Nail H. Ibragimov

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III

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Nail H. Ibragimov

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Volume III

Doctor of Science thesis
Approximate symmetries
Lie groups in mathematical modelling

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Paper 1

Lie groups in some problems of mathematical physics

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Abstarct. This course of lectures is dedicated to applications of Lie groups to various problems of mathematical physics. One of basic topics is development of a general theory of invariants of arbitrary continuous transformation groups in Riemannian spaces. Then we deal with Hadamard's problem on finding differential equations satisfying the Huygens principle. A solution to Hadamard's problem is given in the class of equations possessing a nontrivial conformal group. Finally, we discuss the questions on derivation of conservation laws for differential equations.

In addition to the traditional course of equations of mathematical physics the reader is supposed to be acquainted with foundations of the theory of Lie group analysis of differential equations.

Preface

About a hundred years ago Sophus Lie started to investigate continuous transformation groups. One of the reasons that incited him to develop this

Author's note to the English translation: I edited the translation of Chapters 1, 2 and 5. I also made small changes in these chapters.

new field of mathematics was an attempt to extend the Galois theory for algebraic equations to differential equations. Another reason was to dwell upon properties of transformations in geometry and mechanics.

In the beginning (1869–1871) S. Lie investigated known at the time examples of continuous groups, namely, groups of motions (translations, rotations) and projective transformations in geometry, tangent transformations, etc. He introduced the general notion of continuous transformation groups in 1871, and by 1884 he developed the well-known theory of continuous local groups for the most part. This theory together with its most important applications was represented in the fundamental work of Lie and Engel [91], [92], [94].

From the very beginning the initiation and development of the theory of continuous groups was encouraged by its successful application in various fields of mathematics and mechanics. Already in 1869 Lie noticed that the majority of ordinary differential equations with known methods for their integration admit certain continuous transformation groups. Then he discovered that similar relations exist for partial differential equations of the first order. Later on, together with elaborating the general theory of continuous groups, S. Lie took up systematic investigation of differential equations admitting continuous transformation groups. The results obtained in this field ([89], [90], [95], [91], [92], [94], [131], [137], [109], [110], [111] etc.) led to the modern theory of group properties of differential equations.

Our prime interest will be the following two applications of Lie group theory. The first one is the theory of continuous groups of motions in Riemannian spaces; developed by Killing [82], the theory proved to be extremely efficient both in geometry and in related interdisciplinary issues. The second one is the Noether theorem [107] on existence of conservation laws for the Euler-Lagrange equations of functionals, invariant with respect to continuous transformation groups.

The present lecture notes are devoted to further consideration of issues related to application of Lie group theory in Riemannian geometry and theory of differential equations. The basic problems considered below focus on the following topics:

1. Theory of invariants of arbitrary continuous transformation groups in Riemannian spaces;

2. Group properties of linear and nonlinear differential equations of the second order;

3. The Huygens principle for linear hyperbolic differential equations of the second order;

4. Connections between invariance and conservation laws for differential equations.

We take the notions of groups of *isometric motions* (often called briefly *motions*) and *conformal transformations* in Riemannian spaces as a starting point of our investigation of the first topic. These notions can be formulated as follows. A continuous group G of point transformations in an n-dimensional Riemannian space V_n is called a group of isometric motions in the space V_n if transformations of the group G keep the values of all the components

$$g_{ij}, \quad i, j = 1, \ldots, n,$$

of the metric tensor of the space V_n unaltered. Likewise, a group G of transformations in V_n is called a group of conformal transformations if transformations of the group G preserve the values of all ratios

$$\frac{g_{ij}}{g_{pq}}$$

of the components of the metric tensor of the space V_n. These definitions lead to the following formulation of the problem of invariants of continuous transformation groups in Riemannian spaces.

Given an arbitrary continuous point transformation group G in a Riemannian space V_n. Determine which functions of components of the metric tensor g_{ij} of the space V_n are invariant with respect to all transformations of the group G.

In connection with this problem I introduce the following two basic notions: the *defect* $\delta = \delta(V_n, G)$ and the *invariant family of spaces* $G(V_n)$.

The defect serves to determine the number of all functionally independent invariant functions for the group G of transformations in the space V_n; this number equals to

$$\frac{1}{2}\, n(n+1) - \delta.$$

The invariant family of spaces $G(V_n)$ is the smallest set of n-dimensional Riemannian spaces which is invariant with respect to transformations of the "extended" group $\overline{G}$ (see § 6) and contains the space V_n.

For instance, if G is a group of isometric motions in V_n, then $\delta = 0$ and the invariant family of spaces consists of one space V_n, i.e. $G(V_n) = V_n$. It is also manifest that for groups of conformal transformations $\delta = 1$ and the invariant family of spaces $G(V_n)$ coincides with the family of all spaces conformal to V_n.

Let us turn to the second topic. Although the problem of group classification of partial differential equations of the second order has been repeatedly considered in literature, it is not solved completely yet. At the same time, different problems of geometry and mathematical physics prompt some classes of second-order equations which are of particular interest to researchers. One of these classes consists of equations with two independent variables. Group properties of these equations were investigated by Lie [89] (linear equations) and Medolaghi [104] (linear equations admitting an infinite group). Another important class is composed by linear homogeneous second-order equations

$$g^{ij}(x)u_{ij} + b^i(x)u_i + c(x)u = 0$$

with $n \geq 2$ independent variables $x = (x', \ldots, x^n)$. Group properties of these equations were investigated by L.V. Ovsyannikov [110], [111].

We will dwell upon group properties of *semi-linear* partial differential equations of the second order

$$g^{ij}(x)u_{ij} + b^i(x)u_i + \psi(x, u) = 0$$

with an arbitrary number $n \geq 2$ of independent variables. Coefficients of the equation are supposed to be analytic functions of x. Equations of this type, occurring in relativistic quantum mechanics [124], have been considered by several authors (Jörgens [80], Strauss [128], Lions [99]). In these works, an exceptional role in various respects plays the *nonlinear wave equation*

$$\Box u + u^3 = 0$$

which is invariant, like the usual linear wave equation

$$\Box u = 0,$$

with respect to the group of conformal motions in the flat space V_n. Equations of the second order invariant with respect to the conformal transformations in the space V_n with the metric tensor $g_{ij}(x)$ (where $g_{ij}g^{jk} = \delta_i^k$) will play a significant part in what follows. Therefore, together with investigating group properties of general equations, we will dwell upon these "conformally invariant" equations in different spaces V_n.

Group invariance properties of second-order equations are closely related to the *Huygens principle* which is understood here in the sense of Hadamard's "minor premise" [41], [42], [43], [44]. Namely, a linear hyperbolic differential equation of the second order is said to satisfy the Huygens principle if the solution of the arbitrary Cauchy problem for this equation is defined at every point x by the Cauchy data at the intersection of the initial manifold with the characteristic conoid outgoing from x. The notion of the Huygens principle in the above sense appeared in works of Kirchhoff [83], Beltrami [15], [16], who demonstrated on the basis of the explicit formula for the solution of the Cauchy problem that the wave equation

$\Box u = 0$ satisfies the Huygens principle. Hadamard formulated the problem of finding all linear hyperbolic equations of the second order satisfying the Huygens principle. Until recently only separate examples of such equations have been known [126], [127], [39], [71]. According to Hadamard [41], [42], [43], [44], [45], the Huygens principle holds only for an even number $n \geq 4$ of independent variables. In this connection, the case $n = 4$, having a direct physical meaning, is of special interest. An important result for this case was obtained by Mathisson [102]. Approximately at the same time and independently of M. Mathisson the same result was obtained by L. Asgeirsson [6], A. Douglis [28] but published much later. Another proof of the same result was given by Hadamard [46]. He demonstrated that in the four-dimensional flat space V_4 (this corresponds to the case of equations whose principal part has constant coefficients) only the wave equation satisfies the Huygens principle. This solves Hadamard's problem for the flat space V_4.

We will also consider the case $n = 4$ and demonstrate that in any Riemannian space V_4, having a "non-trivial" conformal group (i.e. the conformal group which is not a group of motions in any space conformal to the space V_4), only the conformally invariant equation satisfies the Huygens principle.

Further, issues related to the well-known Noether theorem [107] on conservation laws for differential equations are considered. The theorem states that if a functional

$$l[u] = \int\limits_{\Omega} \mathcal{L}(x, u, u', \ldots) dx$$

is invariant with respect to an r-parameter continuous transformation group G_r in the space of variables

$$x = (x', \ldots, x^n), \quad u = (u^1, \ldots, u^m),$$

then there exist r independent conservation laws for the Euler-Lagrange equations of the functional $l[u]$. Meanwhile a definite formula is given for calculating the conserved quantities via the Lagrange function $\mathcal{L}$ and coordinates of infinitesimal generators of the group G_r. In what follows we will deal with conservation laws given only by these formulae. Keeping in mind that the Euler-Lagrange equations in this case are also invariant with respect to the group G_r, one can see that the Noether theorem provides the *sufficient* condition for the group G_r, admitted by the Euler-Lagrange equations of the functional $l[u]$, to correspond to r conservation laws. Such condition is the invariance of the value of the functional $l[u]$ on *all* (smooth) functions $u = u(x)$ with respect to the group G_r. Examples demonstrate that such invariance of the functional is not a *necessary* condition for existence of the mentioned conservation laws, and hence, all these conservation laws can be derived from the Noether theorem. This leads to the problem under consideration on finding the *necessary and sufficient* conditions of existence of conservation laws.

The whole material is divided into five chapters. The first chapter introduces the basic notions of the theory of continuous groups, group symmetry properties of differential equations and Riemannian geometry.

The second chapter considers the problem of invariants of continuous transformation groups in Riemannian spaces. The most significant result for the theory is formulated in Theorem 1.12 of Section 8.2. It allows one to calculate the defect $\delta(V_n, G)$ via components of the metric tensor of the space V_n and coordinates of basic infinitesimal generators of the group G. In particular cases it is possible to construct explicitly an invariant family of spaces $G(V_n)$, which allows to construct all invariants. This is illustrated in a number of examples in § 9 and § 10.

The third chapter deals with symmetries of second-order linear and semilinear partial differential equations with several variables. Upon obtaining the determining equations for symmetries in the general case, we turn to considering conformally invariant equations. The theorem on uniqueness of conformally invariant equations in spaces V_4 with nontrivial conformal group (see § 14) proved here is used in the following chapter.

The fourth chapter presents the solution of Hadamard's problem for differential equations with four independent variables when the corresponding spaces V_4 have a nontrivial conformal group. In particular, the chapter includes the case considered by M. Mathisson, because any flat space V_4 has a nontrivial conformal group.

The fifth chapter tackles the problem on conservation laws. The proof of the following statement (see Theorem 1.22 in Section 22.2) holds a central position in this chapter: when the Euler-Lagrange equations of the functional $l[u]$ admit a group G_r, then this group furnishes r conservation laws if and only if the *extremal values* of the functional $l[u]$ are invariant with respect to the group G_r.

Further we mostly use standard notation. In case if a new notation is introduced it is explained in the text. The functions occurring in the text are sufficiently smooth, unless otherwise stated. Note also that all considerations are local.

The author is grateful to L.V. Ovsyannikov for discussing of basic results of the present work in different times.

Chapter 1
Preliminaries

This chapter gives the fundamental notions from the theory of continuous groups, symmetry properties of differential equations and Riemannian geometry.

Literature: Pontryagin [117], Ovsyannikov [109, 110], Eisenhart [31, 29].

§ 1 Continuous groups of transformations

1.1 Local groups

Let us consider a Hausdorff topological space G. The system of local coordinates in space G is the pair (u, φ), consisting of the open set $U \subset G$ and of a topological mapping of φ of the set u on the open set of the r-dimensional Euclidian space R^r. The open set U is referred to as a coordinate vicinity, and the real numbers x^i, $(i = 1, \ldots, r)$, being coordinates of the point $\varphi(x) \in R^r$, are called coordinates of the point $x \in G$ in the system of coordinates under consideration.

Definition 1.1. A topological space G is called a local group if there exist an element (unity) $e \in G$, the neighborhoods U, V (where $V \subset U$) of the element e and if the group operation $U \times U \to G$ is defined so that

1) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in V$.
2) $e \cdot a = a \cdot e = a$ for all $a \in U$.
3) For any $a \in V$ there exists an inverse element $a^{-1} \in U$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$.
4) The mapping $(a, b) \to a \cdot b^{-1}$ is continuous on $U \times V$.

Let us consider a local group G and assume that it has a known system of local coordinates (U, φ), where $e \in U$, and φ is such a topological mapping of the coordinate vicinity U on the open sphere R^2 with the center at the point $0 \in R^2$, that $\varphi(e) = 0$. Let $W \in V$ be such a vicinity of the unity e that the mapping $W \times W \to U$ is known and it is analytical, i.e. the coordinates c^i of the element $c = a \cdot b$ are analytical functions $c^i = \psi^i(a, b)$ $(i = 1, \ldots, r)$ of the coordinates a^i, b^i for the elements $a, b \in W$. Then, one can say that analytical coordinates are introduced in the local group G.

Definition 1.2. A local group G where analytical coordinates are introduced is called an r-dimensional local Lie group and is denoted by G_r.

1.2 Local transformation groups

Consider a mapping

$$f : R^n \times B \to R^n, \tag{1.1}$$

where $B \subset R^2$ is an open sphere with the center $0 \in R^r$, and define the mapping T_a of the space R^n into itself by the equation

$$T_a x = f(x, a); \quad x \in R^n, \ a \in B. \tag{1.2}$$

Definition 1.3. The set G_r of transformations (1.2) is called a continuous r-parameter local transformation group in R^2 if G_r is a local Lie group with respect to the group operation defined by

$$(T_a \cdot T_b)x = T_a(T_b x).$$

For the sake of simplicity we assume that the mapping (1.1) meets the condition $f(x, 0) = x$ so that T_0 is a unity of the group G_r. In what follows, continuous local transformation groups will be simply referred to as groups and will be considered in canonical coordinates.

1.3 Lie's theorem for one-parameter groups

Let the mapping (1.1) determine a one-parameter transformation group G_1 in the space R^n. Let us define the vector field $\xi : R^n \to R^n$ by the formula

$$\xi(x) = \left.\frac{\partial f(x, a)}{\partial a}\right|_{a=0}. \tag{1.3}$$

Theorem 1.1. (*Lie's theorem*). The function $f(x, a)$ satisfies the equation

$$\frac{\partial f}{\partial a} = \xi(f).$$

Conversely, for any continuously differentiable vector field $\xi : R^n \to R^n$, the solution f of the system of first-order ordinary differential equations

$$\frac{df}{da} = \xi(f) \tag{1.4}$$

with the initial condition

$$f|_{a=0} = x \tag{1.5}$$

determines a one-parameter group.

The definition of the vector $\xi(x)$ yields

$$T_a x = x + a\xi(x) + o(a).$$

Therefore, the increment of function $F : R^n \to R^m$ ubnder the transformations of the group G_1 is

$$\Delta F(x) \equiv F(T_a x) - F(x) = aF'(x) \cdot \xi(x) + o(a),$$

where

$$F' = \left\| \frac{\partial F^k}{\partial x^l} \right\|$$

is the derivative of the mapping F. Let us introduce the infinitesimal generator (or simply the *generator*, for the sake of brevity) of the group G_1 :

$$X = \xi^i(x) \frac{\partial}{\partial x^i}, \tag{1.6}$$

acting on functions $F = F(x)$ by the rule

$$XF(x) = \xi^i(x) \frac{\partial F(x)}{\partial x^i} \equiv F'(x) \cdot \xi(x),$$

and write

$$\Delta F = a\, F(x) + o(a).$$

It follows from this equations that

$$\left. \frac{\partial F(T_a x)}{\partial a} \right|_{a=0} = XF(x). \tag{1.7}$$

1.4 Manifolds in Euclidean spaces

Let us accept the following geometrically illustrative definition of a manifold in the Euclidian space consistent with the general definition of a manifold.

Definition 1.4. A set $M \subset R^n$ is called a p-dimensional manifold in the space R^n $(p \le n)$ if for any point $x \in M$ there exist an open manifold $U \subset R$ containing x, the open manifold $V \subset R$ and a diffeomorphism $\varphi : U \to V$ such that

$$\varphi(U \cap M) = \{x \in V : x^{p+1} = \ldots = x^n = 0\}.$$

It follows from this definition that for any point x of the manifold M there exist an open set $U \subset R^n$ containing x, an open set $W \subset R^p$, and a one-to-one differentiable mapping $g : W \to R^n$ (a coordinate system in the vicinity of the point x) such that

$$g(W) = U \cap M \tag{1.8}$$

and

$$\operatorname{rank} g'(y) = p \quad \text{for all} \quad y \in W.$$

Definition 1.5. Let $g : W \to R^n$ be a coordinate system in the vicinity of the point $x = g(y)$ of the p-dimensional manifold M in the space R^n. Let us denote by V_z^m the m-dimensional vector space of m-dimensional vectors $dz \in R^m$ with the origin at the point $z \in R^m$. The condition (1.8) guarantees that the set

$$M_x = \{dx \in V_x^n : \ dx = g'(y)dy, \quad dy \in V_y^p\}$$

is a p-dimensional subspace of the space V_x^n and does not dependent on the choice of a coordinate system. The vector space M_x is called the *tangent space* to the manifold M at the point x. Elements of M_x are termed *tangent vectors* to the manifold M at the point x.

The following theorem will be useful.

Theorem 1.2. Let $\psi : R^n \to R^{n-p}$ be a differentiable mapping such that

$$\operatorname{rank} \psi'(x) = n - p$$

for all $x \in R^n$ satisfying the equation

$$\psi(x) = 0. \tag{1.9}$$

Then the set $M = \psi^{-1}(0)$ of all solutions of the equation (1.9) is a p-dimensional manifold in the space R^n.

1.5 Invariant manifolds

Definition 1.6. A manifold $M \subset R^n$ is referred to as an *invariant manifold* for the transformation group G_r in the space R^n if $Tx \in M$ for any point $x \in M$ and any transformation $T \in G_r$.

Theorem 1.3. Let G_1 be a one-parameter group of transformations in R^n with the vector field ξ defined by Eq. (1.3). A manifold $M \subset R^n$ is an invariant manifold for the group G_1 if and only if the vector $\xi(x)$ belongs, at every point $x \in M$, to the space M_x tangent to M at the point x, i.e.

$$\xi(x) \in M_x \quad \text{for} \quad \text{all} \quad x \in M. \tag{1.10}$$

Corollary 1.1. The manifold M defined by Eq. (1.9) is invariant with respect to the group G_1 with the generator (1.6) if and only if

$$X(\psi)\big|_M = 0. \tag{1.11}$$

1.6 Invariants

A simpler notion of *group invariants* plays an important part in the study of invariant manifolds.

Definition 1.7. A function $J : R^n \to R$ is referred to as an *invariant* of the transformation group G_r in R^n if

$$J(Tx) = J(x)$$

for any point $x \in R^n$ and for any transformation $T \in G_r$.

Theorem 1.4. The necessary and sufficient condition for the function J to be an invariant of the group G_1 with the operator (1.6) is

$$XJ = 0. \tag{1.12}$$

Thus, in order to find the invariants of the group G_1 one should solve the linear homogeneous partial differential equation of the first order

$$\xi^i(x)\frac{\partial J}{\partial x^i} = 0.$$

This equation has $n-1$ functionally independent solutions and the general solution is their arbitrary function.

§ 2 Lie algebras

2.1 Definition of an abstract Lie algebra

Definition 1.8. A *Lie algebra* is a vector space L with a given bilinear multiplication law (the product of the elements $a, b \in L$ is usually denoted by $[ab]$ and is termed the *commutator* of these elements) which satisfies the *skew symmetry* property

$$[ab] = -[ba]$$

and the *Jacobi identity*

$$[[ab]c] + [[bc]a] + [[ca]b] = 0.$$

If the vector space L is finite-dimensional and its dimension is $\dim L = r$, then the corresponding Lie algebra is called an r-dimensional Lie algebra and is denoted by L_r. If $e_1, \ldots, e_r$ is a basis of the vector space L of the Lie algebra L_r then

$$[e_ie_j] = c_{ij}^k e_k,$$

where $c_{ij}^k (i, j, k = 1, \ldots, r)$ are real constants called structure constants of the Lie algebra L_r.

In what follows only finite-dimensional Lie algebras will be considered, unless otherwise stated.

2.2 Lie algebras of r-parameter groups

Let us consider Lie algebras corresponding to continuous transformation groups. Let G_1 be a one-parameter subgroup of the group G_r of transformations (1.2) in the space R^n and assume that ξ is the vector (1.3) corresponding to this one-parameter subgroup. Selecting various one-parameter subgroups of G_r one obtains an r-dimensional vector space L_r of the vectors ξ with usual summation and multiplication by real numbers. The vectors

$$\xi_\alpha(x) = \frac{\partial f(x,a)}{\partial a^\alpha}\bigg|_{a=0}, \quad \alpha = 1,\ldots,r, \tag{2.1}$$

where $a = (a^1,\ldots,a^r)$ is the parameter of the group G_r, can be taken as the basis of the vector space L_r.

Theorem 1.5. The set G_r of transformations (1.2) is an r-parameter continuous local group if and only if the vector space L_r of the vector fields ξ is a Lie algebra with respect to the product defined by the formula

$$[\xi\eta](x) = \eta'(x)\xi(x) - \xi'(x)\eta(x), \tag{2.2}$$

where ξ' and η' are derivatives of the maps ξ and η, respectively.

This theorem simplifies the study of continuous transformation groups by reducing the problem to the study of Lie algebras.

Often it is more convenient to consider a Lie algebra of the corresponding linear operators (1.6) instead of a Lie algebra of the vectors (1.3). In this case, the linear combination $\lambda X + \mu Y$ of the operators

$$X = \xi^i \frac{\partial}{\partial x^i}, \quad Y + \eta^i \frac{\partial}{\partial x^i}$$

corresponds to the linear combination $\lambda\xi + \mu\eta$ of the vectors ξ and η with real constant coefficients λ and μ. The *commutator* of the operators,

$$[XY] = XY - YX, \tag{2.3}$$

where XY is the usual composition of linear operators, corresponds to the multiplication (2.2). In coordinates, the commutator (2.3) is written

$$[XY] = (X\eta^i - Y\xi^i)\frac{\partial}{\partial x^i}\,\cdot \tag{2.4}$$

§ 3 Defect of invariance

3.1 Definition of the defect

Let us assume that G is a group of transformations in the space R^n, and $M \subset R^n$ is a p-dimensional manifold. Consider the problem of a (local) variation of the manifold M under transformations of the group G.

We denote the manifold obtained form M via the transformation $T \in G$ by $T(M)$, and the manifold obtained form M via all transformations of the group G by $G(M)$. Thus:

$$T(M) = \underset{x\in M}{\cup} Tx, \quad G(M) = \underset{T\in G}{\cup} T(M). \tag{3.1}$$

Definition 1.9. (By L.V. Ovsyannikov [112], §17). The number

$$\delta = \dim G(M) - \dim M \tag{3.2}$$

is called the *invariance defect* (or simply *defect*) of the manifold M with respect to the group G.

In order to specify the dependence of the invariance defect δ on the manifold M and the group G we use the notation $\delta = \delta(M, G)$.

Let us assume that the manifold M is given by Eq. (1.9) and the group G_r has generators

$$X_\alpha = \xi^i_\alpha(x)\frac{\partial}{\partial x^i}, \quad \alpha = 1, \ldots, r. \tag{3.3}$$

The following theorem provides a convenient method for calculating the invariance defect of the manifold M with respect to the group G_r.

Theorem 1.6. The invariance defect of the manifold M with respect to the group G_r is given by [113]

$$\delta(M, G_r) = \operatorname{rank} \|X_\alpha \Psi^\sigma\|_M. \tag{3.4}$$

Here the index M means that the rank of the matrix $\|X_\alpha \Psi^\sigma\|$ is calculated at points of the manifold M, and the rank is considered as the *general rank.*

3.2 Partially invariant manifolds

The following theorem extends Theorem 1.3 to r-parameter groups.

Theorem 1.7. A manifold $M \subset R^n$ is an invariant manifold of the group G_r if and only if that vectors $\xi(x)$ of the corresponding Lie algebra L_r are contained in the tangent space M_x at every point $x \in M$:

$$\xi(x) \in M_x \quad \text{for} \quad \text{all} \quad \xi \in L_r, \quad x \in M.$$

Corollary 1.2. The manifold M given by Eq. (1.9) is invariant under the group G_r with generators (3.3) if and only if

$$X_\alpha \Psi|_M = 0 \quad (\alpha = 1, \ldots, r). \tag{3.5}$$

Then we also say that Eq. (1.9) is invariant under the group G_r.

By virtue of Eq. (3.4), one can write the invariant manifold test (3.5) in the form

$$\delta(M, G_r) = 0.$$

If this condition is not satisfied and if $0 < \delta(M, G_r) < n - \dim M$, then the manifold M is called a *partially invariant manifold* of the group G_r. In this case the defect δ is the codimension of the manifold M in the smallest invariant manifold of the group G_r containing M. The latter invariant manifold is $G_r(M)$ by construction.

3.3 Nonsingular invariant manifolds

Theorem 1.8. A function $J : R^n \to R$ is an invariant of the group G_r with generators (3.3) if and only if

$$X_\alpha J = 0, \quad \alpha = 1, \ldots, r. \tag{3.6}$$

Furthermore, if (locally)

$$\operatorname{rank} \|\xi_\alpha^i(x)\| = R,$$

then the group G_r has $t = n - R$ functionally independent invariants $J^1, \ldots, J^t$ (a complete set of invariants), and any other invariant of the group G_r is a function of these basic invariants.

Definition 1.10. An invariant manifold M of the group G_r is said to be nonsingular if (locally)

$$\operatorname{rank} \|\xi_\alpha^i\|_M = \operatorname{rank} \|\xi_\alpha^i\|.$$

In what follows, only nonsingular invariant manifolds given by Eq. (1.9) are considered.

Theorem 1.9. Any nonsingular invariant manifold can be given by an equation of the form

$$\Phi(J^1, \ldots, J^t) = 0, \tag{3.7}$$

where $J^1, \ldots J^t$ is a basis of invariants of the group G_r.

Proof. See [112], Theorem 31 in §14.

Theorem 1.9 allows one to construct all invariant manifolds of the group G_r. To this end, it is sufficient to find a basis of invariants of the group G_r by solving the characteristic systems for Eqs. (3.6) and to consider the general equation of the form (3.7). This procedure for obtaining invariant manifolds is used when basic generators of the group G_r are given. If, vice versa, a manifold M is given then the group G_r leaving invariant the manifold M is is obtained from Eqs. (3.5).

§ 4 Symmetries of differential equations

4.1 Prolongations of groups and their generators

Let $x = (x^1, \ldots, x^n)$ and $u = (u^1, \ldots u^m)$ be independent and dependent variables, respectively, and R^{n+m} be the space of all variables (x, u). We use the usual notation for partial derivatives:

$$u_i^k = \frac{\partial u^k}{\partial x^i} \quad (i = 1, \ldots, n;\ k = 1, \ldots, m),$$

so that the derivative of a map

$$u : R^n \to R^m$$

is the matrix

$$u' = \|u_i^k\|.$$

We will also identify u' with the set of all partial derivatives. If

$$\Phi : R^{n+m} \to R^l$$

and the map

$$F : R^n \to R^l$$

is defined by the equation

$$F(x) = \Phi(x, u(x)),$$

then Φ'_x stands for the "partial" derivative of Φ with respect to x when u is fixed, and Φ'_u denotes the "partial" derivative of Φ with respect to u when x is fixed. In this notation, denoting $\Phi' \equiv F'$, we have:

$$\Phi' = \Phi'_x + \Phi'_u \cdot u'. \tag{4.1}$$

In coordinates, Eq. (4.1) is written

$$D_i(\Phi) = \frac{\partial \Phi}{\partial x^i} + u_i^k \frac{\partial \Phi}{\partial u^k} \cdot$$

Hence, D_i is the differential operator ("total differentiation" in x^i)

$$D_i = \frac{\partial}{\partial x^i} + u_i^k \frac{\partial}{\partial u^k} \cdot$$

Let G_r be a continuous transformation group

$$\begin{aligned} \bar{x} &= f(x, u, a), \quad f(x, u, o) = x, \\ \bar{u} &= \varphi(x, u, a), \quad \varphi(x, u, o) = u \end{aligned} \tag{4.2}$$

in the space R^{n+m}. The transformation (4.2) will also affect the derivatives u_i^k. This yields a continuous transformation group of the points (x, u, u') which is called the "first prolongation" of the group G_r and is denoted by $\widetilde{G}_r$. If G_1 is a one-parameter subgroup of the group G_r with the generator

$$X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^k(x, u) \frac{\partial}{\partial u^k}\,, \tag{4.3}$$

where

$$\xi^i = \left. \frac{\partial f^i(x, u, a)}{\partial a} \right|_{a=0}, \quad \eta^k = \left. \frac{\partial \varphi^k(x, u, a)}{\partial a} \right|_{a=0}, \tag{4.4}$$

then the generator of the corresponding one-parameter subgroup $\widetilde{G}_1$ of the group $\widetilde{G}_r$ has the form

$$\widetilde{X} = X + \zeta_i^k \frac{\partial}{\partial u^k}, \tag{4.5}$$

where $\zeta = \|\zeta_i^k\|$ is given by

$$\zeta = \eta' - u' \cdot \xi'. \tag{4.6}$$

According to Equation (4.1), the "prolongation formula" (4.6) is written:

$$\zeta = \eta'_x + \eta'_u \cdot u' - u'(\xi'_x + \xi'_u \cdot u'),$$

and in the coordinate notation it has the form:

$$\zeta_i^k = \frac{\partial \eta^k}{\partial x^i} + u_i^l \frac{\partial \eta^k}{\partial u^l} - u_j^k \left(\frac{\partial \xi^j}{\partial x^i} + u_i^l \frac{\partial \xi^j}{\partial u^l} \right), \tag{4.7}$$

or

$$\zeta_i^k = D_i(\eta^k) - u_j^k D_i(\xi^j),$$

where $i = 1, \ldots, n;\ k = 1, \ldots, m$. If one considers derivatives up to the order q instead of the first-order derivatives, one obtains a continuous transformation group referred to as a q-th prolongation of the group G_r.

4.2 Groups admitted by differential equations

Let $u^{(\sigma)}$ be the set of all partial derivatives of the order σ of the variables $u^1, \ldots, u^m$ with respect to $x^1, \ldots, x^n$.

Definition 1.11. A system of differential equations

$$F^\nu(x, u, \ldots, u^{(q)}) = 0 \quad (\nu = 1, \ldots, N) \tag{4.8}$$

is said to *admit a group* G_r of transformations (4.2) or to be *invariant* with respect to the group if the manifold in the space of variables $(x, u, \ldots, u^{(q)})$ determined by Eqs. (4.8) is invariant with respect to the qth-order prolongation of the group G_r.

The basic property of the group G_r admitted by equations (4.8) is that any transformation of the group G_r maps every solution of Eqs. (4.8) into a solution of the same equations.

Let us consider the problem of finding a group admitted by a given system of differential equations restricting ourselves to the case of first-order equations

$$F^\nu(x, u, u') = 0 \quad (\nu = 1, \ldots, N). \tag{4.9}$$

The condition of invariance for equations (4.9) with respect to the group G_1 with the infinitesimal generator (4.3) has the form

$$\widetilde{X} F^\nu \big|_{(4.9)} = 0 \quad (\nu = 1, \ldots, N),$$

or, in in the expanded form:

$$\left(\xi^i \frac{\partial F^\nu}{\partial x^i} + \eta^k \frac{\partial F^\nu}{\partial u^k} + \zeta_i^k \frac{\partial F^\nu}{\partial u_i^k} \right) \bigg|_{(4.9)} = 0. \tag{4.10}$$

Substituting the values of ζ^k form (4.7) in Eqs. (4.10) one obtains a system of linear homogeneous differential equations with respect to the unknown functions $\xi^i(x,u)$ and $\eta^k(x,u)$ which is called the *determining equations* for the group G_1 admitted by Eqs. (4.9).

An important property of determining equations is that the complete set of their solutions generates a Lie algebra with respect to the product (2.2). Therefore, as it follows from Theorem 1.5, the family of the corresponding transformations (4.2) is a continuous local group. The resulting group G_r is the *widest transformation group of the form (4.2) admitted by the system of equations (4.10).*

Likewise, one can obtain the determining equations for the group admitted by systems of higher-order differential equations. The procedure requires the qth-order prolongation of the group G_1.

Remark 1.1. The solution of determining equations can lead to an infinite-dimensional Lie algebra L. Then, the system of differential equations under consideration is said to admit an *infinite group.*

§ 5 Riemannian spaces

5.1 Metric tensor and the Christoffel symbols

Let g_{ij} be a symmetric tensor defined on an n-dimensional differentiable manifold M. The manifold M together with the quadratic form

$$ds^2 = g_{ij}(x)dx^i dx^j \tag{5.1}$$

given in a neighborhood of every point of M is called an n-dimensional *Riemannian space* and is denoted by V_n. The tensor g_{ij} is called the *metric tensor* of the space V_n and the form (5.1) is referred to as the *metric form* of the space. The quadratic form (5.1) is independent on the choice of the system of coordinates and defines the "length " ds of the tangent vector $dx = (dx^1, \ldots, dx^n)$ to the manifold M at x.

We will be particularly interested in Riemannian spaces V_n of the signature $(- \cdots - +)$. The latter means that there exists a system of coordinates in a vicinity of every point $x \in V_n$ in which the metric form (5.1) at x is written

$$ds^2 = -(dx^1)^2 - \ldots - (dx^{n-1})^2 + (dx^n)^2.$$

Such spaces V_n are known as spaces of a normal hyperbolic type [41], for in this case linear differential equations of the second order

$$g^{ij}(x)u_{ij} + b^i(x)u_i + c(x)u = 0,$$

where g^{ij} is defined by the equations $g_{ik}g^{kj} = \delta_i^j$, have the normal hyperbolic type.

