-1}$.

§ 9 Integration using quasi-local symmetries

Using quasi-local symmetries and nonlocal transformations, one can extend the class of exact solutions obtained by group methods. First of all, the number of invariant solutions increases due to extension of the symmetry group. Moreover, in general, there appears a possibility of constructing partially invariant solutions for scalar differential equations*.

9.1 Invariant solutions

We will illustrate methods of computation of invariant and partially invariant solutions with respect to quasi-local symmetries by considering, e.g. the following nonlinear heat equation:

$$u_t = \left(\frac{u_1}{1+u^2}\right)_1 . \tag{9.1}$$

According to [102], Equation (9.1) does not have additional point symmetries in comparison with the general case. However, extension of the symmetry algebra occurs due to the quasi-local symmetry (Table 2, $H(w_2) = \arctan w_2$)

$$X_4 = -v\frac{\partial}{\partial x} + (1+u^2)\frac{\partial}{\partial u} \cdot$$

This additional symmetry provides new group invariant solutions.

Since Eq. (9.1) is connected with the filtration equation

$$v_t = \frac{v_1}{1+v_1^2} \tag{9.2}$$

by the substitution $u = v_1$, every invariant solution of rank $\rho = 1$ of Eq. (9.1) can be obtained by differentiation with respect to x of an appropriate invariant solution of Eq. (9.2). Recall that the task of listing all essentially different invariant solutions of rank $\rho = 1$ of Eq. (9.2) reduces (see

*However for the sequence (H) the construction of partially invariant solutions does not lead to new solutions different from the invariant ones (see Section 9.2).

[103], [106]) to construction of an optimal system Θ_1 of inequivalent one-dimensional subalgebras of the Lie algebra with the basis

$$
\begin{gathered}
Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial x}, \quad Y_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} \\
Y_4 = \frac{\partial}{\partial v}, \quad Y_5 = -v\frac{\partial}{\partial x} + x\frac{\partial}{\partial v}\,\cdot
\end{gathered}
\tag{9.3}
$$

The reckoning shows that the system Θ_1 is furnished by the operators

$$
Y_1, Y_2, Y_1 + Y_2, Y_5, Y_1 + Y_5, Y_1 - Y_5, Y_3 + \alpha Y_5, \tag{9.4}
$$

where α is an arbitrary parameter.

The first three operators from (9.4) provide well-known invariant solutions, namely, stationary, spatially homogeneous and travelling waves, respectively. They are easily calculated, e.g. $Y_1 + Y_2$ leads to the solution

$$
v = C_1 - \arcsin(C_2 e^{t-x}).
$$

The invariant solution of Eq. (9.2) obtained by using the operator Y_5 is found from a first order ordinary differential equation (since one of the invariants of the operator Y_5 is the variable t) and has the form

$$
v = \sqrt{C - 2t - x^2}, \quad C = \text{const}. \tag{9.5}
$$

The graph of this solution for $C = 1$ is given in Fig. 1 ($t_1 = 0, t_2 = 0.25$). Differentiating (9.5) with respect to x and setting $v_1 = u$ we get the solution

$$
u = -\frac{x}{\sqrt{C - 2t - x^2}} \tag{9.6}
$$

of (9.1) which is invariant with respect to the nonlocal operator X_4. The question arises of the possibility of obtaining (9.6) as an invariant solution with respect to a point symmetry of Eq. (9.1), i.e. of the existence of constants $\alpha_1, \alpha_2, \alpha_3$, such that the operator (see Table 2, operators X_1, X_2, X_3)

$$
X = \alpha_1 \frac{\partial}{\partial t} + \alpha_2 \frac{\partial}{\partial x} + \alpha_3 \left(2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} \right)
$$

satisfies the invariance condition

$$
X\left(x + u\sqrt{C - 2t - x^2}\,\right)\Big|_{(9.6)} = 0.
$$

This equation yields $\alpha_1 = C,\ \alpha_2 = 0,\ \alpha_3 = -1$, i.e.

$$
X = (C - 2t)\frac{\partial}{\partial t} - x\frac{\partial}{\partial x}\,\cdot
$$

The invariant solutions of (9.1) with respect to this operator have the form $u = \varphi(z), z = x/\sqrt{C - 2t}$ and satisfy the ordinary differential equation

$$z\varphi' = \left(\frac{\varphi'}{1 + \varphi^2}\right)'. \tag{9.7}$$

We have not been able to find the general solution of Eq. (9.7). On the other hand, the invariant solution (9.6) obtained by using the quasi-local symmetry X_4 provides the particular solution $\varphi = -z/\sqrt{1 - z^2}$ to Eq. (9.7). Similar approach can be useful in other cases when the calculation of invariant solutions using point symmetries leads to complicated equations.

For the operator $Y_1 + \varepsilon Y_5$ ($\varepsilon = \pm 1$) we take the independent invariants

$$I_1 = x^2 + v^2, \quad I_2 = \arctan\left(\frac{v}{x}\right) - \varepsilon t$$

and seek the invariant solutions of Eq. (9.2) in the form $I_2 = \varphi(I_1)$, i.e.

$$v = x \tan[\varphi(z) + \varepsilon t], \quad z = x^2 + v^2. \tag{9.8}$$

Substitution of (9.8) in Eq. (9.2) leads to the following system of ordinary differential equations:

$$\varphi' = \frac{\lambda}{2z}, \quad \lambda' = \frac{1}{2}(1 + \lambda^2)\left(\varepsilon - \frac{\lambda}{z}\right). \tag{9.9}$$

The corresponding solution of (9.1) which is invariant with respect to the nonlocal symmetry $X_1 + \varepsilon X_4$ has the form

$$u = \tan[\varphi(z) + \arctan \lambda(z) + \varepsilon t], \quad z = x^2 + v^2, \tag{9.10}$$

where $v(t, x)$ is determined by Eqs. (9.8)-(9.9).

For the operator $Y_3 + \alpha Y_5$ we take the independent invariants

$$I_1 = \frac{v^2 + x^2}{t}, \quad I_2 = \arctan\left(\frac{v}{x}\right) - \frac{\alpha}{2}\ln(v^2 + x^2)$$

and seek the invariant solutions of Eq. (9.2) in the form $I_2 = \varphi(I_1)$, i.e.

$$v = x \tan\left[\varphi(z) + \frac{\alpha}{2}\ln t\right], \quad z = \frac{v^2 + x^2}{t}\cdot$$

Substitution in Eq. (9.2) yields to the following system of ordinary differential equations:

$$\varphi' = \frac{\lambda}{2z}, \quad \lambda' = \frac{1}{2}(1 + \lambda^2)\left(\frac{\alpha}{2} - \frac{\lambda}{2} - \frac{\lambda}{z}\right).$$

The corresponding solution of (9.1) which is invariant with respect to the nonlocal symmetry $X_3 + \alpha X_4$ has the form

$$u = \tan[\varphi(z) + \arctan \lambda(z) + \varepsilon t]. \tag{9.11}$$

The solution (9.10) and (9.11) cannot be obtained as invariant solutions using the Lie point symmetries X_1, X_2, X_3.

Let us consider also the equation of nonlinear filtration

$$v_t = (1 + v_1^2)\exp(\nu \arctan v_1)v_2, \quad \nu \neq 0, \tag{9.12}$$

and find its invariant solutions with respect to the Lie point symmetry

$$Y_5 = -\nu t\frac{\partial}{\partial t} - v\frac{\partial}{\partial v} + x\frac{\partial}{\partial v}\cdot$$

Taking the following basis of invariants of the operator Y_5 :

$$I_1 = v^2 + x^2, \quad I_2 = \frac{1}{\nu}\ln t + \arctan\left(\frac{v}{x}\right)$$

and seeking the invariant solution in the implicit form

$$v = x\tan\left[\varphi(z) - \frac{1}{\nu}\ln t\right], \quad z = v^2 + x^2, \tag{9.13}$$

one obtains a second order ordinary differential equation for $\varphi(z)$. After one integration and substitution $\psi = \tan\left(\frac{1}{\nu}\ln\frac{C-z}{2} - \varphi\right)$ the equation reduces to Abel's equation of the first kind

$$\frac{\psi'}{1+\psi^2} + \frac{\psi}{2z} + \frac{2}{\nu(C-z)} = 0,$$

where C is the constant of integration.

The nonlinear heat equation corresponding to (9.12) is written

$$u_t = \left(\frac{\exp(\nu\arctan u)}{1+u^2}u_1\right)_1 \cdot$$

It describes the heat propagation in a medium, where the coefficient of thermal conductivity dependins non-monotonically on the temperature. For this equation, Eq. (9.13) furnishes the invariant solution

$$u = \tan\left[\varphi(z) + \arctan(2z\varphi') - \frac{1}{\nu}\ln t\right]$$

with respect to the quasi-local symmetry

$$X_4 = -\nu t \frac{\partial}{\partial t} - v \frac{\partial}{\partial x} + (1+u^2)\frac{\partial}{\partial u} \cdot$$

Let us discuss another method (based on nonlocal equivalence transformations) for computing invariant with respect to nonlocal symmetries. Consider the nonlinear heat equation

$$u_t = (u^{-2/3}u_1)_1, \tag{9.14}$$

and find its invariant with respect to the quasi-local symmetry

$$X_5 = w\frac{\partial}{\partial x} - 3uv\frac{\partial}{\partial u} \cdot \tag{9.15}$$

We will first map Eq. (9.14) to the equation

$$\tilde{u}_t = (\tilde{u}^{-4/3}\tilde{u}_1)_1 \tag{9.16}$$

by the nonlocal equivalence transformation (see Section 8.2)

$$x = \tilde{v}, \quad u = \tilde{u}^{-1}. \tag{9.17}$$

The transformation (9.17) carries X_5 (9.14) into the Lie point symmetry

$$\tilde{X}_5 = (1+\tilde{x}^2)\frac{\partial}{\partial \tilde{x}} - 3\tilde{x}\tilde{u}\frac{\partial}{\partial \tilde{u}}$$

of Eq. (9.16). The invariant solution of Eq. (9.16) with respect to $\tilde{X}_5$ is

$$\tilde{u} = \frac{(a^2-4t)^{3/4}}{(1+\tilde{x}^2)^{3/2}} \cdot$$

The transformation (9.17) carries it into the invariant solution

$$u = \frac{(a^2-4t)^{3/2}}{[(a^2-4t)^{3/2} - x^2]^{3/2}} \tag{9.18}$$

of Eq. (9.14) with respect to the operator (9.15). The graph of the solution (9.18) for $a = 2$ is given in Fig. 2 ($t_1 = 0, t_2 = 0.5$). We note that (9.18) is a special case of the invariant solution with respect to the point symmetry

$$X = a^2X_1 - 3X_3 + 3X_4 \equiv (a^2-4t)\frac{\partial}{\partial t} - 3x\frac{\partial}{\partial x} + 3u\frac{\partial}{\partial u} \cdot$$

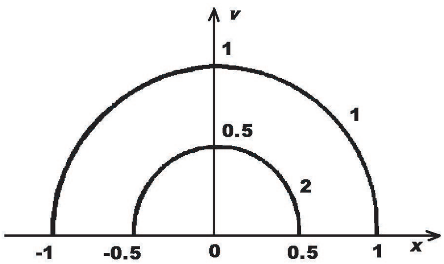

Figure 1:

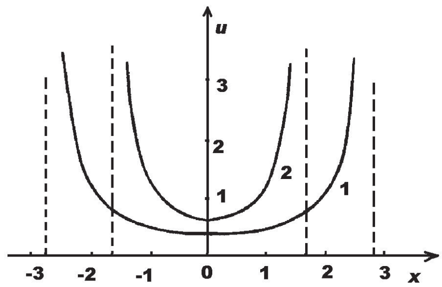

Figure 2:

9.2 Partially invariant solutions

Let us consider the over-determined system

$$v_t = \frac{v_2}{1+v_1^2}, \quad v_1 = u, \quad u_t = \left(\frac{u_1}{1+u^2}\right)_1, \tag{9.19}$$

which admits the Lie algebra with basis

$$Y_1 = \frac{\partial}{\partial t} \quad Y_2 = \frac{\partial}{\partial x}, \quad Y_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v},$$
$$Y_4 = \frac{\partial}{\partial v}, \quad Y_5 = -v\frac{\partial}{\partial x} + x\frac{\partial}{\partial v} + (1+u^2)\frac{\partial}{\partial u}. \tag{9.20}$$

The operators (9.20) are obtained by extending the action of the operators (9.3) admitted by the first equation of the system (9.19) to the variable u (see also Table 2). The problem of listing all essentially different partially-invariant solutions of rank $\rho = 1$ of (9.19) reduces (see [103], [106]) to construction of an optimal system Θ_2 of inequivalent two-dimensional subalgebras of the Lie algebra spanned by (9.20). One can show using (9.4) that Θ_2 is defined by the following pairs of operators:

$$< Y_1, Y_2 >, \ < Y_1, Y_5 >, \ < Y_1, Y_3 + \alpha Y_5 >, \ < Y_2, Y_3 >, \ < Y_3, Y_5 > .$$

To illustrate the method, we take the subalgebra $< Y_1, Y_5 >$. Its invariants are $I_1 = x^2 + v^2, I_2 = (xu - v)/(x + uv)$, and the partially-invariant solution of defect $\delta = 1$ and rank $\rho = 1$ is given by $I_2 = \varphi(I_1)$, i .e.

$$u = \frac{v + x\varphi(z)}{x - v\varphi(z)}, \quad z = x^2 + v^2. \tag{9.21}$$

Substituting (9.21) in the last equation (9.19), taking into account the other two equations and integrating once the resulting ODE for $\varphi(z)$, we get

$$\varphi' = \frac{1+\varphi^2}{2}\left(K - \frac{\varphi}{z}\right), \quad K = \text{const}.$$

The other two equations of (9.19) are equivalent to the integrable system

$$v_t = \frac{Kz}{x - v\varphi}\,, \quad v_1 = \frac{v + x\varphi}{x - v\varphi}\cdot \tag{9.22}$$

We solve Eqs. (9.22) using the change of variables $(t, x, v) \to (t, z, w)$, where $z = x^2 + v^2, w = v$, and obtain the partially-invariant solution

$$v = x\tan(Kt + \lambda(z)), \quad u = \tan(\arctan(\varphi(z)) + Kt + \lambda(z)),$$

$$\lambda' = \pm\frac{\varphi}{2z}, \quad \varphi' = \frac{1+\varphi^2}{2}\left(K - \frac{\varphi}{z}\right), \quad K = \text{const.},$$

to Eqs. (9.19) with respect to the two-parameter group with the generators Y_1, Y_5. However, this solution is invariant with respect to the one-parameter subgroup generated by the operator $Y_1 + KY_5$ [cf. (9.8)-(9.10)], i.e. it is a so-called *reducible partially-invariant solution* [103].

As a matter of fact, all partially-invariant solutions for the sequence (H) are reducible to invariant solutions. Let us prove this statement for partially-invariant solutions with the defect $\delta = 1$. Any such solution can be represented in one of the following three forms:

$$(i)\ u = f(t, x, v), \quad w = g(t, x, v),$$

$$(ii)\ u = f(t, x, w), \quad v = g(t, x, w),$$

$$(iii)\ v = f(t, x, u), \quad w = g(t, x, u).$$

For all three cases the proof is similar. Therefore, let us consider, e.g. the first case (the example considered above belongs to precisely this case). According to Ovsyannikov's reduction theorem [106] it suffices to show that for the function v one gets an integrable system of first order equations which can be solved for the derivatives. Using the equations of the sequence (H) it is easy to show that this is indeed so:

$$v_1 = u = f, \quad v_t = h(v_1)v_2 = h(u)u_1 = h(f)(f_x + ff_v).$$

For the partially invariant solutions with defect $\delta = 2$ the proof is similar.

§ 10 Tables to Chapter 2

In Tables 1 and 2, in the case of a power function $H(w_2)$, we give in the first row the basic extension of the symmetry algebra and then, under numbers (i) and (ii), different subcases of supplementary extension. The shaded areas indicate quasi-local symmetry in Table 2 and nonlocal equivalence transformations in Table 3.

Table 1: Lacunary table

$H(w_t)$	$w_t = H(w_2)$ $\xrightarrow{w_1=v}$	$v_t = h(v_1)v_2$ $\xrightarrow{v_1=u}$	$u_t = (h(u)u_1)_1$
Arbitrary function	$Z_1 = \frac{\partial}{\partial t}$ $Z_2 = \frac{\partial}{\partial x}$ $Z_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2w\frac{\partial}{\partial w}$ $Z_4 = x\frac{\partial}{\partial w}$ $Z_5 = \frac{\partial}{\partial w}$	$Y_1 = \frac{\partial}{\partial t}$ $Y_2 = \frac{\partial}{\partial x}$ $Y_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v}$ $Y_4 = \frac{\partial}{\partial v}$	$X_1 = \frac{\partial}{\partial t}$ $X_2 = \frac{\partial}{\partial x}$ $X_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$
e^{w_2}	$Z_6 = t\frac{\partial}{\partial t} - \frac{x^2}{2}\frac{\partial}{\partial w}$	$Y_5 = t\frac{\partial}{\partial t} - x\frac{\partial}{\partial v}$	$X_4 = t\frac{\partial}{\partial t} - \frac{\partial}{\partial u}$
$\frac{1}{\sigma}w_2^6,\ \sigma \neq 1$ $(i)\sigma = \frac{1}{3}$ $(ii)\sigma = -\frac{1}{3}$	$Z_6 = (1-\sigma)t\frac{\partial}{\partial t} + w\frac{\partial}{\partial w}$ $Z_7 = w\frac{\partial}{\partial x}$ $Z_7 = x^2\frac{\partial}{\partial x} + xw\frac{\partial}{\partial w}$	$Y_5 = (1-\sigma)t\frac{\partial}{\partial t} + v\frac{\partial}{\partial v}$	$X_4 = (1-\sigma)t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$ $X_5 = x^2\frac{\partial}{\partial x} - 3xu\frac{\partial}{\partial u}$
$\ln w_2$	$Z_6 = t\frac{\partial}{\partial t} + (t+w)\frac{\partial}{\partial w}$	$Y_5 = t\frac{\partial}{\partial t} + v\frac{\partial}{\partial v}$	$X_4 = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$
$\arctan w_2$		$Y_5 = x\frac{\partial}{\partial v} - v\frac{\partial}{\partial x}$	
$\frac{1}{\lambda}e^{\lambda \arctan w_2}$ $\lambda > 0$		$Y_5 = x\frac{\partial}{\partial v} - \lambda t\frac{\partial}{\partial t} - v\frac{\partial}{\partial x}$	

Table 2: Classification by quasi-local symmetries

$H(w_2)$	$w_t = H(w_2)$ $\xrightarrow{w_1=v}$	$v_t = h(v_1)v_2$ $\xrightarrow{v_1=u}$	$u_t = (h(u)u_1)_1$
Arbitrary function	$Z_1 = \frac{\partial}{\partial t}$ $Z_2 = \frac{\partial}{\partial x}$ $Z_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2w\frac{\partial}{\partial w}$ $Z_4 = x\frac{\partial}{\partial w}$ $Z_5 = \frac{\partial}{\partial w}$	$Y_1 = \frac{\partial}{\partial t}$ $Y_2 = \frac{\partial}{\partial x}$ $Y_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v}$ $Y_4 = \frac{\partial}{\partial v}$ $--$	$X_1 = \frac{\partial}{\partial t}$ $X_2 = \frac{\partial}{\partial x}$ $X_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$ $--$ $--$
e^{w_2}	$Z_6 = t\frac{\partial}{\partial t} - \frac{x^2}{2}\frac{\partial}{\partial w}$	$Y_5 = t\frac{\partial}{\partial t} - x\frac{\partial}{\partial v}$	$X_4 = t\frac{\partial}{\partial t} - \frac{\partial}{\partial u}$
$\frac{1}{\sigma}w_2^6$, $\sigma \neq 1$ $(i)\sigma = \frac{1}{3}$ $(ii)\sigma = -\frac{1}{3}$	$Z_6 = (1-\sigma)t\frac{\partial}{\partial t} + w\frac{\partial}{\partial w}$ $Z_7 = w\frac{\partial}{\partial x}$ $Z_7 = x^2\frac{\partial}{\partial x} + xw\frac{\partial}{\partial w}$	$Y_5 = (1-\sigma)t\frac{\partial}{\partial t} + v\frac{\partial}{\partial v}$ $Y_6 = w\frac{\partial}{\partial x} - v^2\frac{\partial}{\partial v}$ $Y_6 = x^2\frac{\partial}{\partial x} + (w - xv)\frac{\partial}{\partial v}$	$X_4 = (1-\sigma)t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$ $X_5 = w\frac{\partial}{\partial x} - 3uv\frac{\partial}{\partial u}$ $X_5 = x^2\frac{\partial}{\partial x} - 3xu\frac{\partial}{\partial u}$
$\ln w_2$	$Z_6 = t\frac{\partial}{\partial t} + (t+w)\frac{\partial}{\partial w}$	$Y_5 = t\frac{\partial}{\partial t} + v\frac{\partial}{\partial v}$	$X_4 = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$
$\arctan w_2$	$Z_6 = \left[t + \frac{x^2 - w_1^2}{2}\right]\frac{\partial}{\partial w} - w_1\frac{\partial}{\partial x}$	$Y_5 = x\frac{\partial}{\partial v} - v\frac{\partial}{\partial x}$	$X_4 = (1+u^2)\frac{\partial}{\partial u} - v\frac{\partial}{\partial x}$
$\frac{1}{\lambda}e^{\lambda \arctan w_2}$ $\lambda > 0$	$Z_6 = \lambda t\frac{\partial}{\partial t} + w_1\frac{\partial}{\partial x} - \frac{x^2 - w_1^2}{2}\frac{\partial}{\partial w}$	$Y_5 = \lambda t\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} - x\frac{\partial}{\partial v}$	$X_4 = \lambda t\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}$ $-(1+u^2)\frac{\partial}{\partial u}$

Table 3: Equivalence transformations ($\alpha \neq 0, \beta_1\beta_4 - \beta_2\beta_3 \neq 0$)

$w_t = H(w_2) \quad \xrightarrow{w_1=v}$	$v_t = h(v_1)v_2 \quad \xrightarrow{v_1=u}$	$u_t = (h(u)u_1)_1$
$\tilde{t} = \alpha t + \gamma_1$	$\tilde{t} = at + \gamma_1$	$\tilde{t} = at + \gamma - 1$
$\tilde{x} = \beta_1 x + \beta_2 w_1 + \gamma_2$	$\tilde{x} = \beta_1 x + \beta_2 v + \gamma_2$	$\tilde{x} = \beta_1 x + \beta_2 v + \gamma_2$
$\tilde{w} = \beta_1(\gamma_3 x + \beta_4 w + \frac{1}{2}\beta_3 x^2) + \gamma_4 t + \gamma_5$ $+\beta_2(\beta_3(xw_1 - w) + \gamma_3 w_1 + \frac{1}{2}\beta_4 w_1^2)$	$\tilde{v} = \beta_3 x + \beta_4 v + \gamma_3$	$\tilde{u} = \frac{\beta_3 + \beta_4 u}{\beta_1 + \beta_2 u}$
$\tilde{H} = \frac{\beta_1\beta_4 - \beta_2\beta_3}{\alpha} H + \frac{\gamma_4}{\alpha}$	$\tilde{h} = \frac{(\beta_1 + \beta_2 v_1)^2}{\alpha} h$	$\tilde{h} = \frac{\left(\beta_1 + \beta_2 u\right)^2}{\alpha} h$

Chapter 3
One-dimensional gasdynamic equations

§ 11 Introduction

11.1 Preliminary discussion

One-dimensional gasdynamic equations can be written in the form

$$\begin{aligned} \rho_t + v\,\rho_x + \rho\,v_x &= 0, \\ \rho\,(v_t + v\,v_x) + p_x &= 0, \\ p_t + v\,p_x + A(p,\rho)\,v_x &= 0, \end{aligned} \tag{11.1}$$

where $A(p,\rho)$ is connected with the entropy $S(p,\rho)$ by the equation

$$A = -\rho\,\frac{\partial S/\partial \rho}{\partial S/\partial p}\,. \tag{11.2}$$

In particular, the the *polytropic* gas flows correspond to $A(p,\rho) = \gamma p$ with an arbitrary constant γ.

Until recently, the main result on group analysis in fluid dynamics was Ovsyannikov's classification [101], [103], [106] of gasdynamic equations by admitted groups of point transformations. In particular, for one-dimensional polytropic flows the group theoretic natur of exclusiveness of the adiabatic exponent $\gamma = 3$ has been discovered in [101]. At the same time the value $\gamma = -1$ (the Chaplygin gas [114]) is remarkable from the point of view of integrability, although it has not been singled out in the group classification [101] among other values of γ. This fact hints the existence of "hidden" (non-point) symmetries of the gasdynamic equations (11.1).

In the present chapter we discuss quasi-local symmetries of the equations of planar one- dimensional motion [1], [6], and in particular, we exhibit a hidden symmetry of the Chaplygin gas.

11.2 The LIE sequence

In Euler's variables the one-dimensional adiabatic motion of a gas with planar, cylindrical or spherical waves can be described by the equations

$$\begin{aligned} \rho_t + r^{-n}(\rho v r^n)_r &= 0, \\ v_t + v v_r + \frac{1}{\rho}\, p_r &= 0, \\ p_t + v p_r + \frac{B(p, 1/\rho)}{\rho}\left(v_r + \frac{n}{r} v\right) &= 0, \end{aligned} \tag{E}$$

where

$$B(p, 1/\rho) = \frac{\partial S}{\partial (1/\rho)} \Big/ \frac{\partial S}{\partial p}\cdot \tag{11.3}$$

Here ρ, v, p are the density, velocity, and pressure of the gas, respectively, r is a spatial variable, and S is the entropy which is a given function of the pressure p and the specific volume $1/\rho$. Finally, we have $n = 0$ for planar waves, $n = 1$ for cylindrical waves, and $n = 2$ for spherical waves. In what follows we assume that $B \neq 0$. Note that the functions $A(p, \rho)$ and $B(p, \rho^{-1})$ defined by (11.2) and (11.3), respectively, are connected by the relation $B = \rho A$. In particular, for the polytropic flow $B = \gamma \rho p$.

Recall that passage from Euler's variables (t, r, ρ) to Lagrange's variables (s, y, q) is effected by

$$s = t, \quad y = \int r^n \rho dr, \quad q = \frac{1}{\rho}\cdot$$

As a result, the system (E) is written in Lagrange's variables as follows:

$$
\begin{aligned}
q_s - (r^n v)_y &= 0,\\
v_s + r^n p_y &= 0,\\
p_s + B(p,q)(r^n v)_y &= 0.
\end{aligned}
\tag{L}
$$

For application of the approach developed in Chapter 1, we introduce the intermediate system

$$
\begin{aligned}
R_t + vR_r &= 0,\\
v_t + vv_r + \frac{r^n}{R_r}\,p_r &= 0,\\
p_t + vp_r + \frac{r^n}{R_r}\,B(p, r^n/R_r)\big(v_r + \frac{n}{r}v\big) &= 0
\end{aligned}
\tag{I}
$$

obtained from (E) by the substitution $r^n\rho = R_r$ with subsequent integration with respect to r. The gasdynamic equations in Euler's variables (E) and in Lagrange's variables (L) are obtained from (I) by differential substitutions. Namely, the passage (I) $\to$ (E) is effected by the substitution

$$
\rho = r^{-n} R_r, \tag{11.4}
$$

and (I) $\to$ (L) by the substitution

$$
s = t, \quad y = R, \quad q = \frac{r^n}{R_r}\,\cdot \tag{11.5}
$$

In what follows we consider only the planar waves ($n = 0, r = x$). Then Equations (E) are written

$$
\begin{aligned}
\rho_t + v\,\rho_x + \rho\,v_x &= 0,\\
\rho\,(v_t + v\,v_x) + p_x &= 0,\\
\rho(p_t + v\,p_x) + B(p,\rho^{-1})\,v_x &= 0.
\end{aligned}
\tag{11.6}
$$

The sequence (L) $\longrightarrow$ (I) $\longrightarrow$ (E) (briefly LIE) of three systems of equations related by the transformations (11.4), (11.5), together with the equivalence transformations, is summarized in the case of planar waves in Table 4.

11.3 Equivalence transformations

Using the method described in § 5 we calculate the generators E_i of the equivalence transformations for each term of the sequence LIE. Then for every E_i we find the one-parameter group with the parameter a_i and obtain the multi-parameter equivalence group. Finally, we add discrete equivalence transformations and obtain the general equivalence transformation groups. The result is summarized in the following theorem.

Theorem 5.3. (i) The continuous group $\mathcal{E}_c$ of pointwise equivalence transformations for the system (E) is generated by the operators

$$
\begin{aligned}
&E_1 = \frac{\partial}{\partial t}, \quad E_2 = \frac{\partial}{\partial x}, \quad E_3 = \frac{\partial}{\partial p}, \quad E_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial v}, \\
&E_5 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \quad E_6 = x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} - 2\rho\frac{\partial}{\partial \rho} - 2B\frac{\partial}{\partial B}, \\
&E_7 = \rho\frac{\partial}{\partial \rho} + p\frac{\partial}{\partial p} + 2B\frac{\partial}{\partial B}\cdot
\end{aligned}
\tag{11.7}
$$

The corresponding equivalence transformations are given by

$$
\begin{aligned}
&\tilde{t} = \alpha_1 t + \gamma_1, \quad \tilde{x} = \alpha_2 x + \beta_1 t + \gamma_2, \quad \tilde{v} = \frac{1}{\alpha_1}\big(\alpha_2 v + \beta_1\big), \\
&\tilde{\rho} = \frac{\alpha_1^2\alpha_3}{\alpha_2^2}\rho, \quad \tilde{p} = \alpha_3 p + \gamma_3, \quad \tilde{B} = \frac{\alpha_1^2\alpha_3^2}{\alpha_2^2}B
\end{aligned}
\tag{11.8}
$$

with the coefficients

$$\alpha_1 = a_5, \quad \gamma_1 = a_1 a_5, \quad \alpha_2 = a_5 a_6, \quad \beta_1 = a_4 a_5 a_6,$$

$$\gamma_2 = (a_2 + a_1 a_4) a_5 a_6, \quad \alpha_3 = a_7, \quad \gamma_3 = a_3 a_7,$$

where a_i is the parameter of the group with the generator E_i.

(ii) The group $\mathcal{E}_c$ for the system (I) is generated by the operators

$$
\begin{aligned}
&E_1 = \frac{\partial}{\partial t}, \quad E_2 = \frac{\partial}{\partial x}, \quad E_3 = \frac{\partial}{\partial R}, \quad E_4 = \frac{\partial}{\partial p}, \\
&E_5 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial v}, \quad E_6 = t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} - 2B\frac{\partial}{\partial B}, \\
&E_7 = x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} - R\frac{\partial}{\partial R} - 2B\frac{\partial}{\partial B}, \\
&E_8 = R\frac{\partial}{\partial R} + p\frac{\partial}{\partial p} + 2B\frac{\partial}{\partial B}, \quad E_9 = R\frac{\partial}{\partial x}\cdot
\end{aligned}
\tag{11.9}
$$

The corresponding equivalence transformations are given by

$$\tilde{t} = \alpha_1 t + \gamma_1, \quad \tilde{x} = \alpha_2 x + \beta_1 t + \beta_2 R + \gamma_2, \quad \tilde{v} = \frac{1}{\alpha_1}\big(\alpha_2 v + \beta_1\big),$$

$$\widetilde{R} = \frac{\alpha_1^2\alpha_3}{\alpha_2} R + \gamma_4, \quad \tilde{p} = \alpha_3 p + \gamma_3, \quad \widetilde{B} = \frac{\alpha_1^2\alpha_3^2}{\alpha_2^2} B \tag{11.10}$$

with the coefficients

$$\alpha_1 = a_6, \quad \gamma_1 = a_1 a_6, \quad \alpha_2 = a_6^2 a_7, \quad \beta_1 = a_5 a_6^2 a_7, \quad \beta_2 = \frac{a_8 a_9}{a_7}$$

$$\gamma_2 = \frac{a_3 a_8 a_9}{a_7} + (a_2 + a_1 a_5) a_6^2 a_7, \quad \alpha_3 = a_8, \quad \gamma_3 = a_4 a_8, \quad \gamma_4 = \frac{a_3 a_8}{a_7}\,\cdot$$

(iii) The group $\mathcal{E}_c$ for the system (L) is generated by the operator

$$E_1 = \frac{\partial}{\partial s}\,, \quad E_2 = \frac{\partial}{\partial y}\,, \quad E_3 = \frac{\partial}{\partial p}\,, \quad E_4 = \frac{\partial}{\partial v}\,,$$

$$E_5 = \frac{\partial}{\partial q}\,, \quad E_6 = s\frac{\partial}{\partial s} + y\frac{\partial}{\partial y}\,, \quad E_7 = v\frac{\partial}{\partial v} + p\frac{\partial}{\partial p} + q\frac{\partial}{\partial q}\,, \tag{11.11}$$

$$E_8 = y\frac{\partial}{\partial y} + p\frac{\partial}{\partial p} - q\frac{\partial}{\partial q} + 2B\frac{\partial}{\partial B}\,\cdot$$

The corresponding equivalence transformations are given by

$$\tilde{s} = \delta_1 s + \kappa_1, \quad \tilde{y} = \delta_2 y + \kappa_2, \quad ,\tilde{v} = \frac{\delta_1\delta_3}{\delta_2} v + \kappa_3,$$

$$\tilde{p} = \delta_3 p + \kappa_4, \quad \tilde{q} = \frac{\delta_1^2\delta_3}{\delta_2^2} q + \kappa_5, \quad \tilde{B} = \frac{\delta_2^2}{\delta_1^2} B \tag{11.12}$$

with the coefficients (another form of the coefficients is used in Table 4)

$$\delta_1 = a_6, \quad \kappa_1 = a_1 a_6, \quad \delta_2 = a_6 a_8, \quad \kappa_2 = a_2 a_6 a_8,$$

$$\delta_3 = a_7 a_8, \quad \kappa_3 = a_4 a_7, \quad \kappa_4 = a_3 a_7 a_8, \quad \kappa_5 = \frac{a_5 a_7}{a_8}\,\cdot$$

Adding the corresponding reflections to the continuous equivalence groups listed above, we get the transformations (11.8), (11.10), and (11.12) with arbitrary real coefficients $\alpha_i, \beta_i, \gamma_i, \kappa_i$ satisfying the non-degeneracy condition only. These transformations furnish the general equivalence point transformation groups.

Using differential substitutions relating the equations of the sequence LIE, we transform the pointwise equivalence transformations (11.11) of the system (I) into equivalence transformations of the other two terms of the sequence LIE. This approach does not extend the group of equivalence transformations for the system (L), but it provides supplementary nonlocal equivalence transformations for the system (E). All equivalence groups including nonlocal equivalence transformations are summarized in Table 4.

§ 12 Group classification of the system (I)

12.1 General analysis of the determining equations

We write the system (I) in the form

$$R_t + vR_x = 0,$$

$$v_t + vv_x + \frac{1}{R_x}\, p_x = 0,$$

$$p_t + vp_x + A(p, R_x)v_x = 0,$$

where A is connected with B by the equation

$$R_x\, A(p, R_x) = B(p, R_x^{-1}). \tag{12.1}$$

The equivalence transformations (11.10) change the function A as follows:

$$\tilde{A} = \alpha_3 \left(1 + \frac{\beta_2}{\alpha_2} R_x\right) A. \tag{12.2}$$

Let us write the generator of the point transformation group admitted by the system (I) in the form

$$Y = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \lambda \frac{\partial}{\partial R} + \eta \frac{\partial}{\partial v} + \mu \frac{\partial}{\partial p}$$

The determining equations, upon splitting with respect to "free variables", show that

$$\xi^1 = \xi^1(t), \quad \xi^2 = \xi^2(t, x, R), \quad \lambda = \lambda(R),$$

$$\eta = \eta(t, x, R, v), \quad \mu = \mu(t, R, p),$$

where $\xi^1,\ \xi^2,\ \lambda,\ \eta,\ \mu$ satisfy the following equations:

$$\eta = \xi^2_t - v\xi^1_t + v\xi^2_x, \tag{12.3}$$

$$\lambda_R = \mu_p + 2\xi^1_t - \xi^2_x, \tag{12.4}$$

$$\mu_R = -\eta_t - v\eta_x, \tag{12.5}$$

$$\mu_t = -(\eta_x + R_x + \eta_R)\, A, \tag{12.6}$$

$$\left(\mu_p + 2\xi^1_t - 2\xi^2_x - R_x\xi^2_R\right) R_x \frac{\partial A}{\partial R_x} + \mu\frac{\partial A}{\partial p} = \left(\mu_p + R_x\xi^2_R\right) A. \tag{12.7}$$

One can readily deduce from these equations that the system (I) with an arbitrary function $A(p, R_x)$ admits the five-dimensional Lie algebra L_5 spanned by the operators

$$\begin{gathered} Y_1 = \frac{\partial}{\partial x}, \quad Y_2 = \frac{\partial}{\partial t}, \quad Y_3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + R\frac{\partial}{\partial R}, \\ Y_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial v}, \quad Y_5 = \frac{\partial}{\partial R}\cdot \end{gathered} \tag{12.8}$$

The problem of group classification consists in singling out all particular forms of the function $A(p, R_x)$ when the symmetry algebra L_5 extends. Differentiation of Eq. (12.4) with respect to x yields $\xi^2_{xx} = 0$. Now we substitute in (12.5) and (12.6) the expression (12.3) for η, split the resulting equations with respect to v and obtain $\mu_R = -\xi^2_{tt},\ \xi^1_{tt} = 2\xi^2_{tx},\ \xi^2_{xR} = 0$. These equations yield, in particular, that $\xi^1_{ttt} = 0, \xi^2_{ttx} = 0$. Using these conditions and Eqs. (12.3)-(12.6) we have

$$\begin{gathered} \xi^1 = C_1t^2 + C_2t + C_3, \quad \xi^2 = (C_1t + C_4)x + \alpha(t, R), \\ \lambda = C_5R + C_6, \quad \eta = C_1x + (C_4 - C_2 - C_1t)v + \alpha_t(t, R), \\ \mu = (C_5 - 2C_2 + C_4 - 3C_1t)p + \beta(t, R), \end{gathered} \tag{12.9}$$

where

$$\beta_R = -\alpha_{tt}, \tag{12.10}$$

$$\beta_t = 3C_1p - (C_1 + \alpha_{tR}R_x)\, A. \tag{12.11}$$

The compatibility condition for Eqs. (12.10) and (12.11) yields:

$$\alpha_{ttt} = R_xA\, \alpha_{tRR}. \tag{12.12}$$

Eqs. (12.12), (12.11) and (12.7) provide a system of *classifying relations.* Since $\alpha = \alpha(t, R), A = A(p, R_x)$, Eq. (12.12) singles out the following cases:

$$R_xA = \text{const.}, \quad R_xA \neq \text{const.},$$

or, according to Eq. (12.1), $B(p, R_x^{-1}) = \text{const.}$ and $B(p, R_x^{-1}) \neq \text{const.}$

12.2 The case $R_x A =$ const.

Using the equivalence transformations (11.10), (12.2), one can assume that $R_x A = \varepsilon$, where $\varepsilon = \pm 1$. Then Eqs. (12.12) and (12.10) yield

$$\alpha_{tt} - \varepsilon\alpha_{RR} = K(R),$$

$$\beta = M(t) - \varepsilon\alpha_R - L(R), \quad L'(R) = K(R).$$

Now the equations (12.11) and (12.7) yield $C_1 = 0,\ M'(t) = 0$ and $C_5 = C_2$, respectively. Hence the solution of the determining equations is written

$$\xi^1 = C_2 t + C_3, \quad \xi^2 = C_4 x + a(t,R) + \tau(R),$$

$$\lambda = C_2 R + C_6, \quad \eta = (C_4 - C_2)v + a_t(t,R),$$

$$\mu = (C_4 - C_2)p + C_7 - \varepsilon a_R(t,R),$$

where $\tau(R)$ is an arbitrary function and $a(t,R)$ solves the equation

$$a_{tt} - \varepsilon a_{RR} = 0.$$

Consequently, in the case $R_x A = \varepsilon$ the group G_5 extends to an infinite group generated by the operators Y_2, Y_3, Y_5 from (12.8) and the operators

$$\begin{gathered} Y_6 = x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} + p\frac{\partial}{\partial p}, \quad Y_7 = \frac{\partial}{\partial p}, \quad Y_\tau = \tau(R)\frac{\partial}{\partial x}, \\ Y_a = a(t,R)\frac{\partial}{\partial x} + a_t(t,R)\frac{\partial}{\partial v} - \varepsilon a_R(t,R)\frac{\partial}{\partial p}\cdot \end{gathered} \tag{12.13}$$

12.3 The case $R_x A \neq$ const.

In this case Eq. (12.12) yields $\alpha_{ttt} = 0, \alpha_{tRR} = 0$, and then it follows from Eqs. (12.10) and (12.11) that $\beta_{tR} = 0, \beta_{RRR} = 0, \beta_{ttt} = 0$. Consequently, Eqs. (12.9) are replaced by the following:

$$\xi^1 = C_1 t^2 + C_2 t + C_3, \quad \lambda = C_5 R + C_6,$$

$$\xi^2 = (C_1 t + C_4)x + (C_7 R + C_8)t^2 + (C_9 R + C_{10})t + \tau(R),$$

$$\begin{gathered} \eta = C_1 x + (C_4 - C_2 - C_1 t)v + 2(C_7 R + C_8)t + C_9 R + C_{10}, \\ \mu = (C_4 + C_5 - 2C_2 - 3C_1 t)p - C_7 R^2 - 2C_8 R + \delta(t), \end{gathered} \tag{12.14}$$

where $\tau(R)$ is an arbitrary function and $\delta'''(t) = 0$. Substitution of these expressions in Eq. (12.11) gives the equations

$$C_7 = 0, \quad \delta = C_{11} t + C_{12} \tag{12.15}$$

and the following classifying relation:

$$2C_1p - C_{11} = (C_1 + C_9R_x)\,A. \tag{12.16}$$

It follows from Eq. (12.16) that either

$$C_1 = C_9 = C_{11} = 0, \tag{12.17}$$

or the function $A(p, R_x)$ satisfies the condition

$$3ap + b = (a + cR_x)A \tag{12.18}$$

with constant coefficients a, b, and c not simultaneously equal to zero. First we study the second possibility.

If $a \neq 0$, we reduce Eq. (12.18) by an appropriate equivalence transformation (11.10), (12.2) to the form

$$A(p, R_x) = 3p.$$

Then solving Eqs. (12.16) and (12.7), taking into account Eqs. (12.14) and (12.15), we get

$$\xi^1 = C_1t^2 + C_2t + C_3, \quad \xi^2 = C_1tx + C_4 + C_{10}t + C_{13}, \quad \lambda = C_5R + C_6,$$

$$\eta = C_1x + (C_4 - C_2 - C_1t)v + C_{10}, \quad \mu = (C_4 - 2C_2 + C_5 - 3C_1t)p.$$

Consequently, the symmetries (12.8) are augmented by

$$Y_6 = x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} - R\frac{\partial}{\partial R}, \quad Y_7 = R\frac{\partial}{\partial R} + p\frac{\partial}{\partial p},$$

$$Y_8 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + (x - tv)\frac{\partial}{\partial v} - 3tp\frac{\partial}{\partial p}\cdot$$

If $a = 0$, we arrive at the case $R_xA =$ constant analyzed in Section 12.2.

Let us turn now to the case (12.17). In this case, the function $A(p, R_x)$ is determined from the classifying relation (12.7) which, invoking (12.14), is written in the form

$$[2C_2 - C_4 - C_5 - \tau'(R)R_x]A + [C_5 - C_4 - \tau'(R)R_x]R_xA_{R_x}$$
$$+ [(C_4 - 2C_2 + C_5)p - 2C_8R + C_{12}]A_p = 0. \tag{12.19}$$

Differentiating this equation twice with respect to R, we get

$$(A + R_xA_{R_x})\,\tau'''(R) = 0, \tag{12.20}$$

and hence either $A + R_x A_{R_x} = 0$ or $A + R_x A_{R_x} \neq 0,\ \tau'''(R) = 0$.

In the first case, $A + R_x A_{R_x} = 0$, we have

$$A = \frac{f(p)}{R_x}, \quad f'(p) \neq 0.$$

Substitution in Eq. (12.19) yields $C_8 = 0$ and the classifying relation

$$2(C_2 - C_5)f + [(C_4 - 2C_2 + C_5)p + C_{12}]f' = 0.$$

It follows that if $f(p)$ is an arbitrary function then the symmetries (12.8) are augmented by the operator

$$Y = \tau(R)\frac{\partial}{\partial x} \tag{12.21}$$

with an arbitrary function $\tau(R)$. Further extensions of the symmetry algebra occur for the functions $f(p) = \varepsilon e^p$ and $f(p) = \varepsilon p^\sigma (\varepsilon = \pm 1)$. Namely, the symmetry algebra spanned by 12.8) and (12.21) is extended by the operator

$$Y_6 = x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} - R\frac{\partial}{\partial R} - 2\frac{\partial}{\partial p}$$

for $f(p) = \varepsilon e^p$, and by the operator

$$Y_6 = (2 - \sigma)x\frac{\partial}{\partial x} + (2 - \sigma)v\frac{\partial}{\partial v} + \sigma R\frac{\partial}{\partial R} + 2p\frac{\partial}{\partial p}$$

for $f(p) = \varepsilon p^\sigma$.

Let $A + R_x A_{R_x} \neq 0$. Then Eq. (12.20) yields $\tau'''(R) = 0$, and hence $\tau(R) = C_{13}R^2 + C_{14}R + C_{15}$. Substitution of $\tau(R)$ in Eq. (12.19) and decomposition with respect to R lead to the following equations:

$$C_{13}R_x(A + R_x A_{R_x}) + C_8 A_p = 0, \tag{12.22}$$

$$\begin{aligned}&(2C_2 - C_4 - C_5 - C_{14}R_x)A + (C_5 - C_4 - C_{14}R_x)R_x A_{R_x}\\ &+ [(C_4 - 2C_2 + C_5)p + C_{12}]A_p = 0.\end{aligned} \tag{12.23}$$

It is manifest from Eqs. (12.22), (12.23) that extensions of the symmetry group G_5 with the generators (12.8) occur when $A(p, R_x)$ obeys the equation

$$(a + bR_x)A + (c + bR_x)R_x A_{R_x} + (d - ap)A_p = 0 \tag{12.24}$$

with coefficients a, b, c, and d not simultaneously equal to zero.

Remark 5.5. The classifying relation (12.24) is invariant under the equivalence transformations of the system (I). Namely, after the equivalence transformation (11.10), (12.2), Eq. (12.24) has the same form provided that its coefficients change as follows (cf. Remark 5.1 in Section 6.2):

$$\tilde{a} = \alpha_3 a, \quad \tilde{b} = \frac{\alpha_3\beta_2}{\alpha_2}\, c + \frac{\alpha_1^2\alpha_3^2}{\alpha_2^2}\, b, \quad \tilde{c} = \alpha_3 c, \quad \tilde{d} = d - \gamma_3 a, \tag{12.25}$$

where $\alpha_1, \alpha_2, \alpha_3$ are not equal to zero by virtue of the non-degeneracy the transformation (11.10).

If $a \neq 0, c \neq 0$, one can reduce Eq. (12.24) by the transformation (12.25) to the form $\sigma R_x A_{R_x} - pA_p + A = 0$ ($\sigma \neq 0$), whence

$$A = \frac{p^{1-\sigma}}{R_x} f(p^\sigma R_x). \tag{12.26}$$

If $a = 0,\ c \neq 0$, one can reduce Eq. (12.24) by the transformation (12.25) to the form $R_x A_{R_x} + \delta A_p = 0$ ($\delta = 0$ or 1), whence

$$A = f(p) \tag{12.27}$$

for $\delta = 0$ and

$$A = f\left(\frac{e^p}{R_x}\right) \tag{12.28}$$

for $\delta = 1$.

If $a \neq 0,\ c = 0$, Eq. (12.25) reduces to $(1 - \delta R_x)A - \delta R_x^2 A_{R_x} - pA_p = 0$ ($\delta = 0$ or 1), whence

$$A = pf(R_x)$$

for $\delta = 0$ [this case is included in (12.26) with $\sigma = 0$] and

$$A = \frac{p}{R_x} f\left(pe^{1/R_x}\right) \tag{12.29}$$

for $\delta = 1$.

If $a = 0,\ c = 0,\ b \neq 0$ then, invoking that $A + R_x A_{R_x} \neq 0$, one can reduce Eq. (12.25) to the form $R_x A + R_x^2 A_{R_x} + \varepsilon A_p = 0$ ($\varepsilon = \pm 1$), whence

$$A = \frac{1}{R_x} f\left(p + \frac{\varepsilon}{R_x}\right). \tag{12.30}$$

If $a = 0,\ c = 0,\ b = 0$, then $d \neq 0$ and Eq. (12.25) gives

$$A = f(R_x). \tag{12.31}$$

It remains to find supplementary symmetries for all the cases (12.26)-(12.31). These symmetries are found by solving Eqs. (12.22) and (12.23) and are given in Table 6. Furthermore, we give in Table 5 the result of the group classification of the system (E) presented in [103]. Note, that we replace the function $A(p, \rho)$ by $B(p, \rho^{-1}) = \rho A(p, \rho)$ (see Eq. (12.1)).

§ 13 Preliminary group classification

The group classification of differential equations containing "arbitrary elements" (parameters or functions) is a difficult task. The problem is particularly complicated for systems of equations containing functions of *several* variables as arbitrary elements. In our case an arbitrary element is the function $B(p, q)$ of two variables. The main difficulty is of computational character and is connected with sorting a large number of variants. The *method of preliminary group classification* suggested in this section is helpful in precisely such cases. This approach to group classification opens a new application of the equivalence group. In Sections 13.1 and 13.2 we explain the method on the examples of the systems (E) and (I) and then apply it to the system (L) in § 14.

13.1 Application to the system (E)

It is known ([103], §23.6) that the system (E) with an arbitrary function $B(p, \rho^{-1})$ admits a four-parameter group. This group, called the basic group of the system (E), has the Lie algebra L_4 spanned by the operators

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \quad X_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial v}\cdot \tag{13.1}$$

To pick out the cases of the extension of the Lie algebra L_4 and find the corresponding supplementary operators we will use the continuous equivalence group $\mathcal{E}_c$ of the system (E). We rewrite the generators (11.7) of the group $\mathcal{E}_c$ in the form

$$\begin{gathered} E_1 = \frac{\partial}{\partial x}, \quad E_2 = \frac{\partial}{\partial t}, \quad E_3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \quad E_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial v}, \\ E_5 = \frac{\partial}{\partial p}, \quad E_6 = -\frac{1}{2}x\frac{\partial}{\partial x} - \frac{1}{2}v\frac{\partial}{\partial v} + \rho\frac{\partial}{\partial \rho} + B\frac{\partial}{\partial B}, \\ E_7 = \frac{1}{2}x\frac{\partial}{\partial x} + \frac{1}{2}v\frac{\partial}{\partial v} + p\frac{\partial}{\partial p} + B\frac{\partial}{\partial B} \end{gathered} \tag{13.2}$$

and consider their action on the space of the variables p, ρ, B, i.e.

$$\widetilde{E}_5 = \frac{\partial}{\partial p}, \quad \widetilde{E}_6 = \rho\frac{\partial}{\partial \rho} + B\frac{\partial}{\partial B}, \quad \widetilde{E}_7 = p\frac{\partial}{\partial p} + B\frac{\partial}{\partial B}\cdot \tag{13.3}$$

The operators (13.3) generate the three-parameter group $\tilde{\mathcal{E}}_c$ of transformations in the space of the variables p, ρ, B. Invariant equations with respect to any one-parameter subgroup of the group $\tilde{\mathcal{E}}_c$ define functions $B(p, q)$ with functional arbitrariness such that the corresponding system (E) has an extension of the symmetry algebra L_4. It is manifest that to similar (equivalent) subgroups of the group $\tilde{\mathcal{E}}_c$ furnish equivalent systems (E). Therefore, in order to find different cases of extension of the symmetry algebra L_4 obtained in this way, we have to construct the optimal system Θ_1 of one-dimensional subalgebras of the three-dimensional Lie algebra spanned by (13.3). The reckoning (taking into account the reflection $p \to -p, \rho \to -\rho$) shows that this optimal system is provided by the operators

$$\widetilde{E}_6, \quad \widetilde{E}_7 - \sigma \widetilde{E}_5, \quad \widetilde{E}_5 + \widetilde{E}_6, \quad \widetilde{E}_5, \tag{13.4}$$

where σ is an arbitrary parameter.

Let us find the invariant equations for the operators listed in (13.4). We take the first operator, $\widetilde{E}_6$, solve the equation

$$\widetilde{E}_6(I) \equiv \rho \frac{\partial I}{\partial \rho} + B \frac{\partial I}{\partial B} = 0$$

and obtain two functionally independent invariants, $I_1 = \rho^{-1}B$ and $I_2 = p$. The invariant equation is written $I_1 = f(I_2)$ and yields $B = \rho f(p)$ with an arbitrary function f. For the second operator from (13.4) the equation

$$(\widetilde{E}_7 - \sigma \widetilde{E}_5)(I) \equiv p \frac{\partial I}{\partial p} - \sigma \rho \frac{\partial I}{\partial \rho} + (1 - \sigma) B \frac{\partial I}{\partial B} = 0$$

provides the functionally independent invariants $I_1 = Bp^{\sigma - 1}$ and $I_2 = \rho p^{\sigma}$. Then the equation $I_1 = f(I_2)$ yields $B = p^{1-\sigma} f(\rho p^{\sigma})$. For the third operator from (13.4) the equation

$$(\widetilde{E}_5 + \widetilde{E}_6)(I) \equiv \frac{\partial I}{\partial p} + \rho \frac{\partial I}{\partial \rho} + B \frac{\partial I}{\partial B} = 0$$

yields the invariants $I_1 = \rho^{-1}B$ and $I_2 = \rho^{-1}\mathrm{e}^p$. We see from the invariant equation $I_1 = f(I_2)$ that $B = \rho f(\rho^{-1}\mathrm{e}^p)$. Likewise, the last operator $\widetilde{E}_5$ from (13.4) provides the function $f(\rho)$. Thus, we arrive at the following four distinctly different types of the function $B(p, q)$ with functional arbitrariness when an extension by one of the symmetry algebra L_4 occurs:

$$B = \rho f(p), \quad B = p^{1-\sigma} f(\rho p^{\sigma}), \quad B = \rho f\big(\rho^{-1}\mathrm{e}^p\big), \quad B = f(\rho). \tag{13.5}$$

To find the additional symmetry X_5 in each case listed in (13.5), it suffices to replace the operators $\widetilde{E}_i$ appearing in the system (13.4) by the corresponding operators E_i and take the action of the resulting operators on the variables t, x, ρ, p, v. As a result we have:

$$
\begin{aligned}
&B = \rho f(p): && X_5 = -x\frac{\partial}{\partial x} - v\frac{\partial}{\partial v} + 2\rho\frac{\partial}{\partial \rho}\,,\\
&B = p^{1-\sigma} f(\rho p^{\sigma}): && X_5 = (1+\sigma)\Big[x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v}\Big] + 2\Big[p\frac{\partial}{\partial p} - \sigma\rho\frac{\partial}{\partial \rho}\Big],\\
&B = \rho f\big(\rho^{-1}\mathrm{e}^{p}\big): && X_5 = -x\frac{\partial}{\partial x} - v\frac{\partial}{\partial v} + 2\rho\frac{\partial}{\partial \rho} + 2\frac{\partial}{\partial p}\,,\\
&B = f(\rho): && X_5 = \frac{\partial}{\partial p}\,\cdot
\end{aligned}
\tag{13.6}
$$

We will find now the invariant equations with respect to two-parameter subgroups of the group $\mathcal{E}_c$. The optimal system Θ_2 of two-dimensional subalgebras of the Lie algebra spanned by (13.3) is given by the following pairs of operators (representing the bases of the corresponding subalgebras)*:

$$
< \widetilde{E}_6, \widetilde{E}_7 >, \quad < \widetilde{E}_5, \widetilde{E}_7 - \sigma\widetilde{E}_6 >, \quad < \widetilde{E}_5, \widetilde{E}_6 > . \tag{13.7}
$$

To explain the calculations, let us consider the first pair in (13.7), i.e. the two-dimensional subalgebra spanned by the operators $\widetilde{E}_6$ and $\widetilde{E}_7$. Solving the system of first-order linear partial differential equations

$$
\widetilde{E}_6(I) \equiv \rho\frac{\partial I}{\partial \rho} + B\frac{\partial I}{\partial B} = 0, \quad \widetilde{E}_7(I) \equiv p\frac{\partial I}{\partial p} + B\frac{\partial I}{\partial B} = 0,
$$

we find the only independent invariant $I = B/(p\rho)$. Hence, the invariant equation is written $I = \gamma$, or $B = \gamma p\rho$, where $\gamma = \text{const}$. Thus, for the function $B = \gamma p\rho$ with an arbitrary parameter γ the gasdynamic equations (E) admit an extension of the basic group by two symmetries. To find these supplementary symmetries, we proceed as above in the case of the optimal system Θ_1 and obtain the following two supplementary symmetries:

$$
X_5 = -x\frac{\partial}{\partial x} - v\frac{\partial}{\partial v} + 2\rho\frac{\partial}{\partial \rho}\,, \quad X_6 = x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} + 2p\frac{\partial}{\partial p}\,\cdot
$$

Two remaining subalgebras from (13.7) are treated similarly. Note that for the second subalgebra from (13.7) we have to assume that $\sigma \neq 0$ since the reckoning shows that the subalgebra $< \tilde{E}_5, \tilde{E}_7 >$ does not lead to an extension of the symmetry group. After simple calculations, we get the

---

* *Author's note to this 2006 edition:* They were found by using the approach described in §3 of Paper 1 of the present volume.

functions $B = \gamma\rho$ and $B = \gamma\rho^m$ with $m = (\sigma - 1)/\sigma$. Then we reduce these functions to $B = \rho$ and $B = \varepsilon\rho^m$ ($\varepsilon = \pm 1$), respectively, by appropriate simultaneous dilations of the variables ρ and p. Thus, the invariance under the subalgebras (13.7) leads to following functions $B(p, \rho)$:

$$B = \gamma p\rho, \quad B = \varepsilon\rho^m, \quad B = \rho, \tag{13.8}$$

where ($\varepsilon = \pm 1, m \neq 1$). Now we find the supplementary symmetries for the second and third cases in (13.8) and summarize the result:

$$\begin{aligned}
&B = \gamma p\rho: \ X_5 = -x\frac{\partial}{\partial x} - v\frac{\partial}{\partial v} + 2\rho\frac{\partial}{\partial \rho}, \ X_6 = x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} + 2p\frac{\partial}{\partial p};\\
&B = \varepsilon\rho^m: \ X_5 = \frac{\partial}{\partial p}, \ X_6 = (1+\sigma)\Big[x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v}\Big] + 2\Big[p\frac{\partial}{\partial p} - \sigma\rho\frac{\partial}{\partial \rho}\Big];\\
&B = \rho: \quad X_5 = \frac{\partial}{\partial p}, \quad X_6 = -x\frac{\partial}{\partial x} - v\frac{\partial}{\partial v} + 2\rho\frac{\partial}{\partial \rho}\cdot
\end{aligned} \tag{13.9}$$

The operators (13.9) can be rewritten in the form given in Table 5 by takin appropriate linear combinations with the generators (13.1) of the basic group. Comparison of (13.6) and (13.9) with Table 5 shows that the method of preliminary group classification provides the results of the complete group classification with the exception of $B = 3\rho p$.

13.2 Application to the system (I)

We will write the generators (11.9) of the equivalence group $\mathcal{E}_c$ in the form

$$\begin{gathered}
E_1 = \frac{\partial}{\partial x}, \quad E_2 = \frac{\partial}{\partial t}, \quad E_3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + R\frac{\partial}{\partial R},\\
E_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial v}, \quad E_5 = \frac{\partial}{\partial R},\\
E_6 = \frac{\partial}{\partial p}, \quad E_7 = -x\frac{\partial}{\partial x} - v\frac{\partial}{\partial v} + R\frac{\partial}{\partial R} + 2B\frac{\partial}{\partial B},\\
E_8 = x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} + R\frac{\partial}{\partial R} + 2p\frac{\partial}{\partial p} + 2B\frac{\partial}{\partial B}, \quad E_9 = -R\frac{\partial}{\partial x}\cdot
\end{gathered} \tag{13.10}$$

The first five operators in (13.10) coincide with the operators (12.8), i.e. they span the Lie algebra L_5 of the basic group of the system (I). We find the cases of extensions of the algebra L_5 by means of operators from (13.10).

The action of the operators (13.10) on the variables p, R_x, B yields the four-dimensional Lie algebra $\tilde{L}_4$ spanned by

$$\begin{aligned} \widetilde{E}_6 &= \frac{\partial}{\partial p}, \quad \widetilde{E}_7 = R_x\frac{\partial}{\partial R_x} + B\frac{\partial}{\partial B}, \\ \widetilde{E}_8 &= p\frac{\partial}{\partial p} + B\frac{\partial}{\partial B}, \quad \widetilde{E}_9 = R_x^2\frac{\partial}{\partial R_x}. \end{aligned} \tag{13.11}$$

An optimal system Θ_1 of one-dimensional subalgebras of the algebra $\tilde{L}_4$ is given by the operators

$$\widetilde{E}_7, \quad \widetilde{E}_8 - \sigma\widetilde{E}_7, \quad \widetilde{E}_6 + \tilde{E}_7, \quad \widetilde{E}_8 + \tilde{E}_9, \quad \widetilde{E}_6, \quad \widetilde{E}_6 + \varepsilon\widetilde{E}_9, \quad \widetilde{E}_9,$$

where σ is an arbitrary parameter, $\varepsilon = \pm 1$. Proceedings as in Section 13.1, we obtain the following seven types of the function $B(p, R_x^{-1})$ with functional arbitrariness when an extension by one of the symmetry algebra L_5 occurs:

$$\begin{aligned} &B = R_x f(p): && Y_6 = -x\frac{\partial}{\partial x} - v\frac{\partial}{\partial v} + R\frac{\partial}{\partial R}, \\ &B = p^{1-\sigma}f(R_x p^\sigma): && Y_6 = (\sigma+1)\Big[x\frac{\partial}{\partial x} + (\sigma+1)v\frac{\partial}{\partial v}\Big] \\ &&& \qquad + (1-\sigma)R\frac{\partial}{\partial R} + 2p\frac{\partial}{\partial p}, \\ &B = R_x f(R_x^{-1}\mathrm{e}^p): && Y_6 = x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} - R\frac{\partial}{\partial R} - 2\frac{\partial}{\partial p}, \\ &B = pf\big(pe^{1/R_x}\big): && Y_6 = (x - 2R)\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} + R\frac{\partial}{\partial R} + 2p\frac{\partial}{\partial p}, \\ &B = f(R_x): && Y_6 = \frac{\partial}{\partial p}, \\ &B = f(p + \varepsilon R_x^{-1}): && Y_6 = \frac{\partial}{\partial p} - \varepsilon R\frac{\partial}{\partial x}, \\ &B = f(p): && Y_6 = R\frac{\partial}{\partial x}. \end{aligned} \tag{13.12}$$

The reckoning shows that an optimal system Θ_2 of two-dimensional subalgebras of the algebra $\tilde{L}_4$ with the basis (13.11) is given by

$$\begin{aligned} &< \widetilde{E}_7, \widetilde{E}_8 >, \quad < \widetilde{E}_6, \widetilde{E}_8 + \widetilde{E}_9 >, \quad < \widetilde{E}_6, \widetilde{E}_8 - \sigma\widetilde{E}_7 >, \\ &< \widetilde{E}_6, \widetilde{E}_7 >, \quad < \widetilde{E}_9, \widetilde{E}_6 + \widetilde{E}_7 >, \quad < \widetilde{E}_9, \widetilde{E}_8 - \sigma\widetilde{E}_7 >, \\ &< \widetilde{E}_6, \widetilde{E}_9 >, \quad < \widetilde{E}_7 - \widetilde{E}_8, \widetilde{E}_6 + \varepsilon\widetilde{E}_9 >, \quad < \widetilde{E}_7, \widetilde{E}_9 > . \end{aligned}$$

It leads to the following extensions of the basic group by two symmetries:

$$B=\gamma pR_x:\qquad Y_6=-x\frac{\partial}{\partial x}-v\frac{\partial}{\partial v}+R\frac{\partial}{\partial R}\,,\quad Y_7=R\frac{\partial}{\partial R}+p\frac{\partial}{\partial p}\,;$$

$$B=\varepsilon e^{-1/R_x}:\quad Y_6=\frac{\partial}{\partial p}\,,\quad Y_7=(x-2R)\frac{\partial}{\partial x}+v\frac{\partial}{\partial v}+R\frac{\partial}{\partial R}+2p\frac{\partial}{\partial p}\,;$$

$$B=\varepsilon R_x^m:\ Y_6=\frac{\partial}{\partial p}\,,\ Y_7=(1+\sigma)\Big[x\frac{\partial}{\partial x}+v\frac{\partial}{\partial v}\Big]+(1-\sigma)R\frac{\partial}{\partial R}+2p\frac{\partial}{\partial p}\,;$$

$$B=\varepsilon \mathrm{e}^{-p}:\qquad Y_6=R\frac{\partial}{\partial x}\,,\quad Y_7=x\frac{\partial}{\partial x}+v\frac{\partial}{\partial v}-R\frac{\partial}{\partial R}-2\frac{\partial}{\partial p}\,;$$

$$B=\varepsilon p^{1-\sigma}:Y_6=R\frac{\partial}{\partial x}\,,\ Y_7=(1+\sigma)\Big[x\frac{\partial}{\partial x}+v\frac{\partial}{\partial v}\Big]+(1-\sigma)R\frac{\partial}{\partial R}+2p\frac{\partial}{\partial p}\,;$$

$$B=\varepsilon:\qquad Y_6=\frac{\partial}{\partial p}\,,\quad Y_7=R\frac{\partial}{\partial x},\quad Y_8=x\frac{\partial}{\partial x}+v\frac{\partial}{\partial v}+p\frac{\partial}{\partial p}\,\cdot$$

Remark 5.6. In the first case we use E_7+E_8 for X_7, in the third case we set $m=(\sigma-1)/\sigma$.