Let us interpret the metric form (5.1) as the square of the distance between the infinitesimally close points x and $x + dx$ with the coordinates x^i and $x^i + dx^i$, respectively. Then the length of the curve

$$x^i = x^i(t), \quad t_0 \le t \le t_1 \quad (i = 1, \ldots, n), \tag{5.2}$$

in the space V_n is given by the integral

$$s = \int_{t_0}^{t_1} \mathcal{L} dt, \tag{5.3}$$

where

$$\mathcal{L} = \sqrt{g_{ij}(x)\dot{x}^i\dot{x}^j}, \quad \dot{x}^i = \frac{dx^i(t)}{dt} \quad (i = 1, \ldots, n).$$

If the curve (5.2) is an extremal of the integral (5.3), i.e. is a solution of the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^i}\right) - \frac{\partial \mathcal{L}}{\partial x^i} = 0 \quad (i = 1, \ldots, n), \tag{5.4}$$

it is called a *geodesic curve* connecting the points $x_0 = x(t_0)$ and $x_1 = x(t_1)$ of the space V_n. If we parametrize the curve (5.2) by its arc length s measured from the point x_0, i.e. we set $t = s$, we obtain from (5.4) the following *equations of geodesics* in the space V_n :

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}(x)\frac{dx^j}{ds}\frac{dx^k}{ds} = 0 \quad (i = 1, \ldots, n). \tag{5.5}$$

The reckoning shows that the coefficients of these equations are given by

$$\Gamma^i_{jk} = \frac{1}{2}g^{im}\left(\frac{\partial g_{mj}}{\partial x^k} + \frac{\partial g_{mk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m}\right), \quad i, j, k = 1, \ldots, n. \tag{5.6}$$

They are known as the *Christoffel symbols.*

5.2 The Riemann tensor

By means of Christoffel symbols one can determine the covariant differentiation in the Riemannian space V_n which maps any tensor again to a tensor. We denote the covariant differentiation by a lower index after the comma. In this notation, the covariant derivatives, e.g. of scalars and covariant and contravariant vectors are written as follows:

$$a_{,i} = \frac{\partial a}{\partial x^i}\,;$$

$$a_{i,j} = \frac{\partial a_i}{\partial x^j} - a_k \Gamma^k_{ij}\,;$$

$$a^i_{,j} = \frac{\partial a_i}{\partial x^j} + a^k \Gamma^i_{kj}\,.$$

In what follows, we write only one comma in case of a repeated covariant differentiation, e.g.

$$a_{i,jk} \equiv (a_{i,j})_{,k}\,.$$

For scalars a the repeated covariant differentiation does not depend on the order of differentiation:

$$a_{,ij} = a_{,ji} \quad (i,j = 1,\ldots,n).$$

However, repeated differentiation for tensors depends on the order of differentiation, namely

$$a_{i,jk} = a_{i,kj} + a_l R^l_{ijk},$$

$$a^i_{,jk} = a^i_{,kj} - a_l R^l_{ljk},$$

etc, where

$$R^l_{ijk} = \frac{\partial \Gamma^l_{ik}}{\partial x^j} - \frac{\partial \Gamma^l_{ij}}{\partial x^k} + \Gamma^\lambda_{ik}\Gamma^l_{\lambda j} - \Gamma^\lambda_{ij}\Gamma^l_{\lambda k} \tag{5.7}$$

are the components of a tensor called the *Riemann tensor* and known also as the *Riemann-Christoffel tensor.*

It follows from the above formulae of repeated differentiation that successive covariant differentiations of tensors are permutable if and only if the Riemannian space V_n is flat, i.e

$$R^l_{ijk} = 0 \quad (i,j,k,l = 1,\ldots,n). \tag{5.8}$$

Contracting the indices l and k in the Riemann tensor R^l_{ijk} one obtains the *Ricci tensor*

$$R_{ij} = R^k_{ijk} \quad (i,j = 1,\ldots,n). \tag{5.9}$$

Multiplying the Ricci tensor by g^{ij} and contracting both indices, one obtains the *scalar curvature* of the space V_n :

$$R = g^{ij} R_{ij}. \tag{5.10}$$

Note, that in what follows Riemannian spaces are considered locally.

Chapter 2
Generalized motions in Riemannian spaces

This chapter is dedicated to the problem of invariants of continuous transformation groups in Riemannian spaces. Groups that have at least one invariant will be called groups of generalized motions. The most significant result for the theory is encapsulated in Theorem 1.12.

Literature: Killing [82], Eisenhart [29], Petrov [116], Ibragimov [50, 51, 53, 54].

§ 6 Transformations in Riemannian spaces

6.1 Representation of a metric by a manifold

The following interpretation of the metric tensor of a Riemannian space V_n is convenient for our purposes.

Let x^i and y_{ij} be real variables. Consider real valued functions $g_{ij}(x)$ defined on an open set of the space R^n and satisfying the conditions

$$\det||g_{ij}(x)|| \neq 0, \quad g_{ij}(x) = g_{ji}(x) \quad (i, j = 1, \ldots, n).$$

Let $\mathcal{G}$ be the n-dimensional manifold in the space of the variables x^i, y_{ij} defined by the equations

$$y_{ij} = g_{ij}(x) \quad (i, j = 1, \ldots, n). \tag{6.1}$$

Definition 1.12. The manifold $\overline{\mathcal{G}}$ given by equations

$$y_{ij} = \bar{g}_{ij}(x) \quad (i, j = 1, \ldots, n)$$

is said to be *equivalent* to the manifold $\mathcal{G}$ and written $\overline{\mathcal{G}} \sim \mathcal{G}$ if the system of differential equations

$$\bar{g}_{kl}(f)\frac{\partial f^k}{\partial x^i}\frac{\partial f^l}{\partial x^j} = g_{ij}(x) \quad (i, j = 1, \ldots, n) \tag{6.2}$$

has a continuously differentiable solution

$$f = (f^1, \ldots, f^n),$$

satisfying the condition $\det f' \neq 0$, where f' is a derivative of f.

According to Definition 1.12, the set of all manifolds $\mathcal{G}$ determined by equations (6.1) is separated into classes of equivalent manifolds. Every class of equivalence is termed a *metric tensor* and is denoted by g_{ij}. If $\overline{\mathcal{G}} \sim \mathcal{G}$, then the manifold $\mathcal{G}$ (or Eqs. (6.1) which is the same) is said to define the metric tensor of the Riemannian space V_n in the system of coordinates $\{x\}$, and the manifold $\overline{\mathcal{G}}$ in the system of coordinates $\{\bar{x}\}$ with $\bar{x} = f(x)$, where f is the solution of Eqs. (6.2). When a coordinate system is fixed, a point $x = (x^1, \ldots, x^n)$ can be identified with a point of the Riemannian space V_n.

Let us determine the extent of arbitrariness to which one defines a coordinate system in a Riemannian space V_n by specifying a manifold $\mathcal{G}$ (i.e. the functions $g_{ij}(x); i, j = 1, \ldots, n$). Let us assume that a change of coordinates $\bar{x} = f(x)$ leaves the function $g_{ij}(x)$ defining the manifold $\mathcal{G}$ unaltered. According to Eqs. (6.2), it means that the function f satisfies the equations

$$g_{kl}(f)\frac{\partial f^k}{\partial x^i}\frac{\partial f^l}{\partial x^j} = g_{ij}(x) \quad (i, j = 1, \ldots, n). \tag{6.3}$$

We will see further (see Lemma 1.4 in Section 8.1) that Eqs. (6.3) define transformations preserving the metric form of the Riemannian space V_n with the metric tensor g_{ij}, i.e. the isometric motions in the space V_n. Thus, we have the following result.

Theorem 1.10. A manifold $\mathcal{G}$ given by Eqs. (6.1) defines a system of coordinates in a Riemannian space up to isometric motions.

From a local viewpoint, specifying a Riemannian space is equivalent to specifying its metric tensor. Hence, functions $g_{ij}(x)$ define a Riemannian space V_n in a certain system of coordinates. Therefore, a Riemannian space V_n in a given coordinate system can be identified with a manifold $\mathcal{G}$.

6.2 Transformations of the metric

Let G_1

$$\bar{x}^i = f^i(x, a) \quad (i = 1, \ldots, n) \tag{6.4}$$

be a group of transformations in R^n and let V_n be a Riemannian space with a metric tensor $g_{ij}(x)$. We will consider (6.4) as a group of transformations of points in the space V_n in a given system of coordinates $\{x\}$. Namely, any point $x \in V_n$ with the coordinates x^i is mapped to the point $\bar{x} \in V_n$ with the coordinates $\bar{x}^i$ in the coordinate system $\{x\}$.

Transformations of the group G_1 will also affect vectors tangent to the space V_n. Let us find the corresponding transformation of the lengths of tangent vectors. Let

$$dx = (dx^1, \ldots, dx^n)$$

be the tangent vector at the point $x \in V_n$, and

$$d\bar{x} = (d\bar{x}^1, \ldots, d\bar{x}^n)$$

be the corresponding tangent vector at the point $\bar{x} = f(x)$. The components of the vector $d\bar{x}$ are

$$d\bar{x}^i = \frac{\partial f^i(x,a)}{\partial x^k} dx^k.$$

The length of the vector dx in the space V_n is given by

$$ds^2 = g_{ij}(x) dx^i dx^j,$$

and the length of $d\bar{x}$ is given by

$$d\bar{s}^2 = g_{ij}(\bar{x}) d\bar{x}^i d\bar{x}^j. \tag{6.5}$$

In order to compare the quantities ds^2 and $d\bar{s}^2$ one should express them through components of the differential at one and the same point, e.g. at $\bar{x}$. Due to the invariance of the metric form with respect to a change of coordinates we have:

$$ds^2 = \bar{g}_{ij}(\bar{x}) d\bar{x}^i d\bar{x}^j, \tag{6.5$'$}$$

where $\bar{g}_{ij}(\bar{x})$ are components of the metric tensor in the system of coordinates $\{\bar{x}\}$, defined by Eqs. (6.2). The equations (6.5) and (6.5$'$) yield that the change of the length ds of the tangent vector dx is completely determined by the difference of the functions $g_{ij}(\bar{x})$ and $\bar{g}_{ij}(\bar{x}), (i, j = 1, \ldots, n)$.

6.3 Extension of group actions on metric manifolds

Thus, investigation of the change of the tangent vector length is reduced to investigation of the corresponding transformation of functions $g_{ij}(x)$ under the action of (6.4). It is even more convenient to consider the transformation of the manifold $\mathcal{G}$ given by Eqs. (6.1) rather than that of the functions $g_{ij}(x)$ themselves. It is clear from the previous section that transformations of x^i sould be accompanied by the transformations of the variables $y_{ij}(i, j = 1, \ldots, n)$ as components of a covariant tensor of the second order. Therefore it is convenient to introduce the group $\overline{G}_1$ of extended transformations

$$\begin{aligned} \bar{x}^i &= f^i(x,a), \\ y_{ij} &= \bar{y}_{kl} \frac{\partial f^k(x,a)}{\partial x^i} \frac{\partial f(x,a)}{\partial x^j} \end{aligned} \tag{6.6}$$

in the space of the variables $x^i, y_{ij}(i, j = 1, \ldots, n)$. (It is suggested to verify that the transformations (6.6) corresponding to the group of transformations (6.4) form a local group).

Let us find the generator $\overline{X}$ of the group $\overline{G}_1$. We will write the generator of the group G_1 in the form

$$X = \xi^i(x)\frac{\partial}{\partial x^i}, \tag{6.7}$$

where

$$\xi^i(x) = \frac{\partial f^i(x,a)}{\partial a}\bigg|_{a=0}, \quad i = 1, \ldots, n.$$

According to Eqs. (6.6), the generator $\overline{X}$ of the group $\overline{G}_1$ has the form

$$\overline{X} = \xi^i\frac{\partial}{\partial x^i} + \eta_{ij}\frac{\partial}{\partial y_{ij}},$$

where

$$\eta_{ij} = \frac{\partial \bar{y}_{ij}}{\partial a}\bigg|_{a=0} \quad (i, j = 1, \ldots, n).$$

It is assumed here that the expressions for the quantities $\bar{y}_{ij}$ are obtained from Eqs. (6.6) via x^i, y_{ij} and the group parameter a. Let us assume that these expressions are substituted into Eqs. (6.6). Then, differentiating the resulting identities with respect to the parameter a at $a = 0$ and using the notation (6.7) and the conditions $\bar{y}_{ij}|_{a=0} = y_{ij}$ we obtain

$$\eta_{kl}\delta_i^k\delta_j^l + y_{kl}\frac{\partial \xi^k}{\partial x^i}\delta_j^l + y_{kl}\delta_i^k\frac{\partial \xi^l}{\partial x^j} = \eta_{ij} + y_{kj}\frac{\partial \xi^k}{\partial x^i} + y_{ik}\frac{\partial \xi^k}{\partial x^j} = 0,$$

whence

$$\eta_{ij} = -\left(y_{ik}\frac{\partial \xi^k}{\partial x^j} + y_{kj}\frac{\partial \xi^k}{\partial x^i}\right) \quad (i, j = 1, \ldots, n).$$

Thus, the group $\overline{G}_1$ of the extended transformations (6.6) has the generator

$$\overline{X} = \xi^i(x)\frac{\partial}{\partial x^i} - \left(y_{ik}\frac{\partial \xi^k(x)}{\partial x^j} + y_{kj}\frac{\partial \xi^k(x)}{\partial x^i}\right)\frac{\partial}{\partial y_{ij}}\cdot \tag{6.8}$$

The equation (6.8) implies that if X and Y are generators of two one-parameter groups, then

$$[\overline{X}\ \overline{Y}] = \overline{[X, Y]}.$$

Hence, if G_r is a group of transformations (6.4) with a Lie algebra L_r, then Eqs. (6.6) furnish a group $\overline{G}_r$ of transformations with an isomorphic Lie algebra $\overline{L}_r$. Indeed, if the commutators of a basis $X_\alpha (\alpha = 1, \ldots r)$ of L_r are

$$[X_\alpha X_\beta] = c_{\alpha\beta}^\gamma X_\gamma \quad (\alpha, \beta = 1, \ldots r),$$

then the operators $\overline{X}_\alpha$ given by (6.8) span a Lie algebra $\overline{L}_r$ with the same structure constants $c_{\alpha\beta}^\gamma$ because

$$[\overline{X}_\alpha\ \overline{X}_\beta] = \overline{[X_\alpha, X_\beta]} = c_{\alpha\beta}^\gamma \overline{X}_\gamma \quad (\alpha, \beta = 1, \ldots, r).$$

§ 7 Transformations preserving harmonic coordinates

7.1 Definition of harmonic coordinates

Before considering the issue of invariants of continuous groups of transformations in Riemannian spaces let us demonstrate that the viewpoint, accepted in the above paragraph, on the Riemannian space and groups of transformations in it can be used in investigating transformations preserving harmonic systems of coordinates in Riemannian spaces. Here harmonic coordinates are considered locally [87] omitting conditions on the infinity accepted by Fock [32] in connection with the problem of uniqueness of harmonic coordinates.

Let us consider a simplest problem leading to the notion of harmonic coordinates. Let the scalar function $u(x)$ be defined in the Riemannian space V_n, and

$$\Delta_2 u = g^{ij}u_{,ij} \equiv g^{ij}u_{ij} - g^{jk}\Gamma^i_{jk}u_i$$

be the second differential Beltrami parameter. If V_n is a flat space, then $\Gamma^i_{jk} = 0$ in the Cartesian system of coordinates, so that the first derivatives of u are not included into $\Delta_2(u)$. In a general case, the conditions eliminating the first derivatives of u from $\Delta_2 u$ have the form

$$\Gamma^i \equiv g^{jk}\Gamma^i_{jk} = 0 \quad (i = 1, \ldots, n). \tag{7.1}$$

The values Γ^i, as well as Christoffel symbols Γ^i_{kl}, are not tensor components. Therefore, equations (7.1) represent some conditions selecting special systems of coordinates. Systems of coordinates that meet these conditions are referred to as *harmonic.* It is known [87], that harmonic coordinates exist in any Riemannian space.

Let us write conditions (7.1) in terms of the manifold $\mathcal{G}$ of the space V_n in a more convenient form. Let us transform expressions for Γ^i invoking that covariant derivatives of the metric tensor equal to zero:

$$g^{ij}_{,k} \equiv \frac{\partial g^{ij}}{\partial x^k} + g^{il}\Gamma^j_{lk} + g^{jl}\Gamma^i_{lk} = 0.$$

Assuming that $k = j$ here and summing over j from 1 to n, one obtains

$$\frac{\partial g^{ij}}{\partial x^j} + \frac{1}{\sqrt{|g|}} g^{ij} \frac{\partial\sqrt{|g|}}{\partial x^j} + \Gamma^i = 0, \quad (i = 1, \ldots n).$$

Here we applied the equalities

$$\Gamma^j_{ij} = \frac{\partial \ln\sqrt{|g|}}{\partial x^i} \quad (i = 1, \ldots, n)$$

where $g = \det \|g_{ij}\|$. Thus,

$$\Gamma^i = -\frac{1}{\sqrt{|g|}} \frac{\partial(\sqrt{|g|}g^{ij})}{\partial x^j} \quad (i = 1, \ldots, n),$$

and equations (7.1) take the form

$$\frac{\partial(\sqrt{|g|}g^{ij})}{\partial x^j} = 0 \quad (i = 1 \ldots n). \tag{7.2}$$

Consequently, *a system of coordinates $\{x\}$ in the Riemannian space V_n is said to be harmonic if and only if functions $g_{ij}(x)(i, j = 1, \ldots, n)$ determining the manifold $\mathcal{G}$ (see § 6) satisfy the system of differential equations (7.2).*

7.2 Definition of a group conserving harmonic coordinates

Let us assume that G is a continuous transformation group in the Riemannian space V_n, $\{x\}$ is a harmonic system of coordinates in the space V_n and the manifold $\mathcal{G}$ is defined by equations (6.1) in this system of coordinates. Transformation $\overline{T} \in \overline{G}$ takes the manifold $\mathcal{G}$ over to a manifold $\overline{T}(\mathcal{G}) \sim \mathcal{G}$ with a system of coordinates $\{\bar{x}\}$ corresponding to it. The resulting system of coordinates $\{\bar{x}\}$ is not harmonic in general, for both equations (7.1) and equations (7.2) are not covariant (i.e. not invariant with respect to arbitrary transformations of coordinates). It may occur further, that if there exist two harmonic systems of coordinates in the space V_n, then transformations of the group $\overline{G}$ leave one of the systems of coordinates harmonic and turn the other one into a nonharmonic system of coordinates. Therefore, among all transformation groups in the space V_n we single out those groups that preserve harmonic coordinates as follows.

Definition 1.13. A group G of transformations in the space V_n is called a group preserving harmonic coordinates if with all transformations $\overline{T} \in \overline{G}$ any harmonic system of coordinates in V_n goes into a harmonic system of coordinates again.

In what follows, we find the most extended group G in every space V_n preserving a harmonic system of coordinates. First, let us prove several auxiliary statements.

7.3 First lemma

Lemma 1.1. Let us assume that a manifold M in the space R^n is defined by equations

$$\psi^\sigma(x) = 0 \quad (\sigma = 1, \dots p),$$

let $N \subset M$, and and G_r be a continuous transformation group in the space R^n. If $T_y \in M$ for all $T \in G_r$ and all $y \in R$, then

$$X\psi^\sigma|_N = 0 \quad (\sigma = 1, \dots, p) \tag{7.3}$$

for all infinitesimal generators X of the group G_r.

Proof. It is sufficient to consider a case with a one-parameter group G_1 with the operator X. According to the lemma conditions

$$\psi^\sigma(Ty) \equiv 0 \quad (\sigma = 1, \dots, p)$$

for all points $y \in N$ and all transformations $T \in G_1$. Therefore, according to (1.7),

$$X\psi^\sigma(y) = \left.\frac{\partial(T_a y)}{\partial a}\right|_{a=0} = 0 \quad (\sigma = 1, \dots, p)$$

for every $y \in N$.

7.4 Second lemma

Lemma 1.2. The most extended group G of transformations in an arbitrary Riemannian space V_n, with the extended group $\overline{G}$ admitted by a system of differential equations

$$\frac{\partial(\sqrt{|y|}y^{ij})}{\partial x^j} = 0 \quad (i = 1, \dots n), \tag{7.4}$$

where $y = \det \|y_{ij}\|$, $\|y^{ij}\| = \|y_{ij}\|^{-1}$, $y^{ij} = y^{ji}$, consists of linear transformations

$$\bar{x}^i = a^i_k x^k + b^i \quad (i = 1, \dots n), \tag{7.5}$$

where $a^i_k, b^i (ik = 1, \dots n)$ are arbitrary constants.

Proof. Let us introduce the notation

$$z^{ij} = \sqrt{|y|}y^{ij} \quad (i, j = 1, \dots n),$$

$$\theta^{ij}_k = \frac{\partial z^{ij}}{\partial x^k} \quad (i, j, k = 1, \dots, n),$$

and write Equation (7.4) in the form

$$\theta^{ij}_j = 0 \quad (i = 1, \dots, n). \tag{7.6}$$

Transformations (6.6) of the extended group $\overline{G}$ in variables z^{ij} take the form

$$\bar{x}^i = f^i(x, a), \tag{7.7}$$

$$J\left(\frac{\bar{x}}{x}\right) \bar{z}^{kl} = z^{kl} \frac{\partial f^i(x,a)}{\partial x^k} \frac{\partial f^j(x,a)}{\partial x^l},$$

where $J\left(\frac{\bar{x}}{x}\right)$ is a Jacobian of the transformation $\bar{x}^i = f^i(x, a)$.

Let us find operators of the group $\overline{G}$ of transformations (7.7). Write these operators in the form

$$\overline{X} = \xi^i \frac{\partial}{\partial x^i} + \eta^{ij} \frac{\partial}{\partial z^{ij}},$$

where

$$\eta^{ij} = \frac{\partial \bar{z}^{ij}}{\partial a}\bigg|_{a=0} \qquad \xi^i = \frac{\partial \bar{x}^i}{\partial a}\bigg|_{a=0}.$$

Acting by the operator $\frac{\partial}{\partial a}\big|_{a=0}$ on equalities (7.7), one obtains

$$\xi^i = \frac{\partial f^i(x,a)}{\partial a}\bigg|_{a=0}, \quad \eta^{ij} = z^{ik} \frac{\partial \xi^j}{\partial x^k} = z^{jk} \frac{\partial \xi^i}{\partial x^k} - z^{ij} \frac{\partial \xi^k}{\partial x^k},$$

so that

$$\overline{X} = \xi^i(x) \frac{\partial}{\partial x^i} + \left(z^{ik} \frac{\partial \xi^j(x)}{\partial x^k} + z^{jk} \frac{\partial \xi^i(x)}{\partial x^k} - z^{ij} \frac{\partial \xi^k(x)}{\partial x^k} \right) \frac{\partial}{\partial z^{ij}} \cdot \tag{7.8}$$

Let

$$\widetilde{X} = \overline{X} + \zeta^{ij}_k \frac{\partial}{\partial \theta^{ij}_k}$$

be an operator of a group obtained by dilating the group G to values $\theta^{ij}_k (i, j, k = 1, \dots, n)$. The invariance conditions for equations (7.6) with respect to the group $\overline{G}$ have the form

$$\widetilde{X} \theta^{ij}_j \big|_{(6)} = \zeta^{ij}_j \big|_{(6)} = 0 \quad (i = 1, \dots, n).$$

According to the prolongation formulae of operators to the first derivatives and by virtue of (7.8) one has

$$\zeta^{ij}_k = z^{il} \frac{\partial^2 \xi^j}{\partial x^l \partial x^k} + z^{jl} \frac{\partial^2 \xi^i}{\partial x^l \partial x^k} - z^{ij} \frac{\partial^2 \xi^l}{\partial x^l \partial x^k} + \tag{7.9}$$

$$+ \theta^{il}_k \frac{\partial \xi^j}{\partial x^l} + \theta^{jl}_k \frac{\partial \xi^i}{\partial x^l} - \theta^{ij}_k \frac{\partial \xi^l}{\partial x^l} \theta^{ij}_l \frac{\partial \xi^l}{\partial x^k}$$

so that the invariance conditions for equations (7.6) take the form

$$z^{jk}\frac{\partial^2\xi^i}{\partial x^j\partial x^k}=0 \quad (i=1,\ldots,n). \tag{7.10}$$

The values $x^i, z^{ij}(i,j=1,\ldots,n)$ in equations (7.10) play the part of independent variables. Therefore, these equations are equivalent to equations

$$\frac{\partial^2\xi^i}{\partial x^j\partial x^k}=0 \quad (i,j,k=1,\ldots,n),$$

determining an $n(n+1)$-parameter group of linear transformations (7.5).

7.5 Third lemma

Lemma 1.3. It is necessary and sufficient that equations (7.4) are invariant with respect to the group $\overline{G}$ for the group G of transformations in the space V_n to preserve the harmonic coordinates.

Proof. Let equations (7.4) be invariant with respect to the group $\overline{G}$. Then, any transformation $\overline{T}\in\overline{G}$ takes any solution of equations (7.4) into a solution of the same equations § 4. Thus, if the manifold $\mathcal{G}$, defined by equations $y_{ij}=g_{ij}(x)$ $(ij=1,\ldots,n)$, determines a harmonic system of coordinates in the space V_n so that the functions $g_{ij}(x)$ satisfy equations (7.2), then a manifold $\overline{T}(\mathcal{G})\sim\mathcal{G}$ for any $\overline{T}\in\overline{G}$ will be derfined by equations $\bar{y}_{ij}=\bar{g}_{ij}(\bar{x})$, where functions $\bar{g}_{ij}(\bar{x})$ satisfy equations (7.2) in variables $\bar{x}^i$. It follows that the resulting system of coordinates $\{\bar{x}\}$ is harmonic for any $\overline{T}\in\overline{G}$, i.e. the group G preserves harmonic coordinates in the space V_n.

Let us assume now that the group G preserves harmonic coordinates in the space V_n. We apply Lemma 1.1 and take a manifold in the space of variables $x^i, z^{ij}, \theta^{ij}_k$ $(i,j,k=1,\ldots,n)$ given by equations (7.6) as M, and take the class of all representatives of the metric tensor of the space V_n (i.e. of manifolds $\mathcal{G}$ equivalent to each other) governed by equations (6.1) where $g_{ij}(x)$ satisfy equations (7.2) as $N\subset M$. According to our assumption, the group of transformations in the space of variables $x^i, z^{ij}, \theta^{ij}_k$ $(i,j,k=1,\ldots,n)$, resulting from prolongation of the group $\overline{G}$ to the first derivatives θ^{ij}_k, translates any point of manifold N to a point of M so that we are under the circumstances of Lemma 1.1. Hence, equations (7.3) are satisfied and have the form

$$\widetilde{X}\theta^{ij}_j\big|_N=\zeta^{ij}_j\big|_N=0 \quad (i=1,\ldots,n), \tag{7.11}$$

in the given case. Here the quantities ζ^{ij}_k are calculated according to Equation (7.9).

Note that linear transformations of coordinates translate a harmonic system of coordinates in the space V_n into a harmonic one. On the other hand, one can easily see that a manifold resulting from some manifold $\mathcal{G}$ upon various linear transformations (7.5) coincides (locally) with the space of variables x^i, y_{ij} $(i, j = 1, \ldots, n)$. Therefore, taking a manifold contained in N as $\mathcal{G}$ one can see that the values x^i, z^{ij} $(i, j = 1, \ldots, n)$ play the part of free variables on N. Therefore, upon substitution of values (7.9) of quantities $\mathcal{G}$, equalities (7.11) take the form

$$z^{jk} \frac{\partial^2 \xi^i}{\partial x^j \partial x^k}\bigg|_N = 0 \quad (i = 1, \ldots n),$$

and are equivalent to (7.10). Thus, if a group G preserves harmonic coordinates in space V_n then, equation (7.10) holds, and consequently, equations (7.4) are invariant with respect to the extended group $\overline{G}$.

7.6 Main theorem

Theorem 1.11. The most general group preserving harmonic coordinates in the space V_n for any Riemannian space V_n, consists of linear transformations (7.5).

Proof. The theorem follows from Lemma 1.3 and Lemma 1.2.

§ 8 Groups of generalized motions

8.1 Isometric motions

Given a group G of transformations in a Riemannian space V_n. First, let us find out how to formulate the condition that a group G is a group of motions in the space V_n in terms of the manifold $\mathcal{G}$ in space V_n and the extended group $\overline{G}$. A group G is referred to as a group of motions in the Riemannian space V_n if all transformations of the group G preserve the value of the basic metric form of the space V_n : $d\bar{s}^2 = ds^2$ [82], [29]. According to formulae (6.5) and (6.5′), the group of transformations

$$\bar{x}^i = f^i(x, a) \quad (i = 1 \ldots, n) \tag{8.1}$$

is a group of motions in the space V_n with the metric tensor $g_{ij}(x)$ $(i, j = 1, \ldots, n)$ if and only if

$$g_{ij}(\bar{x}) = \bar{g}_{ij}(\bar{x}) \quad (i, j = 1 \ldots, n) \tag{8.2}$$

for all transformations (8.1).

Let us multiply both parts of equations (8.2) by

$$\frac{\partial f^i}{\partial x^k}\frac{\partial f^j}{\partial x^l}$$

and sum over indices i, j from 1 to n. As a result, invoking equations (6.2) one obtains equations (6.3). If $f'_x \neq 0$, then multiplying equations (6.3) by

$$\frac{\partial (f^{-1})^i}{\partial \bar{x}^k}\frac{\partial (f^{-1})^j}{\partial \bar{x}^l}$$

and summing over i, j form 1 to n one obtains equations (8.2), thus proving the following statement.

Lemma 1.4. A *transformation group (8.1) is a group of isometric motions in the space V_n if and only if the equations (6.3) hold.*

Let G_1 be a one-parameter group of transformations (8.1) with the infinitesimal generator (6.7). If G_1 is a group of isometric motions in the space V_n then by Lemma 1.4 equations (6.3) hold identically with respect to the parameter a. Differentiating these identities with respect to a when $a = 0$ one arrives to Killing equations

$$\xi^k \frac{\partial g_{ij}}{x^k} + g_{ij}\frac{\partial \xi^k}{\partial x^j} + g_{jk}\frac{\partial \xi^k}{\partial x^i} = 0 \quad (i, j = 1 \ldots, n) \tag{8.3}$$

with respect to coordinates $\xi^i(x)$ of the infinitesimal generator X of the group G_1 and components $g_{ij}(x)$ of the metric tensor of the Riemannian space V_n. As it is known in Riemannian geometry, a continuous group G_r of transformations in the space V_n is called a group of motions in the space V_n if and only if the Killing equations (8.3) hold for all infinitesimal generators of the group G_r. These equations are independent of the choice of system of coordinates. Indeed, taking into account that ξ^i represent components of a contravariant vector and using the identities

$$\xi^k \frac{\partial g_{ij}}{\partial x^k} + g_{ij}\frac{\partial \xi^k}{\partial x^j} + g_{jk}\frac{\partial \xi^k}{\partial x^i} \equiv \xi_{i,j} + \xi_{j,i} \quad (i, j = 1 \ldots, n), \tag{8.4}$$

where $\xi_i = g_{ik}\xi^k$ are covariant components of the vector ξ^k, one can write equations (8.3) in the tensor form

$$\xi_{i,j} + \xi_{j,i} = 0 \quad (i, j = 1 \ldots, n). \tag{8.5}$$

Lemma 1.5. In order for the group G of transformations in the Riemannian space V_n with the metric tensor $g_{ij}(x)$ to be a group of motions in this space it is necessary and sufficient that the manifold $\mathcal{G}$ given by equations (6.1) is invariant with respect to the extended group G.

Proof. The necessary and sufficient condition of invariance of the manifold $\mathcal{G}$ with respect to the group $\overline{G}$ consists in satisfaction of equations

$$\overline{X}(g_{ij}(x) - y_{ij})|_{\mathcal{G}} = 0 \quad (i, j = 1, \ldots, n)$$

for all infinitesimal generators (6.8) of the group $\overline{G}$. These equations coincide with the Killing equations (8.5) by virtue of identities (8.4).

8.2 Generalized motions. Defect

Let us consider the general situation. Given a Riemannian space V_n with the metric tensor $g_{ij}(x)$ in a system of coordinates $\{x\}$ and a group G_r of transformations (8.1) in the space V_n. Let us check how the corresponding manifold $\mathcal{G}$ varies with transformations of the extended group $\overline{G}$. Every transformation $\overline{T} \in \overline{G}$ leads to the manifold $\overline{T}(\mathcal{G})$ (see § 3) equivalent to the manifold $\mathcal{G}$. This follows from the definition of equivalence (6.1) and from construction of the group $\overline{G}$. If the group G is a group of motions in the space V_n, then according to Lemma 1.5, $\overline{T}(\mathcal{G}) = \mathcal{G}$ for any $\overline{T} \in \overline{G}$.

In the general case, the set

$$\overline{G}(\mathcal{G}) = \underset{\overline{T}\in\overline{G}}{\cup} \overline{T}(\mathcal{G})$$

of all images $\overline{T}(\mathcal{G})$ of the manifold $\mathcal{G}$ is a manifold in the space of variables $x^i, y_{ij} (i, j = 1, \ldots, n)$, containing the manifold $\mathcal{G}$. As it is mentioned above in § 3, $\overline{G}(\mathcal{G})$ is the smallest invariant manifold of the group $\overline{G}$ containing the manifold $\mathcal{G}$.

Let us obtain a formula for the defect $\delta = \delta(\mathcal{G}, \overline{G})$ of the manifold $\mathcal{G}$ with respect to the group G. Since $\dim \mathcal{G} = n$, definition of the invariance defect of the manifold $\mathcal{G}$ with respect to the group $\overline{G}$ in the given case is as follows:

$$\delta = \dim \overline{G}(\mathcal{G}) - n.$$

Infinitesimal operators of the group G_r are written in the form

$$X_\alpha = \xi^i_\alpha(x) \frac{\partial}{\partial x^i} \quad (\alpha = 1, \ldots, r), \tag{8.6}$$

and $\|\xi_{\alpha i,j}(x) + \xi_{\alpha j,i}(x)\|$ indicates a matrix with columns numbered by the index α and rows numbered by the double subscript ij. Here, indices i and

j after the comma indicate covariant differentiation in the space V_n like in the Killing equations (8.5). The following lemma provides the unknown formula of the defect.

Lemma 1.6.

$$\delta(\mathcal{G}, \overline{G}_r) = \text{rank}\, \|\xi_{\alpha i,j}(x) + \xi_{\alpha j,i}(x)\|. \tag{8.7}$$

Proof. According to Theorem 1.6

$$\delta(\mathcal{G}, \overline{G}_r) = \text{rank}\, \|\overline{X}_\alpha(g_{ij}(x) - y_{ij})\|_{\mathcal{G}}.$$

Substituting the values (6.8) of operators $\overline{X}_\alpha$ here and applying equations (8.4) we arrive to formula (8.7).

Corollary 1.3. If $\overline{\mathcal{G}} \approx \mathcal{G}$, then

$$\delta(\overline{\mathcal{G}}, \overline{G}_r).$$

In other words, the defect δ is independent of the choice of system of coordinates in the space V_n.

Proof. Every column of the matrix $\|\xi_{\alpha i,j} + \xi_{\alpha j,i}\|$ is a covariant tensor of the second rank $\xi_{\alpha i,j} + \xi_{\alpha j,i}$ (α is fixed). Therefore, with the change of coordinates in space V_n, all columns of the matrix undergo linear transformation independent of the number α of the column. Obviously, this does not change the rank of the matrix. According to the formula (8.7) this implies that transformation of the manifold $\mathcal{G}$ to the equivalent manifold $\overline{\mathcal{G}}$ leaves the defect δ unaltered.

Thus, the invariance defect δ of the manifold $\mathcal{G}$ with respect to the group $\overline{G}_r$, being initially set in some system of coordinates, depends only on the space V_n and the group G_r and not on the choice of the system of coordinates indeed. Hence, we can write $\delta = \delta(V_n, G_r)$ and discuss the defect δ of the space V_n *with respect to the group G_r of transformations in the space V_n*.

Invoking that the metric tensor g_{ij} of the space V_n, as well as the infinitesimal generators (8.6) of the group G_r, are independent of the choice of system of coordinates and applying Lemma (1.6) with its Corollary, one arrives to the following result.

Theorem 1.12. Let a group G_r of transformations in the space V_n with the metric tensor g_{ij} have a Lie algebra spanned by infinitesimal generators

$$X_\alpha = \xi^i_\alpha \frac{\partial}{\partial x^i}, \quad \alpha = 1\ldots, r.$$

Then the defect δ of the space V_n with respect to the group G_r is governed by the following formula:

$$\delta(V_n, G_r) = \text{rank}\, \|\xi_{\alpha i,j} + \xi_{\alpha j,i}\|. \tag{8.8}$$

In particular, this theorem together with the Killing equations (8.5) leads to the conclusion that a group G_r is called a group of motions in the space V_n if and only if $\delta(V_n, G_r) = 0$. If

$$\delta(V_n, G_r) < \frac{1}{2}n(n+1).$$

Then, the group G_r is referred to as a *group of generalized motions* in the space V_n.

8.3 Invariant family of spaces

Let us assume that G is a group of generalized motions in the Riemannian space V_n, and $\delta = \delta(V_n, G)$ is the corresponding defect. We suppose that a definite system of coordinates $\{x\}$, with the metric tensor of the space V_n that includes components $g_{i,j}(x)$ $(i,j = 1,\ldots,n)$ and with the manifold $\mathcal{G}$ defined by equations (6.1), is introduced into the space V_n. According to Section 8.2, the smallest invariant manifold of the group G, containing the manifold $\mathcal{G}$, is the manifold $\overline{G}(\mathcal{G})$ with the dimension

$$\dim \overline{G}(\mathcal{G}) = n + \delta. \tag{8.9}$$

Let us consider a manifold $\mathcal{G}^* \subset \overline{G}(\mathcal{G})$ given by equations

$$y_{ij} = g^*_{ij}(x) \quad (i,j = 1,\ldots,n).$$

Here, the manifold $\mathcal{G}^*$ is not necessarily equivalent to $\mathcal{G}$. According to 6.1, the manifold $\mathcal{G}^*$ defines the vector Riemannian space V^*_n in the system of coordinates $\{x\}$. Choosing various manifolds $\mathcal{G}^* \subset \overline{G}(\mathcal{G})$ we obtain a δ-parameter family of n-dimensional Riemannian spaces. The family is independent of the choice of system of coordinates in space V_n by the construction and is referred to as an *invariant family of spaces* for the pair (Y_n, G) and is denoted by $G(V_n)$.

The invariant family of spaces $G(V_n)$ is characterized by the following properties resulting directly from its definition.

1. $V_n \in G(V_n)$.

2. If $V^*_n \in G(V_n)$, then $G(V^*_n) \subset G(V_n)$.

3. $G(V_n)$ is the smallest family of n-dimensional Riemannian spaces with the properties 1 and 2.

The property 2 provides that $\delta(V^*_n, G) \leq \delta(V_n, G)$ for all $V^*_n \in G(V_n)$. In some cases, the following possibilities of a particular interest are realized.

I. There exist such $V_n^* \in G(V_n)$ that $\delta(V_n^*, G) = 0$. In other words, a group G of generalized motions in space V_n is a group of motions in some space $V_n^* \in G(V_n)$. In this case, the group G is called a *trivial* group of generalized motions in V_n.

II. For any $V_n^* \in G(V_n)$, one has $\delta(V_n^*, G) = \delta(V_n, G)$, so that $G(V_n^*) = G(V_n)$ for all $V_n^* \in G(V_n)$.

The second case plays an important part when one has to consider the whole family $G(V_n)$ at once rather than one separate space V_n. We will have to deal with such a situation in sections 10.5 and 14.5, where the group G will be represented by a group of conformal transformations in the space V_n, which naturally emerges when on considers differential equations of the second order.

8.4 Invariants of generalized motions

Let us consider the problem of finding invariants of the group G of transformations in a Riemannian space V_n. According to (6.2), investigation of the change of infinitesimal elements of the space V_n (or, in other word, elements of spaces tangent to V_n) under transformations of the group G is reduced to the investigation of the change of components g_{ij} of the metric tensor. Invariants of the group G of transformations in the space V_n are such functions

$$J = J(g_{ij})$$

of $\frac{1}{2}n(n+1)$ components g_{ij} ($g_{ij} = g_{ji}; i, j = 1, \ldots, n$) of the metric tensor of space V_n, that satisfy the condition

$$J[g_{i,j}(\bar{x})] = J[\bar{g}_{ij}(\bar{x})] \tag{8.10}$$

for all transformations (8.1) of the group G. For example, there are exactly $\frac{1}{2}n(n+1)$ functionally independent invariants $J_{ij} = g_{ij}$ ($i, j = 1, \ldots, n$) for groups of motions, for groups of conformal transformations such invariants are $\frac{1}{2}n(n+1) - 1$ relations of g_{ij}/g_{kl} of components of the metric tensor (see below).

Equation (8.10) has a simple geometric meaning. It signifies that the value of function J at the point $\bar{x} \in V_n$ obtained from the point $x \in V_n$ upon group transformation (8.1) coincides with the value of the same function at the point x. In this case the value at the point x (the right-hand side of equation (8.10)) is written in the system of coordinates $\{\bar{x}\}$ according to considerations above in Section 6.2.

It is convenient to investigate invariants in terms of the manifold $\mathcal{G}$ of the space V_n and the extended group $\overline{G}$ of transformations (6.6. Let us consider a function $J = J(y_{ij})$ and a point p on the manifold $\mathcal{G}$ with the coordinates $x^i, y_{ij} = g_{ij}(x)$ $(i, j = 1, \ldots, n)$. Transformation $\overline{G} \in \overline{G}$ takes the point p into the point $\bar{p} \in \overline{T}(\mathcal{G})$ with the coordinates $\bar{x}^i, \bar{y}_{ij} = \bar{g}_{ij}(\bar{x})$ $(i, j = 1, \ldots, n)$. Equation (8.10) indicates that the value of the function J on the projection of the point $\bar{p}$ into the space of variables $y_{ij}(i, j = 1, \ldots, n)$ coincides with the value of the same function on the projection of the point $Q \in \mathcal{G}$ with the coordinates $\bar{x}^i, y_{ij} = g_{ij}(\bar{x})$ $(i, j = 1, \ldots, n)$. This holds for all $x \in V_n$ and all $T \in G$. Therefore, identifying the projection of a cut of the manifold $\overline{G}(\mathcal{G})$ by a hyperplane $x =$ const. on the space of variables y_{ij} $(i, j = 1, \ldots, n)$ with the cut itself, one can *define the invariant of the group G of transformations in the Riemannian space V_n as such function $J(y_{ij})$, that takes one and the same value at all points of the manifold $\overline{G}(\mathcal{G})$ when $x \in V_n$ is fixed.* This yields the following procedure for finding invariants. According to § 3, the invariant family of spaces $G(V_n)$ represents a family of Riemannian spaces depending on $\delta = \delta(V_n, G)$ of arbitrary functions. Eliminating these arbitrary functions, one finds $\frac{1}{2}n(n + 1) - \delta$ correlations between components of the metric tensor which are the same for all spaces of the family $G(V_n)$. Meanwhile, the number of these invariant correlations cannot be more than $\frac{1}{2}n(n+1)-\delta$, for $\overline{G}(\mathcal{G})$ is the smallest invariant manifold of the group $\overline{G}$ containing the manifold $\mathcal{G}$. The resulting correlations are the unknown invariants. Thus, a group G of transformations in the space V_n has $\frac{1}{2}n(n + 1) - \delta(V_n, G)$ functionally independent invariants $J(g_{ij})$, by means of which one can express any other invariant.

Let us demonstrate the above procedure for obtaining invariants by means of a group of conformal transformations. Given a group of conformal transformations G in space V_n with the metric tensor g_{ij}. As it will be shown in § 9 in the given case, the invariant family of spaces $G(V_n)$ coincides with the family of all spaces V_n^* conformal to the space V_n, and the defect $\delta(V_n, G) = 1$, so that $G(V_n)$ depends on one arbitrary function. This arbitrary function appears in the given case as follows. Let us assume that the space V_n in the system of coordinates $\{x\}$ is determined by the manifold $\mathcal{G}$, governed by the equation $y_{ij} = g_{ij}(x)$ $(i, j = 1, \ldots, n)$, and that the space V_n^* is conformal to the space V_n. Then, one can find such a manifold $\mathcal{G}^*$ determining the space V_n which is defined by equations $y_{ij} = g^*_{ij}(x)$, where

$$g^*_{ij}(x) = \sigma(x)g_{ij}(x), \quad \sigma(x) \neq 0 \quad (i, j = 1, \ldots, n). \tag{8.11}$$

Arbitrary choice of the function σ provides the invariant class $G(V_n)$.

Now let us eliminate the arbitrary function σ from equations (8.11) that determine the invariant class $G(V_n)$. We suppose that, for example,

$g_{11} \neq 0$. Then, equations (8.11) yield $\sigma(x) = g^*_{11}(x)/g_{11}(x)$ when $i = j = 1$. Substituting this value of σ into the remaining equations (8.11) one arrives to the equations

$$\frac{g^*_{ij}(x)}{g^*_{11}(x)} = \frac{g_{ij}(x)}{g_{11}(x)} \quad (i, j = 1, \ldots, n).$$

Hence, the values g_{ij}/g_{11} remain unaltered upon transformation of the space V_n to any conformal space V^*_n and in the given case we have the following invariants, well known in Riemannian geometry:

$$J_{ij} = \frac{g_{ij}}{g_{11}} \quad (i \leq j;\ i = 1, \ldots, n;\ j = 2, \ldots, n).$$

In the following two paragraphs we will dwell upon several examples of groups of generalized motions.

§ 9 Groups of conformal transformations

9.1 Conformal transformations as generalized motions

A group G_r of transformations (8.1) in the space V_n with a basic metric form (5.1) is called a group of conformal transformations in the space V_n if equation $d\bar{s}^2 = \nu(x, a) ds^2$ holds with all transformations of the group G_r when the function $\nu(x, a) \neq 0$. According to 6.2, this equation is equivalent to the fact that

$$g_{ij}(\bar{x}) = \nu(x, a)\bar{g}_{ij}(x) \quad (i, j = 1, \ldots, n)$$

for every transformation (8.1) of the group G_r. Multiplying these equalities by $\frac{\partial f^i}{\partial x^k}\frac{\partial f^j}{\partial x^l}$ and summing over i, j, one obtains

$$g_{ij}[f(x, a)]\frac{\partial f^i(x, a)}{\partial x^k}\frac{\partial f^j(x, a)}{\partial x^l} = \nu(x, a)g_{kl}(x)(k, l = 1, \ldots, n).$$

by virtue of equations (6.2). Now let us differentiate the resulting equalities with respect to parameters a^α ($\alpha = 1, \ldots, r$) when $a = 0$. Finally, we denote $\mu_\alpha(x) = \frac{\partial \nu(x,a)}{\partial a_\alpha}\Big|_{a=0}$ and obtain the *generalized Killing equations*

$$\xi_{\alpha i,j} + \xi_{\alpha j,i} = \mu_\alpha g_{ij} \tag{9.1}$$

$$(\alpha = 1, \ldots, r; i, j = 1, \ldots, n)$$

with respect to infinitesimal generators (8.6) of the group G_r of conformal transformations in the space V_n. Here the same notation as in 8.1 is used.

In the same way as Killing equations (8.5) characterize a group of motions, the generalized Killing equations (9.1) give a necessary and sufficient condition in terms of infinitesimal generators for the group G_r to be a group of conformal transformations in space V_n.

Let us find the defect $\delta(V_n, G_r)$ of a group of conformal transformations. We assume that at least one of the functions $\mu_\alpha(x)$ in equations 9.1 is not vanishing (if all $\mu_\alpha(x) = 0$, then the group G_r is a group of motions, for in this case equations (9.1) coincide with the Killing equations (8.5)). Then, we have $\delta = \operatorname{rank} \|\xi_{\alpha i,j} + \xi_{\alpha j,i}\| = \operatorname{rank} \|\mu_\alpha g_{ij}\| = 1$ according to the formula (8.8). Therefore, according to 8.3, the invariant family of spaces $G_r(V_n)$ represents a one-parameter family of n-dimensional Riemannian spaces in the given case.

Let us demonstrate that the family $G_r(V_n)$ is governed by equations

$$\frac{y_{ip}}{y_{11}} = \frac{g_{ip}(x)}{g_{11}(x)} \quad (i = 1, \ldots, n; p = 2, \ldots n), \tag{9.2}$$

where $g_{ij}(x)$ is the metric tensor of the space V_n (it is assumed that $g_{11}(x) \neq 0$). One can take y_{11} as an arbitrary parameter of the family $G_r(V_n)$. The family of Riemannian spaces determined by equations (9.2) contains the space V_n, for equations (9.2) hold when $y_{ij} = g_{ij}(x)$. Therefore, in order to proof that the invariant lass $G_r(V_n)$ is given by equations (9.2) it is sufficient to show that equations (9.2) determine the invariant manifold of the group $\overline{G}_r$.

Let G_1 be a one-parameter group G_r, $\overline{X}$ be an infinitesimal generator (6.8) of the extended group $\overline{G}_1$ so that functions $\xi^i(x)$ $(i = 1, \ldots, n)$ satisfy the generalized Killing equations (9.1). Let us write equations (9.2) in the form

$$\psi_{ip} \equiv \frac{g_{ip}(x)}{g_{11}(x)} y_{11} - y_{ip} = 0 \quad (i = 1, \ldots, n; p = 2, \ldots, n).$$

The invariance conditions of the manifold given by these equations with respect to the group $\overline{G}_1$ have the form

$$\overline{X}\psi_{ip}\big|_{(9.2)} = 0 \quad (i = 1, \ldots, n; p = 2, \ldots, n).$$

First, let us consider these conditions for $i = 1$. We have

$$\overline{X}\psi_{1p}\big|_{(9.2)} = \left(y_{1k}\frac{\partial \xi^k}{\partial x^p} + y_{pk}\frac{\partial \xi^k}{\partial x^1} - 2\frac{g_{1p}(x)}{g_{11}(x)} y_{1k}\frac{\partial \xi^k}{\partial x^1} \right)\Bigg|_{(9.2)} +$$

$$+\frac{y_{11}}{g_{11}(x)} \left(\xi^k \frac{\partial g_{1p}(x)}{\partial x^k} - \frac{g_{1p}(x)}{g_{11}(x)} \xi^k \frac{\partial g_{11}(x)}{\partial x^k} \right).$$

The right-hand expression in the first braces can be rewritten by means of equation (9.2) in the form

$$\frac{y_{11}}{g_{11}}\left(g_{1k}\frac{\partial\xi^k}{\partial x^p} + g_{pk}\frac{\partial\xi^k}{\partial x^1} - 2\frac{g_{1p}}{g_{11}}g_{1k}\frac{\partial\xi^k}{\partial x^1}\right).$$

Therefore, using the identities (8.4) and equations (9.1) one obtains

$$\overline{X}\psi_{1p}\Big|_{(9.2)} = \frac{y_{11}}{g_{11}^2}\left[g_{11}(\xi_{1,p} + \xi_{p,1}) - 2g_{1p}\xi_{1,1}\right] =$$

$$= \frac{y_{11}}{g_{11}^2}[g_{11}\cdot\mu g_{1p} - g_{1p}\cdot\mu g_{11}] = 0 \quad (p = 2, \ldots, n).$$

Likewise, one obtains equations

$$\overline{X}\psi_{ip}|_{(9.2)} = \left(y_{ik}\frac{\partial\xi^k}{\partial x^p} + y_{pk}\frac{\partial\xi^k}{\partial x^i} - 2\frac{g_{ip}}{g_{11}}y_{1k}\frac{\partial\xi^k}{\partial x^1}\right)\Bigg|_{(9.2)} +$$

$$+\frac{y_{11}}{g_{11}}\left(\xi^k\frac{\partial g_{ip}}{\partial x^k} - \frac{g_{ip}}{\partial x^k} - \frac{g_{ip}}{g_{11}}\xi^k\frac{\partial g_{11}}{\partial x^k}\right) =$$

$$= \frac{y_{11}}{g_{11}^2}\left[g_{11}(\xi_{i,p} + \xi_{p,i}) - 2g_{ip}\xi_{1,1}\right] = 0$$

for values $i = 2, \ldots, n$.

Thus,

$$\overline{X}\psi_{ip}|_{(9.2)} = 0.$$

It means that equations (9.2) determine the invariant manifold of the group $\overline{G}_1$. Since this is true for any one-parameter subgroup of the group G_r, equations (9.2) set the invariant manifold of the group $\overline{G}_r$.

Let us assume that V_n^* is a space of the family $G_r(V_n)$ determined by equations (9.2). If we write equations determining the corresponding manifold $\mathcal{G}^*$ of the space V_n^* in the form

$$y_{ij} = g_{ij}^*(x) \quad (i, j = 1, \ldots, n)$$

and introduce the notation

$$\sigma(x) = \frac{g_{11}^*(x)}{g_{11}(x)},$$

equations (9.2) yield

$$g_{ij}^*(x) = \sigma(x)g_{ij}(x). \tag{9.3}$$

Thus, the metric tensor $g_{ij}^*(x)$ of any space $V_n^* \in G_r(V_n)$ satisfies equations (9.3). On the contrary, if any Riemannian space V_n^* has a metric tensor

$g^*_{ij}(x)$ determined by equations (9.3), then the manifold $y_{ij} = g^*_{ij}(x)$ lies in the manifold defined by equations (9.2). At the same time, an arbitrary function $\sigma(x) \neq 0$ in equations (9.3) corresponds to a one-parameter arbitrariness in equations (9.2). Since equations (9.3) determine the family of all spaces conformal to V_n when an arbitrary function $\sigma(x) \neq 0$, the invariant family of spaces $G_r(V_n)$ for the group G_r of conformal transformations in the space V_n coincides with the family of all spaces conformal to V_n.

9.2 Spaces with trivial and nontrivial conformal group

The invariance defect for a group of conformal transformations in the space V_n is $\delta = I$ (see 9.1). Therefore, in the given case, one of the two possibilities mentioned above at the end of 8.3 is carried out by all means.

If G_r is a group of conformal transformations in the space V_n and the condition (I) in 8.3 holds, then the group G_r is a group of motions in a space V_n^* conformal to the space V_n. In this case, the group G_r is referred to as a *trivial conformal group* in the space V_n. If condition (II) holds, the group G_r is not a group of motions in any space conformal to V_n. Then, the group G_r is called a *nontrivial conformal group* in the space V_n, and the space V_n is termed as a *space with nontrivial conformal group.*

In what follows four-dimensional Riemannian spaces of a normal hyperbolic type with a non-trivial conformal group are considered. The description of these spaces by R.F. Bilyalov will be given here (see [116], Ch.VII, where bibliography is also given). We will formulate Bilyalov's result in terms of contravariant components g^{ij} of the metric tensor, invoking application of Riemannian spaces in investigation of differential equations of the second order with higher coefficients $g^{ij}(x)$. The result is as follows: the space V_4 of the signature $(- - - +)$ has a nontrivial conformal group if and only if a metric tensor of the space is reduced to the form

$$\|g^{ij}\| = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & f(x^4) & \varphi(x^4) & 0 \\ 0 & \varphi(x^4) & h(x^4) & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{9.4}$$

where functions f, φ, h satisfy the condition $fh - \varphi^2 > 0$, by means of transformation of coordinates and shift to a conformal space. In this case, only the values $6, 7, 15$ can take the maximum order of the conformal group.

Let us reduce tensors of the form (9.4) to a more convenient form. First, let us show that instead of the three functions f, φ, h one can take only two. Indeed, noting that $h(x^4) \neq 0$ due to the inequality $fh - \varphi^2 > 0$ and introducing the coordinates

$$\bar{x}^1 = x^1, \bar{x}^2 = x^2, \bar{x}^3 = x^3, \bar{x}^4 = \int h(x^4)dx^4$$

we reduce tensor 9.4 to the form

$$\|g^{ij}\| = \begin{pmatrix} 0 & 0 & 0 & h \\ 0 & f & \varphi & 0 \\ 0 & \varphi & h & 0 \\ h & 0 & 0 & 0 \end{pmatrix}.$$

Then, if we turn to a conformal space with the metric tensor $g^*_{ij} = hg_{ij}$, we obtain the space V^*_4 with a tensor in the form (9.4) with the function $h = 1$ and with arbitrary $f(x^4)$ and $\varphi(x^4)$ satisfying the condition $f - \varphi^2 > 0$.

If we change the coordinates:

$$x^1 = \bar{x}^1 + \bar{x}^4, \quad x^2 = \bar{x}^2, \quad x^3 = \bar{x}^3, \quad x^4 = \bar{x}^1 - \bar{x}^4$$

and introduce the individual notation

$$\bar{x}^1 = x, \quad \bar{x}^2 = y, \quad \bar{x}^3 = z, \quad \bar{x}^4 = t,$$

the above result concerning spaces with a nontrivial conformal group can be finally formulated as follows.

Theorem 1.13. The space V_4 of a normal hyperbolic type has a nontrivial conformal group if and only if there exists a conformal space V^*_n with a metric tensor written in a certain coordinate system in the form

$$\|g^{ij}\| = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -f(x-t) & -\varphi(x-t) & 0 \\ 0 & -\varphi(x-t) & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad f - \varphi^2 > 0. \tag{9.5}$$

The maximum order of the conformal group can assume only one of the three values: $6, 7$ or 15.

9.3 Conformally flat spaces

Further, the following theorem will be useful.

Theorem 1.14. [130] The order of the group of conformal transformations in the space V_n takes its maximum value $\frac{1}{2}(n+1)(n+2)$ if and only if the space V_n is conformal to a flat space.

For the sake of brevity we will term Riemannian spaces conformal to a flat space as *conformally flat spaces.* For our further reference, let us write out basis infinitesimal generators of the group of conformal transformations in conformally flat spaces V_n. Obviously, it is sufficient to consider only a flat space.

Let us assume that a Cartesian system of coordinates is introduced in a flat n-dimensional Riemannian space S_n so that the metric tensor of the space is defined by equations

$$y_{ij} = \delta_{ij} \quad (i, j = 1, \ldots, n),$$

where δ_{ij} is a Kronecker symbol. (We assume that S_n has a positive definite metric form (5.1). When the form is indefinite, e.g. S_n has a normal hyperbolic type, one can obtain a positive definite form by introducing the corresponding number of imaginary coordinates). In this system of coordinates, all Christoffel symbols Γ^i_{jk} are equal to zero, and $\xi_i = \xi^i$. Therefore, the generalized Killing equations (9.2) have the form

$$\frac{\partial \xi^i}{\partial x^j} + \frac{\partial \xi^j}{\partial x^i} = \mu(x)\delta_{ij} \quad (i, j = 1, \ldots, n) \tag{9.6}$$

for every operator

$$X = \xi^i(x)\frac{\partial}{\partial x^i}$$

of the group of conformal transformations in the space S_n. The general solution of these equations is well-known and can be written as follows:

$$\xi^i = A_j(2x^i x^j - |x|^2\delta^{ij}) + a^i_j x^j + bx^i + c^i \quad (i = 1, \ldots, n), \tag{9.7}$$

where

$$|x|^2 = (x')^2 + \ldots + (x^n)^2, \quad a^j_i = -a^i_j,$$

and $b,\ c^i,\ A_i,\ a^i_j\ (i < j)$ are arbitrary constants.

According to the above theorem, the solution of (9.7) depends on $\frac{1}{2}(n+1)(n+2)$ arbitrary constants. If we set consecutively one of these constants equals to 1 and others equal to 0 we obtain the following generators of the group of conformal transformations in the flat space S_n :

$$X_i = \frac{\partial}{\partial x^i}, \quad X_{ij} = x^j\frac{\partial}{\partial x^i} - x^i\frac{\partial}{\partial x^j}\ (i < j),$$

$$Z = x^i\frac{\partial}{\partial x^i}, \quad Y_i = (2x^i x^j - |x|^2\delta^{ij})\frac{\partial}{\partial x^j}, \quad (i, j = 1, \ldots, n). \tag{9.8}$$

Note that in order to obtain the generators of the group of conformal transformations in a flat space with an indefinite metric form it is sufficient to change the corresponding coordinates x^i to imaginary ones in the obove operators. Thus, there is no need to solve Eqs. (9.1) again for spaces with an indefinite form.

§ 10 Other examples

10.1 Generalized motions with defect 2

Let us assume that the coordinates $\xi^i_\alpha(x)$ $(\alpha = 1, \ldots, r)$ of operators (8.6) of the group G_r $(r \geq 2)$ of transformations in the space V_n $(n \geq 2)$ satisfy the conditions

$$\xi_{\alpha k,l} + \xi_{\alpha l,k} = \mu_\alpha g_{kl} \quad (k, l = 1, \ldots, n-1),$$

$$\xi_{\alpha i,n} + \xi_{\alpha n,i} = \nu_\alpha g_{in} \quad (i = 1, \ldots, n), \tag{10.1}$$

for all $\alpha = 1, \ldots, r$.

If $\mu_\alpha(x) = \nu_\alpha(x)(\alpha = 1, \ldots r)$, then equations (10.1) coincide with the generalized Killing equations (9.1) for groups of conformal transformations. Here we consider the case when $\mu_\alpha \neq \nu_\alpha$ for at least one value of α. Moreover, we assume that the vectors $\mu = (\mu_1, \ldots, \mu_r)$ and $\nu = (\nu_1, \ldots, \nu_r)$ are linearly independent. One can easily verify that in this assumptions

$$\operatorname{rank} \|\xi_{\alpha i,j} + \xi_{\alpha j,i}\| = 2,$$

so that $\delta(V_n, G_r) = 2$ according to Eq. (8.8).

10.2 Examples of generalized motions in the flat space

Let us demonstrate that there exist a group with the properties mentioned in Section 10.1. Consider a flat n-dimensional space $S_n (n > 3)$ with the metric tensor $g_{ij} = \delta_{ij}$ $(i, j = 1, \ldots, n)$. Then, the equations (10.1) for every value of α are written in the form

$$\frac{\partial \xi^k}{\partial x^l} + \frac{\partial \xi^l}{\partial x^k} = \mu \delta_{kl} \quad (k, l = 1, \ldots, n), \tag{10.2}$$

$$\frac{\partial \xi^n}{\partial x^k} + \frac{\partial \xi^k}{\partial x^n} = 0 \quad (k = 1, \ldots, n), \tag{10.3}$$

$$\frac{\partial \xi^n}{\partial x^n} = \frac{\nu}{2} \cdot \tag{10.4}$$

Throughout this section the indices k, l run over the values from 1 to $n-1$. In particular, if the summation is assume in these indices, it is also taken over the values from 1 to $n-1$.

Equations (10.2) for $\xi^1, \ldots, \xi^{n-1}$ considered as functions of the variables $x^1, \ldots, x^{n-1}$ have the form (9.6). As for the variable x^n upon which the functions ξ^k depend as well, plays the role of a parameter. Therefore, according to Eqs. (9.7), the general solution to Eqs. (10.2) has the form

$$\xi^k = A_l(x^n)[2x^k x^l - \rho^2 \delta^{kl}] + a^k_l(x^n) x^l + b(x^n) x^k + c^k(x^n) \tag{10.5}$$
$$(k = 1, \ldots, n-1),$$

where

$$\rho^2 = (x^1)^2 + \ldots + (x^{n-1})^2, \quad a^k_l(x^n) = -a^l_k(x^n),$$

and $A_l, a^k_l(k < l), c^k(k, l = 1, \ldots, n-1)$ and b are arbitrary functions of x^n.

The equations (10.3) provide an over-determined system of equations for the function ξ^n. The compatibility conditions of this system are written

$$\frac{\partial}{\partial x^n}\left[\frac{\partial \xi^k}{\partial x^l} - \frac{\partial \xi^l}{\partial x^k}\right] = 0 \quad (k \neq l; k, l = 1, \ldots, n-1).$$

Upon substituting the values

$$\frac{\partial \xi^k}{\partial x^l} - \frac{\partial \xi^l}{\partial x^k} = 4(A_l x^k - A_k x^l) + 2a^k_l$$

obtained from Eqs. (10.5) they become

$$2(\dot{A}_l x^k - \dot{A}_k x^l) + \dot{a}^k_l = 0 \quad (k, l = 1, \ldots, n-1),$$

where the dot indicates differentiation with respect to x^n. It follows:

$$\dot{A}_k = 0, \ \dot{a}^k_l = 0,$$

hence all A_k and a^k_l in (10.5) are constants. Equations (10.3) are easily solved now and yield

$$\xi^n = -\frac{1}{2} b(x^n)\rho^2 - \sum_{k=1}^{n-1} \dot{c}^k(x^n) x^k + f(x)^n,$$

where $f(x^n)$ is an arbitrary function. Thus, the general solution for equations (10.2) and (10.3) has the form

$$\xi^k = A_l(2x^k x^l - \rho^2\delta^{kl}) + a^k_l x^l + b(x^n) + c^k(x^n), \quad k = 1, \ldots, n-1,$$
$$\xi^n = -\frac{1}{2}\dot{b}(x^n)\rho^2 - \sum_{k=1}^{n-1} \dot{c}^k(x^n) x^k + f(x^n). \tag{10.6}$$

The functions $\mu(x)$ and $\nu(x)$ are obtained from equations (10.2) and (10.4), respectively. This completes the solution of Eqs. (10.2)-(10.4).

Remark 1.2. Eqs. (9.1) provide the invariance conditions of the manifold defined by Eqs. (9.2). Therefore, Eqs. (9.1) are the determining equations. Hence, the set of all solutions for Eqs. (9.1) is a Lie algebra with respect to the commutator of the operators X. On the contrary, the set of the operators

$$X = \xi^i \frac{\partial}{\partial x^i}$$

corresponding to the functions (10.6) is not a Lie algebra.

10.3 Particular cases from Section 10.2

Let us extract from the set of solutions (10.6) a subset that spans a Lie algebra. We will set

$$a_l^k = 0, \; c^k = 0 \; (k, l = 1, \ldots, n-1), \; b = 0,$$

and assume that the constants $A_l (l = 1, \ldots, n-1)$ and the function $f(x^n)$ are arbitrary. The operators

$$X_f = f(x^n) \frac{\partial}{\partial x^n}$$

determine an infinite transformation group on a straight line x^n. It is well known from Lie's theory that the maximum order of a finite subgroup of this infinite group equals to three. One can take the operators corresponding to the functions $f = 1, f = x^n, f = (x^n)^2$ as generators of this three-parameter subgroup. Taking arbitrary $A_l (l = 1 \ldots, n-1)$ and the indicated values of the function $f(x^n)$ one obtains the following $(n+2)$-parameter group G_{n+2} with the generators

$$\begin{gathered} X_1 = \frac{\partial}{\partial x^n}, \; X_2 = x^n \frac{\partial}{\partial x^n}, \; X_3 = (x^n)^2 \frac{\partial}{\partial x^n}, \\ Y_k = (2x^k x^l - \rho^2 \delta^{kl}) \frac{\partial}{\partial x^l} \; (k = 1, \ldots, n-1). \end{gathered} \tag{10.7}$$

These operators have the following commutators:

$$\begin{gathered} [X_1 X_2] = X_1, \; [X_1 X_3] = 2X_2, \; [X_2 X_3] = X_3, \\ [X_p Y_k] = 0, \; [Y_k Y_l] = 0 \; (p = 1, 2, 3; k, l = 1, \ldots, n-1), \end{gathered}$$

so that operators (10.7) provide a basis of a Lie algebra indeed.

Substitution of the operators (10.7) in Eq. (8.8) yields

$$\delta(S_n, G_{n+2}) = 2.$$

Hence the group G_{n+2} with the generators (10.7) provides an example of a group satisfying the conditions of Section 10.1.

We will discuss the invariant family of spaces and the invariants of the resulting group at the end of Section 10.5.

10.4 More particular cases from Section 10.2

Consider a particular case of equations (10.2)-(10.4) by letting $\mu = 0$. Then Eqs. (10.5) yield

$$\mu = 4A_l(x^n)x^l + 2b(x^n),$$

so that the condition $\mu = 0$ provides

$$A_l = 0, \ (l = 1, \dots, n-1), \quad b = 0.$$

Substituting these values of A_l, b in Eqs. (10.6) we obtain the following general solution to Eqs. (10.2), (10.3) when $\mu = 0$:

$$\xi^k = a^k_l x^l + c^k(x^n), \quad k = 1, \dots, n-1,$$

$$\xi^n = -\sum_{k=1}^{n-1} \dot{c}^k(x^n)x^k + f(x^n). \tag{10.8}$$

Here $a^k_l, c^k(x^n), f(x^n)$ have the same meaning as in Eqs. (10.6).

The set of the generators with the coordinates (10.8) does not span a Lie algebra (see Remark 1.2 in Section 10.2). One can single out from the set of solutions (10.8) the subsets that compose three Lie algebras given below.