The supplementary operators X_6, X_7 given above can be rewritten as in Table 6 by taking linear combinations with the generators (12.8) of the basic group. The cases $B=\varepsilon e^{-1/R_x}$ and $B=\varepsilon e^{-p}$ are reduced to $B=\varepsilon e^{1/R_x}$ and $B=\varepsilon e^{p}$ from Table 6 by the reflection $p\to -p, R\to -R$. The function $B=\varepsilon$ invariant simultaneously with respect to two subalgebras, $<\tilde{E}_6,\tilde{E}_9>$ and $<\tilde{E}_7-\tilde{E}_8,\tilde{E}_6+\varepsilon\tilde{E}_9>$, therefore in this case we have three additional symmetries. The subalgebra $<\tilde{E}_7,\tilde{E}_9>$ does not lead to additional symmetries. Comparison with Table 6 shows that the method of preliminary group classification provides almost all results of the complete group classification. The case $B=3pR_x$ is an exception as it is for the system (E). Moreover, Table 6 shows that in the cases $B=f(R_x)$ and $B=f(p+\varepsilon/R_x)$ the system (I) admits, along with the operators Y_6 cited above, one more supplementary operator Y_7, whereas for $B=f(p)$ and $B=\varepsilon$ an extension to an infinite group occurs.

§ 14 Preliminary classification of system (L)

The point symmetries the system (L),

$$q_s-v_y=0,\quad v_s+p_y=0,\quad p_s+B(p,q)\,v_y=0,$$

will be written in the form

$$Z=\xi^1\frac{\partial}{\partial s}+\xi^2\frac{\partial}{\partial y}+\eta\frac{\partial}{\partial v}+\mu\frac{\partial}{\partial p}+\lambda\frac{\partial}{\partial q}\,\cdot$$

The determining equations, after decomposition with respect to free variables and preliminary simplifications are written

$$\xi^1 = \xi^1(s), \quad \xi^2 = \xi^2(y), \quad \eta_q = 0, \quad \mu_q = 0, \tag{14.1}$$

$$\eta_s + \mu_y = 0, \quad \lambda_s - \eta_y = 0, \quad \lambda_y + \eta_p = 0, \tag{14.2}$$

$$\mu_s + B\eta_y = 0, \quad B\eta_p - \mu_v = 0, \tag{14.3}$$

$$\lambda_q - B\lambda p - \xi^1_s + \xi^2_y - \eta_v = 0, \tag{14.4}$$

$$\eta_v - \xi^1_s - \mu_p + \xi^2_y = 0, \tag{14.5}$$

$$2B\eta_v - 2B\mu_p + \mu B_p + \lambda B_q = 0. \tag{14.6}$$

One can readily deduce from these equations that the system (L) with an arbitrary function $B(p, q)$ admits the four-dimensional Lie algebra spanned by following operators (generators of the basic group):

$$Z_1 = \frac{\partial}{\partial s}, \quad Z_2 = s\frac{\partial}{\partial s} + y\frac{\partial}{\partial y}, \quad Z_3 = \frac{\partial}{\partial v}, \quad Z_4 = \frac{\partial}{\partial y}\cdot \tag{14.7}$$

In contrast with the systems (I) and (E), in this case it is impossible to solve the determining equations (14.1)-(14.6) in a closed form and find concrete specifications of the function $B(p, q)$ when the system (E) admits an extension of the symmetries (14.7). Therefore we start with a preliminary group classification of the system (L) by the method of § 13. Let us rewrite the generators (11.11) of the equivalence group $\mathcal{E}_c$ for (L) in the form

$$\begin{gathered} E_1 = \frac{\partial}{\partial s}, \quad E_2 = s\frac{\partial}{\partial s} + y\frac{\partial}{\partial y}, \quad E_3 = \frac{\partial}{\partial v}, \quad E_4 = \frac{\partial}{\partial y}, \\ E_5 = \frac{\partial}{\partial p}, \quad E_6 = -y\frac{\partial}{\partial y} + v\frac{\partial}{\partial v} + 2q\frac{\partial}{\partial q} - 2B\frac{\partial}{\partial B}, \\ E_7 = y\frac{\partial}{\partial y} + v\frac{\partial}{\partial v} + 2p\frac{\partial}{\partial p} + 2B\frac{\partial}{\partial B}, \quad E_8 = \frac{\partial}{\partial q} \end{gathered} \tag{14.8}$$

and consider the projection of the operators (14.8) to the space of the variables p, q, and B :

$$\widetilde{E}_5 = \frac{\partial}{\partial p}, \quad \widetilde{E}_6 = q\frac{\partial}{\partial q} - B\frac{\partial}{\partial B}, \quad \widetilde{E}_7 = p\frac{\partial}{\partial p} + B\frac{\partial}{\partial B}, \quad \widetilde{E}_8 = \frac{\partial}{\partial q}\cdot \tag{14.9}$$

Applying to the Lie algebra spanned by (14.9) the standard approach to computation optimal systems and using the reflections $p \mapsto -p, q \mapsto -q$, we arrive at the following optimal system Θ_1 :

$$\widetilde{E}_6,\ \widetilde{E}_7 + \sigma\widetilde{E}_6,\ \widetilde{E}_6 - \widetilde{E}_5,\ \widetilde{E}_7 - \widetilde{E}_8,\ \widetilde{E}_5 - \varepsilon\widetilde{E}_8,\ \widetilde{E}_5,\ \widetilde{E}_8, \tag{14.10}$$

where $\varepsilon = \pm 1$ and σ is an arbitrary parameter. Proceedings as in Section 13.1, we obtain the following seven types of the function $B(p,q)$ with functional arbitrariness when the basic symmetries (14.7) of the system (L) are extended by one symmetry:

$$pf(p\mathrm{e}^q), \quad p^{1-\sigma}f(p^\sigma q^{-1}), \quad \frac{1}{q}f(q\mathrm{e}^p), \tag{14.11}$$

$$\frac{1}{q}f(p), \quad f(q), \quad f(p+\varepsilon q), \quad f(p).$$

In each case, we find the corresponding additional symmetry Z_5 by solving Eqs. (14.1)-(14.6) and arrive at the following result:

$$\begin{aligned}
&B = pf(p\mathrm{e}^q): && Z_5 = y\frac{\partial}{\partial y} + v\frac{\partial}{\partial v} + 2p\frac{\partial}{\partial p} - 2\frac{\partial}{\partial q}\,,\\
&B = p^{1-\sigma}f(p^\sigma q^{-1}): && Z_5 = (1-\sigma)y\frac{\partial}{\partial y} + (1+\sigma)v\frac{\partial}{\partial v} + 2p\frac{\partial}{\partial p} + 2\sigma q\frac{\partial}{\partial q}\,,\\
&B = \frac{1}{q}f(q\mathrm{e}^p): && X_5 = -y\frac{\partial}{\partial y} + v\frac{\partial}{\partial v} - 2\frac{\partial}{\partial p} + 2q\frac{\partial}{\partial q}\,,\\
&B = \frac{1}{q}f(p): && Z_5 = -y\frac{\partial}{\partial y} + v\frac{\partial}{\partial v} + 2q\frac{\partial}{\partial q}\,,\\
&f(q): && Z_5 = \frac{\partial}{\partial p}\,,\\
&B = f(p+\varepsilon q): && Z_5 = \frac{\partial}{\partial p} - \varepsilon\frac{\partial}{\partial q}\,,\\
&B = f(p) && Z_5 = \frac{\partial}{\partial q}\,.
\end{aligned} \tag{14.12}$$

It is plain to see that the possibilities listed in (14.12) coincide with the cases (13.12) of extension of symmetries for the system (I). This is due the fact that the operators (13.11) and (14.9) for the systems (I) and (L), respectively, provide different representatives of a basis of the same Lie algebra. Therefore we shall not use the optimal system Θ_2 since this does not lead to new results compared with the system (I). Instead, we will solve the determining equations (14.1)-(14.6) for the cases listed in (14.11). Note that all functions $B(p,q)$ from (14.11), with the exception of the last of them, obey the condition $B_q \neq 0$.

Therefore, let us first assume that $B_q \neq 0$. Then Eqs. (14.1)-(14.3) and the equation obtained by differentiation of (14.3) with respect to q yield:

$$\begin{gathered}
\xi^1 = \xi^1(s), \quad \xi^2 = \xi^2(y), \quad \eta = \eta(s,v),\\
\mu = \mu(y,p), \quad \lambda = \lambda(y,p,q), \quad \eta_s + \mu_y = 0.
\end{gathered} \tag{14.13}$$

Furthermore, differentiation of (14.5) with respect to p and v leads to the equations $\mu_{pp} = 0$ and $\eta_{vv} = 0$, respectively, whence

$$\eta(s,v) = \alpha(s)\,v + \beta(s), \quad \mu(y,p) = \gamma(y)\,p + \delta(y).$$

Substitution of these expressions into the last equation of (14.13) gives $\alpha'(s) = 0, \gamma'(y) = 0, \quad \beta'(s) + \delta'(y) = 0$. Now using (14.5) and (14.6) we get

$$\xi^1 = C_1 s + C_2, \quad \xi^2 = (C_3 + C_1)y + C_4,$$
$$\eta = C_5 v + C_6 s + C_7, \quad \mu = (C_3 + C_5)p - C_6 y + C_8, \tag{14.14}$$
$$\lambda = 2C_3 \frac{B}{B_q} - \frac{B_p}{B_q}\mu.$$

Finally, splitting Eq. (14.4) with respect to y we arrive at the following two classifying relations:

$$C_6 \left[\left(\frac{B_p}{B_q} \right)_q - B \left(\frac{B_p}{B_q} \right)_p \right] = 0, \tag{14.15}$$

$$[(C_3 + C_5)p + C_8] \left[\left(\frac{B_p}{B_q} \right)_q - B \left(\frac{B_p}{B_q} \right)_p \right] B_q^2 \tag{14.16}$$
$$+ (C_3 - C_5) B B_q \left(B_p - \frac{B_q}{B} \right) - 2C_3 B^2 \left(B_p - \frac{B_q}{B} \right)_q = 0.$$

Now we analyze the classifying relations (14.15)-(14.16) for each function $B(p,q)$ from (14.11) satisfying the condition $B_q \neq 0$. The corresponding supplementary symmetries are easily found from Eqs. (14.14) and are not given in the text. The reader can find them in Table 10. Note that the calculations are similar to all functions from (14.11) satisfying the condition $B_q \neq 0$. Therefore, we will consider in detail only two cases, $B = f(p)/q$ and $B = p^{1-\sigma} f(p^\sigma q^{-1})$, and will give the results for all cases in Table 10.

1°. Let $B = f(p)/q, f'(p) \neq 0$. Eqs. (14.15), (14.16) are written

$$C_6 u' = 0, \quad [(C_3 + C_5)p + C_8]u' + (C_3 + C_5)u = 0, \tag{14.17}$$

where $u = (f' + 1)/f$. If $f(p)$ is an arbitrary function, Eqs. (14.17) yield $C_6 = C_8 = 0, C_3 + C_5 = 0$. Then Eqs. (14.14) give one supplementary operator Z_5 which coincides with that listed in (14.12). Eqs. (14.17) show that further extension of symmetries is possible in the following two cases.

(i) Let $u' = 0$, i.e. $u = k$. Then C_6 and C_8 are arbitrary and Eqs. (14.17) reduce to $(C_3 + C_5)\, u = 0$. The latter equation yields that either

(i_1) $u = 0$ and C_3, C_5 are arbitrary, or

(i_2) $u \neq 0$ and $C_3 + C_5 = 0$.

For (i_1) we have $f = -p$ (up to addition of a constant to p); in this case the five-dimensional symmetry algebra, given by (14.7) and the appropriate operator Z_5 from (14.12), is extended by three operators.

For (i_2) the solution of the equation $u = k$ is $f = C\mathrm{e}^{kp} + \frac{1}{k}$ which can be reduced to $f = 1 + \varepsilon \mathrm{e}^p$, $\varepsilon = \pm 1$, by an equivalence transformation; in this case the symmetry algebra is extended by two operators.

(ii) Let $u' \neq 0$. Then Eqs. (14.17) show that $C_6 = 0$ and that extension of the symmetry algebra is possible if an equation of the form

$$(\alpha p + \beta)u' + \alpha u = 0$$

holds with constant coefficients α, β not simultaneously equal to zero. Since for $\alpha = 0$ we return to the case (i), we assume that $\alpha \neq 0$. Then, using a translation with respect to p, we reduce the above equation to $pu' + u = 0$ and obtain two new cases of extension of the algebra:

$$f = \gamma p + \delta p^{(\gamma+1)/\gamma}, \quad \gamma \neq 0, -1; \quad \delta = 0, \pm 1,$$

and

$$f = -p \ln p.$$

In both cases Eqs. (14.17) yield $C_8 = 0$, and hence the algebra extends by one operator.

2°. When $B = p^{1-\sigma} f(p^\sigma/q)$, Eqs. (14.15)-(14.16), after decomposition of (14.16) with respect to p, lead to the equations

$$C_6[(1-\sigma)fu' + \sigma u f'] = 0, \quad C_8[(1-\sigma)fu' + \sigma u f'] = 0, \tag{14.18}$$

$$[C_3(\sigma+1) + C_5(\sigma-1)]\left(\frac{u}{f}\right)' = 0, \tag{14.19}$$

where

$$u = (1-\sigma)f + \sigma z f' + z^2 \frac{f'}{f}, \quad f = f(z), \quad z = q^{-1}p^\sigma.$$

Extension of the algebra (14.7), (14.2) is possible in three cases:

$$u = 0, \quad u = kf \ (k \neq 0), \quad (1-\sigma)fu' + \sigma u f' = 0 \ (\sigma \neq 1).$$

(i) For $u = 0$ the constants C_3, C_5, C_6, C_8 are arbitrary. The solution to the equation $u = 0$ is written implicitly in the form

$$\left(1 + z^{-1} f\right)^{\sigma - 1} f = C.$$

(ii) If $u = kf$ then

$$\left[(k-1)z^{-1}f - 1\right]^{\sigma+k-1} u^{1-k} = C \quad \text{for} \quad k \neq 1,$$
$$\ln f + \sigma z^{-1} f = C \quad \text{for} \quad k = 1.$$

In this case $C_6 = C_8 = 0$ and C_3, C_5 are arbitrary.

(iii) The equation $(1-\sigma)fu' + \sigma f'u = 0$ after one integration gives

$$kf^{\sigma/(\sigma-1)} = (1-\sigma)f + \sigma z f' + z^2 \frac{f'}{f} \cdot$$

Any solution f of this differential equation leads to extension of the algebra by two operators since in this case C_6, C_8 are arbitrary, while C_3 and C_5 are related by $C_3(\sigma+1) + C_5(\sigma-1) = 0$ and give the operator Z_5 from (14.12).

3°. Let us consider the last function from (14.11), $B = f(p)$ by assuming $f'(p) \neq 0$. Solving Eq. (14.6) and invoking account (14.5), we have:

$$\mu = \frac{2f}{f'}(\xi^2_y - \xi^1_s), \tag{14.20}$$

whence $\mu_v = 0$. Differentiating (14.5) with respect to p and taking into account (14.3), we obtain $\mu_{pp} = 0$. This leads to the classifying equation

$$\left(\frac{2f}{f'}\right)''(\xi^2_y - \xi^1_s) = 0.$$

Let $\left(\frac{2f}{f'}\right)'' \neq 0$. Then $\xi^2_y = \xi^1_s$, whence, using Eqs. (14.1), (14.20), we get $\xi^1 = C_1 s + C_2, \xi^2 = C_1 y + C_3, \mu = 0$. Eqs. (14.1)-(14.3), (14.5) yield $\eta = C_4$. Furthermore, Eq. (14.4) yields the equation $\lambda_q = B\lambda_p$ whose solution, invoking (14.2), is written

$$\lambda = \theta\left(y, q + \int \frac{dp}{f(p)}\right),$$

where θ is an arbitrary function. It follows that the group admitted by the system (L) for $B = f(p)$, $\left(\frac{2f}{f'}\right)'' \neq 0$ is infinite: to the generators (14.7) of the basic group one adds

$$Z_\theta = \theta\left(y, q + \int \frac{dp}{f(p)}\right)\frac{\partial}{\partial q} \cdot \tag{14.21}$$

Let $\left(\frac{2f}{f'}\right)'' = 0$, i.e., $\frac{2f}{f'} = Kp + M$, where K and M are arbitrary constants. Using translation with respect to p, one can restrict oneself to the consideration of two cases, namely $K \neq 0, M = 0$ and $K = 0, M \neq 0$.

If $K \neq 0, M = 0$, then $f(p) = \varepsilon p^\sigma$ $(\varepsilon = \pm 1,\ \sigma \neq 0)$, and Eqs. (14.20) and (14.5) yield

$$\mu = \frac{2}{\sigma}\left(\xi^2_y - \xi^1_s\right) p, \quad \eta = \frac{\sigma - 2}{\sigma}\left(\xi^1_s - \xi^2_y\right) v + b(s, y),$$

where $b(s, y)$ is an arbitrary function. Substituting these expressions in Eqs. (14.2)-(14.3) and decomposing with respect to p and v, we obtain

$$\xi^1 = C_1 s + C_2, \quad \xi^2 = C_3 y + C_4, b = C_5.$$

Consequently, the above expressions for μ and η become

$$\mu = \frac{2}{\sigma}\left(C_3 - C_1\right) p, \quad \eta = \frac{\sigma - 2}{\sigma}(C_1 - C_3)v + C_5.$$

Furthermore, (14.4) leads to the equation

$$\lambda_q - C p^\sigma \lambda_p - \frac{2(\sigma - 1)}{\sigma}(C_1 - C_3) = 0$$

whose solution, in view of (14.2), is written

$$\lambda = \frac{2(\sigma - 1)}{\sigma}(C_1 - C_3)q + \theta\left(y, q + \int \frac{dp}{\varepsilon p^\sigma}\right),$$

where θ is an arbitrary function. It follows that the system (L) with $B = \varepsilon p^\sigma$ has, along with (14.7) and (14.21), the following additional symmetry:

$$Z_5 = \sigma y \frac{\partial}{\partial y} - (\sigma - 2) v \frac{\partial}{\partial v} + 2p \frac{\partial}{\partial p} - 2(\sigma - 1) q \frac{\partial}{\partial q} \cdot$$

If $K = 0, M \neq 0$, then $B = \varepsilon e^p$. Proceeding as above, one can show that in this case the symmetries (14.7), (14.21) are extended by

$$Z_5 = y \frac{\partial}{\partial y} - v \frac{\partial}{\partial v} + 2 \frac{\partial}{\partial p} - 2q \frac{\partial}{\partial q} \cdot$$

4°. Let $B =$ const. Using the dilation one can always assume that $B = \varepsilon$, where $\varepsilon = \pm 1$. Then Eqs. (14.5)-(14.6) yield $\xi^2_y = \xi^1_s$, i.e.

$$\xi^1 = C_1 s + C_2, \quad \xi^2 = C_1 y + C_3. \tag{14.22}$$

By virtue of (14.2) and (14.4), we have $(\varepsilon\mu + \lambda)_s = 0, (\varepsilon\mu + \lambda)_v = 0$, whence $\lambda + \varepsilon\mu = \theta(y, p, q)$ with an arbitrary function θ. Substituting this expression in (14.4) and invoking that $\mu_p = \eta_v$ we get the equation $\theta_q - \varepsilon\theta_p = 0$ with the general solution $\theta = \theta(y, p + \varepsilon q)$. Hence,

$$\lambda = -\varepsilon\mu + \theta(y, p + \varepsilon q). \tag{14.23}$$

Eqs. (14.1)-(14.6) provide the following system for determining η and μ :

$$\eta_s = -\mu_y, \quad \eta_y = -\varepsilon\mu_s, \quad \eta_v = \mu_p, \quad \eta_p = \varepsilon\mu_v, \quad \eta_q = \mu_q = 0. \tag{14.24}$$

Let us take $\varepsilon = 1$ and consider the compatibility conditions for Eqs. (14.24). Part of these conditions are written as the wave equations

$$\mu_{ss} - \mu_{yy} = 0, \quad \mu_{pp} - \mu_{vv} = 0,$$

$$\eta_{ss} - \eta_{yy} = 0, \quad \eta_{pp} - \eta_{vv} = 0,$$

whence

$$\mu = \mu^1(\alpha,\gamma) + \mu^2(\alpha,\delta) + \mu^3(\beta,\gamma) + \mu^4(\beta,\delta),$$
$$\eta = \eta^1(\alpha,\gamma) + \eta^2(\alpha,\delta) + \eta^3(\beta,\gamma) + \eta^4(\beta,\delta),$$

where $\alpha = s - y, \beta = s + y, \gamma = p - v, \delta = p + v$ and μ^i, η^i are arbitrary functions. The remaining compatibility conditions have the form

$$\mu_{ps} = -\mu_{vy}, \quad \mu_{vs} = -\mu_{py},$$

$$\eta_{ps} = -\eta_{vy}, \quad \eta_{vs} = -\eta_{py}$$

and impose additional restrictions on the forms of the functions μ^i, η^i. As a result we have

$$\begin{aligned} \eta &= \psi(s - y, p + v) - \varphi(s + y, p - v), \\ \mu &= \psi(s - y, p + v) + \varphi(s + y, p - v), \end{aligned} \tag{14.25}$$

where ψ and φ are arbitrary functions. Combining Eqs. (14.22), (14.23), and (14.25), we conclude that the system (L) with $B = 1$ has the symmetries (14.7), (14.21), and

$$Z_\varphi = \varphi(s + y, p - v)\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial p} + \frac{\partial}{\partial q}\right),$$

$$Z_\psi = \psi(s - y, p + v)\left(\frac{\partial}{\partial v} + \frac{\partial}{\partial p} + \frac{\partial}{\partial q}\right).$$

In the case $\varepsilon = -1$ the system (14.24), after introducing two complex variables $z_1 = y + is, z_2 = p + iv$, can be written as the condition

$$\frac{\partial\pi}{\partial\bar{z}_1} = 0, \quad \frac{\partial\pi}{\partial\bar{z}_2} = 0$$

of analyticity of the complex function $z(z_1, z_2) = \mu(z_1, z_2) + i\eta(z_1, z_2)$. Consequently, for $B = -1$ one adds to (14.7), (14.21) the following operator:

$$Z_\pi = \operatorname{Re}\pi\left(\frac{\partial}{\partial p} + \frac{\partial}{\partial q}\right) + \operatorname{Im}\pi\frac{\partial}{\partial v}\cdot$$

The results of the preliminary classification are summarized in Table 10.

§ 15 The complete group classification of the system (L)

The case $B_q = 0$, i.e. $B = f(p)$ has been studied completely in § 14. On the other hand, the case $B_q \neq 0$ is analyzed there only in special situations furnished by the method of preliminary group classification. Therefore, we will investigate here the determining equations for the case

$$B_q \neq 0 \tag{15.1}$$

and provide the complete group classification of the system (L).

As shown in § 14, the determining equations (14.1)-(14.6) under the condition (15.1) yield the equations

$$\begin{gathered}\xi^1 = C_1 s + C_2, \quad \xi^2 = (C_1 + C_3)y + C_4, \quad \eta = C_5 v + C_6 s + C_7,\\ \mu = (C_3 + C_5)p - C_6 y + C_8, \quad \lambda = 2C_3\frac{B}{B_q} - \frac{B_p}{B_q}\mu\end{gathered} \tag{15.2}$$

and the following two classifying relations:

$$C_6\left[\left(\frac{B_p}{B_q}\right)_q - B\left(\frac{B_p}{B_q}\right)_p\right] = 0, \tag{15.3}$$

$$\begin{gathered}[(C_3 + C_5)p + C_8]\left[\left(\frac{B_p}{B_q}\right)_q - B\left(\frac{B_p}{B_q}\right)_p\right]B_q^2 +\\ +(C_3 - C_5)BB_q\left(B_p - \frac{B_q}{B}\right) - 2C_3B^2\left(B_p - \frac{B_q}{B}\right)_q = 0.\end{gathered} \tag{15.4}$$

We introduce the notation

$$\mathcal{B} = B_p - \frac{B_q}{B} \tag{15.5}$$

and rewire Eqs. (15.3), (15.4) in the form

$$C_6(B_p\mathcal{B}_q - B_q\mathcal{B}_p) = 0, \tag{15.6}$$

$$[(C_3 + C_5)p + C_8](B_p\mathcal{B}_q - B_q\mathcal{B}_p) + (C_3 - C_5)B_q\mathcal{B} - 2C_3B\mathcal{B}_q = 0. \tag{15.7}$$

15.1 The case $B_p\mathcal{B}_q - B_q\mathcal{B}_p \neq 0$

In this case Eq. (15.6) yields $C_6 = 0$, and Eq. (15.7) shows that an extension of the basic symmetries (14.7) is possible if $B(p,q)$ obeys the equation

$$[(a+b)p+c](B_p\mathcal{B}_q - B_q\mathcal{B}_p) + (a-b)B_q\mathcal{B} - 2aB\mathcal{B}_q = 0 \tag{15.8}$$

with constant coefficients a, b, c, not all zero. The problem naturally splits into three cases: (i) $a+b \neq 0$, (ii) $a+b=0,\ c\neq 0$, and (iii) $a+b=0,\ c=0$.

(i) If $a+b\neq 0$ one can assume $a+b=1, c=0$ by using a dilation and translation of p, and hence carry (15.8) into the equation

$$p(B_p\mathcal{B}_q - B_q\mathcal{B}_p) + (2a-1)B_q\mathcal{B} - 2aB\mathcal{B}_q = 0. \tag{15.9}$$

Eq. (15.7), after substitution of the expression for $B_p\mathcal{B}_q - B_q\mathcal{B}_p$ obtained from (15.9), becomes:

$$\begin{gathered}[(2(a-1)C_3 + 2aC_5)p + 2aC_8]B\mathcal{B}_q \\ -[(2(a-1)C_3 + 2aC_5)p + (2a-1)C_8]B_q\mathcal{B} = 0.\end{gathered} \tag{15.10}$$

Eq. (15.10) shows that if the function $B(p,q)$ satisfies only (15.9) then

$$(a-1)C_3 + aC_5 = 0, \quad C_8 = 0. \tag{15.11}$$

In this case Eqs. (15.2) give the following supplementary operator to (14.7):

$$Z_5 = ay\frac{\partial}{\partial y} - (a-1)v\frac{\partial}{\partial v} + p\frac{\partial}{\partial p} + \frac{2aB - pB_p}{B_q}\frac{\partial}{\partial q}\,. \tag{15.12}$$

Further extension is possible if B satisfies, along with (15.9), an equation

$$(\alpha p + \beta)B\mathcal{B}_q - (\alpha p + \gamma)B_q\mathcal{B} = 0 \tag{15.13}$$

with constant coefficients α, β, γ, not all zero. Let us dwell on Eq. (15.13).

If $\alpha \neq 0$ one can assume $\alpha = 1,\ \beta = 0$ by using an equivalence transformation. Then Eq. (5.12) yields $\mathcal{B} = B^{1+\gamma/p}$. However this expression is in contradiction with the condition (15.1). Indeed, substitution of the above expression for $\mathcal{B}$ in Eq. (15.9) gives $B = \exp\left(2a + \frac{p^2 f'(p)}{\gamma f(p)}\right)$, i.e. $B = B(p)$.

Let $\alpha = 0$. If $\beta = 0$, then $\gamma \neq 0$ and Eq. (15.13) gives $\mathcal{B} = 0$ which does not satisfy the requirement $B_p\mathcal{B}_q - B_q\mathcal{B}_p \neq 0$. If $\beta \neq 0$ one can assume $\beta = 1$ and solve Eqs. (15.13), (15.9) to obtain

$$\mathcal{B} = kB^{\gamma}p^{2a(1-\gamma)-1}. \tag{15.14}$$

The expression (15.14) for $\mathcal{B}$ satisfies the requirement $B_p\mathcal{B}_q - B_q\mathcal{B}_p \neq 0$ provided that $2a(1-\gamma) - 1 \neq 0$. Then Eqs. (15.14) and (15.10) yield

$$(\gamma - 1)[aC_5 + (a-1)C_3] = 0, \quad C_8 = 0.$$

If $\gamma \neq 1$ we obtain the previous case (15.11) with one additional operator (15.12). The case $\gamma = 1$ leads to the extension of (14.7) by two operators,

$$Z_5 = y\frac{\partial}{\partial y} - v\frac{\partial}{\partial v} + \frac{2B}{B_q}\frac{\partial}{\partial q}, \quad Z_6 = v\frac{\partial}{\partial v} + p\frac{\partial}{\partial p} - \frac{pB_p}{B_q}\frac{\partial}{\partial q}\cdot$$

(ii) If $a + b = 0, c \neq 0$, one can assume $a = 1,\ b = -1,\ c = 2$, and write Eq. (15.8) in the form

$$B_p\mathcal{B}_q - B_q\mathcal{B}_p + B_q\mathcal{B} - B\mathcal{B}_q = 0. \tag{15.15}$$

Eq. (15.7), after substitution of the expression for $B_p\mathcal{B}_q - B_q\mathcal{B}_p$ obtained from (15.15), becomes

$$[(C_3+C_5)p + C_8 - 2C_3]B\mathcal{B}_q - [(C_3+C_1)p + C_8 + C_5 - C_3]B_q\mathcal{B} = 0. \tag{15.16}$$

If the function B satisfies only Eq. (15.15), then Eq. (15.16) yields

$$C_5 = -C_3, \quad C_8 = 2C_3, \tag{15.17}$$

and hence the symmetry algebra spanned by (14.7) is extended by one operator

$$Z_5 = y\frac{\partial}{\partial y} - v\frac{\partial}{\partial v} + 2\frac{\partial}{\partial p} + 2\left(\frac{B}{B_q} - \frac{B_p}{B_q}\right)\frac{\partial}{\partial q}\cdot \tag{15.18}$$

Further extension is possible if B, along with (15.15), satisfies Eq. (15.13).

The cases $\alpha \neq 0$ and $\alpha = 0, \beta = 0$ do not satisfy the requirements (15.1) and $B_p\mathcal{B}_q - B_q\mathcal{B}_p \neq 0$, respectively. This is proved as in the case (i) with Eq. (15.9) replaced by (15.15).

For $\alpha = 0, \beta \neq 0$ one can assume $\beta = 1$ and solve Eqs. (15.13), (15.15) to obtain

$$\mathcal{B} = kB^{\gamma}e^{(1-\gamma)p}. \tag{15.19}$$

The requirement $B_p\mathcal{B}_q - B_q\mathcal{B}_p \neq 0$ implies $\gamma \neq 1$. But then Eqs. (15.19) and (15.16) lead to the previous case (15.17).

(iii) Let $a + b = 0, c = 0$. Then Eq. (15.8) yields

$$\mathcal{B} = Bf(p). \tag{15.20}$$

The requirement $B_p\mathcal{B}_q - B_q\mathcal{B}_p \neq 0$ implies that $f'(p) \neq 0$. Furthermore, Eqs. (15.20) and (15.7) yield

$$[(C_3 + C_5)p + C_8]f'(p) + (C_3 + C_5)f(p) = 0.$$

It follows that if $f(p)$ is an arbitrary function, then $C_8 = 0, C_5 = -C_3$, and hence there is one supplementary operator

$$Z_5 = y\frac{\partial}{\partial y} - v\frac{\partial}{\partial v} + \frac{2B}{B_q}\frac{\partial}{\partial q}\cdot \tag{15.21}$$

Further extension is possible if the function $f(p)$ satisfies an equation

$$(\alpha p + \beta)f' + \alpha f = 0$$

with coefficients α, β, not both zero. If $\alpha \neq 0$ then $f = k\,p^{-1}$ up to an equivalence transformation, and hence (15.20) has the form (15.14) with $\gamma = 1$. The case $\alpha = 0$ does not satisfy the requirement $f'(p) \neq 0$.

15.2 The case $B_p\mathcal{B}_q - B_q\mathcal{B}_p = 0$

In this case the coefficients C_6, C_8 are arbitrary and Eq. (15.7) is written

$$(C_3 - C_5)B_q\mathcal{B} - 2C_3B\mathcal{B}_q = 0. \tag{15.22}$$

If B is an arbitrary function satisfying the equation $B_p\mathcal{B}_q - B_q\mathcal{B}_p = 0$ only, then Eq. (15.22) shows that $C_3 = C_5 = 0$. Hence the symmetries (14.7) are extended by two operators:

$$Z_5 = s\frac{\partial}{\partial v} - y\frac{\partial}{\partial p} + y\frac{B_p}{B_q}\frac{\partial}{\partial q}, \quad Z_6 = \frac{\partial}{\partial p} - \frac{B_p}{B_q}\frac{\partial}{\partial q}\cdot \tag{15.23}$$

Further extension is possible if the function B satisfies the equation

$$(a - b)B_q\mathcal{B} - 2aB\mathcal{B}_q = 0 \tag{15.24}$$

with coefficients a, b, not both zero.

If $a \neq 0$ we can assume $a = 1$ and, invoking that $B_p\mathcal{B}_q - B_q\mathcal{B}_p = 0$, obtain:

$$\mathcal{B} = B^{\frac{1-b}{2}}.$$

Substituting this expression in Eq. (15.22), we obtain $C_5 = bC_3$, i.e. the following supplementary operator to (14.7), (15.23):

$$Z_7 = y\frac{\partial}{\partial y} + bv\frac{\partial}{\partial v} + (b+1)p\frac{\partial}{\partial p} + \left(2\frac{B}{B_q} - \frac{B_p}{B_q}(b+1)p\right)\frac{\partial}{\partial q}\cdot$$

If $a = 0$, then we obtain from (15.24) the equation

$$\mathcal{B} = B_p - \frac{B_q}{B} = 0,$$

which corresponds to the maximal extension of symmetry algebra in the case $B_q \neq 0$. Namely, the six-dimensional Lie algebra spanned by (14.7), (15.23) is extended by the following two operators:

$$Z_7 = y\frac{\partial}{\partial y} - v\frac{\partial}{\partial v} + \frac{2}{B_p}\frac{\partial}{\partial q}, \quad Z_8 = v\frac{\partial}{\partial v} + p\frac{\partial}{\partial p} - \frac{p}{B}\frac{\partial}{\partial q}\cdot$$

The results of the complete group classification for the system (L) are collected in Table 7.

§ 16 Computation of quasi-local symmetries

We saw above that transition from Euler's variables to Lagrange's variables leads to considerable extension of symmetries in gas dynamics. This circumstance can be used for obtaining an information about "hidden" symmetries of the gasdynamic equations, e.g. in Euler's variables*.

The computation of hidden (quasi-local) symmetries of the sequence LIE is based on transition formulae given in the first chapter. If we denote the coordinates of the canonical operators for the systems (L), (I), and (E) by f_L, f_I and f_E, respectively, then the transition formulae (4.4) are written

$$f_L^v = f_I^v - \frac{v_x}{R_x} f_I^R, \quad f_L^p = f_I^p - \frac{p_x}{R_x}, f_I^R, \quad f_L^q = -\frac{1}{R_x} D_x\left(\frac{f_I^R}{R_x}\right) \qquad (16.1)$$

and

$$f_E^v = f_I^v, \quad f_E^p = f_I^p, \quad f_E^p = D_x\left(f_I^R\right). \qquad (16.2)$$

To apply them, one should use the following transformations of differentiation operators (4.5):

$$D_y = \frac{1}{R_x} D_x, \quad D_s = D_t - \frac{R_t}{R_x} D_x.$$

In particular, it follows that

$$v_y = \frac{v_x}{R_x}, \quad p_y = \frac{p_x}{R_x}, \quad q_y = -\frac{R_{xx}}{R_x^3}, \qquad (16.3)$$

*Passage from Euler's variables to Lagrange's variables leads to an extension of symmetries in problems of nonlinear thermal conductivity as well [110], [111]. Moreover, this happens also with the equations of planar flows of ideal incompressible fluids [11], [12].

$$v_s = v_t - \frac{R_t}{R_x} v_x, \quad p_s = p_t - \frac{R_t}{R_x} p_x, \quad q_s = -\frac{R_{tx}}{R_x^2} + \frac{R_t R_{xx}}{R_x^3} \cdot$$

To illustrate the transition formulae (16.1), let us carry the operators (14.7) admitted by the system (L) for an arbitrary function $B(p, q)$ into the corresponding operators for the system (I). As to the transition formulae (16.2), they do not require additional comments since they coincide with the prolongation formulae (2.9) on the derivative of R with respect to x.

The first operator from (14.7), $Z_1 = \partial/\partial s$, rewritten in the canonical representation, has the coordinates $f_L^v = v_s, f_L^p = p_s, f_L^q = q_s$. Substituting them in Eqs. (16.1) we obtain the following equations with respect to the coordinates f_I^v, f_I^p, f_I^R of the canonical operator admitted by the system (I):

$$f_I^v - \frac{v_x}{R_x} f_I^R = v_s, \quad f_I^p - \frac{p_x}{R_x} f_I^R = p_s, \quad D_x\left(\frac{f_I^R}{R_x}\right) = -q_s R_x.$$

We integrate the third equation by taking into account the equation $q_s = v_y$ from (L) and the equation $v_y R_x = v_x$ from (16.3), and obtain:

$$f_I^R = -v R_x + C(t) R_x,$$

where $C(t)$ is an arbitrary function. Substituting this expression into the first two equations, taking into account (L) and (16.3), we get

$$f_I^p = -\left[B \frac{v_x}{R_x} + v p_x\right] + C(t) p_x, \quad f_I^v = -\left[\frac{p_x}{R_x} + v v_x\right] + C(t) v_x.$$

Using the system (I), we finally arrive at the following expressions:

$$f_I^p = v_t + C(t) v_x, \quad f_I^p = p_t + C(t) p_x, \quad f_I^R = R_t + C(t) R_x.$$

The corresponding generator of a point transformation group is equal to

$$Y = \frac{\partial}{\partial t} + C(t) \frac{\partial}{\partial x}.$$

To determine the function $C(t)$, we write the condition of invariance of the system (I) with respect to the operator Y and obtain $C'(t) = 0$, i.e. $C = \text{const}$. Since the constant C is arbitrary, the operator Y "decomposes" into two linearly independent operators $\partial/\partial t$ and $\partial/\partial x$. Consequently, the symmetry $Z_1 = \partial/\partial s$ of the system (L) furnishes the following two symmetries for the system (I):

$$Y_1 = \frac{\partial}{\partial x}, \quad Y_2 = \frac{\partial}{\partial t} \cdot$$

For the operator $Z_2 = s\frac{\partial}{\partial s} + y\frac{\partial}{\partial y}$ the transition formulae (16.1) give

$$sv_s + yv_y = f_I^v - \frac{v_x}{R_x} f_I^R, \quad sp_s + yp_y = f_I^p - \frac{p_x}{R_x} f_I^R,$$

$$sq_s + yq_y = -\frac{1}{R_x} D_x \left(\frac{f_I^R}{R_x}\right).$$

Using (L) and (16.3), we rewrite the last equation of this system in the form

$$tv_x + D_x \left(\frac{R}{R_x}\right) - 1 = -D_x \left(\frac{f_I^R}{R_x}\right),$$

whence

$$f_I^R = -R + xR_x - tvR_x + C(t)R_x.$$

Substituting f_I^R in the first two equations and using (L) and (16.3) we get

$$f_I^v = xv_x - t\left(vv_x + \frac{p_x}{R_x}\right) + C(t)v_x,$$

$$f_I^p = xp_x - t\left(vp_x + B\frac{v_x}{R_x}\right) + C(t)p_x.$$

Using the system (I), we present the result as follows:

$$f_I^v = tv_t + xv_x + C(t)v_x,$$

$$f_I^p = tp_t + xp_x + C(t)p_x,$$

$$f_I^R = tR_t + [x + C(t)]R_x - R.$$

The invariance of the system (I) yields $C = \text{const}$. Omitting the immaterial term $C\frac{\partial}{\partial x}$, we conclude that the operator $Z_2 = s\frac{\partial}{\partial s} + y\frac{\partial}{\partial y}$ goes into

$$Y_3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + R\frac{\partial}{\partial R}\cdot$$

The operator $Z_3 = \partial/\partial v$ from (16.1) leads to $Y = C(t)\frac{\partial}{\partial x} + \frac{\partial}{\partial v}$. The invariance of the system (I) yields $C'(t) = 1$, i.e. $C(t) = t + \text{const.}$. Consequently, $Z_3 = \partial/\partial v$ goes into the generator

$$Y_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial v}$$

of the Galilean group. Finally, the operator $Z_4 = \partial/\partial y$ goes into the operator $Y_6 = \partial/\partial R$.

The following examples illustrate the procedure for calculating quasi-local symmetries given in Tables 8 and 9.

Example 5.6. Let $B = f(q)$. It is easy to show that $Z_5 = \partial/\partial p$ admitted in this case by the system (L) goes into $X_5 = \frac{\partial}{\partial p} + C\frac{\partial}{\partial x}$ admitted by the system (E). We dwell in more detail on the operator $Z_6 = s\frac{\partial}{\partial v} - y\frac{\partial}{\partial p}$. Substituting its coordinates $f_L^v = s$, $f_L^p = -y$, $f_L^q = 0$ into the transition formulae (16.1) and taking into account the relation between Lagrange's and Euler's variables, we get the following system of equations for f_I^v, f_I^p, f_I^R :

$$t = f_I^v - \frac{v_x}{R_x} f_I^R, \quad -R = f_I^p - \frac{p_x}{R_x} f_I^R, \quad 0 = -\frac{1}{R_x} D_x \left(\frac{f_I^R}{R_x} \right).$$

The solution of this system is given by

$$f_I^R = C(t)R_x, \quad f_I^p = -R + C(t)p_x, \quad f_I^v = t + C(t)v_x.$$

Now the transition formulae (16.2) and the relation $R_x = \rho$ yield the following coordinates of the canonical operator for the system (E):

$$f_E^v = t + C(t)v_x, \quad f_E^p = -R + C(t)p_x, \quad f_E^\rho = C(t)\rho_x.$$

The corresponding operator written in the variables t, x, v, p, ρ :

$$X = C(t)\frac{\partial}{\partial x} + t\frac{\partial}{\partial v} - R\frac{\partial}{\partial p}$$

is quasi-local symmetry for the system (E) with a nonlocal variable R defined by the equations (see the first equation of the system (I))

$$R_x = \rho, \quad R_t = -\rho v.$$

The invariance of the system (E) with respect to X yields $C'(t) = t$, i.e. $C(t) = \frac{1}{2}t^2 + \text{const}$. Omitting the immaterial term $C\frac{\partial}{\partial x}$, we conclude that the operator $Z_6 = s\frac{\partial}{\partial v} - y\frac{\partial}{\partial p}$ provides the nonlocal symmetry

$$X_6 = \frac{t^2}{2}\frac{\partial}{\partial x} + t\frac{\partial}{\partial v} - R\frac{\partial}{\partial p}$$

for the system (E). The corresponding group of nonlocal transformations

$$t' = t, \quad x' = x + \frac{1}{2}at^2, \quad v' = v + at, \quad p' = p - aR,$$

describes passage to a non-inertial coordinate system moving with a constant acceleration a.

In the following two examples we consider a polytropic gas, $B(p, q) = \frac{\gamma p}{q}$.

Example 5.7. If $\gamma = 3$, the system (E) admits the operator [101]

$$X_7 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + (x - tv)\frac{\partial}{\partial v} - 3tp\frac{\partial}{\partial p} - t\rho\frac{\partial}{\partial \rho}\cdot$$

Let us use the transition formulae (16.1)-(16.3) to get the corresponding operators for the systems (I) and (L). Substituting the coordinates

$$f_E^v = t(v + tv_t + xv_x) - x,$$

$$f_E^p = t(3p + tp_t + xp_x),$$
$$f_E^\rho = t(\rho + t\rho_t + x\rho_x)$$

of the canonical representation of the operator X_7 into (16.2) and noting that $f_E^\rho = D_x(tx\rho - t^2\rho v)$ by virtue of (E), we have

$$f_I^v = t(v + tv_t + xv_x) - x,$$

$$f_I^p = t(3p + tp_t + xp_x),$$
$$f_I^R = txR_x - t^2vR_x.$$

Using these expressions and Eqs. (16.1) and (16.3) we obtain

$$f_L^v = sv + s^2v_s - x, \quad f_L^p = 3sp + s^2p_s, \quad f_L^q = -sq + s^2q_s.$$

In consequence we arrive at the local symmetry

$$Y_8 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + (x - tv)\frac{\partial}{\partial v} - 3tp\frac{\partial}{\partial p}$$

for the system (I) and the quasi-local symmetry

$$Z_7 = s^2\frac{\partial}{\partial s} + (x - sv)\frac{\partial}{\partial v} - 3sp\frac{\partial}{\partial p} + sq\frac{\partial}{\partial q}$$

for the system (L), where the nonlocal variable x is defined by the equations

$$x_v = q, \quad x_s = v.$$

Example 5.8. Let $\gamma = -1$ (the Chaplygin gas). The system (L) admits the operator (see Table 10)

$$Z_7 = s\frac{\partial}{\partial v} - y\frac{\partial}{\partial p} - \frac{yq}{p}\frac{\partial}{\partial q}\cdot$$

In the present case the transition formulae (16.1) are written

$$t = f_I^v - \frac{v_x}{R_x} f_I^R, \quad -R = f_I^p - \frac{p_x}{R_x} f_I^R, \quad \frac{R}{p} = D_x\left(\frac{f_I^R}{R_x}\right). \tag{16.4}$$

Introducing the nonlocal variable Q defined by $D_x(Q) = R/p$ we obtain from Eq. (16.4):

$$f_I^R = R_x Q, \quad f_I^p = -R + p_x Q, \quad f_I^v = t + v_x Q.$$

Now Eqs. (16.2) yield

$$f_E^v = t + v_x Q, \quad f_E^p = -R + p_x Q, \quad f_E^\rho = \frac{\rho R}{p} + \rho_x Q,$$

whence

$$X_7 = -Q\frac{\partial}{\partial x} + t\frac{\partial}{\partial v} - R\frac{\partial}{\partial p} + \frac{\rho R}{p}\frac{\partial}{\partial \rho}\cdot \tag{16.5}$$

The dependence of Q on t is determined by the equation $Q_t + \frac{1}{p}vR + t = 0$ which is the condition of invariance of the system (E) with respect to the operator (16.5). Thus, the gasdynamic equations for the Chaplygin gas in Euler's variables admit the quasi-local symmetry (16.5) with the nonlocal variables R and Q determined by the integrable system

$$R_x = \rho, \quad R_t = -\rho v,$$

$$Q_x = \frac{R}{p}, \quad Q_t = -\frac{vR}{p} - t.$$

Example 5.9. Let $B = f(p)$. In this case the system (L) admits the canonical operator

$$X_\theta = \theta\left(y, q + \int \frac{dp}{f(p)}\right)\frac{\partial}{\partial q} \tag{16.6}$$

with arbitrary function θ. According to (16.1), the coordinates of the corresponding operator Y_θ for the system (I) are determined by the equations

$$D_x\left(\frac{f_I^R}{R_x}\right) = -R_x\theta\left(R, \frac{1}{R_x} + \int\frac{dp}{f(p)}\right),$$

$$f_I^v = \frac{v_x}{R_x} f_I^R, \quad f_I^p = \frac{p_x}{R_x} f_I^R$$

and have the form $f_I^R = -R_x\Theta,\ f_I^p = -p_x\Theta,\ f_I^v = -v_x\Theta$, i.e.

$$Y_\theta = \Theta\frac{\partial}{\partial x}, \tag{16.7}$$

where Θ is a nonlocal variable satisfying the condition $D_x(\Theta) = \theta R_x$. Furthermore, Eqs. (16.1) yield

$$X_\theta = \Theta\frac{\partial}{\partial x} - \rho^2\theta\frac{\partial}{\partial \rho}\cdot \tag{16.8}$$

The condition of invariance of the system (E) with respect to the operator (16.8) yields the equation $\Theta_t + v\Theta_x = 0$. Thus, the symmetry (16.6) of the system (L) leads to the quasi-local symmetries (16.7) and (16.8) for the systems (I) and (E), respectively, where Θ is a nonlocal variable determined by the equations

$$\Theta_x = \rho\theta, \quad \Theta_t + v\Theta_x = 0. \tag{16.9}$$

All quasi-local symmetries from Tables 8 and 9 can be calculated similarly. The additional nonlocal variables given in these tables are defined by the following equations:

$$P_x(a) = \frac{2aB - pB_p}{B_q}R_x + 2a - 1, \quad P_t(a) + vP_x(a) = 0;$$

$$U_x = \frac{B_p}{B_q}RR_x, \quad U_t + vU_x = 0;$$

$$V_x = 2 + 2\frac{B - B_p}{B_q}R_x, \quad V_t + vV_x = 0;$$

$$W_x = \frac{B_p}{B_q}R_x, \quad W_t + vW_x = 0.$$

Remark 5.7. Quasi-local symmetries can degenerate into local symmetries. For example, in the case $B = f(p)$ we have associated with the symmetry (16.6) of the system (L) the quasi-local symmetry (16.7) for the system (I). At the same time the system (I) admits the infinite group of point transformations with the generator

$$Y_\tau = \tau(R)\frac{\partial}{\partial x},$$

where $\tau(R)$ is an arbitrary function. The latter is a special case of the operator Y_θ for $\theta = \theta(y)$. Similar situations arise in the case $B = \varepsilon$ as well.

The calculation of all quasi-local symmetries of the equations of the sequence LIE and their systematization singles out *thirteen basic types* of functions $B(p, q)$ for which the main symmetry algebras are extended by point or quasi-local symmetries. These types are given in Table 9 where the numbering of the types is done according to the principle of increasing dimension of the symmetry algebra. Quasi-local symmetries are indicated by shading and the equations defining the nonlocal variables are given in § 19. Note that the majority of the basic types are determined by differential equations with respect to the function $B(p,q)$.

§ 17 Nonlocal symmetries of first generation

Suppose that for a given sequence of equations connected by Bäcklund transformations all quasi-local symmetries have been calculated. Any quasi-local symmetry contains a nonlocal variable. We call these symmetries and the corresponding variables *quasi-local symmetries and nonlocal variables of the first generation.* In each specific case the question arises of the existence of nonlocal symmetries (depending on nonlocal variables of the first generation) which are different from quasi-local symmetries. To answer this question we include all non-local variables of the first generation in the set of differential variables, add the equations defining these nonlocal variables to the original systems of differential equations of the sequence in question and then calculate the point transformation groups admitted by the extended systems. If this procedure leads to extension of the group we shall speak of nonlocal symmetries of the first generation of the original (unextended) systems of equations.

Example 5.10. Let us find a nonlocal symmetry of the first generation for the Chaplygin gas ($B(p,q) = -pq^{-1}$). We consider the system (I) with $B = -pR_x^{-1}$ and supplement it by the equations

$$U_t - \frac{vR}{p} = 0, \quad U_x + \frac{R}{p}, \quad W_t - \frac{v}{p} = 0, \quad W_x + \frac{1}{p} = 0 \tag{17.1}$$

defining nonlocal variables U and W. The group of point transformations admitted by the extended system (I), (17.1) is generated by the operators

$$Y_1 = \frac{\partial}{\partial x}, \quad Y_2 = \frac{\partial}{\partial t}, \quad Y_3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + R\frac{\partial}{\partial R} - 2U\frac{\partial}{\partial U},$$

$$Y_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial v}, \quad Y_5 = \frac{\partial}{\partial R} + W\frac{\partial}{\partial U}, \quad Y_6 = x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} - R\frac{\partial}{\partial R} + W\frac{\partial}{\partial W},$$

$$Y_7 = R\frac{\partial}{\partial R} + p\frac{\partial}{\partial p} - W\frac{\partial}{\partial W}, \quad Y_8 = \left(\frac{1}{2}t^2 + U\right)\frac{\partial}{\partial x} + t\frac{\partial}{\partial v} - R\frac{\partial}{\partial p},$$

$$Y_9 = W\frac{\partial}{\partial x} + \frac{\partial}{\partial p}, \quad Y_{10} = tW\frac{\partial}{\partial x} + W\frac{\partial}{\partial v} - t\frac{\partial}{\partial p}, \quad Y_{11} = \frac{\partial}{\partial U}, \quad Y_{12} = \frac{\partial}{\partial W}.$$

Comparison with Table 8 shows that a nonlocal symmetry of the first generation corresponds to the operator Y_{10} which in Lagrange's and Euler's variables assumes the form

$$Z_9 = -W\frac{\partial}{\partial v} + \frac{tq}{p}\frac{\partial}{\partial q} + t\frac{\partial}{\partial p}\,, \quad X_9 = tW\frac{\partial}{\partial x} + W\frac{\partial}{\partial v} - t\frac{\partial}{\partial p} + \frac{t\rho}{p}\frac{\partial}{\partial \rho}\,.$$

We arrive at the same result if we begin with the system (L).

§ 18 Second generation of quasi-local symmetries

Now we apply to the extended systems described above the technique of construction of quasi-local symmetries, namely, with the help of transition formulae we construct quasi-local symmetries associated with the nonlocal symmetries of the first generation. If in this way we arrive at quasi-local symmetries generated by new nonlocal variables, we will call the quasi-local symmetries and nonlocal variables of the second generation. Iterating the procedure one can construct a hierarchy of nonlocal variables and symmetries. In the preceding example this procedure breaks off at the first step. The following example illustrates the appearance of quasi-local symmetries of the second generation.

Example 5.11. The system (I) of the sequence LIE in the case $B = f(p)$ admits the quasi-local symmetry (16.7),

$$Y_\theta = \Theta\frac{\partial}{\partial x}\,,$$

with the nonlocal variable Θ defined by the equations

$$\Theta_x = R_x\theta, \quad \Theta_t + v\Theta_x = 0, \tag{18.1}$$

where $\theta = \theta\left(R, \frac{1}{R_x} + \int\frac{dp}{f(p)}\right)$ is an arbitrary function of two variables. In Lagrange's variables the system (18.1) is written

$$\Theta_y = \theta\left(y, q + \int\frac{dp}{f(p)}\right), \quad \Theta_s = 0. \tag{18.2}$$

Let us consider the extended system comprising Eqs. (L) and (18.2):

$$q_s - v_y = 0, \quad v_s + p_y = 0, \quad p_s + f(p)v_y = 0,$$

$$\Theta_y = \theta\left(y, q + \int\frac{dp}{f(p)}\right), \quad \Theta_s = 0 \tag{18.3}$$

and seek an operator admitted by this system in the form

$$Z = \xi^1 \frac{\partial}{\partial s} + \xi^2 \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial v} + \mu \frac{\partial}{\partial p} + \lambda \frac{\partial}{\partial q} + \nu \frac{\partial}{\partial \Theta},$$

where $\xi^1, \xi^2, \eta, \mu, \lambda, \nu$ depend on s, y, v, p, q, Θ. A nontrivial quasi-local symmetry for the extended sequence LIE is given by

$$Z = \frac{1}{\theta_2}(\nu_y + \theta \nu_\theta) \frac{\partial}{\partial q} + \nu \frac{\partial}{\partial \Theta}$$

provided that $\theta_2 \neq 0$, where θ_2 denotes the derivative of θ with respect to the second argument, and $\nu = \nu(y, \Theta)$ is an arbitrary function. To construct quasi-local symmetries of the second generation, it is necessary to carry the operator Z into the corresponding operators Y and X by means of the transition formulae (16.1)-(16.3). Then one obtains the operators

$$Y = \Lambda \frac{\partial}{\partial x}, \quad X = \Lambda \frac{\partial}{\partial x} - \frac{\rho^2}{\theta_2}(\nu_R + \theta \nu_\Theta) \frac{\partial}{\partial \rho}$$

with a new non1ocal variable Λ defined by the equation

$$\Lambda_x = \frac{1}{\theta_2} D_x(\nu), \quad \Lambda_t + v \Lambda_x = 0.$$

Remark 5.8. If $\theta_2 = 1$ then $\Lambda = \nu(R, \Theta)$, and hence

$$Y = \nu(R, \Theta) \frac{\partial}{\partial x} \cdot$$

This operator is a new nonlocal symmetry which generalizes the symmetry (16.7), $Y_\theta = \Theta \frac{\partial}{\partial x}$ (cf. Tables 8 and 9) and furnishes a nonlocal symmetry of the first generation.

§ 19 Tables to Chapter 3

1. In the classification tables we adopt the following convention: at the beginning of each "block" the basic case is given and below with translation to the right we indicate its subcases when further extension of the group occurs. If there are several subcases they are numbered (i), (ii), etc. For example, the next to the last block of Table 5 means that for $B = \rho f(p)$ the system (E) admits one supplementary operator X_5 if $f(p)$ is an arbitrary function, two supplementary operators X_5, X_6 if $f = \gamma p$, and three operators X_5, X_6, X_7 if $\gamma = 3$.

2. k, l, σ, γ are arbitrary real constants; $\varepsilon = \pm 1, \delta = 0, \pm 1$.

3. $\mathcal{B} = B_p - \dfrac{B_q}{B}$.

4. $\pi = \pi(y + is, p + iv)$ is an arbitrary analytic function of two complex variables, $\omega(t, R)$ is an arbitrary harmonic function.

5. Shading indicates quasi-local symmetries.

6. Nonlocal variables are defined by the following equations:

$$\begin{cases} U_x = \rho R B_p B_q^{-1}, \\ U_t + vU_x = 0; \end{cases} \quad \begin{cases} V_x = 2\big[1 + \rho(B - B_p)B_q^{-1}\big], \\ V_t + vV_x = 0; \end{cases}$$

$$\begin{cases} W_x = \rho B_p B_q^{-1}, \\ W_t + vW_x = 0; \end{cases} \quad \begin{cases} \Phi_x = -\rho\varphi, \\ \Phi_t + v\Phi_x + \varphi = 0; \end{cases}$$

$$\begin{cases} \Psi_x = \rho\psi, \\ \Psi_t + v\Psi_x + \psi = 0; \end{cases} \quad \begin{cases} \Theta_x = \rho\theta, \\ \Theta_t + v\Theta_x = 0; \end{cases}$$

$$\begin{cases} \Pi_x = \rho \operatorname{Re}\pi, \\ \Pi_t + v\Pi_x - \operatorname{Im}\pi = 0; \end{cases} \quad \begin{cases} P_x(a) = \rho(2aB - pB_p)B_q^{-1} + 2a - 1, \\ P_t(a) + vP_x(a) = 0. \end{cases}$$

It is assumed that one substitutes in these equations the values of the function $B(p, q)$ from the left column of the corresponding table. Furthermore, φ, ψ and θ are arbitrary functions of two real variables.