The algebra of the dimension

$$r = n + \frac{1}{2}(n-2)(n-3)$$

spanned by the operators

$$X_1 = e^{\lambda x^n}\frac{\partial}{\partial x^1} - \lambda e^{\lambda x^n}x^1\frac{\partial}{\partial x^n} \ (\lambda = \text{ const}),$$

$$X_i = \frac{\partial}{\partial x_i} \ (i = 2, \dots, n),$$

$$X_{ij}x^j\frac{\partial}{\partial x^i} - x^i\frac{\partial}{\partial x^j} \quad (i < j; i, j = 2, \dots, n-1). \tag{10.9}$$

The algebra of the dimension $r = \infty$ spanned by the operators

$$X_k = \frac{\partial}{\partial x^k} \quad (k = 1, \dots, n-1),$$

$$X_{kl} = x^l\frac{\partial}{\partial x^k} - x^k\frac{\partial}{\partial x^l} \quad (k < l; k, l = 1, \dots, n-1),$$

$$X_f = f(x^n)\frac{\partial}{\partial x^n}, \tag{10.10}$$

where $f(x^n)$ is an arbitrary function.

The algebra of the dimension

$$r = (n-1) + \frac{1}{2}(n-2)(n-3)$$

spanned by the operators

$$\begin{aligned} X_1 &= h(x^n)\frac{\partial}{\partial x^1} - h'(x^n)x^1\frac{\partial}{\partial x^n}, \\ X_i &= \frac{\partial}{\partial x^i} \quad (i = 2, \ldots, n-1), \\ X_{kl} &= x^l\frac{\partial}{\partial x^k} - x^k\frac{\partial}{\partial x^l} \quad (k < l; k, l = 1, \ldots, n-1), \end{aligned} \tag{10.11}$$

where $h(x^n)$ is a fixed function and $h'(x^n)$ is its derivative. Depending on the choice of the function $h(x^n)$ one obtains different Lie algebras.

Furthermore, there are Lie algebras with the bases obtained from the operators (10.9) and (10.11) by replacing the variable x^1 by any of the variables $x^2, \ldots, x^{n-1}$, as well as the Lie algebra of the group of isometric motions in S_n corresponding to $\nu = 0$.

The defect of groups corresponding to all extracted algebras is equal to $\delta = 1$. Among these groups there is an infinite group corresponding to an infinite-dimensional Lie algebra spanned by the operators (10.10). One can readily verify that the manifold given by equations

$$y_{ik} = \delta_{ik} \quad (i = 1, \ldots, n; k = 1, \ldots, n-1)$$

is the smallest possible invariant manifold of the extended group $\overline{G}_\infty$ corresponding to the group G_∞ with the generators (10.10). It means that (see Section 8.3) the invariant family of spaces $G_\infty(S_n)$ is the set of the spaces V_n with the metric tensor

$$\begin{aligned} g_{ik} &= \delta_{ik} \quad (i = 1, \ldots, n; k = 1, \ldots, n-1), \\ g_{nn} &= \sigma(x), \end{aligned}$$

where $\sigma(x)$ is an arbitrary function of $x = (x^1, \ldots, x^n)$. The components g_{ik} $(i = 1, \ldots, n; k = 1, \ldots, n-1)$ of the metric tensor provide the invariants of the group G_∞ of transformations in the space S_n. There is no difficulty in understanding the geometrical meaning of these invariants. If we consider an n-dimensional sphere with the radius ρ and the center $x \in S_n$:

$$(dx^1)^2 + \ldots + (dx^n)^2 \leq \rho^2,$$

then the transformations of the group G_∞ turn this sphere into an ellipsoid with the semi axes equal to ρ in the subspace S_{n-1} of the variables $x^1, \ldots, x^{n-1}$ and the semi axis of an arbitrary length in the direction of x^n.

10.5 Generalized motions with given invariants

Let us consider an example on finding a group of generalized motions with given invariants. Consider the flat space S_n with the metric tensor $g_{ij} = \delta_{ij}$ and find a group G of generalized motions with the defect $\delta = 2$, invariants of which are

$$J_{ki} = g_{ki} \quad (k \neq i;\ k = 1, \dots, n-1;\ i = 1, \dots, n),$$
$$J_{kk} = \frac{g_{kk}}{g_{11}} \quad (k = 2, \dots, n-1).$$

According to Section 8.4 the invariant family of spaces $G(S_n)$ is determined by the manifold $\overline{G}(\mathcal{G})$ defined as follows:

$$y_{ki} = 0 \quad (k \neq i;\ k = 1, \dots, n-1;\ i = 1, \dots, n),$$
$$y_{kk} = y_{11} \quad (k = 2, \dots, n-1). \tag{10.12}$$

In order to find the group with the above invariants one has to investigate the invariance conditions of the manifold defined by Eqs. (10.12) with respect to an unknown group $\overline{G}$ with generators $\overline{X}$. These conditions are

$$\overline{X} y_{ki}\big|_{(10.12)} = y_{11}\frac{\partial \xi^k}{\partial x^i} + y_{ii}\frac{\partial \xi^i}{\partial x^k} \quad (k \neq i) \tag{10.13}$$

(there is no summation over the index i in the last term) and

$$\overline{X}(y_{11} - y_{kk})\big|_{(10.12)} = 2y_{11}\left(\frac{\partial \xi^k}{\partial x^k} - \frac{\partial \xi^1}{\partial x^1}\right) \quad (k = 2, \dots, n-1) \tag{10.14}$$

(there is no summation over k in the right-hand side of this equation). Setting expressions (10.12) and (10.13) equal to zero and invoking that y_{11} and y_{nn} are free variables, we obtain invariance conditions of the manifold determined by equations (10.12) with respect to the group $\overline{G}$ in the form of the following equations:

$$\frac{\partial \xi^k}{\partial x^l} + \frac{\partial \xi^l}{\partial x^k} = 0 \quad (k \neq l; k, l = 1, \dots, n-1), \tag{10.15}$$

$$\frac{\partial \xi^k}{\partial x^n} = 0, \quad \frac{\partial \xi^n}{\partial x^k} = 0 \quad (k = 1, \dots, n-1), \tag{10.16}$$

$$2\frac{\partial \xi^k}{\partial x^k} = \mu \quad (k = 1, \dots, n-1), \quad \text{no summation in } k. \tag{10.17}$$

The general solution of equations (10.15)-(10.17) generates a Lie algebra since these equations represent the necessary and sufficient conditions of invariance of some manifold. One can see that the system of equations (10.15), (10.17) coincides with equations (10.2). Therefore, their general solution is given by the formulae (10.5). Substituting (10.5) into equations (10.16) we see that all coefficients

$$A_l, \quad a_l^k \ (k < l), \quad b, \quad c^k \quad (k, l = 1, \ldots, n-1) \tag{10.18}$$

are constants and ξ^n depends only on x^n. Thus, the system of differential equations (10.15)-(10.17) has the following general solution:

$$\begin{aligned} &\xi^k = A_l(2x^k x^l - \rho^2 \delta^{kl}) + a_l^k x^l + bx^k + c^k \quad (k = 1, \ldots, n-1), \\ &\xi^n = f(x^n). \end{aligned} \tag{10.19}$$

This solution depends on

$$\frac{1}{2} n(n+1)$$

arbitrary constants (10.18) and one arbitrary function $f(x^n)$. As a result, we have an infinite group with the generators

$$\begin{aligned} &X_{kl} = x^l \frac{\partial}{\partial x^k} - x^k \frac{\partial}{\partial x^l} \qquad (k < l;\ k, l = 1, \ldots, n-1), \\ &X_k = \frac{\partial}{\partial x^k}\,, \quad Y_k = (2x^k x^l - \rho^2 \delta^{kl}) \frac{\partial}{\partial x^l} \quad (k, = 1, \ldots, n-1), \\ &Z = x^k \frac{\partial}{\partial x^k}\,, \quad X_f = f(x_n) \frac{\partial}{\partial x^n} \cdot \end{aligned} \tag{10.20}$$

The groups with the generators (10.7) and (10.10) considered above are subgroups of the group with the generators (10.20). The groups with the generators (10.7) and (10.20) have one and the same defect $\delta = 2$ and consequently the same invariants. Indeed, all invariants of a group G of transformations in the space V_n are invariants of any subgroup G' of the group G. If

$$\delta(V_n, G) = \delta(V_n, G')$$

then the groups G and G' have one and the same invariants.

Open problems: Classification of Riemannian spaces according to generalized motions

The following problems on local and global classification are important not only for better understanding groups of generalized motions and the global properties of spaces with nontrivial conformal group, but also for geometry of partial differential equations.

Problem 1.1. Classify Riemannian spaces with respect to their local groups of generalized motions.

Such problem in case of groups of isometric motions and conformal transformations has been considered in Riemannian geometry for a long time (see ([116]) and the references therein). In these cases the Killing equations and the generalized Killing equations serve as a starting point in investigation of groups of isometric motions and conformal transformations, respectively.

In case of generalized motions equation (8.8) containing a higher degree of arbitrariness than, e.g., generalized Killing equations (9.8), is the analogue of Killing equations. The thing is that according to § 9, generalized Killing equations (as well as Killing equations for groups of motions) determine not only the defect but also the corresponding invariant family of spaces. Therefore, beginning to consider problems of classification of Riemannian spaces with respect to groups of generalized motions one comes across the arbitrariness connected with the choice of an invariant family of spaces. This complicates the problem to a great extent.

Problem 1.2. Investigate global properties of the Riemannian spaces V_4 of normal hyperbolic type having locally a nontrivial conformal group, i.e. such that there exists a nontrivial local group of conformal transformations in a neighborhood of every point $x \in V_4$. In particular, I think that *if* V_4 *is a topologically 1-connected space with locally nontrivial conformal group, then the dimension of the maximal conformal group is the same at any point in* V_4. Clarify if this hypothesis is correct or false.

Chapter 3
Group analysis of second-order equations

This chapter is dedicated to investigation of symmetry properties of partial differential equations of the second order. Theorem 1.17 on uniqueness of linear conformally invariant equations in four-dimensional spaces of normal hyperbolic type with nontrivial conformal group, proved in Section 14.3, will be used in the next chapter.

Literature: Lie [89], Ovsyannikov [110, 111], Ibragimov [52, 58].

§ 11 Determining equations

11.1 Determining equations for semi-linear equations

Consider the second order *semi-linear* differential equations

$$L[u] \equiv g^{ij}(x)u_{ij} + b^i(x)u_i + \psi(x,u) = 0 \tag{11.1}$$

with $n > 2$ independent variables $x = (x^1, \ldots, x^n)$. We will assume that their coefficients are analytic functions and the matrix $\|g^{ij}(x)\|$ is nondegenerate in some domain of the variables x. In what follows, the usual notation for partial derivatives is used:

$$u_i \equiv \frac{\partial u}{\partial x^i}, \quad u_{ij} \equiv \frac{\partial^2 u}{\partial x^i \partial x^j} \qquad (i,j = 1,\ldots,n).$$

It is convenient to write Eq. (11.1) in the following covariant form:

$$L[u] \equiv g^{ij}(x)u_{,ij} + a^i(x)u_{,i} + \psi(x,u) = 0, \tag{11.2}$$

where indices after the comma indicate covariant differentiation in the Riemannian space V_n with the metric tensor

$$g_{ij}(x), \quad \text{where} \quad \|g_{ij}\| = \|g^{ij}\|^{-1}.$$

Due to the covariance of Eq. (11.2), the quantities

$$a^i = b^i + g^{jk}\Gamma^i_{jk} \quad (i = 1,\ldots,n) \tag{11.3}$$

behave as the components of a contravariant vector in the space V_n [111]. In what follows, all considerations are made in terms of the Riemannian space V_n. The space V_n connected with a linear second-order equation was used by Hadamard [41] in investigation of the Cauchy problem and later by Ovsyannikov in investigation of group properties of linear second-order differential equations [110], [111].

Generators of the group G admitted by equations (11.1) are written in the form

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta(x,u)\frac{\partial}{\partial u}\cdot$$

Let us denote by $\widetilde{\widetilde{X}}$ the prolongation of the operator X to the derivatives of u up to the second order. The invariance condition of Eq. (11.1) with respect to the group G (i.e. the *determining equation*, see § 4) is written in the same form as in the case of a *linear* second-order equation (cf. [110]):

$$\widetilde{\widetilde{X}} L[u] = \lambda L[u], \tag{11.4}$$

where λ is an arbitrary function. Extending to Eq. (11.4) Ovsyannikov's analysis of the determining equation for linear equations ([111], § 27) we arrive at the following statement.

Lemma 1.7. The determining equation (11.4) yields:

$$\lambda = \lambda(x,u), \quad \xi^i = \xi^i(x), \quad \eta = \sigma(x)u + \tau(x).$$

Therefore the admitted operator X has the form

$$X = \xi^i(x)\frac{\partial}{\partial x^i} + [\sigma(x)u + \tau(x)]\frac{\partial}{\partial u}\,, \tag{11.5}$$

where the unknown functions $\xi^i(x)$, $\sigma(x)$, $\tau(x)$ are to be found by solving the determining equation (11.4).

The repeated action of the prolongation procedure given by Eqs. (4.5), (4.7) provide the following *second prolongation* of the operator (11.5):

$$\widetilde{X} = X + \zeta_i\frac{\partial}{\partial u_i}\,, \quad \widetilde{\widetilde{X}} = \widetilde{X} + \zeta_{ij}\frac{\partial}{\partial u_{ij}}\,,$$

where

$$\zeta_i = D_i\big(\sigma(x)u + \tau(x)\big) - u_k D_i(\xi^k) = \sigma u_i + u\sigma_i + \tau_i - u_k\frac{\partial\xi^k}{\partial x_i}$$

and

$$\zeta_{ij} = D_i(\zeta_i) - u_{ik}D_j(\xi^k) = \sigma u_{ij} + u_i\sigma_j + u_j\sigma_i$$
$$+ u\sigma_{ij} + \tau_{ij} - \left(u_{ik}\frac{\partial\xi^k}{\partial x^j} + u_{jk}\frac{\partial\xi^k}{\partial x^i}\right) - u_k\frac{\partial^2\xi^k}{\partial x^i\partial x^j}\cdot$$

Note that in order to calculate the twice prolonged operator $\widetilde{\widetilde{X}}$ one has to apply the prolongation formula (4.7) only to the variables u_i and not to all dependent variables u, u_i in the operator $\widetilde{X}$.

Substituting the resulting expressions of operators $\widetilde{X}$ and $\widetilde{\widetilde{X}}$ into equation (11.4) and collecting the like terms one has

$$\left(\xi^k \frac{\partial g^{ij}}{\partial x^k} - g^{kj}\frac{\partial \xi^i}{\partial x^k} - g^{ki}\frac{\partial \xi^j}{\partial x^k} + \sigma g^{ij}\right) u_{ij}$$
$$+\left(2g^{ij}\sigma_j - g^{jk}\frac{\partial^2 \xi^i}{\partial x^j \partial x^k} - b^j \frac{\partial \xi^i}{\partial x^j} + \sigma b^i + \xi^j \frac{\partial b^i}{\partial x^j}\right) u_i$$
$$+(g^{ij}\sigma_{ij} + b^i\sigma_i)u + X(\psi) + g^{ij}\tau_{ij} + b^i\tau_i$$
$$= \lambda g^{ij}u_{ij} + \lambda b^i u_i + \lambda\psi.$$

All variables x^i, u, u_i, u_{ij} $(i \leq j; i, j = 1, \ldots, n)$ in this equation are independent. Therefore Eq. (11.4) is equivalent to the system of equations

$$\xi^k \frac{\partial g^{ij}}{\partial x^k} - g^{kj}\frac{\partial \xi^i}{\partial x^k} - g^{ki}\frac{\partial \xi^j}{\partial x^k} + \sigma g^{ij} = \lambda g^{ij} \quad (i, j = 1, \ldots, n), \tag{11.6}$$

$$2g^{ij}\sigma_j - g^{jk}\frac{\partial^2 \xi^i}{\partial x^j \partial x^k} - b^j \frac{\partial \xi^i}{\partial x^j} + \sigma b^i + \xi^j \frac{\partial b^i}{\partial x^j} = \lambda b^i \quad (i = 1, \ldots, n), \tag{11.7}$$

$$(g^{ij}\sigma_{ij} + b^i\sigma_i)u + X(\psi) + g^{ij}\tau_{ij} + b^i\tau_i = \lambda\psi. \tag{11.8}$$

This system is obtained by equating the coefficients of u_{ij}, u_i and the terms free of u_i and u_{ij}.

11.2 Covariant form of determining equations

Let us rewrite equations (11.6)-(11.8) in a tensor notation introduced by writing equations (11.1) in the form (11.2). Equations (11.6) provide that $\lambda = \lambda(x)$. Using the notation

$$\mu(x) = \sigma(x) - \lambda(x), \ a_i = g_{ki}a^k, \xi_i = g_{ki}\xi^k$$

and

$$K_{ij} = a_{i,j} - a_{j,i} \ (i, j = 1, \ldots, n),$$

one can write equations (11.6) and (11.7) in the covariant form:

$$\xi_{i,j} + \xi_{j,i} = \mu g_{ij} \ (i, j = 1, \ldots, n) \tag{11.9}$$

and

$$\sigma_{,i} = \frac{2-n}{4}\mu_{,i} - \frac{1}{2}(a_j\xi^j)_{,i} - \frac{1}{2}K_{ij}\xi^j \ (i = 1, \ldots, n), \tag{11.10}$$

respectively [111].

The compatibility conditions for equations (11.10) have the form

$$\sigma_{,ij} = \sigma_{,ji} \quad (i, j = 1, \ldots, n)$$

and yield the following auxiliary system of equations:

$$(K_{il}\xi^l)_{,j} - (K_{jl}\xi^l)_{,i} = 0 \ (i, j = 1, \ldots, n). \tag{11.11}$$

Upon application of the identity ([111], § 28):

$$g^{ij}\sigma_{,ij} + a^i\sigma_{,i} \equiv \xi^i E_{,i} + \mu E,$$

where

$$E = -\frac{1}{2}\left(a^i_{,i} + \frac{1}{2}a^i a_i + \frac{n-2}{2(n-1)}R\right),$$

and R is a scalar curvature of the space V_n, equation (11.8) is rewritten

$$(\xi^i E_{,i} + \mu E)u + (X + \mu - \sigma)\psi + g^{ij}\tau_{,ij} + a^i\tau_{,i} = 0. \tag{11.12}$$

Equations (11.9)-(11.12) represent a system of determining equations of the group G admitted by equation (11.2).

11.3 General discussion of conformal invariance

Suppose that a group G is admitted by equation (11.2). Then, coordinates $\xi^i(x), \sigma(x), \tau(x)$ of infinitesimal generators (11.5) of the group G satisfy all the determining equations (11.9)-(11.12). Since equations (11.9) represent the generalized Killing equations determining a group of conformal transformations, then the "contraction" G° of the group G with the infinitesimal generators

$$X^\circ = \xi^i(x)\frac{\partial}{\partial x^i} \tag{11.13}$$

satisfying equations (11.9) is a subgroup of a group of conformal transformations in the space V_n. One can easily verify that if operators of the form (11.5) generate a Lie algebra, then the corresponding operators (11.2) generate a Lie algebra as well. Meanwhile, the group G° coincides with all the group of conformal transformations in the space V_n if and only if equations (11.11), (11.12) are satisfied identically by virtue of equations (11.9). In case if the group G° coincides with all the group of conformal transformations in V_n we say that equation (11.2) is *conformally invariant.*

Given two equations of the form (11.2) with the higher coefficients $g^{ij}(x)$, $\bar{g}^{ij}(x)$ where $g_{ij}(x)$, $\bar{g}_{ij}(x)$ are elements of the corresponding inverse matrices so that $g_{ik}g^{jk} = \delta^j_i$ and $\bar{g}_{ik}\bar{g}^{jk} = \delta^j_i$. If equations (6.2) are solvable for functions $g_{ij}(x)$ and $\bar{g}_{ij}(x)$ then according to 6.1 one and the same Riemannian space V_n corresponds to the considered equations (11.2) whatever values the coefficients $a^i(x)$ and functions $\psi(x, u)$ take. Equations of the form (11.2), to which one and the same Riemannian space V_n corresponds, will be further referred to as equations in space V_n.

Note that in every Riemannian space V_n there exists at least one conformally invariant equation (11.2). One can easily verify this fact by taking all coefficients a^i equal to zero and setting the function ψ equal to $\frac{n-2}{4(n-1)}Ru$. Then, equations (11.11) and (11.12) are satisfied identically for any operator (11.5) and for the operator (11.5) with the function $\tau(x)$ satisfying $L(\tau) = 0$, respectively. Therefore, operators (11.13) generate all the group of conformal transformations in space V_n so that the constructed equation is conformally invariant. In what follows, we will dwell upon conformally invariant equations in various Riemannian spaces and find all these equations in spaces under consideration V_n.

11.4 Equivalent equations

The problem of group classification of Eqs. (11.2) is simplified if we use the following transformations preserving the form of Eqs. (11.2):

$$\begin{aligned} &\text{a) } \bar{x}^i = f^i(x) \ (i = 1, \dots, n), \quad \bar{u} = \varphi(x)u + h(x), \\ &\text{b) } \overline{L}[u] = \nu(x)L[u)], \end{aligned} \tag{11.14}$$

where

$$\varphi(x) \neq 0, \quad \nu(x) \neq 0, \quad \det\left\| \frac{\partial f^i}{\partial x^j} \right\| \neq 0.$$

Two equations are said to be *equivalent* if one of them is obtained from the other by certain transformations (11.14). Then, all equations of the form (11.2) (or of the form (11.1), which is the same) split into classes of equivalence to be considered further.

§ 12 Linear equations with maximal group

12.1 Standard form of Eq. (11.2) with maximal group

According to 11.3 and § 9, conformally invariant equations in conformally flat spaces V_n are equations (11.2) admitting a group of the maximum order. Since the equivalence transformation (b) in 11.4 corresponds to the shift to a conformal space $\widetilde{V}_n$, we arrive to the fact that if equation (11.2) admits a group of the maximum order it is reduced to the form

$$\Delta u + a^i(x)u_i + \psi(x, u) = 0 \tag{12.1}$$

with some coefficients $a^i(x)$ and the function $\psi(x, u)$. Here Δ indicates the operator

$$\Delta = \sum_{i=1}^{n} \pm \frac{\partial^2}{(\partial x^i)^2}\,,$$

where + and − are allocated in correspondence to the signature of the space V_n. As it was numerously mentioned in the above chapter, one can restrict oneself to consideration of spaces V_n with the positively defined metric form. Therefore, we admit that an n-dimensional Euclidian space corresponds to equation (12.1).

12.2 Determining equations for the linear equation

In the next section, we will be interested in linear equations admitting the group of maximal order. First we note that the general linear equation of the second order

$$g^{ij}(x)u_{,ij} + a^i(x)u_{,i} + c(x)u + f(x) = 0$$

can be reduced to the homogeneous equation

$$L[u] \equiv g^{ij}(x)u_{,ij} + a^i(x)u_{,i} + c(x)u = 0 \tag{12.2}$$

by the equivalence transformation $\bar{u} = u + h(x)$, where $h(x)$ is solution of equation

$$g^{ij}(x)h_{,ij} + a^i(x)h_{,i} + c(x)h + f(x) = 0.$$

Therefore, in what follows, we will consider Eq. (12.2). In order to preserve the homogeneity of Eq. (12.2) we will use, as equivalence transformations, the transformations (11.14) by letting $h \equiv 0$ in (a).

The determining equations (11.9)-(11.11) are independent of the function $\psi(x, u)$ and are the same for both linear and nonlinear equations (11.2). Equation (11.12) contains the function ψ and for the linear equation (12.2) it takes the form

$$(\xi^i H_{,i} + \mu H)u + L[\tau] = 0, \tag{12.3}$$

where

$$H(x) = E(x) + c(x).$$

Variables x and u in equation (12.3) act as independent variables, therefore this equation is equivalent to the following two equations

$$\xi^i H_{,i} + \mu H = 0, \tag{12.4}$$

$$L[\tau] = 0. \tag{12.5}$$

Equations (11.9)-(11.11) and (12.4), (12.5) are determining equations admitted by the linear homogeneous equation (12.2) [110]. Assuming that $\xi^i = 0$ $(i = 1, \ldots, n)$ in these equations, we obtain $\mu = 0, \sigma =$const. Meanwhile, all the determining equations are satisfied. It means that the group G admitted by (12.2) contains the infinite subgroup G^+ with the generators

$$X = u\frac{\partial}{\partial u}, \quad X_\tau = \tau(x)\frac{\partial}{\partial u}\,,$$

where $\tau(x)$ is an arbitrary solution of equation (12.5). This subgroup is an invariant subgroup of the group G [111]. In what follows, instead of the group G, its factor-group G/G^+ is considered with respect to the invariant subgroup G^+. Correspondingly, the function $\sigma(x)$ in operators (11.5) is determined with the accuracy up to the constant summand and the function $\tau(x) = 0$. Therefore, in case of linear equations (12.2), the infinitesimal generator has the form

$$X = \xi^i(x)\frac{\partial}{\partial x^i} + \sigma(x)u\frac{\partial}{\partial u}\,\cdot \tag{12.6}$$

12.3 Conformally invariant equations

Let us find all linear conformally invariant equations of the second order in the space $V_n (n \geq 3)$ having a group of conformal transformations of the maximum order $\frac{1}{2}(n+1)(n+2)$. According to the above reasoning, certain equivalence transformations take these equations to the form

$$\Delta u = a^i(x)u_i + c(x)u = 0. \tag{12.7}$$

In order to find all conformally invariant equations among equations (12.7), one has to determine all coefficients a^i, c satisfying the determining equations.

Equations (11.9) for a flat space have already been considered in § 9. Their general solution ξ^i is known and leads to operators (9.8). In order to find the remaining unknown functions (functions σ in operator (12.6) and coefficients a^i, c of equation (12.7)), one has to investigate the remaining determining equations (11.10), (11.11) and (12.4).

First let us consider equations (11.11). By definition of a covariant derivative 5.2

$$(K_{il}\xi^l)_{,j} = \frac{\partial(K_{il}\xi^l)}{\partial x^j} - K_{pl}\xi^l\Gamma^p_{ij}.$$

Therefore, one can also write equations (11.11) in the form

$$\frac{\partial(K_{il}\xi^l)}{\partial x^j} - \frac{\partial(K_{jl}\xi^l)}{\partial x^i} = 0 \quad (i, j = 1, \dots, n).$$

Let us rewrite these equations in a more convenient form for further calculations using the identities

$$\frac{\partial K_{ij}}{\partial x^l} + \frac{\partial K_{jl}}{\partial x^i} + \frac{\partial K_{li}}{\partial x^j} = 0 \quad (i, j, l = 1, \dots, n),$$

that follow immediately from definition of the tensor K_{ij}. Invoking these identities, one can write equations (11.11) in the form

$$\xi^l\frac{\partial K_{ij}}{\partial x^l} + K_{lj}\frac{\partial \xi^l}{\partial x^i} + K_{il}\frac{\partial \xi^l}{\partial x^j} = 0 \quad (i, j = 1, \dots, n). \tag{12.8}$$

Since coordinates ξ^i of the operator (12.6) are known, one can derive coefficients a^i of equation (12.7), which is invariant with respect to the conformal group, from equation (12.8). With this purpose we substitute values of ξ^i sequentially for operators (9.8) of the conformal group. First let us take the operators of translation $X_l = \frac{\partial}{\partial x^l}$. For these operators $\xi^k_l = \delta^k_l$, where the index l under ξ^kindicates the number of the operator X_l. Therefore, in this case equations (12.8) have the form

$$\frac{\partial K_{ij}}{\partial x^l} = 0 \quad (i, j, l = 1, \dots, n).$$

Taking into account these conditions and writing equations (12.8) for the operator

$$Z = x^l\frac{\partial}{\partial x^l}$$

with the coordinates $\xi^l = x^l (l = 1, \dots, n)$ we obtain

$$K_{lj}\delta^l_i + K_{il}\delta^l_j = 0,$$

whence

$$K_{ij} = 0 \quad (i, j = 1, \dots, n). \tag{12.9}$$

As it is well known, satisfaction of equations (12.9) is the necessary and sufficient condition for

$$a_i(x) = \frac{\partial\varphi(x)}{\partial x^i} \quad (i = 1, \dots, n).$$

On the other hand, the equivalence transformation

$$\overline{L}[u] \equiv e^{-\nu} L[ue^{\nu}] = 0$$

of equation (11.2) transforms coefficients a_i of this equation according to the formula

$$\bar{a}_i = a_i + 2\frac{\partial \nu}{\partial x^i} \quad (i = 1, \ldots, n).$$

If we take $\nu = -\frac{\varphi}{2}$, we obtain an equation of the form (12.7) with the coefficients $\bar{a}^i = 0$. Suppose that such equivalence transformation has been already carried out so that in equation (12.7) $a^i = 0$ $(i = 1, \ldots, n)$.

Now let us consider equation (12.4) with the translation operators $X_i = \frac{\partial}{\partial x^i}$. One has $\mu = 0$ for these operators by virtue of equations (11.9). Therefore, equation (12.4) has the form

$$\frac{\partial H}{\partial x^i} = 0 \quad (i = 1, \ldots, n).$$

Taking the dilation generator $Z = x^i \frac{\partial}{\partial x^i}$ and using the resulting conditions, one obtains from (12.4):

$$H = 0,$$

since by virtue of equations (11.9) $\mu \neq 0$ for the operator Z. By definition of the function H (see 12.2), one has $H = c(x)$ for equation (12.7) with the coefficients $a^i = 0 (i = 1, \ldots, n)$ so that in equation (12.7) $c = 0$.

Thus, if equation (12.7) is conformally invariant, it is equivalent to the equation

$$\Delta u = 0. \tag{12.10}$$

The above reasoning holds for the case of the space V_n of an arbitrary signature.

It remains to consider equations (11.10) determining the function $\sigma(x)$ in operator (12.6). Substitution of the coordinates ξ^i of operators of translation X_k, rotation X_{ij}, and dilation Z into equations (11.9) shows that $\mu =$const. for these operators. Therefore, equations (11.10), that in the given case have the form

$$\frac{\partial \sigma}{\partial x^i} = \frac{2-n}{4} \frac{\partial \mu}{\partial x^i} \, ,$$

provide that $\sigma =$const. for the mentioned operators. This constant can be chosen arbitrarily for according to 12.2 the function $\sigma(x)$ is determined with the accuracy up to the constant summand. Taking into account this arbitrariness we take $\sigma = 0$ for all operators of translation and rotation, and $\sigma = \frac{2-n}{2}$ for the dilation generator, respectively. The choice is made for the sake of uniformity with the case of nonlinear conformally invariant equations, see the following section. Equations (11.9) provide $\mu = 4x^i$ for operators Y_i, so that in this case $\sigma = (2-n)x^i$+const. Let the arbitrary constant be equal to zero and $\sigma = (2-n)x^i$.

Thus, the group G, admitted by equation (12.10), has the following basis operators

$$X_i = \frac{\partial}{\partial x^i},$$

$$X_{ij} = x^j \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^j} \quad (i < j),$$

$$Y_i = (2x^i x^j - \|x\|^2 \delta^{ij}) \frac{\partial}{\partial x^j} + (2-n)x^i u \frac{\partial}{\partial u}, \ (i, j = 1, \ldots, n)$$

$$Z = x^i \frac{\partial}{\partial x^i} + \frac{2-n}{2} u \frac{\partial}{\partial u} \cdot \tag{12.11}$$

The above result concerning linear differential equations admitting a maximum order group can be formulated as follows.

Theorem 1.15. Any linear conformally invariant differential equation of the second order in the space $V_n (n \geq 3)$ with a conformal group of the maximum order $\frac{1}{2}(n+1)(n+2)$ is equivalent to equation (12.10). The factor group G/G^+ of the group G admitted by equation (12.10) over the infinite invariant subgroup G^+ represents $\frac{1}{2}(n+1)(n+2)$-parameter group with the basis infinitesimal generators (12.11).

Note that the condition $n > 2$ is of importance. If $n = 2$ the conformal group is infinite, therefore we do not consider this case directly. Group classification of linear equations of the second order with $n = 2$ was made by S. Lie [89].

§ 13 Nonlinear equations with maximal group

13.1 Preliminaries

Let us consider nonlinear equations of the form (11.1) and find among them all conformally invariant equations in space V_n $(n \geq 3)$ with a group of conformal transformations of a maximum order. According to Section 12.1, we will consider equations of the form (12.1). The reasoning used in Section 12.3 while solving the determining equations (11.11) for the linear equation

(12.7) holds in the case of the equation (12.1) as well. Therefore one can assume that $a^i = 0$ ($i = 1, \ldots, n$) in Eq. (12.1) and consider the equation

$$\Delta u + \psi(x, u) = 0 \tag{13.1}$$

with an arbitrary function $\psi(x, u)$.

Let us find all nonlinear equations (with respect to the function $\psi(x, u)$) for which equation (13.1) is conformally invariant. In order to determine such functions ψ one has to investigate equation (11.12) and operators (11.2) will run all the set of operators of a conformal group of transformations in a flat space V_n.

Function E (11.2) occurring in equation (11.12) is equal to zero in the given case. Therefore equation (11.12) has the form

$$\xi^i \frac{\partial \psi}{\partial x^i} + (\sigma u + \tau)\frac{\partial \psi}{\partial u} + (\mu - \sigma)\psi + \Delta\tau = 0, \tag{13.2}$$

and our problem is reduced to investigation of the latter equation.

13.2 Classifying equations

Let us write equation (13.2) for all infinitesimal generators of the group of conformal transformations in a flat space V_n. According to formulae (9.8) and equations (11.9) and (11.10), operators (11.5) have the following form (there is no need to consider the rotation operators here):

$$X_i = \frac{\partial}{\partial x^i} + (\sigma_i u + \tau_i)\frac{\partial}{\partial u},$$

$$Y_i = (2x^i x^k - |x|^2\delta^{ik})\frac{\partial}{\partial x^k} + [(\sigma_{n+i} + (2-n)x^i)u + \tau_{n+i}]\frac{\partial}{\partial u} \quad (i = 1, \ldots, n),$$

$$Z = x^i \frac{\partial}{\partial x^i} + (\sigma_o u + \tau_o)\frac{\partial}{\partial u}, \tag{13.3}$$

where $\sigma_\alpha (\alpha = 0, 1, \ldots, 2n)$ are arbitrary constants and τ_α are arbitrary functions of x.

Substituting operators (13.3) into equation (13.2) we obtain

$$\frac{\partial \psi}{\partial x^i} + (\sigma_i u + \tau_i)\frac{\partial \psi}{\partial u} - \sigma_i\psi + \Delta\tau_i = 0 \quad (i = 1, \ldots n), \tag{13.4}$$

$$x^i \frac{\partial \psi}{\partial x^i} + (\sigma_o u + \tau_o)\frac{\partial \psi}{\partial u} + (2 - \sigma_o)\psi + \Delta\tau_o = 0, \tag{13.5}$$

$$(2x^i x^k - |x|^2\delta^{ik})\frac{\partial \psi}{\partial x^k} + [(\sigma_{n+i} + (2-n)x^i)u + \tau_{n+i}]\frac{\partial \psi}{\partial u} +$$

$$[(2+n)x^i - \sigma_{n+i}]\psi + \Delta\tau_{n+i} = 0 \quad (i=1,\ldots,n). \tag{13.6}$$

If we multiply every equation (13.4) by x^i with the corresponding number i and make summation with respect to i from 1 to n, and then eliminate $x^i \frac{\partial\psi}{\partial x^i}$ from the resulting equality and equation (13.5), we arrive to equation

$$[(x^i\sigma_i - \sigma_o)u + x^i\tau_i - \tau_o]\frac{\partial\psi}{\partial u} + (\sigma_o - x^i\sigma_i - 2)\psi +$$

$$+(x^i\Delta\tau_i - \Delta\tau_o) = 0. \tag{13.7}$$

instead of equation (13.5). Further, by virtue of equations (13.5) and (13.4), one has

$$(2x^ix^k - |x|^2\delta^{ik})\frac{\partial\psi}{\partial x^k} = [|x|^2(\sigma_i u + \tau_i) - 2x^i(\sigma_o u \tau_o)]\frac{\partial\psi}{\partial u} +$$

$$+(2x^i(\sigma_o - 2) - |x|^2\sigma_i)\psi + |x|^2\Delta\tau_i - 2x^i\Delta\tau_o \quad (i=1,\ldots,n).$$

Therefore equations (13.6) can be written in the form

$$[(|x|^2\sigma_i + (2-n-2\sigma_o)x^i + \sigma_{n+i})u + |x|^2\tau_i - 2x^i\tau + \tau_{n+i}]\frac{\partial\psi}{\partial u} - \tag{13.8}$$

$$-[|x|^2\sigma_i + (2-n-2\sigma_o)x^i + \sigma_{n+i}]\psi + |x|^2\Delta\tau_i - 2x^i\Delta\tau_o + \Delta\tau_{n+i} = 0$$

13.3 Separating the classifying equations into two cases

Let us solve equations (13.4), (13.7) and (13.8). Write equations (13.7) and (13.8) in the form

$$A\frac{\partial\psi}{\partial u} - (\rho+2)\psi + B = 0 \tag{13.9}$$

$$A_i\frac{\partial\psi}{\partial u} - \rho_i\psi + B_i = 0 \quad (i=1,\ldots,n), \tag{13.10}$$

respectively. Here

$$\rho(x) = x^i\sigma_i - \sigma_o,\ q(x) = x^i\tau_i - \tau_o,\ B(x) = x^i\Delta\tau_i - \Delta\tau_o,$$

$$\rho_i(x) = |x|^2\sigma_i + (2-n-2\sigma_o)x^i + \sigma_{n+i},$$

$$g_i(x) = |x|^2\tau - 2x^i\tau_o + \tau_{n+i},$$

$$B_i(x) = |x|^2\Delta\tau_i - 2x^i\Delta\tau_o + \Delta\tau_{n+i} \quad (i=1,\ldots,n), \tag{13.11}$$

$$A = \rho(x)u + q(x),$$

$$A_i(x) = \rho_i(x)u + g_i(x) \quad (i=1,\ldots,n). \tag{13.12}$$

One can assume that $A \neq 0$ in what follows. Indeed, if $A = 0$, then by definition (13.12) values $A, \rho(x) = 0$. Then equation (13.9) provides $\psi = \frac{1}{2}B(x)$, i.e. we arrive to the case of linear equations considered in the previous section.

Let us consider two cases separately:

1°. $A_i = 0 \quad (i = 1, \ldots, n)$

2°. $A_i \neq 0$ at least for one value of i.

13.4 Solution of classifying equations in the first case

Let us consider the first case. According to formulae (13.12) and equations (13.10), the following equations hold when $A_i = 0$:

$$\rho_i = 0,\ g_i = 0,\ B_i = 0 \quad (i = 1, \ldots, n),$$

or, according to notation (13.10)

$$|x|^2\sigma_i + (2 - n - 2\sigma_o)x^i + \sigma_{n+i} = 0 \quad (i = 1, \ldots, n), \tag{13.13}$$

$$|x|^2\tau_i - 2x^i\tau_o + \tau_{n+i} = 0 \quad (i = 1, \ldots, n), \tag{13.14}$$

$$|x|^2\Delta\tau_i - 2x^i\Delta\tau_o + \Delta\tau_{n+i} = 0 \quad (i = 1, \ldots, n). \tag{13.15}$$

Equations (13.13), by virtue of the constancy $\sigma_\alpha (\alpha = 0, 1, \ldots, 2n)$, provide

$$\sigma_o = \frac{2 - n}{2},\ \sigma_\alpha = 0 \quad (\alpha = 1, \ldots, 2n). \tag{13.16}$$

Therefore, one can write equations (13.4) and (13.9) in the form

$$\frac{\partial\psi}{\partial x^i} = -\frac{1}{A}\tau_i\left(\frac{n+2}{2}\psi - B\right) - \Delta\tau_i \quad (i = 1, \ldots, n) \tag{13.17}$$

and

$$\frac{\partial\psi}{\partial u} = \frac{1}{A}\left(\frac{n+2}{2}\psi - B\right), \tag{13.18}$$

respectively. The consistency conditions for equations (13.7) and (13.8) have the form

$$\left(\frac{n+2}{2}\psi - B\right)\left(\frac{n-2}{2}\tau_i + \frac{\partial q}{\partial x^i}\right) + A\left(\frac{n+2}{2}\Delta\tau_i + \frac{\partial B}{\partial x^i}\right) = 0 \quad (i = 1, \ldots, n),$$

whence, when the function $\psi(xu)$ is nonlinear,

$$\frac{n-2}{2}\tau_i + \frac{\partial q}{\partial x^i} = 0 \quad (i = 1, \ldots, n) \tag{13.19}$$

and

$$\frac{n+2}{2}\Delta\tau_i + \frac{\partial B}{\partial x^i} = 0 \quad (i = 1, \ldots, n)$$

with respect to functions τ_i. Introducing the notation

$$\varphi = \frac{2}{2-n}q \equiv \frac{2}{2-n}(x^i\tau_i - \tau_o), \tag{13.20}$$

one can write equations (13.19) in the form

$$\tau_i = \frac{\partial \varphi}{\partial x^i} \quad (i = 1, \ldots, n). \tag{13.21}$$

Equations (13.20) and (13.21) yield

$$\Delta\varphi = \frac{2}{2-n}(2\Delta\varphi + x^i\Delta\tau_i - \Delta\tau_o),$$

whence

$$B \equiv x^i\Delta\tau_i - \Delta\tau_o = -\frac{n+2}{2}\Delta\varphi.$$

Equation (13.18) takes the form

$$\frac{\partial\psi}{\partial u} = \frac{n+2}{n-2}\frac{\psi + \Delta\varphi}{u - \varphi}\,\cdot$$

The general solution of this equation is given by the formula

$$\psi = K(x)(u-\varphi)^{\frac{n+2}{n-2}} - \Delta\varphi, \tag{13.22}$$

where $K(x)$ is an arbitrary function. Substituting formula (13.22) into equations (13.17) one obtains

$$(u-\varphi)^{\frac{n+2}{n-2}} \cdot \frac{\partial K}{\partial x^i} = 0 \quad (i = 1\ldots, n),$$

whence K =const.

Thus, in the case 1° the conformally invariant equation (13.1) is equivalent to equation

$$\Delta u + K(u-\varphi)^{\frac{n+2}{n-2}} - \Delta\varphi = 0$$

with the arbitrary function $\varphi(x)$ and the constant K. Finally, the equivalence transformation

$$\bar{u} = u - \varphi(x)$$

provides the following equation

$$\Delta u + Ku^{\frac{n+2}{n-2}} = 0. \tag{13.23}$$

One can easily verify that equation (13.23) admits a group with the basis operators (12.11).

13.5 Solution of classifying equations in the second case

Let us consider the case 2°. To be specific, it is assumed that $A_i \neq 0$. Eliminating $\frac{\partial\psi}{\partial u}$ from equations (13.9) and (13.10) when $i = 1$ one obtains

$$[A_1(\rho+2) - A\rho_1]\psi + AB_1 - A_1B = 0. \tag{13.24}$$

Differentiation with respect to u yields

$$(\rho_1(\underline{\rho}+2) - \rho\rho_1)\psi + (A_1(\rho+2) - A\rho_1)\frac{\partial\psi}{\partial u} + \rho B_1 - B\rho_1 = 0.$$

Substituting here the values of $A_1\frac{\partial\psi}{\partial u}$ and $A\frac{\partial\psi}{\partial u}$ from equations (13.10) and (13.9), respectively, one obtains

$$\rho_1\psi - B_1 = 0.$$

Then, equation (13.10) leads to $\psi = \psi(x)$, provided that $i = 1$ and invoking the condition $A_1 \neq 0$.

Thus, the case 2° leads only to a linear function ψ.

13.6 Formulation of the result

Results of § 12, § 13 can be formulated in the following theorem.

Theorem 1.16. Any conformally invariant equation (11.1) in the space $V_n (n \geq 3)$ with a conformal group of the maximum order $\frac{1}{2}(n+1)(n+2)$ is equivalent to the equation

$$\Delta_{(K)}u \equiv \Delta u + Ku^{\frac{n+2}{n-2}} = 0, \tag{13.25}$$

where Δ is the Laplace operator and K is an arbitrary constant. Equation (13.25) admits a group with basis infinitesimal generators (12.11).

Remark 1.3. If the constant $K \neq 0$, then a conformal group with operators (12.11) is the most extended group admitted by equation (13.25). In this case, the constant K can be considered equal to either $+1$ or -1 depending on the sign of K. Indeed, one has only to consider the equation resulting from transformation $\bar{u} = |K|^{\frac{n-2}{4}}u$ instead of equation (13.25) to which it is equivalent. When $n = 6 + 8l$, $l = 0, 1 \ldots$ (and only with such n) the constant $K \neq 0$ can be reduced to $+1$ using the corresponding dilation of u.

§ 14 Conformally invariant equations

14.1 General conformally invariant equation

It was demonstrated in § 12 that equation (12.10) is the only linear conformally invariant equation of the second order (with the accuracy to equivalence transformation) in a flat space V_n, which is also conformally flat. Let us find out which equations (12.2) are conformally invariant when the space V_n is not conformally flat. To this end we have to investigate equations (11.11) and (12.4) that impose limitations on coordinates $\xi^i (i = 1, \ldots, n)$, in addition to generalized Killing equations (11.9), if equation (12.2) is set. However, we assume that operators of a conformal group are given, i.e. solutions of equation (11.9), and we consider equations (11.11) and (12.4) as differential equations to determine coefficients a^i and c of the conformally invariant equations (12.2).

Equations (11.11) and (12.4) are linear and homogeneous with respect to functions K_{ij} and H. Therefore, these equations have a trivial solution

$$K_{ij} = 0 \quad (i, j = 1, \ldots, n), \quad H = 0. \tag{14.1}$$

As it was mentioned in 12.3, when $K_{ij} = 0$, equivalence transformations can reduce coefficients a^i of equations (12.2) to zero. On the other hand, according to 11.2 and 12.2 one has

$$H = c - \frac{1}{2}\left(a^i_{,i} + \frac{1}{2}a^i a_i + \frac{n-2}{2(n-1)}R\right), \tag{14.2}$$

for equation (12.2), so that when $a^i = 0$

$$H = c - \frac{n-2}{4(n-1)}R.$$

Thus, the following lemma holds.

Lemma 1.8. If equation (12.2) satisfies conditions (14.1), then it is conformally invariant and equivalent to equation

$$\Delta u \equiv g^{ij}u_{,ij} + \frac{n-2}{4(n-1)}Ru = 0. \tag{14.3}$$

Remark 1.4. The notation Δ in equation (14.3) is due to the fact that this operator, corresponding in case of Euclidian space with the Laplace operator, is a natural generalization of the Laplace operator (or a wave operator in case of the space V_n with the signature $- \ldots -+$) on Riemannian spaces [57].

14.2 Conformally invariant equations in spaces with trivial conformal group

In general case, Eq. (14.3) is not the only linear conformally invariant equation in the space V_n. Let, e.g. the space V_n be a space with a trivial conformal group. Select a space $\widetilde{V}_n$ conformal to the space V_n, where the conformal group coincides with a group of motions so that in equations (11.9) $\mu = 0$. Consider equation (12.2) in the space $\widetilde{V}_n$ with the coefficients

$$a^i = 0(i = 1, \ldots, n), \quad c = \frac{n-2}{4(n-1)} R + \lambda,$$

where λ is an arbitrary constant. The resulting equation

$$\widetilde{\Delta} u + \lambda u = 0 \tag{14.4}$$

is conformally invariant. Indeed, for equation (14.4) $H = \lambda$ =const., so that by virtue of condition $\mu = 0$, equation (12.4) holds; equations (11.11) obviously hold as well.

Let us assume that the spaces V_n and $\widetilde{V}_n$ have metric tensors $g_{ij}(x)$, and $\tilde{g}_{ij}(x) = \sigma(x) g_{ij}(x)$ where $\sigma(x) \neq 0$, respectively. Multiplying equation (14.4) by the function $\sigma(x)$ and writing the resulting equation as equation (12.2) in space V_n (in this case only coefficients a^i change), one obtains a conformally invariant equation in space V_n. When $\lambda \neq 0$ this equation is not equivalent to equation (14.3) in space V_n. Indeed, equivalence transformations modify the values K_{ij} as tensor components, and function H is multiplied by a nonzero function. Therefore, one has

$$H \neq 0, \quad K_{ij} = 0,$$

for the resulting equation in space V_n, so that conditions (14.1) do not hold for this equation though they hold for equation (14.3).

Thus, in any space V_n with a trivial conformal group, there exist a linear conformally invariant equation of the second order, other than equation (14.3).

14.3 Conformally invariant equations in spaces with nontrivial conformal group

One has a different situation for spaces with nontrivial conformal group. In particular, the following theorem holds.

Theorem 1.17. Any linear conformally invariant equation in every Riemannian space V_4 of normal hyperbolic type with nontrivial conformal group is equivalent to equation (14.3).

Proof. We have to prove that if equation (12.2) in space V_4 with nontrivial conformal group is conformally invariant, then conditions (14.1) hold for this equation. As it was mentioned in 14.2, equations (14.1) are invariant with respect to all equivalence transformations. Therefore, according to Theorem 1.14, it is sufficient to consider spaces V_4 with a metric tensor g^{ij} of the form (9.5). Let us introduce the coordinates

$$x^1 = x + t,\ x^2 = y,\ x^3 = z,\ x^4 = x - t.$$

One can easily verify that infinitesimal generators of a group of conformal transformations in the indicated spaces include the following four operators (the remaining operators are not required here):

$$X_i = \frac{\partial}{\partial x^i}\ (i = 1, 2, 3), \quad X_4 = 2x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2} + x^3\frac{\partial}{\partial x^3}\cdot \tag{14.5}$$

Let us solve equations (11.11), written in the form (12.8), for operators (14.5). Taking operators $X_i (i = 1, 2, 3)$ one obtains

$$\frac{\partial K_{ij}}{\partial x^l} = 0\ (i, j = 1, \ldots, 4; l = 1, 2, 3) \tag{14.6}$$

By virtue of equations (14.6), operator X_4, for which $\xi^4 = 0$, yields

$$K_{lj}\frac{\partial \xi^l}{\partial x^i} + K_{il}\frac{\partial \xi^l}{\partial x^j} = 0 \quad (i, j = 1\ \ldots, 4)$$

or, upon substituting the values of coordinates ξ^l of the operator X_4,

$$2(K_{1j} + \delta_i^1 + K_{i1}\delta_j^1) + \sum_{\alpha=2}^{3}(K_{\alpha j}\delta_i^\alpha + K_{i\alpha}\delta_j^\alpha) = 0.$$

These equations can be also written in the form

$$2K_{ij} + K_{1j}\delta_i^1 + K_{i1}\delta_j^1 - K_{4j}\delta_i^4 - K_{i4}\delta_j^4 = 0\ (i, j = 1, \ldots, 4),$$

whence, invoking the skew symmetry of the tensor K_{ij}, one has

$$K_{ij} = 0 \quad (i, j = 1, \ldots, 4).$$

Now let us solve equation (12.4). We have $\mu = 0$ for operators $X_i (i = 1, 2, 3)$, therefore equation (12.4) for these operators takes the form

$$\frac{\partial H}{\partial x^i} = 0 \quad (i = 1, 2, 3).$$

Using these equations we write equations (12.4) for operator X_4 in the form

$$\mu H = 0.$$

Substituting coordinates ξ^i of the operator X_4 into generalized Killing equations one obtains that $\mu \neq 0$. Therefore,

$$H = 0.$$

Thus, conditions (14.1) hold for any conformally invariant equation (12.2) in the considered spaces V_n.

14.4 Relationship between conformally invariant equations in conformal spaces

The relations between operators Δ in spaces conformal to each other can be of help in investigating equation (14.3). These relations are given by the following theorem.

Theorem 1.18. Let us assume that the two spaces V_n and $\widetilde{V}_n$ are conformal to each other and have metric tensors

$$g_{ij}(x) \text{ and } \tilde{g}_{ij}(x) = e^{2\theta(x)} g_{ij}(x) \quad (i, j = 1, \ldots, n),$$

respectively, and operators Δ and $\widetilde{\Delta}$ are calculated in spaces V_n and $\widetilde{V}_n$ by formula (14.3). Then,

$$\widetilde{\Delta} u = e^{-\frac{n+2}{2}\theta} \Delta(u e^{\frac{n-2}{2}\theta}). \tag{14.7}$$

Proof. We have

$$e^{-\frac{n+2}{2}\theta} \Delta \left(u e^{\frac{n-2}{2}\theta}\right) \equiv$$

$$e^{-\frac{n+2}{2}\theta} g^{ij} \left(u e^{\frac{n-2}{2}\theta}\right)_{,ij} + \frac{n-2}{4(n-1)} e^{-2\theta} R u =$$

$$e^{-2\theta} g^{ij} \left\{ u_{ij} + (n-2) u_i \theta_j + \frac{1}{2}(n-2) u \theta_{ij} + \frac{1}{4}(n-2)^2 u \theta_i \theta_j - \right.$$

$$\left. - \Gamma^k_{ij} u_k - \frac{1}{2}(n-2) u \Gamma^k_{ij} \theta_k \right\} + \frac{n-2}{4(n-1)} e^{-2\theta} R u. \tag{14.8}$$

Using the formulae ([29], § 28)

$$g^{ij} = e^{2\theta} \tilde{g}^{ij},$$

$$
\begin{aligned}
\Gamma^k_{ij} &= \widetilde{\Gamma}^k_{ij} - \delta^k_i \theta_j - \delta^k_j \theta_i + \tilde{g}_{ij}\tilde{g}^{kl}\theta_l, \qquad (14.9)\\
R &= e^{2\theta}[\widetilde{R} - 2(n-1)\tilde{g}^{ij}(\theta_{ij} - \widetilde{\Gamma}^k_{ij}\theta_k) +\\
&\quad (n-1)(n-2)\tilde{g}^{ij}\theta_i\theta_j],
\end{aligned}
$$

connecting the corresponding values in spaces V_n and $\widetilde{V}_n$, one can write the right-hand side of equation (14.8) in the following form:

$$
\begin{aligned}
&\tilde{g}^{ij}\{u_{ij} - \widetilde{\Gamma}^k_{ij}u_k + nu_i\theta_j - \tilde{g}_{ij}\tilde{g}^{kl}u_k\theta_l +\\
&+\frac{1}{2}(n-2)u(\theta_{ij} + \frac{1}{2}(n-2)\theta_i\theta_j - \widetilde{\Gamma}^k_{ij}\theta_k + 2\theta_i\theta_j - \tilde{g}_{ij}\tilde{g}^{kl}\theta_l\theta_k)\} +\\
&\frac{n-2}{4(n-1)}\left\{\widetilde{R} - 2(n-1)\tilde{g}^{ij}(\theta_{ij} - \tilde{\Gamma}^k_{ij}\theta_k) + (n-1)(n-2)\tilde{g}^{ij}\theta_i\theta_j\right\}u.
\end{aligned}
$$

The resulting expression equals

$$
\tilde{g}^{ij}(u_{ij} - \widetilde{\Gamma}^k_{ij}u_k) + \frac{n-2}{4(n-1)}\widetilde{R}u \equiv \widetilde{\Delta}u,
$$

so that equation (14.7) holds.

14.5 Nonlinear conformally invariant equations

The nonlinear equation

$$
\Delta_{(K)}u \equiv g^{ij}u_{,ij} + \frac{n-2}{4(n-1)}Ru + Ku^{\frac{n+2}{n-2}} = 0, \qquad (14.10)
$$

generalizing equation (13.25) to an arbitrary Riemannian space, as well as linear equation (14.3), is conformally invariant for any space V_n. Meanwhile, infinitesimal generators of the group admitted by equation (14.10) have the form (12.6) with the function

$$
\sigma = \frac{2-n}{4}\mu,
$$

where μ is determined by equations (11.9). One can readily verify that determining equations (11.10)–(11.12) are satisfied, i.e. that equation (14.10) is conformally invariant. The constant K in equation (14.10) can be regarded to take one of the following three values $0, 1, -1$, similarly to the case of equation (13.25).

Operators $\Delta_{(K)}$ in spaces conformal to each other are also connected by equation of the form (14.7). Indeed, using equation (14.7) and definition of operator $\Delta_{(K)}$ in equation (14.10), we obtain

$$
\widetilde{\Delta}_{(K)}u = e^{-\frac{n+2}{2}\theta}\Delta_{(K)}\left(ue^{\frac{n-2}{2}\theta}\right).
$$

Open problem: Semi-linear conformally invariant equations

Problem 1.3. Find all *semi-linear* conformally invariant equations of the form (11.1) in every Riemannian space V_4 of normal hyperbolic type with nontrivial conformal group.

Solution of this problem would be an addition to results concerning conformally invariant equations given in § 13 and § 14.

Chapter 4
The Huygens principle

In this chapter, the emphasis on a connection between the conformal invariance and the Huygens principle on existence of a rear wave front in the light propagation in curved space-times. Using this connection, Hadamard's problem on the Huygens principle is solved in four-dimensional Riemannian spaces with nontrivial conformal group. An explicit solution to the Cauchy problem is given for conformally invariant equations.

Literature: Hadamard [41, 46], Mathisson [102], Asgeirsson [6], Douglis [28], Stellmacher [126, 127], Günther [38, 39], Ibragimov and Mamontov [71], Ibragimov [57, 58, 59].

§ 15 Hadamard's criterion

15.1 Definition of the Huygens principle

Let us consider the following Cauchy problem for a linear hyperbolic equation of the second order

$$L[u] \equiv g^{ij}(x)u_{,ij} + a^i(x)u_{,i} + c(x)u = 0 \tag{15.1}$$

with n independent variables $x = (x^1, \ldots, x^n)$. Find the solution $u = u(x)$ of equation (15.1) that satisfies given values

$$u\big|_M, \quad u_\nu\big|_M \tag{15.2}$$

on an $(n-1)$-dimensional space-like manifold M. I use here the notation

$$u_\nu = \frac{\partial u}{\partial \nu}$$

for the derivative of u in the direction of the normal ν to the manifold M, while $u|_M$ and $u_\nu|_M$ indicate the values of the corresponding functions on the manifold M. The functions (15.2) are called the *initial data,* or the *Cauchy data,* and the manifold M is termed an *initial manifold* or a manifold bearing the initial data.

As in the previous chapter, V_n indicates an n-dimensional Riemannian space with the metric tensor $g_{ij}(x)$, where g_{ij} are elements of the matrix $\|g_{ij}\| = \|g^{ij}\|^{-1}$. Equation (15.1) will be called an equation in the space V_n. Since in this chapter only hyperbolic equations (15.1) are considered, the corresponding spaces V_n are spaces of a normal hyperbolic type.

Hadamard's theory [41] of the Cauchy problem for equation (15.1) is called a *local* theory meaning that it is limited by consideration of such domain U of points x, that any two points of the domain U can be connected by a unique geodesic (in the space V_n) totally belonging to the domain U. In what follows, all considered points x belong to such domain U by default.

Hadamard's formula for solving the Cauchy problem provides that in a general case the value of the solution u of an arbitrary Cauchy problem for equation (15.1) at the point x_o depends on the values of initial data in that part M_o of the manifold M which lies inside the characteristic conoid with the apex at the point x_o.

Definition 1.14. If solution of an arbitrary Cauchy problem for equation (15.1) at every point x_o depends on initial data only on the intersection of the characteristic conoid with the apex at the point x_o with the initial manifold M (i.e. on the border M_o), then equation (15.1) is said to *satisfy the Huygens principle* or that the *Huygens principle holds for equation (15.1).*

15.2 Formulation of Hadamard's criterion

Hadamard [41] proved that when n is odd, the Huygens principle does not hold for any equation of the form (15.1). This principle can not hold also when $n = 2$ [42]. In this connection, note that equations with two independent variables, considered by Hörnich [49] as equations satisfying the Huygens principle, have peculiarities in the initial manifold. Therefore, the Cauchy problem for these equations cannot be considered with arbitrary initial data. Thus, the Huygens principle does not hold in the above sense.

For even $n \geq 4$, Hadamard [41] derived the necessary and sufficient condition of validity of the Huygens principle in terms of an elementary solution of equation

$$L^*[v] \equiv g^{ij}v_{,ij} + a^{*i}v_{,i} + c^*v = 0 \tag{15.3}$$

conjugate to equation (15.1). The elementary solution (in Hadamard's sense) of equation (15.3) when $n \geq 4$ is even and the pole is at the point $x_o = (x_o^1 \ldots, x_o^n)$ has the form

$$v = \frac{V}{\Gamma^{\frac{n-2}{2}}} - W \lg \Gamma. \tag{15.4}$$

Here $V(x_o, x)$ and $W(x_o, x)$ are analytical functions with respect to x, $\Gamma(x_o, x)$ is the square of the geodesic distance between the points $x_o, x \in V_n$. The Hadamard's criterion of validity of the Huygens principle consists in fulfilment of the equation

$$W(x_o, x) = 0 \tag{15.5}$$

for all points x_o, x.

Further we consider the case $n = 4$. Therefore, let us write the Hadamard criterion for this case more explicitly using the formula that determines function V in elementary solution (15.4). For $n = 4$ this function is determined by the following formula ([41] § 62-68):

$$V = \exp\left\{-\frac{1}{4}\int_{x_o}^{x}(L^*[\Gamma] - c^*\Gamma - 8)\frac{ds}{s}\right\}, \tag{15.6}$$

where integration is made along the geodesic connecting the point x_o and x, and $\Gamma(x_o, x)$ is considered as a function of x. Equations expressing the function W via the function V [41] provide that, when $n = 4$, equation (15.5) is equivalent to the condition that $V(x_o, x)$, being a function of x, satisfies the conjugate equation (15.3) at all points x lying on the characteristic conoid with the apex at x_o. However, the characteristic conoid with the apex at the point x_o for equation (15.1) consists of all points x satisfying equation

$$\Gamma(x_o, x) = 0. \tag{15.7}$$

Thus, the Hadamard criterion of validity of the Huygens principle for equation (15.1) with $n = 4$ is written in the form of the equality

$$L^*[V]\Big|_{\Gamma(x_o,x)=0} = 0, \tag{15.8}$$

which must hold for all points x_o.

In what follows, the Hadamard criterion is used in the form of equation (15.8).

§ 16 Geodesic distance

16.1 Introduction

According to formula (15.6), in order to use the Hadamard criterion in the form (15.8) one has to know the function $\Gamma(x_0, x)$ which is the square of the geodesic distance between the points $x_o, x \in V_4$. We intend to use the Hadamard criterion in case when the space V_4, corresponding to equation (15.1), has a nontrivial conformal group. We can limit our consideration to spaces V_4 with the metric tensor g^{ij} of the form (9.5) because the Huygens principle is invariant with respect to equivalence transformations (see 12.2) of equation (16.1) [41], [46]. In this section the function $\Gamma(x_o, x)$ is calculated for spaces V_4 with the metric tensor (9.5).

16.2 Outline of the approach

The general approach for computing the geodesic distance is as follows. Let us assume that the point $x_o \in V_n$ is fixed, and the geodesic line passing through the fixed point x_o and an arbitrary point $x \in V_n$ is parametrized by means of the length of the arc s counted from the point x_o. Note that we consider such vicinity of the point x_o where any point is connected with x_o by the only geodesic. Then coordinates x^i of the point x are functions $x^i = x^i(s)$ $(i = 1, \ldots, n)$ satisfying the system of differential equations (see 5.1)

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}(x)\frac{dx^j}{ds}\frac{dx^k}{ds} = 0 \quad (i = 1, \ldots, n) \tag{16.1}$$

and the initial conditions

$$x^i|_{s=0} = x^i_o, \quad \left.\frac{dx^i}{ds}\right|_{s=0} = \alpha^i \quad (i = 1, \ldots, n). \tag{16.2}$$

Here Γ^i_{jk} are Christoffel symbols of the space V_n and the constant vector $\alpha = (\alpha^1, \ldots, \alpha^n)$ satisfies the condition

$$g_{ij}(x_o)\alpha^i\alpha^j = 1, \tag{16.3}$$

resulting from definition (5.1) of the arc length in the space V_n. Indeed, one has

$$dx^i = \frac{dx^i(s)}{ds}ds \quad (i = 1, \ldots, n)$$

along the geodesic parametrized by means of the arc length. Substituting these vales of differentials into formula (5.1), reducing the resulting equality by ds^2 and then assuming that $s = 0$, one obtains equation (16.3) by virtue of conditions (16.2).

Let

$$x^i = x^i(s; x_o, \alpha) \quad (i = 1, \ldots, n) \tag{16.4}$$

be the solution of the problem (16.1), (16.2). The values α^i can be obtained (see, e.g., de Rham [25], § 27) as functions

$$\alpha^i = \psi^i(s; x_o, x) \quad (i = 1, \ldots, n) \tag{16.5}$$

from equations (16.4) when $|\alpha|$ are sufficiently small. Substituting the values (16.5) of quantities α^i into condition (16.3) and solving the resulting equation

$$g_{ij}(x_o)\psi^i(s; x_o, x)\psi^j(s; x_o, x) = 1 \tag{16.6}$$

with respect to s, one obtains the square of the geodesic distance

$$\Gamma(x_o, x) = [s(x_o x)]^2$$

between the points x_o and x in the space V_n.

16.3 Equations of geodesics in spaces with nontrivial conformal group

Let us turn back to spaces V_4 with the metric tensor (9.5). The following notation is used further:

$$x_o = (\xi, \eta, \zeta, \tau), \quad x = (x, y, z, t), \quad \alpha = (\alpha, \beta, \gamma, \delta),$$

$$f_o = f(\xi - \tau), \quad f = f(x - t), \ldots, \tag{16.7}$$

$$\Delta = \det \|g^{ij}\| = \varphi^2 - f.$$

According to formulae (5.6) the following non-zero Christoffel symbols (and symbols that differ by interchange of lower indices) are obtained

$$\Gamma^1_{22} = \frac{1}{2}\left(\frac{1}{\Delta}\right)', \quad \Gamma^1_{23} = -\frac{1}{2}\left(\frac{\varphi}{\Delta}\right)', \quad \Gamma^1_{33} = \frac{1}{2}\left(\frac{\varphi^2}{\Delta}\right)',$$

$$\Gamma^2_{12} = -\Gamma^2_{24} = \frac{1}{2}\left[\Delta\left(\frac{1}{\Delta}\right)' + \frac{\varphi\varphi'}{\Delta}\right], \quad \Gamma^2_{13} = -\Gamma^2_{34} = -\varphi\Gamma^2_{12} - \frac{\varphi'}{2},$$

$$\Gamma^3_{12} = -\Gamma^3_{24} = \frac{\varphi'}{2\Delta}, \quad \Gamma^3_{13} = -\Gamma^3_{34} = -\frac{\varphi\varphi'}{2\Delta},$$

$$\Gamma^4_{ij} = \Gamma^1_{ij} \quad (i, j = 2, 3), \tag{16.8}$$

where the prime indicates differentiation with respect to the argument $\varphi'(\sigma) = \frac{d\varphi(\sigma)}{d\sigma}$ etc. By virtue of formulae (16.8), one can write equations (16.1) in the form

$$\frac{d^2x}{ds^2} + \frac{1}{2}\left(\frac{1}{\Delta}\right)'\left(\frac{d\varphi}{ds} - \varphi\frac{dz}{ds}\right)^2 + \frac{\varphi'}{\Delta}\left(\frac{d\varphi}{ds} - \varphi\frac{dz}{ds}\right)\frac{dz}{ds} = 0,$$
$$\frac{d}{ds}\left[\frac{1}{\Delta}\left(\frac{d\varphi}{ds} - \varphi\frac{dz}{ds}\right)\right] = 0,$$
$$\frac{d^2z}{ds^2} + \frac{1}{\Delta}\left(\frac{d\varphi}{ds} - \varphi\frac{dz}{ds}\right)\frac{d\varphi}{ds} = 0,$$
$$\frac{d^2(x-t)}{ds^2} = 0, \tag{16.9}$$

where

$$\frac{d\varphi}{ds} = \varphi'\frac{d(x-t)}{ds} \quad \text{etc.}$$

16.4 Solution of equations of geodesics

Let us solve the system of equations (16.9). Using the initial conditions (16.2) and notation (16.7) one obtains

$$x - t = \xi - \tau + (\alpha - \delta)s, \tag{16.10}$$

$$\varphi\frac{dz}{ds} - \frac{dy}{ds} = \frac{\gamma\varphi_o - \beta}{\Delta_o}\Delta \tag{16.11}$$

from the second and the fourth equations of the system (16.9). Thus

$$\frac{d^2z}{ds^2} = \frac{\gamma\varphi_o - \beta}{\Delta_o}\frac{d\varphi}{ds},$$

so that

$$\frac{ds}{dz} = \frac{1}{\Delta_o}\left[(\gamma\varphi_o - \beta)\varphi + \beta\varphi_o - \gamma f_o\right] \tag{16.12}$$

and

$$\frac{d\varphi}{ds} = \frac{1}{\Delta_o}[(\gamma\varphi_o - \beta)f + (\beta\varphi_o - \gamma f_o)]. \tag{16.13}$$

Due to equality (16.10), solutions of equations (16.13) and (16.12) have the form

$$y = \eta + a(F - F_o) + b(\Phi - \Phi_o) \tag{16.14}$$

and

$$z = \zeta + a(\Phi - \Phi_o) + b(\alpha - \delta)s, \tag{16.15}$$

respectively, where

$$a = \frac{\gamma\varphi_o - \beta}{(\alpha - \delta)\Delta_o}, \quad b = \frac{\beta\varphi_o - \gamma f_o}{(\alpha - \delta)\Delta_o}, \tag{16.16}$$

and F, Φ are primitives for functions f and φ, respectively. Here, as well as in § 3, the following notation is used: $F_o = F(\xi - \tau)$, $F = F(x - t)$ etc.