Table 4: The LIE sequence

	System of equations	Equivalence transformation	
E	$\rho_t + v\rho_x + \rho v_x = 0,$ $\rho(v_t + vv_x) + p_x = 0,$ $\rho(p_t + vp_x) + B(p, \rho^{-1})v_x = 0.$	$\tilde{t} = \alpha_1 t + \gamma_1,\ \tilde{x} = \alpha_2 x + \beta_1 t$ $\boldsymbol{+\beta_2 R} + \gamma_2,$ $\tilde{v} = \frac{\alpha_2}{\alpha_1} v + \frac{\beta_1}{\alpha_1},\ \tilde{p} = \alpha_3 p + \gamma_3,$ $\tilde{\rho} = \frac{\alpha_1^2\alpha_3\rho}{\alpha_2^2 \boldsymbol{+\alpha_2\beta_2\rho}},\ \tilde{B} = \frac{\alpha_1^2\alpha_3^2}{\alpha_2^2} B.$	(E) ↑ $\rho = R_x$
I	$R_t + vR_x = 0,$ $R_x(v_t + vv_x) + p_x = 0,$ $R_x(p_t + vp_x) + B(p, R_x^{-1})v_x = 0.$	$\tilde{t} = \alpha_1 t + \gamma_1,$ $\tilde{x} = \alpha_2 x + \beta_1 t + \beta_2 R + \gamma_2,$ $\tilde{v} = \frac{\alpha_2}{\alpha_1} v + \frac{\beta_1}{\alpha_1},\ \tilde{p} = \alpha_3 p + \gamma_3,$ $\tilde{R} = \frac{\alpha_1^2\alpha_3}{\alpha_2} R + \gamma_4,\ \tilde{B} = \frac{\alpha_1^2\alpha_3^2}{\alpha_2^2} B.$	(I)
L	$q_s - v_y = 0,$ $v_s + p_y = 0,$ $p_s + B(p, q)v_y = 0$	$\tilde{s} = \alpha_1 s + \gamma_1,$ $\tilde{y} = \frac{\alpha_1^2\alpha_3}{\alpha_2} y + \gamma_4,$ $\tilde{v} = \frac{\alpha_2}{\alpha_1} v + \frac{\beta_1}{\alpha_1},$ $\tilde{p} = \alpha_3 p + \gamma_3,$ $\tilde{q} = \frac{\alpha_2^2}{\alpha_1^2\alpha_3} q + \frac{\alpha_2\beta_2}{\alpha_1^2\alpha_3},$ $\tilde{B} = \frac{\alpha_1^2\alpha_3^2}{\alpha_2^2} B.$	$s = t,\ y = R,\ q = R_x^{-1}$ ↓ (L)

Table 5: Classification of equations (E)

$B(p, \rho^{-1})$	Type	Admissible operators
Arbitrary functions		$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$ $X_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial v}$
$p^{1-\sigma} f(\rho p^{\sigma})$	I	$X_5 = -(\sigma+1)t\frac{\partial}{\partial t} + (\sigma+1)v\frac{\partial}{\partial v} + 2p\frac{\partial}{\partial p} - 2\sigma\rho\frac{\partial}{\partial \rho}$
$\rho f(\rho^{-1} e^{p})$	II	$X_5 = x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} - 2\rho\frac{\partial}{\partial \rho} - 2\frac{\partial}{\partial p}$
$\rho f(p)$ $f = \gamma p$ $f = 3p$	III IV VII	$X_5 = x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} - 2\rho\frac{\partial}{\partial \rho}$ $X_6 = p\frac{\partial}{\partial p} + \rho\frac{\partial}{\partial \rho}$ $X_7 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} - 3tp\frac{\partial}{\partial p} + (x - tv)\frac{\partial}{\partial v} - t\rho\frac{\partial}{\partial \rho}$
$f(\rho)$ $f = \varepsilon\rho^{\sigma+1}$	V VI	$X_5 = \frac{\partial}{\partial p}$ $X_6 = (\sigma-1)t\frac{\partial}{\partial t} - (\sigma-1)v\frac{\partial}{\partial v} - 2\sigma p\frac{\partial}{\partial p} - 2\rho\frac{\partial}{\partial \rho}$

Table 6: Classification of equations (I)

$B(p, R_x^{-1})$	Type	Admissible operators
Arbitrary function		$Y_1 = \frac{\partial}{\partial x}$ $Y_2 = \frac{\partial}{\partial t}$ $Y_3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + R\frac{\partial}{\partial R}$ $Y_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial v}$ $Y_5 = \frac{\partial}{\partial R}$
$pf(pe^{R_x^{-1}})$	I	$Y_5 = t\frac{\partial}{\partial t} + 2R\frac{\partial}{\partial x} - v\frac{\partial}{\partial v} - 2p\frac{\partial}{\partial p}$
$p^{1-\sigma}f(R_x p^\sigma)$	I	$Y_5 = -(\sigma+1)t\frac{\partial}{\partial t} + (\sigma+1)v\frac{\partial}{\partial v} - 2\sigma R\frac{\partial}{\partial R} + 2p\frac{\partial}{\partial p}$
$R_x f(R_x^{-1}e^p)$	II	$Y_6 = x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} - R\frac{\partial}{\partial R} - 2\frac{\partial}{\partial p}$
$R_x f(p)$ $(i)\ f = \gamma p$ $f = 3p$ $(ii)\ f = 1$	III IV VII VI	$Y_6 = x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} - R\frac{\partial}{\partial R}$ $Y_7 = x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} + p\frac{\partial}{\partial p}$ $Y_8 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + (x - tv)\frac{\partial}{\partial v} - 3tp\frac{\partial}{\partial p}$ $Y_7 = \frac{\partial}{\partial p}$ $Y_8 = \frac{1}{2}t^2\frac{\partial}{\partial x} + t\frac{\partial}{\partial v} - R\frac{\partial}{\partial p}$
$f(R_x)$ $(i)\ \varepsilon e^{R_x^{-1}}$ $(ii)\ \varepsilon R_x^{\sigma+1}$	V VI VI	$Y_6 = \frac{\partial}{\partial p}$ $Y_7 = t^2\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial v} - 2R\frac{\partial}{\partial p}$ $Y_8 = t\frac{\partial}{\partial t} - 2R\frac{\partial}{\partial x} - v\frac{\partial}{\partial v} - 2p\frac{\partial}{\partial p}$ $Y_8 = (\sigma-1)t\frac{\partial}{\partial t} - (\sigma-1)v\frac{\partial}{\partial v} - 2\sigma p\frac{\partial}{\partial p} - 2R\frac{\partial}{\partial R}$
$f(p + \varepsilon R_x^{-1})$	V	$Y_6 = \varepsilon R\frac{\partial}{\partial x} - \frac{\partial}{\partial p}$ $Y_7 = (t^2 + \varepsilon R^2)\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial v} - 2R\frac{\partial}{\partial p}$
$f(p)$ $(i) \varepsilon e^p$ $(ii)\ \varepsilon p^\sigma,\ \sigma \neq 0$ $(iii)\ 1$ $(iv)\ -1$	IX X XI XII XIII	$Y_\tau = \tau(R)\frac{\partial}{\partial x}$ $Y_6 = x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} - R\frac{\partial}{\partial R} - 2\frac{\partial}{\partial p}$ $Y_5 = (2-\sigma)x\frac{\partial}{\partial x} + (2-\sigma)v\frac{\partial}{\partial v} + \sigma R\frac{\partial}{\partial R} + 2p\frac{\partial}{\partial p}$ $Y_\kappa = \kappa(t+R)\frac{\partial}{\partial x} + \kappa'(t+R)\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial p}\right)$ $Y_\chi = \chi(t-R)\frac{\partial}{\partial x} + \kappa'(t-R)\left(\frac{\partial}{\partial v} + \frac{\partial}{\partial p}\right)$ $Y_\omega = \omega(t,R)\frac{\partial}{\partial x} + \omega_t(t,R)\frac{\partial}{\partial v} + \omega_R(t,R)\frac{\partial}{\partial p}$

Table 7: Classification of equations (L)

$B(p,q)$	Type	Admissible operators
Arbitrary function		$Z_1 = \frac{\partial}{\partial s}$ $Z_2 = s\frac{\partial}{\partial s} + y\frac{\partial}{\partial y}$ $Z_3 = \frac{\partial}{\partial v}$ $Z_4 = \frac{\partial}{\partial y}$
$(2a-1)B_q\mathcal{B} - 2aB\mathcal{B}_q$ $+p(B_p\mathcal{B}_q - B_q\mathcal{B}_p) = 0$	I	$Z_5 = ay\frac{\partial}{\partial y} + (1-a)v\frac{\partial}{\partial v} + p\frac{\partial}{\partial p}$ $+(2aB - pB_p)B_q^{-1}\frac{\partial}{\partial q}$
$B_p\mathcal{B}_q - B_q\mathcal{B}_p$ $+B_q\mathcal{B} - B\mathcal{B}_q = 0$	II	$Z_5 = y\frac{\partial}{\partial y} - v\frac{\partial}{\partial v} + 2\frac{\partial}{\partial p}$ $+2(B - B_p)B_q^{-1}\frac{\partial}{\partial q}$
$\mathcal{B} = Bf(p),\ f'(p) \neq 0$	III	$Z_5 = \frac{\partial}{\partial v} - v\frac{\partial}{\partial v} + 2BB_q^{-1}\frac{\partial}{\partial q}$
$\mathcal{B} = \frac{kB}{p},\ k \neq 0$	IV	$Z_6 = v\frac{\partial}{\partial v} + p\frac{\partial}{\partial p} - pB_pB_q^{-1}\frac{\partial}{\partial q}$
$B_p\mathcal{B}_q - B_q\mathcal{B}_p = 0$	V	$Z_5 = s\frac{\partial}{\partial v} - y\frac{\partial}{\partial p} + yB_pB_q^{-1}\frac{\partial}{\partial q}$ $Z_6 = \frac{\partial}{\partial p} - -B_pB_q^{-1}\frac{\partial}{\partial q}$
$\mathcal{B} = B^{(1-b)/2}$	VI	$Z_7 = y\frac{\partial}{\partial y} + bv\frac{\partial}{\partial v} + (b+1)p\frac{\partial}{\partial p}$ $+(2BB_q^{-1} - (b+1)pB_pB_q^{-1})\frac{\partial}{\partial q}$
$\mathcal{B} = 0$	VIII	$Z_8 = v\frac{\partial}{\partial v} + p\frac{\partial}{\partial p} - pB^{-1}\frac{\partial}{\partial q}$
$f(p)$	IX	$Z_\theta = \theta\left(y, q + \int\frac{dp}{f(p)}\right)\frac{\partial}{\partial q}$
$(i)\ \varepsilon e^p$	X	$Z_5 = y\frac{\partial}{\partial y} - v\frac{\partial}{\partial v} + 2\frac{\partial}{\partial p} - 2q\frac{\partial}{\partial q}$
$(ii)\ \varepsilon p^\sigma,\ \sigma \neq 0$	XI	$Z_5 = \sigma y\frac{\partial}{\partial y} + (2-\sigma)v\frac{\partial}{\partial v} + 2p\frac{\partial}{\partial p}$ $-2(\sigma-1)q\frac{\partial}{\partial q}$
$(iii)\ 1$	XII	$Z_\varphi = \varphi(s+y, p-v)\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial p} + \frac{\partial}{\partial q}\right)$ $X_\psi = \psi(s-y, p+v)\left(\frac{\partial}{\partial v} + \frac{\partial}{\partial p} - \frac{\partial}{\partial q}\right)$
$(iv)\ -1$	XIII	$Z_\pi = Re\pi\left(\frac{\partial}{\partial p} + \frac{\partial}{\partial v}\right) + Im\pi\frac{\partial}{\partial v}$

Table 8: Point and quasi-local symmetries of the sequence LIE

$B(p,q)$	Type	Admissible operators		
		L	I	E
Arbitrary		–	$Y_1 = \frac{\partial}{\partial x}$	$X_1 = \frac{\partial}{\partial x}$
function		$Z_1 = \frac{\partial}{\partial s}$	$Y_2 = \frac{\partial}{\partial t}$	$X_2 = \frac{\partial}{\partial t}$
		$Z_2 = s\frac{\partial}{\partial s} + y\frac{\partial}{\partial y}$	$Y_3 = t\frac{\partial}{\partial x} + x\frac{\partial}{\partial x} + R\frac{\partial}{\partial R}$	$X_3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$
		$Z_3 = \frac{\partial}{\partial v}$	$Y_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial v}$	$X_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial v}$
		$Z_4 = \frac{\partial}{\partial y}$	$Y_5 = \frac{\partial}{\partial R}$	–
$pf(p\varepsilon^q)$	I	$Z_5 = -s\frac{\partial}{\partial s} - v\frac{\partial}{\partial v} + 2p\frac{\partial}{\partial p} - 2\frac{\partial}{\partial q}$	$Y_6 = -t\frac{\partial}{\partial t} - 2R\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} + 2p\frac{\partial}{\partial p}$	$X_5 = -t\frac{\partial}{\partial t} - 2R\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} + 2p\frac{\partial}{\partial p} + 2\rho^2\frac{\partial}{\partial \rho}$
$(i)(k-f)^{1-k} =$	VI	$Z_6 = -s\frac{\partial}{\partial p} - \frac{k-1}{(k-f)p}\frac{\partial}{\partial q}$	$Y_7 = -w\frac{\partial}{\partial x} + \frac{\partial}{\partial p}$	$X_6 = -w\frac{\partial}{\partial x} + \frac{\partial}{\partial p} B_p B^{-1}\rho^2\frac{\partial}{\partial \rho}$
$= lfz,\ k \neq 0,1$		$Z_7 = \frac{\partial}{\partial v} - y\frac{\partial}{\partial p} + \frac{y}{p}\frac{k-1}{k-t}\frac{\partial}{\partial q}$	$Y_8 = \left(\frac{t^2}{2} + U\right)\frac{\partial}{\partial x} + t\frac{\partial}{\partial v} - R\frac{\partial}{\partial p}$	$X_7 = \left(\frac{t^2}{2} + U\right)\frac{\partial}{\partial x} + t\frac{\partial}{\partial v} - R\frac{\partial}{\partial p} - U_x\rho\frac{\partial}{\partial \rho}$
$(ii)\ f^{-1} + \ln f = l +$	V	$Z_6 = y\frac{\partial}{\partial y} + p\frac{\partial}{\partial p} + \frac{(2-k)f-1}{(k-1)f}\frac{\partial}{\partial q}$	$Y_7 = p(1)\frac{\partial}{\partial x} + p\frac{\partial}{\partial p} + R\frac{\partial}{\partial R}$	$X_6 = p(1)\frac{\partial}{\partial x} + p\frac{\partial}{\partial p} + (1 - P_x(1))\rho\frac{\partial}{\partial \rho}$
$+(k-1)\ln z,\ k \neq 0$				
$k = 0,$ i.e.	VIII	$Z_7 = \frac{\partial}{\partial p} - \frac{1}{pf}\frac{\partial}{\partial q}$	$Y_8 = -w\frac{\partial}{\partial x} + \frac{\partial}{\partial p}$	$X_7 = -w\frac{\partial}{\partial x} + \frac{\partial}{\partial p} + B_p B_q^{-1}\rho^2\frac{\partial}{\partial \rho}$
$e^{1/f} = l/(zf)$		$Z_8 = s\frac{\partial}{\partial v} - y\frac{\partial}{\partial p} + \frac{y}{p^f}\frac{\partial}{\partial q}$	$Y_8 = \left(\frac{t^2}{2} + U\right)\frac{\partial}{\partial x} + t\frac{\partial}{\partial v} - R\frac{\partial}{\partial p}$	$X_8 = \left(\frac{t^2}{2} + U\right)\frac{\partial}{\partial x} + t\frac{\partial}{\partial v} - R\frac{\partial}{\partial p} - U_x\rho\frac{\partial}{\partial \rho}$
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$B(p,q)$		Admissible operators		
		L	I	E
$p^{1-\sigma}f(p^\sigma/q)$	I	$Z_5 = -(\sigma+1)s\frac{\partial}{\partial s} - 2\sigma y\frac{\partial}{\partial y}$ $+(\sigma+1)v\frac{\partial}{\partial v} + 2p\frac{\partial}{\partial p} + 2\sigma q\frac{\partial}{\partial q}$	$Y_6 = -(\sigma+1)t\frac{\partial}{\partial t} + (\sigma+1)v\frac{\partial}{\partial v}$ $-2\sigma R\frac{\partial}{\partial R} + 2p\frac{\partial}{\partial p}$	$X_5 = -(\sigma+1)t\frac{\partial}{\partial t} + (\sigma+1)v\frac{\partial}{\partial v}$ $+2p\frac{\partial}{\partial p} - 2\sigma\rho\frac{\partial}{\partial \rho}$
(i_1) $\left(k\frac{f}{z}-1\right)^{\sigma+k}$ $= lf^k,\ k \neq 0,-1$ (i_2) $z\ln f + \sigma f = lz$	V	$Z_6 = v\frac{\partial}{\partial v} + p\frac{\partial}{\partial p} + \frac{(k+1)\sigma f+(1-\sigma)z}{(k+\sigma)f}q\frac{\partial}{\partial q}$	$Y_7 = (x+P(0))\frac{\partial}{\partial x}$ $+v\frac{\partial}{\partial v} + p\frac{\partial}{\partial p}$	$X_6 = (x+P(0))\frac{\partial}{\partial x} + v\frac{\partial}{\partial v}$ $+p\frac{\partial}{\partial p} + pB_pB_q^{-1}\rho^2\frac{\partial}{\partial \rho}$
(ii) $\sigma zf' + z^2f'$	VI	$Z_6 = \frac{\partial}{\partial p} + \frac{q}{p}\frac{k\sigma f^{\sigma/(\sigma-1)+(1-\sigma)z}}{kf^{\sigma/(\sigma-1)}+(\sigma-1)f}\frac{\partial}{\partial q}$	$Y_7 = -w\frac{\partial}{\partial x} + \frac{\partial}{\partial p}$	$X_6 = -w\frac{\partial}{\partial x} + \frac{\partial}{\partial p} + B_pB_q^{-1}\rho^2\frac{\partial}{\partial \rho}$
$= kf^{\sigma/(\sigma-1)}$ $+(\sigma-1)f,\ k \neq 0$		$Z_7 = 3\frac{\partial}{\partial v} - \frac{yq}{p}\frac{k\sigma f^{\sigma/(\sigma-1)}+(1-\sigma)z}{kf^{\sigma/(\sigma-1)}+(\sigma-1)f}\frac{\partial}{\partial q} - y\frac{\partial}{\partial p}$	$Y_8 = \left(\frac{t^2}{2}+U\right)\frac{\partial}{\partial x} + t\frac{\partial}{\partial v} - R\frac{\partial}{\partial p}$	$X_7 = \left(\frac{t^2}{2}+U\right)\frac{\partial}{\partial x} + t\frac{\partial}{\partial v}$ $-R\frac{\partial}{\partial p} - RB_pB_q^{-1}\rho^2\frac{\partial}{\partial \rho}$
$k=0,$ i.e. $f\left(1-\frac{f}{z}\right)^{\sigma-1} = l$	VIII	$Z_8 = v\frac{\partial}{\partial v} + p\frac{\partial}{\partial p} + \frac{\sigma f+(1-\sigma)z}{\sigma f}q\frac{\partial}{\partial q}$	$Y_9 = [x+P(0)]\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} + p\frac{\partial}{\partial p}$	$X_8 = [x+P(0)]\frac{\partial}{\partial x} + v\frac{\partial}{\partial v}$ $+p\frac{\partial}{\partial p} + pB_pB_q^{-1}\rho^2\frac{\partial}{\partial \rho}$
$\frac{1}{q}f(qe^p)$	II	$Z_5 = -y\frac{\partial}{\partial y} + v\frac{\partial}{\partial v} - 2\frac{\partial}{\partial p} + 2q\frac{\partial}{\partial q}$	$Y_6 = x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} - R\frac{\partial}{\partial R} - Z\frac{\partial}{\partial p}$	$X_5 = x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} - 2\rho\frac{\partial}{\partial \rho} - 2\frac{\partial}{\partial p}$
$f(1-kf)^{\frac{1-k}{k}} = lz,$ $k \neq 0;1$	VI	$Z_6 = \frac{\partial}{\partial p} - \frac{q}{f}\frac{1-kf}{1-k}\frac{\partial}{\partial q}$	$Y_7 = -W\frac{\partial}{\partial x} + \frac{\partial}{\partial p}$	$X_6 = -W\frac{\partial}{\partial x} + \frac{\partial}{\partial p} + B_pB_q^{-1}\rho^2\frac{\partial}{\partial \rho}$
		$Z_7 = s\frac{\partial}{\partial v} - y\frac{\partial}{\partial p} + \frac{yq}{f}\frac{1-kf}{1-k}\frac{\partial}{\partial q}$	$Y_8 = \left(\frac{t^2}{2}+U\right)\frac{\partial}{\partial x} + t\frac{\partial}{\partial v} - R\frac{\partial}{\partial p}$	$X_7 = \left(\frac{t^2}{2}+U\right)\frac{\partial}{\partial x} + t\frac{\partial}{\partial v}$ $-R\frac{\partial}{\partial p} - U_x\rho\frac{\partial}{\partial \rho}$
$k \to 0,$ i.e. $fe^{-f} = lz$	VIII	$Z_8 = y\frac{\partial}{\partial y} + p\frac{\partial}{\partial p} + \frac{2-p-2f}{f}q\frac{\partial}{\partial q}$	$Y_9 = P(1)\frac{\partial}{\partial x} + p\frac{\partial}{\partial p} + R\frac{\partial}{\partial R}$	$X_8 = P(1)\frac{\partial}{\partial x} + p\frac{\partial}{\partial p}$ $+[1-P_x(1)]\rho\frac{\partial}{\partial \rho}$
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$B(p,q)$	Type	Admissible operators		
		L	I	E
$\frac{1}{q}f(p)$	III	$Z_5 = -y\frac{\partial}{\partial y} + v\frac{\partial}{\partial v} + 2q\frac{\partial}{\partial q}$	$Y_6 = x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} - R\frac{\partial}{\partial R}$	$X_5 = x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} - 2\rho\frac{\partial}{\partial \rho}$
$(i_1)\ \gamma p + \delta p^{\frac{\delta+1}{\delta}}$, $\gamma \neq 0, -1$ $(i_2)\ -p\ln p$	IV	$Z_8 = v\frac{\partial}{\partial v} + p\frac{\partial}{\partial p} + \left(\frac{\gamma+1}{\gamma} - \frac{p}{f}q\frac{\partial}{\partial q}\right)$	$Y_7 = (x + P(0))\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} + p\frac{\partial}{\partial p}$	$X_6 = (x + P(0))\frac{\partial}{\partial x} + v\frac{\partial}{\partial v}$ $+p\frac{\partial}{\partial p} + pB_pB_q^{-1}\rho^2\frac{\partial}{\partial \rho}$
$(ii)\ 1 + \delta e^p$	VI	$Z_6 = \frac{\partial}{\partial p} + \frac{\delta e^p}{1+\delta e^p}q\frac{\partial}{\partial q}$	$Y_7 = -W\frac{\partial}{\partial x} + \frac{\partial}{\partial p}$	$X_6 = -W\frac{\partial}{\partial x} + \frac{\partial}{\partial p} + B_pB_q^{-1}\rho^2\frac{\partial}{\partial \rho}$
		$Z_7 = s\frac{\partial}{\partial v} - y\frac{\partial}{\partial p} - \frac{\delta e^p}{1+\delta e^p}yq\frac{\partial}{\partial q}$	$Y_8 = \left(\frac{t^2}{2} + U\right)\frac{\partial}{\partial x} + t\frac{\partial}{\partial v} - R\frac{\partial}{\partial p}$	$X_7 = \left(\frac{t^2}{2} + U\right)\frac{\partial}{\partial x} + t\frac{\partial}{\partial v}$ $-R\frac{\partial}{\partial p} - U_x\rho\frac{\partial}{\partial \rho}$
$(iii)\ 3p$	VII	$Z_6 = v\frac{\partial}{\partial v} + p\frac{\partial}{\partial p} + q\frac{\partial}{\partial q}$	$Y_7 = x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} + p\frac{\partial}{\partial p}$	$X_6 = x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} + p\frac{\partial}{\partial p} - \rho\frac{\partial}{\partial \rho}$
		$Z_7 = s^2\frac{\partial}{\partial s} + (x - sv)\frac{\partial}{\partial v}$ $-3sp\frac{\partial}{\partial p} + sq\frac{\partial}{\partial q}$	$Y_8 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x}$ $+(x - tv)\frac{\partial}{\partial v} - 3tp\frac{\partial}{\partial p}$	$X_7 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} - 3tp\frac{\partial}{\partial p}$ $+(x - tv)\frac{\partial}{\partial v} - t\rho\frac{\partial}{\partial \rho}$
$(iv)\ -p$	VIII	$Z_6 = v\frac{\partial}{\partial v} + p\frac{\partial}{\partial p} + q\frac{\partial}{\partial q}$	$Y_7 = x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} + p\frac{\partial}{\partial p}$	$X_5 = x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} + p\frac{\partial}{\partial p} - \rho\frac{\partial}{\partial \rho}$
		$Z_7 = \frac{\partial}{\partial p} + \frac{q}{p}\frac{\partial}{\partial q}$	$Y_8 = -W\frac{\partial}{\partial x} + \frac{\partial}{\partial p}$	$X_7 = -W\frac{\partial}{\partial x} + \frac{\partial}{\partial p} + B_pB_q^{-1}\rho^2\frac{\partial}{\partial \rho}$
		$Z_8 = s\frac{\partial}{\partial v} - y\frac{\partial}{\partial p} - \frac{yq}{p}\frac{\partial}{\partial q}$	$Y_9 = \left(\frac{t^2}{2} + U\right)\frac{\partial}{\partial x} + t\frac{\partial}{\partial v} - R\frac{\partial}{\partial p}$	$X_8 = \left(\frac{t^2}{2} + U\right)\frac{\partial}{\partial x} + t\frac{\partial}{\partial v} - R\frac{\partial}{\partial p} - U_x\rho\frac{\partial}{\partial \rho}$
$f(q)$	V	$Z_5 = \frac{\partial}{\partial p}$	$Y_6 = \frac{\partial}{\partial p}$	$X_5 = \frac{\partial}{\partial p}$
		$Z_6 = s\frac{\partial}{\partial v} - y\frac{\partial}{\partial p}$	$Y_7 = t^2\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial v} - 2R\frac{\partial}{\partial p}$	$X_6 = t^2\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial v} - 2R\frac{\partial}{\partial p}$
$(i)\ \varepsilon e^q$	VI	$Z_7 = -s\frac{\partial}{\partial s} + v\frac{\partial}{\partial v} + 2p\frac{\partial}{\partial p} + 2\frac{\partial}{\partial q}$	$Y_8 = -t\frac{\partial}{\partial t} + 2R\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} + 2p\frac{\partial}{\partial p}$	$X_7 = -t\frac{\partial}{\partial t} + 2R\frac{\partial}{\partial x} + v\frac{\partial}{\partial v}$ $+2p\frac{\partial}{\partial p} - 2\rho^2\frac{\partial}{\partial \rho}$
$(ii)\ \varepsilon q^{(\sigma+1)}$	VI	$Z_7 = (\sigma - 1)s\frac{\partial}{\partial s} - 2y\frac{\partial}{\partial y}$ $-(\sigma - 1)v\frac{\partial}{\partial v} - 2\sigma p\frac{\partial}{\partial p} + 2q\frac{\partial}{\partial q}$	$Y_8 = (\sigma - 1)t\frac{\partial}{\partial t} - (\sigma - 1)v\frac{\partial}{\partial v}$ $-2\sigma p\frac{\partial}{\partial p} - 2R\frac{\partial}{\partial R}$	$X_7 = (\sigma - 1)t\frac{\partial}{\partial t} - (\sigma - 1)v\frac{\partial}{\partial v}$ $-2\sigma p\frac{\partial}{\partial p} - 2\rho\frac{\partial}{\partial p}$

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$B(p,q)$	Type	Admissible operators		
		L	I	E
$f(p+\varepsilon q)$	V	$Z_5=\frac{\partial}{\partial p}-\varepsilon\frac{\partial}{\partial q}$	$Y_6=-\varepsilon R\frac{\partial}{\partial x}+\frac{\partial}{\partial p}$	$X_5=-\varepsilon R\frac{\partial}{\partial x}+\frac{\partial}{\partial p}+\varepsilon\rho^2\frac{\partial}{\partial\rho}$
		$Z_6=s\frac{\partial}{\partial v}-y\frac{\partial}{\partial p}+\varepsilon y\frac{\partial}{\partial q}$	$Y_7=\frac{t^2+\varepsilon R^2}{2}\frac{\partial}{\partial x}+t\frac{\partial}{\partial v}-R\frac{\partial}{\partial p}$	$X_6=\frac{t^2+\varepsilon R^2}{2}\frac{\partial}{\partial x}+t\frac{\partial}{\partial v}-R\frac{\partial}{\partial p}-\varepsilon R\rho^2\frac{\partial}{\partial\rho}$
$(i)\varepsilon\ln f+f^{-1}$ $=k(p+\varepsilon q)+l$	VI	$Z_7=-y\frac{\partial}{\partial y}+v\frac{\partial}{\partial v}+\frac{2}{kf}(\varepsilon-f)\frac{\partial}{\partial q}$	$Y_8=[x+P(0)-P(1)]\frac{\partial}{\partial x}$ $+v\frac{\partial}{\partial v}-R\frac{\partial}{\partial R}$	$X_7=[x+P(0)-P(1)]\frac{\partial}{\partial x}+v\frac{\partial}{\partial v}+$ $2BB_q^{-1}\rho^2\frac{\partial}{\partial\rho}$
$(ii)\ \varepsilon f-\ln f$ $=k(p+\varepsilon q)+l$	VI	$Z_7=y\frac{\partial}{\partial y}+v\frac{\partial}{\partial v}+2p\frac{\partial}{\partial p}$ $+\left(\frac{2\varepsilon}{k}\ln f+2q+2\varepsilon l-\frac{2\varepsilon}{k}\right)\frac{\partial}{\partial q}$	$Y_8=[x+2P(0,5)]\frac{\partial}{\partial x}+v\frac{\partial}{\partial v}$ $+2p\frac{\partial}{\partial p}+R\frac{\partial}{\partial R}$	$X_7=[x+2P(0,5)]\frac{\partial}{\partial x}+v\frac{\partial}{\partial v}$ $+2p\frac{\partial}{\partial p}-2p_x(0,5)\rho\frac{\partial}{\partial\rho}$
$(iii)\ \frac{\varepsilon}{1+\sigma}f^{1+\sigma}-\frac{1}{\sigma}f^{\sigma}$ $=k(p+\varepsilon q)+l,$ $\sigma\neq 0,-1$	V	$Z_7=y\frac{\partial}{\partial y}+(2\sigma+1)v\frac{\partial}{\partial v}+$ $2(\sigma+1)p\frac{\partial}{\partial p}$ $+\left[2(\sigma+1)(q+\varepsilon l)+\frac{2\varepsilon}{k\sigma}f^{\sigma}\right]\frac{\partial}{\partial q}$	$Y_8=[(2\sigma+1)(x+P(0))$ $+P(1)]\frac{\partial}{\partial x}+(2\sigma+1)v\frac{\partial}{\partial v}$ $+2(\sigma+1)p\frac{\partial}{\partial p}+R\frac{\partial}{\partial R}$	$X_7=[(2\sigma+1)(x+P(0)+P(1))]\frac{\partial}{\partial x}$ $+(2\sigma+1)v\frac{\partial}{\partial v}+2(\sigma+1)p\frac{\partial}{\partial p}$ $-\left[2\sigma+(2\sigma+1)P_x(0)+P_x(1)\right]\rho\frac{\partial}{\partial\rho}$
$f(p)$	IX	$Z_0=\theta\left(y,q+\int\frac{dp}{f(p)}\right)\frac{\partial}{\partial q}$	$Y_8=\theta\frac{\partial}{\partial x}$	$X_0=\theta\frac{\partial}{\partial x}-\rho^2\theta\frac{\partial}{\partial p}$
$(i)\varepsilon e^{p}$	X	$Z_5=y\frac{\partial}{\partial y}-v\frac{\partial}{\partial v}+z\frac{\partial}{\partial p}-2q\frac{\partial}{\partial q}$	$Y_\delta=-x\frac{\partial}{\partial x}-v\frac{\partial}{\partial v}+R\frac{\partial}{\partial R}+2\frac{\partial}{\partial p}$	$X_5=-x\frac{\partial}{\partial x}-v\frac{\partial}{\partial v}+2\frac{\partial}{\partial p}+2\rho\frac{\partial}{\partial\rho}$
$(ii)\varepsilon p^{\sigma},\sigma\neq 0$	XI	$Z_5=\sigma y\frac{\partial}{\partial y}+(2-\sigma)v\frac{\partial}{\partial v}$ $+2p\frac{\partial}{\partial p}-2(\sigma-1)q\frac{\partial}{\partial q}$	$Y_6=(2-\sigma)x\frac{\partial}{\partial x}+(2-\sigma)v\frac{\partial}{\partial v}$ $+\sigma R\frac{\partial}{\partial R}+2p\frac{\partial}{\partial p}$	$X_5=(2-\sigma)x\frac{\partial}{\partial x}+(2-\sigma)v\frac{\partial}{\partial v}$ $+2p\frac{\partial}{\partial p}+2(\sigma-1)\rho\frac{\partial}{\partial\rho}$
$(iii)\ 1$	XII	$Z_4=\varphi(s+y,p+v)\left(\frac{\partial}{\partial v}-\frac{\partial}{\partial p}+\frac{\partial}{\partial q}\right)$	$Y_\varphi=\Phi\frac{\partial}{\partial x}-\varphi\left(\frac{\partial}{\partial v}-\frac{\partial}{\partial p}\right)$	$X_\varphi=\Phi\frac{\partial}{\partial x}-\varphi\left(\frac{\partial}{\partial v}-\frac{\partial}{\partial p}-\rho^2\frac{\partial}{\partial\rho}\right)$
		$Z_\psi=\psi(s-y,p+v)\left(\frac{\partial}{\partial v}\frac{\partial}{\partial p}-\frac{\partial}{\partial q}\right)$	$Y_\psi\frac{\partial}{\partial x}-\psi\left(\frac{\partial}{\partial v}+\frac{\partial}{\partial p}\right)$	$X_\psi=\Psi\frac{\partial}{\partial x}-\psi\left(\frac{\partial}{\partial v}+\frac{\partial}{\partial p}+\rho^2\frac{\partial}{\partial\rho}\right)$
$(iv)\ -1$	XIII	$Z_\pi=\mathrm{Re}\pi\left(\frac{\partial}{\partial p}+\frac{\partial}{\partial p}\right)+\mathrm{Im}\pi\frac{\partial}{\partial v}$	$Y_\pi=\Pi\frac{\partial}{\partial x}+\mathrm{Im}\pi\frac{\partial}{\partial v}+\mathrm{Re}\pi\frac{\partial}{\partial p}$	$X_\pi=\Pi\frac{\partial}{\partial x}+\mathrm{Im}\pi\frac{\partial}{\partial v}+\mathrm{Re}\pi\left(\frac{\partial}{\partial p}-\rho^2\frac{\partial}{\partial\rho}\right)$

Table 9: Basic types of equations with extended symmetry groups

Type	Condition for $B(p,q)$	Admissible operators	
		L	E
	Arbitrary function	$Z_1 = \frac{\partial}{\partial s}$ $Z_2 = s\frac{\partial}{\partial s} + y\frac{\partial}{\partial y}$ $Z_3 = \frac{\partial}{\partial v}$ $Z_4 = \frac{\partial}{\partial y}$	$X_1 = \frac{\partial}{\partial x}$ $X_2 = \frac{\partial}{\partial t}$ $X_3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$ $X_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial v}$
I	$p(B_p\mathcal{B}_q - B_q\mathcal{B}_p)$ $+(2a-1)B_q\mathcal{B} - 2aB\mathcal{B}_q = 0$	$Z_5 = ay\frac{\partial}{\partial y} + (1-a)v\frac{\partial}{\partial v} + p\frac{\partial}{\partial p}$ $+(2ab - pB_p)B_q^{-1}\frac{\partial}{\partial q}$	$X_5 = [(1-a)x + P(a)]\frac{\partial}{\partial x} + (1-a)v\frac{\partial}{\partial v}$ $+p\frac{\partial}{\partial p} - (2aB - pB_p)B_q^{-1}\rho^2\frac{\partial}{\partial \rho}$
II	$B_p\mathcal{B}_q - B_q\mathcal{B}_p + B_q\mathcal{B} - B\mathcal{B}_q = 0$	$Z_5 = y\frac{\partial}{\partial y} - v\frac{\partial}{\partial v} + 2\frac{\partial}{\partial p} + 2(B - B_p)B_q^{-1}\frac{\partial}{\partial q}$	$X_5 = (U - x)\frac{\partial}{\partial x} - v\frac{\partial}{\partial v} + 2\frac{\partial}{\partial p} + (2 - \rho RB_pB_q^{-1})\rho\frac{\partial}{\partial \rho}$
III	$\mathcal{B} = Bf(p),\ f'(p) \neq 0$	$Z_5 = y\frac{\partial}{\partial y} - v\frac{\partial}{\partial v} + 2BB_q\frac{\partial}{\partial q}$	$X_5 = [P(1) - P(0) - x]\frac{\partial}{\partial x} - v\frac{\partial}{\partial v} - 2BB_q^{-1}\rho^2\frac{\partial}{\partial \rho}$
IV	$\mathcal{B} = \frac{kB}{p},\ k \neq 0$	$Z_5 = y\frac{\partial}{\partial y} - v\frac{\partial}{\partial v} + 2BB_q^{-1}\frac{\partial}{\partial q}$ $Z_6 = v\frac{\partial}{\partial v} + p\frac{\partial}{\partial p} - pB_pB_q^{-1}\frac{\partial}{\partial q}$	$X_5 = [P(1) - P(0) - x]\frac{\partial}{\partial x} - v\frac{\partial}{\partial v} - 2BB_q^{-1}\rho^2\frac{\partial}{\partial \rho}$ $X_6 = [x + P(0)]\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} + p\frac{\partial}{\partial p} + pB_pB_q^{-1}\rho^2\frac{\partial}{\partial \rho}$
V	$B_p\mathcal{B}_q - B_q\mathcal{B}_p = 0$	$Z_5 = s\frac{\partial}{\partial v} - y\frac{\partial}{\partial p} + yB_pB_q^{-1}\frac{\partial}{\partial q}$ $Z_6 = \frac{\partial}{\partial p} - B_pB_q^{-1}\frac{\partial}{\partial q}$	$X_5 = \left(\frac{t^2}{2} + v\right)\frac{\partial}{\partial x} + t\frac{\partial}{\partial v} - R\frac{\partial}{\partial p} - 2[1 + (B - B_p)B_q^{-1}\rho]\rho\frac{\partial}{\partial \rho}$ $X_6 = -w\frac{\partial}{\partial x} + \frac{\partial}{\partial p} + B_pB_q^{-1}\rho^2\frac{\partial}{\partial \rho}$
VI	$\mathcal{B} = B^{\frac{1-b}{2}}$	$Z_5 = s\frac{\partial}{\partial v} - y\frac{\partial}{\partial p} + yB_pB_q^{-1}\frac{\partial}{\partial q}$ $Z_6 = \frac{\partial}{\partial p} - B_pB_q^{-1}\frac{\partial}{\partial q}$ $Z_7 = y\frac{\partial}{\partial y} + bv\frac{\partial}{\partial v} + (b+1)p\frac{\partial}{\partial p}$ $+[2B - (b+1)pB_p]B_q^{-1}\frac{\partial}{\partial q}$	$X_5 = \left(\frac{t^2}{2} + v\right)\frac{\partial}{\partial x} + t\frac{\partial}{\partial v} - R\frac{\partial}{\partial p} - 2[1 + (B - B_p)B_q^{-1}\rho]\rho\frac{\partial}{\partial \rho}$ $X_6 = -w\frac{\partial}{\partial x} + \frac{\partial}{\partial p} + B_pB_q^{-1}\rho^2\frac{\partial}{\partial \rho}$ $X_7 = [bx + bP(0) + P(1)]\frac{\partial}{\partial x} + bv\frac{\partial}{\partial v}$ $+(b+1)p\frac{\partial}{\partial p} - [2B - (b+1)pB_p]B_q^{-1}\rho^2\frac{\partial}{\partial \rho}$

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Type	Condition for $B(p,q)$	Admissible operators L	Admissible operators E
VII	$B = 3p/q$	$Z_5 = -y\frac{\partial}{\partial y} + v\frac{\partial}{\partial v} + 2q\frac{\partial}{\partial q}$ $Z_6 = v\frac{\partial}{\partial v} + p\frac{\partial}{\partial p} + q\frac{\partial}{\partial q}$ $Z_7 = s^2\frac{\partial}{\partial s} + (x - sv)\frac{\partial}{\partial v} - 3sp\frac{\partial}{\partial p} + sq\frac{\partial}{\partial q}$	$X_5 = x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} - 2\rho\frac{\partial}{\partial \rho}$ $X_6 = x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} + p\frac{\partial}{\partial p} - \rho\frac{\partial}{\partial \rho}$ $X_7 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} - 3tp\frac{\partial}{\partial p} + (x - tv)\frac{\partial}{\partial v} - t\rho\frac{\partial}{\partial \rho}$
VIII	$\mathcal{B} = 0$	$Z_5 = s\frac{\partial}{\partial v} - y\frac{\partial}{\partial p} + yB_pB_q^{-1}\frac{\partial}{\partial q}$ $Z_6 = \frac{\partial}{\partial p} - B_pB_q^{-1}\frac{\partial}{\partial q}$ $Z_7 = y\frac{\partial}{\partial y} + p\frac{\partial}{\partial p} + (2B - pB_p)B_q^{-1}\frac{\partial}{\partial q}$ $Z_8 = v\frac{\partial}{\partial v} + p\frac{\partial}{\partial p} - pB^{-1}\frac{\partial}{\partial q}$	$X_5 = \left(\frac{t^2}{2} + v\right)\frac{\partial}{\partial x} + t\frac{\partial}{\partial v} - R\frac{\partial}{\partial p} - 2[1 + (B - B_p)B_q^{-1}\rho]\rho\frac{\partial}{\partial \rho}$ $X_6 = -w\frac{\partial}{\partial x} + \frac{\partial}{\partial p} + B_pB_q^{-1}\rho^2\frac{\partial}{\partial \rho}$ $X_7 = P(1)\frac{\partial}{\partial x} + p\frac{\partial}{\partial p} - [2B - pB_p]B_q^{-1}\rho^2\frac{\partial}{\partial \rho}$ $X_8 = [x + P(0)]\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} + p\frac{\partial}{\partial p} + pB^{-1}\rho^2\frac{\partial}{\partial \rho}$
IX	$f(p)$	$Z_\theta = \theta\left(y, q, + \int\frac{dp}{f(p)}\right)\frac{\partial}{\partial q}$	$X_\theta = \Theta\frac{\partial}{\partial x} - \rho^2\theta\frac{\partial}{\partial \rho}$
X	εe^p	$Z_\theta = \theta\left(y, q + \int\frac{dp}{f(p)}\right)\frac{\partial}{\partial q}$ $Z_5 = y\frac{\partial}{\partial y} - v\frac{\partial}{\partial v} + 2\frac{\partial}{\partial p} - 2q\frac{\partial}{\partial q}$	$X_\theta = \Theta\frac{\partial}{\partial x} - \rho^2\theta\frac{\partial}{\partial \rho}$ $X_5 = -x\frac{\partial}{\partial x} - v\frac{\partial}{\partial v} + 2\frac{\partial}{\partial p} + 2\rho\frac{\partial}{\partial \rho}$
XI	εp^σ	$Z_\theta = \theta\left(y, q + \int\frac{dp}{f(p)}\right)\frac{\partial}{\partial q}$ $Z_5 = \sigma y\frac{\partial}{\partial y} + (2 - \sigma)v\frac{\partial}{\partial v} + 2p\frac{\partial}{\partial p} - 2(\sigma - 1)q\frac{\partial}{\partial q}$	$X_\theta = \Theta\frac{\partial}{\partial x} - \rho^2\theta\frac{\partial}{\partial \rho}$ $X_5 = (2 - \sigma)x\frac{\partial}{\partial x} + (2 - \sigma)v\frac{\partial}{\partial v} + 2p\frac{\partial}{\partial p} + 2(\sigma - 1)\rho\frac{\partial}{\partial \rho}$
XII	1	$Z_\theta = \theta\left(y, q + \int\frac{dp}{f(p)}\right)\frac{\partial}{\partial q}$ $Z_\varphi = \varphi(s + y, p - v)\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial p} + \frac{\partial}{\partial q}\right)$ $Z_\psi = \psi(s - y, p + v)\left(\frac{\partial}{\partial v} + \frac{\partial}{\partial p} - \frac{\partial}{\partial q}\right)$	$X_\theta = \Theta\frac{\partial}{\partial x} - \rho^2\theta\frac{\partial}{\partial \rho}$ $X_\varphi = \Phi\frac{\partial}{\partial x} - \varphi\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial p} - \rho^2\frac{\partial}{\partial \rho}\right)$ $X_\psi = \Psi\frac{\partial}{\partial x} - \psi\left(\frac{\partial}{\partial v} + \frac{\partial}{\partial p} + \rho^2\frac{\partial}{\partial \rho}\right)$
XIII	-1	$Z_\theta = \theta\left(y, q + \int\frac{dp}{f(p)}\right)\frac{\partial}{\partial q}$ $Z_\pi = \mathrm{Re}\pi\left(\frac{\partial}{\partial p} + \frac{\partial}{\partial q}\right) + \mathrm{Im}\pi\frac{\partial}{\partial v}$	$X_\theta = \Theta\frac{\partial}{\partial x} - \rho^2\theta\frac{\partial}{\partial \rho}$ $X_\pi = \Pi\frac{\partial}{\partial x} + \mathrm{Im}\pi\frac{\partial}{\partial v} + \mathrm{Re}\pi\left(\frac{\partial}{\partial p} - \rho^2\frac{\partial}{\partial \rho}\right)$

Table 10: Preliminary classification of equations (L)

$B(p,q)$	Type	Admissible operators
Arbitrary function		$Z_1=\frac{\partial}{\partial s}$, $Z_2=s\frac{\partial}{\partial s}+y\frac{\partial}{\partial y}$ $Z_3=\frac{\partial}{\partial v}$, $Z_4=\frac{\partial}{\partial y}$
$pf(pe^q)$	I	$Z_5=-s\frac{\partial}{\partial s}+v\frac{\partial}{\partial v}-2p\frac{\partial}{\partial p}-2\frac{\partial}{\partial q}$
(i) $(k-f)^{1-k}$	VI	$Z_6=\frac{\partial}{\partial p}-\frac{k-1}{(k-1)p}\frac{\partial}{\partial q}$
$=lfz,\ k\neq 0;1$		$Z_7=s\frac{\partial}{\partial v}-y\frac{\partial}{\partial p}+\frac{(k-1)y}{(k-f)p}\frac{\partial}{\partial q}$
(ii) $\frac{1}{f}+\ln f=l$	V	$Z_6=y\frac{\partial}{\partial y}+p\frac{\partial}{\partial p}+\frac{(2-k)f-1}{(k-1)f}\frac{\partial}{\partial q}$
$+(k-1)\ln z,\ k\neq 0$		
$k=0$, i.e.	VIII	$Z_7=\frac{\partial}{\partial p}-\frac{1}{pf}\frac{\partial}{\partial q}$
$e^{1/f}=l/(zf)$		$Z_8=s\frac{\partial}{\partial p}-y\frac{\partial}{\partial p}+\frac{y}{pf}\frac{\partial}{\partial q}$
$p^{1-\sigma}f\left(\frac{p^\sigma}{q}\right)$	I	$Z_5=-(\sigma+1)s\frac{\partial}{\partial s}-2\sigma y\frac{\partial}{\partial y}$ $+(\sigma+1)v\frac{\partial}{\partial v}+2p\frac{\partial}{\partial p}+2\sigma q\frac{\partial}{\partial q}$
(i_1) $\left(k\frac{f}{z}-1\right)^{\sigma+k}=lf^k$, $k\neq 0,-1$		$Z_6=v\frac{\partial}{\partial v}+p\frac{\partial}{\partial p}+\frac{(k+1)\sigma f+(1-\sigma)z}{(k+\sigma)f}q\frac{\partial}{\partial q}$
(i_2) $\ln f+\sigma\frac{f}{z}=l$		
(ii) $\sigma zf'+z^2\frac{f'}{f}=kf^{\frac{\sigma}{\sigma-1}}$	VI	$Z_6=\frac{\partial}{\partial p}+\frac{q}{p}\frac{k\sigma f^{\frac{\sigma}{\sigma-1}}+(1-\sigma)z}{kf^{\sigma/(\sigma-1)}}\frac{\partial}{\partial q}$
$+(\sigma-1)f,\ k\neq 0$		$Z_7=s\frac{\partial}{\partial v}-y\frac{\partial}{\partial p}-\frac{yq}{p}\frac{k\sigma f^{\sigma/(\sigma-1)}+(1-\sigma)z}{kf^{\sigma/(\sigma-1)}+(\sigma-1)f}\frac{\partial}{\partial q}$
$k=0$, i.e. $f\left(1+\frac{f}{z}\right)^{\sigma-1}=l$	VIII	$Z_8=v\frac{\partial}{\partial v}+p\frac{\partial}{\partial p}+\frac{\sigma f+(1-\sigma)z}{\sigma f}q\frac{\partial}{\partial q}$
$\frac{1}{q}f(qe^p)$	II	$Z_5=-y\frac{\partial}{\partial y}+v\frac{\partial}{\partial v}-2\frac{\partial}{\partial p}+2q\frac{\partial}{\partial q}$
$f(1-kf^{\frac{1-k}{k}}=lz,\ k\neq 0;1$	VI	$Z_6=\frac{\partial}{\partial p}-\frac{q}{f}\frac{1-kf}{1-k}\frac{\partial}{\partial q}$
		$Z_7=s\frac{\partial}{\partial v}-y\frac{\partial}{\partial p}+\frac{yq}{f}\frac{1-kf}{1-k}\frac{\partial}{\partial q}$
$k\to 0$, i.e. $fe^{-f}=lz$	VII	$Z_8=y\frac{\partial}{\partial y}+p\frac{\partial}{\partial p}+\frac{2-p-2f}{f}q\frac{\partial}{\partial q}$
$\frac{1}{q}f(p)$	III	$Z_5=-y\frac{\partial}{\partial y}+v\frac{\partial}{\partial v}+2q\frac{\partial}{\partial q}$
(i_1) $\gamma p+\delta p^{\frac{\gamma+1}{\gamma}},\gamma\neq 0,-1$	IV	$Z_6=v\frac{\partial}{\partial v}+p\frac{\partial}{\partial p}+\left(\frac{\gamma+1}{\gamma}-\frac{p}{f}\right)q\frac{\partial}{\partial q}$
(i_2) $-p\ln p$		
(ii) $1+\delta e^p$	VI	$Z_6=\frac{\partial}{\partial p}+\frac{\delta e^p}{1+\delta e^p}q\frac{\partial}{\partial q}$, $Z_7=s\frac{\partial}{\partial v}-yZ_6$
(iii) $-p$	VIII	$Z_6=v\frac{\partial}{\partial v}+p\frac{\partial}{\partial p}+q\frac{\partial}{\partial q}$
		$Z_7=\frac{\partial}{\partial p}+\frac{q}{p}\frac{\partial}{\partial q}$, $Z_8=s\frac{\partial}{\partial v}-y\frac{\partial}{\partial p}-\frac{yq}{p}\frac{\partial}{\partial q}$

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$B(p,q)$	Type	Admissible operators
$f(q)$	V	$Z_5=\frac{\partial}{\partial p}\,,\ Z_6=s\frac{\partial}{\partial v}-y\frac{\partial}{\partial p}$
$(i)\ \varepsilon e^q$	VI	$Z_7=-s\frac{\partial}{\partial s}+v\frac{\partial}{\partial v}+2p\frac{\partial}{\partial p}+2\frac{\partial}{\partial q}$
$(ii)\ \varepsilon q^{-(\sigma+1)}$	VI	$Z_7=(\sigma-1)s\frac{\partial}{\partial s}-2y\frac{\partial}{\partial y}-(\sigma-1)v\frac{\partial}{\partial v}$ $-2\sigma p\frac{\partial}{\partial p}+2q\frac{\partial}{\partial q}$
$f(p+\varepsilon q)$	V	$Z_5=\frac{\partial}{\partial p}-\varepsilon\frac{\partial}{\partial q}$
		$Z_6=s\frac{\partial}{\partial v}-y\frac{\partial}{\partial p}+\varepsilon y\frac{\partial}{\partial q}$
$(i)\ \varepsilon\ln f+\frac{1}{f}=l$ $+k(p+\varepsilon q)$	VI	$Z_7=-y\frac{\partial}{\partial y}+v\frac{\partial}{\partial v}+\frac{2}{kf}(\varepsilon-f)\frac{\partial}{\partial q}$
$(ii)\ \varepsilon f-\ln f=l$ $+k(p+\varepsilon q)$	VI	$Z_7=y\frac{\partial}{\partial y}+v\frac{\partial}{\partial v}+2p\frac{\partial}{\partial p}$ $+\left(\frac{2\varepsilon}{k}\ln f+2q+2\varepsilon l-\frac{2\varepsilon}{k}\right)\frac{\partial}{\partial q}$
$(iii)\ \frac{\varepsilon}{1+\sigma}f^{1+\sigma}-\frac{1}{\sigma}f^{\sigma}=l$ $+k(p+\varepsilon q),\ \sigma\neq 0,-1$	VI	$Z_7=y\frac{\partial}{\partial y}+(2\sigma+1)v\frac{\partial}{\partial v}+2(\sigma+1)p\frac{\partial}{\partial p}$ $+\left[2(\sigma+1)(q+\varepsilon l)+\frac{2\varepsilon}{k\sigma}f^{\sigma}\right]\frac{\partial}{\partial q}$
$f(p)$	IX	$Z_\theta=\theta\left(y,q+\int\frac{dp}{f(p)}\right)\frac{\partial}{\partial q}$
$(i)\ \varepsilon e^p$	X	$Z_5=y\frac{\partial}{\partial y}-v\frac{\partial}{\partial v}+2\frac{\partial}{\partial p}-2q\frac{\partial}{\partial q}$
$(ii)\ \varepsilon p^{\sigma},\ \sigma\neq 0$	XI	$Z_5=\sigma y\frac{\partial}{\partial y}+(2-\sigma)v\frac{\partial}{\partial v}+2p\frac{\partial}{\partial p}$ $-2(\sigma-1)q\frac{\partial}{\partial q}$
$(iii)\ 1$	XII	$Z_\varphi=\varphi(s+y,p-v)\left(\frac{\partial}{\partial v}-\frac{\partial}{\partial p}+\frac{\partial}{\partial q}\right)$
		$X_\psi=\psi(s-y,p+v)\left(\frac{\partial}{\partial v}+\frac{\partial}{\partial p}-\frac{\partial}{\partial q}\right)$
$(iv)\ -1$	XIII	$Z_\pi=\mathrm{Re}\pi\left(\frac{\partial}{\partial p}+\frac{\partial}{\partial v}\right)+\mathrm{Im}\pi\frac{\partial}{\partial v}$

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Paper 6

Approximate symmetries

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ABSTRACT. A theory, based on the new concept of an approximate group, is developed for approximate group analysis of differential equations with a small parameter. An approximate Lie theorem is proved that enables one to construct approximate symmetries that are stable under small perturbations of the differential equations. The use of the algorithm is illustrated in detail by examples: approximate symmetries of nonlinear wave equations are considered along with a broad class of evolution equations that includes the Korteweg-de Vries and Burgers-Korteweg-de Vries equations.

Tables: 2. Bibliography: 4 titles.

Introduction

The methods of classical group analysis enable one to distinguish among all equations of mathematical physics the equations that are remarkable with respect to their symmetry properties (see, for example, [91], [106], and [58]). Unfortunately, any small perturbation of an equation disturbs the group admitted, and this reduces the practical value of these "refined" equations and of group-theoretic methods in general. Therefore, it became necessary to work out group analysis methods that are stable under small perturbations of the differential equations. In this article we develop such a method that is based on the concepts of an approximate group of transformations and approximate symmetries.

The following notation is used: $z = (z^1, ..., z^N)$ is the independent variable; ε is a small parameter; all functions are assumed to be jointly analytic in their arguments; the vector expression $\xi \frac{\partial}{\partial z}$ is used, along with $\xi^k \frac{\partial}{\partial z^k}$ for expressions of the type

$$\sum_1^N \xi^k \frac{\partial}{\partial z^k} \, .$$

Everywhere below, $\theta_p(z, \varepsilon)$ denotes an infinitesimally small function of order $\varepsilon^{p+1}, p \geq 0$, i.e., $\theta_p(z, \varepsilon) = o(\varepsilon^p)$, where this equality (in the case of functions analytic in a neighborhood of $\varepsilon = 0$) is equivalent to any of the following conditions:

$$\lim_{\varepsilon \to 0} \frac{\theta_p(z, \varepsilon)}{\varepsilon^p} = 0,$$

or there exists a constant $C > 0$ such that

$$|\theta_P(z, \varepsilon)| \leq C|\varepsilon|^{p+1},$$

or there exists a function $\varphi(z, \varepsilon)$ analytic in a neighborhood of $\varepsilon = 0$ such that

$$\theta_p(z, \varepsilon) = \varepsilon^{p+1} \varphi(z, \varepsilon). \tag{1}$$

Furthermore, the approximate equality $f \approx g$ means the equation

$$f(z, \varepsilon) = g(z, \varepsilon) + o(\varepsilon^p)$$

for some fixed value of $p \geq 0$. The following notation is also used in § 5: t and x are independent variables, u is a differentiable variable with successive derivatives (with respect to x) $u_1, u_2, \ldots$, i.e.

$$u_{\alpha+1} = D(u_\alpha), \quad u_0 = u, \quad D = \frac{\partial}{\partial x} + \sum_{\alpha \geq 0} u_{\alpha+1} \frac{\partial}{\partial u_\alpha} \, .$$

We denote by $\mathcal{A}$ the space of differentiable functions, i.e. analytic functions of any finite number of variables $t, x, u, u_1, \ldots$. We also use the notation

$$f_t = \frac{\partial f}{\partial t} \, , \; f_x = \frac{\partial f}{\partial x} \, , \; f_\alpha = \frac{\partial f}{\partial u_\alpha} \, , \; f_* = \sum_{\alpha \geq 0} f_\alpha D^\alpha.$$

Below we use the following variant of the theorem on continuous dependence of the solution of the Cauchy problem on the parameters.

Theorem 6.1. Suppose that the functions $f(z, \varepsilon)$ and $g(z, \varepsilon)$, which are analytic in a neighborhood of the point $(z_0, 0)$, satisfy the condition

$$g(z, \varepsilon) = f(z, \varepsilon) + o(\varepsilon^p)$$

and let $z = z(t, \varepsilon)$ and $\tilde{z} = \tilde{z}(t, \varepsilon)$ be the solutions of the respective problems

$$\frac{dz}{dt} = f(z, \varepsilon), \quad z|_{t=0} = \alpha(\varepsilon)$$

and

$$\frac{d\tilde{z}}{dt} = g(\tilde{z}, \varepsilon), \quad \tilde{z}|_{t=0} = \beta(\varepsilon),$$

where $\alpha(0) = \beta(0) = z_0$ and $\beta(\varepsilon) = \alpha(\varepsilon) + o(\varepsilon^p)$. Then

$$\tilde{z}(t, \varepsilon) = z(t, \varepsilon) + o(\varepsilon^p).$$

We consider the approximate Cauchy problem

$$\frac{dz}{dt} \approx f(z, \varepsilon), \tag{2}$$

$$z|_{t=0} \approx \alpha(\varepsilon), \tag{3}$$

which is defined as follows. The approximate differential equation (2) is understood as a family of differential equations

$$\frac{dz}{dt} = g(z, \varepsilon) \quad \text{with} \quad g(z, \varepsilon) \approx f(z, \varepsilon); \tag{4}$$

the approximate initial condition (3) is understood similarly, namely,

$$z|_{t=0} = \beta(\varepsilon) \quad \text{with} \quad \beta(\varepsilon) \approx \alpha(\varepsilon). \tag{5}$$

The approximate equality in (4) and (5) has the same degree of accuracy p as in (2) and (3). According to Theorem 6.1, the solutions of all the problems of the form (4), (5) coincide to within $o(\varepsilon^p)$. Therefore, the *solution of the approximate Cauchy problem* (2), (3) is defined to be the solution of any of the problems (4), (5), considered to within $o(\varepsilon^p)$. Theorem 6.1 gives us the uniqueness (with the indicated accuracy) of this solution.

§ 1 One-parameter approximate groups

Let $z' = g(z, \varepsilon, a)$ be given (local) transformations forming a one-parameter group with respect to a, so that

$$g(z, \varepsilon, 0) = z, \quad g(g(z, \varepsilon, a), \varepsilon, b) = g(z, \varepsilon, a + b), \tag{1.1}$$

and depending on the small parameter ε. Suppose that $f \approx g$, i.e.,

$$g(z, \varepsilon, a) = g(g, \varepsilon, a) + o(\varepsilon^p). \tag{1.2}$$

Together with the points z' we introduce the "close" points $\tilde{z}$ defined by

$$\tilde{z} = f(z, \varepsilon, a). \tag{1.3}$$

It is easy to show by substituting (1.2) in (1.1) that (1.3) gives an approximate group in the sense of the following definition.

Definition 6.1. The transformations (1.3), or

$$z' \approx f(z, \varepsilon, a), \tag{1.4}$$

form a one-parameter *approximate transformation group* with respect to the parameter a if

$$f(z, \varepsilon, 0) \approx z, \tag{1.5}$$

$$f(f(z, \varepsilon, a), \varepsilon, b) \approx f(z, \varepsilon, a + b), \tag{1.6}$$

and the condition $f(z, \varepsilon, a) \approx z$ for all z implies that $a = 0$.

The main assertions about the infinitesimal description of local Lie groups remain true upon passing to approximate groups, with the exact equalities replaced by approximate equalities.

Theorem 6.2. (an approximate Lie theorem). Suppose that the transformations (1.4) form an approximate group with the tangent vector field

$$\xi(z, \varepsilon) \approx \left.\frac{\partial f(z, \varepsilon, a)}{\partial a}\right|_{a=0}. \tag{1.7}$$

Then the function $f(z, \varepsilon, a)$ satisfies

$$\frac{\partial f(z, \varepsilon, a)}{\partial a} \approx \xi(f(z, \varepsilon, a), \varepsilon). \tag{1.8}$$

Conversely, for any (smooth) function $\xi(z, \varepsilon)$ the solution (1.4) of the approximate Cauchy problem

$$\frac{dz'}{da} \approx \xi(z', \varepsilon), \tag{1.9}$$

$$z'|_{a=0} \approx z \tag{1.10}$$

determines an approximate one-parameter group with group parameter a.

Remark 6.1. Equation (1.9) will be called the approximate Lie equation.

Proof. Suppose that $f(z,\varepsilon,a)$ gives an approximate group of transformations (1.4). The (1.6) takes the form

$$f(f(z,\varepsilon,a),\varepsilon,0)+\frac{\partial f(f(z,\varepsilon,a),\varepsilon,b)}{\partial b}\Big|_{b=0}\cdot b+o(b)$$
$$\approx f(z,\varepsilon,a)+\frac{\partial f(z,\varepsilon,a)}{\partial a}\cdot b+o(b)$$

after the principal terms with respect to b are singled out. The approximate equation (1.8) is obtained from this by transforming the left-hand side with the help of (1.5) and (1.7), dividing by b, and passing to the limit as $b\to 0$.

Conversely, suppose that the function (1.4) is a solution of the approximate problem (1.9), (1.10). To prove that $f(z,\varepsilon,a)$ gives an approximate group it suffices to verify the approximate equality (1.6),

$$f(f(z,\varepsilon,a),\varepsilon,b)\approx f(z,\varepsilon,a+b).$$

Denote by $x(b,\varepsilon)$ and $y(b,\varepsilon)$ the left-hand and right-hand side of (1.6), regarded (for fixed z and a) as functions of (b,ε). By (1.9), they satisfy the same approximate Cauchy problem:

$$\frac{\partial x}{\partial b}\approx \xi(x,\varepsilon),\quad x|_{b=0}\approx g(z,\varepsilon,a),$$
$$\frac{\partial y}{\partial b}\approx \xi(y,\varepsilon),\quad y|_{b=0}\approx g(z,\varepsilon,a).$$

Therefore Theorem 6.1 furnishes the approximate equation $x(b,\varepsilon)\approx y(b,\varepsilon)$, i.e the group property (1.6).

§ 2 An algorithm for constructing an approximate group

The construction of an approximate group from a given infinitesimal operator is implemented on the basis of the approximate Lie theorem. To show how to solve the approximate Lie equation (1.9) we consider first the case $p=1$.

We seek the approximate group of transformations

$$z'\approx f_0(z,a)+\varepsilon f_1(z,a), \tag{2.1}$$

determined by the infinitesimal operator

$$X=(\xi_0(z)+\varepsilon\xi_1(z))(\partial/\partial z). \tag{2.2}$$

The corresponding approximate Lie equation

$$\frac{d(f_0 + \varepsilon f_1)}{da} \approx \xi_0(f_0 + \varepsilon f_1) + \varepsilon \xi_1(f_0 + \varepsilon f_1)$$

can be rewritten as the system

$$\frac{df_0}{da} \approx \xi_0(f_0), \quad \frac{df_1}{da} \approx \xi_0'(f_0) f_1 + \xi_1(f_0)$$

after singling out the principal terms with respect to ε, where ξ_0' is the derivative of ξ_0. The initial condition $z'|_{a=0} \approx z$ gives us that $f_0|_{a=0} \approx z$ and $f_1|_{a=0} \approx 0$.

Thus, according to the definition of a solution of the approximate Cauchy problem (§ 1), to construct the approximate (to within $o(\varepsilon)$) group (2.1) from the given infinitesimal operator (2.2) it suffices to solve the following (exact) Cauchy problem:

$$\frac{df_0}{da} = \xi_0(f_0), \quad \frac{df_1}{da} = \xi_0'(f_0) f_1 + \xi_1(f_0), \quad f_0|_{a=0} = z, \quad f_1|_{a=0} = 0. \tag{2.3}$$

Example 6.1. Suppose that $N = 1$ and $X = (1 + \varepsilon x)(\partial/\partial x)$. The corresponding problem (2.3)

$$\frac{df_0}{da} = 1, \quad \frac{df_1}{da} = f_0, \quad f_0|_{a=0} = z, \quad f_1|_{a=0} = 0$$

is easily solved, and gives us $f_0 = x + a$ and $f_1 = xa + a^2/2$. Consequently, the approximate group is determined by

$$x' \approx x + a + (xa + a^2/2)\varepsilon.$$

This formula is clearly the principal term in the Taylor series expansion with respect to ε of the exact group

$$x' = xe^{a\varepsilon} + \frac{e^{a\varepsilon - 1}}{\varepsilon} = (x + a) + a\left(x + \frac{a}{2}\right)\varepsilon + \frac{a^2}{2}\left(x + \frac{a}{3}\right)\varepsilon^2 + \ldots,$$

generated by the operator $X = (1 + \varepsilon x)(\partial/\partial x)$.

Example 6.2. Let us find the approximate group of transformations

$$x' \approx f_0^1(x, y, a) + \varepsilon f_1^1(x, y, a), \quad y' \approx f_0^2(x, y, a) + \varepsilon f_1^2(x, y, a)$$

determined by the operator

$$X = (1 + \varepsilon x^2)\frac{\partial}{\partial x} + \varepsilon xy \frac{\partial}{\partial y}$$

in the (x, y)-plane. After solving problem (2.3)

$$\frac{df_0^1}{da} = 1, \quad \frac{df_0^2}{da} = 0, \quad \frac{df_1^1}{da} = (f_0^1)^2, \quad \frac{df_1^2}{da} = f_0^1 f_0^2,$$

$$f_0^1|_{a=0} = x, \quad f_0^2|_{a=0} = y, \quad f_1^1|_{a=0} = 0, \quad f_1^2|_{a=0} = 0,$$

one obtains

$$x' \approx x + a + (x^2 a + x a^2 + a^3/3)\varepsilon, \quad y' \approx y + (xya + ya^2/2)\varepsilon.$$

To construct an approximate (to within $o(\varepsilon^p)$) group for arbitrary p we need a formula for the principal (with respect to ε) part of a function of the form $F(y_0 + \varepsilon y_1 + ... + \varepsilon^p y_p)$. By Taylor's formula,

$$F(y_0 + \varepsilon y_1 + \ldots + \varepsilon^p y_p) = F(y_0) + \sum_{|\sigma|=1}^{p} \frac{1}{\sigma!} F^{(\sigma)}(y_0)(\varepsilon y_1 + \ldots + \varepsilon^p y_p)^\sigma + o(\varepsilon^p), \tag{2.4}$$

where

$$F^{(\sigma)} = \frac{\partial^{|\sigma|} F}{(\partial z^1)^{\sigma_1} \ldots (\partial z^N)^{\sigma_N}}, \quad (\varepsilon y_1 + \ldots + \varepsilon^p y_p)^\sigma = \prod_{k=1}^{N} (\varepsilon y_1^k + \ldots + \varepsilon^p y_p^k)^k, \tag{2.5}$$

$\sigma = (\sigma_1, \ldots, \sigma_N)$ is a multi-index, $|\sigma| = \sigma_1 + ... + \sigma_N, \sigma! = \sigma!...\sigma_N!$, and the indices $\sigma_1, ..., \sigma_N$ run from 0 to p. In the last expression we single out the terms up to order ε^p :

$$\begin{aligned} \prod_{k=1}^{N} (\varepsilon y_1^k + \cdots + \varepsilon^p y_p^k)^{\sigma_k} &= \prod_{k=1}^{N} \left(\sum_{i_1, \ldots, i_{\sigma_k}=1}^{p} y_{i_1}^k \ldots y_{i_{\sigma_k}}^k \varepsilon^{i_1 + \cdots + i_{\sigma_k}} \right) \\ \approx \prod_{k=1}^{N} \left(\sum_{\nu_k=\sigma_k}^{p} \varepsilon^{\nu_k} \sum_{i_1 + \ldots + i_{\sigma_k} = \nu_k} y_{i_1}^k \ldots y_{i_{\sigma_k}}^k \right) &\equiv \prod_{k=1}^{N} \sum_{\nu_k=\sigma_k}^{p} \varepsilon^{\nu_k} y_{(\nu_k)}^k \\ \approx \sum_{j=|\sigma|} \varepsilon^j \left(\sum_{\nu_1 + \cdots + \nu_N = j} y_{(\nu_1)}^1 \ldots y_{(\nu_N)}^N \right) &\equiv \sum_{j=|\sigma|}^{p} \varepsilon^j \sum_{|\nu|=j} y_{(\nu)}. \end{aligned} \tag{2.6}$$

Here the notation is

$$y_{(\nu_k)}^k \equiv \sum_{i_1} + \cdots + i_{\sigma_k = \nu_k} y_{i_1}^k \ldots y_{i_{\sigma_k}}^k, \quad y_{(\nu)} = y_{(\nu_1)}^1 \ldots y_{(\nu_N)}^N, \tag{2.7}$$

where the indices $i_1, ..., i_{\sigma_k}$ run from 0 to p, and $\nu = \nu(\sigma) = (\nu_1, ..., \nu_N)$ is a multi-index associated with the multi-index σ in such a way that if the index σ_s in σ is equal to zero, then the corresponding index ν_s is absent in ν, and each of the remaining indices ν_k takes values from σ_k to p; for example, for $\sigma = (0, \sigma_2, \sigma_2, 0, ..., 0)$ with $\sigma_2, \sigma_3 \neq 0$ we have that $\nu = (\nu_2, \nu_3)$, so that $y_{(\nu)} = y^2_{(\nu_2)} y^3_{(\nu_3)}$.