By virtue of equations (16.11) and (16.12) the first equation of the system (16.9) takes the form

$$\frac{d^2x}{ds^2} + \frac{\gamma\varphi_o - \beta}{\Delta_o^2}\left[\frac{1}{2}(\gamma\varphi_o - \beta)f' + (\beta\varphi_o - \gamma f_o)\varphi'\right] = 0,$$

whence,

$$x = \xi + (\alpha + \beta a - \frac{\alpha - \delta}{2}a^2 f_o)s - \frac{1}{2}a^2(F - F_o) - ab(\Phi - \Phi_o). \tag{16.17}$$

Substituting expressions (16.10) for $x - t$ into formulae (16.14), (16.15), (16.17), and the value of x resulting from (16.17) into equation (16.10), one arrives at the form (16.4) of solution of the system of equations (16.9) with the initial conditions (16.2).

16.5 Computation of the geodesic distance

Let us write equation (16.3) now. The inverse matrix for (9.5) has the form

$$\|g_{ij}\| = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{\Delta} & -\frac{\varphi}{\Delta} & 0 \\ 0 & -\frac{\varphi}{\Delta} & \frac{f}{\Delta} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \Delta = \varphi^2 - f.$$

Therefore, substituting elements g_{ij} of this matrix into equation (16.3) and using the notation (16.16) one obtains

$$g_{ij}(x_o)\alpha^i\alpha^j = -\alpha^2 + \delta^2 + \frac{1}{\Delta_o}(\beta^2 - 2\beta\gamma\varphi_o + \gamma^2 f_o) =$$

$$= -(\alpha - \delta)(\alpha + \delta + \beta a + \gamma b),$$

so that equality (16.3) takes the form

$$-(\alpha - \delta)(\alpha + \delta + \beta a + \gamma b) = 1. \tag{16.18}$$

Equations (16.16) provide

$$\beta = (\alpha - \delta)(af_o + b\varphi_o), \quad \gamma = (\alpha - \delta)(a\varphi_o + b), \tag{16.19}$$

and formulae (16.14) and (16.15) yield

$$a = \frac{1}{A}[(y - \eta)(\alpha - \delta)s - (z - \zeta)(\Phi - \Phi_o)],$$

$$b = \frac{1}{A}[(z - \zeta)(F - F_o) - (y - \eta)(\Phi - \Phi_o)], \tag{16.20}$$

where

$$A = (F - F_o)(\alpha - \delta)s - (\Phi - \Phi_0)^2. \tag{16.21}$$

When s is small formula (16.21) provide

$$A = -\Delta_o(\alpha - \delta)^2 s^2 + o(s^2). \tag{16.22}$$

On the other hand, equation (16.18) provides that $\alpha - \delta \neq 0$, and non-degeneracy of the matrix $\|g^{ij}\|$ entails that $\Delta_o \neq 0$. Thus, formula (16.22) demonstrates that function A is non-vanishing when $s \neq 0$ is sufficiently small.

Equalities (16.19) provide that

$$\beta a + \gamma b = (\alpha - \delta)(a^2 f_o + 2ab\varphi_o + b^2),$$

and equations (16.17) and (16.19) yield

$$2\alpha = 2\frac{x - \xi}{s} - (\alpha - \delta)(a^2 f_o + 2ab\varphi_o) + \frac{1}{s}[a^2(F - F_o) + 2ab(\Phi - \Phi_o)].$$

Therefore writing $\alpha + \delta = 2\alpha - (\alpha - \delta)$, one has

$$\alpha + \delta + \beta a + \gamma b =$$

$$= \frac{1}{s}[2(x - \xi) - (\alpha - \delta)s + a^2(F - F_o) + 2ab(\Phi - \Phi_o) + b^2(\alpha - \delta)s].$$

Substitute the values of $(\alpha - \delta)s$ from formula (16.10) and the values of a and b from formula (16.20) into the resulting expression for $\alpha + \delta + \beta a + \gamma b$. One has

$$\alpha + \delta + \beta a + \gamma b = \frac{1}{s}\left\{x - \xi + t - \tau + \frac{1}{A}[(y - \eta)^2(x - \xi - t + \tau) - \right.$$

$$\left. -2(y - \eta)(z - \zeta)(\Phi - \Phi_o) + (z - \zeta)^2(F - F_o)]\right\}.$$

Substitute this value $\alpha+\delta+\beta a+\gamma b$ and the value for $\alpha-\delta$ resulting from (16.10) into equation (16.18) and multiply the latter by s^2. Finally we arrive to the following formula of the geodesic distance $\Gamma = s^2$:

$$\begin{aligned}\Gamma(x_o, x) &= x-\tau)^2-(x-\xi)^2\\ &-\frac{x-\xi-t+\tau}{(x-\xi-t+\tau)(F-F_o)-(\Phi-\Phi_o)^2}\Big[(x-\xi-t+\tau)(y-\eta)^2\\ &-2(\Phi-\Phi_o)(y-\eta)(z-\zeta)+(F-F_o)(z-\zeta)^2\Big].\end{aligned} \tag{16.23}$$

§ 17 Conformal invariance and the Huygens principle

17.1 Validity of Huygens' principle in spaces with nontrivial conformal group

Let us demonstrate that the Huygens principle holds for the equation (14.3) in every space V_4 with the metric tensor (9.5), which means in every space V_4 of normal hyperbolic type with nontrivial conformal group. In doing so we will verify the Hadamard criterion in the form (15.8).

First let us rewrite equation (14.3) in spaces under consideration V_4 in a convenient form. Find the scalar curvature of these spaces. One has

$$R = g^{ij}R_{ij} = R_{44}-R_{11}-fR_{22}-2\varphi R_{23}-R_{33}.$$

Let us calculate the Ricci tensor (5.9) by formula

$$R_{ij}=\frac{\partial\Gamma^k_{ik}}{\partial x^j}-\frac{\Gamma^k_{ij}}{\partial x^k}+\Gamma^l_{ik}\Gamma^k_{jl}-\Gamma^l_{ij}\Gamma^k_{lk}$$

and using the relations (see (16.8))

$$\Gamma^i_{4j}-\Gamma^i_{1j}\quad \frac{\partial\Gamma^k_{ij}}{\partial x^4}=-\frac{\partial\Gamma^k_{ij}}{\partial x^1}\quad (i,j,k=1,\dots,4),$$

one has

$$R_{11}=R_{44},\quad R_{22}=R_{23}=R_{33}=0.$$

Therefore

$$R=0$$

and equation (14.3) in the spaces V_4 has the form

$$g^{ij}u_{,ij} \equiv g^{ij}u_{ij} - g^{ij}\Gamma^k_{ij}u_k = 0. \tag{17.1}$$

Equations (16.8) provide the following components of values $\Gamma^k = g^{ij}\Gamma^k_{ij}$ ($k = 1, \ldots, 4$):

$$\Gamma^1 = \Gamma^4 =$$

$$-\frac{1}{2}f\left(\frac{1}{\Delta}\right)^1 + \varphi\left(\frac{\varphi}{\Delta}\right)^1 - \frac{1}{2}\left(\frac{\varphi^2}{\Delta}\right)^1 = \frac{1}{2}(\varphi^2 - f)\left(\frac{1}{\Delta}\right)^1 =$$

$$= -\frac{1}{2}\frac{\Delta'}{\Delta} = [\ln(-\Delta)^{-1/2}]^1,$$

$$\Gamma^2 = \Gamma^3 = 0.$$

According to 12.3, upon equivalence transformation

$$\overline{L}[u] = e^{-\nu}L[ue^{\nu}]$$

with the function

$$\nu = \ln(-\Delta)^{1/4},$$

equation (17.1) with the coefficients $a^i = 0 (i = 1, \ldots, 4)$ turns into an equivalent equation with the coefficients

$$\bar{a}^i = \Gamma^i \quad (i = 1, \ldots, 4),$$

i.e. into equation

$$g^{ij}u_{,ij} + \bar{a}^i u_{,i} = g^{ij}u_{ij} = 0.$$

If we substitute here the values (9.5) of coefficients g^{ij} and use Theorem (1.17), the following lemma holds.

Lemma 1.9. Every linear conformally invariant equation of the second order in space V_4 of a normal hyperbolic type having a nontrivial conformal group is equivalent to equation

$$L[u] \equiv u_{tt} - u_{xx} - f(x-t)u_{yy} - 2\varphi(x-t)u_{yz} - u_{zz} = 0. \tag{17.2}$$

In what follows coefficients f and φ of equation (17.2) are considered to be smooth functions satisfying the hyperbolic condition $f - \varphi^2 > 0$.

Theorem 1.19. Let us assume that a Riemannian space of normal hyperbolic type has a nontrivial conformal group and that equation (15.1) is conformally invariant in this space. Then equation (15.1) satisfies the Huygens principle.

Proof. According to Lemma 1.9 it is sufficient to consider equation (17.2).

Let us find the function V in an elementary solution (15.4) according to formula (15.6). As one can easily verify, operator L in equation (17.2) is self-adjoint ($L^* = L$), and $c^* = 0$. Therefore,

$$L^*[\Gamma] - c^*\Gamma - 8 = L[\Gamma] - 8.$$

Note that the function Γ, determined by formula (16.23), has the form

$$\Gamma = (t-\tau)^2 - (x-\xi)^2 + g(x-t, y, z).$$

Thus,

$$\Gamma_{tt} - \Gamma_{xx} = 4.$$

Invoking this equality, one has

$$L[\Gamma] = 8 = 2\frac{x-\xi-t+\tau}{(x-\xi-t+\tau)(F-F_o) - (\Phi-\Phi_o)^2}\times$$
$$[(x-\xi-t+\tau)f - 2(\Phi-\Phi_o)\varphi + (F-F_o)] - 4.$$

Making use of the function A, defined by formula (16.21), one can write the resulting equation for $L[\Gamma]-8$ along the geodesic line in the form

$$L[\Gamma] - 8 = 2s\frac{d\ln A}{ds} - 4.$$

Hence,

$$-\frac{1}{4}\int\limits_{x_o}^{x}(L[\Gamma]-8)\frac{\partial\sigma}{\sigma} = -\frac{1}{2}\int\limits_{0}^{s}\left(\frac{d\ln A}{d\sigma} - \frac{2}{\sigma}\right)d\sigma = \ln\frac{\sigma}{\sqrt{A}}\bigg|_0^s,$$

where the parameter s corresponds to the variable point $x = (x, y, z, t)$. According to the formula (16.22)

$$\lim_{\sigma\to 0}\frac{\sqrt{A(\sigma)}}{\sigma} = \lim_{\sigma\to 0}\frac{\sqrt{-\Delta_o(\alpha-\delta)^2\sigma^2 + o(\sigma)^2}}{\sigma} = (\alpha-\delta)\sqrt{-\Delta_o}.$$

Therefore,

$$\ln\frac{\sigma}{\sqrt{A}}\bigg|_0^s = \ln\frac{s}{\sqrt{A(s)}} + \ln[(\alpha-\delta)\sqrt{-\Delta_o}],$$

so that

$$-\frac{1}{4}\int\limits_{x_o}^{x}(L[\Gamma]-8)\frac{d\sigma}{\sigma} = \ln\left[(\alpha-\sigma)s\sqrt{\frac{-\Delta_o}{A(s)}}\right].$$

Substituting here the values $A(s)$ and $(\alpha - \delta)s$ from formulae (16.21) and (16.10), respectively, and invoking the notation (16.7), according to formula (15.6) one obtains the unknown function V in the following form:

$$V = (x - \bar{\xi} - t + \tau)\times$$
$$\sqrt{\frac{f(x-\tau) - \varphi^2(\xi - \tau)}{(x - \xi - t + \tau)[F(x-t) - F(\xi - \tau)] - [\Phi(x-t) - \Phi(\xi - \tau)]^2}}. \quad (17.3)$$

When $\varphi = 0, f = 1$ this formula naturally leads to the function $V = 1$ known for the wave equation.

Function V, determined by formula (17.3) has the form $V = V(x - t, \xi - \tau)$. Since equation (17.2) is self-adjoint, the Hadamard criterion (15.8) holds.

Beginning with Hadamard's work [41] the following issue has been discussed in literature over and over again, namely existence of equations of the form (15.1) with four independent variables not equivalent (see 11.4) to the wave equation for which the Huygens principle holds. Therefore, it is of interest to find out with what functions f and φ equation (17.2), which according to Theorem 1.19 satisfies the Huygens principle with any f and φ, is not equivalent to the wave equation. This problem is considered in the following subsection for equation (17.2) with the function $\varphi = 0$. In this case it is possible to single out all functions $f(x - t)$ for which equation (17.2) is not equivalent to the wave equation. For arbitrary equations (17.2) it is also possible to obtain the necessary and sufficient conditions of equivalence of equation (17.2) to the wave equation in the form of some system of differential equations with respect to the functions f and φ. With this purpose one should act similarly to the case $\varphi = 0$, see Section 17.2. However we will limit our consideration with a more simple case mentioned above. Actually this topic is closely connected with a more general problem of classification of equations (17.2) formulated in the end of this chapter.

17.2 Discussion of the equivalence to the classical wave equation

Equivalence transformations (see 11.4) of equation (15.1) take the corresponding Riemannian space V_n to a conformal space at most. Therefore, equation (17.2) is equivalent to the classical wave equation

$$\Box u \equiv u_{tt} - u_{xx} - u_{yy} - u_{zz} = 0$$

if and only if the space V_4 with the metric tensor (9.5) is conformally flat. The necessary and sufficient condition for that is that the tensor of conformal curvature of the space V_4 equals to zero ([29] § 28). This tensor, denoted by C^l_{ijk}, is expressed by means of the tensor of space curvature by the following formula:

$$C^l_{ijk} = R^l_{ijk} + \frac{1}{n-2}(\delta^l_j R_{ik} - \delta^l_k R_{ij} + g_{ik}R^l_j - g_{ij}R^l_k) + \frac{R}{(n-1)(n-2)}(\delta^l_k g_{ij} - \delta^l_j g_{ik}) \quad (i,j,k,l = 1,\ldots,n). \tag{17.4}$$

Let us consider equations (17.2) when $\varphi = 0$. In this case, according to formulae (16.8), only the following Christoffel symbols are other than zero for the corresponding space V_4 :

$$\Gamma'_{22} = \Gamma^4_{22} = -\frac{1}{2}\left(\frac{1}{f}\right)', \quad \Gamma^2_{22} = -\Gamma^2_{24} = \frac{1}{2}f\left(\frac{1}{f}\right)'.$$

The prime here, as well as in § 16, indicates differentiation with respect to the argument.

In order to make tensor C^l_{ijk} equal to zero it is necessary to make all its components equal to zero. First, let us consider one component of the tensor e.g. C^1_{212}. Using expressions for the Richi tensor and scalar curvature of the space V_4 given in 17.1, one has

$$C^1_{212} = R^1_{212} + \frac{1}{2f}R_{11}.$$

Equations (5.7) and (5.9) in the considered case yield

$$R^1_{212} = \frac{1}{2f}R_{11} = h\left\{(\ln\sqrt{h})^{11} + (\ln\sqrt{h})^{12}\right\},$$

where $h = \frac{1}{f}$. Therefore, equality $C^1_{212} = 0$ has the form of the following differential equation of the second order for the function h :

$$(\ln\sqrt{h})^{11} + (\ln\sqrt{h})^{12} = 0.$$

This equation has the general solution

$$h(\sigma) = (a + b\sigma)^2$$

with arbitrary constants a and b.

Calculation of all components of the curvature tensor R^l_{ijk} of the space V_4 demonstrates that when $f(\sigma) = (a + b\sigma)^{-2}$ and $\varphi = 0$, all components of the Riemann-Christoffel tensor R^l_{ijk} are equal to zero. Then, obviously,

$$C^l_{ijk} = 0 \quad (i,j,k,l = 1,\ldots,4).$$

Thus, *equation (17.2) with the function $\varphi = 0$ is equivalent to the wave equation if and only if*

$$f(x-t) = [a + b(x-t)]^{-2},$$

where a and b are arbitrary constants satisfying the condition $a^2 + b^2 \neq 0$.

17.3 Solution of Hadamard's problem

The problem on finding all equations (15.1) satisfying the Huygens principle was formulated by J. Hadamard [41]. It is known as *Hadamard's problem.* The following theorem proved in [57] (see also [58]) solves Hadamard's problem in the spaces V_4 with nontrivial conformal group.

Theorem 1.20. Equation (15.1) in every space V_4 of a normal hyperbolic type with a nontrivial conformal group satisfies the Huygens principle if and only if this equation is conformally invariant, i.e. equivalent to equation (17.2).

Proof. We have already proved that every conformally invariant equation (15.1) in spaces V_4 with nontrivial conformal group satisfies the Huygens principle (Theorem 1.19). Therefore, we have only to prove that if the space V_4 of normal hyperbolic type has a nontrivial conformal group, and equation (15.1) in this space satisfies the Huygens principle, then this equation is conformally invariant. According to Lemma 1.8, for this purpose it is sufficient to prove that if equation (15.1) in the space V_4 satisfies the Huygens principle, then conditions (14.1) hold for this equation.

Let us proceed from the Hadamard criterion (15.8). Let us assume that it holds for equations (15.1) with four independent variables. Then, this equation satisfies the following conditions:

$$H = 0.$$

Function H is determined via coefficients of equation (15.1) by formula (14.2) and

$$g^{pq}(R_{jip,q} + C^m_{ijq}L_{mp}) =$$

$$= \frac{5}{2}g^{pq}\left(K_{pi}K_{qj} - \frac{1}{4}g_{ij}g^{ml}K_{pm}K_{ql}\right) \quad (i, j = 1, \ldots, 4), \tag{17.5}$$

where

$$L_{ij} = -R_{ij} + \frac{1}{6}Rg_{ij}, \quad R_{jip} = L_{jp,i} - L_{ji,p},$$

and C_{ijq}, R_{ij} and R are a tensor of conformal curvature (17.4), the Richi tensor (9.5), and a scalar curvature (5.10) of the space V_4 corresponding to equation (15.1), respectively. Let us demonstrate that in case of spaces V_4 with nontrivial conformal group equations (17.5) provide that $K_{ij} = 0$ $(i, j = 1, \ldots, 4)$. This will prove the theorem for together with the equality $H = 0$, which as it was mentioned before is one of necessary conditions for validity of the Huygens principle, we will have all equations (14.1).

According to Theorem 1.19 conditions (17.5) for equation (17.2) hold. On the other hand, all values K_{ij} for equation (17.2) are equal to zero since equation (17.2) is obtained from equation (14.3), satisfying the conditions (14.1), by equivalence transformations. Hence, the right-hand sides of equations (17.5) are equal to zero. However, in this case the tensor in the left-hand side of equations (17.5), independent of minor terms of equation (15.1), is equal to zero for all spaces V_4 with nontrivial conformal group. Thus, if some equation (15.1) in a space V_4 with nontrivial conformal group satisfies the Huygens principle, then, according to conditions (17.5), the following equalities hold for this equation:

$$g^{pq}(K_{pi}K_{qj} - \frac{1}{4}g_{ij}g^{ml}K_{pm}K_{ql}) = 0 \quad (i, j = 1, \dots, 4),$$

or, upon substitution of the value (9.5) of the tensor g^{ij}, equalities

$$\begin{gathered} K_{1i}K_{1j} + fK_{2i}K_{2j} + \varphi(K_{2i}K_{3j} + K_{3i}K_{2j}) + \\ + K_{3i}K_{3j} - K_{4i}K_{4j} + Ng_{ij} = 0 \quad (i, j = 1, \dots, 4). \end{gathered} \tag{17.6}$$

Here, the notation

$$N = \frac{1}{2}\left\{fK_{12}^2 + K_{13}^2 - \Delta K_{23}^2 + 2\varphi K_{12}K_{13} - K_{14}^2 - fK_{24}^2 - 2\varphi K_{24}K_{34} - K_{34}^2\right\}.$$

is introduced. Writing equations (17.6) consecutively for values of indices

$$(i, j) = (1, 2), (2, 2), (2, 3), (3, 3), (4, 4),$$

one obtains the following system of equations

$$fK_{12}^2 + 2\bar{\varphi}K_{12}K_{13} + K_{13}^2 - K_{14}^2 - N = 0, \tag{17.7}$$

$$K_{12}^2 + K_{23}^2 - K_{24}^2 + \frac{1}{\Delta}N = 0, \tag{17.8}$$

$$K_{12}K_{13} - K_{24}K_{34} - \varphi K_{23}^2 - \frac{\varphi}{\Delta}N = 0, \tag{17.9}$$

$$K_{13}^2 + fK_{23}^2 - K_{34}^2 + \frac{f}{\Delta}N = 0, \tag{17.10}$$

$$K_{14}^2 + fK_{24}^2 + 2\varphi K_{24}K_{34} + K_{34}^2 + N = 0. \tag{17.11}$$

Let us eliminate the value N first from equations (17.8) and (17.9) and then from equations (17.8) and (17.10). As a result we obtain

$$K_{12}K_{13} - K_{24}K_{34} = \varphi\{K_{24}^2 - K_{12}^2\} \tag{17.12}$$

and

$$K_{34}^2 - K_{13}^2 = f\{K_{24}^2 - K_{12}^2\}. \tag{17.13}$$

Subtracting the left-hand side of equation (17.7) form the left-hand side of equation (17.11) and invoking equalities (17.12) and (17.13) one obtains

$$K_{14}^2 - \Delta \cdot K_{23}^2 = 0.$$

Whence, by virtue of the condition $\Delta < 0$,

$$K_{14} = K_{23} = 0. \tag{17.14}$$

Therefore, equality (17.7) takes the form

$$(K_{13} + \varphi K_{12})^2 - \Delta \cdot K_{24}^2 = 0,$$

whence, as above,

$$K_{13} = -\varphi K_{12}, \quad K_{24} = 0. \tag{17.15}$$

Substituting these values K_{13} and K_{24} into equation (17.13), one obtains

$$K_{34}^2 - \Delta \cdot K_{12}^2 = 0,$$

so that $K_{12} = K_{34} = 0$. Now equations (17.14), (17.15) and the skew-symmetry condition $K_{ij} = -K_{ji}$ of the tensor K_{ij} provide the necessary equalities

$$K_{ij} = 0 \quad (i, j = 1, \ldots, 4).$$

In connection with the proved theorem note the following. Examples drawn by Stellmacher [126], [127] demonstrate that in case of $n > 4$ independent variable there is no one-to-one connection between the conformal invariance of an equation and validity of the Huygens principle for this equation. Namely, the mentioned examples by Stellmacher represent equations (15.1) in a flat space $V_n (n \geq 6)$ satisfying the Huygens principle but not equivalent to the wave equation in V_n, which means that according to Theorem (12.3) they are not invariant with respect to the group of conformal transformations in V_n. Nevertheless, one can observe the connection of the Huygens principle with the invariance properties of the considered equations in this case either. Thus, when $n = 6$, the equation considered by Stellmachr has the form

$$\Box u - \frac{8}{(1 - r^2)^2} u = 0,$$

where

$$\Box u \equiv u_{tt} - \sum_{i=1}^{5} u_{ii}, \quad r^2 = t^2 - \sum_{i=1}^{5} (x^i)^2,$$

i.e. it belongs to the class of equations that are invariant with respect to groups of motions of the maximum order [111]. Passing over to dimensions $n > 6$ one comes across equations satisfying the Huygens principle with even weaker invariance properties. This is demonstrated by the corresponding Stellmacher examples.

§ 18 Solution of the Cauchy problem

18.1 Reduction to a particular Cauchy problem

In this section we will solve the Cauchy problem

$$L[u] = 0,\ u|_{t=0} = g(x, y, z),\ u_t|_{t=0} = h(x, y, z) \tag{18.1}$$

for equation (17.2) with smooth initial data g and h. As in the case with the wave equation, we can restrict our consideration by the Cauchy problem in a particular form

$$L[u] = 0,\ u|_{t=0} = 0,\ u_t|_{t=0} = h(x, y, z). \tag{18.2}$$

Indeed, for the operator under consideration

$$L[u] = u_{tt} - u_{xx} - f(x - t)u_{yy} - 2\varphi(x - t)u_{yz} - u_{zz} \tag{18.3}$$

the following identity holds

$$L\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)[u] \equiv \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)L[u]. \tag{18.4}$$

Therefore, if v and w are solutions of the Cauchy problem (18.2) with the data $v_t|_{t=0} = g$, $w_t|_{t=0} = g_x - h$, respectively, then function

$$u = v_t + v_x - w \tag{18.5}$$

is the solution of the Cauchy problem in a general form (18.1).

18.2 Alternative form of Poisson's solution for the classical wave equation

Let us choose $f = 1, \varphi = 0$ in operator (18.3), i.e. take the classical wave operator

$$\Box u = u_{tt} - u_{xx} - u_{yy} - u_{zz}, \tag{18.6}$$

and consider the Cauchy problem

$$\Box u = 0, \quad u|_{t=0} = 0, \quad u_t|_t = h(x, y, z).$$

It is well known that its solution is given by Poisson's formula

$$u(x, y, z, t) = \frac{1}{4\pi t} \int\int_{s_t} h ds, \tag{18.7}$$

where s_t is a sphere of the radius t with the center at the point (x, y, z). Introducing polar coordinates on the plane of variables (y, z) for the points $(\xi, \eta\zeta)$ lying on the sphere s_t, one has

$$\xi = \xi,$$

$$\eta = y + \rho \cos\theta$$
$$\zeta = z + \rho \sin\theta,$$

where coordinates (ξ, ρ, θ) satisfy the conditions

$$(\xi - x)^2 + \rho^2 = t^2; \quad x - t \leq \xi \leq x + t, \quad 0 \leq \theta \leq 2\pi.$$

In these coordinates

$$ds = td\xi \, d\theta,$$

so that the Poisson formula (18.7) is written in the form

$$u(x, y, z, t) = \tag{18.8}$$

$$\frac{1}{4\pi} \int_{x-t}^{x+t} d\xi \int_0^{2\pi} h\left(\xi, y + \sqrt{t^2 - (\xi - x)^2} \cos\theta, z + \sqrt{t^2 - (\xi - x)^2} \sin\theta\right) d\theta.$$

We will obtain now a similar formula for the arbitrary operator (18.3).

18.3 Fourier transform and solution of the particular Cauchy problem

First let us solve the Cauchy problem (18.2) acting formally without any explanations. Then, we will verify in Sec. 18.5 that the resulting formula gives the solution of the problem under consideration indeed.

The Fourier transformation

$$\hat{u}(x, \lambda, \mu, t) = \frac{1}{2\pi} \int_{R^2} e^{-i(\lambda y + \mu z)} u(x, y, z, t) dy dz,$$

takes the Cauchy problem (18.2) into the Cauchy problem

$$\hat{L}[\hat{u}] = 0,\ \hat{u}|_{t=0} = 0, \quad \hat{u}_t|_{t=0} = \hat{h}(x, \lambda, \mu) \tag{18.9}$$

for the operator

$$\hat{L}[\hat{u}] = \hat{u}_{tt} - \hat{u}_{xx} + (f\lambda^2 + 2\varphi\lambda\mu + \mu^2)\hat{u} \tag{18.10}$$

with the independent variables x and t. Here λ and μ are considered as parameters.

Upon change of variables

$$\bar{x} = -\frac{1}{2}[\lambda^2 F(x-t) + 2\lambda\mu\Phi(x-t) + \mu^2(x-t)],$$

$$\bar{t} = \frac{1}{2}(x+t)$$

equation $\hat{L}[\hat{u}] = 0$ turns into equation

$$\hat{u}_{\bar{x}\bar{t}} + \hat{u} = 0$$

with the known Riemannian function

$$R(\bar{\xi}, \bar{\tau}; \bar{x}, \bar{t}) = J_o(\sqrt{4(\bar{t} - \bar{\tau})(\bar{x} - \bar{\xi})}),$$

where J_o is the Bessel function. Here, as well as in previouos sections F and Φ indicate the primary functions for f and φ, respectively. Turning back to the previous variables x and t, we obtain the following Riemannian function for equation $\hat{L}[\hat{u}] = 0$:

$$R(\xi, \tau; x, t) =$$

$$J_o(\sqrt{(x - \xi + t - \tau)[\lambda^2(F_o - F) + 2\lambda\mu(\Phi_o - \Phi) + \mu^2(\xi - x - \tau + t)]}),$$

where

$$F_o = F(\xi - \tau), \quad F = F(x-t) \text{ etc.}$$

Having the Riemann function one can solve the Cauchy problem (18.9) using the formula

$$\hat{u}(x, \lambda, \mu, t) = \frac{1}{2}\int\limits_{x-t}^{x+t} \hat{h}(\xi, \lambda, \mu) R(\xi, 0; x, t) d\xi.$$

Upon substitution of the value $\hat{u}(\xi, \lambda, \mu)$, this formula is written in the form

$$\hat{u}(x, \lambda, \mu, t) = \frac{1}{4\pi}\int\limits_{x-t}^{x+t} R(\xi, 0; x, t) \int\limits_{R^2} e^{-i(\lambda\eta + \mu\zeta)} h(\xi, \eta, \zeta) d\eta d\zeta. \tag{18.11}$$

18.4 Inverse Fourier transform of the solution

The inverse Fourier transformation of the formula (18.11) provides the solution of the initial Cauchy problem (18.2) in the following form

$$u(x,y,z,t) = \frac{1}{4\pi}\int\limits_{x-t}^{x+t} d\xi \int\limits_{R^2} I\cdot h(\xi,\eta,\zeta)d\eta d\zeta, \tag{18.12}$$

where

$$I = \frac{1}{2\pi}\int\limits_{R^2} e^{-i[\lambda(\eta-y)+\mu(\zeta-z)]} J_o(k\sqrt{Q(\lambda,\mu)})d\lambda d\mu.$$

Here $Q(\lambda,\mu)$ indicates the quadratic form

$$Q(\lambda,\mu) = \alpha^2\lambda^2 + 2b\lambda\mu + c^2\mu^2$$

with the coefficients

$$a^2 = F(\xi) - F(x-t),\ b = \Phi(\xi) - \Phi(x-t),\ c^2 = \xi - (x-t),$$

and

$$k = \sqrt{x+t-\xi}.$$

Let us reduce the integral I to a more convenient form. The condition

$$-\Delta(\sigma) = f(\sigma) - \varphi^2(\sigma) > 0$$

of hyperbolic property of the operator (18.3) entails that the quadratic form of λ and μ is determined by the formula

$$q(\sigma;\lambda,\mu) = f(\sigma)\lambda^2 + 2\varphi(\sigma)\lambda\mu + \mu^2$$

and $-\Delta(\sigma)$ has a discriminant and is positively defined. Hence, the quadratic form

$$Q(\lambda,\mu) = \int\limits_{x-t}^{\xi} q(\sigma;\lambda,\mu)d\sigma,$$

contained in the integral I, is also positively defined. Then, the discriminant of the quadratic form $Q(\lambda,\mu)$ equal to $a^2c^2 - b^2$ is positive. Keeping this in mind, make the change of variables $\lambda,\mu;\eta,\zeta$:

$$\lambda = \frac{1}{a}\left(\bar\lambda - \frac{b}{\sqrt{a^2c^2-b^2}}\bar\mu\right),\ \mu = \frac{a}{\sqrt{a^2c^2-b^2}}\bar\mu$$

and

$$\bar{\eta} - \bar{y} = \frac{1}{a}(\eta - y),$$

$$\bar{\zeta} - \bar{z} = \frac{1}{\sqrt{a^2c^2 - b^2}} \left[a(\zeta - z) - \frac{b}{a}(\eta - y) \right]. \tag{18.13}$$

In these variables

$$Q = \bar{\lambda}^2 + \bar{\mu}^2$$

and

$$\lambda(\eta - y) + \mu(\zeta - z) = \bar{\lambda}(\bar{\eta} - \bar{y}) + \bar{\mu}(\bar{\zeta} - \bar{z}),$$

$$d\eta\, d\zeta = \sqrt{a^2c^2 - b^2} d\bar{\eta}\, d\bar{\zeta}, \quad d\lambda\, d\mu = \frac{d\bar{\lambda}\, d\bar{\mu}}{\sqrt{a^2c^2 - b^2}} \cdot$$

Let us make use of the well-known formula of Fourier transformation of spherically symmetric functions and properties of the Bessel function J_o. Making some standard calculations one obtains

$$I = \frac{1}{\sqrt{a^2c^2 - b^2}} \int_0^\infty J_o(kr) J_o(\rho r) r dr = \frac{\delta(k - \rho)}{\rho\sqrt{a^2c^2 - b^2}},$$

where δ is the delta-function and

$$\rho = \sqrt{(\bar{\eta} - \bar{y})^2 + (\bar{\zeta} - \bar{z})^2}.$$

Using the resulting value of the integral I, one can readily calculate the inner integral of formula (18.12). For this purpose it is convenient to shift to polar coordinates ρ, θ on the plane of variables $\bar{\eta}, \bar{\zeta}$:

$$\bar{\eta} - \bar{y} = \rho \cos\theta, \ \bar{\zeta} - \bar{z} = \rho \sin\theta.$$

Substituting here the values (18.13) of the quantities $\bar{\eta} - \bar{y}$ and $\bar{\zeta} - \bar{z}$ one obtains the functions $\eta(\rho, \theta)$ and $\zeta(\rho, \theta)$. Now one can calculate the mentioned integral. One has

$$\int_{R^2} I \cdot h(\xi, \eta, \zeta) d\eta\, d\zeta =$$

$$= \int_0^{2\pi} d\theta \int_0^\infty I \cdot h(\xi, \eta(\rho, \theta), \zeta(\rho, \theta)) \sqrt{a^2c^2 - b^2} \rho d\rho =$$

$$= \int_0^{2\pi} h(\xi, \eta(k, \theta), \zeta(k, \theta)) d\theta.$$

Determining the values of functions $\eta(k,\theta)$, $\zeta(k,\theta)$ from formulae (18.13) one obtains

$$\int\limits_{R^2} I\cdot h(\xi,\eta,\zeta)d\eta\, d\zeta =$$

$$= \int\limits_0^{2\pi} h(\xi, y + ak\cos\theta, z + \frac{b}{a}k\cos\theta + \frac{\sqrt{a^2c^2-b^2}}{a}k\sin\theta)d\theta.$$

Using this expression and introducing the notation

$$A = \sqrt{(x+t-\xi)[F(\xi)-F(x-t)]},$$

$$B = \frac{x+t-\xi}{A}[\Phi(\xi)-\Phi(x-t)],\ C = \sqrt{t^2-(x-\xi)^2-B^2}, \qquad (18.14)$$

one can write formula (18.12) for solution of the Cauchy problem (18.2) in the following final form:

$$u(x,y,z,t) =$$

$$T[h] \equiv \frac{1}{4\pi}\int\limits_{x-t}^{x+t} d\xi \int\limits_0^{2\pi} h(\xi, y + A\cos\theta, z + B\cos\theta + C\sin\theta)d\theta. \qquad (18.15)$$

When $f = 1, \varphi = 0$ formula (18.15) obviously coincides with the Poisson formula (18.8).

18.5 Verification of the solution

Let us verify now that formula (18.15) determines the solution of the Cauchy problem (18.2) indeed. One meets no difficulty in verifying the satisfaction of initial conditions

$$u|_{t=0} = 0,\ u_t|_{t=0} = h(x,y,z)$$

for the function u determined by formula (18.15) and we will not dwell on that.

Let us make a preliminary change of variables

$$\alpha = x - t, \quad \beta = x + t$$

in order to check whether equations

$$L[u] = 0$$

hold. Writing operator (18.3) in these variables one obtains

$$L[u] = 4u_{\alpha\beta} + f(\alpha)u_{yy} + 2\varphi(\alpha)u_{yz} + u_{zz}. \tag{18.16}$$

The formula (18.15) with the variables α, β takes the form

$$u(\alpha, y, z, \beta) = \frac{1}{4\pi}\int\limits_{\alpha}^{\beta} d\xi \int\limits_{0}^{2\pi} h(\xi, y + A\cos\theta, z + B\cos\theta + C\sin\theta)d\theta,$$

where, according to notation (18.14),

$$A = \sqrt{(\beta-\xi)[F(\xi) - F(\alpha)]},$$

$$B = \frac{\beta-\xi}{A}[\Phi(\xi) - \Phi(\alpha)],\ C = \sqrt{(\beta-\xi)(\xi-\alpha) - \beta^2}.$$

One can check that functions A, B, and C satisfy the equations

$$A_\alpha A_\beta = -\frac{1}{4}f(\alpha),\ B_\beta = \frac{B}{A}A_\beta,\ C_\beta = \frac{C}{A}A_\beta,$$

$$A_\alpha B_\beta + A_\beta B_\alpha = -\frac{1}{2}\varphi(\alpha),\ B_\alpha B_\beta + C_\alpha C_\beta = -\frac{1}{4}, \tag{18.17}$$

$$A_{\alpha\beta} = \frac{A_\alpha A_\beta}{A},\ B_{\alpha\beta} = \frac{B_\alpha B_\beta}{B},\ C_{\alpha\beta} = \frac{C_\alpha C_\beta}{C}\cdot$$

Now we can act by operator (18.16) on the function $u(\alpha, y, z, \beta)$ resulting from the formula (18.15). One has

$$u_{\alpha\beta} = \frac{1}{4\pi}\int\limits_{\alpha}^{\beta}\{S + \int\limits_{0}^{2\pi}(h_{yy}A_\alpha A_\beta\cos^2\theta + h_{yz}[(A_\alpha C_\beta +$$

$$A_\beta C_\alpha)\cos\theta\sin\theta + (A_\alpha B_\beta + A_\beta B_\alpha)\cos^2\theta] + h_{zz}[B_\alpha B_\beta\cos^2\theta +$$

$$(B_\alpha C_\beta + B_\beta C_\alpha)\cos\theta\sin\theta + C_\alpha C_\beta\sin^2\theta])d\theta\}d\xi, \tag{18.18}$$

where

$$S = \int\limits_{0}^{2\pi}[h_y A_{\alpha\beta}\cos\theta + h_z(B_{\alpha\beta}\cos\theta + C_{\alpha\beta}\sin\theta)]d\theta.$$

Integrating by parts:

$$\int\limits_{0}^{2\pi} h_y\cos\theta d\theta = \int\limits_{0}^{2\pi} h_y d(\sin\theta) = -\int\limits_{0}^{2\pi}\sin\theta d(h_y) =$$

$$\int_0^{2\pi}[h_{yy}A\sin^2\theta + h_{yz}(B\sin^2\theta - C\sin\theta\cos\theta)]d\theta$$

and doing the same with the second term of the integral S one transforms the integral S to the form

$$S = \int_0^{2\pi}\{h_{yy}AA_{\alpha\beta}\sin^2\theta + h_{yz}[(AB_{\alpha\beta} + BA_{\alpha\beta})\sin^2\theta -$$

$$-(AC_{\alpha\beta} + CA_{\alpha\beta})\cos\theta\cdot\sin\theta] + h_{zz}[BB_{\alpha\beta}\sin^2\theta -$$

$$-(BC_{\alpha\beta} + CB_{\alpha\beta})\cos\theta\cdot\sin\theta + CC_{\alpha\beta}\cos^2\theta]\}d\theta.$$

Substituting the resulting expression of the integral S in (18.18) and using equalities (18.17), one obtains

$$4u_{\alpha\beta} = \frac{1}{4\pi}\int_\alpha^\beta d\xi\int_0^{2\pi} -(f(\alpha)h_{yy} + 2\varphi(\alpha)h_{yz} + h_{zz})d\theta. \tag{18.19}$$

The function $u(\alpha, y, z, \beta)$ is readily differentiated with respect to the variables y and z. Therefore, acting by the operator (18.16) on the function $u(\alpha, y, z, \beta)$ and using Eq. (18.19), one obtains

$$L[u] = 0.$$

It should be noted that peculiarities occurring in (18.18) are integrable, so that all the above operations are true.

Thus, one can see that the function $u(x, y, z, t)$, determined by formula (18.15), is the solution of the Cauchy problem (18.2) indeed.

18.6 Solution of the general Cauchy problem

The formulae (18.5) and (18.15) demonstrate that solution of the Cauchy problem (18.1) for equation (17.2) has the form

$$u = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)T[q] - T\left[\frac{\partial q}{\partial x} - h\right] \tag{18.20}$$

with the same operator T as in formula (18.15).

The resulting formulae let to find solution of the Cauchy problem for all equations (15.1) when $n = 4$, the corresponding space V_4 has a nontrivial conformal group, and equation (15.1) satisfies the Huygens principle. Indeed, we have demonstrated (Theorem 1.20) that every such equation is equivalent to equation (17.2), for which the solution of the Cauchy problem is known. However, equivalence transformations (see Sec. 11.4) consist of a change of independent variables, a linear substitution of the function u and a shift to a conformal space. Therefore, the solution of the Cauchy problem for indicated equations (15.1) with $n = 4$, satisfying the Huygens principle, is provided by formula (18.20) by the corresponding change of variables x and y and application of Theorem 1.18.

§ 19 Spaces with trivial conformal group

19.1 Example of a space with trivial conformal group

In previous sections we considered equations (15.1) in four-dimensional Riemannian spaces of normal hyperbolic type with nontrivial conformal group. We could see that in every such space there exists the only linear equation of the second order satisfying the Huygens principle. Now, let us draw examples of spaces V_4 of a normal hyperbolic type with a *trivial* conformal group, where none equation of the form (15.1) satisfies the Huygens principle.

Let us take a space V_4 with the metric form

$$ds^2 = -(1+t)dx^2 - dy^2 - dz^2 + dt^2, \quad t \geq 0. \tag{19.1}$$

First let us demonstrate that this space has a nontrivial conformal group.

Denoting

$$x^1 = x, \ x^2 = y, \ x^3 = z, \ x^4 = t,$$

as usually, we write the generalized Killing equations for the operator

$$X = \xi^i \frac{\partial}{\partial x^i}$$

of a group of conformal transformations in this space in the form of the following two systems of equations:

$$\frac{\partial \xi^2}{\partial z} + \frac{\partial \xi^3}{\partial y} = \frac{\partial \xi^2}{\partial t} - \frac{\partial \xi^4}{\partial y} = \frac{\partial \xi^3}{\partial t} - \frac{\partial \xi^4}{\partial z} = 0,$$

$$\frac{\partial \xi^2}{\partial y} = \frac{\partial \xi^3}{\partial z} = \frac{\partial \xi^4}{\partial t} = \frac{\mu}{2} \tag{19.2}$$

and

$$(1+t)\frac{\partial \xi^1}{\partial y} + \frac{\partial \xi^2}{\partial x} = (1+t)\frac{\partial \xi^1}{\partial z} + \frac{\partial \xi^3}{\partial x} = (1+t)\frac{\partial \xi^1}{\partial t} - \frac{\partial \xi^4}{\partial x} = 0,$$

$$\frac{\partial \xi^1}{\partial x} = \frac{\mu}{2} - \frac{1}{2(1+t)}\xi^4. \tag{19.3}$$

System (19.2) has the form of generalized Killing equations in a flat three-dimensional space of variables (y, z, t). Therefore, (see § 9) system (19.2) has the following general solution

$$\xi^2 = A_1(y^2 - z^2 + t^2) + 2A_2yz + 2A_3yt + By + C_1z + C_2t + k_1,$$

$$\xi^3 = 2A_1yz + A_2(z^2 - y^2 + t^2) + 2A_3zt + Bz - C_1y + C_3t + k_2,$$

$$\xi^4 = 2A_1yt + 2A_2zt + A_3(y^2 + z^2 + t^2) + Bt + C_2y + C_3z + k_3, \tag{19.4}$$

where $A_i, C_i, k_i (i = 1, 2, 3)$ and B are arbitrary functions of x. One can determine these functions by means of the system (19.3).

Substituting expressions (19.4) into system (19.3) and investigating the conditions of convergence of the resulting equations with respect to the function ξ^1, one obtains

$$A_i = 0 \quad (i = 1, 2, 3), \quad C_2 = C_3 = 0,$$

$$k_i = \text{const} \quad (i = 1, 2, 3), \quad B = k_3.$$

Using these correlations, one arrives at the general solution of equations (19.2), (19.3)

$$\xi^1 = \frac{1}{2}Bx + k_1, \quad \xi^2 = By + Cz + k_2,$$

$$\xi^3 = Bz - Cy + k_3, \quad \xi^4 = B(1+t),$$

depending on the five arbitrary constants B, C, k_i $(i = 1, 2, 3)$.

Thus, the space V_4 with the metric form (19.1) has a five-parameter group G_5 of conformal transformations. One can take

$$X_1 = \frac{\partial}{\partial x}, \ X_2 = \frac{\partial}{\partial y}, \ X_3 = \frac{\partial}{\partial z}, \ X_4 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z},$$

$$X_5 = \frac{1}{2}x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + (1+t)\frac{\partial}{\partial t} \tag{19.5}$$

as basis operators of the group. It means that the space V_4 is a space with a trivial conformal group. Indeed, according to § 9, spaces V_4 of normal hyperbolic type with nontrivial conformal group has a conformal group depending on $6, 7$ or 15 parameters.

One can readily indicate the space $\widetilde{V}_4$, conformal to the considered space V_4, where the group G_5 with the basis infinitesimal generators (19.5) is a group of motions. Note that if operator X with the coordinates $\xi^i (i =$

$1, \ldots, n)$ is an operator of a one-parameter subgroup of a group of conformal transformations in a space V_n with the metric tensor $g_{ij}(x)$, so that the generalized Killing equations with some function μ

$$\xi_{i,j} + \xi_{j,i} = \mu g_{i,j} \quad (i, j = 1, \ldots, n)$$

hold, then operator X, with transition to a conformal space $\widetilde{V}_n$ with a metric tensor

$$\tilde{g}_{ij}(x) = e^{\sigma(x)} g_{ij}(x) \quad (i, j = 1, \ldots, n), \tag{19.6}$$

will satisfy the generalized Killing equations in the space $\widetilde{V}_n$, and the corresponding function $\tilde{\mu}$ equals

$$\tilde{\mu} = \mu + \xi^i \frac{\partial \sigma}{\partial x^i} \cdot$$

Therefore, the space V_n represents a space with a trivial conformal group if and only if equations

$$\xi^i_\alpha(x) \frac{\partial \sigma(x)}{\partial x^i} + \mu_\alpha(x) = 0 \quad (\alpha = 1, \ldots, r) \tag{19.7}$$

are conjugate for all operators

$$X_\alpha = \xi^i_\alpha \frac{\partial}{\partial x^i} \quad (\alpha = 1, \ldots, r)$$

of the group G_r of conformal transformations in the space V_n. If equations (19.7) are conjugate, the group G_r is a group of motions in a space $\widetilde{V}_n$ with the metric tensor (19.6), where $\sigma(x)$ is the solution of the system of equations (19.7).

In our case equation (19.7), written for operators X_1, X_2, X_3 of the set of operators (19.5), leads to the conditions $\sigma = \sigma(t)$. Meanwhile, equation (19.7) for the operator X_4 holds identically and for the operator X_5 has the form

$$(1 + t) \frac{d\sigma}{dt} + 2 = 0.$$

Eliminating the constant addend which is not important here, one can write the solution of the resulting equation in the form

$$\sigma = \ln(1 + t)^{-2}.$$

Thus, the group G_5 with the basis infinitesimal generators (19.5) is a group of motions in the space $\widetilde{V}_4$ with the metric form

$$ds^2 = \frac{1}{(1 + t)^2} [-(1 + t) dx^2 - dy^2 - dz^2 + dt^2].$$

19.2 Invalidity of Huygens' principle

Now, let us demonstrate that there exist no equations of the form (15.1) satisfying the Huygens principle in space V_4 with the metric form (19.1). With this purpose, we make use of equations (17.5). Calculating the tensor

$$S_{ij} = g^{pq}(R_{ijp,q} + C^m_{ijq} L_{mp}) \quad (i, j = 1, \ldots, 4) \tag{19.8}$$

in the left-hand side of equations (17.5) for the metric form (19.1), one obtains that

$$S_{11} = 4\tau(1+t)S_{44}, \quad S_{22} = S_{33} = \frac{5}{3}S_{44},$$
$$S_{44} = -\frac{1}{48(1+t)^4}\,. \tag{19.9}$$

Equations (17.5) for the values of indices

$$(i, j) = (1, 1), (2, 2), (3, 3), (4, 4),$$

respectively, will have the form

$$\frac{1}{1+t}(K^2_{12} + K^2_{13} - K^2_{14}) - N = -\frac{94}{5}S_{44}, \tag{19.10}$$

$$\frac{1}{1+t}K^2_{12} + K^2_{23} - K^2_{24} - N = -\frac{2}{3}S_{44}, \tag{19.11}$$

$$\frac{1}{1+t}K^2_{13} + K^2_{23} - K^2_{34} - N = -\frac{2}{3}S_{44}, \tag{19.12}$$

$$\frac{1}{1+t}K^2_{14} + K^2_{24} + K^2_{34} + N = -\frac{2}{5}S_{44}, \tag{19.13}$$

where

$$N = \frac{1}{2}\left[\frac{1}{1+t}(K^2_{12} + K^2_{13} - K^2_{14}) + K^2_{23} - K^2_{24} - K^2_{34}\right].$$

Subtracting term by term, one obtains from (19.11) and (19.12)

$$\frac{1}{1+t}K^2_{12} + K^2_{34} = \frac{1}{1+t}K^2_{13} + K^2_{24}. \tag{19.14}$$

Addition of equations (19.11) and (19.12) term by term and substitution of the quantities of expressions for N yields

$$K^2_{23} + \frac{1}{1+t}K^2_{14} = -\frac{4}{3}S_{44}. \tag{19.15}$$

In view of equation (19.14), equations (19.10) and (19.13) provide

$$\frac{1}{1+t}K_{12}^2 + K_{34}^2 = -\frac{48}{5}S_{44}. \tag{19.16}$$

Using equalities (19.10), (19.15), (19.16) and invoking that due to formulae (19.9) $S_{44} \neq 0$, one arrives to the contradictory condition $67S_{44} = 141S_{44}$.

Thus, one can see that whatever value of K_{ij} is chosen, the necessary conditions (17.5) of validity of the Huygens principle for equation (15.1) in the space V_4 with the basic metric form (19.1) do not hold. Hence, none equation of the second order satisfies the Huygens principle in the space under consideration V_4.

19.3 Schwarzschild space

Let us consider the well-known in physics Schwarzschild space as a next example. Making the corresponding choice of the system of units, one can write the metric form of the space in spherical coordinates as follows:

$$ds^2 = -\left(1 - \frac{1}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) + \left(1 - \frac{1}{r}\right) dt^2.$$

The Schwarzschild space is a space V_4 of a normal hyperbolic type with a trivial conformal group. A group of conformal transformations in the space corresponds to a four-parameter group of motions generated by rotation in a three-dimensional space of variables (x, y, z) and translation with respect to time $t = x^4$.

The Schwarzschild space satisfies equation

$$R_{ij} = 0 \quad (i, j = 1, \ldots, 4).$$

Therefore, tensor S_{ij} for this space is also equal to zero according to formula (19.8). Bearing this in mind one can demonstrate that the only solution of equations (17.5) in this case is $K_{ij} = 0$ $(i, j = 1, \ldots, 4)$. It means that if the Schwarzschild space contains equation (15.1) satisfying the Huygens principle then conditions (14.1) should hold for this equation, i.e. this equation is equivalent to equation (14.3). However, the Hadamard criterion (15.8) for equation (14.3) in the Schwarzschild space demonstrates that condition (15.8) does not hold for this equation. This follows also from results of McLenaghan [103]. Thus, the Schwarzschild space gives us another example of the space V_4 with a trivial conformal group where equations of the form (15.1) satisfying the Huygens principle do not exist.

19.4 General discussion

In case when spaces V_4 have a nontrivial conformal group, we know all linear equations of the second order in these spaces satisfying the Huygens principle. On the other hand, we do not know any example of the space V_4 with a trivial conformal group containing a second order equation satisfying the Huygens principle. We have already drawn examples of spaces V_4 where such equations do not exist and the number of these examples can be increased. In this connection the following question is of interest. Do four-parameter Riemannian spaces of a normal hyperbolic type with a trivial conformal group containing an equation of the form (15.1) satisfying the Huygens principle exist?

The negative answer to this question for spaces V_4, satisfying the condition

$$R_{ij} = 0 \quad (i, j = 1, \ldots, 4),$$

was given by McLenaghan [103]. The question remains open for the general case.

Open problem: Classification of spaces with nontrivial conformal group

In 17.2 we have found a family of equations (17.2) equivalent to the wave equation which also has the form (17.2). Generally speaking, equations (17.2) contain equations equivalent to each other. In order to find out how broad the class of known second order equations satisfying the Huygens principle is it is desirable to exclude from the class of equations (17.2) only equations not equivalent to each other. Thus, the following classification problem arises.

Problem 1.4. Divide the family of equations (17.2) into equivalence classes.

Note that there are at least three equations among (17.2) that are not equivalent to each other. Indeed, let us assume that $\varphi = 0$ in (17.2) and choose the function $f(\sigma)$ equal to $1, e^{\sigma}, e^{-\sigma^2}$ consecutively. As the solution of generalized Killing equations demonstrates, the resulting spaces V_4 have conformal groups of the order $15, 7$ and 6, respectively. It means that the resulting spaces V_4 are not conformal to each other and hence, the corresponding equations (17.2) are not equivalent to each other. Thus, the number of classes of equivalence is not less than 3.

Chapter 5
Invariant variational problems and conservation laws

This chapter is dedicated to the theory of conservation laws for differential equations admitting continuous transformation groups. Theorem 1.22 (Section 22.2) holds a central position in the theory. It establishes a correspondence between existence of conservation laws for the Euler-Lagrange equations with a given functional and the invariance of extremal values of this functional with respect to a continuous transformation group admitted by the Euler-Lagrange equations.

Literature: Noether [107], Bessel-Hagen [17], Hill [47], Ibragimov [56] (see also Candotti, Palmieri and Vitale [20]).

§ 20 Conservation laws

20.1 Definition of conservation laws

Consider a system of differential equations

$$F_1(x, u, u', \ldots) = 0, \ldots, F_N(x, u, u', \ldots) = 0, \tag{20.1}$$

where $x = (x^1, \ldots, x^n)$, $u = (u^1, \ldots, u^m)$, and u' is the collection of the first-order partial derivatives

$$u_i^k \equiv \frac{\partial u^k}{\partial x^i} \quad (i = 1, \ldots, n; k = 1, \ldots, m).$$

The dots indicate that Eq. (20.1) can contain derivatives of a higher order.

Definition 1.15. The system of differential equations (20.1) is said to have a conservation law if there exists an n-dimensional vector $A = (A^1, \ldots, A^n)$ with the components

$$A^i = A^i(x, u, u', \ldots), \quad i = 1, \ldots, n,$$

that satisfies the condition

$$\operatorname{div} A \equiv D_i(A^i) = 0 \tag{20.2}$$

at any solution $u = u(x)$ of Eqs. (20.1).

Here and in what follows D_i indicates the operator of total differentiation with respect to the variable x^i (see also Section 4.1):

$$D_i = \frac{\partial}{\partial x^i} + u_i^k \frac{\partial}{\partial u^k} + u_{ij}^k \frac{\partial}{\partial u_j^k} + \cdots \quad (i = 1, \ldots, n).$$

Let us assume that one of the independent variables, e.g. x^1, in Eqs. (20.1) is time t and write Eq. (20.2) in the form

$$D_t(A^1) + D_2(A^2) + \cdots + D_n(A^n) = 0. \tag{20.2*}$$

Proposition 1.1. Eq. (20.2*) implies existence of a function $E(t, u, u', \ldots)$ which does not vary with time on any solution $u = u(x)$ of Eqs. (20.1).

Proof. Consider an $(n-1)-$dimensional tube domain Ω in the $n-$dimensional space $\mathbb{R}^n$ of the variables $x = (t, x^2, \ldots, x^n)$ given by

$$\Omega = \left\{ x \in R^n : \sum_{i=2}^{n} (x^i)^2 = r^2, t_1 \leq t \leq t_2 \right\},$$

where r, t_1 and t_2 are given constants such that $r > 0,\ t_1 < t_2$. Let S be the boundary of Ω and let ν be the unit outward normal to the surface S. Applying the divergence theorem to the domain Ω and using Equation (20.2) one obtains:

$$\int_S A \cdot \nu \quad d\sigma = \int_\Omega \operatorname{div} A = 0. \tag{20.3}$$

Assuming that the components A^i of the vector A evaluated at solutions of Eqs. (20.1) decrease rapidly enough at the space infinity and letting $r \to \infty$, we can neglect the integral over the cylindrical surface in the left-hand side of Eq. (20.3). In order to obtain integrals over the bases of the cylinder Ω note that at the lower base of the cylinder $(t = t_1)$ we have

$$A \cdot \nu = -A^1|_{t=t_1},$$

and at the upper base $(t = t_2)$ we have

$$A \cdot \nu = A^1|_{t=t_2}.$$

Therefore, Eq. (20.3) entails that the function $A^1(x, u(x), u'(x), \ldots)$ satisfies the condition

$$\int_{R^{n-1}} A^1 dx^2 \ldots dx^n \bigg|_{t=t_1} = \int_{R^{n-1}} A^1 dx^2 \ldots dx^n \bigg|_{t=t_2}$$

for any solution $u(x)$ of the system (20.1). Since t_1 and t_2 are arbitrary, the above equation means that the function

$$E = \int\limits_{R^{n-1}} A^1(x, u(x), u'(x), \ldots)dx^2 \cdots dx^n \tag{20.4}$$

is independent of time for any solution of Eqs. (20.1), i.e.

$$D_t(E)\Big|_{(20.1)} = 0. \tag{20.5}$$

This completes the proof.

Note that in the one-dimensional case (i.e. $n = 1$) the equations (20.2) and (20.5) coincide (see § 24 - § 26). Moreover, a conservation law for Eqs. (20.1) is often identified with Eq. (20.5) due to a physical significance of the latter. In what follows we will consider conservation laws in the form (20.2). If there exist p linearly independent vectors satisfying the condition (20.2) the system (20.1) is said to have p independent conservation laws.

Remark 1.5. If

$$A^1\Big|_{(20.1)} = \tilde{A}^1 + D_2(h^2) + \cdots + D_n(h^n)$$

the conservation equation (20.2*) can be equivalently rewritten in the form

$$D_t(\tilde{A}^1) + D_2(\tilde{A}^2) + \cdots + D_n(\tilde{A}^n) = 0 \tag{20.2**}$$

with

$$\tilde{A}^2 = A^2 + D_t(h^2), \quad \ldots, \quad \tilde{A}^n = A^n + D_t(h^n)$$

because, e.g.

$$D_t D_2(h^2) = D_2 D_t(h^2).$$

If $\tilde{A}^1 = 0$, the corresponding physical conserved quantity $\tilde{E}$ defined by Eq. (20.4) with A^1 replaced by $\tilde{A}^1$ vanishes. Therefore the conservation law (20.2*) is *trivial* from the physical view point.

20.2 Historical notes

A general constructive method of determining conservation laws for arbitrary systems of differential equations (20.1) does not exist*. Therefore, one has to make special investigation in each case.

*Excepting the *direct method* based on the definition (20.2) of conservation laws. It was used in 1798 by Laplace. He applied the direct method to Kepler's problem in celestial mechanics and found a new vector-valued conserved quantity (see [88], Book II, Chap. III, Eqs. (P)) known as Laplace's vector.

It was observed already in the 19th century that conservation laws in classical mechanics can be found using symmetry properties of the considered mechanic system, i.e. group properties of differential equations of particle motions. On the other hand, it seems quite impossible to develop a general theory leading to construction of conservation laws proceeding only from symmetry properties of differential equations*. It turns out that it becomes possible if equations (20.1) are obtained, like equations of mechanics, from the variational principle.

Klein [84] was the first to pay serious attention to connection between properties of invariance of variational problems and conservation laws. This question was investigated in a general form in Noether' work [107]. Inspecting the condition of the invariance of variational integrals with respect to local continuous group of transformations of the independent of dependent variables, E. Noether obtained the conservation laws of the form (20.2) for corresponding Euler-Lagrange equations. This result is known in the literature as Noether's theorem, more specifically, *Noether's first theorem.*

Note, that the case when the variational integral is invariant with respect to an infinite group containing one or several arbitrary functions of all independent variables $x^1, \ldots, x^n$ is singled out specially in Noether's work. This case leads to dependence of some Euler-Lagrange equations on the remaining equations instead of conservation laws (*Noether's second theorem*). This case is omitted in further consideration.

The next section contains my own proof of Noether's theorem. It differs from the proof given by E. Noether and is based essentially on invariance properties of differential equations rather than on techniques of variations used by Noether.

§ 21 Noether's theorem

21.1 Euler-Lagrange equations

For the sake of simplicity, let us limit our consideration by variational integrals involving derivatives of the first order only:

$$l[u] = \int_{\Omega} \mathcal{L}(x, u, u')dx. \tag{21.1}$$

This limitation is not of a fundamental nature and further results are easily extended to the general case when the Lagrangian $\mathcal{L}$ can depend on

**Author's note to this 2008 edition:* Such theory has been developed recently in [70].

derivatives of any finite order. On the other hand, functionals of the form (21.1) are of special interest for physical applications, for the most part of equations considered there are deduced from the variational principle with action integrals of the form (21.1).

Considering extremal values of the functional (21.1) with an arbitrary area of integration $\Omega \subset R^n$ one obtains the Euler-Lagrange equations

$$D_i\left(\frac{\partial \mathcal{L}}{\partial u_l^k}\right) - \frac{\partial \mathcal{L}}{\partial u^k} = 0 \quad (k = 1, \ldots, m), \tag{21.2}$$

which is a system of m equations of less than second order with respect to functions $u^k (k = 1, \ldots, m)$ with independent variables $x^i (i = 1, \ldots, n)$. In what follows *any solution of the Euler-Lagrange equations* (21.2) *is called an extremal of the functional* (21.1).

21.2 Invariant functionals

Let G_r be an r-parameter local group of transformations

$$\bar{x} = f(x, u, a),$$
$$\bar{u} = \varphi(x, u, a), \tag{21.3}$$

where

$$f = (f^1, \ldots, f^n), \quad \varphi = (\varphi^1, \ldots, \varphi^m),$$

and let the group parameter $a = (a^1, \ldots, a^r)$ be such that

$$f(x, u, 0) = x, \quad \varphi(x, u, 0) = u.$$

A basis of generators of the group G_r are written in the form

$$X_\alpha = \xi_\alpha^i(x, u)\frac{\partial}{\partial x^i} + \eta_\alpha^k(x, u)\frac{\partial}{\partial u^k} \quad (\alpha = 1, \ldots, r) \tag{21.4}$$

with

$$\xi_\alpha^i = \left.\frac{\partial f^i}{\partial a^\alpha}\right|_{a=0}, \quad \eta_\alpha^k = \left.\frac{\partial \varphi^k}{\partial a^\alpha}\right|_{a=0} \quad (i = 1, \ldots, n;\ k = 1, \ldots, m).$$

An equation

$$u = u(x), \tag{21.5}$$

where $u(x)$ is a vector-function function with given components $u^k(x)$ $(k = 1, \ldots, m)$, defines an n-dimensional manifold in the space R^{n+m} of variables (x, u). The transformation (21.3) maps this manifold to an n-dimensional manifold given by the equation

$$\bar{u} = \bar{u}(\bar{x}).$$

The specific form of the function $\bar{u}(\bar{x})$ can be obtained by substituting the function (21.5) in Eqs. (21.3), but this orm is not of our interest for the moment. *The integral*

$$\int_{\bar{\Omega}} \mathcal{L}(\bar{x}, \bar{u}, \bar{u}')d\bar{x}, \tag{21.6}$$

is called the transformed value of the functional (21.1). Here the domain of integration $\overline{\Omega}$ is obtained form the domain Ω by transformations (21.3). Note that the domain $\overline{\Omega}$ also depends on the choice of the function (21.5) if the function f in transformations (21.3) depends on u. We will have to take this into account in our further considerations.

Definition 1.16. The functional (21.1) is said to be invariant with respect to the group G_r if for all transformations (21.3) of the group and all functions (21.5) the following equality is fulfilled irrespective of the choice of the domain of integration Ω :

$$\int_{\Omega} \mathcal{L}(x, u, u')dx = \int_{\bar{\Omega}} \mathcal{L}(\bar{x}, \bar{u}, \bar{u}')d\bar{x}. \tag{21.7}$$

21.3 Alternative proof of Noether's theorem

The lemma that we are going to prove now sets the necessary and sufficient condition of invariance of the functional (21.1) with respect to the group of transformations (21.3) in terms of the generators (21.4) of this group and the Lagrange function $\mathcal{L}$. It has been proved by Noether [107] using the technique of variations. An alternative proof of the lemma has been given in [56]. It is presented below in a modified form.

Lemma 1.10. The functional (21.1) is invariant with respect to the group G_r with the generators (21.4) if and only if the following equalities hold:

$$(\eta^k_\alpha - u^k_i \xi^i_\alpha)\frac{\delta \mathcal{L}}{\delta u^k} + D_i(A^i_\alpha) = 0 \quad (\alpha = 1, \ldots, r), \tag{21.8}$$

where

$$\frac{\delta \mathcal{L}}{\delta u^k} = \frac{\partial \mathcal{L}}{\partial u^k} - D_i\left(\frac{\partial \mathcal{L}}{\partial u^k_i}\right) \quad (k = 1, \ldots, m),$$

and

$$A^i_\alpha = (\eta^k_\alpha - u^k_j \xi^j_\alpha) \frac{\partial \mathcal{L}}{\partial u^k_i} + \mathcal{L}\xi^i_\alpha \quad (i = 1, \ldots, n;\ \alpha = 1, \ldots, r). \tag{21.9}$$

Proof. By using the change of variables $\bar{x} \to x$ in the integral (21.6) (formulae (21.3) and (21.5)) we reduce the integration over the domain $\overline{\Omega}$ to the integration over the domain Ω :

$$\int_{\overline{\Omega}} \mathcal{L}(\bar{x}, \bar{u}, \bar{u}') d\bar{x} = \int_{\Omega} \mathcal{L}(\bar{x}, \bar{u}, \bar{u}') J\left(\frac{\bar{x}}{x}\right) dx,$$

where

$$J\left(\frac{\bar{x}}{x}\right) = \det \|D_i f^j\|$$

is the Jacobian of the change of variables. Then, due to arbitrariness of the domain Ω, Eq. (21.7) can be written in the following equivalent form:

$$\mathcal{L}(x, u, u') dx = \mathcal{L}(\bar{x}, \bar{u}, \bar{u}') J\left(\frac{\bar{x}}{x}\right) dx. \tag{21.10}$$

Let us prolong the group G_r of transformations (21.3) up to the group $\widetilde{G}_r$ of transformations of the variables (x, u, u', dx) as follows. The transformations (21.3) are extended to the first derivatives u^k_i according to the usual prolongation procedure described in Section 4.1, while the transformation of the volume element dx is defined by the formula

$$d\bar{x} = J\left(\frac{\bar{x}}{x}\right) dx.$$

The generators of the group $\widetilde{G}_r$ are obtained by extending extending the generators (21.4) of the group G_r,

$$X_\alpha = \xi^i_\alpha(x, u) \frac{\partial}{\partial x^i} + \eta^k_\alpha(x, u) \frac{\partial}{\partial u^k}\,,$$

and have the following form:

$$\widetilde{X}_\alpha = X_\alpha + \zeta^k_{\alpha i} \frac{\partial}{\partial u^k_i} + D_i(\xi^i_\alpha) dx \frac{\partial}{\partial dx} \quad (\alpha = 1, \ldots, r),$$

for, according to the rule of differentiation of determinants, we have

$$\left.\frac{\partial d\bar{x}}{\partial a^\alpha}\right|_{a=0} = \left.\frac{\partial J\left(\frac{\bar{x}}{x}\right)}{\partial a^\alpha}\right|_{a=0} dx = D_i(\xi^i_\alpha) dx.$$

The coordinates $\zeta^k_{\alpha i}$ of the operators $\widetilde{X}_\alpha$ are determined by the prolongation formulae (4.7):

$$\zeta^k_{\alpha i} = D_i(\eta^k_\alpha) - u^k_j D_i(\xi^j_\alpha).$$

It is obvious now that Eq. (21.10) indicates that the function

$$\mathcal{L}(x, u, u')dx$$

is an invariant of the group $\widetilde{G}_r$ (see 1.6). Therefore, the following infinitesimal invariance test should be satisfied for this function (see Section 3.3):

$$\widetilde{X}_\alpha(\mathcal{L}(x, u, u')dx) = 0 \quad (\alpha = 1, \ldots, r).$$

Substituting the expression for operators $\widetilde{X}_\alpha$ and its coordinates $\zeta^k_{\alpha i}$ we have:

$$\xi^i_\alpha \frac{\partial \mathcal{L}}{\partial x^i} + \eta^k_\alpha \frac{\partial \mathcal{L}}{\partial u^k} + [D_i(\eta^k_\alpha) - u^k_j D_i(\xi^j_\alpha)] \frac{\partial \mathcal{L}}{\partial u^k_i} + \mathcal{L} D_i(\xi^i_\alpha) = 0 \ (\alpha = 1, \ldots, r)$$

It can be verified that the following identities hold:

$$\xi^i_\alpha \frac{\partial \mathcal{L}}{\partial x^i} + \eta^k_\alpha \frac{\partial \mathcal{L}}{\partial u^k} + \left[D_i(\eta^k_\alpha) - u^k_j D_i(\xi^j_\alpha)\right] \frac{\partial \mathcal{L}}{\partial u^k_i} + \mathcal{L} D_i(\xi^i_\alpha) \equiv$$

$$(\eta^k_\alpha - u^k_j \xi^j_\alpha) \frac{\delta \mathcal{L}}{\delta u^k} + D_i(A^i_\alpha) \quad (\alpha = 1, \ldots, r), \tag{21.11}$$

where A^i_α are given by (21.9). These identities lead to Eqs. (21.8).