Substituting (2.6) into (2.4) and interchanging the summations over σ and j, we get the following formula for the principal part:

$$F(y_0+\varepsilon y_1+\ldots+\varepsilon^p y_p) = F(y_0)+\sum_{j=1}^{p} \varepsilon^j \sum_{|\sigma|=1}^{j} \frac{1}{\sigma!} F^{(\sigma)}(y_0) \sum_{|\nu|=j} y_{(\nu)} + o(\varepsilon^p), \quad (2.8)$$

where the notation in (2.5) and (2.7) has been used. For example,

$$F(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \varepsilon^3 y_3) =$$

$$= F(y_0) + \varepsilon \sum_{k=1}^{N} \frac{\partial F(y_0)}{\partial z^k} y_1^k$$

$$+\varepsilon^2 \left(\sum_{k=1}^{N} \frac{\partial F(y_0)}{\partial z^k} y_2^k + \frac{1}{2} \sum_{k=1}^{N} \sum_{l=1}^{N} \frac{\partial^2 F(y_0)}{\partial z^k \partial z^l} y_1^k y_1^l \right)$$

$$+\varepsilon^3 \left(\sum_{k=1}^{N} \frac{\partial F(y_0)}{\partial z^k} y_3^k + \frac{1}{2} \sum_{k=1}^{N} \sum_{l=1}^{N} \frac{\partial^2 F(y_0)}{\partial z^k \partial z^l} \times (y_1^k y_2^l + y_1^l y_2^k) \right.$$

$$\left. + \frac{1}{3!} \sum_{k=1}^{N} \sum_{l=1}^{N} \sum_{m=1}^{N} \frac{\partial^2 F(y_0)}{\partial z^k \partial z^l \partial z^m} y_1^k y_1^l y_1^m \right) + \sigma(\varepsilon^3).$$

We also need a generalization of (2.8) for the expression

$$\sum_{i=0}^{p} \varepsilon^i F_i(y_0 + \varepsilon y_1 + \cdots + \varepsilon^p y_p).$$

Applying (2.8) to each function F_i and introducing for brevity the notation

$$\tau_{j,i} = \sum_{|\sigma=1|}^{j} \frac{1}{\sigma!} F_i^{(\sigma)}(y_0) \sum_{|\nu|=j} y_{(\nu)},$$

we have that

$$\sum_{i=0}^{p} \varepsilon^i F_i(y_0 + \varepsilon y_1 + \cdots + \varepsilon^p y_p) \approx \sum_{i=0}^{p} \varepsilon^i \left[F_i(y_0) + \sum_{j=1}^{p} \varepsilon^j \tau_{j,i} \right]$$

$$\approx \sum_{i=0}^{p} \varepsilon^i F_i(y_0) + \sum_{i=0}^{p-1} \sum_{j=1}^{p-i} \varepsilon^{i+j} \tau_{j,i}$$

to within $o(\varepsilon^p)$. The standard transformations are used to order the last term with respect to powers of ε :

$$\sum_{i=0}^{p-1} \sum_{j=1}^{p-i} \varepsilon^{i+j} \tau_{j,i} = \sum_{i=0}^{p-1} \sum_{l=i+1}^{p} \varepsilon^l \tau_{l-i,j} = \sum_{l=1}^{p} \varepsilon^l \sum_{i=0}^{l-1} \tau_{l-i,j} = \sum_{l=1}^{p} \varepsilon^l \sum_{j=1}^{l} \tau_{j,l-j}.$$

As a result, we arrive at the following generalization of (2.8):

$$\sum_{i=0}^{p} \varepsilon^i F_i(y_0 + \varepsilon y_1 + \ldots + \varepsilon^p y_p)$$

$$\approx F_0(y_0) + \sum_{i=1}^{p} \varepsilon^i \left[F_i(y_0) + \sum_{j=1}^{i} \sum_{|\sigma|=1}^{j} \frac{1}{\sigma!} F_{i-j}^{(\sigma)}(y_0) \sum_{|\nu|=j} y_{(\nu)} \right] \tag{2.9}$$

with the same notation as in (2.5) and (2.7).

We now return to the construction of an approximate group to within $o(\varepsilon^p)$ with an arbitrary p. For the infinitesimal operator

$$X = [\xi_0(z) + \varepsilon \xi_1(z) + \ldots + \varepsilon^p \xi_p(z)] \frac{\partial}{\partial z}$$

the approximate group of transformations

$$z' \approx f_0(z, a) + \varepsilon f_1(z, a) + \ldots + \varepsilon^p f_p(z, a) \tag{2.10}$$

is determined by the approximate Lie equation

$$\frac{d}{da}(f_0 + \varepsilon f_1 + \ldots + \varepsilon^p f_p) \approx \sum_{i=0}^{p} \varepsilon^i \xi_i(f_0 + \varepsilon f_1 + \ldots + \varepsilon^p f_p). \tag{2.11}$$

Transforming the right-hand side of this equation according to (2.9) and equating the coefficients of like powers of ε, we get the system of equations (in the notation of (2.5) and (2.7))

$$\frac{df_0}{da} = \xi_0(f_0), \tag{2.12}$$

$$\frac{df_i}{da} = \xi_i(f_0) + \sum_{j=1}^{i} \sum_{|\sigma|=1}^{j} \frac{1}{\sigma!} \xi_{i-j}^{(\sigma)}(f_0) \sum_{|\nu|=j} f_\nu, \quad i = 1, \ldots, p, \tag{2.13}$$

which is equivalent to the approximate equation (2.11).

Accordingly, the problem of constructing the approximate group (2.10) reduces to the solution of system (2.12), (2.13) under the initial conditions

$$f_0|_{a=0} = z, \quad f_i|_{a=0}, \quad i = 1, \ldots, p. \tag{2.14}$$

For clarity we write out the first few equations of system (2.12), (2.13):

$$\begin{aligned}
\frac{df_0}{da} &= \xi_0(f_0),\\
\frac{df_1}{da} &= \sum_{k=1}^{N} \frac{\partial \xi_0(f_0)}{\partial z^k} f_1^k + \xi_1(f_0), \qquad (2.15)\\
\frac{df_2}{da} &= \sum_{k=1}^{N} \frac{\partial \xi_0(f_0)}{\partial z^k} f_2^k + \frac{1}{2}\sum_{k=1}^{N}\sum_{l=1}^{N} \frac{\partial^2 \xi_0(f_0)}{\partial z^k \partial z^l} f_l^k f_1^l + \sum_{k=1}^{N} \frac{\partial \xi_1(f_0)}{\partial z^k} f_l^k + \xi_2(f_0).
\end{aligned}$$

Example 6.3. Let us write out system (2.12), (2.13) for the operator

$$X = (1 + \varepsilon x^2)\frac{\partial}{\partial x} + \varepsilon xy \frac{\partial}{\partial y}$$

in Example 6.2. In this case $N = 2, z = (x, y)$ and

$$\begin{aligned}
&f_k = (f_k^1, f_k^2), \quad k = 0, 1, \ldots, p,\\
&\xi_0 = (1, 0), \quad \xi_1 = (x^2, xy), \quad \xi_l = 0 \ (l \geq 2).
\end{aligned}$$

Eqs. (2.15) yield:

$$\begin{gathered}
df_0^1/da = 1, \quad df_0^2/da = 0;\\
df_1^1/da = (f_0^1)^2, \quad df_1^2/da = f_0^1 f_0^2;\\
df_2^1/da = 2 f_0^1 f_1^1, \quad df_2^2/da = f_0^2 f_1^1 + f_0^1 f_1^2.
\end{gathered}$$

For $i \geq 3$ equation (2.13) simplifies because of the special form of the vector ξ. Namely, since $\xi_0 =$ const. and $\xi_l = 0$ for $l \geq 2$, only terms with $j = i - 1$ are present on the right-hand side of (2.13), and the latter can be written in the form

$$\frac{df_i}{da} = \sum_{|\sigma|=1}^{i-1} \frac{1}{\sigma!} \xi_1^{(\sigma)}(f_0) \sum_{|\nu|=i-1} f_{(\nu)}.$$

A further simplification of these equations has to do with the form of the vector ξ_1 : since $\xi_1^1 = x^2$ and $\xi_1^2 = xy$, only $\sigma = (1, 0)$ and $\sigma = (2, 0)$ are used in the expression for the first component of the equations under consideration, and only σ equal to$(1,0),(0,1)$, and $(1,1)$ in the expression for the second component. As a result we have the following recurrence system:

$$\frac{df^1}{da} = 2f_0^1 f_{i-1}^1 + \sum_{i_1+i_2=i-1} f_{i_1}^1 f_{i_2}^1,$$

Example 6.4. We compute the approximate group of transformations of order ε^p generated by the operator

$$X = (1+\varepsilon x)\frac{\partial}{\partial x}$$

in Example 6.1. In this case system (2.12), (2.13) takes the form

$$\frac{df_0}{da} = 1, \quad \frac{df_i}{da} = f_{i-1}, \quad i = 1,\ldots,p,$$

and, under the initial conditions (2.14), gives us

$$f_i = \frac{xa^i}{i!} + \frac{a^{i+1}}{(i+1)!}, \quad i = 0,\ldots,p.$$

The corresponding approximate group of transformations is determined by

$$x' \approx \sum_{i=0}^{p} \frac{a^i}{i!}\left(x + \frac{a}{i+1}\right)\varepsilon^i.$$

§ 3 A criterion for approximate invariance

Definition 6.2. The approximate equation

$$F(z,\varepsilon) \approx 0 \tag{3.1}$$

is said to be invariant with respect to the approximate group of transformations $z' \approx f(z,\varepsilon,a)$ if

$$F(f(z,\varepsilon,a),\varepsilon) \approx 0 \tag{3.2}$$

for all $z = (z^1,...,z^N)$ satisfying (3.1).

Theorem 6.3. Suppose that the function

$$F(z,\varepsilon) = (F^1(z,\varepsilon),...,F^n(z,\varepsilon)), \quad n < N,$$

which is jointly analytic in the variables z and ε, satisfies the condition

$$\operatorname{rank} F'(z,0)|_{F(z,0)=0} = n, \tag{3.3}$$

where $F'(z,\varepsilon) = \|\partial F^\nu(z,\varepsilon)/\partial z^i\|$ for $\nu = 1,...,n$ and $i = 1,...,N$. For the approximate equation (3.1)

$$F(z,\varepsilon) = o(\varepsilon^p)$$

to be invariant under the approximate group of transformations

$$z' = f(z,\varepsilon,a) + o(\varepsilon^p)$$

with infinitesimal operator

$$X = \xi(z,\varepsilon)\frac{\partial}{\partial z}, \quad \xi = \frac{\partial f}{\partial a}|_{a=0} + o(\varepsilon^p), \tag{3.4}$$

it is necessary and sufficient that

$$XF(z,\varepsilon)|_{(3.1)} = o(\varepsilon^p). \tag{3.5}$$

Proof. *Necessity.* Suppose that condition (3.2) for invariance of the approximate equation (3.1) holds:

$$F(f(z,\varepsilon,a),\varepsilon)|_{(3.1)} = o(\varepsilon^p).$$

By differentiation with respect to a at $a = 0$, this yields (3.5).

Sufficiency. Suppose now that (3.5) holds for a function $F(z,\varepsilon)$ satisfying (3.3). Let us prove the invariance of the approximate equation (3.1). To do this we introduce the new variables

$$y^1 = F^1(z,\varepsilon), ..., \; y^n = F^n(z,\varepsilon), \; y^{n+1} = H^1(z,\varepsilon), ..., \; y^N = H^{N-n}(z,\varepsilon)$$

instead of $z^1, ..., z^N$, choosing $H^1(z,\varepsilon), ..., H^{N-n}(z,\varepsilon)$ so that the functions $F^1, ..., F^n, H^1, ..., H^{N-n}$ are functionally independent (for sufficiently small ε this is possible in view of condition (3.3)). In the new variables the original approximate equation (3.1), the operator (3.4), and condition (3.5) take the respective forms

$$y^\nu = \theta^\nu_p(y,\varepsilon), \quad \nu = 1,\dots,n, \tag{3.6}$$

$$X = \eta^i(y,\varepsilon)\frac{\partial}{\partial y^i}, \quad \text{where} \quad \eta^i \approx \xi^j(x,\varepsilon)\frac{\partial y^i(x,\varepsilon)}{\partial x^j}, \tag{3.7}$$

$$\eta^\nu(\theta^1_p,\dots,\theta^n_p,y^{n+1},\dots,y^N) = o(\varepsilon) \quad \nu = 1,\dots,n, \tag{3.8}$$

where $\theta_p^\nu = o(\varepsilon^p)$ (see Eq. (1)). By Theorem 6.2, the transformations of the variables y are determined from the approximate Cauchy problem

$$\frac{dy'^\nu}{da} \approx \eta^\nu(y'^1, \ldots, y'^n, y'^{n+1}, \ldots, y'^N, \varepsilon), \quad y'^\nu|_{a=0} = \theta_p^\nu(y, \varepsilon),$$

$$\frac{dy'^k}{da} \approx \eta^k(y'^1, \ldots, y'^n, y'^{n+1}, \ldots, y'^N, \varepsilon), \quad y'^k|_{a=0} = y^k, \quad k = n+1, \ldots, N,$$

where the initial conditions for the first subsystem are written with (3.1) taken into account. According to Theorem 6.1, the solution of this problem is unique (with the accuracy under consideration) and has the form

$$y' = (\theta_p^1, ..., \theta_p^n, y^{n+1}, ..., y^N)$$

in view of (3.8). Returning to the old variables, we get that

$$F^\nu(z, \varepsilon) = o(\varepsilon^p), \nu = 1, ..., n,$$

i.e., the approximate equation (3.2). The theorem is proved.

Example 6.5. Let $N = 2, z = (x, y)$, and $p = 1$. We consider the approximate group of transformations (see Example 6.2 in § 2)

$$x' \approx x + a + \left(x^2 a + x a^2 + \frac{1}{3} a^3\right) \varepsilon, \quad y' \approx y + \left(xya + \frac{1}{2} y a^2\right) \varepsilon \tag{3.9}$$

with the generator

$$X = (1 + \varepsilon x^2)\frac{\partial}{\partial x} + \varepsilon xy \frac{\partial}{\partial y} \cdot \tag{3.10}$$

Let us show that the approximate equation

$$F(x, y, \varepsilon) \equiv y^{2+\varepsilon} - \varepsilon x^2 - 1 = o(\varepsilon) \tag{3.11}$$

is invariant with respect to the transformations (3.10). We first verify the invariance of (3.11), following Definition 6.2. For this it is convenient to rewrite (3.11), while preserving the necessary accuracy, in the form

$$\tilde{F}(x, y, \varepsilon) \equiv y^2 - \varepsilon(x^2 - y^2 \ln y) - 1 \approx 0. \tag{3.12}$$

After the transformation (3.9) we have that

$$\tilde{F}(x', y', \varepsilon) = y'^2 + \varepsilon(x'^2 - y'^2 \ln y') - 1$$

$$\approx y^2 - \varepsilon(x^2 - y^2 \ln y) - 1 + \varepsilon(2xa + a^2)(y^2 - 1)$$

$$= \tilde{F}(x, y, \varepsilon) + \varepsilon(2xa + a^2)[\tilde{F}(x, y, \varepsilon) + \varepsilon(x^2 - y^2 \ln y)]$$

$$= [1 + \varepsilon(2ax + a^2)]\tilde{F}(x, y, \varepsilon) + o(\varepsilon),$$

which implies the necessary equality (3.2): $\tilde{F}(x', y', \varepsilon)|_{(3.12)} \approx 0$.

The function $F(x, y, \varepsilon)$ satisfies condition (3.3) of Theorem 6.3; therefore, the variance can be established also with the help of the infinitesimal criterion (3.5). the operator (3.10) we have that

$$XF = (2 + \varepsilon)\varepsilon x y^{2+\varepsilon} - 2\varepsilon x(1 + \varepsilon x^2)$$

$$= 2\varepsilon x(y^{2+\varepsilon} - 1) + o(\varepsilon) = 2\varepsilon x F + o(\varepsilon),$$

so that the satisfaction of the invariance criterion (3.5) is obvious.

According to Theorem 6.3, the construction of the approximate group leaving the equation $F(z, \varepsilon) \approx 0$ invariant reduces to the solution of the determining equation

$$XF(z, \varepsilon)\Big|_{F\approx 0} \approx 0 \tag{3.13}$$

for the coordinates $\xi^k(z, \varepsilon)$ of the operator

$$X = \xi \frac{\partial}{\partial z} \cdot$$

To solve the determining equation (3.13) to within $o(\varepsilon^p)$ it is necessary to represent z, F, and ξ^k in the form

$$z \approx y_0 + \varepsilon y_1 + \cdots + \varepsilon^p y_p,$$

$$F(z, \varepsilon) \approx \sum_{i=0}^{p} \varepsilon^i F_i(z), \tag{3.14}$$

$$\xi^k(z, \varepsilon) \approx \sum_{i=0}^{p} \varepsilon^i \xi_i^k(z), \tag{3.15}$$

substitute them in XF, and single out their principal terms. We have

$$XF = \xi^k \frac{\partial F}{\partial z^k}$$

$$= \left[\sum_{i=0}^{p} \varepsilon^i \xi_i^k (y_0 + \varepsilon y_1 + \ldots + \varepsilon^p y_p)\right] \cdot \left[\sum_{j=0}^{p} \frac{\partial}{\partial z^k} F_j (y_0 + \varepsilon y_1 + \ldots + \varepsilon^p y_p)\right].$$

Using (2.9) and the notation

$$A_i^k \xi_i^k(y_0) + \sum_{j=1}^{i} \sum_{|\sigma|=1}^{j} \frac{1}{|\sigma!|} (\xi_{i-j}^k)^{(\sigma)}(y_0) \sum_{|\nu|=j} y_{(\nu)}, \tag{3.16}$$

$$B_{j,k} = \frac{\partial F_j(y_0)}{\partial z^k} + \sum_{i=1}^{j} \sum_{|\omega|=1}^{i} \frac{1}{|\omega!|} \left(\frac{\partial F_{j-1}}{\partial z^k} \right)^{(\omega)} (y_0) \sum_{|\mu|=i} y_{(\mu)}, \tag{3.17}$$

we get that

$$XF = \left[\xi_0^k(y_0) + \sum_{i=1}^{p} \varepsilon^i A_i^k \right] \cdot \left[\frac{\partial F_0(y_0)}{\partial z^k} + \sum_{j=1}^{p} \varepsilon^j B_{j,k} \right],$$

which implies

$$XF = \xi_0^k(y_0) \frac{\partial F_0(y_0)}{\partial z^k} + \varepsilon \left[\xi_0^k(y_0) B_{1,k} + A_1^k \frac{\partial f_0(y_0)}{\partial z^k} \right]$$
$$+ \sum_{s=2}^{p} \left[\xi_0^k(y_0) B_{s,k} + A_s^k \frac{\partial F_0(y_0)}{\partial z^k} + \sum_{i+j=s} A_i^k B_{j,k} \right]. \tag{3.18}$$

Combining (3.13)-(3.18) and (2.9), we arrive at the following form of the determining equation:

$$\begin{gathered} \xi_0^k(y_0) \frac{\partial F_0(y_0)}{\partial z^k} = 0, \quad \xi_0^k B_{1,k} + A_1^k \frac{\partial F_0(y_0)}{\partial z^k} = 0, \\ \xi_0^k(y_0) B_{l,k} \frac{\partial F_0(y_0)}{\partial z^k} + \sum_{i+j=l} A_i^k B_{j,k} = 0, \quad l = 2, \dots, p; \end{gathered} \tag{3.19}$$

these equations hold on the set of all $y_0, ..., y_p$ satisfying the system

$$F_0(y_0) = 0, \quad F_i(y_0) + \sum_{j=1}^{i} \sum_{|\sigma|=1}^{j} \frac{1}{|\sigma!|} F_{i-j}^{(\sigma)}(y_0) \sum_{|\nu|=j} y_{(\nu)}, \quad i = 1, \dots, p, \tag{3.20}$$

which is equivalent to the approximate equation (3.1). Thus, the problem of solving the approximate determining equation (3.13) has been reduced to the solution of the system of exact equations (3.19), (3.20).

We write the determining equations for $p = 1$. Equations (3.19) and (3.20) give us*

$$\xi_0^k(y_0) \frac{\partial F_0(y_0)}{\partial z^k} = 0, \tag{3.21}$$

*Here, as everywhere in this section, the following notation is used for brevity:

$$y_1^l \frac{\partial}{\partial z^l} \left(\xi_0^k \frac{\partial F_0(y_0)}{\partial \partial z^k} \right) = \sum_{l=1}^{N} \sum_{k=1}^{N} y_1^l \frac{\partial}{\partial z_1^l} \frac{\partial}{\partial z^l} \left(\xi_0^k(z) \frac{\partial f_0(z)}{\partial z^k} \right) \Big|_{z=y_0}.$$

$$\xi_1^k(y_0)\frac{\partial F_0(y_0)}{\partial z^k} + \xi_0^k(y_0)\frac{\partial F_1(y_0)}{\partial z^k} = 0 + y_1^l \frac{\partial}{\partial z^l}\left(\xi_0^k(y_0)\frac{\partial F_0(y_0)}{\partial z^k}\right) = 0 \quad (3.22)$$

under the conditions

$$F_0(y_0) = 0, \quad F_1(y_0) + y_1^l \frac{\partial F_0(y_0)}{\partial z^l} = 0. \quad (3.23)$$

Example 6.6. Let us again consider the approximate equation (3.11)

$$F(x, y, \varepsilon) \equiv y^{2+\varepsilon} - \varepsilon x^2 - 1 = o(\varepsilon)$$

from Example 6.5. In the notation of (3.14) (see also (3.12)) we have

$$F_0(x, y) = y^2 - 1, \quad F_1(x, y) = y^2 \ln y - x^2.$$

Since $y > 0$, the equations (3.23) imply that $y_0 = 1$ and $y_1 = x_0^2/2$, and the determining equations (3.21) and (3.22) can be written in the form

$$\begin{gathered} \xi_0^2(x_0, y_0) = 0, \quad \frac{\partial \xi_0^2(x_0, y_0)}{\partial x} = 0, \\ y_0\xi_1^2(x_0, y_0) - x_0\xi_0^1(x_0, y_0) - x_0\xi_0^1(x_0, y_0) + \frac{x_0^2}{2}\frac{\partial \xi_0^2(x_0, y_0)}{2} = 0 \end{gathered} \quad (3.24)$$

after splitting with respect to x_1 and substituting $y_1 = x_0^2/2$. Any operator

$$X = [\xi_0^1(x, y) + \varepsilon\xi_1^1(x, y)]\frac{\partial}{\partial x} + [\xi_0^2(x, y) + \varepsilon\xi_1^2(x, y)]\frac{\partial}{\partial y}$$

with coordinates satisfying (3.24) with $y_0 = 1$ and arbitrary values of x_0 generates an approximate group leaving (3.11) invariant (to within $o(\varepsilon)$). For example,

$$X_1 = x\frac{\partial}{\partial x} + 2(y - 1)\frac{\partial}{\partial y}, \quad X_2 = xy\frac{\partial}{\partial x} + (y^2 - 1)\frac{\partial}{\partial y}$$

are such operators, along with (3.10).

Remark 6.2. If some variables z^k do not enter in the equation $F(z, \varepsilon) \approx 0$, then it is unnecessary to represent z^k in the form $\sum_{i\geq 0} y_i^k \varepsilon^i$ in the determining equation (3.19).

Table 1: Group classification of the equations (4.4)

	$\varphi(u)$	ξ_0^1	ξ_0^2	η_0
	Arbitrary function	$C_1t + C_2$	$C_1x + C_3$	0
1	ku^σ	$C_1t + C_2$	$C_3x + C_4$	$\frac{2}{\sigma}(C_3 - C_1)u$
2	$ku^{-\frac{4}{3}}$	$C_1t + C_2$	$C_3x^2 + C_4x + C_5$	$-\frac{3}{2}(2C_3x + C_4 - C_1)u$
3	ku^{-4}	$C_1t^2 + C_2t + C_3$	$C_4x + C_5$	$\big(C_1t + \frac{1}{2}(C_2 - C_4)\big)\, u$
4	$k\mathrm{e}^u$	$C_1t + C_2$	$C_3x + C_4$	$2(C_3 - C_1)$
$k = \pm 1, \sigma$ is an arbitrary parameter, and $C_1, \ldots, C_5 = \text{const.}$				

§ 4 Approximate symmetries of the equation $u_{tt} + \varepsilon u_t = (\varphi(u)u_x)_x$

The approximate symmetries (understood either as admissible approximate groups or as their infinitesimal operators) of differential equations can be computed according to the algorithm in § 3 with the use of the usual technique for prolongation of the infinitesimal operators by the necessary derivatives. Below, we consider approximate symmetries of first order ($p = 1$) and classify according to such symmetries second-order equations

$$u_{tt} + \varepsilon u_t = (\varphi(u)u_x)_x, \quad \varphi \neq \text{const.}, \tag{4.1}$$

with a small parameter, which arise in various applied problems (see, for example, [10]). The infinitesimal operator of an approximate symmetry is sought in the form

$$X = (\xi_0^1 + \varepsilon\xi_1^1)\frac{\partial}{\partial t} + (\xi_0^2 + \varepsilon\xi_1^2)\frac{\partial}{\partial x} + (\eta_0 + \varepsilon\eta_1)\frac{\partial}{\partial u}\,\cdot \tag{4.2}$$

The coordinates ξ and η of the operator (4.2) depend on t, x, and u and occur in the determining equations (3.21) and (3.22), in which

$$z = (t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}), \quad F_0 = u_{tt} - (\varphi(u)u_x)_x, \quad F_1 = u_t;$$

according to Remark 6.2 (see § 3), it suffices to carry out the decomposition $z = y_0 + \varepsilon y_1$ only for the differentiable variable (since t and x do not appear explicitly in (4.1)): $u = u_0 + \varepsilon u_1, u_x = (u_0)_x + \varepsilon(u_1)_x$, and so on.

Equation (3.21) is the determining equation for the operator

$$X^0 = \xi_0^1\frac{\partial}{\partial t} + \xi_0^2\frac{\partial}{\partial x} + \eta_0\frac{\partial}{\partial u}, \tag{4.3}$$

admitted by the zero approximate of equation (4.1), i.e., by the equation

$$u_{tt} = (\varphi(u)u_x)_x, \quad \varphi \neq \text{const.} \tag{4.4}$$

Consequently, the first step in the classification of the equations (4.1) according to approximate symmetries is the classification of the equations (4.4) according to exact symmetries. The second step is to solve the determining equation (3.22) with known F_0 and with values ξ_0^1, ξ_0^2, η_0 of the coordinates of the operator (4.3).

A group classification of the equations (4.4) (according to exact point symmetries) was obtained in [10], and its result can be written in the form of Table 1 with the use of dilations and translations.

We now pass to the second step in constructing approximate symmetries. Let us begin with an arbitrary function $\varphi(u)$. Substituting in (3.22) the values

$$\xi_0^1 = C_1 t + C_2, \quad \xi_0^2 = C_1 x + C_3, \quad \eta_0 = 0$$

we get that

$$C_1 = 0, \quad \xi_1^1 = K_1 t + K_2, \quad \xi_1^2 = K_1 x + K_3, \quad \eta_1 = 0, \quad K_i = \text{const.}$$

We now observe that equation (4.1) admits together with any admissible (exactly or approximately) operator X also the operator εX; such operators will be assumed to be inessential and omitted. In particular, the operators $\varepsilon(\partial/\partial t)$ and $\varepsilon(\partial/\partial x)$ are inessential, so that the constants K_2 and K_3 in the solution of the determining equation (3.22) can be set equal to zero. Thus, for an arbitrary function $\varphi(u)$ equation (4.1) admits three essential approximate symmetry operators, corresponding to the constants C_2, C_3, and K_1. The remaining cases in Table 1 are analyzed similarly. The result is summarized in Table 2, where for convenience in comparing approximate symmetries with exact ones we have given the operators admitted by equations (4.4) exactly, and those admitted by (4.1) exactly and approximately.

Note. In Table 2 bases of the admitted algebras are given for the exact symmetries, and generators for them are given for the approximate symmetries: a basis for the corresponding algebra is obtained by multiplying the generators bye and discarding the terms of order ε^2. For example, for $\varphi(u) = ku^{-4/3}$ equation (4.4) admits a 5-dimensional algebra, and (4.1) admits a 4-dimensional algebra of exact symmetries and a 10−dimensional

Table 2: Comparative table of exact and approximate symmetries

			Symmetries for (4.1)	
	$\varphi(u)$	Symmetries for (4.4)	Exact	Approximate
	Arbitrary function	$X_1^0 = \frac{\partial}{\partial t}\,,\ X_2^0 = \frac{\partial}{\partial x}\,,$ $X_3^0 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$	$Y_1 = X_1^0,$ $Y_2 = X_2^0$	$X_1 = X_1^0, \quad X_2 = X_2^0,$ $X_3 = \varepsilon X_3^0$
1	ku^σ	$X_4^0 = \sigma x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}$	$Y_3 = X_4^0$	$\widetilde{X}_3 = X_3^0 + \frac{\varepsilon t}{\sigma+4}\left(\frac{2}{\sigma}t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}\right),$ $X_4 = X_4^0$
2	$ku^{-4/3}$	$X_4^0 = 2x\frac{\partial}{\partial x} - 3u\frac{\partial}{\partial u}\,,$ $X_5^0 = x^2\frac{\partial}{\partial x} - 3xu\frac{\partial}{\partial u}$	$Y_3 = X_4^0,$ $Y_4 = X_5^0$	$\widetilde{X}_3 = X_3^0 - \frac{\varepsilon}{4}\left(t^2\frac{\partial}{\partial t} + 3tu\frac{\partial}{\partial u}\right),$ $X_4 = X_4^0, \quad X_5 = X_5^0$
3	ku^{-4}	$X_4^0 = 2x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}\,,$ $X_5^0 = t^2\frac{\partial}{\partial t} + tu\frac{\partial}{\partial u}$	$Y_3 = X_4^0,$ $Y_4 = X_5^0$	$X_4 = X_4^0,$ $X_5 = \varepsilon X_5^0$
4	ke^u	$X_4^0 = x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial u}$	$Y_3 = X_4^0$	$\widetilde{X}_3 = X_3^0 + \varepsilon t\left(\frac{t}{2}\frac{\partial}{\partial t} - \frac{\partial}{\partial u}\right),$ $X_4 = X_4^0$

algebra of approximate symmetries with basis

$$X_1 = \frac{\partial}{\partial t}\,, \quad X_2 = \frac{\partial}{\partial x}\,, \quad \widetilde{X}_3 = \Big(t - \frac{1}{4}\varepsilon t^2\Big)\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - \frac{3}{4}\varepsilon t u\frac{\partial}{\partial u}\,,$$
$$X_4 = 2x\frac{\partial}{\partial x} - 3u\frac{\partial}{\partial u}\,,\ X_5 = x^2\frac{\partial}{\partial x} - 3xu\frac{\partial}{\partial u}\,,\ X_6 = \varepsilon X_1,\ X_7 = \varepsilon X_2,$$
$$X_8 = \varepsilon\Big(t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}\Big), \quad X_9 = \varepsilon X_4, \quad X_{10} = \varepsilon X_5.$$

§ 5 Approximate symmetries of the equation $u_t = h(u)u_1 + \varepsilon H$

We Consider the class of evolution equations of the form

$$u_t = h(u)u_1 + \varepsilon H, \quad H \in \mathcal{A}, \tag{5.1}$$

which contains, in particular, the Korteweg-de Vries equation, the Burgers-Korteweg-de Vries equation, etc.

Theorem 6.4. Equation (5.1) approximately (with any degree of accuracy) inherits all the symmetries of the Hopf equation

$$u_t = h(u)u_1. \tag{5.2}$$

Namely, any canonical Lie-Bäcklund operator [58]

$$X^0 = f^0\frac{\partial}{\partial u} + \cdots$$

admitted by (5.2) gives rise to an approximate (of arbitrary order p) symmetry for (5.1) determined by the coordinate

$$f = \sum_{i=0}^{p}\varepsilon^i f^i, \quad f^i \in \mathcal{A}, \tag{5.3}$$

of the canonical operator

$$X = f\frac{\partial}{\partial u} + \cdots\,.$$

Proof. The approximate symmetries (5.3) of equation (5.1) are found from the determining equation (3.19), which in this case takes the form

$$f_t^0 - h(u)f_x^0 + \sum_{\alpha\geq 0}[D^\alpha(hu_1) - hu_{1+\alpha}]f_\alpha^0 - h'(u)u_1 f^0 = 0, \tag{5.4}$$

$$f_t^i - h(u)f_x^i + \sum_{\alpha\geq 0}[D^\alpha(hu_1) - hu_{1+\alpha}]f_\alpha^i - h'(u)u_1 f^i$$
$$= \sum_{\alpha\geq 0}[D^\alpha(f^{i-1})H_\alpha - f_\alpha^{i-1}D^\alpha(H)], i = 1, \ldots, p. \tag{5.5}$$

Equation (5.4) in f^0 is a determining equation for finding the exact group of transformations admitted by (5.2). Let f^0 be an arbitrary solution of (5.4) that is a differentiable function of order $k \geq 0$ and let H be a differentiable function of order $n \geq 1$, i.e.,

$$f_0 = f^0(t, x, u, ..., u_{k_0}), \quad H = H(t, x, u, ..., u_n).$$

We look for a solution f^1 of (5.5) in the form of a differentiable function of order $k_1 = n + k_0 - 1$. Then (5.5) is a linear first-order partial differential equation in the function f^1 of the $k_1 + 3$ arguments $t, x, u, u_1, ..., u_{k_1}$, and is hence solvable. Substitution of any solution $f^1(t, x, u, u_1, ..., u_{k_1},)$ in the right-hand side of (5.5) with $i = 2$ shows that f^2 can be found in the form of a differentiable function of order $k_2 = n + k_1 - 1$, and the corresponding equation for f^2 is solvable. The rest of the coefficients $f^i, i = 3, ..., p$, in (5.3) are determined recursively from (5.5). The theorem is proved.

It follows from Theorem 6.4 that, in particular, any point symmetry of (5.2) determined by the infinitesimal operator

$$Y = \theta(t, x, u)\frac{\partial}{\partial t} + [\varphi(x+tu, u) - t\psi(x+tu, u) - u\theta(t, x, u)]\frac{\partial}{\partial x} + \psi(x+tu, u)\frac{\partial}{\partial u}$$

with arbitrary functions φ, ψ and θ, or with the corresponding canonical Lie-Bäcklund operator with coordinate

$$f^0 = [\varphi(x + tu, u) - t\psi(x + tu, u)]u_1 - \psi(x + tu, u),$$

is approximately inherited by equation (5.1). For example, the Burgers-Korteweg-de Vries equation

$$u_t + uu_1 + \varepsilon(au_3 + bu_2) \tag{5.6}$$

to within $o(\varepsilon^2)$ admits the operator

$$f_u = \varphi(u)u_1 + \varepsilon(a\varphi' u_3 + 2a\varphi'' u_1 u_2 + \frac{1}{2}a\varphi''' u_1^3 + b\varphi' u_2 + b\varphi'' u_1^2)$$
$$+\varepsilon^2\Big(\frac{3}{5}a^2\varphi'' u_5 + \frac{5}{4}ab\varphi'' u_4 + \frac{1}{10}ab\varphi''\frac{u_2 u_3}{u_1} - \frac{1}{20}ab\varphi''\frac{u_2^3}{u_1^2} + \frac{2}{3}b^2\varphi'' u_3$$
$$+\frac{9}{5}a^2\varphi''' u_1 u_4 + 3a^2\varphi'' u_2 u_3 + \frac{7}{2}ab\varphi''' u_1 u_3 + \frac{23}{10}ab\varphi''' u_2^2 \tag{5.7}$$

$$+\frac{5}{3}b^2\varphi'''u_1u_2+\frac{23}{10}a^2\alpha^{IV}u_1^2u_3+\frac{31}{10}a^2\varphi^{IV}u_1u_2^2+\frac{15}{4}ab\varphi^{IV}u_1^2u_2$$
$$+\frac{1}{2}b^2\varphi^{IV}u_1^3+\frac{8}{5}a^2\varphi^{V}u_1^3u_2+\frac{1}{2}ab\varphi^{V}u_1^4+\frac{1}{8}a^2\varphi^{VI}u_1^5\Big)+o(\varepsilon^2).$$

Setting $a=1$ and $b=0$ in (5.7), we get a second-order approximate symmetry for the Korteweg-de Vries equation

$$u_t=uu_1+\varepsilon u_3. \tag{5.8}$$

We remark that in this case the coefficient f^k of the approximate symmetry (5.3) is a differential function of order $2k+1$ containing derivatives of φ of order $\geq k$. This implies that if $\varphi(u)$ is a polynomial, then the approximate symmetry becomes an exact Lie-Bäcklund symmetry; then we can set $\varepsilon=1$ and get exact symmetries of the equation

$$u_t=u_3+uu_1. \tag{5.9}$$

For example, for $p=2$ and $\varphi=u^2$ Eq. (5.7) yields (cf. [58], Section 18.2)

$$f_u=u^2u_1+4u_1u_2+2uu_3+\frac{6}{5}u_5.$$

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Paper 7

Preliminary group classification of equations $v_{tt} = f(x, v_x)v_{xx} + g(x, v_x)$

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A classification is given of equations $V_{tt} = f(v, v_x)v_{xx} + g(x, v_x)$ admitting an extension by one of the principal Lie algebra of the equation under consideration. The paper is one of few applications of a new algebraic approach to the problem of group classification: the method of preliminary group classification. The result of the work is a wide class of equations summarized in Table 2.

I. Introduction

The first general solution of the problem of group classification was given by Sophus Lie for an extensive class of second-order partial differential equations with two independent variables. In his paperl he gave a complete

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group classification of linear equations

$$A(x,y)u_{yy} + 2B(x,y)u_{xy} + C(x,y)u_{xx} \\ + a(x,y)u_y + b(x,y)u_x + c(x,y)u = 0. \tag{1.1}$$

An essential part of the classification was the utilization of equivalence transformations of Eq. (1.1) , i.e., arbitrary changes of independent variables

$$\bar{x} = f(x,y), \quad \bar{y} = g(x,y) \tag{1.2}$$

and linear transformations of the dependent one,

$$\bar{u} = \varphi(x)u, \quad \varphi \neq 0. \tag{1.3}$$

In another paper Lie accomplished the group classification of nonlinear equations of the form

$$u_{xy} = f(u). \tag{1.4}$$

Here, he again essentially used equivalence transformations.

Ames et al., [10] motivated by a number of physical problems, investigated group properties of quasilinear hyperbolic equations of the form

$$u_{tt} = [f(u)u_x]_x, \tag{1.5}$$

or

$$v_{tt} = f(v_x)v_{xx}, \tag{1.5a}$$

where

$$u = v_x. \tag{1.6}$$

Later, this investigation was generalized in [118] to equations of the form

$$u_{tt} = [f(x,u)u_x]_x \tag{1.7}$$

or

$$v_{tt} = f(x,v_x)v_{xx}, \tag{1.7a}$$

and in [119] to equations

$$u_{tt} = [f(u)u_x + g(x,u)]_x \tag{1.8}$$

or

$$v_{tt} = [f(v_x)v_{xx} + g(x,v_x). \tag{1.8a}$$

In this paper we investigate the problem of group classification of equations of the general form

$$v_{tt} = f(x,v_x)v_{xx} + g(x,v_x). \tag{1.9}$$

The study of Eq. (1.9) is stimulated not only by physical examples mentioned in [10], [118], [119] but also by other examples: nonlinear nonhomogeneous vibrating string, nonlinear telegraph equation, etc.

We remark, that because of a nonpoint character of the transformation (1.6), group properties of Eqs. (1.5), (1.7), and (1.8) are not completely identical to group properties of the corresponding Eqs. (1.5a), (1.7a), and (1.8a).

II. Invariance transformations and principal Lie algebra

Following the well-known monographs [106], [58], [99], [8], [9], [22], [113] on these arguments we write the invariance condition for Eq. (1.9) as

$$X_{(2)}[v_{tt} - f(x, v_x)v_{xx} - g(x, v_x)]|_{(1.9)} = 0. \tag{2.1}$$

Here, $X_{(2)}$ is the second prolongation of the infinitesimal operator

$$X = \xi^i(t, x, v)\frac{\partial}{\partial t} + \xi^2(t, x, v)\frac{\partial}{\partial x} + \eta(t, x, v)\frac{\partial}{\partial v} \tag{2.2}$$

obtained by the following prolongation formulas:

$$X_{(2)} = X + \zeta_1\frac{\partial}{\partial v_t} + \zeta_2\frac{\partial}{\partial v_x} + \zeta_{11}\frac{\partial}{\partial v_{tt}} + \zeta_{22}\frac{\partial}{\partial v_{xx}} \tag{2.3}$$

where

$$\begin{aligned} \zeta_1 &= D_t(\eta) - v_t D_t(\xi^1) - v_x D_t(\xi^2), \\ \zeta_2 &= D_x(\eta) - v_t D_x(\xi^1) - v_x D_x(\xi^2), \\ \zeta_{11} &= D_t(\zeta_1) - v_{tt} D_t(\xi^1) - v_{tx} D_t(\xi^2), \\ \zeta_{22} &= D_x(\zeta_2) - v_{tx} D_x(\xi^1) - v_{xx} D_x(\xi^2). \end{aligned} \tag{2.1$'$}$$

The operators D_t, and D_x denote the total derivatives with respect to t and x :

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + v_t\frac{\partial}{\partial v} + v_{tt}\frac{\partial}{\partial v_t} + v_{tx}\frac{\partial}{\partial v_x} + \cdots, \\ D_x &= \frac{\partial}{\partial x} + v_x\frac{\partial}{\partial v} + v_{tx}\frac{\partial}{\partial v_t} + v_{xx}\frac{\partial}{\partial v_x} + \cdots. \end{aligned} \tag{2.4}$$

The term "total" is to distinguish D_t, and D_x from partial derivatives $\partial/\partial t$ and $\partial/\partial x$.

So, after substituting (2.2) and (2.3) in (2.1) we obtain the following determining equation:

$$[\zeta_{11} - f\zeta_{22} - v_{xx}(\xi^2 f_x + \zeta_2 f_{v_x}) - \xi^2 g_x - \zeta_2 g_{v_x}]|_{(1.9)} = 0. \tag{2.5}$$

In the case of arbitrary f and g it follows

$$\xi^2 = 0, \quad \zeta_2 = 0, \quad \zeta_{22} = \zeta_{11} = 0 \tag{2.6}$$

or

$$\xi^1 = c_1, \quad \xi^2 = 0, \quad \eta = c_2 + c_3 t. \tag{2.7}$$

Therefore, for arbitrary $f(x, v_x)$ and $g(x, v_x)$ Eq. (1.9) admits the three-dimensional Lie algebra L_3 with the basis

$$X_t = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial v}, \quad X_3 = t\frac{\partial}{\partial v}\cdot \tag{2.8}$$

We call L_3 the principal Lie algebra for Eq. (1.9). So, the remaining part of the group classification is to specify the coefficients f and g such that Eq. (1.9) admits an extension of the principal algebra L_3. Usually, the group classification is obtained by inspecting the determining equation. But in our case the complete solution of the determining equation (2.5) is a wasteful venture. Therefore, we don't solve the determining equation but, instead we obtain a partial group classification of Eq. (1.9) via the so-called method of preliminary group classification.

This method was suggested in [7] and applied when an equivalence group is generated by a finite-dimensional Lie algebra $L_{\mathcal{E}}$. The essential part of the method is the classification of all nonsimilar subalgebras of $L_{\mathcal{E}}$. Actually, the application of the method is simple and effective when the classification is based on finite-dimensional equivalence algebra $L_{\mathcal{E}}$.

III. Equivalence transformations

An equivalence transformation is a non-degenerate change of the variables t, x, v taking any equation of the form (1.9) into an equation of the same form, generally speaking, with different $f(x, v_x)$ and $g(x, v_x)$. The set of all equivalence transformations forms an equivalence group $\mathcal{E}$. We shall find a continuous subgroup $\mathcal{E}_c$ of it making use of the infinitesimal method [106].

We introduce the local notation $f = f^1, g = f^2$ and seek for an operator of the group $\mathcal{E}_c$ in the form

$$Y = \xi^1\frac{\partial}{\partial t} + \xi^2\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial v} + \mu^k\frac{\partial}{\partial f^k} \tag{3.1}$$

from the invariance conditions of Eq. (1.9) written as the system:

$$v_{tt} - f^1 v_{xx} - f^2 = 0,$$

$$f_t^k = f_v^k = f_{v_t}^k = 0 \tag{3.2}$$

Here, v and f^k are considered as differential variables: v on the space (t,x) and f^k on the extended space (t,x,v,v_t,v_x). The coordinates ξ^1,ξ^2,η of the operator (3.1) are sought as functions of t,x,v while the coordinates, μ^k are sought as functions of t,x,v,v_t,v_x,f^1,f^2.

The invariance conditions of the system (3.2) are

$$\widetilde{Y}(v_{tt} - f^1 v_{xx} - f^2) = 0, \tag{3.3}$$

$$\tilde{Y}(f_t^k) = \tilde{Y}(f_v^k) = \tilde{Y}(f_{v_t}^k) = 0 \quad (k = 1,2), \tag{3.3a}$$

where $\tilde{Y}$ is the prolongation of the operator (3.1):

$$\tilde{Y} = Y + \zeta_1 \frac{\partial}{\partial v_t} + \zeta_2 \frac{\partial}{\partial v_x} + \zeta_{11} \frac{\partial}{\partial v_{tt}} + \zeta_{22} \frac{\partial}{\partial v_{xx}}$$

$$+\omega_1^k \frac{\partial}{\partial f_t^k} + \omega_0^k \frac{\partial}{\partial f_v^k} + \omega_{01}^k \frac{\partial}{\partial f_{v_t}^k} \cdot \tag{3.4}$$

The coefficients $\zeta_1,\zeta_2,\zeta_{11},\zeta_{22}$ are given in (2.3) and the other coefficients of (3.4) are obtained by applying the prolongation procedure to differential variables f^k with independent variables (t,x,v,v_t,v_x). For instance

$$\omega_1^k = \tilde{D}_t(\mu^k) - f_t^k \tilde{D}_t(\xi^1) - f_x^k \tilde{D}_t(\xi^2) - f_v^k \tilde{D}_t(\eta) - f_{v_t}^k \tilde{D}_t(\zeta_1) - f_{v_x}^k \tilde{D}_t(\zeta_2), \tag{3.5}$$

where

$$\tilde{D}_t = \frac{\partial}{\partial t} + f_t^k \frac{\partial}{\partial f^k} \cdot \tag{3.6}$$

In view of Eq. (3.2) we have

$$\tilde{D}_t = \frac{\partial}{\partial t} \cdot \tag{3.6a}$$

We obtain the coefficients ω_0 and ω_{01} from (3.5) replacing the operator $\tilde{D}_t$ by the operators

$$\tilde{D}_v = \frac{\partial}{\partial v} + f_v^k \frac{\partial}{\partial f^k} \tag{3.7}$$

and

$$\tilde{D}_{v_t} = \frac{\partial}{\partial v_t} + f_{v_t}^k \frac{\partial}{\partial f^k}, \tag{3.8}$$

respectively. In view of Eqs. (3.2) they become

$$\tilde{D}_v = \frac{\partial}{\partial v}, \tag{3.7a}$$

$$\tilde{D}_{v_t} = \frac{\partial}{\partial v_t}\,, \tag{3.8a}$$

So, we have the following prolongation formulas:

$$\begin{aligned} \omega_1^k &= \mu_t^k - f_x^k \xi_t^2 - f_{v_x}^k (\zeta_2)_t, \\ \omega_0^k &= \mu_v^k - f_x^k \xi_v^2 - f_{v_x}^k (\zeta_2)_v, \\ \omega_{01}^k &= \mu_{v_t}^k - f_{v_x}^k (\zeta_2)_{v_t}. \end{aligned} \tag{3.9}$$

After (3.4), the invariance conditions (3.3a) give rise to

$$\omega_1^k = \omega_0^k = \omega_{01}^k = 0, \quad k = 1, 2. \tag{3.10}$$

So, taking into account Eqs. (3.9) and the fact that (3.10) must hold for every f^1 and f^2, we obtain

$$\begin{aligned} \mu_t^k = \mu_v^k &= \mu_{v_t}^k = 0, \\ \xi_t^2 &= \xi_v^2 = 0, \\ (\zeta_2)_t = (\zeta_2)_v &= (\zeta_2) v_t = 0. \end{aligned} \tag{3.11}$$

After easy calculations we find from (3.11)

$$\begin{aligned} \xi^1 = \xi^1(t), &\quad \xi^2 = \xi^2(x) \\ \eta = c_1 v + F(x) + H(t), &\quad c_1 = \text{const.} \\ \mu^k = \mu^k(x, v_x, f^1, f^2) &\quad (k = 1, 2). \end{aligned} \tag{3.12}$$

The remaining invariance condition of (3.3), after (3.4) can be written as,

$$\zeta_{11} - \mu^1 v_{xx} - f^1 \zeta_{22} - \mu^2 = 0. \tag{3.13}$$

From (3.13), taking into account (3.12), and introducing the relation $v_{tt} = f^1 v_{xx} + f^2$ to eliminate $v_t t$, it follows

$$\begin{aligned} (\xi^1)'' v_t + \{[c_1 - 2(\xi^1)'] f^1 - \mu^1 - [c_1 - 2(\xi^2)'] f^1\} v_{xx} + [c_1 - 2(\xi^1)'] f^2 \\ + H'' - f^1 F'' + f^1 v_x (\xi^2)'' - \mu^2 = 0. \end{aligned} \tag{3.14}$$

Since in Eq. (3.14) the quantities v, v_t, v_x, and v_{xx} are considered to be independent variables it follows

$$\begin{aligned} \xi^1 = c_2 t + c_3, &\quad \xi^2 = \varphi(x), \\ \eta = c_1 v + F(x) &+ c_4 t^2 + c_5 t, \end{aligned} \tag{3.15}$$

$$\mu^1 = 2(\varphi' - c_2)f,$$
$$\mu^2 = (c_1 - 2c_2)g + 2c_4 + (\varphi'' v_x - F'')f,$$

with constants c_1, c_2, c_3, c_4, c_5, and two arbitrary functions $\varphi(x)$ and $F(x)$.

We summarize: The class of Eqs. (1.9) has an infinite continuous group of equivalence transformations generated by the following infinitesimal operators:

$$\begin{gathered} Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial v}, \quad Y_3 = t\frac{\partial}{\partial v}, \\ Y_4 = x\frac{\partial}{\partial v}, \quad Y_5 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2v\frac{\partial}{\partial v}, \\ Y_6 = t\frac{\partial}{\partial t} - 2f\frac{\partial}{\partial f} - 2g\frac{\partial}{\partial g}, \\ Y_7 = t^2\frac{\partial}{\partial v} + 2\frac{\partial}{\partial g}, \\ Y_\varphi = \varphi\frac{\partial}{\partial x} + 2\varphi' f\frac{\partial}{\partial f} + \varphi'' v_x f\frac{\partial}{\partial g}, \\ Y_F = F\frac{\partial}{\partial v} - F'' f\frac{\partial}{\partial g} \cdot \end{gathered} \tag{3.16}$$

Moreover, in the group of equivalence transformations are included also discrete transformations, i.e., reflections

$$t \mapsto -t, \tag{3.17}$$

$$x \mapsto -x, \tag{3.18}$$

$$v \mapsto -v, \quad g \mapsto -g. \tag{3.19}$$

Remark 7.1. The operator Y_2 from (3.16) is included in the set of operators y_F as the particular case $F = 1$. The reason for individualizing Y_2 is that it is part of the principal Lie algebra L_3 [see (2.8)].

IV. Sketch of the method of preliminary group classification

One can observe in many applications of group analysis that most of extensions of the principal Lie algebra admitted by the equation under consideration are taken from the equivalence algebra $L_{\mathcal{E}}$. We call these extensions $\mathcal{E}$-extensions of the principal Lie algebra.

The classification of all nonequivalent equations (with respect to a given equivalence group $G_{\mathcal{E}}$,) admitting $\mathcal{E}$-extensions of the principal Lie algebra is called a preliminary group classification. Here, $G_{\mathcal{E}}$ is not necessarily the largest equivalence group but, it can be any subgroup of the group of all equivalence transformations.

The method is clarified here by means of its application to Eq.(1.9).

As we said in Sec. II, an application of the method is effective and simple when it is based on a finite-dimensional equivalence algebra.

So, we can take any finite-dimensional subalgebra (desirable as large as possible) of an infinite-dimensional algebra with basis (3.16) and use it for a preliminary group classification. We select the subalgebra L1o spanned on the following operators:

$$
\begin{aligned}
&Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial v}, \quad Y_3 = t\frac{\partial}{\partial v},\\
&Y_4 = \frac{\partial}{\partial x}, \quad Y_5 = x\frac{\partial}{\partial v},\\
&Y_6 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2v\frac{\partial}{\partial v},\\
&Y_7 = -\frac{t}{2}t^2\frac{\partial}{\partial t} + f\frac{\partial}{\partial f} + g\frac{\partial}{\partial g},\\
&Y_8 = \frac{t^2}{2}\frac{\partial}{\partial v} + \frac{\partial}{\partial g},\\
&Y_9 = v\frac{\partial}{\partial v} + g\frac{\partial}{\partial g},\\
&Y_{10} = \frac{x^2}{2}\frac{\partial}{\partial v} - f\frac{\partial}{\partial g}\,.
\end{aligned}
\tag{4.1}
$$

The coefficients f and g of Eq. (1.9) depend on the variables x and v_x. Therefore, we construct prolongations of operators (4.1) to the variable u” and take their projections on the space (x, v_x, f, g).

The prolongations are

$$
\begin{aligned}
&\tilde{Y}_1 = \frac{\partial}{\partial t}, \quad \tilde{Y}_2 = \frac{\partial}{\partial v}, \quad \tilde{Y}_3 = t\frac{\partial}{\partial v},\\
&\tilde{Y}_4 = \frac{\partial}{\partial x}, \quad \tilde{Y}_5 = x\frac{\partial}{\partial v} + \frac{\partial}{\partial v_x},\\
&\tilde{Y}_6 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2v\frac{\partial}{\partial v} + v_x\frac{\partial}{\partial v_x},
\end{aligned}
$$

$$\tilde{Y}_7 = -\frac{t}{2}t^2\frac{\partial}{\partial t} + f\frac{\partial}{\partial f} + g\frac{\partial}{\partial g}, \tag{4.2}$$

$$\tilde{Y}_8 = \frac{t^2}{2}\frac{\partial}{\partial v} + \frac{\partial}{\partial g},$$

$$\tilde{Y}_9 = v\frac{\partial}{\partial v} + g\frac{\partial}{\partial g} + v_x\frac{\partial}{\partial v_x},$$

$$\tilde{Y}_{10} = \frac{x^2}{2}\frac{\partial}{\partial v} - f\frac{\partial}{\partial g} + x\frac{\partial}{\partial v_x}\cdot$$

The nonzero projections of (4.2) are

$$Z_1 = pr(\tilde{Y}_4), \quad Z_2 = pr(\tilde{Y}_5), \quad Z_3 = pr(\tilde{Y}_6)$$
$$Z_4 = pr(\tilde{Y}_7), \quad Z_2 = pr(\tilde{Y}_5), \quad , Z_= pr(\tilde{Y}_9) \tag{4.3}$$
$$Z_7 = pr(\tilde{Y}_{10}),$$

or

$$Z_1 = \frac{\partial}{\partial x}, \quad Z_2 = \frac{\partial}{\partial v_x},$$
$$Z_3 = x\frac{\partial}{\partial x} + v_x\frac{\partial}{\partial v_x}, \quad Z_4 = f\frac{\partial}{\partial f} + g\frac{\partial}{\partial g}\,,$$
$$Z_5\frac{\partial}{\partial g}, \quad Z_6 = g\frac{\partial}{\partial g} + v_x\frac{\partial}{\partial v_x},$$
$$Z_7 = x\frac{\partial}{\partial v_x} - f\frac{\partial}{\partial v_g}\cdot \tag{4.3a}$$

We denote by L_7 the Lie algebra with the basis (4.3a).

The essence of the method is contained in the following statements.

Proposition 7.1. Let L_m be an m-dimensional subalgebra of the algebra L_7. Denote by $Z^{(i)}i = 1, ..., m$ a basis of L_m and by $Y^{(i)}$ the elements of the algebra L_{10} such that $Z^{(i)} = pr(\tilde{Y}^{(i)})$, i.e., if

$$Z^{(i)} = \sum_{\alpha=1}^{7} e_i^\alpha Z_\alpha \tag{4.4}$$

then by (4.1)-(4.3)

$$Z^{(i)}e_i^1 Y_4 + e_i^2 Y_4 + 2e_i^2 Y_5 + \cdots + e_i^7 Y_{10}. \tag{4.4a}$$

If equations

$$f = \Phi(x, v_x)v_{xx} + \Gamma(x, v_x) \tag{4.5}$$

are invariant with respect to the algebra L_m then the equation

$$v_{tt} = \Phi(x, v_x)v_{xx} + \Gamma(x, v_x) \tag{4.6}$$

admits the operators

$$X^{(i)} = \quad \text{projection} \quad \text{of} \quad Y^{(i)} \quad \text{on} \quad (t, x, v). \tag{4.4b}$$

Proposition 7.2. Let Eq. (4.6) and the equation

$$v_{tt} = \Phi'(x, v_x)v_{xx} + \Gamma'(x, v_x) \tag{4.6$'$}$$

be constructed according to Proposition 7.1 via subalgebras L_m and L'_m, respectively. If L_m and L'_m are similar subalgebras in L_{10} then Eqs. (4.6), (4.6$'$) are equivalent with respect to the equivalence group G_{10} generated by L_{10}.

According to these propositions the problem of preliminary group classification of Eq. (1.9) with respect to the finite-dimensional subalgebra L_{10} of the main equivalence algebra (3.16) is reduced to the algebraic problem of constructing of nonsimilar subalgebras of L_7, or optimal systems of subalgebras [106]. Actually we can consider only one and two-dimensional subalgebras because for subalgebras $L_m, m \geq 3$, there are no invariant equations (4.5).

In this paper we completely solve the problem of preliminary group classification with respect to one-dimensional subalgebras. The case of two-dimensional subalgebras will be considered elsewhere.

V. Adjoint group for algebra L_7

Let G be a Lie group, with L its Lie algebra. Each element $T \in G$ yields inner automorphism $T_a \to TT_aT^{-1}$ of the group G. Every automorphism of the group G induces an automorphism of its Lie algebra L. The set of all these automorphisms of L is a local Lie group called the group of inner automorphisms of the algebra L, or the adjoint group G^A. The Lie algebra of G^A is the adjoint algebra L^A of the algebra L, defined as follows. Let $X \in L$. The linear mapping ad $X : X \to [\xi, X_1]$ is an automorphism of L, called inner derivation of the Lie algebra L. The set L^A of all inner derivations ad $X (X \in L)$ together with the Lie bracket $[adX_1, adX_2] = ad[X_1, X_2]$ is a Lie algebra, called the adjoint algebra of L. Clearly, the adjoint algebra L^A is the Lie algebra of the adjoint group G^A. Two subalgebras in L are conjugate (or similar) if there is a transformation from G^A which takes one subalgebra into the other. The collection of pairwise nonconjugate s-dimensional subalgebras is called an optimal system of order s and denoted by θs. The algorithm becomes clear by the following calculations of the adjoint algebra and the adjoint group for the algebra L_7 with the basis (4.3a).

Denote by A elements of the algebra adL_7. According to what is stated above one can take operators

$$A_\alpha = [Z_\alpha, Z_\beta]\frac{\partial}{\partial Z_\beta} \tag{5.1}$$

as a basis of the algebra adL_7.

Using the table of commutators (Table 1) we get

$$\begin{gathered} A_1 = Z_1\frac{\partial}{\partial Z_3} + Z_2\frac{\partial}{\partial Z_7}, \quad A_2 = Z_2\frac{\partial}{\partial Z_3} + Z_2\frac{\partial}{\partial Z_6}, \\ A_3 = -\left(Z_1\frac{\partial}{\partial Z_1} + Z_2\frac{\partial}{\partial Z_2}\right), \quad A_4 = -Z_5\frac{\partial}{\partial Z_5}, \\ A_5 = Z_5\left(\frac{\partial}{\partial Z_4} + \frac{\partial}{\partial Z_6}\right), \\ A_6 = -\left(Z_2\frac{\partial}{\partial Z_2} + Z_5\frac{\partial}{\partial Z_5} + Z_7\frac{\partial}{\partial Z_7}\right), \\ A_7 = -Z_2\frac{\partial}{\partial Z_1} + Z_7\frac{\partial}{\partial Z_6}\cdot \end{gathered} \tag{5.2}$$

The infinitesimal operator A_1 generates the following one-parameter group of linear transformations:

$$\begin{gathered} Z_1' = Z_1, \quad Z_2' = Z_2, \quad Z_3' = Z_3 + a_1 Z_1, \quad Z_4' = Z_4, \\ Z_5' = Z_5, \quad Z_6' = Z_6, \quad Z_7' = Z_7 + a_1 Z_2, \end{gathered}$$

which is represented by the matrix

$$M_1(a_1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ a_1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & a_1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Following the same procedure we obtain the matrices $M_2(a_2), \ldots, M_7(a_7)$ associated to the infinitesimal operators $A_2, ..., A_7$, respectively.

Here $0 < a_3, a_4, a_6 < +\infty$ and $-\infty < a_1, a_2, a_5, a_7 < +\infty$. We do not write all these matrices because for our purposes we need only their product:

$$M = M(a_1)\ldots M(a_7) = \begin{pmatrix} a_3 & -a_3a_7 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_3a_6 & 0 & 0 & 0 & 0 & 0 \\ a_1a_3 & a_2a_3a_6 - a_1a_3a_7 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & a_5a_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_4a_6 & 0 & 0 \\ 0 & a_2a_3a_6 & 0 & 0 & a_5a_6 & 1 & a_7 \\ 0 & a_1a_3a_6 & 0 & 0 & 0 & 0 & a_6 \end{pmatrix}$$

Actually it is preferable to work not with the operators $Z_1, ..., Z_7$, but with coordinates of the decomposition

$$Z = \sum_1^7 {}_ie^i Z_i \tag{5.3}$$

of $Z \in L_7$, i.e., with the vectors

$$e = (e^1, e^2, \ldots, e^7). \tag{5.4}$$

Vector e is transformed by means of the transposed matrix M^T of M and after the transformation has the following coordinates:

$$\begin{aligned} &\bar{e}^1 = a_3(e^1 + a_1e^3), \\ &\bar{e}^2 a_3[-a_7e^1 + a_6e^2 + (a_2a_6 - a_1a_7)e^3 + a_2a_6e^6 + a_1a_6e^7], \\ &\bar{e}^3 = e^3, \quad \bar{e}^4 = e^4, \quad \bar{e}^6 = e^6, \\ &\bar{e}^5 = a_6[a_5e^4 + a_4e^5a_5e^6], \\ &\bar{e}^7 = a_7e^6 + a_6e^7. \end{aligned} \tag{5.5}$$

These transformations give rise to the adjoint group of the algebra L_7. We remind that a_1, a_2, a_5, and a_7 are arbitrary real parameters, while a_3, a_4, and a_6 are real positive parameters. We use also the reflections (3.18), (3.19) which give rise to the following transformations of operators (4.3a):

$$Z_1 \mapsto -Z_1, \quad Z_2 \mapsto -Z_2 \tag{3.18$'$}$$

$$Z_2 \mapsto -Z_2, \quad Z_5 \mapsto -Z_5, \quad Z_7 \mapsto -Z_7. \tag{3.19$'$}$$

Table 1: Table of commutators

	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7
Z_1	0	0	Z_1	0	0	0	Z_2
Z_2	0	0	Z_2	0	0	Z_2	0
Z_3	$-Z_1$	$-Z_2$	0	0	0	0	0
Z_4	0	0	0	0	$-Z_5$	0	0
Z_5	0	0	0	Z_5	0	Z_5	0
Z_6	0	$-Z_2$	0	0	$-Z_5$	0	$-Z_7$
Z_7	$-Z_2$	0	0	0	0	Z_7	0

VI. Construction of the optimal system of one-dimensional subalgebras of L_7

The construction of the optimal system of one-dimensional subalgebras of L_7 can be carried out using a very simple natural approach. Namely, we simplify any given vector (5.4) $e = (e^1, ..., e^7)$ by means of transformation (5.5) and reflections (3.18′), (3.19′) and divide the obtained vectors into nonequivalent classes; in any class we select a representative having as simple form as possible. After this brief and somewhat vague description, we proceed to the calculations.

First, we remark that transformations (5.5) leave invariant the components e^3, e^4, and e^6 of the vector under consideration. Therefore, we have to look over all for possibilities for e^3, e^4, and e^6 and in every case to simplify other components by means of transformations (5.5). So, we will start by the case

$$e^3 \neq 0, e^4 \neq 0, e^6 \neq 0. \tag{6.1}$$

In this case we get

$$\bar{e}^1 = 0, \bar{e}^7 = 0 \tag{6.2}$$

by putting in formulas (5.5)

$$a_1 = -e^1/e^3, \quad a_6 = 1, \quad a_7 = -e^7/a^6. \tag{6.3}$$

So, after the transformation (5.5) with the values (6.3) of the parameters a_1, a_6, a_7 (other parameters are arbitrary) any vector e is transformed to

$$(0, e^2, e^3, e^4, e^5, e^6, 0), \tag{6.4}$$

provided that conditions (6.1) are valid. We further simplify the vector (6.4) by means of transformations (5.5) with $a_1 = a_7 = 0$.

After this transformation the vector (6.4) goes to the vector $\bar{\mathbf{e}}$ with components

$$\begin{aligned} \bar{e}_1 &= \bar{e}_7 = 0, \\ \bar{e}^3 = e^3, \quad \bar{e}^4 &= e^4, \quad \bar{e}^6 = e^6 \\ \bar{e}^2 &= a_3 a_6 [e^2 + a_2(e^3 + e^6)], \\ \bar{e}^5 &= a_6[a_5(e^4 + e^6) + a_4 e^5]. \end{aligned} \tag{6.5}$$

Formulas (6.5) point out that we have to distinguish the following four subcases:

$$e^3 + e^6 \neq 0, \quad e^4 + e^6 \neq 0, \tag{6.6}$$

$$e^3 + e^6 \neq 0, \quad e^4 + e^6 = 0, \tag{6.7}$$

$$e^3 + e^6 = 0, \quad e^4 + e^6 \neq 0, \tag{6.8}$$

$$e^3 + e^6 = 0, \quad e^4 + e^6 = 0, \tag{6.9}$$

If (6.6) is valid we put

$$a_2 = -e^2/(e^3 + e^6), \tag{6.10}$$

$$a_4 = 1, \quad a_5 = -e^5/(e^4 + e^6) \tag{6.11}$$

to obtain

$$\bar{\mathbf{e}} = (0, 0, e^3, e^4, 0, e^6, 0). \tag{6.12}$$

Using the fact that any infinitesimal operator is defined up to a constant factor we can write the vector (6.12) in the form

$$\bar{\mathbf{e}} = (0, 0, \alpha, \beta, 0, 1, 0), \quad \alpha \neq 0, -1, \quad \beta \neq 0, -1. \tag{6.12a}$$

If (6.7) is valid we take a_2 from (6.10) to obtain

$$\bar{\mathbf{e}} = (0, 0, e^3, -e^6, a_4 a_6 e^5, e^6, 0).0, -1, \quad \beta \neq 0, -1. \tag{6.13}$$

Here, when $e^5 \neq 0$ we can get $a_4 a_6 e^5 = e^6$ using an arbitrary factor $a_4 a_6$ and the reflection (3.19′) , and obtain [see the passage from (6.12) to (6.12a)]

$$\bar{\mathbf{e}} = (0, 0, \alpha, -1, 1, 1, 0), \quad \alpha \neq 0, -1.0, -1, \quad \beta \neq 0, -1. \tag{6.13a}$$

When $e^5 = 0$ we get from (6.13)

$$\bar{\mathbf{e}} = (0, 0, \alpha, -1, 0, 1, 0), \quad \alpha \neq 0, -1.0, -1, \quad \beta \neq 0, -1. \tag{6.13b}$$

If (6.8) is valid we take (6.11) to obtain

$$\bar{\mathbf{e}} = (0, a_3 a_6 e^2, -e^6, e^4, 0, e^6, 0).0, -1, \quad \beta \neq 0, -1. \tag{6.14}$$

Following the same procedure as in the case (6.7) we obtain from (6.14) two different vectors:

$$\bar{\mathbf{e}} = (0, 1, -1, \beta, 0, 1, 0), \beta \neq 0, -1 \tag{6.14a}$$

and

$$\bar{\mathbf{e}} = (0, 0, -1, \beta, 0, 1, 0), \beta \neq 0, -1. \tag{6.14b}$$

If (6.9) is valid we put $a_6 = 1$ and have

$$\bar{\mathbf{e}} = (0, a_3, e_2, -e^6, -e^6, a_4 a^5, e^6, 0). \tag{6.15}$$

Here we use arbitrary positive factors a_3, a_4 and both reflections $(3.18')$, $(3.19')$ and obtain from (6.15) the following four different vectors:

$$\bar{\mathbf{e}} = (0, 1, -1, -1, 1, 1, 0), \tag{6.15a}$$

$$\bar{\mathbf{e}} = (0, 0, -1, -1, 1, 1, 0), \tag{6.15b}$$

$$\bar{\mathbf{e}} = (0, 1, -1, -1, 0, 1, 0), \tag{6.15c}$$

$$\bar{\mathbf{e}} = (0, 0, -1, -1, 0, 1, 0). \tag{6.15d}$$

We summarize: Any vector (5.4) that satisfies the conditions (6.1) is equivalent to vectors (6.12a), (6.13a), (6.13b), (6.14a), (6.14b) and (6.15a)-(6.15d). These vectors give rise [via formulas (5.3) and (5.4)] to the following nonequivalent operators:

$$\begin{gathered} \alpha Z_3 + \beta Z_4 + Z_6, \quad \alpha \neq 0, \ \beta \neq 0, \\ \alpha Z_3 - Z_4 + Z_5 + Z_6, \quad \alpha \neq 0, \\ Z_2 - Z_3 + \beta Z_4 + Z_6, \quad \beta \neq 0, \\ Z_2 - Z_3 - Z_4 + Z_5 + Z_6. \end{gathered} \tag{6.16}$$

Here, we changed the restrictions on the parameters α and β in order to compact the operators; for example, the vector (6.15c) is included in formula (6.14b) when we cancel the condition $\beta \neq -1$.

Now we pass from the case (6.1) to the second case

$$e^3 \neq 0, \quad e^4 \neq 0, \quad e^6 = 0. \tag{6.17}$$

Analysis of this case gives the following nonequivalent operators

$$\alpha Z_3 + Z_4 + Z_7, \quad \alpha Z_3 + Z_4, \quad \alpha \neq 0. \tag{6.18}$$

The other cases are

$$\begin{aligned} & e^3 \neq 0, \quad e^4 = 0, \quad e^6 \neq 0, \\ & Z_2 - Z_3 + Z_6, \quad \alpha Z_3 + Z_6, \quad \alpha \neq 0; \end{aligned} \tag{6.19}$$

$$\begin{aligned} & e^3 = 0, \quad e^4 \neq 0, \quad e^6 \neq 0, \\ & Z_4 + Z_5 - Z_6, \quad Z_1 - Z_4 + Z_5 + Z_6, \\ & \beta Z_4 + Z_6, \quad Z_1 + \beta Z_4 + Z_6, \quad \beta \neq 0; \end{aligned} \tag{6.20}$$

$$\begin{aligned} & e^3 \neq 0, \quad e^4 = e^6 = 0, \\ & Z_3, \quad Z_3 + Z_5, \quad Z_3 + Z_7, \\ & Z_3 + Z_5 + Z_7, \quad Z_3 + Z_5 - Z_7, \end{aligned} \tag{6.21}$$

$$\begin{aligned} & e^4 \neq 0, \quad e^3 = e^6 = 0, \\ & Z_4, \quad Z_1 + Z_4, \quad Z_2 + Z_4, \quad Z_4 + Z_7, \\ & Z_1 + Z_4 + Z_7, \quad Z_2 + Z_4 + Z_7; \end{aligned} \tag{6.22}$$

$$\begin{aligned} & e^3 = e^4 = 0, \quad e^6 \neq 0, \\ & Z_6, \quad Z_1 + Z_6; \end{aligned} \tag{6.23}$$

$$\begin{aligned} & e^3 = e^4 = e^6 = 0, \\ & Z_1, \quad Z_2, \quad Z_5, \quad Z_7, \quad Z_1 + Z_5, \quad Z_1 + Z_7, \quad Z_2 + Z_5, \\ & Z_2 + Z_7, \quad Z_5 + Z_7, \quad Z_5 - Z_7, \quad Z_1 + Z_5 + Z_7, \\ & Z_1 + Z_5 - Z_7, \quad Z_2 + Z_5 + Z_7, \quad Z_2 + Z_5 - Z_7. \end{aligned} \tag{6.24}$$

We summarize the results of Eqs. (6.16), (6.18) - (6.24) to obtain the following optimal system of one-dimensional subalgebras of L_7 :

$$\begin{aligned} & Z^{(1)} = Z_1, \quad Z^{(2)} = Z_2, \quad Z^{(3)} = Z_3, \qquad (6.25) \\ & Z^{(4)} = Z_4 + \alpha Z_3, \quad Z^{(5)} = Z_5, \\ & Z^{(6)} = Z_6 + \alpha Z_3 + \beta Z_4, \quad Z^{(7)} = Z_7, \\ & Z^{(8)} = Z_1 + Z_4, \quad Z^{(9)} = Z_1 + Z_5, \\ & Z^{(10)} = Z_1 + Z_6 + \beta Z_4, \quad Z^{(11)} = Z_1 + Z_7, \\ & Z^{(12)} = Z_2 + Z_4, \quad Z^{(13)} = Z_2 + Z_5, \\ & Z^{(14)} = Z_2 + Z_7, \quad Z^{(15)} = Z_3 + Z_5, \end{aligned}$$

$$
\begin{aligned}
&Z^{(16)} = Z_3 + Z_7, \quad Z^{(17)} = Z_5 + Z_7,\\
&Z^{(18)} = Z_5 - Z_7, \quad Z^{(19)} = Z_1 + Z_4 + Z_7,\\
&Z^{(20)} = Z_1 + Z_5 + Z_7, \quad Z^{(21)} = Z_1 + Z_5 - Z_7,\\
&Z^{(22)} = Z_2 + Z_4 + Z_7, \quad Z^{(23)} = Z_2 + Z_5 + Z_7,\\
&Z^{(24)} = Z_2 + Z_5 - Z_7, \quad Z^{(25)} = \alpha Z_3 + Z_4 + Z_7,\\
&Z^{(26)} = Z_3 + Z_5 + Z_7, \quad Z^{(27)} = Z_3 + Z_5 - Z_7,\\
&Z^{(28)} = Z_1 - Z_4 + Z_5 + Z_6, \quad Z^{(29)} = Z_2 - Z_3 + \beta Z_4 + Z_6,\\
&Z^{(30)} = \alpha Z_3 - Z_4 + Z_5 + Z_6, \quad Z^{(31)} = Z_2 - Z_3 - Z_4 + Z_5 + Z_6.
\end{aligned}
$$

Here, α and β are arbitrary constants.

VII. Equations admitting an extension by one of the principal Lie algebra

Now we apply Propositions 7.1 and 7.2 to the optimal system (6.25) and obtain all nonequivalent equations (1.9) admitting $\mathcal{E}$-extensions of the principal Lie algebra L_3 by one, i.e., equations of the form (1.9) such that they admit, together with the three basic operators (2.8) of L_3, also a fourth operator X_4. For every case, when this extension occurs, we indicate the corresponding coefficients f and g and the additional operator X_4.

We clarify the algorithm of passing from operators (6.25) to f, g, and X_4 by the following examples. For the first example we take the last operator from (6.25):

$$
\begin{aligned}
Z^{(31)} &= Z_2 - Z_3 - Z_4 + Z_5 + Z_6\\
&= -x\frac{\partial}{\partial x} + \frac{\partial}{\partial v_x} - f\frac{\partial}{\partial f} + \frac{\partial}{\partial g}\,.
\end{aligned}
\tag{7.1}
$$

Invariants are found from the equations

$$
-\frac{dx}{x} = dv_x = -\frac{df}{f} = dg
$$

and can be taken in the form

$$
I_1 = v_x + \ln|x|, \quad I_2 = f/x, \quad I_3 = g - v_x. \tag{7.2}
$$

From the invariance equations taken in the form

$$
I_2 = \Phi(I_1), \quad I_3 = \Gamma(I_1) \tag{7.3}
$$

it follows

$$f = x\Phi(\lambda), \quad g = v_x + \Gamma(\lambda), \tag{7.4}$$

where $\lambda = I_7$. From the formulas (4.4a)-(4.4) applied to the operator

$$X_4 = t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} - (t^2 + 2x - 2v)\frac{\partial}{\partial v}\cdot \tag{7.5}$$

So, the equation

$$v_{tt} = x\Phi(v_x + \ln|x|)\, v_{xx} + \Gamma(v_x + \ln|x|) + v_x \tag{7.6}$$

admits the four-dimensional algebra L_4 generated by the operators (2.8) and (7.5).

For the second example we take the operator

$$Z^{(5)} = Z_5 = \frac{\partial}{\partial g}\cdot \tag{7.7}$$

Invariants of this operator are

$$I_1 = x, I_2 = v_x, I_3 = f. \tag{7.8}$$

In this case there are no invariant equations of the form (4.5) because the necessary condition for existence of invariant solutions (see [106], Sec. 19.3) is not satisfied, i.e., invariants (7.8) cannot be solved with respect to f and g.

After similar calculations applied to all operators (6.25) we obtain the following result (Table 2) of the preliminary group classification of equation (1.9) admitting an extension L_4 of the principal Lie algebra L_3.

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Table 2: The result of the classification ($\sigma = \frac{1}{\alpha}$, $\gamma = \frac{\beta}{\alpha}$, and Φ and Γ are arbitrary functions of λ)

N	Z	Invariant λ	Equation	Additional operator X_4
1	$Z^{(1)}$	v_x	$v_{tt} = \Phi v_{xx} + \Gamma$	$\frac{\partial}{\partial x}$
2	$Z^{(2)}$	x	$v_{tt} = \Phi v_{xx} + \Gamma$	$x\frac{\partial}{\partial v}$
3	$Z^{(3)}$	v_x/x	$v_{tt} = \Phi v_{xx} + \Gamma$	$t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2v\frac{\partial}{\partial v}$
4	$Z^{(4)}_{\alpha\neq 0}$	v_x/x	$v_{tt} = x^{\sigma}\{\Phi v_{xx} + \Gamma\}$	$\left(1 - \frac{\sigma}{2}\right) t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2v\frac{\partial}{\partial v}$
5	$Z^{(6)}_{\alpha=0}$	x	$v_{tt} = v_x^{\beta}\{\Phi v_{xx} + \Gamma v_x\}$	$\beta t\frac{\partial}{\partial t} - 2v\frac{\partial}{\partial v}$
6	$Z^{(6)}_{\alpha\neq 0}$	$v_x/x^{\sigma+1}$	$v_{tt} = x^{\gamma}\{\Phi v_{xx} + x^{\sigma}\Gamma\}$	$(2-\gamma)t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + 2(\sigma+2)\frac{\partial}{\partial v}$
7	$Z^{(7)}$	x	$v_{tt} = \Phi v_{xx} - x^{-1}\Phi v_x + \Gamma$	$x^2\frac{\partial}{\partial v}$
8	$Z^{(8)}$	v_x	$v_{tt} = e^{x}\{\Phi v_{xx} + \Gamma\}$	$t\frac{\partial}{\partial t} - 2\frac{\partial}{\partial x}$
9	$Z^{(9)}$	v_x	$v_{tt} = \Phi v_{xx} + \Gamma + x$	$2\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial v}$
10	$Z^{(10)}$	$e^{-x}v_x$	$v_{tt} = v_x^{\beta}\{\Phi v_{xx} + \Gamma v_x\}$	$\beta t\frac{\partial}{\partial t} - 2\frac{\partial}{\partial x} - 2v\frac{\partial}{\partial v}$
11	$Z^{(11)}$	$x^2 - 2v_x$	$v_{tt} = \Phi v_{xx} + \Gamma - x\Phi$	$2\frac{\partial}{\partial x} + x^2\frac{\partial}{\partial v}$
12	$Z^{(12)}$	x	$v_{tt} = e^{v_x}\{\Phi v_{xx} + \Gamma\}$	$t\frac{\partial}{\partial t} - 2x\frac{\partial}{\partial v}$
13	$Z^{(13)}$	x	$v_{tt} = \Phi v_{xx} + \Gamma + v_x$	$(t^2 + 2x)\frac{\partial}{\partial v}$
14	$Z^{(14)}$	x	$v_{tt} = \Phi v_{xx} + \Gamma - (x+1)^{-1}\Phi v_x$	$(x^2 + 2x)\frac{\partial}{\partial v}$
15	$Z^{(15)}$	v_x/x	$v_{tt} = \Phi v_{xx} + \Gamma + \ln\|x\|$	$2t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + (t^2 + 4v)\frac{\partial}{\partial v}$
16	$Z^{(16)}$	$\frac{v_x}{x} - \ln\|x\|$	$v_{tt} = \Phi v_{xx} + \Gamma - \Phi\ln\|x\|$	$2t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + (x^2 + 4v)\frac{\partial}{\partial v}$
17	$Z^{(17)}$	x	$v_{tt} = \Phi v_{xx} + (1-\Phi)x^{-1}v_x + \Gamma$	$(t^2 + x^2)\frac{\partial}{\partial v}$
18	$Z^{(18)}$	x	$v_{tt}\Phi v_{xx} - (1+\Phi)x^{-1}v_x + \Gamma$	$(t^2 - x^2)\frac{\partial}{\partial v}$
19	$Z^{(19)}$	$x^2 - 2v_x$	$v_{tt} = e^{x}\{\Phi v_{xx} - x\Phi + \Gamma\}$	$t\frac{\partial}{\partial t} - 2\frac{\partial}{\partial x} - x^2\frac{\partial}{\partial v}$
20	$Z^{(20)}$	$x^2 - 2v_x$	$v_{tt} = \Phi v_{xx} + (1-\Phi)x + \Gamma$	$2\frac{\partial}{\partial x} + (t^2 + x^2)\frac{\partial}{\partial v}$
21	$Z^{(21)}$	$x^2 + 2v_x$	$v_{tt} = \Phi v_{xx} + (1+\Phi)x + \Gamma$	$2\frac{\partial}{\partial x} + (t^2 - x^2)\frac{\partial}{\partial v}$
22	$Z^{(22)}$	x	$v_{tt} = e^{\frac{v_x}{x+1}}\{\Phi v_{xx} - \frac{\Phi}{x+1}v_x + \Gamma\}$	$t\frac{\partial}{\partial t} - x(x+2)\frac{\partial}{\partial v}$
23	$Z^{(23)}$	x	$v_{tt} = \Phi v_{xx} - \frac{\Phi-1}{x+1}v_x + \Gamma$	$(t^2 + x^2 + 2x)\frac{\partial}{\partial v}$
24	$Z^{(24)}$	x	$v_{tt} = \Phi v_{xx} + \frac{\Phi+1}{1-x}v_x + \Gamma$	$(t^2 - x^2 + 2x)\frac{\partial}{\partial v}$
25	$Z^{(25)}_{\alpha=0}$	x	$v_{tt} = e^{\frac{v_x}{x}}\{\Phi v_{xx} - \ln\|x\|\Phi + \Gamma\}$	$t\frac{\partial}{\partial t} - x^2\frac{\partial}{\partial v}$
26	$Z^{(25)}_{\alpha\neq 0}$	$\frac{v_x}{x} - \sigma\ln\|x\|$	$v_{tt} = x^{\sigma}\{\Phi v_{xx} - \sigma\ln\|x\|\Phi + \Gamma\}$	$(2-\sigma)t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x}$ $+(\sigma x^2 + 4v)\frac{\partial}{\partial v}$
27	$Z^{(26)}$	$\frac{v_x}{x} - \ln\|x\|$	$v_{tt} = \Phi v_{xx} + (1-\Phi)\ln\|x\| + \Gamma$	$2t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + (t^2 + x^2 + 4v)\frac{\partial}{\partial v}$
28	$Z^{(27)}$	$\frac{v_x}{x} + \ln\|x\|$	$v_{tt} = \Phi v_{xx} + (1+\Phi)\ln\|x\| + \Gamma$	$2t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + (t^2 - x^2 + 4v)\frac{\partial}{\partial v}$
29	$Z^{(28)}$	$e^{-x}v_x$	$v_{tt} = e^{-x}\Phi v_{xx} + \Gamma + x$	$t\frac{\partial}{\partial t} + 2\frac{\partial}{\partial x} + (t^2 + 2v)\frac{\partial}{\partial v}$
30	$Z^{(29)}$	$v_x + \ln\|x\|$	$v_{tt} = x^{-\beta}\{\Phi v_{xx} + x^{-1}\Gamma\}$	$(\beta+2)t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + 2(v-x)\frac{\partial}{\partial v}$
31	$Z^{(30)}_{\alpha=0}$	x	$v_{tt} = \Phi v_x^{-1}v_{xx} + \Gamma + \ln\|v_x\|$	$t\frac{\partial}{\partial t} + (t^2 + 2v)\frac{\partial}{\partial v}$
32	$Z^{(30)}_{\alpha\neq 0}$	$x^{-(1+\sigma)}v_x$	$v_{tt} = \frac{\Phi}{x^{\sigma}}v_{xx} + \Gamma + \sigma\ln\|x\|$	$(2+\sigma)t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x}$ $+[\sigma(t^2 + 2v) + 4v]\frac{\partial}{\partial v}$
33	$Z^{(31)}$	$v_x + \ln\|x\|$	$v_{tt} = x\Phi v_{xx} + \Gamma + v_x$	$t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} - (t^2 + 2x - 2v)\frac{\partial}{\partial v}$

Paper 8

A simple method for group analysis and its application to a model of detonation

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A simple procedure is suggested to obtain extensions of the principal Lie algebra of a given family of equations by using the equivalence algebra. An application to a qualitative model of detonation is performed.

Introduction

It is well known that the problem of group classification of a given family of equations (containing arbitrary parameters or functions) is more complicated than the problem of calculation of a symmetry group for a given equation.

Recently, a simple approach was suggested in [7] for a partial solution of group classification based on an equivalence group, or its Lie algebra, called equivalence algebra and denoted $L_{\mathcal{E}}$. The approach was called a *method of preliminary group classification.*

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The program of realizing this simplified approach to, the problem of the group classification was undertaken in our paper [71] and applied to a wide class of nonlinear wave equations that appear in many physical problems.

The method of preliminary group classification has been hinted by the observation that in applications of group analysis most of the extensions of principal Lie algebra $L_{\mathcal{P}}$ (algebra admitted for every equation of the family of equations under consideration) are taken from equivalence algebra $L_{\mathcal{E}}$. These extensions are called $\mathcal{E}$ extensions.

In this paper we use the equivalence algebra in order to simplify the calculation of symmetry groups without solving the determining equation. If we calculate the equivalence algebra, $\mathcal{E}$ extensions for equations of a given family are obtained by solving simple algebraic equations only.

An important feature of the approach is that the construction of an equivalence algebra for a given family of I equations is reduced to calculation of a symmetry algebra for a given system of partial differential equations (PDE's), where arbitrary functions do not appear (see [71]).

In Secs. 8 and 8 we discuss this approach in detail by focusing on the first-order system

$$u_t + \frac{dp}{dx} = 0, \quad v_t = g, \tag{1.1}$$

where p and g are arbitrary functions of u and v, and

$$\frac{dp}{dx} := p_u u_x + p_v v_x.$$

In Sec. 8 we give a sketch of a qualitative model of detonation [36] and in Sec. 8 we apply our method to this model.

The Appendix is written for those who want to follow the details of calculations.

II. Equivalence algebra and notations

An equivalence transformation in our case is a non-degenerate change of variables t, x, u, v, taking any system of the form (1.1) into a system of the same form, generally speaking, with different functions $p(u, v)$ and $g(u, v)$.

We consider a continuous group of equivalence transformations and seek for its generator

$$Y = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta^1 \frac{\partial}{\partial u} + \eta^2 \frac{\partial}{\partial v} + \mu^1 \frac{\partial}{\partial p} + \mu^2 \frac{\partial}{\partial g}, \tag{2.1}$$

from the invariance conditions of Eqs (1.1) written as the following system:

$$u_t + p_u u_x + p_v v_x = 0, \quad u_t = g, \quad p_t = p_x = 0, \quad g_t = g_x = 0. \tag{2.2}$$

Here, the coordinates ξ and η of the operator (2.1) are sought as functions of t, x, v, u while the coordinates μ are sought as functions of t, x, u, v, p, g.

The invariance conditions of the system (2.2) are

$$\widetilde{Y}(u_t + p_u u_x + p_v v_x) = 0, \quad \widetilde{Y}(v_t - g) = 0, \tag{2.3}$$

$$\widetilde{Y} p_t = 0, \quad \widetilde{Y} p_x = 0, \quad \widetilde{Y} g_t = 0, \quad \widetilde{Y} g_x = 0, \tag{2.4}$$

where $\widetilde{Y}$ is the prolongation of the operator (2.1).

In order to write the prolongation formulas, we introduce necessary notations. First we emphasize that the symmetry transformations and their generators, for differential equations (1.1), act on the space $(\mathbf{x}, \mathbf{u})$ of independent variables,

$$x = (x^1, x^2), \quad x^1 := t, \quad x^2 := x, \tag{2.5}$$

and dependent variables,

$$u = (u^1, u^2), \quad u^1 := u, \quad u^2 := v, \tag{2.6}$$

In contrast to this equivalence transformations and their generator (2.1) act on the space $(\mathbf{y}, \mathbf{f})$ of four independent variables,

$$\mathbf{y} = (\mathbf{x}, \mathbf{y}), \tag{2.7}$$

and dependent variables,

$$\mathbf{f} = (f^1, f^2), \quad f^1 := f, \quad f^2 := g, \tag{2.8}$$

These notations allow us to put $\widetilde{Y}$ in a compact form and to clearly distinguish symmetry and equivalence generators. Namely, we write a symmetry operator as

$$X = \xi^i \frac{\partial}{\partial x} + \eta^i \frac{\partial}{\partial u^i} \tag{2.9}$$

and its first prolongation as

$$X_{(1)} = X + \zeta^i_j \frac{\partial}{\partial u^i_j}, \tag{2.10}$$

with

$$\zeta^i_j = D_j(\eta^i) - u^i_k D_j(\xi^k), \tag{2.10$'$}$$

where

$$D_j := \frac{\partial}{\partial x_j} + u^i_j \frac{\partial}{\partial u^i} \cdot \tag{2.11}$$

As for equivalence generator (2.1), we write it in the form

$$Y = \nu^\alpha \frac{\partial}{\partial y^\alpha} + \mu^i \frac{\partial}{\partial f^i}, \tag{2.12}$$

where

$$\nu := (\xi, \eta). \tag{2.13}$$

so that

$$Y = X + \mu^i \frac{\partial}{\partial f^i} \cdot \tag{2.12$'$}$$

Now we can give a compact form for the prolongation of (2.12):

$$\widetilde{Y} = Y + \zeta^i_j \frac{\partial}{\partial u^i_j} + \omega^i_\alpha \frac{\partial}{\partial f^i_\alpha} \cdot \tag{2.14}$$

The coordinates ζ^i_j of operator (2.14) are given by formula (2.10′) while

$$\omega^i_\alpha = \widetilde{D}_\alpha(\mu^i) - f^i_\beta \widetilde{D}_\alpha(\nu^\beta), \quad i = 1, 2, \quad \alpha = 1, \ldots, 4, \tag{2.15}$$

where

$$\widetilde{D}_\alpha := \frac{\partial}{\partial y^\alpha} + f^i_\alpha \frac{\partial}{\partial f^i} \cdot \tag{2.16}$$

It is worthwhile to note that the differential operators defined by (2.11) and (2.16) are similar, but act on different spaces.

We also note that the prolongation formula (2.14) can be rewritten as

$$\tilde{Y} = X_{(1)} + \mu_i \frac{\partial}{\partial f^i} + \omega^i_\alpha \frac{\partial}{\partial f^i_\alpha}\,, \tag{2.14$'$}$$

in accordance with (2.12′).

Now, taking into account our notations, we substitute the prolongation formulas (2.14) in the invariance conditions given by Eqs. (2.3) and (2.4). The general solution of these equations, obtained in the Appendix [cf. (A16)] gives rise to the equivalence operator for the system (1.1):

$$\begin{gathered} Y = (C_1 + C_2 t)\frac{\partial}{\partial t} + (C_3 + C_4 x)\frac{\partial}{\partial x} + (C_5 + C_6 u)\frac{\partial}{\partial u} + \\ + \sigma(v)\frac{\partial}{\partial v} + [(C_4 + C_6 - C_2)p + C_7]\frac{\partial}{\partial p} + [\sigma'(v) - C_2]g\frac{\partial}{\partial g} \cdot \end{gathered} \tag{2.17}$$

Here $C_1, ..., C_7$ are arbitrary constants and $\sigma(v)$ is an arbitrary function.

So the equivalence algebra $L_\mathcal{E}$ is infinite dimensional and can be written as the direct sum

$$L_\mathcal{E} = L_7 \oplus L_\sigma,$$

where L_7 is spanned by the seven operators

$$\begin{aligned} &Y_1 = \frac{\partial}{\partial t}\,, \quad Y_2 = \frac{\partial}{\partial t} - p\frac{\partial}{\partial p} - g\frac{\partial}{\partial g}\,,\\ &Y_3 = \frac{\partial}{\partial x}\,, \quad Y_4 = x\frac{\partial}{\partial x} + p\frac{\partial}{\partial p}\,, \quad Y_5 = \frac{\partial}{\partial u}\,,\\ &Y_6 = u\frac{\partial}{\partial u} + p\frac{\partial}{\partial p}\,, \quad Y_7 = \frac{\partial}{\partial p}\,, \end{aligned} \tag{2.18}$$

and L_σ is the infinite-dimensional subalgebra of operators

$$Y_\sigma = \sigma(v)\frac{\partial}{\partial v} + \sigma'(v)g\frac{\partial}{\partial g}\,\cdot \tag{2.19}$$

III. Projections and principal lie algebra

We introduce the following projections of the equivalence operator (2.1):

$$\begin{aligned} &\mathrm{pr}_{(\mathbf{x},\mathbf{u})}(Y) = X \equiv \xi^1\frac{\partial}{\partial t} + \xi^2\frac{\partial}{\partial x} + \eta^1\frac{\partial}{\partial u} + \eta^2\frac{\partial}{\partial v}\,,\\ &\mathrm{pr}_{(\mathbf{u},\mathbf{f})}(Y) = Z \equiv \eta^1\frac{\partial}{\partial u} + \eta^2\frac{\partial}{\partial v} + \mu^1\frac{\partial}{\partial p} + \mu^2\frac{\partial}{\partial g}\,\cdot \end{aligned} \tag{3.1}$$

The significance of these projections is defined by the following simple (but important for applications) statements.

Proposition 8.1. An operator X belongs to the principal Lie algebra $L_{\mathcal{P}}$ for the system (1.1) iff

$$X = \mathrm{pr}_{(\mathbf{x},\mathbf{u})}(Y) \tag{3.2}$$

with an equivalence generator Y, such that

$$\mathrm{pr}_{(\mathbf{u},\mathbf{f})}(Y) = 0. \tag{3.3}$$

Proposition 8.2. Let Y be an equivalence generator. The operator

$$X = \mathrm{pr}_{(\mathbf{x},\mathbf{u})}(Y), \tag{3.4}$$

is a symmetry operator for the system (1.1) with functions

$$p = p(u,v), \quad g = g(u,v), \tag{3.5}$$

iff the equations (3.5) are invariant under the group generated by

$$Z = \mathrm{pr}_{(\mathbf{u},\mathbf{f})}(Y). \tag{3.6}$$

The proof of these statements is almost immediate. In fact, to prove Proposition 8.1 we recall that the principal Lie algebra consists of all the operators (2.9) admitted by system (1.1) for any $p(u,v)$ and $g(u,v)$. Therefore the principal Lie algebra is the subalgebra of the equivalence algebra, such that any operator Y of this subalgebra leaves invariant equations $p = p(u,v)$ and $g = g(u,v)$. It follows that p, g, u, and v are invariant with respect to Y. It means that $\eta^i = 0$ and $\mu^i = 0$, or

$$\mathrm{pr}_{(\mathbf{u},\mathbf{f})}(Y) = 0.$$

Proposition 8.2 can be easily proved in a similar way.

Example 8.1. Let us find the principal Lie algebra via Proposition 8.1. We have, for the general operator (2.17) of the equivalence algebra,

$$\mathrm{pr}_{(\mathbf{u},\mathbf{f})}(Y) = (C_5+C_6u)\frac{\partial}{\partial u}+\sigma(v)\frac{\partial}{\partial v}+[(C_4+C_6-C_2)p+C_7]\frac{\partial}{\partial p}+[\sigma'(v)-C_2]g\frac{\partial}{\partial g}\,\cdot$$

Therefore from Eq. (3.3) we get

$$C_5 + C_6u = 0, \quad \sigma(v) = 0,$$
$$(C_4 + C_6 - C_2)p + C_7 = 0, \quad \sigma'(v) - C_2 = 0,$$

which give

$$C_2 = 0, \quad C_4 = 0, \quad C_5 = 0, \quad C_6 = 0, \quad C_7 = 0, \quad \sigma(v) = 0.$$

So

$$Y = C_1\frac{\partial}{\partial t} + C_3\frac{\partial}{\partial x}\,,$$

and the principal Lie algebra $L_{\mathcal{P}}$ is two dimensional and spanned by

$$X_1 = \frac{\partial}{\partial t}\,, \quad X_2 = \frac{\partial}{\partial x}\,\cdot \tag{3.7}$$

Example 8.2. Let us take the two-dimensional Abelian algebra L_2 spanned by the following operators (3.6):

$$\begin{aligned} Z_3 &= (1+v^2)\frac{\partial}{\partial v} + 2vg\frac{\partial}{\partial g}\,, \\ Z_4 &= u\frac{\partial}{\partial u} + (k+1)p\frac{\partial}{\partial p} - g\frac{\partial}{\partial g}\,\cdot \end{aligned} \tag{3.8}$$

The operator Z_3 corresponds to the equivalence operator (2.17) with $C_1 = \ldots = C_7 = 0$ and $\sigma(v) = 1 + v^2$, while the operator Z_4 is obtained by letting $C_2 = 1, C_4 = k+1, C_6 = 1, C_5 = C_7 = 0$, and $\sigma(v) = 0$.

So, here we look for Eqs. (1.1) admitting an extension of the principal algebra $L_{\mathcal{P}}$ by two-dimensional algebra with the basis

$$X_3 = (1+v^2)\frac{\partial}{\partial v}, \quad X_4 = t\frac{\partial}{\partial t} + (k+1)x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}\cdot \tag{3.9}$$

According to Proposition 8.2, we have to find invariant equations (3.5) for algebra L_2 with the basis (3.8). For this algebra we have the following two functionally independent invariants:

$$A = pu^{-(k+1)}, \quad B = \frac{ug}{1+v^2},$$

and invariant equations can be written by putting A and B to be arbitrary constants:

$$p = Au^{k+1}, \quad g = B\frac{1+v^2}{u}, \quad A, B = \text{const.} \tag{3.10}$$

So the specialization

$$u_t + A(k+1)u^k u_x = 0, \quad v_t = B\frac{1+v^2}{u} \tag{3.11}$$

of the system (1.1) admits the algebra L_4 spanned by the operators (3.7) and (3.9).

Now we inspect if this algebra L_4 is the greatest $\mathcal{E}$- extension of the principal algebra $L_{\mathcal{P}}$. For this we write the invariance conditions of Eqs. (3.10) with respect to the general operator Z defined in (3.1). These conditions are written as

$$(C_4 + C_6 - C_2)u^{k+1} + C_7 = (k+1)C_5 u^k + (k+1)C_6 u^{k+1},$$

$$(\sigma' - C_2)(1+v^2) = -C_6(1+v^2) + 2v\sigma$$

and give, in the case of arbitrary k,

$$C_5 = C_7 = 0, \quad C_4 = C_2 + kC_6,$$

$$\sigma = [L + (C_2 - C_6)\arctan v](1+v^2), \quad L = \text{const.}$$

So the system (3.11) admits the following operator:

$$X = (C_1 + C_2 t)\frac{\partial}{\partial t} + [C_3 + (C_2 + kC_6)x]\frac{\partial}{\partial x} + C_6 u\frac{\partial}{\partial u}$$

$$+[L + (C_2 - C_6)\arctan v](1+v^2)\frac{\partial}{\partial v}\cdot$$

Therefore the largest $\mathcal{E}$ extension of $L_{\mathcal{P}}$ is given by the operators (3.9) and

$$X_5 = kx\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} - (1+v^2)\arctan v\frac{\partial}{\partial v}\cdot$$

IV. Sketch of a qualitative model in detonation

As is well known when we consider the model of a binary reacting mixture of inviscid compressible gases by assuming that flow is adiabatic and neglecting diffusive effects, the governing system consists of the following equations:

$$\dot{\rho} + \rho u_x = 0, \quad \dot{u} + \frac{1}{\rho}\frac{d\hat{p}}{dx} = 0, \quad \dot{e} - \frac{\hat{p}}{\rho^2}\dot{\rho} = 0, \quad \dot{\lambda} = \hat{r}, \tag{4.1}$$

where the independent variables t and x represent the time and space, and $\rho, u, e,$ and λ represent, respectively, density, particle velocity, internal energy, and reaction progress variable ($0 \leq \lambda \leq 1$) given by the mass fraction of the product.

The pressure $\hat{p}$ and reaction rate $\hat{r}$ are given as constitutive functions, describing the mixture

$$\hat{p} = \hat{p}(\rho, e, \lambda), \quad \hat{r} = \hat{r}(\rho, e, \lambda). \tag{4.2}$$

The qualitative model, to which we apply the simplified approach exposed in previous sections, is defined by the system of equations

$$\rho_t + \frac{d\hat{p}}{dx} = 0, \quad \lambda_t = \hat{r} \tag{4.3}$$

with

$$\hat{p} = \hat{p}(\rho, \lambda), \quad \hat{r} = \hat{r}(\rho, \lambda). \tag{4.3$'$}$$

This system is a simplified mockup of the physical system (4.1) and can be regarded as a prototype of Euler's equations describing reactive compressible adiabatic flows.

This model was introduced by Fickett in [35] and widely discussed in [36] (also see [48], [37], [109]). The number of original field variables is reduced to ρ and λ, while the first three original equations of the "real" model are replaced by the first equation in (4.3).

The equation of progress of reaction is retained, but in a simplified form. Also, the equation of state and rate equation are retained, but they depend only on two arguments instead of three. Some variations are introduced in these equations in order to adapt them to some classes of phenomena. Here we consider an irreversible and exothermic reaction. In this case it is usual to assume the following form of the state equation for pressure:

$$\hat{p}(\rho, \lambda) = \frac{1}{2}(\rho^2 + q\lambda), \quad q = \text{const.}, \tag{4.4}$$

where $q > 0$ is the heat of reaction.

Hall and Ludford [48] generalized this form of $\hat{p}$ by assuming

$$\hat{p} = \frac{1}{2}[\rho^2 + qf(\lambda)]$$

and specialized the function f in the form

$$f(\lambda) = \lambda^n \quad (n \text{ is a positive integer}).$$

In agreement with this, here we also consider the following form of $\hat{p}$:

$$\hat{p} = \frac{1}{2}[\rho^2 + q(2\lambda - \lambda^2)]. \tag{4.5}$$

A rate equation, the most general form of which was considered in [36], is given by

$$\hat{r} = kR(\rho)G(\lambda),$$

where k is a rate multiplier and $G(\lambda)$ is the depletion factor. Following [36] we put $G(\lambda) = (1-\lambda)^m$, where m is a real positive number. So we write

$$\hat{r} = k(1-\lambda)^m R(\rho). \tag{4.6}$$

When we take $R(\rho) = 1$ the rate function assumes the following form:

$$\hat{r} = k(1-\lambda)^m.$$

In this case the second equation in (4.3) is decoupled and the reaction proceeds independently of ρ.

V. Application of the method to system (4.3)

We first rewrite the projection Z [cf. (3.1)] of the equivalence operator (2.17) in physical notations:

$$\begin{aligned} Z = (C_5 + C_6\rho)\frac{\partial}{\partial\rho} + \sigma(\lambda)\frac{\partial}{\partial\lambda} + [(C_4 + C_6 - C_2)\hat{p} + C_7]\frac{\partial}{\partial\hat{p}} \\ +[\sigma'(\lambda) - C_2]\hat{r}\frac{\partial}{\partial\hat{r}}\,. \end{aligned} \tag{5.1}$$

Here we consider the following two cases for the form of constitutive equations (4.3′).

Case 1:

$$\hat{p} = \frac{1}{2}(\rho^2 + q\lambda), \tag{5.2}$$

$$\hat{r} = k(1-\lambda)^m R(\rho). \tag{5.3}$$

The invariance condition for Eq. (5.2) is written

$$Z\Big[\hat{p}-\frac{1}{2}(\rho^2+q\lambda)\Big]_{(5.2)}$$

$$=\Big[(C_4+C_6-C_2)\hat{p}+C_7\big]-C_5\rho-C_6\rho^2-\frac{q}{2}\sigma\Big]_{(5.2)}$$

$$=\frac{1}{2}(C_4-C_6-C_2)\rho^2+\frac{q}{2}(C_4+C_6-C_2)\lambda+C_7-C_5\rho-\frac{q}{2}\sigma=0.$$

It follows that

$$C_4=C_2+C_6,\quad C_5=0,\quad \sigma=2C_6\lambda+\frac{2}{q}C_7. \tag{5.4}$$

The invariance condition for (5.3), by using (5.4), is written as

$$Z\Big[\hat{r}-k(1-\lambda)^m R(\rho)\Big]_{(5.3)}=(2C_6-C_2)(1-\lambda)^m R(\rho)$$

$$+2m(1-\lambda)^{m-1}\Big(C_6\lambda+\frac{C_7}{q}\Big)R(\rho)-C_6(1-\lambda)^m\rho R'(\rho)=0. \tag{5.5}$$

It follows that a nontrivial extension of the principal algebra $L_{\mathcal{P}}$ exists only if

$$C_7=-qC_6 \tag{5.6}$$

$$\rho R'(\rho)=l,\quad l=\text{const.} \tag{5.7}$$

We substitute the solution

$$R(\rho)=h\rho^l,\quad h=\text{const.}, \tag{5.8}$$

of Eq. (5.7) into Eq. (5.5) to obtain

$$C_2=(2-2m-6)C_6. \tag{5.9}$$

We use Proposition 8.2 and summarize the equations (5.4), (5.6), (5.8) and (5.9) as follows.

The system (4.3) with the constitutive equations (5.2) and (5.3) admits an $\mathcal{E}$- extension of the principal algebra $L_{\mathcal{P}}$ [cf. (3.7)] iff

$$\hat{p}=\frac{1}{2}(\rho^2+q\lambda),\quad \hat{r}=K(1-\lambda)^m\rho^l\quad (K=kh).$$

The corresponding system (4.3),

$$\rho_t+\rho\rho_x+\frac{q}{2}\lambda_x=0,\quad \lambda_t=K(1-\lambda)^m\rho^l, \tag{5.10}$$

has the symmetry algebra spanned by

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \tag{5.11}$$

$$X_3 = (2-2m-l)t\frac{\partial}{\partial t} + (3-2m-l)x\frac{\partial}{\partial x} + \rho\frac{\partial}{\partial \rho} + 2(\lambda-1)\frac{\partial}{\partial \lambda}\cdot$$

Some results of [109] are also related to this case.

Case 2: Here we take

$$\hat{p} = \frac{1}{2}[\rho^2 + q(2\lambda - \lambda^2)], \tag{5.12}$$

and $\hat{r}$ of the same form (5.3) as in the previous case.

The invariance condition of Eq. (5.12) is written

$$Z\Big[\hat{p} - \frac{1}{2}\big(\rho^2 + q(2\lambda - \lambda^2)\big)\Big]_{(5.12)} = 0$$

and implies

$$C_5 = 0, \quad C_4 = C_2 + C_6, \quad \sigma = \frac{1}{1-\lambda}\,[C_6(2\lambda - \lambda^2) + C_7].$$

The invariance condition of (5.3) yields

$$C_7 - C_6, \quad \sigma = C_6(\lambda - 1),$$
$$R(\rho) = h\rho^l, \quad C_2 = (1-m-l)C_6.$$

Now we summarize.

The system (4.3) with constitutive equations of the form (5.12) and (5.3) admits an $\mathcal{E}$- extension of the principal algebra $L_{\mathcal{P}}$ if the function $R(\rho)$ is of the form (5.8). The corresponding system (4.3),

$$\rho_t + \rho\rho_x + q(1-\lambda)\lambda_x = 0, \quad \lambda_t = K(1-\lambda)^m\rho^l, \tag{5.13}$$

has a symmetry algebra L_3 spanned by (3.7) and

$$X_3 = (1-m-l)t\frac{\partial}{\partial t} + (2-m-l)x\frac{\partial}{\partial x} + \rho\frac{\partial}{\partial \rho} + (\lambda-1)\frac{\partial}{\partial \lambda}\cdot \tag{5.14}$$

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Appendix: Calculation of $L_{\mathcal{E}}$

Here, a detailed solution of the determining equations for the equivalence algebra $L_{\mathcal{E}}$ of Eqs. (1.1) is offered.

After (2.14), Eqs. (2.3) and (2.4) are written as

$$\zeta_1^1 + p_u\zeta_2^1 + p_v\zeta_2^2 + \omega\frac{1}{3}u_x + \omega_4^1 v_x = 0, \quad \zeta_1^2 - \mu^2 = 0; \tag{A1}$$

$$\omega_1^1 = 0, \quad \omega_2^1 = 0, \quad \omega_1^2 = 0, \quad \omega_2^2 = 0. \tag{A2}$$

We solve these determining equations, taking into account Eqs. (2.2).

According to the prolongation formula (2.10′), we have

$$\zeta_1^1 = D_t(\eta^1) - u_t D_t(\xi^1) - u_x D_t(\xi^2) \tag{A3}$$

$$= \eta_t^1 + u_t + u_t\eta_u^1 - u_t(\xi_t^1 + u_t\xi_u^1 + v_t\xi_v^1) - u_x(\xi_t^2 + u_t\xi_u^2 + v_t\xi_v^2),$$

$$\zeta_2^1 = D_x(\eta^1) - u_t D_x(\xi^1) - u_x D_x(\xi^2)$$

$$= \eta_x^1 + u_x\eta_u^1 + v_x\eta_v^1 - u_t(\xi_x^1 + u_x\xi_u^1 + v_x\xi_v^1) - u_x(\xi_x^2 u_x\xi_u^2 + v_x\xi_u^2 + v_x\xi_v^2),$$

$$\zeta_1^2 = D_t(\eta^2) - v_t D_t(\xi^1) - v_x D_t(\xi^2)$$

$$= \eta_t^2 + u_t\eta_u^2 + v_t\eta_v^2 - v_t(\xi_t^1 + u_t\xi_u^1 + v_t\xi_v^1) - v_x(\xi_t^2 + u_t\xi_u^2 + v_t\xi_v^2),$$

$$\zeta_2^2 = D_x(\eta^2) - v_t D_x(\xi^1) - v_x D_x(\xi^2)$$

$$= \eta_x^2 + u_x\eta_u^2 + v_x\eta_v^2 - v_t(\xi_x^1 + u_x\xi_u^1 + v_x\xi_v^1) - v_x(\xi_x^2 + u_t\xi_u^2 + v_x\xi_v^2).$$

In view of equations $p_t = p_x = g_t = g_x = 0$ from (2.2), the differentiations (2.16) are reduced to

$$\widetilde{D}_t = \frac{\partial}{\partial t}, \quad \widetilde{D}_u = \frac{\partial}{\partial u} + p_u\frac{\partial}{\partial p} + g_u\frac{\partial}{\partial g},$$

$$\widetilde{D}_x = \frac{\partial}{\partial x}, \quad \widetilde{D}_v = \frac{\partial}{\partial v} + p_v\frac{\partial}{\partial p} + g_v\frac{\partial}{\partial g},$$

and the prolongation formulas (2.15) become

$$\begin{aligned}\omega_1^1 &= \widetilde{D}_t(\mu^1) - p_t\widetilde{D}_t(\xi^1) - p_x\tilde{D}_t(\xi^2) - p_u\widetilde{D}_t(\eta^1) - p_v\widetilde{D}_t(\eta^2)\\ &= \mu_t^1 - p_u\eta_t^1 - p_v\eta_t^2,\end{aligned} \tag{A4}$$

$$\begin{aligned}\omega_2^1 &= \widetilde{D}_x(\mu^1) - p_t\widetilde{D}_x(\xi^1) - p_x\tilde{D}_x(\xi^2) - p_u\widetilde{D}_x(\eta^1) - p_v\widetilde{D}_x(\eta^2)\\ &= \mu_x^1 - p_u\eta_x^1 - p_v\eta_x^2,\end{aligned} \tag{A5}$$

$$\begin{aligned}\omega_1^2 &= \widetilde{D}_t(\mu^2) - g_t\widetilde{D}_t(\xi^1) - g_x\tilde{D}_t(\xi^2) - g_u\widetilde{D}_t(\eta^1) - g_v\widetilde{D}_t(\eta^2)\\ &= \mu_t^2 - g_u\eta_t^1 - g_v\eta_t^2,\end{aligned} \tag{A6}$$

$$\begin{aligned}\omega_2^2 &= \widetilde{D}_x(\mu^2) - g_t\widetilde{D}_x(\xi^1) - g_x\tilde{D}_x(\xi^2) - g_u\widetilde{D}_x(\eta^1) - g_v\widetilde{D}_x(\eta^2)\\ &= \mu_t^2 - g_u\eta_x^1 - g_v\eta_x^2,\end{aligned} \tag{A7}$$

$$\begin{aligned}\omega_3^1 &= \widetilde{D}_u(\mu^1) - p_t\widetilde{D}_u(\xi^1) - p_x\tilde{D}_u(\xi^2) - p_u\widetilde{D}_u(\eta^1) - p_v\widetilde{D}_u(\eta^2)\\ &= \mu_u^1 + p_u\mu_p^1 + g_u\mu_g^1 - p_u\eta_u^1 - p_v\eta_u^2,\end{aligned} \tag{A8}$$

$$\begin{aligned}\omega_4^1 &= \widetilde{D}_v(\mu^1) - p_t\widetilde{D}_v(\xi^1) - p_x\widetilde{D}_v(\xi^2) - p_u\widetilde{D}_v(\eta^1) - p_v\widetilde{D}_v(\eta^2)\\ &= \mu_v^1 + p_v\mu_p^1 + g_v\mu_g^1 - p_u\eta_v^1 - p_v\eta_v^2.\end{aligned} \tag{A9}$$

First we solve Eqs. (A2), which, after (A4)-(A7), are written as

$$\mu_t^1 - p_u\eta_t^1 - p_v\eta_t^2 = 0, \quad \mu_x^1 - p_u\eta_x^1 - p_v\eta_x^2 = 0,$$

$$\mu_t^2 - g_u\eta_t^1 - g_v\eta_t^2 = 0, \quad \mu_x^1 - g_u\eta_x^1 - g_v\eta_x^2 = 0.$$

Since here μ^i, η^i are independent on p_u, p_v, g_u, g_v, it follows that

$$\mu_t^i = \mu_x^i = 0, \quad \eta_t^i = \eta_x^i = 0, \quad i = 1, 2. \tag{A10}$$

After (A10) and (2.2), the second equation in (A1) is written as

$$\begin{aligned}&\mu^2(u, v, p, g) = \zeta_1^2\\ &= (p_u u_x + p_v v_x)(g\xi_u^1 + v_x\xi_u^2 - \eta_u^2) + (\eta_v^2 + \xi_t^1 - v_x\xi_v^2)g - \xi_v^1\, g^2 - \xi_t^2\, v_x\end{aligned}$$

and implies immediately that

$$g\xi_u^1 + v_x\xi_u^2 - \eta_u^2 = 0$$

or

$$\xi_u^1 = 0, \quad \xi_u^2 = 0, \quad \eta_u^2 = 0, \tag{A11}$$

and also

$$\xi_t^2 = 0, \quad \xi_v^2 = 0. \tag{A12}$$

Thus μ^2 is equal to

$$\mu^2 = (\eta_v^2 - \xi_t^1)g - \xi_v^1 g^2, \tag{A13}$$

and, according to (A10) and (A11), depends on v and g only. Differentiation of (A13) with respect to t and x gives rise to

$$\xi_{tt}^1 = \xi_{tx}^1 = \xi_{tv}^1 = \xi_{xv}^1 = 0. \tag{A14}$$

It follows from Eqs. (A10)-(A14) that

$$\begin{aligned}&\xi^1 = Ct + \alpha(x) + \beta(v), \quad \xi^2 = \gamma(x),\\ &\eta^1 = \eta^1(u, v), \quad \eta^2 = \sigma(v)\\ &\mu^1 = \mu^1(u, v, p, g), \quad \mu^2 = [\sigma'(v) - C]g - \beta'(v)g^2.\end{aligned} \tag{A15}$$

Consider now the remaining determining equation, namely the first equation in (A1).

After (2.2) and (A15), the formulas (A3), (A8), and (A9) give rise to

$$\zeta_1^1 = (p_u u_x + p_v v_x)[C - \eta_u^1 + \beta'(v)g] + g\eta_v^1,$$

$$\zeta_2^1 = u_x\eta_u^1 + v_x\eta_v^1 + (p_u u_x + p_v v_x)[\alpha'(x) + \beta'(v)v_x] - \gamma'(x)u_x,$$

$$\zeta_2^2 = \sigma'(v)v_x - [\alpha'(x) + \beta'(v)v_x]g - \gamma'(x)v_x,$$

$$\omega_3^1 = \mu_u^1 + p_u\mu_p^1 + g_u\mu_g^1 - p_u\eta_u^1,$$

$$\omega_4^1 = \mu_v^1 + p_v\mu_p^1 + g_v\mu_g^1 - p_u\eta_v^1 - p_v\sigma'(v).$$

We substitute these expressions into the first equation (A1), and after easy calculations find the following general solution of the determining equations:

$$\begin{aligned} &\xi^1 = C_1 + C_2 t, \quad \xi^2 = C_3 + C_4 x,\\ &\eta^1 = C_5 + C_6 u, \quad \eta^2 = \sigma(v),\\ &\mu^1 = (C_4 + C_6 - C_2)p + C_7,\\ &\mu^2 = [\sigma'(v) - C_2]g. \end{aligned} \tag{A16}$$

Paper 9

Group analysis - a microscope of mathematical modelling. I: Galilean relativity in diffusion models

N. H. Ibragimov

Original unabridged manuscript partially published in [59], [63] and [64].

Principles of invariance and symmetry play a significant part in philosophy, mathematics, physics, engineering and life sciences.

In particular, Felix Klein [77] noted that the special theory of relativity is, in fact, a theory of invariants of the Lorentz group.

The aim of this series of papers is to extend Klein's idea and to clarify a significance of Lie group analysis in mathematical modelling in general. In these papers, however, no adherence to general theory is attempted, the emphasis is rather on physically relevant examples. The series comprises three parts, namely,

Part I: The Galilean relativity in diffusion models,

Part II: Dynamics in the de Sitter space,

Part III: Comments on explanation of Mercury's anomaly

Physical effects hinted by Lie group analysis and discussed in this series are relatively small and difficult to observe. But small effects are sometimes of fundamental significance for the theory, specifically if we deal with description of real world phenomena.

§ 1 Introduction

The invariance of the heat equation with respect to the Galilean transformation was discovered by S. Lie [85] in the case of one spatial variable and by J.A. Goff [44] in the case of three spatial variables. However, a physical significance of the Galilean invariance has not been elucidated until recently.

I enunciated in [59] the principle of Galilean relativity in diffusion problems. In what follows, I provide a systematic development of this approach by considering the linear thermal diffusion. The Galilean relativity expresses the independence of the fundamental physical law of heat balance upon a choice of inertial frames. It is shown that the derivation of the heat conduction equation can be based totally on the Galilean relativity principle, without using Fourier's law.

A physical consequence of the validity of this principle is that the phenomenological temperature, defined by Fourier's law, depends upon a choice of an inertial frame.

Finally, it is shown that the thermal diffusion with an arbitrary initial distribution of temperature is determined uniquely by the Galilean relativity principle, without using the differential equation of heat conduction.

The concluding section is devoted to nonlinear diffusion type equations.

1.1 Some comments on special relativity

Let us discuss the idea of Lorentz invariance in physics by considering the following well-known examples.

Consider, in the space (x, y, z, t), the Lorentz transformation describing the motion of a reference frame with a constant velocity V along the x axis:

$$x' = \frac{x + Vt}{\sqrt{1-\beta^2}}\,, \quad y' = y, \quad z' = z, \quad t' = \frac{t + x(V/c^2)}{\sqrt{1-\beta^2}}\,, \tag{1.1}$$

where $\beta^2 = V^2/c^2$ and $c \approx 3 \times 10^{10}$cm/sec. is the light velocity in vacuum.

Eqs. (1.1) determine, e.g. the transformation of velocities. Consider the particular case of motion parallel to the x axis. If an observer at rest in the original reference frame (x, y, z, t) detects the velocity u, then an observer at rest in the inertial frame (x', y', z', t') moving with velocity V along the x axis will detect the velocity

$$u' = \frac{u + V}{1 + u(V/c^2)}\,. \tag{1.2}$$

Likewise, one finds the transformation law of the volume Ω of a body:

$$\Omega' = \Omega\sqrt{1-\beta^2}. \tag{1.3}$$

It follows that the particle density ρ undergoes the transformation

$$\rho' = \frac{\rho}{\sqrt{1-\beta^2}}. \tag{1.4}$$

The transformation law (1.4) is an immediate consequence of (1.3) and the "physically natural" assumption that the number $\rho d\Omega$ of particles in a given volume $d\Omega$ is Lorentz-invariant.

From point of view of an observer in the moving reference frame, u, Ω, ρ and u', Ω', ρ' can be considered as *proper* and *effective* velocity, volume and density, respectively.

If the velocity V is significantly smaller than the velocity of light, then the Lorentz transformation (1.1) is written approximately as the *Galilean transformation*

$$x' = x + Vt, \quad y' = y, \quad z' = z, \quad t' = t, \tag{1.5}$$

whereas Eqs. (1.2) and (1.3)-(1.4) lead to the *Galilean addition of velocities*

$$u' = u + V \tag{1.6}$$

and the laws of Galilean invariance of volume and density

$$\Omega' = \Omega, \quad \rho' = \rho, \tag{1.7}$$

respectively.

We observe the Galilean addition of velocities (1.6) in everyday life while travelling, e.g. by train. When the train sets out from a station, it is difficult to perceive which of two trains, ours or a nearby train, is moving. This is the commonly known Galilean relativity.

Unlike the transformation of the velocity, the law of Lorentz diminution (1.4) of the density ρ is mainly of a theoretical value. Indeed, it is difficult to imagine how one might practically detect the *effective* density ρ'.

1.2 Galilean group in classical mechanics

The *Galilean transformations* provide a group theoretic background of classical and continuum mechanics based on Newton's laws of motion. In numerous models of fluid mechanics, one deals with finite or infinite dimensional extensions of the Galilean group represented in the space of physical variables (density, pressure, velocity vector etc.). Consequently, fluid mechanics is, from group theoretic point of view, a classification theory of extensions of the Galilean group, their representations and invariants.

If a mechanical system is composed of particles obeying Newton's laws of motion, then the system will be described by a set of ordinary differential equations of second order in which time is the independent variable. These equations are invariant under the Galilean transformation. In particular, this applies to systems consisting of arbitrarily large number of particles, e.g., fluids. In fluid mechanics, the molecular character of the fluid is of no direct interest and one can approximate a system by a continuum. Then one is primarily interested in distribution of field quantities such as *density, pressure* and *velocity* of fluid elements. It is remarkable that the *hydrodynamical approximation* also obeys the principle of *Galilean relativity.* That is, the differential equations are invariant under the Galilean transformation (1.5) provided that the field quantities undergo a suitable transformation.

Consider, e.g. the one-dimensional gasdynamic equations

$$\begin{aligned} &\rho_t + u\rho_x + \rho u_x = 0,\\ &\rho(u_t + uu_x) + p_x = 0,\\ &p_t + up_x + A(p,\rho)u_x = 0, \end{aligned} \tag{1.8}$$

where ρ, p and u are the density, pressure and velocity, respectively, $A(p,\rho)$ is an arbitrary function. The equations (1.8) are invariant under the Galilean transformations (1.5)-(1.6) and the identity transformation $\rho' = \rho,\ p' = p$ for the density and pressure.

1.3 Does temperature depend upon motion?

The usual approach to modelling of diffusion processes does not bear the Galilean invariance of resulting differential equations. This invariance has been revealed for the first time in 1881 by S. Lie in his paper [85] on the group classification of linear second-order partial differential equations.

Consider the one-dimensional heat conduction equation:

$$u_t = u_{xx} \tag{1.9}$$

The symmetry group of the heat equation (1.9) contains, in particular, the Galilean transformation (cf. (1.5))

$$\bar{t} = t, \quad \bar{x} = x + 2at \tag{1.10}$$

describing the motion of the reference frame by the velocity $V = 2a$. Eq. (1.9) is invariant under the transformation (1.10) provided that the temperature u undergoes the transformation

$$\bar{u} = ue^{-(ax+a^2t)}. \tag{1.11}$$

Thus, Lie's result provides us with the positive answer to the question whether the temperature in diffusion problems depends upon a motion of inertial frames. To the best of my knowledge, a physical significance of this result was not elucidated until recently. It seemed that the thermal diffusion, unlike the classical and continuum mechanics, did not bear the Galilean relativity as an essential principle.

On the other hand, it has been shown in [59] that the Galilean invariance determines uniquely the fundamental solution and hence allows one to solve the problem of heat diffusion with an arbitrary initial temperature without using the differential equation of heat conduction. This is due to the fact that the heat equation itself is uniquely determined by the requirement of the Galilean invariance.

The present paper contains a discussion of both mathematical and physical aspects of the Galilean principle in thermal diffusion problems.

§ 2 Physical Postulates

The behavior of physical systems in diffusion processes is described by neglecting the molecular character of the system. The elements of this idealized system are assumed to be unaffected by molecular fluctuations regardless of how small a volume is being considered.

Let us discuss a steady heat diffusion process in a homogeneous medium of an arbitrary dimension n. We assume that the *density* ρ of the medium, its *specific heat* c and *thermal conductivity* k are positive constants. Let us isolate, in the medium, an arbitrary domain Ω and denote by $\partial\Omega$ and ν the boundary of Ω and the unit outer normal to the boundary, respectively. Denote by u the absolute temperature, so that $u = u(t, \boldsymbol{x})$ is the temperature field defined for any time t and $\boldsymbol{x} \in \Omega$.

After J.B.J. Fourier's 1811 paper on the theory of heat conduction and his famous 1822 book *Théorie analitique de la chaleur*, the mathematical model of thermal diffusion is usually based on the following physical laws of heat balance in arbitrary domains Ω.

First Postulate. The quantity of heat Q in the domain Ω is proportional to the mass of the domain and to its temperature:

$$Q(t) = \int_\Omega \rho c u(t, \boldsymbol{x}) d\Omega. \tag{2.1}$$

Second Postulate. Heat diffuses from a higher to a lower temperature, and the heat flow is proportional to the gradient of temperature.

Third Postulate. For any domain Ω, the rate of change of heat content dQ/dt in Ω is equal to the difference between influx and efflux of heat through the surface $\partial\Omega$. According to (2.1) and the Second Postulate, this postulate is written:

$$\int_\Omega \rho c \frac{\partial u}{\partial t} d\Omega = \int_{\partial\Omega} k\nabla u \cdot \nu dS. \tag{2.2}$$

The above three postulates do not include the invariance under Galilean transformations. In other words, they do not specify whether the physical laws of heat balance depend upon a choice of inertial frames. Therefore, I complete the Fourier laws by the following fourth postulate expressing the Galilean principle in diffusion problems.

Fourth Postulate. The heat balance equation is Galilean invariant. Namely, the Galilean transformations (1.10), accompanied by a suitable linear transformation of the temperature u, leave the balance equation (2.2) unaltered.

The following statement shows that the fourth postulate determines how the temperature T behaves in different inertial frames.

Theorem 9.1. In order that the invariance Fourth Postulate hold it is necessary and sufficient that the Galilean transformation

$$\overline{\boldsymbol{x}} = \boldsymbol{x} + t\boldsymbol{V} \tag{2.3}$$

be accompanied by the following transformation of temperature:

$$\bar{u} = ue^{-\alpha(2\boldsymbol{x}\cdot\boldsymbol{V} + t|\boldsymbol{V}|^2)}. \tag{2.4}$$

Here $\boldsymbol{V} = (V^1, \ldots, V^n)$ is any constant vector, and the positive constant

$$\alpha = \frac{k}{4c\rho}$$

is known as the *thermal diffusivity* of the medium.

Proof. Let the balance equation (2.2) be invariant under the transformation (2.3) and

$$\bar{u} = f(t, \boldsymbol{x}, \boldsymbol{V})u \tag{2.5}$$

with an unknown coefficient $f(t, \boldsymbol{x}, \boldsymbol{V})$ to be found from the invariance condition of Eq. (2.2). For twice continuously differentiable functions $u(t, \boldsymbol{x})$

the surface integral in the right hand side of Eq. (2.2) can be rewritten by the divergence theorem in the form

$$\int_{\partial\Omega} k\nabla u \cdot \nu dS = \int_{\Omega} k\text{div}(\nabla u) d\Omega.$$

Hence, Eq. (2.2) is equivalent to the differential equation of heat conduction

$$u_t = 4\alpha\Delta u \tag{2.6}$$

since the domain Ω is arbitrary. The usual infinitesimal invariance test of Eq. (2.6) under the n-parameter group of transformations (2.3), (2.5) with the group parameters $V^1, \ldots, V^n$ yields

$$f = e^{-\alpha(2\boldsymbol{x}\cdot\boldsymbol{V}+t|\boldsymbol{V}|^2)}.$$

Hence, the transformation law (2.4) is a necessary condition for the validity of the fourth postulate.

One can verify by straightforward computation that the transformations (2.3)–(2.4) leave invariant the integral equation (2.2) for any continuously differentiable $u(t, \boldsymbol{x})$ and any domain Ω. Hence the transformation law (2.4) is sufficient for the validity of the fourth postulate. This completes the proof. See also § 3.

Thus, according to the Galilean principle, if an observer at rest detects the temperature field

$$u = u(t, \boldsymbol{x}),$$

an observer moving with the constant velocity $\boldsymbol{V}$ will detect field in his local coordinate system $\{\overline{\boldsymbol{x}}\}$ the following *effective* temperature:

$$\bar{u} = e^{\alpha(t|\boldsymbol{V}|^2-2\boldsymbol{x}\cdot\boldsymbol{V})}u(t, \overline{\boldsymbol{x}} - t\boldsymbol{V}).$$

§ 3 Derivation of diffusion equations from Galilean principle

3.1 Semi-scalar representation of the Galilean group

The Galilean group comprises the time translations, the *isometric motions*, i.e. the translations and rotations, in the space $\mathbb{R}^n$ of variables $\boldsymbol{x} = (x^1, ..., x^n)$, and the n-dimensional generalization of the Galilean transformation (1.10). The respective generators of this transformations are:

$$X_0 = \frac{\partial}{\partial t}, \quad X_i = \frac{\partial}{\partial x^i}, \quad X_{ij} = x^j\frac{\partial}{\partial x^i} - x^i\frac{\partial}{\partial x^j}, \quad i, j = 1, ..., n, \tag{3.1}$$

and

$$Y_i^0 = 2t\frac{\partial}{\partial x^i}\,, \quad i = 1, ..., n. \tag{3.2}$$

It was shown by Lie [85] and, in a more general context, by Ovsyannikov [103], [106] that a point symmetry group for *linear* partial differential equations with one dependent variable u and any number of independent variables $z = \{z^i\}$ has a special form. Namely, the transformations of the independent variables do not depend upon u while the dependent variable u remains unaltered or undergoes a linear transformation, i.e.

$$z'^i = f^i(z, a), \quad u' = g(z, a)u + h(z, a). \tag{3.3}$$

We will deal with *homogeneous linear* equations. In this case we can let $h(z, a) = 0$, and hence consider infinitesimal symmetries of the form

$$X = \xi^i(z)\frac{\partial}{\partial z^i} + \mu(z)u\frac{\partial}{\partial u}\,. \tag{3.4}$$

where

$$\xi^i(z) = \frac{\partial f^i(z, a)}{\partial a}\Big|_{a=0}, \quad \mu(z) = \frac{\partial g(z, a)}{\partial a}\Big|_{a=)}\,.$$

Furthermore, any linear homogeneous equation admits the dilation of the dependent variable, i.e. has the infinitesimal symmetry

$$T_1 = u\frac{\partial}{\partial u}\,\cdot \tag{3.5}$$

Therefore, we will add T_1 to the operators (3.1), (3.2) and call the corresponding extended transformation group again the *Galilean group* and denote it by $\mathcal{G}$.

Definition 9.1. Let the group of transformations (3.3) do not change the dependent variable u, i.e. $u' = u$. Then u is said to be *invariant* or *scalar*. Thus, transformation groups with a scalar u are given by generators (3.4) with $\mu(z) = 0$.

The geometric uniformity of a medium means its invariance under isometric motions in $\mathbb{R}^n$. Therefore, it is natural to assume that *steady diffusion processes in homogeneous media* are governed by functions u that are invariant under the time translations as well as translations and rotations in $\mathbb{R}^n$. The situation is described by the following definition.

Definition 9.2. A linear representation of the Galilean group $\mathcal{G}$ is an extension of the action of $\mathcal{G}$ by linear transformations of the variable u. A linear representation is said to be *semi-scalar* if u is invariant under the time translations and isometric motions in $\mathbb{R}^n$. That is, a semi-scalar representation is determined by a Lie algebra spanned by the operators (3.1) and by the operators of the form:

$$T_1 = u\frac{\partial}{\partial u}\,, \quad Y_i = t\frac{\partial}{\partial x^i} + \mu^i(t,\boldsymbol{x})u\frac{\partial}{\partial u}\,, \quad i,j = 1,...,n. \tag{3.6}$$

In the particular case $\mu^i(t,\boldsymbol{x}) \equiv 0$, $i = 1,\ldots,n$, the generators (3.1), (3.6) define what is called the *scalar representation* of the Galilean group. The scalar representation is naturally identified with $\mathcal{G}$.

Theorem 9.2. There exist precisely two different semi-scalar linear representations of the Galilean group. One of them is the scalar representation $\mathcal{G}$ and the other is a proper semi-scalar representation with the generators

$$X_0 = \frac{\partial}{\partial t}\,, \quad X_i = \frac{\partial}{\partial x^i}\,, \quad X_{ij} = x^j\frac{\partial}{\partial x^i} - x^i\frac{\partial}{\partial x^j}\,,$$

$$T_1 = u\frac{\partial}{\partial u}\,, \quad Y_i = 2t\frac{\partial}{\partial x^i} - x^i u\frac{\partial}{\partial u}\,, \quad i,j = 1,...,n. \tag{3.7}$$

Proof. According to Definition 9.2, the operators (3.1), (3.6) generate a semi-scalar representation of the group $\mathcal{G}$ if and only if the linear span L of the operators (3.1) and (3.6) is a Lie algebra. This condition is satisfied if

$$[X_\alpha, Y_i] \in L, \quad \alpha = 0,1,\ldots,n; \quad i = 1,\ldots,n, \tag{3.8}$$

and

$$[X_{ij}, Y_k] \in L, \quad i,j,k = 1,\ldots,n. \tag{3.9}$$

We have

$$[X_0, Y_i] = \frac{\partial}{\partial x^i} + \frac{\partial \mu^i}{\partial t}\,u\,\frac{\partial}{\partial u}\,, \quad [X_k, Y_i] = \frac{\partial \mu^i}{\partial x^k}\,u\,\frac{\partial}{\partial u}\,, \quad i,k = 1,\ldots,n. \tag{3.10}$$

It follows from (3.10) and (3.8):

$$[X_0, Y_i] = X_i + C_0^i T_1, \quad [X_k, Y_i] = C_k^i T_1,$$

or

$$\frac{\partial \mu^i}{\partial t} = C_0^i, \quad \frac{\partial \mu^i}{\partial x^k} = C_k^i$$

where C_α^i are arbitrary constants for $\alpha = 0,1,\ldots,n$ and $i = 1,\ldots,n$. Hence, up to the addition of T_1 to Y_i, we have

$$Y_i = t\frac{\partial}{\partial x^i} + (C_0^i t + C_k^i x^k)u\frac{\partial}{\partial u}\,, \quad i,j = 1,...,n. \tag{3.11}$$

Substitution of the operator (3.11) into the conditions (3.9) yields

$$C_0^i = 0, \quad C_k^i = C\delta_k^i,$$

where C is an arbitrary constant and δ_k^i is the Kronecker symbol. If $C = 0$ one arrives at the scalar representation $\mathcal{G}$. In the case $C \neq 0$ one can set $C = -\frac{1}{2}$ by choosing a suitable scaling of t or $\boldsymbol{x}$. This proves the theorem.

Definition 9.3. The group with the generators (3.7) is called the *heat representation of the Galilean group* and is denoted by $\mathcal{H}$.

Thus, according to Theorem 9.2, there are exactly two distinctly different semi-scalar linear representations of the Galilean group, namely, the scalar representation $\mathcal{G}$ and the heat representation $\mathcal{H}$.

3.2 Extension by scaling transformations

In mechanics, scaling transformations (dilations) play an essential role. The generator of an arbitrary dilation can be taken in the form

$$T = at\frac{\partial}{\partial t} + b^1x^1\frac{\partial}{\partial x^1} + ... + b^nx^n\frac{\partial}{\partial x^n} \tag{3.12}$$

with arbitrary constant coefficients a and b^i. The dilation of u can be eliminated by virtue of the operator T_1 given in (3.5). The extension of the heat representation $\mathcal{H}$ of the Galilean group by scaling transformations is described by the following theorem.

Theorem 9.3. The Lie algebra spanned by the operators (3.7) admits an extension by a generator of dilations (3.12). The extension is unique and is obtained by adding to (3.7) the operator

$$T_2 = 2t\frac{\partial}{\partial t} + x^i\frac{\partial}{\partial x^i}\,. \tag{3.13}$$

Proof. Let L be the linear span of the operators (3.7) and an operator (3.12) with undetermined coefficients a, b^i. The Lie algebra condition requires that

$$[T, X_{ij}] \in L, \quad [T, Y_i] \in L. \tag{3.14}$$

We have (no summation with respect to the repeated indices i, j):

$$[T, X_{ij}] = (b^j - b^i)(x^j\frac{\partial}{\partial x^i} + x^i\frac{\partial}{\partial x^j}), \quad i, j = 1, ..., n.$$

Therefore the first equation (3.14) yields

$$b^1 = \cdots = b^n = b.$$

Hence,

$$T = at\frac{\partial}{\partial t} + bx^i\frac{\partial}{\partial x^i}\,.$$

Now we have

$$[T, Y_i] = 2(a-b)t\frac{\partial}{\partial x^i} - bx^i u\frac{\partial}{\partial u}\,,$$

and the second equation (3.14) yields

$$a = 2b.$$

Thus, we have arrived at the operator (3.13).

Definition 9.4. The group with the generators (3.7) and (3.13) is termed the *extended heat representation* of the Galilean group. This extension of the group $\mathcal{H}$ by scaling transformations is denoted by $\mathcal{S}$.

3.3 Diffusion equations

Diffusion processes are described, in the linear approximation, by second-order partial differential equations. Bearing this in mind, let us find all linear second-order equations

$$a^{\alpha\beta}(t, \boldsymbol{x})u_{\alpha\beta} + b^{\alpha}(t, \boldsymbol{x})u_{\alpha} + c(t, \boldsymbol{x})u = 0 \tag{3.15}$$

satisfying the Galilean invariance principle. Here $\alpha, \beta = 0, ..., n,$ and $x^0 = t$. The subscripts α, β denote the derivations with respect x^α, x^β.

Theorem 9.4. Equation (3.15) is invariant under the scalar representation of the Galilean group $\mathcal{G}$ if and only if it is the Helmholtz equation

$$\Delta u + \beta u = 0, \quad \beta = \text{const.} \tag{3.16}$$

Proof. We have to investigate the invariance of Eq. (3.15) under the generators (3.1), (3.2) and (3.3). The invariance under the translation generators X_α $(\alpha = 0, ..., n)$ shows that Eq. (3.15) has the constant coefficients $a^{\alpha\beta}, b^\alpha, c$. This equation is invariant under (3.3) as well because it is homogeneous. Thus, we have to determine the constant coefficients of Eq. (3.15) from the invariance test under X_{ij} and Y_i^0.

Consider the operator X_{ij}. Its prolongation to the first and second derivatives has the form

$$X_{ij} = x^j \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^j} + u_j \frac{\partial}{\partial u_i} - u_i \frac{\partial}{\partial u_j} + (\delta^i_k u_{\alpha j} + \delta^i_\alpha u_{kj} - \delta^j_k u_{\alpha i} - \delta^j_\alpha u_{ki}) \frac{\partial}{\partial u_{\alpha k}}$$

where $i, j, k = 1, ..., n$ and $\alpha = 0, ..., n$. Hence:

$$X_{ij}(a^{\alpha\beta} u_{\alpha\beta} + b^\alpha u_\alpha + cu) = a^{\alpha i} u_{\alpha j} + a^{ik} u_{kj} - a^{\alpha j} u_{\alpha i} - a^{jk} u_{ki} + b^i u_j - b^j u_i \,.$$

Using this expression and the well-known invariance condition one obtains

$$a^{0i} = 0, \quad a^{ij} = a\delta^{ij}, \quad b^i = 0, \quad i, j = 1, \ldots, n,$$

and hence arrives at the following equation:

$$Au_{tt} + a\Delta u + bu_t + cu = 0, \quad A, a, b, c = \text{const.} \tag{3.17}$$

Now we test Eq. (3.17) for the invariance under (3.2). The second prolongation of the operator (3.2) has the form

$$Y^0_i = t\frac{\partial}{\partial x^i} - u_i \frac{\partial}{\partial u_t} - 2u_{ti} \frac{\partial}{\partial u_{tt}} - u_{ij} \frac{\partial}{\partial u_{tj}} \cdot$$

Therefore, the invariance conditions

$$Y^0_i (Au_{tt} + a\Delta u + bu_t + cu) = 0, \quad i = 1, ..., n,$$

are written

$$-2Au_{ti} - bu_i = 0, \quad i = 1, ..., n,$$

and yield $A = 0, b = 0$. Setting $\beta = c/a$, we obtain Eq. (3.16).

Theorem 9.5. Equation (3.15) is invariant under the heat representation $\mathcal{H}$ of the Galilean group if and only if it has the form

$$u_t = \Delta u + \beta u, \tag{3.18}$$

where Δ is the n-dimensional Laplacian in $\mathbb{R}^n$ and $\beta = \text{const}$.

Proof. We have to investigate the invariance of Eq. (3.15) under the generators (3.7) of the group $\mathcal{H}$. As shown in the proof of the previous theorem, the invariance under the translation generators X_α $(\alpha = 0, ..., n)$ and the rotation generators X_{ij} reduces Eq. (3.15) to the form (3.17). Thus, it remains to satisfy the invariance conditions of Eq. (3.17) under Y_i from (3.7). The second prolongation of the operator Y_i has the form

$$Y_i = 2t\frac{\partial}{\partial x^i} - x^i u\frac{\partial}{\partial u} - (x^i u_t + 2u_i)\frac{\partial}{\partial u_t} - (\delta^i_k u + x^i u_k)\frac{\partial}{\partial u_k}$$
$$-(x^i u_{tt} + 4u_{ti})\frac{\partial}{\partial u_{tt}} - (\delta^i_j u_t + x^i u_{tj} + 2u_{ij})\frac{\partial}{\partial u_{tj}} - (\delta^i_k u_j + \delta^i_j u_k - x^i u_{jk})\frac{\partial}{\partial u_{jk}}\cdot$$

The invariance conditions

$$Y_i(Au_{tt} + a\Delta u + bu_t + cu) = 0, \quad i = 1, ..., n,$$

yield

$$A = 0, \quad a + b = 0.$$

By setting $\beta = c/a$, one arrives at Equation (3.18), thus completing the proof of the theorem.

3.4 Heat equation

Theorem 9.6. The extended heat representation $\mathcal{S}$ of the Galilean group is a symmetry group for the heat equation

$$u_t = \Delta u. \tag{3.19}$$

Furthermore, Eq. (3.19) is the only linear second-order equation (3.15) admitting the group $\mathcal{S}$.

Proof. In virtue of Theorem 9.5, it suffices to identify those diffusion equations (3.18) invariant under the scaling transformations with the generator (3.13). After simple calculations one obtains from this invariance condition that $\beta = 0$, and hence arrives at Eq. (3.19).

Remark 9.1. The maximal Lie algebra admitted by the n-dimensional heat equation (3.19) is spanned by (found for $n = 3$ in [44])

$$X_0 = \frac{\partial}{\partial t}, \quad X_i = \frac{\partial}{\partial x^i}, \quad X_{ij} = x^j\frac{\partial}{\partial x^i} - x^i\frac{\partial}{\partial x^j}, \quad Y_i = 2t\frac{\partial}{\partial x^i} - x^i u\frac{\partial}{\partial u},$$
$$T_1 = u\frac{\partial}{\partial u}, \quad T_2 = 2t\frac{\partial}{\partial t} + x^i\frac{\partial}{\partial x^i}, \quad Z = t^2\frac{\partial}{\partial t} + tx^i\frac{\partial}{\partial x^i} - \frac{1}{4}(2nt + |\boldsymbol{x}|^2)u\frac{\partial}{\partial u}$$

and by the infinite-dimensional ideal consisting of the operators

$$X_\tau = \tau(t, \boldsymbol{x})\frac{\partial}{\partial u},$$

where $\tau(t, \boldsymbol{x})$ is an arbitrary solution of Equation (3.19).

3.5 Addition to the 2006 edition

The results of § 3 have been extended to the Lorentz group in [68]. It is proved there that there exist two non-similar semi-scalar linear representations of the Lorentz group. One of them is the *scalar representation* with the usual generators augmented by the generator T_1 of the scaling transformation, namely:

$$X_0 = \frac{\partial}{\partial t}\,, \quad X_i = \frac{\partial}{\partial x^i}\,, \quad X_{ij} = x^j \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^j}\,,$$

$$T_1 = u\frac{\partial}{\partial u}\,, \quad Y_i = t\frac{\partial}{\partial x^i} + \frac{x^i}{c^2}\frac{\partial}{\partial t}\,, \quad i,j = 1,2,3,$$

where c is the light velocity. The other representation is the proper *semi-scalar representation* defined by the generators

$$X_0 = \frac{\partial}{\partial t}\,, \quad X_i = \frac{\partial}{\partial x^i}\,, \quad X_{ij} = x^j \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^j}\,, \quad T_1 = u\frac{\partial}{\partial u}\,,$$

$$Y_i = t\frac{\partial}{\partial x^i} + \frac{x^i}{c^2}\frac{\partial}{\partial t} - \alpha x^i u\frac{\partial}{\partial u}\,, \quad \alpha = \text{const.} \neq 0, \quad i,j = 1,2,3.$$

The most general linear second-order equation admitting the proper semi-scalar representation of the Lorentz group has the form

$$\frac{1}{c^2}u_{tt} = \Delta u - \alpha u_t + \beta u, \quad \alpha, \beta = \text{const.}$$

It is manifest that by letting here $c \to \infty$ we arrive at Equation (3.18).

§ 4 Solution of the Cauchy problem using Galilean principle

The Galilean principle, specifically the invariance under the extended representation $\mathcal{S}$ of the Galilean group can be used, e.g. for constructing the fundamental solution and solving the Cauchy problem. This approach is essentially simpler than the commonly used Fourier transform method. Another advantage of the new approach is its independence on a choice of coordinate systems. Therefore the method can be applied to differential equations with variable coefficients as well*.

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* *Author's note to this 2006 edition:* A general invariance theory of fundamental solutions for parabolic and hyperbolic equations is presented in [65], Chap. 3. See also Paper 21 in Vol. I of these "Selected Works".

4.1 Fundamental solution of the Cauchy problem

Recall that the distribution $u = E(t, \boldsymbol{x})$ is called the fundamental solution of the Cauchy problem for the heat equation (3.19) if it solves Eq. (3.19)

$$u_t - \Delta u = 0, \quad t > 0, \tag{4.1}$$

and satisfies the following special initial condition:

$$u\big|_{t=0} = \delta(\boldsymbol{x}), \tag{4.2}$$

where $\delta(\boldsymbol{x})$ is Dirac's δ-function and

$$u\big|_{t=0} = \lim_{t\to+0} E(t, \boldsymbol{x}).$$

The theory of distributions reduces the solution of an arbitrary Cauchy problem to calculation of the fundamental solution. Namely, if the fundamental solution $E(t, \boldsymbol{x})$ is known, the solution $u(t, \boldsymbol{x})$ of the Cauchy problem

$$u_t - \Delta u = 0 \quad (t > 0), \quad u\big|_{t=0} = u_0(\boldsymbol{x}) \tag{4.3}$$

with an arbitrary initial data $u_0(\boldsymbol{x})$ is given by the convolution:

$$u(t, \boldsymbol{x}) = E * u_0 \equiv \int\limits_{\mathbf{R}^n} u_0(\boldsymbol{y}) E(t, \boldsymbol{x} - \boldsymbol{y}) d\boldsymbol{y}, \quad t > 0, \tag{4.4}$$

provided that the convolution (4.4) exists.

4.2 Symmetry of the initial condition

Theorem 9.7. The maximal subgroup of the extended heat representation $\mathcal{S}$ of the Galilean group leaving invariant the initial condition (4.2) is generated by the operators

$$\begin{aligned} X_{ij} &= x^j \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^j}\,, \\ Y_i &= 2t\frac{\partial}{\partial x^i} - x^i u \frac{\partial}{\partial u}\,, \quad i, j = 1, ..., n, \\ T_2 - nT_1 &= 2t\frac{\partial}{\partial t} + x^i \frac{\partial}{\partial x^i} - nu\frac{\partial}{\partial u}\,. \end{aligned} \tag{4.5}$$

Proof. The invariance of the initial condition (4.2) assumes, in particular, the invariance of the initial manifold $t = 0$ and of the support $\boldsymbol{x} = 0$ of the

δ-function. Thus, we write the invariance test of the equations $t = 0, \boldsymbol{x} = 0$ with respect to the linear combination with constant coefficients,

$$X = c^0 X_0 + c^i X_i + c^{ij} X_{ij} + l^i Y_i + k^1 T_1 + k^2 T_2,$$

of the generators (3.7), (3.13) of the group $\mathcal{S}$. Inspection of the invariance condition shows that $c^0 = c^i = 0$, i.e. the translation generators X_0 and X_i has been eliminated. Hence, the operators (3.7), (3.13) are restricted to X_{ij}, Y_i, T_1 and T_2. Equation (4.2) is manifestly invariant under the operators X_{ij} and Y_i, but it is not invariant under the two-dimensional algebra spanned by T_1, T_2. Therefore, we inspect the infinitesimal invariance test for the linear combination

$$(T_1 + kT_2)\big|_{t=0} = x^i \frac{\partial}{\partial x^i} + ku \frac{\partial}{\partial u}, \quad k = \text{const}.$$

Under this operator, the variable u and the δ-function are subjected to the infinitesimal transformations

$$\overline{u} \approx u + aku, \quad \overline{\delta} \approx \delta - an\delta.$$

It follows, that

$$\overline{u} - \overline{\delta} = u - \delta + a(ku + n\delta) + o(a),$$

and that

$$(\overline{u} - \overline{\delta})\big|_{u=\delta} = a(k+n)\delta + o(a).$$

Hence, the invariance condition is written $k + n = 0$ and we arrive at the operators (4.5).

Remark 9.2. Since the differential equation (4.1) admits the group $\mathcal{S}$, the operators (4.5) are admitted by both the differential equation (4.1) and the initial condition (4.2). In other words, the group with the generators (4.5) is a symmetry group of the special Cauchy problem (4.1)–(4.2).

Remark 9.3. Eqs. (4.1)–(4.2) admit, in addition to (4.5), the generator

$$Z = t^2 \frac{\partial}{\partial t} + tx^i \frac{\partial}{\partial x^i} - \frac{1}{4}(2nt + |\boldsymbol{x}|^2) u \frac{\partial}{\partial u}$$

of projective transformations (see Remark 9.1). However, this excess symmetry is not used in what follows.

4.3 Calculation of the fundamental solution

Now we can show that the invariance under the heat representation of the Galilean group determines the fundamental solution uniquely. Moreover, computation of the fundamental solution does not require integration of the heat equation. Namely, the following statement holds.

Theorem 9.8. There exists one and only one function, namely

$$E(t, \boldsymbol{x}) = (2\sqrt{\pi t})^{-n}\, \mathrm{e}^{-|\boldsymbol{x}|^2/(4t)}, \tag{4.6}$$

which is invariant under the group with the generators (4.5) and satisfies the initial condition (4.2). The function (4.6) solves Eq. (4.1) as well. Hence it is the fundamental solution of the Cauchy problem for the heat equation.

Proof. Let us first notice that the functionally independent invariants for the operators X_{ij} from (4.5) are t, r, u, where

$$r = |\boldsymbol{x}| \equiv \sqrt{(x^1)^2 + \cdots + (x^n)^2}.$$

Now we restrict the action of the operators Y_i from (4.5) to functions of these invariants and arrive at the operators

$$Y_i = x^i\Big(2\frac{t}{r}\frac{\partial}{\partial r} - u\frac{\partial}{\partial u}\Big).$$

For them, the independent invariants are

$$t \quad \text{and} \quad p = u\,\mathrm{e}^{r^2/(4t)}.$$

The last operator from (4.5) is written in these variables in the form:

$$Z_1 - nZ_2 = 2t\frac{\partial}{\partial t} - np\frac{\partial}{\partial p}\,.$$

It has one functionally independent invariant, namely $J = t^{n/2}p$. Hence, the function

$$J = (\sqrt{t})^n u\,\mathrm{e}^{r^2/(4t)}$$

is the only independent invariant for the operators (4.5). Accordingly, the general form of the function $u = \phi(t, \boldsymbol{x})$ which is invariant under the operators (4.5) is given by $J = C$, whence

$$u = C(\sqrt{t})^{-n}\mathrm{e}^{-r^2/(4t)}, \quad C = \text{const.}$$

Invoking the following equation known from theory of distributions:

$$\lim_{t\to+0}\frac{1}{(2\sqrt{\pi t})^n}\,\mathrm{e}^{-r^2/(4t)}=\delta(\boldsymbol{x}),$$

we obtain from the initial condition (4.2) that

$$C=(2\sqrt{\pi})^{-n}.$$

Thus, we have proved the uniqueness of the invariant function satisfying the condition (4.2) and arrived at the fundamental solution (4.6).

Remark 9.4. It is worth noting that the proof provides also a practical algorithm for constructing the fundamental solution. Furthermore, using Equation (4.6), we conclude that the Poisson formula

$$u(t,\boldsymbol{x})=\frac{1}{(2\sqrt{\pi t})^n}\int\limits_{\mathbf{R}^n}u_0(y)\mathrm{e}^{-|\boldsymbol{x}-\boldsymbol{y}|^2/(4t)}d\boldsymbol{y},\quad t>0, \tag{4.7}$$

for the solution of the Cauchy problem (4.3) has been obtained solely by the invariance principle, without integrating the heat equation.

Remark 9.5. According to the above construction, the fundamental solution (4.6) has the remarkable property to be independent on a choice of inertial frames.

§ 5 Nonlinear diffusion type equations

Definition 9.5. An evolutionary equation

$$u_t=F(x,u,u_x,u_{xx}) \tag{5.1}$$

is said to be diffusion type if it is invariant under the heat representation $\mathcal{H}$ of the Galilean group (see Definition 9.3).

Theorem 9.9. The most general diffusion type equation (5.1) has the form

$$u_t=\Phi\left[\left(\frac{u_x}{u}\right)_x\right]. \tag{5.2}$$

Proof. The reckoning shows that the generators (3.7) of the of the group $\mathcal{H}$ in the case $n=1$ have the following two functionally independent invariants depending on u,u_t,u_x,u_{xx} :

$$J_1=u_t-uxx,\quad J_2=\frac{u_{xx}}{u}-\frac{u_x^2}{u^2}\equiv\left(\frac{u_x}{u}\right)_x.$$

Hence, the most general regular invariant equation is given by $J_1=\Phi(J_2)$, i.e. by (5.2).

Theorem 9.10. The most general diffusion type equation that is invariant under the extended heat representation $\mathcal{S}$ of the Galilean group with the generators (3.7), (3.7) (see Definition 9.4) has the form

$$u_t = k\, u_{xx} + (1-k)\, \frac{u_x^2}{u}\,, \quad k = \text{const.} \tag{5.3}$$

Proof. Using the invariance test under the generators (3.7), (3.7) of the group $\mathcal{S}$, one can show that the group $\mathcal{S}$ has only one functionally independent invariants depending on u, u_t, u_x, u_{xx}, namely:

$$J = \frac{uu_t - u_x^2}{uu_{xx} - u_x^2}\,.$$

The most general regular invariant equation is given by $J = k$, i.e. by (5.3).

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Paper 10

Group analysis - a microscope of mathematical modelling. II: Dynamics in de Sitter space

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§ 1 The de Sitter space

> Two astronomers who live in the de Sitter world and have different de Sitter clocks might have an interesting conversation concerning the real or imaginary nature of some world events.
>
> F. Klein [78]

The present work provides a first step towards the relativistic mechanics in the de Sitter space. It is based on an approximate representation of the de Sitter group. Namely, the de Sitter group is considered as a perturbation of the Poincaré group by a small curvature. In order to illustrate a utility of the approximate approach, a derivation of exact transformations of the de Sitter group is presented, then an independent and simple computation of the first-order approximate representation of this group is suggested. Modified relativistic conservation laws for the free motion of a particle and unusual properties of neutrinos in the de Sitter space are discussed.

1.1 Introduction

In 1917, de Sitter [30] suggested a solution for Einstein's equations of general relativity and discussed in several papers its potential value for astronomy. This solution is a four-dimensional space-time of constant Riemannian curvature and is called the de Sitter space.

Felix Klein [78] gave a remarkably complete projective-geometric analysis of the de Sitter metric in the spirit of his Erlangen program. Earlier, Klein mentioned (see [77], p. 287) that "what modern physicists call the theory of relativity is the theory of invariants of the four-dimensional space-time x, y, z, t (the Minkowski space) with respect to a certain group, namely the Lorentz group" and suggested to identify the notions "theory of relativity" and "theory of invariants of a group of transformations" thus obtaining numerous types of theories of relativity. The *de Sitter universe* (the space-time of a constant curvature) admitting, like the Minkowski space, a ten-parameter group of isometric motions, *the de Sitter group*, served Klein as a perfect illustration. Klein's idea consisted in using the de Sitter group for developing the relativity theory comprising, along with the light velocity, another empirical constant, namely the curvature of the universe. The latter would encapsulate all three possible types of spaces with constant curvature: elliptic, hyperbolic (the Lobachevsky space), and parabolic (the Minkowski space as a limiting case of zero curvature).

The following opinions of the world authorities in this field give a comprehensive idea of utility and complexity in developing relativistic mechanics in the de Sitter universe.

Dirac [31]: *"The equations of atomic physics are usually formulated in terms of space-time of the space of the special theory of relativity. . . . Nearly all of the more general spaces have only trivial groups of operations which carry the spaces over into themselves, so they spoil the connexion between physics and group theory. There is an exception, however, namely the de Sitter space (without no local gravitational fields). This space is associated with a very interesting group, and so the study of the equations of atomic physics in this space is of special interest, from mathematical point of view."*

Synge ([117], Chapter VII): *"The success of the special theory of relativity in dealing with those phenomena which do not involve gravitation suggests that, if we are to work instead with a de Sitter universe, the curvature K must be very small in comparison with significant physical quantities of like dimensions (K has the dimension of sec^{-2}). Without good reason one does not feel inclined to complicate the simplicity of the Minkowski space-time by introducing curvature. Nevertheless the de Sitter universe is interesting in itself. It opens up new vistas, introducing us to the idea that* *space may be finite, and this seems to satisfy some mental need in us, for infinity is one of those things which we find difficulty in comprehending."*

Gürsey [47]: *"Since the de Sitter group gives such a close approximation to the empirical Poincaré group and, in addition, is based on the structure of the observed universe in accordance with Mach's principle, there seems now sufficient motivation for the detailed study of this group. What can we expect from such study? The curvature of our universe being as small as it is, can we hope to obtain any results not already given by the Lorentz group? A tentative answer is that we should be optimistic for two reasons. Firstly, the translation group no longer being valid in the de Sitter space, we shall lose the corresponding laws of energy and momentum conservation. The deviations from these laws with the usual definitions of energy and momentum based on the Poincaré group should manifest themselves in the cosmological scale. These laws, however, will be replaced by the laws of the conservation of observables corresponding to the new displacement operators which in the de Sitter group have taken the place of translations. It is in the light of these new definitions that such vague motions as the creation of matter in the expanding universe should be discussed. Secondly, we should see if new results are implied for elementary particles. Because the structure of the de Sitter group is so widely different from the Poincaré group its representations will have a totally different character, so that the concept of particle, that we have come to associate with the representations of a kinematical group will need revision. Another point worth investigating is the possibility of having new symmetry principles connected with the global properties of the de Sitter group."*

One of obstructing factors is that in the de Sitter group usual shifts of space-time coordinates are substituted by transformations of a rather complex form (see Section 1.3). Therefore invariants and invariant equations also become much more complicated as compared to Lorentz-invariant equations. In order to simplify the theory of dynamics in the de Sitter space by preserving its qualitative characteristics, an approximate group approach is employed here. The approach is based on recently developed theory of approximate groups with a small parameter [14]. In this connection the de Sitter group is considered as a perturbation of the Poincaré group by a small constant curvature K (cf. [72]). The *approximate representation* of the de Sitter group obtained in this way can be dealt with as easily as the Poincaré group. To elicit possible new effects specific for dynamics in the de Sitter universe it is sufficient to calculate first-order perturbations with respect to the curvature of the universe because, according to cosmological data, the curvature is a small constant $K \sim 10^{-54}$ cm^{-2}.

1.2 Notation from Riemannian geometry

We will denote by V_n an n-dimensional Riemannian space with the metric

$$ds^2 = g_{ij}(x)dx^i dx^j, \tag{1.1}$$

where $x = (x^1, \ldots, x^n)$ and $i, j = 1, \ldots, n$. We use the convention on summation in repeated indices. It is assumed that the matrix $\|g_{ij}(x)\|$ is symmetric, $g_{ij}(x) = g_{ji}(x)$, and is non-degenerate, i.e. $\det\|g_{ij}(x)\| \neq 0$ in a generic point $x \in V_n$. Hence, there exists the inverse matrix $\|g_{ij}(x)\|^{-1}$ with the entries denoted by $g^{ij}(x)$. By definition of an inverse matrix, one has $g_{ij}g^{jk} = \delta_i^k$.

The length s of a curve $x^i = x^i(\sigma)$ $(\sigma_1 \leq \sigma \leq \sigma_2)$ in V_n joining the points $x_1, x_2 \in V_n$, so that $x^i(\sigma_1) = x_1^i,\ x^i(\sigma_2) = x_2^i$, is given by the integral

$$s = \int_{\sigma_1}^{\sigma_2} L d\sigma, \tag{1.2}$$

where

$$L = \sqrt{g_{jk}(x)\dot{x}^j\dot{x}^k} \quad \dot{x}^j = \frac{dx^j(\sigma)}{d\sigma}\cdot \tag{1.3}$$

If the curve has an extremal length, i.e. it provides an extremum to the integral (1.2), it is called a *geodesic* joining the points x_1 and x_2 in the space V_n. The condition to be a geodesic is thus given by the Euler-Lagrange equations with the Lagrangian (1.3):

$$\frac{d}{d\sigma}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0, \quad i = 1, \ldots, n. \tag{1.4}$$

If we set $\sigma = s$, where s is the arc length of the curve measured from the point x_1, then the equations (1.4) of geodesics are written

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}(x)\frac{dx^j}{ds}\frac{dx^k}{ds} = 0, \quad i = 1, \ldots, n, \tag{1.5}$$

where the coefficients Γ^i_{jk}, known as the *Christoffel symbols,* are given by

$$\Gamma^i_{jk} = \frac{1}{2}g^{il}\left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right). \tag{1.6}$$

It is manifest from (1.6) that $\Gamma^i_{jk} = \Gamma^i_{kj}$.

With the aid of Christoffel symbols one defines the covariant differentiation of tensors on the Riemannian space V_n. Covariant differentiation takes tensors again into tensors. Covariant derivatives, e.g. of a *scalar* a, and of covariant and contravariant vectors a_i and a^i are defined as follows:

$$a_{,k} = \frac{\partial a}{\partial x^k} \cdot$$

$$a_{i,k} = \frac{\partial a_i}{\partial x^k} - a_j \Gamma^j_{ik} \cdot$$

$$a^i_{,k} = \frac{\partial a^i}{\partial x^k} + a^j \Gamma^i_{jk} \cdot$$

The covariant differentiation will be indicated by a subscript preceded by a comma. For repeated covariant differentiation we use only one comma, e.g. $a_{i,jk}$ denotes the second covariant derivative of a covariant vector a_i.

The covariant differentiation of higher-order covariant, contravariant and mixed tensors is obtained merely by iterating the above formulae. For example, in the case of second-order tensors we have

$$a_{ij,k} = \frac{\partial a_{ij}}{\partial x^k} - a_{il}\Gamma^l_{jk} - a_{lj}\Gamma^l_{ik}\,,$$

$$a^{ij}_{\ \ ,k} = \frac{\partial a^{ij}}{\partial x^k} + a^{il}\Gamma^j_{lk} + a^{lj}\Gamma^i_{lk}\,,$$

$$a^i_{j,k} = \frac{\partial a^i_j}{\partial x^k} + a^l_j \Gamma^i_{lk} - a^i_l \Gamma^l_{jk} \cdot$$

For scalars the double covariant derivative does not depend on the order of differentiation, $a_{,jk} = a_{,kj}$. Indeed,

$$a_{,jk} = \frac{\partial^2 a}{\partial x^j \partial x^k} - \Gamma^i_{jk}\frac{\partial a}{\partial x^i}\,, \quad a_{,kj} = \frac{\partial^2 a}{\partial x^k \partial x^j} - \Gamma^i_{kj}\frac{\partial a}{\partial x^i}\,, \tag{1.7}$$

and the equation $\Gamma^i_{jk} = \Gamma^i_{kj}$ yields that $a_{,jk} = a_{,kj}$. However, this similarity with the usual differentiation is violated, in general, when dealing with vectors and tensors of higher order. Namely, one can prove that

$$\begin{aligned} a_{i,jk} &= a_{i,kj} + a_l R^l_{ijk}, \\ a^i_{,jk} &= a^i_{,kj} - a^l R^i_{ljk}, \end{aligned} \tag{1.8}$$

etc. Here

$$R^l_{ijk} = \frac{\partial \Gamma^l_{ik}}{\partial x^j} - \frac{\partial \Gamma^l_{ij}}{\partial x^k} + \Gamma^m_{ik}\Gamma^l_{mj} - \Gamma^m_{ij}\Gamma^l_{mk} \tag{1.9}$$

is a mixed tensor of the fourth order called the *Riemann tensor.* It is also used in the form of the covariant tensor of the fourth order defined by

$$R_{hikl} = g_{hl} R^l_{ijk}. \tag{1.10}$$

According to Eqs, (1.8), the successive covariant differentiations of tensors in V_n permute only if the Riemann tensor vanishes identically:

$$R^l_{ijk} = 0, \qquad i, j, k, l = 1, \dots, n. \tag{1.11}$$

The Riemannian spaces satisfying the condition (1.11) are said to be *flat.* The flat spaces are characterized by the following statement:

The metric form (1.1) of a Riemannian space V_n can be reduced by an appropriate change of variables x to the form

$$ds^2 = (dx^1)^2 + \cdots + (dx^p)^2 - (dx^{p+1})^2 - \cdots - (dx^n)^2$$

in a neighborhood of a regular point x_0 (not only at the point x_0) if and only if the Riemann tensor R^l_{ijk} of V_n vanishes.

Contracting the indices l and k in the Riemann tensor (1.9), one obtains the following tensor of the second order called the *Ricci tensor:*

$$R_{ij} \overset{\text{def}}{=} R^k_{ijk} = \frac{\partial \Gamma^k_{ik}}{\partial x^j} - \frac{\partial \Gamma^k_{ij}}{\partial x^k} + \Gamma^m_{ik}\Gamma^k_{mj} - \Gamma^m_{ij}\Gamma^k_{mk}. \tag{1.12}$$

Finally, multiplication of the Ricci tensor by g^{ij} followed by contraction yields the *scalar curvature* of the space V_n:

$$R = g^{ij} R_{ij}. \tag{1.13}$$

1.3 Spaces of constant Riemannian curvature

Recall the notions of *curvature* and *spaces of constant curvature* introduced by Riemann [112] (for a detailed discussion, see [34], Sections 25-27).

A pair of contravariant vector-fields λ^i and μ^i in a Riemannian space V_n is called an *orientation* in V_n. Riemann introduced a so-called *geodesic surface* S at a point $x \in V_n$ as a locus of geodesics through x in the directions

$$a^i(x) = \tau\lambda^i(x) + \sigma\mu^i(x)$$

with parameters τ and σ. Then the *curvature* K of V_n at $x \in V_n$ is defined as the Gaussian curvature of S. The *Riemannian curvature* K can be expressed via Riemann's tensor $R_{ijkl} = g_{im} R^m_{jkl}$ as follows:

$$K = \frac{R_{ijkl}\lambda^i\mu^j\lambda^k\mu^l}{(g_{ik}g_{jl} - g_{il}g_{jk})\lambda^i\mu^j\lambda^k\mu^l}\,.$$

Definition 10.1. A Riemannian space V_n with the metric (1.1) is called a space of a *constant Riemannian curvature* if K is constant, i.e.

$$R_{ijkl} = K(g_{ik}g_{jl} - g_{il}g_{jk}), \quad K = \text{const.}$$

The following form of the line element (I keep Riemann's notation) in the spaces of constant curvature:

$$ds = \frac{1}{1+\frac{\alpha}{4}\sum x^2}\sqrt{\sum dx^2} \tag{1.14}$$

was obtained by Riemann ([112], Section II.4), whose name it bears.

In the case of an arbitrary signature, the Riemannian spaces V_n of constant curvature are characterized by the condition that their metric (1.1) can be written in appropriate coordinates in the form (see [34], Section 27):

$$ds^2 = \frac{1}{\theta^2}\sum_{i=1}^{n}\epsilon_i (dx^i)^2, \quad \theta = 1 + \frac{K}{4}\sum_{i=1}^{n}\epsilon_i (x^i)^2, \tag{1.15}$$

where each $\epsilon_i = \pm 1$, in agreement with the signature of V_n, and $K = \text{const}$. Eq. (1.15) is also referred to as Riemann's form of the metric of the spaces of constant curvature.

1.4 Killing vectors in spaces of constant curvature

The generators

$$X = \xi^i(x)\frac{\partial}{\partial x^i} \tag{1.16}$$

of the *isometric motions* in the space V_n with the metric (1.1) are determined by the *Killing equations*

$$\xi^l \frac{\partial g_{ij}}{\partial x^l} + g_{il}\frac{\partial \xi^l}{\partial x^j} + g_{jl}\frac{\partial \xi^l}{\partial x^i} = 0, \ i,j = 1,\ldots,n. \tag{1.17}$$

The solutions $\xi = (\xi^1,\ldots,\xi^n)$ of Eqs. (1.16) are often referred to as the *Killing vectors.* Let us write Eqs. (1.17) for the metric (1.15). We have:

$$g_{ij} = \frac{1}{\theta^2}\,\epsilon_i\delta_{ij}, \quad \frac{\partial g_{ij}}{\partial x^l} = -\frac{K}{\theta^3}\,\epsilon_l x^l \epsilon_i \delta_{ij}, \tag{1.18}$$

where δ_{ij} is the Kronecker symbol. The indices i and l in the expressions $\epsilon_i\delta_{ij}$ and $\epsilon_l x^l$, respectively, are fixed (no summation in these indices). Upon substituting the expressions (1.18), Eqs. (1.17) assume the following form:

$$\epsilon_i\delta_{il}\frac{\partial \xi^l}{\partial x^j} + \epsilon_j\delta_{jl}\frac{\partial \xi^l}{\partial x^i} - \frac{K}{\theta}\,\epsilon_i\delta_{ij}\sum_{l=1}^{n}\epsilon_l x^l \xi^l = 0,$$

or (no summation in i and j)

$$\epsilon_i\frac{\partial \xi^i}{\partial x^j} + \epsilon_j\frac{\partial \xi^j}{\partial x^i} - \frac{K}{\theta}\,\epsilon_i\delta_{ij}\sum_{l=1}^{n}\epsilon_l x^l \xi^l = 0, \quad i,j = 1,\ldots,n. \tag{1.19}$$

The spaces V_n of constant Riemannian curvature are distinguished by the remarkable property that they are the only spaces possessing the largest group of isometric motions, i.e. the maximal number $\frac{n(n+1)}{2}$ of the Killing vectors (see, e.g. [34], Section 71). Namely, integration of Eqs. (1.19) yields the $\frac{n(n+1)}{2}$ - dimensional Lie algebra spanned by the operators

$$X_i = \Big(\frac{K}{2}x^i x^j + (2-\theta)\epsilon_i\delta^{ij}\Big)\frac{\partial}{\partial x^j}\,, \tag{1.20}$$

$$X_{ij} = \epsilon_j x^j \frac{\partial}{\partial x^i} - \epsilon_i x^i \frac{\partial}{\partial x^j} \quad (i<j),$$

where $i,j = 1,\ldots,n$, and δ^{ij} is the Kronecker symbol. In the expressions $\epsilon_i x^i$ and $\epsilon_i\delta^{ij}$ the index i is fixed (no summation). Likewise, there is no summation in the index j in the expression $\epsilon_j x^j$.

Remark 10.1. The direct solution of the Killing equations (1.19) requires tedious calculations. Therefore, I give in Section 3.2 a simple method based on our theory of continuous approximate transformation groups.

1.5 Spaces with positive definite metric

The spaces V_n of constant Riemannian curvature can be represented as hypersurfaces in the $(n+1)$ - dimensional Euclidean space $\mathbb{R}^{n+1}$. Let us dwell on four-dimensional Riemannian spaces V_4.

The surface of the four-dimensional sphere with the radius ρ :

$$\zeta_1^2 + \zeta_2^2 + \zeta_3^2 + \zeta_4^2 + \zeta_5^2 = \rho^2 \tag{1.21}$$

in the five-dimensional Euclidean space $\mathbb{R}^5$ represents the Riemannian space V_4 with the positive definite metric having the constant curvature

$$K = \rho^{-2}. \tag{1.22}$$

Introducing the coordinates x^μ on the sphere by means of the stereographic projection:

$$x^\mu = \frac{2\zeta_\mu}{1+\zeta_5\rho^{-1}}\,, \quad \mu = 1,\ldots,4, \tag{1.23}$$

one can rewrite the metric of the space V_4 in Riemann's form (1.15):

$$ds^2 = \frac{1}{\theta^2}\sum_{i=1}^{4}(dx^\mu)^2, \tag{1.24}$$

where

$$\theta = 1 + \frac{K}{4}\sigma^2, \quad \sigma^2 = \sum_{\mu=1}^{4}(x^\mu)^2. \tag{1.25}$$

Remark 10.2. The inverse transformation to (1.23) has the form

$$\zeta_\mu = \frac{x^\mu}{\theta}, \quad \frac{\zeta_5}{\rho} = \frac{1}{\theta}\left(1 - \frac{K}{4}\sigma^2\right). \tag{1.26}$$

It is obtained by adding

$$\frac{K}{4}\sigma^2 = \frac{1 - \zeta_5\rho^{-1}}{1 + \zeta_5\rho^{-1}}$$

to the equation (1.23).

Let us consider the generators

$$X = \xi^\mu(x)\frac{\partial}{\partial x^\mu}$$

of the group of isometric motions in the space V_4 of constant curvature with the positive definite metric (1.24). In this case the Killing equations

$$\xi^\alpha \frac{\partial g_{\mu\nu}}{\partial x^\alpha} + g_{\mu\alpha}\frac{\partial \xi^\alpha}{\partial x^\nu} + g_{\nu\alpha}\frac{\partial \xi^\alpha}{\partial x^\mu} = 0$$

have the form (1.19) with all $\epsilon_i = +1$, i.e.

$$\frac{\partial \xi^\mu}{\partial x^\nu} + \frac{\partial \xi^\nu}{\partial x^\mu} - \frac{K}{\theta}(x \cdot \xi)\delta_{\mu\nu} = 0, \quad \mu, \nu = 1, \ldots, 4, \tag{1.27}$$

where $x \cdot \xi = \sum\limits_{\alpha=1}^{4} x^\alpha \xi^\alpha$ is the scalar product of the vectors x and ξ.

According to (1.20), Eqs. (1.27) have 10 linearly independent solutions that provide the following 10 generators:

$$X_1 = \left[1 + \frac{K}{4}\big(2(x^1)^2 - \sigma^2\big)\right]\frac{\partial}{\partial x^1} + \frac{K}{2}x^1\left[x^2\frac{\partial}{\partial x^2} + x^3\frac{\partial}{\partial x^3} + x^4\frac{\partial}{\partial x^4}\right],$$

$$X_2 = \left[1 + \frac{K}{4}\big(2(x^2)^2 - \sigma^2\big)\right]\frac{\partial}{\partial x^2} + \frac{K}{2}x^2\left[x^1\frac{\partial}{\partial x^1} + x^3\frac{\partial}{\partial x^3} + x^4\frac{\partial}{\partial x^4}\right],$$

$$X_3 = \left[1 + \frac{K}{4}\big(2(x^3)^2 - \sigma^2\big)\right]\frac{\partial}{\partial x^3} + \frac{K}{2}x^3\left[x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2} + x^4\frac{\partial}{\partial x^4}\right],$$

$$X_4 = \left[1 + \frac{K}{4}\big(2(x^4)^2 - \sigma^2\big)\right]\frac{\partial}{\partial x^4} + \frac{K}{2}x^4\left[x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2} + x^3\frac{\partial}{\partial x^3}\right],$$

$$X_{\mu\nu} = x^\nu\frac{\partial}{\partial x^\mu} - x^\mu\frac{\partial}{\partial x^\nu} \quad (\mu < \nu, \quad \mu = 1,2,3;\ \nu = 1,2,3,4), \tag{1.28}$$

where $\sigma^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2$ defined in (1.25).

Let us verify that, e.g. the coordinates

$$\xi^1 = 1 + \frac{K}{4}\big[(x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2\big],$$
$$\xi^2 = \frac{K}{2}x^1x^2, \quad \xi^3 = \frac{K}{2}x^1x^3, \quad \xi^4 = \frac{K}{2}x^1x^4$$

of the operator X_1 satisfy the Killing equations. We write

$$\xi^\mu = \Big(1 - \frac{K}{4}\sigma^2\Big)\delta_{1\mu} + \frac{K}{2}x^1x^\mu, \quad \mu = 1, \ldots, 4,$$

and obtain

$$\sum_{\mu=1}^{4} x^\mu \xi^\mu = \Big(1 + \frac{K}{4}\,\sigma^2\Big)x^1, \quad \frac{\partial \xi^\mu}{\partial x^\nu} = \frac{K}{2}(x^1\delta_{\mu\nu} + x^\mu\delta_{1\nu} - x^\nu\delta_{1\mu}).$$

Whence,

$$x \cdot \xi = \theta x^1, \quad \frac{\partial \xi^\mu}{\partial x^\nu} + \frac{\partial \xi^\nu}{\partial x^\mu} = Kx^1\delta_{\mu\nu},$$

and hence the Killing equations (1.27) are obviously satisfied.

The six operators $X_{\mu\nu}$ from (1.28) correspond to the rotational symmetry of the metric (1.24) inherited from the rotational symmetry of the sphere (1.21) written in the variables (1.23). The other four operators in (1.28) can also be obtained as a result of the rotational symmetry of the sphere (1.21). To this end one can utilize the substitution (1.23) and rewrite the rotation generators on the plane surfaces $(\zeta_1, \zeta_5), \ldots, (\zeta_4, \zeta_5)$ in the variables x. Unlike this external geometrical construction, the Killing equations give an internal definition of the group of motions independent of embedding V_4 in the five-dimensional Euclidean space.

Remark 10.3. The structure constants c^p_{mn} of the Lie algebra L_{10} spanned by the operators (1.28) are determined by the commutators

$$(X_\mu, X_\nu) = KX_{\mu\nu}, \quad (X_{\mu\nu}, X_\mu) = X_\nu,$$
$$(X_{\mu\nu}, X_\alpha) = 0 \quad (\alpha \neq \mu, \alpha \neq \nu),$$
$$(X_{\mu\nu}, X_{\alpha\beta}) = \delta_{\mu\alpha}X_{\nu\beta} + \delta_{\nu\beta}X_{\mu\alpha} - \delta_{\mu\beta}X_{\nu\alpha} - \delta_{\nu\alpha}X_{\mu\beta}.$$

It follows that the determinant of the matrix $A_{mn} = c^q_{mp}c^p_{nq}$ is equal to $-6K^4$, and hence does not vanishes if $K \neq 0$. Thus, by E. Cartan's theorem (see e.g. [33], Section 45), the group of isometric motions of the metric (1.24) with $K \neq 0$ is semi-simple. This fact is essential in investigation of representations of the de Sitter group (see [47] and the references therein).

Exercise 10.1. Deduce the operators (1.28) by both suggested approaches, namely by the change of variables (1.23) in the rotation generators of the sphere (1.21), and by solving the Killing equations (1.27).

Remark 10.4. It is manifest from Riemann's form (1.15) that the spaces of constant Riemannian curvature are conformally flat. In particular, the space V_4 with the metric (1.24) is conformal to the four-dimensional Euclidean space and has the metric tensor

$$g_{\mu\nu} = \frac{1}{\theta^2}\delta_{\mu\nu}, \quad g^{\mu\nu} = \theta^2\delta^{\mu\nu}, \quad \mu,\nu = 1,\ldots,4. \tag{1.29}$$

Its Christoffel symbols $\Gamma^\alpha_{\mu\nu}$ and the scalar curvature R are given by

$$\Gamma^\alpha_{\mu\nu} = \frac{K}{2\theta}(\delta_{\mu\nu}x^\alpha - \delta^\alpha_\mu x^\nu - \delta^\alpha_\nu x^\mu), \quad R = -12K. \tag{1.30}$$

In Eqs. (1.29), (1.30), $\delta_{\mu\nu}, \delta^{\mu\nu}$ and δ^ν_μ denote the Kronecker symbol.

1.6 Geometric realization of the de Sitter metric

In order to arrive at the de Sitter space V_4 we set

$$x^1 = x, \quad x^2 = y, \quad x^3 = z, \quad x^4 = ct\sqrt{-1}, \tag{1.31}$$

where c is the velocity of light in vacuum. Then, according to Eqs. (1.23), the variable ζ_4 is imaginary whereas $\zeta_1, \zeta_2, \zeta_3$ are real variables. As for the fifth coordinate, ζ_5, it is chosen from the condition that the ratio ζ_5/ρ is a real quantity. Hence, either both variables ζ_5 and ρ are real, or both of them are purely imaginary. If both ζ_5 and ρ are real, Eq. (1.21) yields that the de Sitter metric is represented geometrically by the surface (see [31])

$$\zeta_1^2 + \zeta_2^2 + \zeta_3^2 - \zeta_4^2 + \zeta_5^2 = \rho^2. \tag{1.32}$$

If both ζ_5 and R are imaginary, we have the surface

$$\zeta_1^2 + \zeta_2^2 + \zeta_3^2 - \zeta_4^2 - \zeta_5^2 = -\rho^2. \tag{1.33}$$

Inserting (1.31) in (1.24) and denoting $-ds^2$ by ds^2 in accordance with the usual physical convention to interpret ds as the interval between *world events,* we obtain the following *de Sitter metric:*

$$ds^2 = \frac{1}{\theta^2}(c^2dt^2 - dx^2 - dy^2 - dz^2), \tag{1.34}$$

where

$$\theta = 1 - \frac{K}{4}(c^2t^2 - x^2 - y^2 - z^2). \tag{1.35}$$

Eqs. (1.22), (1.32) and (1.33) show that the de Sitter space can have positive or negative curvature depending on the choice of the surface (1.32) or (1.33), respectively. The geometry of these surfaces is discussed by F. Klein [78].

1.7 Generators of the de Sitter group

Rewriting the operators (1.28) in the variables x, y, z, t defined by Eqs. (1.31) we obtain the following generators of the *de Sitter group:*

$$X_1 = \Big[1 + \frac{K}{4}(x^2 - y^2 - z^2 + c^2t^2)\Big]\frac{\partial}{\partial x} + \frac{K}{2}\,x\Big[y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + t\frac{\partial}{\partial t}\Big],$$

$$X_2 = \Big[1 + \frac{K}{4}(y^2 - x^2 - z^2 + c^2t^2)\Big]\frac{\partial}{\partial y} + \frac{K}{2}\,y\Big[x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} + t\frac{\partial}{\partial t}\Big],$$

$$X_3 = \Big[1 + \frac{K}{4}(z^2 - x^2 - y^2 + c^2t^2)\Big]\frac{\partial}{\partial x} + \frac{K}{2}\,z\Big[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t}\Big],$$

$$X_4 = \frac{1}{c^2}\Big[1 - \frac{K}{4}(c^2t^2 + x^2 + y^2 + z^2)\Big]\frac{\partial}{\partial t} - \frac{K}{2}t\Big[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\Big]$$

$$X_{ij} = x^j\frac{\partial}{\partial x^i} - x^i\frac{\partial}{\partial x^j}\,, \quad Y_i = t\frac{\partial}{\partial x^i} + \frac{1}{c^2}x^i\frac{\partial}{\partial t}\,, \qquad i, j = 1, 2, 3. \quad (1.36)$$

In other words, the *de Sitter group* is the group of isometric motions of the metric (1.34). If $K = 0$ the de Sitter group coincides with the Poincaré group (i.e. the non-homogeneous Lorentz group). Let $K \neq 0$. Then the operators X_{ij} and Y_i in (1.36) generate the homogeneous Lorentz group (the rotations and the Lorentz transformations). However, the operators X_1, X_2, X_3, X_4 in (1.36) are essentially more complicated than the simple generators $\frac{\partial}{\partial x}\,,\ \frac{\partial}{\partial y}\,,\ \frac{\partial}{\partial z}\,,\ \frac{\partial}{\partial t}\,,$ of the space and time translations in the Poincaré group. The operators X_1, X_2, X_3 and X_4 in (1.36) can be considered as the generators of the "generalized space-translations" and the "generalized time-translations", respectively.

§ 2 The de Sitter group

Attempting to find the group transformations $x'^\mu = f^\mu(x, a)$ by solving the Lie equation

$$\frac{dx'^\mu}{da} = \xi^\mu(x'), \quad x'^\mu|_{a=0} = x^\mu, \quad \mu = 1, \ldots, 4, \qquad (2.1)$$

for X_1 or X_4 from (1.36), it is hard to disagree with Synge's opinion cited above. Nevertheless, let us check how these transformations look like.

2.1 Conformal transformations in $\mathbb{R}^3$

Let us consider a relatively simple problem on determining the one-parameter group of conformal transformations in the Euclidian space $\mathbb{R}^3$ generated by the operator

$$X=(y^2+z^2-x^2)\frac{\partial}{\partial x}-2xy\frac{\partial}{\partial y}-2xz\frac{\partial}{\partial z}\cdot \tag{2.2}$$

In this case the Lie equations (2.1) are written

$$\begin{aligned} \frac{dx'}{da}&=y'^2+z'^2-x'^2, & x'\big|_{a=0}&=x,\\ \frac{dy'}{da}&=-2x'y', & y'\big|_{a=0}&=y,\\ \frac{dz'}{da}&=-2x'z', & z'\big|_{a=0}&=z. \end{aligned} \tag{2.3}$$

To simplify the integration of Eqs. (2.3), one can introduce canonical variables u,v,w, where u,v are two functionally independent invariants, i.e. two different solutions of the equation

$$X(f)\equiv(y^2+z^2-x^2)\frac{\partial f}{\partial x}-2xy\frac{\partial f}{\partial y}-2xz\frac{\partial f}{\partial z}=0, \tag{2.4}$$

and w is a solution of the equation

$$X(w)=1. \tag{2.5}$$

Then rewriting the operator X in the new variables by the formula

$$X=X(u)\frac{\partial}{\partial u}+X(v)\frac{\partial}{\partial v}+X(w)\frac{\partial}{\partial w} \tag{2.6}$$

and invoking that $X(u)=X(v)=0,\ X(w)=1$, one obtains

$$X=\frac{\partial}{\partial w}\cdot$$

Hence, the transformation of our group has the form

$$u'=u, \qquad v'=v, \qquad w'=w+a. \tag{2.7}$$

The solution of Eqs. (2.3) is obtained by substituting in (2.7) the expressions for u,v,w via x,y,z, and the similar expressions for u',v',w' via x',y',z'. Thus, the problem has been reduced to determination of u and v from Eq. (2.4), and w from Eq. (2.5). Let us carry out the calculations.

Calculation of invariants u and v requires the determination of two independent first integrals of the characteristic system for Equation (2.4):

$$\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} \cdot$$

The second equation of this system, $dy/y = dz/z$, yields the invariant

$$u = \frac{z}{y} \cdot$$

Substituting $z = uy$ into the first equation of the characteristic system one obtains

$$\frac{dx}{x^2 - (1+u^2)y^2} = \frac{dy}{2xy}$$

or, setting $p = x^2$,

$$\frac{dp}{dy} = \frac{p}{y} - (1+u^2)y.$$

Taking into account that u is constant on the solutions of the characteristic system, we readily integrate this first-order linear equation and obtain

$$p = vy - (1+u^2)y^2, \quad v = \text{const.} \tag{2.8}$$

Expressing the constant of integration v from (2.8) and substituting $p = x^2$, $u = z/y$, we obtain the second invariant

$$v = \frac{r^2}{y}, \quad r^2 = x^2 + y^2 + z^2.$$

In order to solve Eq. (2.5) we rewrite the operator (2.2) in the variables u, v, y using the transformation formula similar to (2.6):

$$X = X(u)\frac{\partial}{\partial u} + X(v)\frac{\partial}{\partial v} + X(y)\frac{\partial}{\partial y}$$

Since $X(u) = 0, X(v) = 0,\ X(y) = -2xy$, and Eq. (2.8) upon substitution $p = x^2$ yields $x = \sqrt{vy - (1+u^2)y^2}$, the operator (2.2) becomes

$$X = -2y\sqrt{vy - (1+u^2)y^2}\,\frac{\partial}{\partial y} \cdot$$

Therefore Equation (2.5) takes the form

$$\frac{dw}{dy} = -\frac{1}{2y\sqrt{vy - (1+u^2)y^2}}$$

whence, ignoring the constant of integration, we obtain

$$w = \sqrt{\frac{1}{vy} - \frac{1+u^2}{v^2}} = \frac{x}{r^2}\,.$$

Now we substitute the resulting expressions

$$u = \frac{z}{y}, \quad v = \frac{r^2}{y}, \quad w = \frac{x}{r^2}$$

and the similar expressions for the primed variables in Eqs. (2.7) and obtain

$$\frac{z'}{y'} = \frac{z}{y}, \quad \frac{r'^2}{y'} = \frac{r^2}{y}, \quad \frac{x'}{r'^2} = \frac{x}{r^2} + a.$$

Solution of these equations with respect to x', y', z', furnishes the following transformations of the conformal group with the generator (2.2):

$$\begin{aligned} x' &= \frac{x + ar^2}{1 + 2ax + a^2r^2}\,, \\ y' &= \frac{y}{1 + 2ax + a^2r^2}\,, \\ z' &= \frac{z}{1 + 2ax + a^2r^2}\,. \end{aligned} \tag{2.9}$$

2.2 Inversion

An alternative approach to calculation of the conformal transformations (2.9) is based on Liouville's theorem [93] which states that any conformal mapping in the three-dimensional Euclidian space is a composition of translations, rotations, dilations and the *inversion**

$$x_1 = \frac{x}{r^2}, \quad y_1 = \frac{y}{r^2}, \quad z_1 = \frac{z}{r^2}\,. \tag{2.10}$$

The inversion (2.10) is the reflection with respect to the unit sphere. It preserves the angles, i.e. represents a particular case of conformal transformations. It carries the point (x, y, z) with the radius r to the point (x_1, y_1, z_1) with the radius $r_1 = 1/r$ (therefore it is also called a *transformation of reciprocal radii*) and hence, the inverse transformation

$$x = \frac{x_1}{r_1^2}, \quad y = \frac{y_1}{r_1^2}, \quad z = \frac{z_1}{r_1^2} \tag{2.10$'$}$$

*Liouville's theorem was generalized to multi-dimensional spaces by Sophus Lie. A detailed discussion is available in [90], Ch. 10, and [19], § 1.

coincides with (2.10). If the inversion (2.10) is denoted by S, then the latter property means that $S^{-1} = S$. According to Liouville's theorem, the one-parameter transformation group (2.9) can be obtained merely as a conjugate group from a group of translations H along the x-axis. Indeed, if one introduces the new variables (2.10′) in the generator of translations $X_1 = \partial/\partial x_1$ according to the formula (cf. (2.6))

$$X = SX_1 \equiv X_1(x)\frac{\partial}{\partial x} + X_1(y)\frac{\partial}{\partial y} + X_1(z)\frac{\partial}{\partial z}, \tag{2.11}$$

then by virtue of

$$X_1(x) = \frac{y_1^2 + z_1^2 - x_1^2}{r_1^4} = y^2 + z^2 - x^2,$$

$$X_1(y) = -2\frac{x_1 y_1}{r_1^4} = -2xy, \quad X_1(z) = -2xz$$

one obtains the operator (2.2). Therefore, the group G with the generator (2.2) is obtained from the translation group

$$H: \ x_2 = x_1 + a, \quad y_2 = y_1, \quad z_2 = z_1 \tag{2.12}$$

by the conjugation

$$G = SHS. \tag{2.13}$$

Indeed, mapping the point (x, y, z) to the point (x_1, y_1, z_1) by means of the inversion (2.10), we write the transformation (2.12) in the original variables

$$x_2 = \frac{x + ar^2}{r^2}, \quad y_2 = \frac{y}{r^2} \quad z_2 = \frac{z}{r^2}\,\cdot \tag{2.14}$$

Now according to (2.13) let us make one more inversion

$$x' = \frac{x_2}{r_2^2}, \quad y' = \frac{y_2}{r_2^2} \quad z' = \frac{z_2}{r_2^2}\,\cdot$$

Substituting here the expressions (2.14) and the expression

$$r_2^2 = \frac{(x + ar^2)^2 + y^2 + z^2}{r^4} = \frac{1 + 2ax + a^2r^2}{r^2}\,,$$

we arrive precisely to the transformations (2.9) of the group G.

Remark 10.5. Note, that the inversion, like any conformal transformation, is admitted by the Laplace equation. Specifically, the Laplace equation

$$\Delta u \equiv u_{xx} + u_{yy} + u_{zz} = 0$$

is invariant under the transformation (2.10) of the independent variables supplemented by the following transformation of the dependent variable u :

$$u_1 = ru. \tag{2.15}$$

Thus, Equations (2.10), (2.15) define a symmetry transformation

$$x_1 = \frac{x}{r^2}, \quad y_1 = \frac{y}{r^2}, \quad z_1 = \frac{z}{r^2}, \quad u_1 = ru, \tag{2.16}$$

of the Laplace equation known as the *Kelvin transformation.*

2.3 Bateman's transformations

As mentioned in Remark 10.5, the inversion (specifically, the Kelvin transformation (2.16)) is admitted by the Laplace equation and is widely used in potential theory for elliptic equations. Upon obvious modification, inversion can also be adjusted to the wave equation. Bateman [19], [20] found several other transformations admitted by the wave equation that differ from the inversion and the Lorentz transformations. We will consider here the following transformation (it differs from Bateman's form given in [19], p. 76, only by notation and the constant coefficient α):

$$\bar{x} = \frac{\alpha}{2}\frac{\sigma^2 - 1}{x - ct}, \quad \bar{y} = \alpha\frac{y}{x - ct}, \quad \bar{z} = \alpha\frac{z}{x - ct}, \quad \bar{t} = \frac{\alpha}{2c}\frac{\sigma^2 + 1}{x - ct}, \tag{2.17}$$

where

$$\sigma^2 = r^2 - c^2t^2.$$

Remark 10.6. Bateman's transformation (2.17) supplemented by the change of the dependent variable

$$\bar{u} = (x - ct)u$$

is admitted by the wave equation

$$u_{tt} - c^2\Delta u = 0$$

and hence is a conformal transformation in the Minkowski space-time.

We will use Bateman's transformation (2.17) for determining transformations of the de Sitter group generated by a linear combination of the operators X_1 and X_4 from (1.36). Let us rewrite the generator $X = \partial/\partial x$ of translation group in the variables (2.17). In accordance with the formula of change of variables (2.6) we determine the action of the operator X on variables (2.17), and by virtue of the equation

$$\bar{x} - c\bar{t} = -\frac{\alpha}{x - ct} \tag{2.18}$$

arrive at

$$\overline{X} = \frac{\alpha}{2}\left[1 + \frac{1}{\alpha^2}(\bar{x} - c\bar{t})^2 - \frac{1}{\alpha^2}\bar{y}^2 - \frac{1}{\alpha^2}\bar{z}^2\right]\frac{\partial}{\partial\bar{x}} + \frac{1}{\alpha}\bar{y}(\bar{x} - c\bar{t})\frac{\partial}{\partial\bar{y}}$$
$$+\frac{1}{\alpha}\bar{z}(\bar{x} - c\bar{t})\frac{\partial}{\partial\bar{z}} + \frac{\alpha}{2c}\left[1 - \frac{1}{\alpha^2}(\bar{x} - c\bar{t})^2 - \frac{1}{\alpha^2}\bar{y}^2 - \frac{1}{\alpha^2}\bar{z}^2\right]\frac{\partial}{\partial\bar{t}}\cdot$$

Assuming that $K > 0$ and letting $\alpha = 2/\sqrt{K}$ we obtain

$$\overline{X} = \frac{1}{\sqrt{K}}\Big(\overline{X}_1 + c\overline{X}_4\Big),$$

where $\overline{X}_1, \overline{X}_4$ are the corresponding operators from (1.36) written in terms of the variables $\bar{x}, \bar{y}, \bar{z}, \bar{t}$. Thus, the Bateman transformation B :

$$\bar{x} = \frac{\sigma^2 - 1}{\sqrt{K}(x - ct)}, \quad \bar{y} = \frac{2y}{\sqrt{K}(x - ct)},$$
$$\bar{z} = \frac{2z}{\sqrt{K}(x - ct)}, \quad \bar{t} = \frac{\sigma + 1}{c\sqrt{K}(x - ct)}, \tag{2.19}$$

where $\sigma^2 = x^2 + y^2 + z^2 - c^2t^2$, transforms the operator of translation $X = \partial/\partial x$ into

$$BX \equiv \overline{X} = \frac{1}{\sqrt{K}}\Big(\overline{X}_1 + c\overline{X}_4\Big). \tag{2.20}$$

We can get rid of the factor $1/\sqrt{K}$ by replacing the parameter $\bar{a}$ in the group with the generator (2.20) by the parameter

$$a = \bar{a}/\sqrt{K}. \tag{2.21}$$

Let us invert the transformation (2.19). Substituting $x - ct$ given by (2.18) into the expressions for $\bar{y}$ and $\bar{z}$ we obtain

$$y = -\frac{\bar{y}}{\bar{x} - c\bar{t}}, \quad z = -\frac{\bar{z}}{\bar{x} - c\bar{t}}\cdot \tag{2.22}$$

Then (2.19) shows that the expression $\bar{\sigma}^2 = \bar{x}^2 + \bar{y}^2 + \bar{z}^2 - c^2\bar{t}^2$ has the form

$$\bar{\sigma}^2 = -\frac{4(x + ct)}{K(x - ct)}\cdot \tag{2.23}$$

Equations (2.18) and (2.23) yield

$$x - ct = -\frac{2}{\sqrt{K}(\bar{x} - c\bar{t})}, \quad x + ct = \frac{\sqrt{K}\bar{\sigma}^2}{2(\bar{x} - c\bar{t})},$$

whence

$$x = \frac{K\bar{\sigma}^2 - 4}{4\sqrt{K}(\bar{x} - c\bar{t})}, \quad t = \frac{K\bar{\sigma}^2 + 4}{4c\sqrt{K}(\bar{x} - c\bar{t})}\cdot \tag{2.24}$$

Equations (2.22) and (2.24) define the inverse transformation B^{-1} :

$$x_1 = \frac{K\sigma^2 - 4}{4\sqrt{K}(x - ct)}, \quad y_1 = -\frac{y}{x - ct},$$

$$z_1 = -\frac{z}{x - ct}, \quad t_1 = \frac{K\sigma^2 + 4}{4c\sqrt{K}(x - ct)} \tag{2.25}$$

for the transformation (2.19).

2.4 Calculation of de Sitter group transformations

I provide here the computation of finite transformations of a one-parameter subgroup of the de Sitter group using Bateman's transformation. It is useful to compare these calculations with the calculation of approximate group transformations given below in Section 3.3.

Equation (2.20) shows that the group G with the infinitesimal operator $X_1 + cX_4$ and the group H of translations along the x-axis are conjugated by means of the Bateman transformation:

$$G = BHB^{-1}.$$

Therefore one can use the following procedure for calculating the one-parameter transformation group generated by $X_1 + cX_4$. First one moves the point (x, y, z, t) to the position (x_1, y_1, z_1, t_1) by applying the map B^{-1} and writes the translation

$$x_2 = x_1 + \bar{a}, \quad y_2 = y_1, \quad z_2 = z_1, \quad t_2 = t_1$$

in the original variables in the form

$$x_2 = \frac{K\sigma^2 - 4 + 4\bar{a}\sqrt{K}(x - ct)}{4\sqrt{K}(x - ct)}, \quad y_2 = -\frac{y}{x - ct},$$

$$z_2 = -\frac{z}{x - ct}, \quad t_2 = \frac{K\sigma^2 + 4}{4c\sqrt{K}(x - ct)}\cdot$$

Then one applies Bateman's transformation (2.19) to the point (x_2, y_2, z_2, t_2). To this end, one has to evaluate the expressions

$$\sigma_2^2 = x_2^2 + y_2^2 + z_2^2 - c^2 t_2^2$$

and

$$\sqrt{K}\,(x_2 - ct_2).$$

Using the change of the parameter (2.21), the above expressions are written

$$\sigma_2^2 = \frac{1}{x - ct}\left[-(x + ct) + 2a\left(\frac{K}{4}\sigma^2 - 1\right) + Ka^2(x - ct)\right]$$

and

$$\sqrt{K}\,(x_2 - ct_2) = \frac{-2 + Ka(x - ct)}{x - ct},$$

respectively. These equations together with the representation (2.19) of Bateman's transformation B furnish ultimately the point

$$(x', y', z', t') = B(x_2, y_2, z_2, t_2)$$

with the coordinates

$$\begin{aligned} x' &= \frac{x + a - \frac{K}{4}[a\sigma^2 + 2a^2(x - ct)]}{1 - \frac{K}{2}a(x - ct)}, \quad y' = \frac{y}{1 - \frac{K}{2}a(x - ct)}, \\ z' &= \frac{z}{1 - \frac{K}{2}a(x - ct)}, \quad t' = \frac{t + \frac{a}{c} - \frac{K}{4c}[a\sigma^2 + 2a^2(x - ct)]}{1 - \frac{K}{2}a(x - ct)}. \end{aligned} \tag{2.26}$$

In order to check that the transformation (2.26) leads to the generator $X_1 + cX_4$, one can single out in (2.26) the terms that are linear in a :

$$\begin{aligned} x' &\approx x + \left[1 + \frac{K}{4}\big((x - ct)^2 - y^2 - z^2\big)\right]a, \quad y' \approx y + \frac{K}{2}y(x - ct)a, \\ z' &\approx z + \frac{K}{2}z(x - ct)a, \quad t' \approx t + \frac{1}{c}\left[1 - \frac{K}{4}\big((x - ct)^2 + y^2 + z^2\big)\right]a, \end{aligned} \tag{2.27}$$

and compare the result with the coordinates of the operator $X_1 + cX_4$.

The transformations of the whole de Sitter group, but in a realization of the metric different from (1.34), are presented in [128], Chapter 13, § 3. A matrix representation of this group in the metric (1.34) is available in [47].

§ 3 Approximate representation of the de Sitter group

In this section, the de Sitter group is considered as a perturbation of the Poincaré group by a small constant curvature K. The *approximate representation* of the de Sitter group obtained here can be dealt with as easily as the Poincaré group. A brief introduction to the theory of approximate transformation groups given in Section 3.1 is sufficient for our purposes.

3.1 Approximate groups

The concept of approximate transformation groups was suggested in [14]. Here the necessary minimum of information about approximate groups is given. For a detailed presentation of the theory of approximate transformation groups, see, e.g. [15] (Paper 6 in this volume).

Let $f(x,\varepsilon)$ and $g(x,\varepsilon)$ be analytical functions, where $x = (x^1, \ldots, x^n)$ and ε is a small parameter. We say that the functions f and g are approximately equal and write $f \approx g$ if $f - g = o(\varepsilon)$. The same notation is used for functions depending additionally on a parameter a (group parameter).

Consider a vector-function $f(x,a,\varepsilon)$ with components

$$f^1(x,a,\varepsilon), \ldots, f^n(x,a,\varepsilon)$$

satisfying the "initial conditions" $f^i(x,0,\varepsilon) \approx x^i$ $(i = 1, \ldots, n)$. We will write these conditions in the vector form

$$f(x,0,\varepsilon) \approx x.$$

Furthermore, we assume that the function f is defined and smooth in a neighborhood of $a = 0$ and that, in this neighborhood, $a = 0$ is the only solution of the equation

$$f(x,a,\varepsilon) \approx x.$$

Definition 10.2. An *approximate transformation*

$$x' \approx f(x,a,\varepsilon) \tag{3.1}$$

in $\mathbb{R}^n$ is the set of all invertible transformations

$$x' = g(x,a,\varepsilon)$$

with $g(x,a,\varepsilon) \approx f(x,a,\varepsilon)$.

Definition 10.3. We say that (3.1) is a one-parameter *approximate transformation group* with the group parameter a if

$$f(f(x,a,\varepsilon),b,\varepsilon) \approx f(x,a+b,\varepsilon). \tag{3.2}$$

Let us write the transformations (3.1) with the precision $o(\varepsilon)$ in the form

$$x' \approx f_0(x,a) + \varepsilon f_1(x,a) \tag{3.3}$$

and denote

$$\xi_0(x) = \frac{\partial f_0(x,a)}{\partial a}\Big|_{a=0}, \quad \xi_1(x) = \frac{\partial f_1(x,a)}{\partial a}\Big|_{a=0}. \tag{3.4}$$

Theorem 10.1. If (3.3) obeys the approximate group property (3.2) then the following *approximate Lie equation* holds:

$$\frac{d(f_0+\varepsilon f_1)}{da} = \xi_0(f_0+\varepsilon f_1) + \varepsilon\xi_1(f_0+\varepsilon f_1) + o(\varepsilon). \tag{3.5}$$

Conversely, given any smooth vector-function $\xi(x,\varepsilon) \approx \xi_0(x) + \varepsilon\xi_1(x) \not\approx 0$, the solution $x' \approx f(x,a,\varepsilon)$ of the *approximate Cauchy problem*

$$\frac{dx'}{da} \approx \xi(x',\varepsilon), \tag{3.6}$$

$$x'\big|_{a=0} \approx x \tag{3.7}$$

determines a one-parameter approximate transformation group with the parameter a.

The proof of this theorem is similar to the proof of Lie's theorem for exact groups [14, 15, 16]. However, we have to specify the notion of the approximate Cauchy problem. The approximate differential equation (3.6) is considered here as a family of differential equations

$$\frac{dz'}{da} = \tilde{\xi}(x',\varepsilon) \tag{3.8}$$

with the right-hand sides $\tilde{\xi}(x,\varepsilon) \approx \xi(x,\varepsilon)$. Likewise, the approximate initial condition (3.7) is treated as the set of equations

$$x'\big|_{a=0} = \alpha(x,\varepsilon), \quad \alpha(x,\varepsilon) \approx x. \tag{3.9}$$

Definition 10.4. We define the solution of the approximate Cauchy problem (3.6), (3.7) as a solution to any problem (3.8), (3.9) considered with the precision $o(\varepsilon)$.

Proposition 10.1. The solution of the approximate Cauchy problem given by Definition 10.4 is unique.

Proof. Indeed, according to the theorem on continuous dependence on parameters of the solution to Cauchy's problem, solutions for all problems of the form (3.8), (3.9) coincide with the precision $o(\varepsilon)$.

Definition 10.4 shows that in order to find solution of the approximate Lie equation (3.5) with the initial condition (3.7) it suffices to solve the following exact Cauchy problem:

$$\frac{df_0}{da} = \xi_0(f_0), \quad f_0\big|_{a=0} = x, \tag{3.10}$$

$$\frac{df_1}{da} = \sum_{k=1}^{n} \frac{\partial \xi_0(f_0)}{\partial x^k} f_1^k + \xi_1(f_0), \quad f_1\big|_{a=0} = 0.$$

Equations (3.10) are obtained from (3.5) by separating principal terms with respect to ε.

Example 10.1. To illustrate the method, let us find the approximate transformation group

$$x' = x_0' + \varepsilon x_1', \quad y' = y_0' + \varepsilon y_1'$$

on the (x, y) plane defined by the generator

$$X = (1 + \varepsilon x^2)\frac{\partial}{\partial x} + 2\varepsilon xy\frac{\partial}{\partial y}\cdot$$

The above operator X is a one-dimensional analogue of the operator X_1 from (1.36). We have:

$$\xi_0 = (1, 0), \quad \xi_1 = (x^2, 2xy),$$

and the Cauchy problem (3.10) is written in the form

$$\frac{dx_0'}{da} = 1, \quad \frac{dy_0'}{da} = 0, \quad x_0'\big|_{a=0} = x, \quad y_0'\big|_{a=0} = y; \tag{3.11}$$

$$\frac{dx_1'}{da} = (x_0')^2, \quad \frac{dy_1'}{da} = 2x_0'y_0', \quad x_1'\big|_{a=0} = y_1'\big|_{a=0} = 0. \tag{3.12}$$

Eqs. (3.11) yield $x_0' = x + a,\ y_0' = y$, and hence Eqs. (3.12) take the form

$$\frac{dx_1'}{da} = (x + a)^2, \quad \frac{dy_1'}{da} = 2y(x + a), \quad x_1'|_{a=0} = y_1'|_{a=0} = 0,$$

whence

$$x_1' = x^2 a + x a^2 + \frac{1}{3} a^3, \quad y_1' = 2xya + ya^2.$$

Thus, the approximate transformation group is given by

$$x' = x + a + (x^2 a + x a^2 + \frac{1}{3} a^3)\varepsilon, \quad y' = y + (2xya + ya^2)\varepsilon. \tag{3.13}$$

It is instructive to compare the approximate transformations with the corresponding exact transformation group. The latter is obtained by solving the (exact) Lie equations

$$\frac{dx'}{da} = 1 + \varepsilon x'^2, \quad \frac{dy'}{da} = 2\varepsilon x' y', \quad x'\big|_{a=0} = x, \quad y'\big|_{a=0} = y.$$

These equations yield:

$$x' = \frac{\sin(\delta a) + \delta x \cos(\delta a)}{\delta\big[\cos(\delta a) - \delta x \sin(\delta a)\big]}, \quad y' = \frac{y}{\big[\cos(\delta a) - \delta x \sin(\delta a)\big]^2}, \tag{3.14}$$

where $\delta = \sqrt{\varepsilon}$. The approximate transformation (3.13) can be obtained from (3.14) by singling out the principal linear terms with respect to ε.

3.2 Simple method of solution of Killing's equations

Now we will carry out the program of an approximate representation of the de Sitter group. Let us first apply the approximate approach to differential equations outlined in Section 3.1 to the Killing equations (1.27) for the metric (1.24). Setting

$$\xi = \xi_0 + K\xi_1 + o(K), \tag{3.15}$$

we obtains from (1.27) the *approximate Killing equations*

$$\frac{\partial \xi_0^\mu}{\partial x^\nu} + \frac{\partial \xi_0^\nu}{\partial x^\mu} + K\left[\frac{\partial \xi_1^\mu}{\partial x^\nu} + \frac{\partial \xi_1^\nu}{\partial x^\mu} - (x \cdot \xi_0)\delta_{\mu\nu}\right] \approx 0, \tag{3.16}$$

whence (cf. Eqs. (3.10))

$$\frac{\partial \xi_0^\mu}{\partial x^\nu} + \frac{\partial \xi_0^\nu}{\partial x^\mu} = 0, \tag{3.17}$$

and

$$\frac{\partial \xi_1^\mu}{\partial x^\nu} + \frac{\partial \xi_1^\nu}{\partial x^\mu} - (x \cdot \xi_0)\delta_{\mu\nu} = 0. \tag{3.18}$$

Equations (3.17) define the group of isometric motions in the Euclidian space, namely rotations and translations, so that

$$\xi_0^\mu = a_\nu^\mu x^\nu + b^\mu \tag{3.19}$$

with constant coefficients a^μ_ν and b^μ. The coefficients a^μ_ν skew-symmetric, i.e. $a^\mu_\nu = -a^\nu_\mu$, therefore we have

$$x \cdot \xi_0 = b \cdot x \equiv \sum_{\mu=1}^{4} b^\mu x^\mu. \tag{3.20}$$

Due to the obvious symmetry of Eqs. (3.18) it is sufficient to find their particular solution letting $b = (1, 0, 0, 0)$. Then Eq. (3.20) yields $x \cdot \xi_0 = x^1$ and Eqs. (3.18) take the form

$$\frac{\partial \xi_1^\mu}{\partial x^\nu} + \frac{\partial \xi_1^\nu}{\partial x^\mu} = x^1 \delta_{\mu\nu}. \tag{3.21}$$

Solving Eqs. (3.21) with $\mu = \nu$ one obtains

$$\xi_1^1 = \frac{1}{4}(x^1)^2 + \varphi^1(x^2, x^3, x^4), \quad \xi_1^2 = \frac{1}{2}x^1x^2 + \varphi^2(x^1, x^3, x^4),$$

$$\xi_1^3 = \frac{1}{2}x^1x^2 + \varphi^2(x^1, x^2, x^4), \quad \xi_1^4 = \frac{1}{2}x^1x^4 + \varphi^4(x^1, x^2, x^3). \tag{3.22}$$

Using Eqs. (3.22), we arrive at the following Eqs. (3.21) with $\mu = 1$:

$$\frac{\partial \varphi^1}{\partial x^\nu} + \frac{\partial \varphi^\nu}{\partial x^1} + \frac{1}{2}x^\nu = 0, \quad \nu = 2, 3, 4,$$

whence

$$\frac{\partial^2 \varphi^1}{(\partial x^\nu)^2} + \frac{1}{2} = 0, \quad \nu = 2, 3, 4.$$

A particular solution to the latter equations is given by

$$\varphi^1 = -\frac{1}{4}\big[(x^2)^2 + (x^3)^2 + (x^4)^2\big]. \tag{3.23}$$

One can easily verify that substitution of (3.23) and $\varphi^2 = \varphi^3 = \varphi^4 = 0$ in (3.22) yields a solution to Eqs. (3.21). Finally, combining Eqs. (3.22), (3.23), (3.15) and Eqs. (3.19) with $a^\mu_\nu = 0, b = (1, 0, 0, 0)$, one obtains the following solution of the approximate Killing equations (3.16):

$$\xi^1 = 1 + \frac{K}{4}[(x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2], \ \xi^\nu = \frac{K}{2}x^1x^\nu, \quad \nu = 2, 3, 4. \tag{3.24}$$

The vector (3.24) gives the operator X_1 from (1.28) and hence, according to Section 1.5, it is an exact solution of the Killing equations (1.27). The fact that the approximate solution coincides with the exact solution of the Killing equations is a lucky accident and has no significance in what follows. Of importance for us is the simplification achieved by the approximate approach to solving the Killing equations. All other operators (1.28) can be obtained by renumbering the coordinates in (3.24). The transition from (1.28) to the generators of the de Sitter group has been given in Section 1.7.

3.3 Derivation of the approximate representation of the de Sitter group

It is sufficient find the approximate representation of the de Sitter group by takin one typical generator of the group, e.g. (1.36). Let us take $\varepsilon = K/4$ as a small parameter and write the coordinates of the first operator (1.36),

$$X_1 = \big(1 + \varepsilon[x^2 - y^2 - z^2 + c^2t^2]\big)\frac{\partial}{\partial x} + 2\varepsilon\, x\left(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + t\frac{\partial}{\partial t}\right), \quad (3.25)$$

in the form $\xi = \xi_0 + \varepsilon\xi_1$. Then

$$\xi_0 = (1,0,0,0), \quad \xi_1 = (x^2 - y^2 - z^2 + c^2t^2, 2xy, 2xz, 2xt).$$

In order to find the corresponding approximate group transformations

$$x' = x'_0 + \varepsilon x'_1, \quad y' = y'_0 + \varepsilon y'_1, \quad z' = z'_0 + \varepsilon z'_1, \quad t' = t'_0 + \varepsilon t'_1$$

we have to solve the Cauchy problem (3.10) comprising, in our case, the equations

$$\begin{aligned} &\frac{dx'_0}{da} = 1, \qquad \frac{dy'_0}{da} = 0, \qquad \frac{dz'_0}{da} = 0, \qquad \frac{dt'_0}{da} = 0,\\ &x'_0|_{a=0} = x, \quad y'_0|_{a=0} = y, \quad z'_0|_{a=0} = z, \quad t'_0|_{a=0} = t, \end{aligned} \quad (3.26)$$

and

$$\begin{aligned} &\frac{dx'_1}{da} = x'^2_0 - y'^2_0 - z'^2_0 + c^2t'^2_0,\\ &\frac{dy'_1}{da} = 2x'_0y'_0, \quad \frac{dz'_1}{da} = 2x'_0z'_0, \quad \frac{dt'_1}{da} = 2x'_0t'_0,\\ &x'_1|_{a=0} = y'_1|_{a=0} = z'_1|_{a=0} = t'_1|_{a=0} = 0. \end{aligned} \quad (3.27)$$

In order to solve the system (3.26), (3.26), we proceed as in Example 10.1. Namely, we solve Eqs. (3.26) and obtain

$$x'_0 = x + a, \quad y'_0 = y, \quad z'_0 = z, \quad t'_0 = t. \quad (3.28)$$

Then we substitute the expressions (3.28) in the differential equations from (3.27) and arrive at the following equations:

$$\frac{dx'_1}{da} = x^2 - y^2 - z^2 + c^2t^2 + 2xa + a^2,$$

$$\frac{dy'_1}{da} = 2xy + 2ya, \quad \frac{dz'_1}{da} = 2xz + 2za, \quad \frac{dt'_1}{da} = 2xt + 2ta.$$

Their integration by using the initial conditions yields

$$
\begin{aligned}
x_1' &= (x^2 - y^2 - z^2 + c^2t^2)a + xa^2 + \frac{1}{3}a^3,\\
y_1' &= 2xya + ya^2,\\
z_1' &= 2xza + za^2,\\
t_1' &= 2xta + ta^2.
\end{aligned}
\tag{3.29}
$$

Combining the equations (3.28), (3.29) and substituting $\varepsilon = K/4$, we obtain the one-parameter approximate transformation group

$$
\begin{aligned}
x' &= x + a + \frac{K}{4}[(x^2 - y^2 - z^2 + c^2t^2)a + xa^2 + \frac{1}{3}a^3],\\
y' &= y + \frac{K}{4}(2xya + ya^2),\\
z' &= z + \frac{K}{4}(2xza + za^2),\\
t' &= t + \frac{K}{4}(2xta + ta^2)
\end{aligned}
\tag{3.30}
$$

with the first generator (1.36):

$$
X_1 = \left(1 + \frac{K}{4}[x^2 - y^2 - z^2 + c^2t^2]\right)\frac{\partial}{\partial x} + \frac{K}{2}x\left(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + t\frac{\partial}{\partial t}\right).
$$

The transformations (3.30) can be called a *generalized translation* in the de Sitter universe. In the case $K = 0$ they coincide with the usual translation $x' = x + a$ along the x-axis, whereas for a small curvature $K \neq 0$ they have little deviation from the translation.

Remark 10.7. Integration of the exact Lie equations for the generator (3.25) yields the following exact one-parameter group:

$$
\begin{aligned}
\overline{x} &= \frac{2\sqrt{\varepsilon}\,x\,\cos(2a\sqrt{\varepsilon}) + (1 - \varepsilon\sigma^2)\sin(2a\sqrt{\varepsilon})}{\sqrt{\varepsilon}(1 + \varepsilon\sigma^2) + \sqrt{\varepsilon}(1 - \varepsilon\sigma^2)\cos(2a\sqrt{\varepsilon}) - 2\varepsilon\,x\,\sin(2a\sqrt{\varepsilon})},\\
\overline{y} &= \frac{2y}{1 + \varepsilon\sigma^2 + (1 - \varepsilon\sigma^2)\cos(2a\sqrt{\varepsilon}) - 2x\sqrt{\varepsilon}\sin(2a\sqrt{\varepsilon})},\\
\overline{z} &= \frac{2z}{1 + \varepsilon\sigma^2 + (1 - \varepsilon\sigma^2)\cos(2a\sqrt{\varepsilon}) - 2x\sqrt{\varepsilon}\sin(2a\sqrt{\varepsilon})},\\
\overline{t} &= \frac{2t}{1 + \varepsilon\sigma^2 + (1 - \varepsilon\sigma^2)\cos(2a\sqrt{\varepsilon}) - 2x\sqrt{\varepsilon}\sin(2a\sqrt{\varepsilon})},
\end{aligned}
\tag{3.31}
$$

where $\sigma^2 = x^2 + y^2 + z^2 - c^2t^2$.

Exercise 10.2. Find the approximate group with the generator X_4 of the generalized time-translations from (1.36).

The following exercise is useful for clarifying specific properties of the Bateman transformations.

Exercise 10.3. Find an approximate group generated by $X_1 + cX_4$ and compare the result with Eqs. (2.26) in Section 2.4.

§ 4 Motion of a particle in de Sitter space

4.1 Introduction

The free motion of a particle with mass m in the space-time V_4 with the metric

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$$

is determined by the Lagrangian

$$\mathcal{L} = -mc\sqrt{g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu}, \tag{4.1}$$

where $x = (x^1, \ldots, x^4)$ is the four-vector and $\dot{x} = dx/ds$ is its derivative with respect to the arc length s measured from a fixed point x_0. In other words, the free particle moves along geodesic curves in V_4 defined by the equations (1.5),

$$\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu}(x)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0, \quad \lambda = 1, \ldots, 4, \tag{4.2}$$

where the coefficients $\Gamma^\lambda_{\mu\nu}$ are the *Christoffel symbols* given by (1.6):

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\alpha}\left(\frac{\partial g_{\alpha\mu}}{\partial x^\nu} + \frac{\partial g_{\alpha\mu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right), \quad \lambda, \mu, \nu = 1, 2, 3, 4. \tag{4.3}$$

The factor $-mc$ in (4.1) appears due to special relativity.

By using the Noether theorem, we associate with the generators

$$X = \xi^\mu(x)\frac{\partial}{\partial x^\mu} \tag{4.4}$$

of the isometric motions in V_4 the following conserved quantities for the equations of motion (1.5) (see [58], Section 23.1):

$$T = mcg_{\mu\nu}\,\xi^\mu\,\frac{dx^\nu}{ds}\,\cdot \tag{4.5}$$

4.2 Relativistic conservation laws

The relativistic conservation laws are obtained by applying the above procedure to the Minkowski space with the metric

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2. \tag{4.6}$$

The group of isometric motions in the Minkowski space is the Lorentz group with the generators (cf. (1.36))

$$X_0 = \frac{\partial}{\partial t}\,, \ X_i = \frac{\partial}{\partial x^i}\,, \ X_{ij} = x^j\frac{\partial}{\partial x^i} - x^i\frac{\partial}{\partial x^j}\,, \ Y_i = t\frac{\partial}{\partial x^i} + \frac{1}{c^2}x^i\frac{\partial}{\partial t} \tag{4.7}$$

where $i, j = 1, 2, 3.$

Let us apply the conservation formula (4.5) to the generators (4.7). We will write Eq. (4.5) in the form

$$T = mc\sum_{\mu,\nu=0}^{3} g_{\mu\nu}\,\xi^\mu\,\frac{dx^\nu}{ds}\,, \tag{4.8}$$

where

$$x^0 = t,\ x^1 = x,\ x^2 = y,\ x^3 = z. \tag{4.9}$$

We will denote the spatial vector by $\boldsymbol{x} = (x^1, x^2, x^3)$. Accordingly, the physical velocity $\boldsymbol{v} = d\boldsymbol{x}/dt$ is a three-dimensional vector $\boldsymbol{v} = (v^1, v^2, v^3)$. We also use the usual symbols $(\boldsymbol{x}\cdot\boldsymbol{v})$ and $\boldsymbol{x}\times\boldsymbol{v}$ for the scalar and vector products, respectively.

Writing Eq. (4.6) in the form

$$ds = c\sqrt{1-\beta^2}\,dt, \quad \beta^2 = |\boldsymbol{v}|^2/c^2\,, \tag{4.10}$$

we obtain from (4.1) the following *relativistic Lagrangian,* specifically, the Lagrangian for the free motion of a particle in the special theory of relativity:

$$\mathcal{L} = -mc^2\sqrt{1-\beta^2}. \tag{4.11}$$

Furthermore, we have:

$$\frac{dx^0}{ds} = \frac{1}{c\sqrt{1-\beta^2}}\,, \quad \frac{dx^i}{ds} = \frac{v^i}{c\sqrt{1-\beta^2}}\,, \quad i = 1, 2, 3. \tag{4.12}$$

Now the conserved quantity (4.9) is written:

$$T = \frac{m}{\sqrt{1-\beta^2}}\left[c^2\xi^0 - (\boldsymbol{\xi}\cdot\boldsymbol{v})\right], \tag{4.13}$$

where

$$(\boldsymbol{\xi} \cdot \boldsymbol{v}) = \sum_{i=1}^{3} \xi^i v^i.$$

Let us take the generator X_0 from (4.7). Substituting its coordinates $\xi^0 = 1, \xi^i = 0$ in (4.13) we obtain Einstein's *relativistic energy:*

$$E = \frac{mc^2}{\sqrt{1-\beta^2}}\,. \tag{4.14}$$

Likewise, substituting in (4.13) the coordinates of the operators X_i and X_{ij}, we arrive at the *relativistic momentum*

$$\boldsymbol{p} = \frac{m\boldsymbol{v}}{\sqrt{1-\beta^2}} \tag{4.15}$$

and the *relativistic angular momentum*

$$\mathbf{M} = \boldsymbol{x} \times \boldsymbol{p}, \tag{4.16}$$

respectively.

The generators Y_i of the Lorentz transformations give rise to the vector

$$\mathbf{Q} = \frac{m(\boldsymbol{x} - t\boldsymbol{v})}{\sqrt{1-\beta^2}}\,. \tag{4.17}$$

In the case of N-body problem, conservation of the vector (4.17) furnishes the relativistic center-of-mass theorem.

4.3 Conservation laws in de Sitter space

In the notation (4.9), the de Sitter space metric (1.34) is written

$$ds^2 = \frac{1}{\theta^2}\big(c^2dt^2 - |d\boldsymbol{x}|^2\big), \quad \theta = 1 - \frac{K}{4}\big(c^2t^2 - |\boldsymbol{x}|^2\big), \tag{4.18}$$

and Eqs. (4.10), (4.11), (4.12) and (4.13) are replaced by

$$ds = \frac{c}{\theta}\sqrt{1-\beta^2}\,dt, \quad \beta^2 = |\boldsymbol{v}|^2/c^2, \tag{4.19}$$

$$\mathcal{L} = -mc^2\theta^{-1}\sqrt{1-\beta^2}\,, \tag{4.20}$$

$$\frac{dx^0}{ds} = \frac{\theta}{c\sqrt{1-\beta^2}}\,, \quad \frac{dx^i}{ds} = \frac{\theta v^i}{c\sqrt{1-\beta^2}}\,, \quad i = 1,2,3, \tag{4.21}$$

and

$$T = \frac{m}{\theta\sqrt{1-\beta^2}}\Big[c^2\xi^0 - (\boldsymbol{\xi}\cdot\boldsymbol{v})\Big], \tag{4.22}$$

respectively.

Let us apply (4.22) to the generalized time-translation generator X_4 from (1.36) written in the form

$$X_0 = \Big[1 - \frac{K}{4}(c^2t^2 + |\boldsymbol{x}|^2)\Big]\frac{\partial}{\partial t} - \frac{K}{2}c^2t\,x^i\frac{\partial}{\partial x^i}\,\cdot \tag{4.23}$$

Substituting the coordinates

$$\xi^0 = 1 - \frac{K}{4}(c^2t^2 + |\boldsymbol{x}|^2), \quad \xi^i = -\frac{K}{2}c^2tx^i, \quad i = 1, 2, 3,$$

of the operator (4.23) in (4.22), we obtain the conserved quantity

$$T_0 = \frac{mc^2}{\theta\sqrt{1-\beta^2}}\Big[1 - \frac{K}{4}(c^2t^2 + |\boldsymbol{x}|^2) + \frac{K}{2}t(\boldsymbol{x}\cdot\boldsymbol{v})\Big].$$

One can readily verify that the following equation holds:

$$\frac{1}{\theta}\Big[1 - \frac{K}{4}(c^2t^2 + |\boldsymbol{x}|^2)\Big]$$

Therefore, T_0 the yields the *energy of a free particle in the de Sitter space:*

$$\mathcal{E} = \frac{mc^2}{\sqrt{1-\beta^2}}\big[1 + \frac{K}{2\theta}(t\boldsymbol{v} - \boldsymbol{x})\cdot\boldsymbol{x}\big]. \tag{4.24}$$

If $K = 0$, we have $\mathcal{E} = E$, where E is the relativistic energy (4.14). If $K \neq 0$, (4.24) yields in the linear approximation with respect to K :

$$\mathcal{E} \approx E\big[1 - \frac{K}{2}(\boldsymbol{x} - t\boldsymbol{v})\cdot\boldsymbol{x}\big]$$

Similar calculations for the operators X_i from (1.36) provide the momentum

$$\boldsymbol{\mathcal{P}} = \frac{m}{\theta\sqrt{1-\beta^2}}\big[(2-\theta)\boldsymbol{v} - \frac{K}{2}(c^2t - \boldsymbol{x}\cdot\boldsymbol{v})\boldsymbol{x}\big]. \tag{4.25}$$

In the linear approximation with respect to K, it is written:

$$\boldsymbol{\mathcal{P}} \approx \boldsymbol{p} - \frac{K}{2}E\big[t(\boldsymbol{x} - t\boldsymbol{v}) + \frac{1}{c^2}(\boldsymbol{x}\times\boldsymbol{v})\times\boldsymbol{x}\big]$$

where E and $\boldsymbol{p}$ are the relativistic energy (4.14) and momentum (4.15), respectively. It is manifest that $\boldsymbol{\mathcal{P}} = \boldsymbol{p}$ if $K = 0$.

By using the infinitesimal rotations X_{ij} and the generators Y_i of the Lorentz transformations from (1.36), one obtains the angular momentum

$$\boldsymbol{\mathcal{M}} = \frac{1}{2-\theta}(\boldsymbol{x} \times \boldsymbol{\mathcal{P}})$$

and the vector

$$\boldsymbol{\mathcal{Q}} = \frac{m(\boldsymbol{x} - t\boldsymbol{v})}{\theta\sqrt{1-\beta^2}},$$

respectively.

4.4 The Kepler problem in de Sitter space

Let us extend the Lagrangian (4.20) of the free motion of a particle to Kepler's problem in the de Sitter space. We will require that the generalized Lagrangian possesses the basic properties of the Lagrangian in the classical Kepler problem, namely, its invariance with respect to the rotations and time translations. Furthermore, we require that when $K = 0$ and $\beta^2 \to 0$, the generalized Lagrangian assumes the classical value

$$L = \frac{1}{2} m|\boldsymbol{v}|^2 + \frac{\alpha}{|\boldsymbol{x}|}, \quad \alpha = \text{const.} \tag{4.26}$$

Therefore, starting from (4.20), we seek the Lagrangian of Kepler's problem in the de Sitter space in the form

$$\mathcal{L} = -mc^2\theta^{-1}\sqrt{1-\beta^2} + \frac{\alpha}{|\boldsymbol{x}|}\theta^s(1-\beta^2)^l \tag{4.27}$$

with undetermined constants s and l. The action integral

$$\int \mathcal{L}dt \tag{4.28}$$

with the Lagrangian (4.27) is invariant with respect to rotations. Hence, the invariance test (see [58])

$$\widetilde{X}_0(\mathcal{L}) + D_t(\xi^0)\mathcal{L} = 0 \tag{4.29}$$

of the action integral (4.28) with respect to the generalized time-translations with the generator (4.23),

$$X_0 = \Big[1 - \frac{K}{4}(c^2t^2 + |\boldsymbol{x}|^2)\Big]\frac{\partial}{\partial t} - \frac{K}{2}c^2 t\, x^i \frac{\partial}{\partial x^i},$$

will be the only additional condition for determining the unknown constants k and l in (4.27). Here $\widetilde{X}_0$ is the prolongation of the operator X_0 to $\boldsymbol{v}$ and ξ^0 is its coordinate at $\frac{\partial}{\partial t}$:

$$\xi^0 = 1 - \frac{K}{4}(c^2t^2 + |\boldsymbol{x}|^2).$$

The reckoning shows that

$$\widetilde{X}_0(\mathcal{L}) + D_t(\xi^0)\mathcal{L} = \frac{\alpha K}{2|\boldsymbol{x}|}\theta^s(1-\beta^2)^l\Big[(1-2l)\frac{x\cdot v}{c^2} + st\Big]. \tag{4.30}$$

Therefore, Eq. (4.29) yields that $s = 0$ and $l = \frac{1}{2}$. This proves the following.

Theorem 10.2. The Lagrangian (4.27) is invariant under the group of generalized time-translations if and only if it has the form

$$\mathcal{L} = \Big[- mc^2\theta^{-1} + \frac{\alpha}{|\boldsymbol{x}|}\Big]\sqrt{1-\beta^2}. \tag{4.31}$$

Remark 10.8. A similar theorem on the uniqueness of an invariant Lagrangian is not valid in the Minkowsky space. Indeed, as follows from (4.30), for $K = 0$ Eq. (4.29) is satisfied for any Lagrangian (4.27). Thus, Theorem 10.2 manifests a significance of the curvature K.

Let us check that the Lagrangian (4.31) assumes the classical value (4.26) when $K = 0$ and $\beta^2 \to 0$. Taking in (4.31) the approximation

$$\sqrt{1-\beta^2} \approx 1 - \beta^2 = 1 - \frac{1}{2}\frac{|\boldsymbol{v}|^2}{c^2}$$

and letting $\beta^2 \to 0$, we obtain

$$\mathcal{L} = -mc^2 + \frac{1}{2}m|\boldsymbol{v}|^2 + \frac{\alpha}{|\boldsymbol{x}|}\,\cdot$$

Ignoring the constant term $-mc^2$, we arrive at (4.26).

§ 5 Wave equation in de Sitter space

Recall the definition of the wave equation in curved space-times given in [52], [53]. According to this definition, the wave equation in the space-time V_4 with the metric $ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$, where $x = (x^1, \ldots, x^t)$, is given by

$$\Box u \equiv g^{\mu\nu}u_{,\mu\nu} + \frac{1}{6}Ru = 0, \tag{5.1}$$

where R is the scalar curvature of V_4 and $u_{,\mu\nu}$ is the second-order covariant derivative. Upon substituting the expression (1.7) for the covariant differentiation, the wave operator defined by Eq. (5.1) is written

$$\Box u = g^{\mu\nu}\left(\frac{\partial^2 u}{\partial x^\mu \partial x^\nu} - \Gamma^\lambda_{\mu\nu}\frac{\partial a}{\partial x^\lambda}\right) + \frac{1}{6}Ru. \tag{5.2}$$

Characteristic property of Eq. (5.1) is that it is the only conformally invariant equation in V_4 provided that V_4 has a nontrivial conformal group. Moreover, in V_4 with a nontrivial conformal group, (5.1) is the only equation obeying the Huygens principle [54].

The wave operator (5.2) in the de Sitter space with the metric (1.34),

$$ds^2 = \frac{1}{\theta^2}(c^2dt^2 - dx^2 - dy^2 - dz^2), \tag{5.3}$$

can be written explicitly by using the expressions (1.30) and the change of variables (1.31). However, it is simpler to use the equation (see [52])

$$\widetilde{\Box} u = H^{-3}\Box(Hu) \tag{5.4}$$

connecting the wave operators $\Box$ and $\widetilde{\Box}$ in the the conformal spaces V_4 and $\widetilde{V}_4$ with metric tensors $g_{\mu\nu}$ and $\widetilde{g}_{\mu\nu}$, respectively, related by the equation

$$\widetilde{g}_{\mu\nu}(x) = H^2(x)g_{\mu\nu}(x), \quad \mu,\nu = 1,\ldots,4. \tag{5.5}$$

In the case of the metric (5.3) we have $H = \theta^{-1}$ and $\Box$ is the usual wave operator in the Minkowski space. Therefore, using Eq. (5.4) we obtain the following wave operator in the de Sitter space:

$$\widetilde{\Box} u = \theta^3\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial u^2}\right)\left(\frac{u}{\theta}\right). \tag{5.6}$$

Hence, we arrive at the following statement.

Theorem 10.3. The solution to the wave equation

$$\widetilde{\Box} u = 0 \tag{5.7}$$

in the de Sitter space is given by

$$u = \theta\, v, \tag{5.8}$$

where v is the solution of the usual wave equation,

$$v_{tt} - c^2(u_{xx} + u_{yy} + u_{zz}) = 0, \tag{5.9}$$

and

$$\theta = 1 - \frac{K}{4}(c^2t^2 - x^2 - y^2 - z^2). \tag{5.10}$$

§ 6 Neutrinos in de Sitter space

In the present section, a specific symmetry of the Dirac equations for particles with zero mass (neutrinos) in the de Sitter space-time is considered in the framework of the theory of approximate groups. A theoretical conclusion of this section is that, in the de Sitter universe with a constant curvature $K \neq 0$, a massless neutrino splits into two "massive" neutrinos. The question remains open of whether this conclusion has a real physical significance and of how one can detect the splitting of neutrinos caused by curvature.

6.1 Two approximate representations of Dirac's equations in de Sitter space

Dirac's equations in the de Sitter space [31] for particles with zero mass (neutrinos) can be written in the linear approximation with respect to the curvature K as follows:

$$\gamma^\mu \frac{\partial \phi}{\partial x^\mu} - \frac{3}{4} K (x \cdot \gamma) \phi = 0. \tag{6.1}$$

Here γ^μ are the usual Dirac matrices in the Minkowski space, ϕ is a four-dimensional complex vector, and

$$(x \cdot \gamma) = \sum_{\mu=1}^{4} x^\mu \gamma^\mu.$$

The following propositions can be proved by direct calculations.

Proposition 10.2. The substitution

$$\psi = \phi - \frac{3}{8} K |x|^2 \phi. \tag{6.2}$$

reduces Eq. (6.1), in the first order of precision with respect to K, to the Dirac equation in the Minkowski space,

$$\gamma^\mu \frac{\partial \psi}{\partial x^\mu} = 0. \tag{6.3}$$

Note that Eq. (6.3), due to homogeneity, is invariant under the following transformation of the variables x:

$$y^\mu = i x^\mu, \quad i = \sqrt{-1}. \tag{6.4}$$

Therefore, by setting

$$\phi(x) = \varphi(y), \tag{6.5}$$

we can rewrite Eq. (6.2) in the form

$$\psi = \varphi + \frac{3}{8}K|y|^2\varphi. \tag{6.6}$$

The invariance of Eq. (6.3) with respect to the transformation (6.4) means that

$$\gamma^\mu \frac{\partial \psi}{\partial y^\mu} = 0. \tag{6.7}$$

Substitution of (6.6) into Eq. (6.7) yields:

$$\gamma^\mu \frac{\partial \varphi}{\partial y^\mu} + \frac{3}{4}K(y \cdot \gamma)\varphi = 0. \tag{6.8}$$

Eq. (6.8) coincides with the Dirac equation (6.1) in the de Sitter space-time with the curvature $(-K)$.

Proposition 10.3. The combined system of equations (6.1), (6.8) inherits all symmetries of the usual Dirac equation (6.3). Namely, the system (6.1), (6.8) is invariant under the approximate representation of the de Sitter group and under the transformation (6.4) - (6.5). Moreover, it is conformally invariant in the first order of accuracy with respect to K.

6.2 Splitting of neutrinos by curvature

The above calculations show that the Dirac equations in the de Sitter universe have the following peculiarities due to curvature.

6.2.1 Effective mass

The equation (6.1) can be regarded as a Dirac equation

$$\gamma^\mu \frac{\partial \phi}{\partial x^\mu} + m\phi = 0 \tag{6.9}$$

with the variable "effective" mass

$$m = -\frac{3}{4}K(x \cdot \gamma) \tag{6.10}$$

Then, in the framework of the usual relativistic theory, neutrinos will have *small but nonzero* mass. It follows from Proposition 10.3 that the "massive" neutrinos move with velocity of light and obey the Huygens principle on existence of a sharp rear front.

6.2.2 Splitting of neutrinos

Since the equations (6.3) and (6.7) coincide, there is no preference between two transformations (6.2) and (6.6), and hence between Eqs. (6.1) and (6.8). Consequently, a "massless" neutrino given by Eq. (6.3) splits into two "massive" neutrinos. These "massive" neutrinos have "effective" masses

$$m_1 = -\frac{3}{4}K(x \cdot \gamma)$$

and

$$m_2 = \frac{3}{4}K(x \cdot \gamma)$$

and are described by the equations (6.1) and (6.8), respectively. These two particles are distinctly different if and only if $K \neq 0$. One of them, namely given by the equation (6.1), can be regarded as a *proper neutrino* and the other given by the equation (6.8) as an *antineutrino*.

6.2.3 Neutrino as a compound particle

I suggest the following interpretation of the above mathematical observations. In the de Sitter universe with a curvature $K \neq 0$, a neutrino is a compound particle, namely *neutrino–antineutrino* with the total mass

$$m = m_1 + m_2 = 0. \tag{6.11}$$

It is natural to assume that only the first component of the compound is observable and is perceived as a massive neutrino. The counterpart to the neutrino provides the validity of the zero-mass-neutrino model and has the real nature in the *antiuniverse* with the curvature $(-K)$. A physical relevance of this model can be manifested, however, only by experimental observations.

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Paper 11

Seven miniatures on group analysis

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Experience gained from solving concrete problems and reading lectures on various aspects of mathematical physics has convinced me that quite a few phenomena (diffusion, wave, etc.) can be modelled directly in group-theoretical terms. Differential equations, conservation laws, solutions to boundary value problems, and so forth can be obtained as immediate consequences of group invariant principles. It is this idea that has inspired the major part of the present paper. Various results on group analysis are presented here in the form of relatively independent, self-contained short stories, so that the exposition is concise and the reader does not have to read the entire paper.

§ 1 The Galilean group and diffusion

The invariance principle with respect to the Galilean group can be used to describe heat propagation in the linear approximation and to replace the Fourier law (or the Nernst law, which describes diffusion in solutions etc.). It is shown in [59] (see also [61]) that the fundamental solution can be directly derived from the invariance principle and the heat equation itself plays only an auxiliary role. The reasoning is as follows.

First, consider the Galilean group for the case in which space is one-dimensional. This is a three-parameter group formed by the translations with respect to time t and the space variable x and by the Galilean transformations, which describe the passage to steadily moving coordinate systems. We add a dependent variable u to the independent variables t and x. It will denote the temperature, and, by definition, will be transformed in passing to a moving (with constant velocity $2a$) coordinate system according to the law $\bar{u} = ue^{-(ax+a^2t)}$. This extension of the Galilean group yields the three-parameter family of transformations, generated by the operators

$$X_0 = \frac{\partial}{\partial t}, \quad X_1 = \frac{\partial}{\partial x}, \quad Y = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}\cdot$$

This family is not a group since the linear span of these operators is not closed with respect to the commutator. The closure of this vector space yields the four-dimensional Lie algebra with the basis

$$X_0 = \frac{\partial}{\partial t}, \quad X_1 = \frac{\partial}{\partial x}, \quad T_1 = u\frac{\partial}{\partial u}, \quad Y = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}\cdot \tag{1.1}$$

Thus, in the one-dimensional theory of heat conductivity (diffusion) the Galilean group is realized as the four-parameter group formed by the translations with respect to t and x, the Galilean transformations, and dilatations of u. In [59] this realization was named the *heat representation of the Galilean group.*

Lemma 11.1. The Lie algebra with the basis (1.1) can be extended to a five-dimensional algebra by adding operators of the dilatation group (scale transformations of t and x). This extension is unique and is obtained by adding the operator

$$T_2 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} \tag{1.2}$$

to the basis (1.1).

Proof. Let us add a dilatation operator of general form $T = \alpha t\frac{\partial}{\partial t} + \beta x\frac{\partial}{\partial x}$ to the basis (1.1) and write down the conditions that the resultant space is closed with respect to the commutator. We obtain $\alpha = 2$ and $\beta = 1$.

Theorem 11.1. Any linear second-order equation that admits the Lie algebra with basis (1.1) has the form

$$u_t = u_{xx} + cu, \quad c = \text{const.} \tag{1.3}$$

Furthermore, the only second order equation admitting the extended algebra with the basis (1.1), (1.2) is the heat equation

$$u_t = u_{xx}. \tag{1.4}$$

Proof. The general linear equation that admits the operators X_0, X_1, and T_1 in (1.1) is the equation

$$Au_{xx} + Bu_{xt} + Cu_{tt} + au_x + bu_t + cu = 0$$

with constant coefficients. It remains to write out the criterion for this equation to be invariant with respect to Y, and then with respect to T_2.

Remark 11.1. Equation (1.4) also admits the group of projective transformations with the operator

$$Z = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} - \frac{1}{4}(2t + x^2)u\frac{\partial}{\partial u}\cdot \tag{1.5}$$

The generalization to the multi-dimensional case is obvious. Thus, the heat equation

$$u_t = \Delta u \tag{1.6}$$

with n space variables $x = (x^1, ..., x^n)$ admits the Lie algebra with the basis

$$\begin{gathered} X_0 = \frac{\partial}{\partial t}, \quad X_i = \frac{\partial}{\partial x^i}, \quad X_{ij} = x^j\frac{\partial}{\partial x^i} - x^i\frac{\partial}{\partial x^j}, \\ T_1 = u\frac{\partial}{\partial u}, \quad T_2 = 2t\frac{\partial}{\partial t} + x^i\frac{\partial}{\partial x^i}, \quad Y_i = 2t\frac{\partial}{\partial x^i} - x^i u\frac{\partial}{\partial u}, \\ Z = t^2\frac{\partial}{\partial t} + tx^i\frac{\partial}{\partial x^i} - \frac{1}{4}(2nt + r^2)u\frac{\partial}{\partial u}, \quad i, j = 1, \ldots, n. \end{gathered} \tag{1.7}$$

In conjunction with $X = \varphi(t, x)\frac{\partial}{\partial u}$ [where $\varphi(t, x)$ is an arbitrary solution to (1.6)] this algebra furnishes the maximum symmetry algebra of Eq. (1.6).

Theorem 11.2. (see [59], [61]). The equation

$$u_t - \Delta u = \delta(t, x) \tag{1.8}$$

with the Dirac δ-function on the right-hand side admits the algebra with basis $X_{ij}, Y_i, T_2 - nT_1, Z$. The fundamental solution to the heat equation is an invariant solution with respect to the group generated by that algebra.

§ 2 On the Newton-Cotes potential

Consider the motion of a particle of mass m in a central potential field. The Lagrangian function is

$$L = \frac{m}{2}v^2 + U(r), \tag{2.1}$$

where $v^2 = |\boldsymbol{v}|^2,\ \boldsymbol{v} = \dot{\boldsymbol{x}} \equiv d\boldsymbol{x}/dt$. The equation of motion is written

$$m\dot{\boldsymbol{v}} = \frac{U'(r)}{r}\boldsymbol{x}. \tag{2.2}$$

Since scale transformations are important in mechanics, let us find out for what potentials $U(r)$ the group of dilatations with the generator

$$T = kt\frac{\partial}{\partial t} + x^i\frac{\partial}{\partial x^i}, \quad k = \text{const.} \tag{2.3}$$

satisfies the Noether theorem on conservation laws [i.e., when (2.3) is a Noether symmetry].

It is convenient to represent the generator (2.3) as the canonical Lie-Bäcklund operator (see [58])

$$X = (x^i - ktv^i)\frac{\partial}{\partial x^i} \tag{2.4}$$

and to consider the infinitesimal coordinate transformation $\tilde{\boldsymbol{x}} = \boldsymbol{x} + \delta\boldsymbol{x}$, where

$$\delta\boldsymbol{x} = (\boldsymbol{x} - kt\boldsymbol{v})a. \tag{2.5}$$

The differentiation of (2.5) with respect to t yields the velocity increment

$$\delta\boldsymbol{v} = \big((1-k)\boldsymbol{v} - kt\dot{\boldsymbol{v}}\big)a. \tag{2.6}$$

For small coordinate and velocity variations $\delta\boldsymbol{x}$ and $\delta\boldsymbol{v}$ the main part of increment of the Lagrangian (2.1) is equal to

$$\delta L = m\boldsymbol{v}\cdot\delta\boldsymbol{v} + \frac{U'(r)}{r}\boldsymbol{x}\cdot\delta\boldsymbol{x}. \tag{2.7}$$

As applied to this case, the well-known Noether theorem asserts that *if the increment of the Lagrangian is the total derivative*

$$\delta L = \frac{dF}{dt}, \tag{2.8}$$

then equation (2.2) has the integral

$$J = m\boldsymbol{v}\cdot\delta\boldsymbol{x} - F. \tag{2.9}$$

Theorem 11.3. The condition (2.8) is valid for the Lagrangian (2.1) and for the infinitesimal transformations (2.5) and (2.6) of the dilatation group if and only if

$$U = -\frac{\alpha}{r^2}, \quad \alpha = \text{const.} \tag{2.10}$$

Moreover, we have $k = 2$, and consequently, the equation

$$m\ddot{\boldsymbol{x}} = 2\alpha\frac{\boldsymbol{x}}{r^4} \tag{2.11}$$

admits the dilatation group with the operator

$$T = 2t\frac{\partial}{\partial t} + x^i\frac{\partial}{\partial x^i}\cdot \tag{2.12}$$

Equation (2.11) also admits the group of projective transformations with the generator

$$Z = t^2\frac{\partial}{\partial t} + tx^i\frac{\partial}{\partial x^i}\cdot \tag{2.13}$$

Both operators satisfy the condition (2.8), and the formula (2.9) provides the corresponding integrals of motion

$$J_1 = 2tE - m\boldsymbol{x}\cdot\boldsymbol{v}, \quad J_2 = 2t^2E - m\boldsymbol{x}\cdot(2t\boldsymbol{v} - \boldsymbol{x}), \tag{2.14}$$

where $E = \frac{m}{2}v^2 + \frac{\alpha}{r^2}$ is the energy.

Proof. With (2.5), (2.6), and (2.2) taken into account, the formula (2.7) can be rewritten as

$$\delta L = \frac{d}{dt}\big[(1-k)m\boldsymbol{x}\cdot\boldsymbol{v} - 2ktU\big]a + ka(rU' + 2U).$$

Thus, we should find out under which conditions the expression $rU' + 2U$ is a total derivative. This is possible if and only if its variational derivative vanishes, i.e.

$$\frac{\delta}{\delta x^i}(rU' + 2U) = \frac{\partial}{\partial x^i}(rU' + 2U) = (rU' + 2U)'\,\frac{x^i}{r} = 0.$$

Hence we have the second-order equation $(rU' + 2U)' = 0$ for the unknown function $U(r)$. Its solution, to within an additive constant, is the potential (2.10). The other statements of the theorem can be verified by standard computations (also see [58], Sec. 25.2).

The central potential (2.10) possesses a number of specific properties (we ascribe this to its projective invariance), and that is why it occurs in various problems in Newtonian mechanics. Newton [97] (see also [98]) considered the motion of a body under the action of a central force proportional to the inverse cube of the distance. Later, such motion was investigated in more detail by Cotes (Roger Cotes, 1682-1716, English astronomer and mathematician, who prepared the second edition of "Principia") in his "Harmonia Mensurarum."

There is also a remarkable connection between the Newton-Cotes potential (2.10) and a monoatomic gas. This connection is realized by the conservation laws (2.14) (see [58], Sec. 25.2), which are also inherited by the Boltzmann equations [23].

§ 3 The Lie-Bäcklund group instead of Newton's apple

> Had not Newton lain in the garden under an apple tree and had not an apple suddenly fallen on his head, we might be still unaware of the motion of celestial bodies and about a great number phenomena, which depend upon it.
>
> L. Euler

As far as I know, the "law of inverse squares" of Newton's gravitation theory has not yet been derived from symmetry principles in literature. In this section (which is, surely, only of methodical significance) such an attempt is made. The main point here is to state Kepler's first empirical law in group-theoretical terms.

It was shown as early as by Laplace [82] that Kepler's first law (the planets move along ellipses with the Sun in one of the focuses) is a direct corollary of the conservation law for the vector*

$$\boldsymbol{A} = \boldsymbol{v} \times \boldsymbol{M} + \alpha \frac{\boldsymbol{x}}{r}, \quad \big(\boldsymbol{M} = m\boldsymbol{x} \times \boldsymbol{v}\big) \tag{3.1}$$

for the equation of motion

$$m\dot{\boldsymbol{v}} = \alpha \frac{\boldsymbol{x}}{r^3} \tag{3.2}$$

in the central field with potential

$$U = -\frac{\alpha}{r}, \quad \alpha = \text{const.} \tag{3.3}$$

In addition, Eq. (3.2) admits the infinitesimal Lie-Bäcklund transformation $\tilde{\boldsymbol{x}} = \boldsymbol{x} + \delta\boldsymbol{x}$ with the vector-parameter $\boldsymbol{a} = (a^1, a^2, a^3)$, where

$$\delta\boldsymbol{x} = 2(\boldsymbol{a}\cdot\boldsymbol{x})\boldsymbol{v} - (\boldsymbol{a}\cdot\boldsymbol{v})\boldsymbol{x} - (\boldsymbol{x}\cdot\boldsymbol{v})\boldsymbol{a}. \tag{3.4}$$

The *Hermann-Bemoulli-Laplace vector* (3.1) can be obtained from the Noether theorem according to formula (2.9) (see [58], Sec. 25.2).

The cited Lie-Bäcklund symmetry is a natural generalization of the rotational symmetry of the Kepler problem. To visualize this fact, it suffices to note that an infinitesimal transformation from the rotational group is

*According to [45], this vector appeared in the literature as a constant of integration for the orbit equation in 1710 in the publications by Jacob Hermann, a student of brothers Bernoulli, and by Johann Bernoulli.

determined by the increment $\delta\boldsymbol{x} = \boldsymbol{x} \times \boldsymbol{a}$ and that the expression (3.4) can be represented as the, symmetrized double vector product of the vectors $\boldsymbol{x}, \boldsymbol{v}$, and $\boldsymbol{q}$:

$$\delta\boldsymbol{x} = [(\boldsymbol{x} \times \boldsymbol{v}) \times \boldsymbol{a}] + [\boldsymbol{x} \times (\boldsymbol{v} \times \boldsymbol{a})]. \tag{3.5}$$

The symmetry with respect to the Lie-Bäcklund group with infinitesimal increment (3.5) is just Kepler's first law expressed in group-theoretical terms.

Theorem 11.4. The condition (2.8) for the applicability of the Noether theorem is valid for a central field $U(r)$ and the increment (3.5) if and only if the potential has the form (3.3).

Proof. The reasoning is the same as in the preceding section. In this case we have

$$\delta L = 2\left[\frac{d}{dt}\left((\boldsymbol{x}\cdot\boldsymbol{a})U\right) - (\boldsymbol{v}\cdot\boldsymbol{a})(rU' + U)\right].$$

Hence the condition (2.8) takes the form

$$\frac{\delta\Phi}{\delta\boldsymbol{x}} \equiv \frac{\partial\Phi}{\partial\boldsymbol{x}} - \frac{d}{dt}\frac{\partial\Phi}{\partial\boldsymbol{v}} = 0,$$

where $\Phi = (\boldsymbol{v}\cdot\boldsymbol{a})(rU' + U)$, and so

$$\frac{\delta\Phi}{\delta\boldsymbol{x}} = (rU' + U)'(\boldsymbol{v}\cdot\boldsymbol{a})\frac{\boldsymbol{x}}{r} - \frac{d}{dt}\left(rU' + U\right)\boldsymbol{a} = \frac{1}{r}\left(rU' + U\right)'\,\boldsymbol{v} \times (\boldsymbol{x} \times \boldsymbol{a}) = 0.$$

It follows that $(rU' + U)' = 0$, which implies the potential (3.3) to within an unessential additive constant.

§ 4 Is the parallax of Mercury's perihelion consistent with the Huygens principle?

In the concluding "Common scholium" of his "Principia," Newton wrote: "The gravity to the Sun is the gravity to its isolated particles and it is reduced with the distance to the Sun. This reduction is proportional to the square of the distance even up to the Saturn orbit, which follows from the fact that planets' aphelions are at rest, and even up to the farthest comets' aphelions provided that these aphelions are at rest. But so far I have not been able to derive the reason for these gravity properties from the phenomena, and I do not devise any hypotheses... It is enough to know that gravity really exists, acts according to the laws stated and is sufficient to explain all motions of celestial bodies and of the sea."

About 150 years later it was however found that the planets' aphelions (or perihelions) are not at rest but are slowly moving. For example, the observed parallax of Mercury is only about 43″ per century. However, small effects are sometimes of fundamental significance for the theory, all the more so if we deal with description of real phenomena. Specifically, the anomaly in the motion of Mercury has not been given a satisfactory explanation on the basis of Newton's gravitation law, despite the efforts of greatest scientists. The explanation given by Einstein in 1915 was the first experimental justification of the general relativity theory. All these facts are well known, and a wonderful critical survey can be found in [128], Chap. 8, Sec. 6. Here we consider a point that has not apparently been taken into account so far.

Einstein's explanation of the parallax of Mercury's perihelion is based on the assumption that the space near the Sun is not plane but has the Schwarzschild metric (1916), previously found by Einstein to within the second approximation. But in passing from the plane Minkowsky space-time to the Schwarzschild metric the Huygens principle is violated. This principle implies the existence of rear front for sonic, light, and other waves that carry localized perturbations. This is because the Minkowsky space belongs to the family of Riemann spaces with nontrivial conformal group, whereas the Schwarzschild space has a trivial conformal group and therefore does not satisfy the Huygens principle (see [55], Chap. 4 or [58], Chap. 2).

Thus, in connection with the theoretical explanation of the observed anomaly in the planets' motion a new problem arises. It can be stated as the following alternative.

1. The explanation by passing to the Schwarzschild metric is adequate to the phenomenon in the approximation required. Then the Huygens principle is not valid, and hence sonic, light, and other signals undergo distortions. And we should estimate the distortion level from the view point of possible observation.

2. The Huygens principle holds in the real world. Then we should explain the parallax of Mercury's perihelion without any contradiction with this principle. This task requires thorough physical analysis of the equations of motion for a particle in a curved space-time with nontrivial conformal group. The problem is facilitated by the fact that such spaces can be described completely. Namely, any space-time with nontrivial conformal group is defined in an appropriate coordinate system by the metric ([58], Sec. 8.5)

$$ds^2 = e^{\mu(x)}[(dx^0)^2 - (dx^1)^2 - (dx^2)^2 - 2f(x^1 - x^0)dx^2dx^3 - g(x^1 - x^0)(dx^3)^2],$$

where f and g are functions only of $x^1 - x^0$, such that $g - f^2 > 0$, and the function $\mu(x)$ can depend on all variables x.

§ 5 Integration of ordinary differential equations with a small parameter admitting an approximate group

The theory of approximate transformation groups [17] can be used to classify and integrate differential equations with a small parameter. Consider an example that illustrates the method of integrating ordinary differential equations in the framework of regular perturbations of symmetry groups.

The second-order equation

$$y'' - x - \varepsilon y^2 = 0 \tag{5.1}$$

with small parameter ε does not have any exact symmetries, and hence cannot be integrated by Lie group methods. However, it admits the following approximate symmetry generators (here we do not dwell on the question whether Eq. (5.1) has any other approximate symmetries):

$$\begin{aligned} X_1 &= \frac{2}{3}\,\varepsilon x^3 \frac{\partial}{\partial x} + \Big[1 + \varepsilon\Big(yx^2 + \frac{11}{60}\,x^5\Big)\Big]\frac{\partial}{\partial y}, \\ X_2 &= \frac{1}{6}\,\varepsilon x^4 \frac{\partial}{\partial x} + \Big[x + \varepsilon\Big(\frac{1}{3}\,yx^3 + \frac{7}{180}\,x^6\Big)\Big]\frac{\partial}{\partial y}\,. \end{aligned} \tag{5.2}$$

The operators (5.2) span a two-dimensional Abelian Lie algebra and generate a two-parameter approximate transformation group [17].

The method of successive reduction of order (see [61], Chap. 2) provides the following technique for approximate integration of Eq. (5.1) by means of the approximate symmetries (5.2). In what follows all equations should be interpreted as approximate equations to within $o(\varepsilon)$.

The change of variables

$$t = y - \varepsilon\left(\frac{1}{2}x^2y^2 + \frac{11}{60}yx^5\right), \quad u = x - \frac{2}{3}\varepsilon yx^3 \tag{5.3}$$

transforms X_1 into the translation operator $X_1 = \frac{\partial}{\partial t}$. The transformed equation (5.1) for the function $u(t)$ reads

$$u'' + u(u')^3 + \varepsilon\left[3u^2u' + \frac{1}{6}(u^2u')^2 - \frac{11}{60}(u^2u')^3\right] = 0$$

and does not depend upon t. Therefore, it can be integrated by the standard substitution $u' = p(u)$, which yields

$$p' + up^2 + \varepsilon\left(3u^2 + \frac{1}{6}u^4p - \frac{11}{60}u^6p^2\right) = 0. \tag{5.4}$$

Let us now integrate Eq. (5.4) by using the second operator in (5.2). To this end we rewrite the operator X_2 in terms of the new variables t, u defined by (5.3), extend the resultant operator to the variable $p = u'$ and consider its action in the space of the variables (u, p). As a result, we obtain the following approximate symmetry for Eq. (5.4):

$$\widetilde{X}_2 = \frac{\varepsilon}{2} u^4 \frac{\partial}{\partial u} + \left[p^2 + \varepsilon\left(2u^3 p - \frac{13}{14} u^5 p^2\right)\right] \frac{\partial}{\partial p}\,. \tag{5.5}$$

The change of variables

$$z = u + \varepsilon \frac{u^4}{2p}, \quad q = -\frac{1}{p} + \varepsilon\left(\frac{u^3}{p^2} - \frac{13}{15}\frac{u^5}{p}\right) \tag{5.6}$$

takes the operator (5.5) into the translation operator $\widetilde{X}_2 = \frac{\partial}{\partial q}$ and Eq. (5.4) into the explicitly integrable form

$$q'(z) + z + \frac{11}{60}\varepsilon z^6 = 0.$$

Integration yields:

$$q = -\frac{1}{2} z^2 - \frac{11}{420}\varepsilon z^7 + C, \quad C = \text{const}. \tag{5.7}$$

If we substitute the expressions (5.6) for z and q into (5.7), and then resolve the obtained equation as $p = f(u)$, a single quadrature yields the solution $t = \int f(u) du$. Then the solution of the original equation (5.1) can be found via the change of variables (5.3).

§ 6 Specific features of group modelling in the de Sitter world

> Two astronomers who live in the de Sitter world and have different de Sitter clocks might have an interesting conversation concerning the real or imaginary nature of some world events.
>
> F.Klein [78]

This section is a sketch for future more detailed work on some new approaches and effects that are possible if the curvature of our universe is small but nonzero. Here we will discuss only the following two features of the transition from the Minkowsky geometry to the de Sitter world (space-time of constant curvature).

The first feature is the possibility to treat the de Sitter group in terms of the theory of approximate groups and to consider it as the perturbation of the Poincaré group by introducing a small constant curvature. Indeed, according to the cosmological data, the curvature of our universe is so small (about $10^{-54} cm^{-2}$) that it suffices to calculate only the first-order perturbations. The resultant Lie equations can easily be solved and provide an approximate representation of the de Sitter group [60]. This permits us to simplify the formulas of the exact theory dramatically. The second feature is connected with the use of a special complex transformation, which, in the case of the Minkowsky space, is a very simple conformal transformation, admitted by the Dirac equation with zero mass and is not significant there. But if the curvature is not zero, then the transformation becomes a non-trivial equivalence transformation on the collection of spaces with constant curvature. The addition of this transformation to the de Sitter group results in an interesting combination of the three possible types of spaces of constant curvature: elliptic, hyperbolic (Lobachevsky spaces), and parabolic (Minkowsky spaces considered as the limit case in which the curvature is zero). The metric of the de Sitter space is

$$dr^2 = \left(1 + \frac{K}{4}(r^2 - c^2t^2)\right)^{-2} (c^2dt^2 - dx^2 - dy^2 - dz^2), \tag{6.1}$$

where

$$r^2 = x^2 + y^2 + z^2, \quad K = \text{const.} \tag{6.2}$$

With the standard notation $(x^1, x^2, x^3, x^4) = (x, y, z, ict), \quad ds = id\tau$ we have

$$ds^2 = \left(1 + \frac{K}{4}\sigma^2\right)^{-2} \sum_{\mu=1}^{4} (dx^\mu)^2, \quad \sigma^2 = \sum_{\mu=1}^{4} (x^\mu)^2 \tag{6.3}$$

The motions of the metric (6.3) form the de Sitter group, which differs from the Poincaré group in that simple translations of the coordinates x^μ are replaced by more complicated "generalized translations". For example, generalized translation with respect to the coordinate x^1 has the generator

$$\begin{aligned} X_1 = &\left(1 + \frac{K}{4}\Big[(x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2\Big]\right)\frac{\partial}{\partial x^1} \\ &+ \frac{K}{2}x^1\left(x^2\frac{\partial}{\partial x^2} + x^3\frac{\partial}{\partial x^3} + x^4\frac{\partial}{\partial x^4}\right), \end{aligned} \tag{6.4}$$

and the translation itself has the form

$$\tilde{x}^1 = 2\frac{x^1\cos(a\sqrt{K}) + (1-\frac{K}{4}\sigma^2)\frac{1}{\sqrt{K}}\sin(a\sqrt{K})}{1+\frac{K}{4}\sigma^2 + (1-\frac{K}{4}\sigma^2)\cos(a\sqrt{K}) - x^1\sqrt{K}\sin(a\sqrt{K})}\,,$$
$$\tilde{x}^l = \frac{x^l}{1+\frac{K}{4}\sigma^2 + (1-\frac{K}{4}\sigma^2)\cos(a\sqrt{K}) - x^1\sqrt{K}\sin(a\sqrt{K})}\,, \tag{6.5}$$

where a is the group parameter, and $l = 2, 3, 4$.

If the constant curvature K is small, one can use the theory of approximate groups [60]. The approximate Lie equation for the operator (6.4) can be solved easily. As a result, we obtain the following simple approximate representation of the generalized translation (6.5) (see the detailed calculation in [60]):

$$\tilde{x}^1 = x^1 + a + \frac{K}{4}\Big\{[(x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2]a + x^1a^2 + \frac{a^3}{3}\Big\} + o(K),$$

$$\tilde{x}^l = x^l + \frac{K}{4}x^l(2ax^1 + a^2) + o(K), \quad l = 2, 3, 4. \tag{6.6}$$

The free motion of a particle in the de Sitter world is described by the Lagrangian

$$L = -mc^2\theta^{-1}\sqrt{1-\beta^2} \tag{6.7}$$

with

$$\theta = 1 + \frac{K}{4}(r^2 - c^2t^2), \quad \beta^2 = \frac{|\boldsymbol{v}|^2}{c^2}\,,$$

where m and $\boldsymbol{v}$ are the mass and the velocity of the particle, respectively. We write $\boldsymbol{x} = (x^1, x^2, x^3)$, so that $\boldsymbol{v} = d\boldsymbol{x}/dt$. Starting from the formula (6.7), we shall find the Lagrangian for Kepler's problem in the de Sitter world. The invariance of the classical Kepler problem with respect to the rotations and time translations, the known limit value of the potential (3.3) for $K = 0$ and $\beta^2 \to 0$, and formula (6.7) will serve as heuristics in our consideration. Hence we seek the Lagrangian in the form

$$L = -mc^2\theta^{-1}\sqrt{1-\beta^2} + \frac{\alpha}{r}\theta^s(1-\beta^2)^n; \quad s, n = \text{const}. \tag{6.8}$$

The action integral $\int L dt$ is invariant with respect to rotations. Therefore, the invariance with respect to the generalized time translations with the operator

$$X_4 = \frac{1}{c^2}\left[1 - \frac{K}{4}(c^2t^2 + r^2)\right]\frac{\partial}{\partial t} - \frac{K}{2}t\sum_{l=1}^{3}x^l\frac{\partial}{\partial x^l} \tag{6.9}$$

[which is obtained from (6.4) by replacing x^1 with x^4] will be the only additional condition.

Theorem 11.5. The action integral $\int Ldt$ with the Lagrangian (6.8) is invariant with respect to the group with the operator (6.9) if and only if $s=0$ and $n=\frac{1}{2}$, i.e. when

$$L=-mc^2\theta^{-1}\sqrt{1-\beta^2}+\frac{\alpha}{r}\sqrt{1-\beta^2}. \qquad (6.10)$$

Proof. The required invariance condition is [58]

$$[\widetilde{X}_4+D_t(\xi)]L=0, \qquad (6.11)$$

where X_4 is the extension of operator (6.9) to v, and ξ is the coordinate of this operator at $\frac{\partial}{\partial t}\cdot$ The calculation gives

$$[\widetilde{X}_4+D_t(\xi)]L=\frac{\alpha K}{2r}\theta^s(1-\beta^2)^n[(1-2n)\frac{x\cdot v}{c^2}+st]. \qquad (6.12)$$

Therefore, the statement of the theorem follows from (6.11).

Remark 11.2. A similar theorem on the uniqueness of an invariant Lagrangian is not valid in the Minkowsky space. Indeed, as follows from (6.12), for $K=0$ condition (6.11) identically holds for the Lagrangian (6.8) with any n. Thus, the theorem proved is an effect characteristic of nonzero curvature.

The spinor analysis in a curvilinear space has been developed from various points of view by many authors. A good exposition of its techniques and a general review of literature on the topic are given in [26]. According to [26], Sec. 2, the Dirac equation in the metric (6.3) can be rewritten in the form

$$\left(1+\frac{K}{4}\sigma^2\right)\gamma^\mu\frac{\partial\psi}{\partial x^\mu}-\frac{3}{4}K(x\cdot\gamma)\psi+m\psi=0, \quad m=\text{const.}, \qquad (6.13)$$

where $\gamma^\mu(\mu=1,...,4)$ are the usual four-row Dirac matrices in the Minkowsky space and $x{\cdot}\gamma$ denotes the four-dimensional inner product $(x{\cdot}\gamma)=\sum_{\mu=1}^4 x^\mu\gamma^\mu$. Here we are interested only in the equation for neutrino ($m=0$) in the linear approximation with respect to K :

$$\gamma^\mu\frac{\partial\psi}{\partial x^\mu}-\frac{3}{4}K(x\cdot\gamma)\psi=0. \qquad (6.14)$$

Equation (6.13) admits the de Sitter group, whose action on the wave function ψ is defined as follows. Let us write out the infinitesimal transformation of the de Sitter group as $\delta x=a\xi$. Then $\delta\psi=aS\psi$, where

$$S=\frac{1}{8}\sum_{\mu,\nu=1}^4\frac{\partial\xi^\mu}{\partial x^\nu}(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu-3\delta_{\mu\nu})+\frac{3}{4}K\left(1+\frac{K}{4}\sigma^2\right)^{-1}(x\cdot\xi). \quad (6.15)$$

Let us now return to Eq. (6.14). It can be reduced (in the linear approximation with respect to K) to the common Dirac equation

$$\gamma^\mu \frac{\partial \varphi}{\partial x^\mu} = 0 \tag{6.16}$$

by the change

$$\varphi = \psi(x) \exp\left[\frac{1}{2}\left(1 + \frac{K}{4}\sigma^2\right)^{-3}\right]. \tag{6.17}$$

Equation (6.16) is invariant with respect to the transformation

$$\tilde{x}^\mu = ix^\mu, \quad i = \sqrt{-1}. \tag{6.18}$$

Therefore, instead of (6.17), we choose the representation of φ as

$$\varphi = \chi(\tilde{x}) \exp\left[\frac{1}{2}\left(1 + \frac{K}{4}\sigma^2\right)^{-3}\right]. \tag{6.19}$$

From (6.16) we obtain the equation (the bar over $\tilde{x}$ is omitted)

$$\gamma^\mu \frac{\partial \chi}{\partial x^\mu} + \frac{3}{4}K(x \cdot \gamma)\chi = 0, \tag{6.20}$$

which coincides with Eq. (6.14) in the de Sitter space with curvature of the opposite sign.

In the Minkowsky space the transformation (6.18) takes timelike intervals into spacelike ones and vice versa. The same is true of the de Sitter space, with the simultaneous change of sign of the space curvature. Indeed, assigning the subscript K to the interval (6.3) and using (6.18), we have

$$ds^2_{(K)} = -d\bar{s}^2_{(-K)}. \tag{6.21}$$

Formulas (6.17) and (6.19) can be interpreted as the "splitting" of a neutrino into two neutrinos, that are described by equations (6.14) and (6.20) and differ only if $K \neq 0$. System of equations (6.14), (6.20) admits the approximate transformation group whose infinitesimal transformations are defined by the increments

$$\delta x = a\xi, \quad \delta\psi = aS\psi \quad \delta\chi = aT\chi \tag{6.22}$$

where the vector $\xi = (\xi^1, \ldots, \xi^4)$ belongs to the 15-dimensional Lie algebra of the group of conformal transformations of the Minkowsky space (e.g., see [58]) and the matrices S and T are given by the formulas (see (6.15))

$$\begin{aligned} S &= \frac{1}{8}\sum_{\mu,\nu=1}^{4} \frac{\partial \xi^\mu}{\partial x^\nu}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu - 3\delta_{\mu\nu}) + \frac{3}{4}K(x \cdot \xi), \\ T &= \frac{1}{8}\sum_{\mu,\nu=1}^{4} \frac{\partial \xi^\mu}{\partial x^\nu}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu - 3\delta_{\mu\nu}) - \frac{3}{4}K(x \cdot \xi). \end{aligned} \tag{6.23}$$

§ 7 Two-dimensional Zabolotskaya-Khokhlov equation coincides with the Lin-Reissner-Tsien equation

The group admitted by the equation

$$\varphi_x \varphi_{xx} + 2\varphi_{tx} - \varphi_{yy} = 0 \tag{7.1}$$

describing a nonsteady potential gas flow with transonic velocities [92], has been calculated in [94]. The Lie algebra of this group is infinite-dimensional and contains five arbitrary functions of t. On the other hand, in [125] the infinite-dimensional symmetry algebra of the equation

$$\frac{\partial^2}{\partial q_1^2}\left(\frac{u^2}{2}\right) - \frac{\partial^2 u}{\partial q_1 \partial q_2} + \frac{\partial^2 u}{\partial q_3^2} = 0, \tag{7.2}$$

which is the two-dimensional version of the Zabolotskaya-Khokhlov equation [129] known in nonlinear acoustics, has been found in [125]. This algebra contains three arbitrary functions of the variable q_2. The comparison of the two algebras suggests the possibility of a nonpoint (since the dimensions of the symmetry algebras are different) correspondence between equations (7.1) and (7.2). In fact, these equations can easily be identified by introducing a potential. Namely, if we set $x = -q_1, t = -2q_2$, and $y = q_3$ in Eq. (7.1), differentiate it with respect to q_1 and denote

$$u = \frac{\partial \varphi}{\partial q_1}, \tag{7.3}$$

then we obtain Eq. (7.2). The passage from the symmetry algebra of Eq. (7.1) to the algebra for Eq. (7.2) can easily be accomplished by applying the formulas from ([58], Sec. 19.4) to the differential substitution (7.3).

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Paper 12

Perturbation methods in group analysis: Approximate exponential map

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Recently, a new direction in symmetry analysis of differential equations has been developed. This approach, based on the concept of approximate groups, is used for tackling differential equations with a small parameter and for approximate representations of Lie groups.

In the classical Lie group theory, the one-parameter group with a given infinitesimal generator is represented by the exponential map. The present paper is an introduction to the topic with the emphasis on the role of the exponential map in the theory of approximate groups.

Introduction

The initiation and subsequent development of the theory of approximate transformation groups were inspired by the following two chief circumstances.

A variety of differential equations recognized as mathematical models in engineering and physical sciences, involve empirical parameters or constitutive laws. Therefore coefficients of model equations are defined approximately with an inevitable error. Consequently, differential equations depending on a small parameter are of frequent occurrence in applications.

Unfortunately, any small perturbation of coefficients of a differential equation disturbs its symmetry properties, and this reduces the practical value of group theoretical methods.

Instability of Lie symmetry groups is the first circumstance that has led us [14] to the concept of approximate groups.

The second factor is that, in practical applications, *Lie group analysis may come across unjustified complexities.*

The second circumstance is illustrated by the following example. Consider the de Sitter space-time with the metric form

$$ds^2 = -\Big(1 + \varepsilon\sigma^2\Big)^{-2} \sum_{\mu=1}^{4} (dx^\mu)^2,$$

where

$$\sigma^2 = \sum_{\mu=1}^{4} (x^\mu)^2, \quad (x^1, x^2, x^3, x^4) = (x, y, z, ict),$$

$i = \sqrt{-1}$, $c = 2.99793 \times 10^{10}$ cm/sec is the velocity of light in empty space, and $\varepsilon = K/4$ with K denoting the curvature of the de Sitter universe. According to cosmological data, the curvature K is a small constant ($\sim 10^{-54}$ cm^{-2}), hence ε can be treated as a small parameter. We assume here that $\varepsilon \geq 0$. The *de Sitter group* (i.e., the group of isometric motions in the de Sitter space-time) differs from the *Poincaré group* (i.e., the group of isometries in the Minkowski space-time) in that the usual translations of space-time coordinates x^μ are replaced by more complicated transformations, the so-called "generalized translations" in the de Sitter space-time. The generalized translation, e.g. along the x^1 axis has the infinitesimal generator

$$X = \Big(1+\varepsilon[(x^1)^2-(x^2)^2-(x^3)^2-(x^4)^2]\Big)\frac{\partial}{\partial x^1}+2\varepsilon x^1\Big(x^2\frac{\partial}{\partial x^2}+x^3\frac{\partial}{\partial x^3}+x^4\frac{\partial}{\partial x^4}\Big).$$

The corresponding group transformations (with the group parameter a) have the form

$$\bar{x}^1 = 2\frac{x^1\cos(2a\sqrt{\varepsilon}) + (1-\varepsilon\sigma^2)\dfrac{1}{2\sqrt{\varepsilon}}\sin(2a\sqrt{\varepsilon})}{1+\varepsilon\sigma^2+(1-\varepsilon\sigma^2)\cos(2a\sqrt{\varepsilon}) - 2x^1\sqrt{\varepsilon}\sin(2a\sqrt{\varepsilon})},$$

$$\bar{x}^j = 2\frac{x^j}{1+\varepsilon\sigma^2+(1-\varepsilon\sigma^2)\cos(2a\sqrt{\varepsilon}) - 2x^1\sqrt{\varepsilon}\sin(2a\sqrt{\varepsilon})}, \ j = 2, 3, 4.$$

However, since ε is small, it is sufficient to use an approximate expression of these transformations, e.g. by expanding them in powers of ε and considering only the leading terms of the first order. That is, to consider the de Sitter group as a perturbation of the Poincaré group by the curvature K. Then the result is rather simple, viz.

$$\bar{x}^1 \approx x^1 + a + \varepsilon\Big([(x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2]a + x^1a^2 + \frac{1}{3}a^3\Big),$$

$$\bar{x}^j \approx x^j + \varepsilon(2ax^1 + a^2)x^j, \quad j = 2, 3, 4.$$

The question naturally arises of how to calculate this perturbation directly, without using the complicated formula of group transformations. The theory of approximate groups gives a method of calculation. Namely, one can use the *approximate Lie equations*. Furthermore, this approximation inherits the group property of the exact transformation group in the precision prescribed.

The present paper focuses on construction of one-parameter approximate groups, in the first order of precision, by using the exponential map. The main result is formulated in Theorem 12.2 and states the following.

Given an operator

$$X = X_0 + \varepsilon X_1$$

with a small parameter ε, *where*

$$X_0 = \xi_0^i(x)\frac{\partial}{\partial x^i}\,, \quad X_1 = \xi_1^i(x)\frac{\partial}{\partial x^i}\,,$$

the corresponding approximate group of transformations

$$\bar{x}^i = \bar{x}_0^i + \varepsilon\bar{x}_1^i, \quad i = 1, \ldots, n,$$

is determined by the following formulae:

$$\bar{x}_0^i = \mathrm{e}^{aX_0}(x^i), \quad \bar{x}_1^i = \langle\!\langle aX_0, aX_1\rangle\!\rangle(\bar{x}_0^i), \quad i = 1, \ldots, n,$$

where

$$\mathrm{e}^{aX_0} = 1 + aX_0 + \frac{a^2}{2!}X_0^2 + \frac{a^3}{3!}X_0^3 + \cdots$$

and

$$\langle\!\langle aX_0, aX_1\rangle\!\rangle = aX_1 + \frac{a^2}{2!}[X_0, X_1] + \frac{a^3}{3!}[X_0, [X_0, X_1]] + \cdots\,.$$

In other words, the approximate operator $X = X_0 + \varepsilon X_1$ generates the following one-parameter approximate group of transformations in $\mathbb{R}^n$:

$$\bar{x}^i = \big(1 + \varepsilon\langle\!\langle aX_0, aX_1\rangle\!\rangle\big)\mathrm{e}^{aX_0}(x^i), \quad i = 1, \ldots, n.$$

The above approximate representation of the de Sitter group is readily obtained by applying this theorem to the generator X of the generalized translations:

$$X = X_0 + \varepsilon X_1$$

where

$$X_0 = \frac{\partial}{\partial x^1}$$

and

$$X_1 = \Big((x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2\Big)\frac{\partial}{\partial x^1} + 2x^1\Big(x^2\frac{\partial}{\partial x^2} + x^3\frac{\partial}{\partial x^3} + x^4\frac{\partial}{\partial x^4}\Big).$$

The calculations are given in Section 3.2, Example 12.9.

§ 1 Preliminaries on Lie groups

A brief sketch of the Lie equations and the exponential map is given here to draw a parallel between the classical and approximate group theories.

1.1 Continuous one-parameter groups

Let $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$. Consider a one-parameter family of invertible transformations T_a:

$$\bar{x} = f(x, a), \tag{1.1}$$

or in coordinates,

$$\bar{x}^i = f^i(x, a), \quad i = 1, \ldots, n.$$

Given $a \in U$, the transformation T_a carries the point $x \in \mathbb{R}^n$ to the point $\bar{x} \in \mathbb{R}^n$.

Here the parameter a ranges over all real numbers from a neighborhood $U \subset R$ of $a = 0$, and we impose the condition that (1.1) is the identity transformation if and only if $a = 0$, i.e.

$$f(x, 0) = x, \tag{1.2}$$

and, conversely, the equation $f(x, a) = x$ with $a \in U$ implies $a = 0$.

Definition 12.1. A set G of transformations (1.1) is called a *continuous one-parameter group of transformations in* $\mathbb{R}^n$ if the functions $f^i(x, a)$ satisfy the condition (1.2) and the *group property*

$$f^i(f(x, a), b) = f^i(x, c), \quad i, \ldots, n, \tag{1.3}$$

for all $a, b \in U$, where $c \in U$ is a certain (smooth) function of a and b:

$$c = \phi(a, b), \tag{1.4}$$

such that the equation

$$\phi(a, b) = 0 \tag{1.5}$$

has a unique solution $b \in U$ for any $a \in U$. Given a, the solution b of Eq. (1.5) is denoted by a^{-1}. The function $\phi(a, b)$ is termed a *group composition law.*

According to Definition 12.1, a continuous group G contains the (unique) identity transformation $I = T_0$. Further, the group property (1.3) means that any two transformations $T_a, T_b \in G$ carried out one after the other result in a transformation which also belongs to G:

$$T_b T_a = T_c, \quad c = \phi(a, b) \in U,$$

for any $a, b \in U$. The solvability of the equation (1.5), together with the group property (1.3), provides the inverse transformation $T_a^{-1} = T_{a^{-1}} \in G$ to $T_a \in G$:

$$T_{a^{-1}} T_a = T_a T_{a^{-1}} = I$$

for any $a \in U$.

The group parameter a is said to be *canonical* if the composition law is $\phi(a, b) = a + b$, i.e., if the group property has the form

$$f^i(f(x, a), b) = f^i(x, a + b), \quad i = 1, \ldots, n. \tag{1.6}$$

Given an arbitrary composition law (1.3), there exists the canonical parameter $\tilde{a}$. It is defined by the formula

$$\tilde{a} = \int_0^a \frac{da}{A(a)} \ ,$$

where

$$A(a) = \frac{\partial \phi(a, b)}{\partial b}\Big|_{b=0}.$$

Example 12.1. Let $n = 1$, and let $\bar{x} = x + ax$. This is a one-parameter group with the composition law $\phi(a, b) = a + b + ab$. Here $A(a) = 1 + a$ and hence the canonical parameter is

$$\tilde{a} = \int_0^a \frac{da}{1 + a} = \ln(1 + a).$$

In this paper we shall adopt the canonical parameter when referring to one-parameter groups as well as approximate groups.

1.2 Group generator. Lie equations

Let G be a group of transformations (1.1) with a function $f(x,a)$ satisfying the initial condition (1.2) and the group property (1.6). The infinitesimal transformation of the group G is the main linear part (in a) of (1.1), viz.

$$\bar{x}^i \approx x^i + a\xi^i(x), \tag{1.7}$$

where

$$\xi^i(x) = \frac{\partial f^i(x,a)}{\partial a}\Big|_{a=0}, \quad i = 1,\dots,n.$$

The first-order linear differential operator

$$X = \xi^i(x)\frac{\partial}{\partial x^i} \tag{1.8}$$

is known as the *infinitesimal operator* or *generator* of the group G. S. Lie called it a *symbol* of the infinitesimal transformation (1.7). One-parameter groups are determined by their infinitesimal generators according to the following Lie's theorem.

Theorem 12.1. Given an infinitesimal transformation (1.7) or its symbol (1.8), the solution $\bar{x} = f(x,a)$ of the system of ordinary differential equations

$$\frac{d\bar{x}^i}{da} = \xi^i(\bar{x}), \quad i = 1,\dots,n, \tag{1.9}$$

with the initial conditions

$$\bar{x}^i\big|_{a=0} = x^i, \quad i = 1,\dots,n \tag{1.10}$$

determines a one-parameter group of transformations (1.1).

First-order ordinary differential equations (1.9) (sometimes the equations (1.9) together with the initial conditions (1.10)) are known as the *Lie equations.*

Example 12.2. Consider, in the (x,y) plane, the infinitesimal transformation

$$\overline{x} \approx x + ax^2, \quad \overline{y} \approx y + axy$$

with the symbol

$$X = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}\cdot$$

Here the Lie equations (1.9) have the form

$$\frac{d\bar{x}}{da} = \bar{x}^2, \quad \frac{d\bar{y}}{da} = \bar{x}\bar{y}\,.$$

Integrating,

$$\bar{x} = \frac{1}{C_1 - a}, \quad \bar{y} = \frac{C_2}{C_1 - a},$$

where C_1 and C_2 are arbitrary constants. The initial conditions (1.10) provide

$$x = \frac{1}{C_1}, \quad y = \frac{C_2}{C_1},$$

whence

$$C_1 = \frac{1}{x}, \quad C_2 = \frac{y}{x} \cdot$$

Thus the one-parameter group of transformations has the form

$$\bar{x} = \frac{x}{1 - ax}, \quad \bar{y} = \frac{y}{1 - ax} \cdot \tag{1.11}$$

1.3 The exponential map

The solution of the Lie equations (1.9) –(1.10) can be represented explicitly by the exponential map,

$$\bar{x}^i = \mathrm{e}^{aX}(x^i), \quad i = 1, \ldots, n, \tag{1.12}$$

where the exponent is given by the infinite sum:

$$\mathrm{e}^{aX} = 1 + aX + \frac{a^2}{2!}X^2 + \frac{a^3}{3!}X^3 + \cdots . \tag{1.13}$$

Example 12.3. The exponential map (1.12) is written in the (x, y) plane in the form:

$$\bar{x} = e^{aX}(x), \quad \bar{y} = e^{aX}(y), \tag{1.14}$$

where e^{aX} is given by the series (1.13). Let us apply the exponential map to the generator

$$X = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}$$

considered in Example 12.2. We have:

$$X(x) = x^2, \quad X^2(x) = X(X(x)) = X(x^2) = 2!x^3, \quad X^3(x) = 3!x^4, \ldots .$$

These equations hint the general formula

$$X^n(x) = n!x^{n+1}, \quad n = 1, 2, \ldots .$$

The proof is given by induction:

$$X^{n+1}(x) = X(n!x^{n+1}) = (n+1)!x^2x^n = (n+1)!x^{n+2}.$$

It follows:

$$e^{aX}(x) = x + ax^2 + \cdots + a^n x^{n+1} + \cdots .$$

One can rewrite the right-hand side of this equation in the form

$$x(1 + ax + \cdots + a^n x^n + \cdots) = \frac{x}{1 - ax}$$

where the well known Taylor expansion of the function $(1 - ax)^{-1}$ is used provided that $|ax| < 1$. Hence,

$$e^{aX}(x) = \frac{x}{1 - ax} \cdot$$

Similarly,

$$X(y) = xy, \quad X^2(y) = X(xy) = yX(x) + xX(y) = y(x^2) + y(xy) = 2!yx^2,$$

$$X^3(y) = 2![yX(x^2) + x^2X(y)] = 2![y(2x^3) + x^2(xy)] = 3!yx^3.$$

This hints the general formula

$$X^n(y) = n!yx^n, \quad n = 1, 2, \ldots$$

that can be readily verified by induction:

$$X^{n+1}(y) = n!X(yx^n) = n![nyx^{n+1} + x^n(xy)] = (n + 1)!yx^{n+1}.$$

It follows:

$$e^{aX}(y) = y + ayx + a^2yx^2 + \cdots + a^n yx^n + \cdots$$

$$= y(1 + ax + \cdots + a^n x^n + \cdots) = \frac{y}{1 - ax} \cdot$$

Thus, we arrive at the transformation (1.11):

$$\bar{x} = \frac{x}{1 - ax}\,, \quad \bar{y} = \frac{y}{1 - ax} \cdot$$

§ 2 One-parameter approximate transformation groups

For a detailed discussion of the material presented here and of the theory of multi-parameter approximate transformation groups, see [65], Ch. 2.

2.1 Notation and definition

In what follows, functions $f(x,\varepsilon)$ of n variables $x = (x^1, \ldots, x^n)$ and a parameter ε are considered locally in a neighbourhood of $\varepsilon = 0$. These functions are continuous in the x's and ε, as are also their derivatives to as high an order as enters in the subsequent discussion.

If a function $f(x,\varepsilon)$ satisfies the condition

$$\lim_{\varepsilon\to 0} \frac{f(x,\varepsilon)}{\varepsilon^p} = 0,$$

it is written (after E. Landau, *Vorlesungen über Zahlentheorie*, vol. 2, 1927)

$$f(x,\varepsilon) = o(\varepsilon^p).$$

Then f is said to be *of order less than* ε^p. If

$$f(x,\varepsilon) - g(x,\varepsilon) = o(\varepsilon^p),$$

the functions f and g are said to be *approximately equal* (with an error $o(\varepsilon^p)$) and written

$$f(x,\varepsilon) = g(x,\varepsilon) + o(\varepsilon^p),$$

or, briefly

$$f \approx g$$

when there is no ambiguity.

The approximate equality defines an equivalence relation, and we join functions into equivalence classes by letting the functions $f(x,\varepsilon)$ and $g(x,\varepsilon)$ to be members of the same class if and only if $f \approx g$. Given a function $f(x,\varepsilon)$, let

$$f_0(x) + \varepsilon f_1(x) + \cdots + \varepsilon^p f_p(x)$$

be the approximating polynomial of degree p in ε obtained via the Taylor series expansion of $f(x,\varepsilon)$ in powers of ε about $\varepsilon = 0$. Then any function $g \approx f$ (in particular, the function f itself) has the form

$$g(x,\varepsilon) \approx f_0(x) + \varepsilon f_1(x) + \cdots + \varepsilon^p f_p(x) + o(\varepsilon^p).$$

Consequently the function

$$f_0(x) + \varepsilon f_1(x) + \cdots + \varepsilon^p f_p(x)$$

is called a *canonical representative of the equivalence class of functions containing* f.

Thus, the equivalence class of functions $g(x,\varepsilon) \approx f(x,\varepsilon)$ is determined by the ordered set of $p+1$ functions

$$f_0(x),\ f_1(x),\ \ldots,\ f_p(x).$$

In the theory of approximate transformation groups we consider ordered sets of smooth vector–functions depending on x's and a group parameter a, viz.

$$f_0(x,a),\ f_1(x,a),\ \ldots,\ f_p(x,a)$$

with coordinates

$$f_0^i(x,a),\ f_1^i(x,a), \ldots f_p^i(x,a), \quad i = 1,\ldots,n.$$

Let us define the one-parameter family G of *approximate transformations*

$$\overline{x}^i \approx f_0^i(x,a) + \varepsilon f_1^i(x,a) + \cdots + \varepsilon^p f_p^i(x,a), \quad i = 1,\ldots,n, \tag{2.1}$$

of points $x = (x^1,\ldots,x^n) \in \mathbb{R}^n$ into points $\bar{x} = (\bar{x}^1,\ldots,\bar{x}^n) \in \mathbb{R}^n$ as the class of invertible transformations

$$\bar{x} = f(x,a,\varepsilon) \tag{2.2}$$

with vector–functions $f = (f^1,\ldots,f^n)$ such that

$$f^i(x,a,\varepsilon) \approx f_0^i(x,a) + \varepsilon f_1^i(x,a) + \cdots + \varepsilon^p f_p^i(x,a).$$

Here a is a real parameter, and the following condition is imposed:

$$f(x,0,\varepsilon) \approx x.$$

Furthermore, it is assumed that the transformation (2.2) is defined for any value of a from a small neighborhood of $a = 0$, and that, in this neighborhood, the equation $f(x,a,\varepsilon) \approx x$ yields $a = 0$.

Definition 12.2. *The set G of transformations (2.1) is called a one-parameter approximate transformation group if*

$$f(f(x,a,\varepsilon),b,\varepsilon) \approx f(x,a+b,\varepsilon)$$

for all transformations (2.2).

Remark 12.1. Here, unlike the classical Lie group theory, f does not necessarily denote the same function at each occurrence. It can be replaced by any function $g \approx f$ (see the next example).

Example 12.4. Let us take $n = 1$ and consider the following two functions:

$$f(x,a,\varepsilon) = x + a(1 + \varepsilon x + \frac{1}{2}\varepsilon a)$$

and

$$g(x,a,\varepsilon) = x + a(1 + \varepsilon x)(1 + \frac{1}{2}\varepsilon a).$$

They are equal in the first order of precision, viz.

$$g(x,a,\varepsilon) = f(x,a,\varepsilon) + \varepsilon^2\varphi(x,a), \quad \varphi(x,a) = \frac{1}{2}a^2x,$$

and satisfy the approximate group property. Indeed,

$$f(g(x,a,\varepsilon),b,\varepsilon) = f(x,a+b,\varepsilon) + \varepsilon^2\phi(x,a,b,\varepsilon),$$

where

$$\phi(x,a,b,\varepsilon) = \frac{1}{2}a(ax + ab + 2bx + \varepsilon abx).$$

2.2 Approximate group generator

The generator of the approximate group G of transformations (2.2) is the class of first-order linear differential operators

$$X = \xi^i(x,\varepsilon)\frac{\partial}{\partial x^i} \tag{2.3}$$

such that

$$\xi^i(x,\varepsilon) \approx \xi^i_0(x) + \varepsilon\xi^i_1(x) + \cdots + \varepsilon^p\xi^i_p(x),$$

where the vector fields $\xi_0, \xi_1, \ldots, \xi_p$ are given by

$$\xi^i_\nu(x) = \frac{\partial f^i_\nu(x,a)}{\partial a}\Big|_{a=0}, \quad \nu = 0,\ldots,p; \; i = 1,\ldots,n.$$

In what follows, an approximate group generator is written as

$$X \approx \left(\xi^i_0(x) + \varepsilon\xi^i_1(x) + \ \ldots \ + \varepsilon^p\xi^i_p(x)\right)\frac{\partial}{\partial x^i}\cdot$$

It is written also in a specified form, viz.

$$X = \xi^i(x,\varepsilon)\frac{\partial}{\partial x^i} \equiv \left(\xi^i_0(x) + \varepsilon\xi^i_1(x) + \cdots + \varepsilon^p\xi^i_p(x)\right)\frac{\partial}{\partial x^i}\cdot \tag{2.4}$$

Remark 12.2. In theoretical discussions, approximate equalities are considered with an error $o(\varepsilon^p)$ of an arbitrary order $p \geq 1$. However, in the most of applications the theory is simplified by letting $p = 1$. The assumption $p = 1$ is adopted in what follows.

2.3 Approximate Lie equations

Consider one-parameter approximate groups in the first order of precision.

Let

$$X = X_0 + \varepsilon X_1 \tag{2.5}$$

be a given approximate operator, where

$$X_0 = \xi_0^i(x)\frac{\partial}{\partial x^i}\,, \quad X_1 = \xi_1^i(x)\frac{\partial}{\partial x^i}\,\cdot$$

The corresponding approximate group of transformations of points x into points $\bar{x} = \bar{x}_0 + \varepsilon \bar{x}_1$ with the coordinates

$$\bar{x}^i = \bar{x}_0^i + \varepsilon \bar{x}_1^i \tag{2.6}$$

is determined by the following equations:

$$\frac{d\bar{x}_0^i}{da} = \xi_0^i(\bar{x}_0), \quad \bar{x}_0^i\big|_{a=0} = x^i, \tag{2.7}$$

$$\frac{d\bar{x}_1^i}{da} = \sum_{k=1}^{n} \frac{\partial \xi_0^i(x)}{\partial x^k}\bigg|_{x=\bar{x}_0} \bar{x}_1^k + \xi_1^i(\bar{x}_0), \quad \bar{x}_1^i\big|_{a=0} = 0, \tag{2.8}$$

where $i = 1, \ldots, n$. The equations (2.7)–(2.8) were derived in [14] and called the *approximate Lie equations*.

2.4 Solution of approximate Lie equations

An approach to the solution of approximate Lie equations is illustrated by the following simple examples.

Example 12.5. Let $n = 1$ and let

$$X = (1 + \varepsilon x)\frac{\partial}{\partial x}\,\cdot$$

Here $\xi_0(x) = 1,\ \xi_1(x) = x,$ and Eqs. (2.7)–(2.8) are written:

$$\frac{d\bar{x}_0}{da} = 1, \quad \bar{x}_0\big|_{a=0} = x,$$

$$\frac{d\bar{x}_1}{da} = \bar{x}_0, \quad \bar{x}_1\big|_{a=0} = 0.$$

Its solution has the form

$$\bar{x}_0 = x + a, \quad \bar{x}_1 = ax + \frac{a^2}{2}\,\cdot$$

Hence, the approximate group is given by

$$\bar{x} \approx x + a + \varepsilon\Big(ax + \frac{a^2}{2}\Big).$$

Example 12.6. Let $n = 2$ and let

$$X = (1 + \varepsilon x^2)\frac{\partial}{\partial x} + \varepsilon xy\frac{\partial}{\partial y}\,\cdot$$

Here $\xi_0(x, y) = (1, 0),\ \xi_1(x, y) = (x^2, xy),$ and Eqs. (2.7)–(2.8) are written:

$$\frac{d\bar{x}_0}{da} = 1, \qquad \frac{d\bar{y}_0}{da} = 0, \qquad \bar{x}_0\big|_{a=0} = x,\ \bar{y}_0\big|_{a=0} = y,$$

$$\frac{d\bar{x}_1}{da} = (\bar{x}_0)^2,\ \frac{d\bar{y}_1}{da} = \bar{x}_0\bar{y}_0,\ \bar{x}_1\big|_{a=0} = 0,\ \bar{y}_1\big|_{a=0} = 0.$$

The integration yields:

$$\bar{x} \approx x + a + \varepsilon\Big(ax^2 + a^2x + \frac{a^3}{3}\Big), \quad \bar{y} \approx y + \varepsilon\Big(axy + \frac{a^2}{2}y\Big).$$

§ 3 Approximate exponential map

3.1 Main theorem

Let X and Y be linear differential operators of the first order. Consider the exponential map

$$\mathrm{e}^X = \sum_{k=0}^{\infty}\frac{1}{k!}X^k \equiv 1 + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \cdots .$$

The differential of the exponential map is a linear mapping given by the following infinite sum (see, e.g. [25], Ch. III, § 4.3):

$$\sum_{k=0}^{\infty}\frac{1}{(k+1)!}(\mathrm{ad}\,X)^k\,,$$

where $\mathrm{ad}\,X$ is the *inner derivation* by X. It is known also as the *map adjoint to* X and is defined by the linear mapping

$$\mathrm{ad}\,X(Y) = [X, Y]$$

with the usual Lie bracket (commutator)

$$[X, Y] = XY - YX.$$

Definition 12.3. Let us define the differential operator of the infinite order $\langle\!\langle X, Y\rangle\!\rangle$ by the formal infinite sum

$$\langle\!\langle X, Y\rangle\!\rangle = \sum_{k=0}^{\infty} \frac{1}{(k+1)!}(\operatorname{ad} X)^k Y.$$

By substituting

$$\operatorname{ad} X(Y) = [X, Y], \quad (\operatorname{ad} X)^2(Y) = [X, [X, Y]], \ldots$$

we write it as follows:

$$\langle\!\langle X, Y\rangle\!\rangle = Y + \frac{1}{2!}[X, Y] + \frac{1}{3!}[X, [X, Y]] + \frac{1}{4!}[X, [X, [X, Y]]] + \cdots . \quad (3.1)$$

Theorem 12.2. The solution

$$(\bar{x}_0; \bar{x}_1) = (\bar{x}_0^1, \ldots, \bar{x}_0^n; \bar{x}_1^1, \ldots, \bar{x}_1^n)$$

to the approximate Lie equations (2.7)–(2.8) is given by the formulae:

$$\bar{x}_0^i = \mathrm{e}^{aX_0}(x^i), \quad \bar{x}_1^i = \langle\!\langle aX_0, aX_1\rangle\!\rangle(\bar{x}_0^i), \quad i = 1, \ldots, n, \quad (3.2)$$

where

$$\langle\!\langle aX_0, aX_1\rangle\!\rangle = aX_1 + \frac{a^2}{2!}[X_0, X_1] + \frac{a^3}{3!}[X_0, [X_0, X_1]] + \cdots . \quad (3.3)$$

In other words, the approximate operator (2.5),

$$X = X_0 + \varepsilon X_1$$

generates the following one-parameter approximate group of transformations in $\mathbb{R}^n$:

$$\bar{x}^i = \big(1 + \varepsilon\langle\!\langle aX_0, aX_1\rangle\!\rangle\big)\mathrm{e}^{aX_0}(x^i), \quad i = 1, \ldots, n. \quad (3.4)$$

Proof. According to the definition of the approximate group generator (see Sections 2.2 and 3.1), we substitute

$$X_0 + \varepsilon X_1$$

into the definition (1.13) of the exponent:

$$\mathrm{e}^{a(X_0+\varepsilon X_1)} = 1 + a\big(X_0 + \varepsilon X_1\big) + \frac{a^2}{2!}\big(X_0 + \varepsilon X_1\big)^2 + \frac{a^3}{3!}\big(X_0 + \varepsilon X_1\big)^3 + \cdots ,$$

and single out the sum of terms of the first degree in ε. Then we obtain the following:

$$
\begin{aligned}
\mathrm{e}^{a(X_0+\varepsilon X_1)} \approx 1 + aX_0 + \frac{a^2}{2!}X_0^2 + \frac{a^3}{3!}X_0^3 + \cdots \\
+\varepsilon\Big\{aX_1 + \frac{a^2}{2!}\big(X_0X_1 + X_1X_0\big) + \frac{a^3}{3!}\big(X_0^2X_1 + X_0X_1X_0 + X_1X_0^2\big) \\
+\frac{a^4}{4!}\big(X_0^3X_1 + X_0^2X_1X_0 + X_0X_1X_0^2 + X_1X_0^3\big) + \cdots\Big\}.
\end{aligned}
\tag{3.5}
$$

By using the identities

$$
X_0X_1 = X_1X_0 + [X_0, X_1],
$$

$$
X_0^2X_1 + X_0X_1X_0 = 2X_1X_0^2 + 3[X_0, X_1]X_0 + [X_0, [X_0, X_1]], \ldots
$$

one can rewrite (3.5) in the form:

$$
\begin{aligned}
\mathrm{e}^{a(X_0+\varepsilon X_1)} \approx 1 + aX_0 + \frac{a^2}{2!}X_0^2 + \frac{a^3}{3!}X_0^3 + \cdots \\
+\varepsilon\Big\{aX_1\big(1 + aX_0 + \frac{a^2}{2!}X_0^2 + \frac{a^3}{3!}X_0^3 + \cdots\big) \\
+\frac{a^2}{2!}[X_0, X_1]\big(1 + aX_0 + \frac{a^2}{2!}X_0^2 + \frac{a^3}{3!}X_0^3 + \cdots\big) \\
\frac{a^3}{3!}[X_0, [X_0, X_1]]\big(1 + aX_0 + \frac{a^2}{2!}X_0^2 + \frac{a^3}{3!}X_0^3 + \cdots\big) + \cdots\Big\}.
\end{aligned}
$$

Thus, in virtue of (3.1):

$$
\mathrm{e}^{a(X_0+\varepsilon X_1)} \approx \big(1 + \varepsilon\langle\!\langle aX_0, aX_1\rangle\!\rangle\big)\mathrm{e}^{aX_0}. \tag{3.6}
$$

Hence, the exponential map (1.12) written for the operator (2.5) in the first order of precision with respect to ε has the form (3.4). Taking into account (2.6), one obtains the formulae (3.2) thus proving the theorem.

3.2 Examples

Example 12.7. Let us use Theorem 12.2 in Example 12.5. Here,

$$
X_0 = \frac{\partial}{\partial x}, \quad X_1 = x\frac{\partial}{\partial x}.
$$

Therefore

$$
X_0(x) = 1, \ X_0^2(x) = X_0^3(x) = \cdots = 0,
$$

and

$$[X_0, X_1] = \frac{\partial}{\partial x} = X_0,$$

$$[X_0, [X_0, X_1]] = [X_0, X_0] = 0, \ldots .$$

Consequently,

$$\bar{x}_0 = \mathrm{e}^{aX_0}(x) = x + a,$$

and

$$\langle\!\langle aX_0, aX_1 \rangle\!\rangle = (ax + \frac{a^2}{2!})\frac{\partial}{\partial x},$$

whence

$$\bar{x}_1 = \langle\!\langle aX_0, aX_1 \rangle\!\rangle(\bar{x}_0) = (ax + \frac{a^2}{2!})\frac{\partial}{\partial x}(x + a) = ax + \frac{a^2}{2!}.$$

Hence,

$$\bar{x} \approx x + a + \varepsilon\Big(ax + \frac{a^2}{2}\Big).$$

Example 12.8. Let us use Theorem 12.2 in Example 12.6. Here,

$$X_0 = \frac{\partial}{\partial x}, \quad X_1 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}.$$

Therefore,

$$\bar{x}_0 = \mathrm{e}^{aX_0}(x) = x + a,$$

and

$$[X_0, X_1] = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$$

$$[X_0, [X_0, X_1]] = 2\frac{\partial}{\partial x},$$

$$[X_0, [X_0, [X_0, X_1]]] = 0, \ldots .$$

Consequently,

$$\langle\!\langle aX_0, aX_1 \rangle\!\rangle = aX_1 + \frac{a^2}{2!}(2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}) + 2\frac{a^3}{3!}\frac{\partial}{\partial x}$$

$$= (ax^2 + a^2x + \frac{a^3}{3})\frac{\partial}{\partial x} + (axy + \frac{a^2}{2}y)\frac{\partial}{\partial y}.$$

Whence

$$\bar{x}_1 = \langle\!\langle aX_0, aX_1 \rangle\!\rangle(\bar{x}_0) = (ax^2 + a^2x + \frac{a^3}{3})\frac{\partial}{\partial x}(x + a),$$

$$\bar{y}_1 = \langle\!\langle aX_0, aX_1 \rangle\!\rangle(\bar{y}_0) = (axy + \frac{a^2}{2}y)\frac{\partial}{\partial y}(y).$$

Hence,

$$\bar{x}_1 = ax^2 + a^2x + \frac{a^3}{3}\,, \quad \bar{y}_1 = axy + \frac{a^2}{2}y.$$

We thus arrive at the result of Example 12.6:

$$\overline{x} \approx x + a + \varepsilon\Big(ax^2 + a^2x + \frac{a^3}{3}\Big),$$

$$\overline{y} \approx y + \varepsilon\Big(axy + \frac{a^2}{2}y\Big).$$

Example 12.9. Consider now the generator of the generalized translation along the x^1 axis in the de Sitter space-time (see Introduction):

$$X = \Big(1+\varepsilon[(x^1)^2-(x^2)^2-(x^3)^2-(x^4)^2]\Big)\frac{\partial}{\partial x^1}+2\varepsilon x^1\Big(x^2\frac{\partial}{\partial x^2}+x^3\frac{\partial}{\partial x^3}+x^4\frac{\partial}{\partial x^4}\Big).$$

Here $X = X_0 + \varepsilon X_1$ with

$$X_0 = \frac{\partial}{\partial x^1}\,,$$

$$X_1 = \Big((x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2\Big)\frac{\partial}{\partial x^1} + \ 2x^1\Big(x^2\frac{\partial}{\partial x^2} + x^3\frac{\partial}{\partial x^3} + x^4\frac{\partial}{\partial x^4}\Big).$$

The operator X_0 generates the translation group:

$$\bar{x}_0^1 = x^1 + a, \quad \bar{x}_0^j = x^j, \ j = 2, 3, 4.$$

Now we calculate the differential of the exponential map by Eq. (3.1) applied to the operators X_0, X_1. We have:

$$[X_0, X_1] = 2x^1\Big(x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2} + x^3\frac{\partial}{\partial x^3} + x^4\frac{\partial}{\partial x^4}\Big),$$

$$[X_0, [X_0, X_1]] = 2\frac{\partial}{\partial x^1}\,, \quad [X_0, [X_0, [X_0, X_1]]] = 0, \ldots .$$

Consequently, the formula (3.3) takes the form:

$$\langle\!\langle aX_0, aX_1\rangle\!\rangle = \Big([(x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2]a + x^1a^2 + \frac{1}{3}a^3\Big)\frac{\partial}{\partial x^1}$$

$$+(2ax^1 + a^2)\Big(x^2\frac{\partial}{\partial x^2} + x^3\frac{\partial}{\partial x^3} + x^4\frac{\partial}{\partial x^4}\Big).$$

Therefore (3.2) yields

$$\bar{x}^1_1 = \langle\!\langle aX_0, aX_1\rangle\!\rangle(\bar{x}^1_0) = [(x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2]a + x^1a^2 + \frac{1}{3}a^3,$$

$$\bar{x}^j_1 = (2ax^1 + a^2)x^j, \quad j = 2, 3, 4.$$

We thus arrive at the approximate transformation given in Introduction:

$$\bar{x}^1 \approx \bar{x}^1_0 + \varepsilon x^1_1 = x^1 + a + \varepsilon\Big([(x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2]a + x^1a^2 + \frac{1}{3}a^3\Big),$$

$$\bar{x}^j \approx x^j_0 + \varepsilon x^j_1 = x^j + \varepsilon(2ax^1 + a^2)x^j, \quad j = 2, 3, 4.$$

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Paper 13

Discussion of Lie's nonlinear superposition theory

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§ 1 Lie's theorem on nonlinear superposition

It is a very interesting problem to seek, together with E. Vessiot [121] and A. Guldberg [46], all systems

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = f^i(t,x), \quad i = 1,\ldots,n, \tag{1.1}$$

whose general solutions $x = (x^1,\ldots,x^n)$ *can be expressed via* m *particular solutions* $x_1 = (x_1^1,\ldots,x_1^n),\ldots,x_m = (x_m^1,\ldots,x_m^n)$ *in the form*

$$x^i = \varphi^i(x_1,\ldots,x_m;\ C_1,\ldots,C_n), \quad i = 1,\ldots,n. \tag{1.2}$$

S. Lie, 1893

Lie (see [88], [87], [89]) solved the problem by proving the following theorem*.

Theorem 13.1. Equations (1.1) possess a nonlinear superposition if and only if they have the form (discovered by Lie [86])

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = T_1(t)\xi_1^i(x) + \cdots + T_r(t)\xi_r^i(x), \quad i = 1,\ldots,n, \tag{1.3}$$

*The reader can find a detailed presentation of Lie's theory in [59].

whose coefficients $\xi^i_\alpha(x)$ satisfy the condition that the operators

$$X_\alpha = \xi^i_\alpha(x)\frac{\partial}{\partial x^i}\,, \quad \alpha = 1,\ldots,r, \tag{1.4}$$

span a Lie algebra L_r of a finite dimension r termed the Vessiot-Guldberg-Lie algebra for equation (1.1). The number m of necessary particular solutions is estimated by

$$nm \geq r. \tag{1.5}$$

Superposition formulae (1.2) are defined implicitly by the equations

$$J_i(x, x_1, \ldots, x_m) = C_i,\ \ i = 1,\ldots,n, \tag{1.6}$$

where J_i are functionally independent (with respect to $x^1,\ldots,x^n$) invariants of the $(m+1)$-point representation

$$V_\alpha = X_\alpha + X^{(1)}_\alpha + \cdots + X^{(m)}_\alpha \tag{1.7}$$

of the operators (1.4).

In the present talk, I illustrate Lie's theorem by several examples.

§ 2 Examples on Lie's theorem

Example 13.1. Consider the homogeneous linear equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x.$$

Here $r = 1$ and the operator (1.4) has the form

$$X = x\frac{\mathrm{d}}{\mathrm{d}x}\cdot$$

We take the two-point representation (1.7) of X :

$$V = x\frac{\partial}{\partial x} + x_1\frac{\partial}{\partial x_1}$$

and its invariant $J(x, x_1) = x/x_1$. Equation (1.6) has the form $x/x_1 = C$. Hence, $m = 1$ and the formula (1.2) is the linear superposition $x = Cx_1$.

Example 13.2. The simplest generalization (1.3) of Example 13.2 is the equation with separated variables:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = T(t)h(x).$$

Here $r = 1$ and (1.4) is

$$X = h(x)\frac{\mathrm{d}}{\mathrm{d}x}\cdot$$

Taking the two-point representation V of X,

$$V = h(x)\frac{\partial}{\partial x} + h(x_1)\frac{\partial}{\partial x_1},$$

and integrating the characteristic system $\mathrm{d}x/h(x) = \mathrm{d}x_1/h(x_1)$, one obtains the invariant $J(x, x_1) = H(x) - H(x_1)$, where $H(x) = \int(1/h(x))\mathrm{d}x$. Equation (1.6) has the form $H(x) - H(x_1) = C$. Hence, $m = 1$ and the formula (1.2) provides the nonlinear superposition

$$x = H^{-1}(H(x_1) + C).$$

Example 13.3. The non-homogeneous linear equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x + B(t)$$

has the form (1.3) with $T_1 = B(t)$ and $T_2 = A(t)$. The Vessiot-Guldberg-Lie algebra (1.4) is an L_2 spanned by the operators

$$X_1 = \frac{\mathrm{d}}{\mathrm{d}x}, \quad X_2 = x\frac{\mathrm{d}}{\mathrm{d}x}\cdot$$

Substituting $n = 1$ and $r = 2$ in $nm \geq r$, we see that the expression (1.2) for the general solution requires at least two ($m = 2$) particular solutions. In fact, this number is sufficient. Indeed, let us take the three-point representation (1.7) of the basic operators X_1 and X_2 :

$$V_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, \quad V_2 = x\frac{\partial}{\partial x} + x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2},$$

and show that they admit one invariant. To find it, we first solve the characteristic system for the equation $V_1(J) = 0$, namely, $\mathrm{d}x = \mathrm{d}x_1 = \mathrm{d}x_2$. Integration yields two independent invariants, e.g. $u = x - x_1$ and $v = x_2 - x_1$. Hence, the common invariant $J(x, x_1, x_2)$ for two operators, V_1 and V_2, can be obtained by taking it in the form $J = J(u, v)$ and solving the equation $\widetilde{V}_2(J(u,v)) = 0$, where the action of V_2 is restricted to the space of the variables u, v by using the formula $\widetilde{V}_2 = V_2(u)\partial/\partial u + V_2(v)\partial/\partial v$. Noting, that $V_2(u) = x - x_1 \equiv u$ and $V_2(v) = x_2 - x_1 \equiv v$, we have

$$\widetilde{V}_2 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\cdot$$

Hence the invariant is $J(u,v) = u/v$, or $J(x, x_1, x_2) = (x - x_1)/(x_2 - x_1)$. Thus, equation (1.6) is written $(x - x_1)/(x_2 - x_1) = C$. Hence, (1.2) is the linear superposition:

$$x = x_1 + C(x_2 - x_1) \equiv (1 - C)x_1 + Cx_2.$$

Example 13.4. Consider the Riccati equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = P(t) + Q(t)x + R(t)x^2. \tag{2.1}$$

Here the Vessiot-Guldberg-Lie algebra is L_3 spanned by

$$X_1 = \frac{\mathrm{d}}{\mathrm{d}x}\,, \quad X_2 = x\frac{\mathrm{d}}{\mathrm{d}x}\,, \quad X_3 = x^2\frac{\mathrm{d}}{\mathrm{d}x}\cdot \tag{2.2}$$

We take the four-point representation of the operators (2.2),

$$V_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}\,, \quad V_2 = x\frac{\partial}{\partial x} + x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}\,,$$

$$V_3 = x^2\frac{\partial}{\partial x} + x_1^2\frac{\partial}{\partial x_1} + x_2^2\frac{\partial}{\partial x_2} + x_3^2\frac{\partial}{\partial x_3}\,, \tag{2.3}$$

and find its invariant

$$J = \frac{(x - x_2)(x_3 - x_1)}{(x_1 - x)(x_2 - x_3)}.$$

The equation $J = C$ gives the well-known nonlinear superposition.

Example 13.5. Theorem 13.1 associates with any Lie algebra a system of differential equations admitting a superposition of solutions. Consider, as an illustrative example, the three-dimensional algebra spanned by

$$X_1 = \frac{\partial}{\partial x}\,, \quad X_2 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\,, \quad X_3 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}\cdot \tag{2.4}$$

The system (1.3) corresponding to the operators (2.4) is written*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = T_1(t) + 2T_2(t)x + T_3(t)x^2, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = T_2(t)y + T_3(t)xy. \tag{2.5}$$

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*The operators (2.4) span a subalgebra of the eight-dimensional Lie algebra of the projective group on the plane. Accordingly, the first equation of the system (2.5) is the Riccati equation (2.1) with $P = T_1, Q = 2T_2, R = T_3$.

In other words, the operators (2.4) span the Vessiot-Guldberg-Lie algebra L_3 for the system (2.5). The estimation $nm \geq r$ with $n = 2, r = 3$ determines the minimum $m = 2$ of necessary particular solutions. Consequently, we take the three-point representation of the operators (2.4):

$$\begin{aligned}
V_1 &= \frac{\partial}{\partial x} + \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\,,\\
V_2 &= 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2x_1\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial y_1} + 2x_2\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial y_2}\,,\\
V_3 &= x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + x_1^2\frac{\partial}{\partial x_1} + x_1y_1\frac{\partial}{\partial y_1} + x_2^2\frac{\partial}{\partial x_2} + x_2y_2\frac{\partial}{\partial y_2}\cdot
\end{aligned}$$

The operator V_1 provides five invariants:

$$y,\ y_1,\ y_2,\ z_1 = x_1 - x,\ z_2 = x_2 - x_1.$$

Restricting V_2 to these invariants, one obtains the dilation generator

$$\widetilde{V}_2 = 2z_1\frac{\partial}{\partial z_1} + 2z_2\frac{\partial}{\partial z_2} + y\frac{\partial}{\partial y} + y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2}\cdot$$

Its independent invariants are $u_1 = z_2/z_1, u_2 = y^2/(x_1 - x), u_3 = y_1^2/(x_1 - x)$, and $u_4 = y_2^2/(x_1 - x)$. Substituting the expression for z_1 and z_2, one obtains the following basis of the common invariants for V_1 and V_2 :

$$u_1 = \frac{x_2 - x_1}{x_1 - x}\,, \quad u_2 = \frac{y^2}{x_1 - x}\,, \quad u_3 = \frac{y_1^2}{x_1 - x}\,, \quad u_4 = \frac{y_2^2}{x_1 - x}\cdot$$

It remains to find the restriction $\widetilde{V}_3$ of V_3 to the above invariants by the formula

$$\widetilde{V}_3 = V_3(u_1)\frac{\partial}{\partial u_1} + \cdots + V_3(u_4)\frac{\partial}{\partial u_4}\cdot$$

The reckoning shows that

$$V_3(u_1) = \frac{(x_2 - x_1)(x - x_2)}{x - x_1} \equiv (x_1 - x)(1 + u_1)u_1,\ V_3(u_3) = y_1^2 \equiv (x_1 - x)u_3,$$

$$V_3(u_2) = -y^2 \equiv -(x_1 - x)u_2,\ V_3(u_4) = \frac{x + x_1 - 2x_2}{x - x_1}\,y_2^2 \equiv (x_1 - x)(1 + 2u_1)u_4.$$

Hence,

$$\widetilde{V}_3 = \Big(x_1 - x\Big)\bigg((1 + u_1)u_1\frac{\partial}{\partial u_1} - u_2\frac{\partial}{\partial u_2} + u_3\frac{\partial}{\partial u_3} + (1 + 2u_1)u_4\frac{\partial}{\partial u_4}\bigg).$$

Consequently, the equation $\widetilde{V}_3\psi(u_1,\dots,u_4)=0$ is equivalent to

$$(1+u_1)u_1\frac{\partial\psi}{\partial u_1}-u_2\frac{\partial\psi}{\partial u_2}+u_3\frac{\partial\psi}{\partial u_3}+(1+2u_1)u_4\frac{\partial\psi}{\partial u_4}=0,$$

whence, by solving the characteristic system

$$\frac{\mathrm{d}u_1}{(1+u_1)u_1}=-\frac{\mathrm{d}u_2}{u_2}=\frac{\mathrm{d}u_3}{u_3}=\frac{\mathrm{d}u_4}{(1+2u_1)u_4}\,,$$

one obtains the following three independent invariants:

$$\psi_1=u_2u_3\equiv\frac{y^2y_1^2}{(x_1-x)^2}\,,\ \psi_2=\frac{u_1u_2}{1+u_1}\equiv\frac{(x_2-x_1)y^2}{(x_1-x)(x_2-x)}\,,$$
$$\psi_3=\frac{u_4}{(1+u_1)u_1}\equiv\frac{(x_1-x)y_2^2}{(x_2-x_1)(x_2-x)}\,\cdot$$

Hence, the general nonlinear superposition (1.6), involving two particular solutions, (x_1,y_1) and (x_2,y_2), is written

$$J_1(\psi_1,\psi_2,\psi_3)=C_1,\quad J_2(\psi_1,\psi_2,\psi_3)=C_2,\tag{2.6}$$

where J_1 and J_2 are arbitrary functions of three variables such that their Jacobian with respect to x,y does not vanish identically. Letting, e.g. $J_1=\sqrt{\psi_1}$ and $J_2=\sqrt{\psi_2\psi_3}$, i.e. specifying (2.6) in the form

$$\frac{yy_1}{x_1-x}=C_1,\quad\frac{yy_2}{x_2-x}=C_2,$$

one arrives at the following representation of the general solution via two particular solutions:

$$x=\frac{C_1x_1y_2-C_2x_2y_1}{C_1y_2-C_2y_1}\,,\quad y=\frac{C_1C_2(x_2-x_1)}{C_1y_2-C_2y_1}\,\cdot$$

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