Theorem 1.21. Let the functional (21.1) be invariant with respect to the group G_r with the generators (2.4). Then the Euler-Lagrange equations (21.1) have r linearly independent conservation laws (20.2), where the vectors A_α $(\alpha = 1, \ldots, r)$ have the components A^i_α determined by (21.9).

Proof. Eqs. (21.8) yield that

$$\left(\text{div } A_\alpha\right)\big|_{(21.2)} \equiv \left[D_i(A^i_\alpha)\right]_{(21.2)} = 0 \quad (\alpha = 1, \ldots, r) \tag{21.12}$$

The linear independence of the operators (21.4) imply that the vectors A_α $(\alpha = 1, \ldots, r)$ are linearly independent (see also [107]).

Remark 1.6. Inspection of the transformed values (21.6) of the functional (21.1) shows that one can have conservation laws even when Eq. (21.10) does not hold. Namely, according to Bessel-Hagen [17], it was noticed by E. Noether that it becomes possible if Eqs. (21.8) are replaced by equaions of the form

$$(\eta^k_\alpha - u^k_i \xi^i_\alpha) \frac{\delta \mathcal{L}}{\delta u^k} + D_i(A^i_\alpha) = D_i(B^i_\alpha) \quad (\alpha = 1, \ldots, r) \tag{21.13}$$

with some vectors $B_\alpha = (B_\alpha^1, \ldots, B_\alpha^n)$ $(\alpha = 1, \ldots, r)$ depending on x, u, u'. Then one has the conservation equation

$$\big[\text{div } (A_\alpha - B_\alpha)\big]_{(21.2)} = 0 \quad (\alpha = 1, \ldots, r) \tag{21.14}$$

instead of (21.12).

It follows from the proof of Lemma 1.10, that if such a vector B exists for a given operator

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^k \frac{\partial}{\partial u^k},$$

it is obtained from equation

$$\left(X + \zeta_i^k \frac{\partial}{\partial u_i^k}\right) \mathcal{L} + \mathcal{L} D_i(\xi^i) = D_i(B^i). \tag{21.15}$$

For instance, if condition

$$\mathcal{L}(\bar{x}, \bar{u}, \bar{u}') J\left(\frac{\bar{x}}{x}\right) = M(a)\mathcal{L}(x, u, u') + N(a; x, u, u') \tag{21.16}$$

holds instead of Eq. (21.10), then differentiation of Eq. (21.16) with respect to the group parameter a at $a = 0$ leads to the equation

$$\left(X + \zeta_i^k \frac{\partial}{\partial u_i^k}\right) \mathcal{L} + \mathcal{L} D_i(\xi^i) = \mu \mathcal{L}(x, u, u') + \nu(x, u, u'), \tag{21.17}$$

where

$$\mu = \frac{dM(a)}{da}\bigg|_{a=0}, \quad \nu(x, u, u') = \frac{\partial N(a; x, u, u')}{\partial a}\bigg|_{a=0}.$$

Thus, existence of the vector B satisfying the condition (21.15) in this case is possible if the right-hand side of Eq. (21.17) has the form of divergence. This kind of situations are considered in § 28.

§ 22 The basic theorem

22.1 Invariance of functionals is sufficient but not necessary for conservation laws

If the functional (21.1) is invariant with respect to the group G_r of transformations (21.3), then the Euler-Lagrange equations (21.2) are also invariant with respect to this group. Therefore one can say that Noether's theorem sets the *sufficient condition* for validity of conservation laws (21.12) with the vectors (21.9) for Euler-Lagrange equations (21.2) admitting a group with generators (21.4). The invariance of the corresponding functional (21.1) is this conditions.

Examples show that the invariance of the functional is not a *necessary* condition for validity of conservation laws. One of such examples is the Dirac equation considered in § 29. Hence, the Noether theorem provides not all the conservation laws (21.12) for Eqs. (21.2) admitting a group G_r with generators (21.4) because the conservation equation (21.12) for vectors (21.9) can be satisfied while the functional (21.1) will not be invariant with respect to the group G_r. The problem question then arises on finding the *necessary and sufficient* condition under which the vectors (21.9) satisfy the conservation equation (21.12). This condition has been found in [56] and is discussed in the following section.

22.2 Necessary and sufficient condition for conservation laws

Theorem 1.22. Let the Euler-Lagrange equations (21.2) admit the group G_r with the generators (21.4). The necessary and sufficient condition for the vectors A_α with the components (21.9) to satisfy the conservation equation (21.12) is the invariance of the *extremal values* of the functional (21.1) with respect to the group G_r.

Proof. The invariance of the extremal values of the functional $l[u]$ with respect to the group G_r means that Eq. (21.7) is satisfied for all extremals, i.e. for all functions (21.5) solving the Euler-Lagrange equations (21.2). Therefore let us write the necessary and sufficient condition for the invariance of the differential equations (21.2) and the extremal values of the functional (21.1) with respect to the group G_r in terms of the generators (21.4).

Let us rewrite Eq. (21.10) in the following equivalent form:

$$\mathcal{L}(x, u, u') = \mathcal{L}(\bar{x}, \bar{u}, \bar{u}')J\left(\frac{\bar{x}}{x}\right), \tag{22.1}$$

where $J\left(\frac{\bar{x}}{x}\right)$ is the same Jacobian as in Eq. (21.10).

Since in our case the differential equations (21.2) are, generally speaking, equations of the second order, we prolong the group G_r of transformations (21.3) up on the to the second-order derivatives of $u = u(x)$. Then we introduce an additional variable Λ and extend the action of the group G_r to the new variable Λ by the equation

$$\bar{\Lambda}J\left(\frac{\bar{x}}{x}\right) = \Lambda. \tag{22.2}$$

The resulting group is denoted by $\widetilde{G}_r$. It acts on the space of the variables (x, u, u', u'', Λ). In this notation, the invariance of Eqs. (21.2) and of the extremal values of the functional (21.1) with respect to the group G_r means that the system of equations

$$\frac{\delta \mathcal{L}}{\delta u^k} = 0 \quad (k = 1, \ldots, m), \tag{22.3}$$

$$\Lambda = \mathcal{L}(x, u, u') \tag{22.4}$$

define an invariant manifold of the group $\widetilde{G}_r$ in the space of the variables (x, u, u', u'', Λ). Let us write the invariance criterion (3.5) for this manifold. To this end we have to find the generators of the group $\widetilde{G}_r$.

In order to reduce calculations in finding the generators, we observe some peculiarities of the considered manifold. First of all, we see that the variable Λ is not contained in Eqs. (22.3), and hence Eq. (22.4) does not influence the invariance conditions for the differential equations (22.3). Therefore the invariance conditions (3.5) for Eqs. (22.3) can be considered separately from Eq. (22.4). These conditions are already satisfied because, according to the conditions of the theorem, the Euler-Lagrange equations (22.3) admit the group G_r. Hence, the invariance conditions for the simultaneous equations (22.3), (22.4) with respect to the group $\widetilde{G}_r$ take the following form:

$$\widetilde{X}_\alpha(\mathcal{L}(x, u, u') - \Lambda)\big|_{(22.3),(22.4)} = 0 \quad (\alpha = 1, \ldots, r), \tag{22.5}$$

where $\widetilde{X}_\alpha (\alpha = 1, \ldots, r)$ are the unknown generators of the group $\widetilde{G}_r$.

Now we see the second peculiarity of the situation. It consists in the fact that there is no need to make the second prolongation of the group G_r since, according to Eqs. (22.5), it is sufficient to know the first prolongation only. The second prolongation has been required only for formulating the invariance condition for the extremal values of the functional (21.1) in terms of the manifold defined by Eqs. (22.3), (22.4). Using this peculiarity we write the generators of the group $\widetilde{G}_r$ in the form

$$\widetilde{X}_\alpha = X_\alpha + \zeta^k_{\alpha i} \frac{\partial}{\partial u^k_i} + \lambda_\alpha \frac{\partial}{\partial \Lambda} \quad (\alpha = 1, \ldots, r),$$

where $\zeta^k_{\alpha i}$ are given by Eqs. (4.7) for every $\alpha = 1, \ldots, r$. Furthermore, the coefficients

$$\lambda_\alpha = \frac{\partial \bar{\Lambda}}{\partial a^\alpha}\Big|_{a=0} \quad (\alpha = 1, \ldots, r)$$

are obtained by differentiating both parts of Eq. (22.2) with resect to a^α at $a = 0$. Using the rule for differentiating determinants and noting that only the left-hand side of Eq. (22.2) depends on the parameter a we get

$$\left[\frac{\partial\bar{\Lambda}}{\partial a^\alpha} J\left(\frac{\bar{x}}{x}\right)\right]_{a=0} + \left[\bar{\Lambda}\frac{\partial J\left(\frac{\bar{x}}{x}\right)}{\partial a^\alpha}\right]_{a=0} = \lambda_\alpha + \Lambda D_i(\xi^i_\alpha) = 0.$$

Thus,

$$\lambda_\alpha = -\Lambda D_i(\xi^i_\alpha) \quad (\alpha = 1, \ldots, r),$$

and the generators of the group $\widetilde{G}_r$ take the following final form:

$$\widetilde{X}_\alpha = X_\alpha + [D_i(\eta^k_\alpha) - u^k_j D_i(\xi^j_\alpha)]\frac{\partial}{\partial u^k_i} - \Lambda D_i(\xi^i_\alpha)\frac{\partial}{\partial \Lambda}, \tag{22.6}$$

where X_α are the generators (21.4) of the group G_r.

Using the expressions (22.6) of the operators $\widetilde{X}_\alpha$ and the identities (21.11), one has

$$\widetilde{X}_\alpha(\mathcal{L} - \Lambda)\big|_{(22.4)} = (\eta^k_\alpha - u^k_j \xi^j_\alpha)\frac{\delta\mathcal{L}}{\delta u^k} + D_i(A^i_\alpha). \tag{22.7}$$

Therefore, the necessary and sufficient conditions (22.5) for the invariance of the extremal values of the functional (21.1) with respect to the group G_r take the form

$$D_i(A^i_\alpha)|_{(22.3)} = 0 \quad (\alpha = 1, \ldots, r),$$

i.e. coincide with the conservation equations (21.12). This proves our theorem.

Remark 1.7. Independence of conservation laws given by Noether's theorem follows from the linear independence of the operators (21.4) and from Eqs. (21.8). Under the conditions of Theorem 1.22, Eqs. (21.8) do not necessarily hold. As a result, not all conservation laws derived by means of this theorem have to be independent. However, one can claim that the number of independent conservation laws in the given case is no less than in the Noether theorem. Indeed, all conservation laws resulting from the Noether theorem are also obtained from Theorem 1.22.

§ 23 Inverse Noether theorem

23.1 Degenerate and non-degenerate functionals

Theorem 1.22 answers the question about the nature of conservation laws (21.12) completely. They arise as a result of invariance of extremal values of the functional $l[u]$. Nevertheless, it is of interest to find out for which functionals (21.1) Theorem 1.22 gives the same conservation laws as Noether's theorem. In other words, when the invariance of the functional is not only sufficient but also necessary for the validity of the conservation equations (21.12). If this property is met, I will say that the *inverse Noether theorem* holds for the corresponding Euler-Lagrange equations (21.2).

Examples of differential equations for which the inverse Noether theorem holds are equations of a free particle motion in classical mechanics, the wave equation etc. This can be verified by inspecting all conservation laws of the form (21.12) for the mentioned equations. However, one can use one general theorem proved in [56] that gives many equations for which the inverse Noether theorem holds. First, let us introduce the following definition.

Definition 1.17. The functional (21.1) is said to be *non-degenerate* if all equations in the system of Euler-Lagrange equations (21.2) are second-order equations, and *degenerate* otherwise.

Remark 1.8. Likewise, functionals are divided into degenerate and non-degenerate if the corresponding Lagrangian depends on partial derivatives of the order $q > 1$. Namely, a functional $l[u]$ is said to be *non-degenerate* if all equations in the corresponding system of Euler-Lagrange equations have the order $q + 1$, and *degenerate* if at least one of Euler-Lagrange equations for the functional $l[u]$ is an equation of order $\leq q$.

The equations of mechanics and the wave equation mentioned above are Euler-Lagrange equations of non-degenerate functionals. An example of a degenerate functional is given by the Dirac equations (see § 29).

23.2 Inverse Noether theorem in non-degenerate case

The theorem mentioned in the previous section is formulated as follows.

Theorem 1.23. Let $l[u]$ be a non-degenerate functional of the form (21.1) and let the corresponding Euler-Lagrange equations (21.2) admit a group G_r with the generators (21.4). Then the functional $l[u]$ is invariant with respect to the group G_r if and only if the vectors (21.9) satisfy the conservation equation (21.12).

Proof. Let us use the notation and results obtained when proving Theorem 1.22. As it was demonstrated, the invariance of the functional $l[u]$ with respect to the group G_r is equivalent to the invariance of the manifold defined by Eq. (22.4) with respect to the group $\widetilde{G}_r$ with the infinitesimal generators (22.6). Therefore, it is sufficient to prove that in case of non-degenerate functionals the following identities hold:

$$\widetilde{X}_\alpha(\mathcal{L}-\Lambda)\big|_{(22.4)} \equiv \widetilde{X}_\alpha(\mathcal{L}-\Lambda)|_{(22.4,22.3)} \tag{23.1}$$

Indeed, vanishing of the left-hand sides of (23.1) is the necessary and sufficient condition of invariance of the manifold given by equation (22.4) with respect to the group $\widetilde{G}_r$. On the other hand, according to equality (22.7), the right-hand sides of (23.1) are equal to

$$D_i(A^i_\alpha)\big|_{(21.2)} \quad (\alpha = 1, \ldots, r).$$

Therefore, the validity of the identity (23.1) entails that satisfaction of the conservation laws (21.12) is the necessary and sufficient condition of invariance of the functional (21.1) with respect to the group G_r.

Let us demonstrate that equalities (23.1) are really satisfied in the considered case. The formulae (22.7) and identities (21.11) demonstrate that expressions

$$\widetilde{X}_\alpha(\mathcal{L}-\Lambda)\big|_{(22.4)} \quad (\alpha = 1, \ldots, r)$$

can depend only on the variables x, u, u'. On the other hand, the non-degenerate character of the functional $l[u]$ means that all variables x, u, u' act as free variables on the manifold defined by the Euler-Lagrange equations (21.2) in the space of variables (x, u, u', u''). This, manifestly ensures the validity of the identity (23.1), thus proving the theorem.

In the following sections contain a number of examples on finding the conservation laws by means of the theorems proved above. The majority of these examples deal with Euler-Lagrange equations of non-degenerate functionals. A case of a degenerate functional is considered in § 29.

§ 24 Classical mechanics

24.1 Free motion of a particle

Let us begin with the well-known conservation laws in classical mechanics.

The Lagrangian for a freely moving particle with a mass m has the form

$$\mathcal{L} = \frac{1}{2}m\sum_{k=1}^{3}(\dot{x}^k)^2. \tag{24.1}$$

Here, time t is the independent variable, and the coordinates $\vec{x} = (x^1, x^2, x^3)$ of the particle are the dependent variables. The differentiation with respect to t is denoted by the dot, so that we can write $v^k = \dot{x}^k$, where the vector $\vec{v} = (\dot{x}^1, \dot{x}^2, \dot{x}^3)$ is the velocity of the particle. The Euler-Lagrange equations (21.2) in the given case are the second-order equations

$$m\ddot{x}^k = 0 \quad (k = 1, 2, 3), \tag{24.2}$$

and hence the functional (21.1) with the Lagrangian (24.1) is non-degenerate.

Writing the generators (21.4) in the form

$$X = \xi \frac{\partial}{\partial t} + \eta^k \frac{\partial}{\partial x^k} \tag{24.3}$$

and substituting in (21.9), we obtain the following expression for computing conserved quantities:

$$A = m \sum_{k=1}^{3} \dot{x}^k (\eta^k - \frac{1}{2}\xi\, \dot{x}^k). \tag{24.4}$$

Let us consider the group G_7 composed by the translations of the coordinates x^k $(k = 1, 2, 3)$ and of the time t, as well as by the rotations of the position vector $\vec{x} = (x^1, x^2, x^3)$ of the particle. The group G_7 is obviously admitted by Eqs. (24.2). The generators of the group G_7 are

$$\begin{aligned} &X_k = \frac{\partial}{\partial x^k} \quad (k = 1, 2, 3), \quad X_4 = \frac{\partial}{\partial t}\,, \\ &X_{kl} = x^l \frac{\partial}{\partial x^k} - x^k \frac{\partial}{\partial x^l} \quad (k < l;\ k, l = 1, 2, 3). \end{aligned} \tag{24.5}$$

24.2 Computation of conserved quantities

Let us write out the conserved quantities (24.4) for the operators (24.5).

(a) **Translations of space coordinates**

The translation along the x^1 axis is generated by the operator X_1. Substituting its coordinates

$$\xi = 0, \quad \eta^1 = 1, \quad \eta^2 = \eta^3 = 0$$

in (24.4) we obtain the conserved quantity

$$A = mv^1.$$

Taking the translations along all axes of coordinates we obtain the vector of the momentum

$$\vec{p} = m\vec{v} \tag{24.6}$$

as a conserved quantity.

Thus, the invariance with respect to translations of coordinates yields the *conservation of the momentum.*

(b) **Time translation**

The time translation has the generator X_4 with

$$\xi = 1, \quad \eta^k = 0, \quad (k = 1, 2, 3).$$

Accordingly, the formula (24.4) yields the following conserved quantity:

$$E = \frac{1}{2} m \sum_{k=1}^{3} (v^k)^2, \tag{24.7}$$

which is the energy of the particle. Here the quantity $E = -A$ is taken instead of A. Thus, the invariance with respect to the time translation is connected with the *conservation of the energy.*

(c) **Rotations**

First, let us consider the one-parameter group of rotations round the x^3 axis. The generator of this subgroup is

$$X_{12} = x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2} \cdot$$

The substitution in (24.4) yields the conserved quantity

$$M_3 = x^2 p^1 - x^1 p^2.$$

Considering the rotations round the remaining axes one can see that the group of rotations with the generators X_{kl} $(k < l)$ leads to the *conservation of the angular momentum*

$$\overrightarrow{M} = \vec{p} \times \vec{x}, \tag{24.8}$$

where $\vec{p} \times \vec{x}$ is the vector product of the vectors $\vec{p}$ and $\vec{x}$.

§ 25 Relativistic mechanics

25.1 Lagrangian of a particle in a curved space-time

It is accepted in theoretical physics that a free particle in a curved space-time, i.e. in a Riemannian space V_4 of normal hyperbolic type moves along geodesic curves. If the basic metric form of a space V_4 has the form

$$ds^2 = g_{ij}(x)dx^i dx^j, \tag{25.1}$$

and the particle under consideration has the mass m, then a free motion of the particle in the space V_4 corresponds to the Lagrangian

$$\mathcal{L} = -mc\sqrt{g_{ij}(x)\dot{x}^i\dot{x}^j}, \tag{25.2}$$

where c is a constant equal to the light velocity in vacuum and the dot in $\dot{x}^i$ indicates the differentiation with respect to the parameter σ of a curve

$$x^i = x^i(\sigma) \quad (i = 1, \ldots, 4)$$

in the space V_4. According to Section 5.1, the Euler-Lagrange equations (21.2) with the Lagrangian (25.2) are the equations of geodesic lines in V_4.

The parameter σ taken along the trajectory of the particle is the independent variable in the given case (so that $n = 1$), and the coordinates $x^i (i = 1, \ldots, 4)$ of the particle are the dependent variables. Like in the previous section we are dealing with a non-degenerate functional here.

Since in the considered case the functional $l[u]$ is equal to the integral of the element of the length of the arc ds in the space V_4, one should take a group of motions in the space V_4 (see 8.1) as a group G_r with respect to which the functional is invariant. Therefore, the operators (21.4) are

$$X = \eta^i(x)\frac{\partial}{\partial x^i}, \tag{25.3}$$

where the functions $\eta^i(x)$ $(i = 1, \ldots, 4)$ satisfy the Killing equations (8.3).

In this case, the formula (21.9) for determining the conserved quantities can be also simplified. One has

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^i} = -\frac{mc}{\sqrt{g_{kl}\dot{x}^k\dot{x}^l}}\, g_{ij}\, \dot{x}^j \quad (i = 1, \ldots, 4).$$

Parametrizing the curves in the space V_4 by means of the arc length s one has $g_{kl}\dot{x}^k\dot{x}^l = 1$ according to the Eq. (25.1). Then Eq. (21.9) is written:

$$A = -mcg_{ij}\dot{x}^i\eta^j. \tag{25.4}$$

25.2 Motion of a particle in the Minkowski space-time

Now let us turn to conservation laws of relativistic mechanics. The Minkowski space-time is the space V_4 with the metric form

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2. \tag{25.5}$$

I will use the following notation of variables:

$$x^1 = x,\ x^2 = y,\ x^3 = z,\ x^4 = t.$$

In what follows, we will use a connection the components

$$\dot{x}^i = \frac{dx^i}{ds} \quad (i = 1, \ldots, 4)$$

of the four-velocity with the components

$$v^\mu = \frac{dx^\mu}{dt} \quad (\mu = 1, 2, 3)$$

of the physical velocity $\vec{v}$. In order to obtain this connection, let us write the interval ds determined by Eq. (25.5) in the form

$$ds = c\sqrt{1-\beta}dt,$$

where the notation

$$\beta = \frac{1}{c^2}\sum_{\mu=1}^{3}(v^\mu)^2$$

is introduced. Substituting the expression of the interval ds into the definition of the four-velocity $\dot{x}^i$ one obtains the desired connection:

$$\dot{x}^\mu = \frac{v^\mu}{c\sqrt{1-\beta}} \quad (\mu = 1, 2, 3), \quad \dot{x}^4 = \frac{1}{c\sqrt{1-\beta}}\, \cdot \tag{25.6}$$

The group of motions in the space V_4 with the metric form (25.5) is the 10-parameter Lorentz group having the following basis of generators:

$$\begin{aligned} X_i &= \frac{\partial}{\partial x^i} \quad (i = 1, \ldots, 4),\\ X_{\mu\nu} &= x^\nu\frac{\partial}{\partial x^\mu} - x^\mu\frac{\partial}{\partial x^\nu} \quad (\mu < \nu),\\ X_{\mu 4} &= x^4\frac{\partial}{\partial x^\mu} + \frac{1}{c^2}x^\mu\frac{\partial}{\partial x^4} \quad (\mu, \nu = 1, 2, 3). \end{aligned} \tag{25.7}$$

Note that here and in what follows the Roman letters $i, j, k, \ldots$ run over the values 1, 2, 3, 4, whereas the Greek letters $\mu, \nu, \ldots$ run over the values from 1, 2, 3, unless it is otherwise stated.

25.3 Computation of relativistic conserved quantities

Let us compute the conserved quantities (25.4) corresponding to the operators (25.7).

(a) **Translations of space coordinates**

The generators X_μ of the translations of the spacial coordinates have the coordinates $\eta^j_\mu = \delta^j_\mu$. Accordingly, the formulae (25.4) provide the following conserved quantities:

$$A_\mu = -mcg_{ij}\dot{x}^i\delta^j_\mu = -mcg_{i\mu}\dot{x}^i = mc\dot{x}^\mu = \frac{mv^\mu}{\sqrt{1-\beta}}\,\cdot$$

Thus, we have obtained the conservation of *relativistic momentum*

$$\vec{p}_o = \frac{m\vec{v}}{\sqrt{1-\beta}}\,\cdot \tag{25.8}$$

(b) **Time translation**

The generator X_4 of the time translations has the coordinates $\eta^j = \delta^j_4$, and (25.4) yields the conserved quantity

$$A = -mcg_{ij}\dot{x}^i\delta^j_4 = -mc^3\dot{x}^4 = -\frac{mc^2}{\sqrt{1-\beta}}\,\cdot$$

We have obtained the conservation of *relativistic energy*

$$\mathcal{E}_o = \frac{mc^2}{\sqrt{1-\beta}}\,\cdot \tag{25.9}$$

(c) **Rotations**

The generator X_{12} of the rotations around the z-axis in the space of variables x, y, z has the coordinates

$$\eta^1 = x^2, \quad \eta^2 = -x^1, \quad \eta^3 = \eta^4 = 0.$$

The substitution in (25.4) yields the conserved quantity

$$A_{12} = -mc(-\dot{x}^1x^2 + \dot{x}^2x^1) = p^1_o x^2 - p^2_o x^1.$$

Calculating the conserved quantities (25.4) for all rotations, one obtains the conservation of the *relativistic angular momentum*

$$\overrightarrow{M}_o = \vec{p}_o \times \vec{x}, \tag{25.10}$$

which has the same form as in classical mechanics. Here again

$$\vec{x} = (x^1, x^2, x^3).$$

(d) **Lorentz transformations**

If we substitute the coordinates of the operator X_{14} in (25.4), we obtain the conserved quantity

$$A = -mc(-\dot{x}^1 x^4 + \dot{x}^4 x^1) = \frac{m}{\sqrt{1-\beta}}(v^1 t - x^1).$$

It is manifest that using all generators $X_{\mu 4}$ of the Lorentz transformations we arrive at following conserved vector:

$$\overrightarrow{Q}_o = \frac{m}{\sqrt{1-\beta}}(\vec{x} - \vec{v}t). \tag{25.11}$$

Noting that $\mathcal{E}_o$, and hence the expression

$$\frac{\mathcal{E}_o}{c^2} = \frac{m}{\sqrt{1-\beta}},$$

is a constant of motion (i.e. a conserved quantity), we can take the vector

$$\vec{q}_o = \vec{x} - \vec{v}t$$

as the conserved quantity instead of the vector (25.11).

If there are several particles that do not interact, the corresponding conservation law is known as the center-of-mass theorem. Thus, the invariance with respect to the Lorentz transformations leads to the *center-of-mass theorem.* As one can see from the expression for the conserved vector $\vec{q}_o$, in the case of one particle the center-of-mass theorem is equivalent to the statement on uniformity of the particle motion.

§ 26 Particle in space of constant curvature

26.1 Symmetries

Let us find conservation laws for a free motion of a particle in a space with a constant curvature K =const. Spaces V_n of constant curvature (and only such spaces) have group of isometric motions G_r of a maximum order [29]

$$r = \frac{1}{2}n(n+1).$$

Therefore, like in relativistic mechanics, there exist 10 independent conservation laws for a free motion of a particle in the space V_4 with a constant curvature K. If $K = 0$, the conservation laws will obviously coincide with conservation laws of relativistic mechanics.

Let us write the metric of a space V_4 of constant curvature of normal hyperbolic type in Riemann's form

$$ds^2 = \frac{1}{\Phi^2}(c^2dt^2 - dx^2 - dy^2 - dz^2), \tag{26.1}$$

where

$$\Phi = 1 + \frac{K}{4}r^2, \quad r^2 = c^2t^2 - x^2 - y^2 - z^2.$$

Denoting $x^1 = x,\ x^2 = y,\ x^3 = z,\ x^4 = t$ and $\dot{x}^i = \frac{dx^i}{ds}$, we get from (26.1):

$$ds = \frac{c}{\Phi}\sqrt{1-\beta}dt,$$

where

$$\beta = \frac{|\vec{v}|^2}{c^2} = \frac{1}{c^2}\sum_{\mu=1}^{3}(v^\mu)^2,$$

and $\vec{v}$ is the physical velocity with the components

$$v^\mu = \frac{dx^\mu}{dt} \quad (\mu = 1, 2, 3).$$

The components $\dot{x}^i$ of the four-velocity $\dot{x}^i$ and the components v^μ of the three-dimensional physical velocity are connected by the relations

$$\dot{x}^\mu = \frac{\Phi}{c\sqrt{1-\beta}}\, v^\mu \ (\mu = 1, 2, 3), \quad x^4 = \frac{\Phi}{c\sqrt{1-\beta}}\cdot \tag{26.2}$$

It is known [82] that the following 10 operators generate the group of isometric motions in the space V_4 with the metric form (26.1):

$$\begin{aligned}
X_\mu &= \left(\frac{K}{2}x^\mu x^i + (\Phi - 2)\delta^{\mu i}\right)\frac{\partial}{\partial x^i}\,,\\
X_4 &= \frac{K}{2}x^4 x^\mu \frac{\partial}{\partial x^\mu} + \frac{1}{c^2}\left(\Phi + \frac{K}{2}\rho^2\right)\frac{\partial}{\partial x^4}\,,\\
X_{\mu\nu} &= x^\nu \frac{\partial}{\partial x^\mu} - x^\mu \frac{\partial}{\partial x^\nu} \qquad (\mu < \nu),\\
X_{\mu 4} &= x^4 \frac{\partial}{\partial x^\mu} + \frac{1}{c^2}x^\mu \frac{\partial}{\partial x^4} \qquad (\mu, \nu = 1, 2, 3),
\end{aligned} \tag{26.3}$$

where $\rho^2 = x^2 + y^2 + z^2$.

The operators X_i $(i = 1, \ldots, 4)$ from (26.3) differ from the generators of translations given in § 25. Nevertheless, the conserved quantities resulting due to the generators X_i will be termed the *momentum* and *energy*.

26.2 Conserved quantities

Let us compute the conserved quantities (25.4) corresponding to the operators (26.3).

(a) Momentum

The conservation of momentum corresponds to the operators X_μ with the coordinates

$$\eta_\mu^\nu = (\Phi - 2)\delta^{\mu\nu} + \frac{K}{2}\, x^\mu x^\nu, \quad \eta_\mu^4 = \frac{K}{2}\, x^\mu x^4.$$

Using the relations (26.2), one obtains the following conserved quantities by the formula (25.4):

$$\begin{aligned} A_\mu &= \frac{m}{\Phi\sqrt{1-\beta}} \left(\sum_{\nu=1}^{3} v^\nu \eta_\mu^\nu - c^2 \eta_\mu^4 \right) \\ &= \frac{m}{\Phi\sqrt{1-\beta}} \left[(\Phi - 2) v^\mu + \frac{K}{2} \left(\sum_{\nu=1}^{3} x^\nu v^\nu - c^2 t \right) x^\mu \right] \\ &= \left(1 - \frac{2}{\Phi} \right) p_o^\mu + \frac{K}{2\Phi} \left(\vec{x} \cdot \vec{p}_o - \mathcal{E}_o t \right) x^\mu, \end{aligned}$$

where $\vec{p}_o$ and $\mathcal{E}_o$ are the relativistic momentum and energy defined by the equations (25.8) and (25.9), respectively, $\vec{x} \cdot \vec{p}_o$ is the scalar product of three-dimensional vectors $\vec{x} = (x^1, x^2, x^3)$ and $\vec{p}_o = (p_o^1, p_o^2, p_o^3)$:

$$\vec{x} \cdot \vec{p}_o = \sum_{\nu=1}^{3} x^\nu p_o^\nu.$$

Thus, the momentum of a free particle in the space V_4 with the constant curvature K has the form

$$\vec{p}_K = \frac{m}{\sqrt{1-\beta}} \left[\left(\frac{2}{\Phi} - 1 \right) \vec{v} - \frac{K}{2\Phi} (\vec{x} \cdot \vec{v} - c^2 t) \vec{x} \right] \tag{26.4}$$

or

$$\vec{p}_K = \left(\frac{2}{\Phi} - 1 \right) \vec{p}_o - \frac{K}{2\Phi} (\vec{x} \cdot \vec{p}_o - \mathcal{E}_o t) \vec{x}.$$

If the curvature $K = 0$, Eq. (26.4) provides the previous relativistic momentum $\vec{p}_o$.

(b) **Energy**

The conservation of energy corresponds to the operator X_4 with the coordinates

$$\eta^\mu = \frac{K}{2} x^4 x^\mu, \quad \eta^4 = \frac{1}{c^2}\left(\Phi + \frac{K}{2}\rho^2\right).$$

The conserved quantity (25.4) in this case has the form

$$\begin{aligned} A &= \frac{mc}{\Phi^2}\left(\sum_{\mu=1}^{3} \dot{x}^\mu \eta^\mu - c^2 \dot{x}^4 \eta^4\right) \\ &= \frac{m}{\Phi\sqrt{1-\beta}}\left[\frac{K}{2}\vec{x}\cdot\vec{v}t - \left(\Phi + \frac{K}{2}\rho^2\right)\right] \\ &= -\frac{m}{\sqrt{1-\beta}}\left[1 + \frac{K}{2\Phi}\vec{x}\cdot(\vec{x} - \vec{v}t)\right]. \end{aligned}$$

Multiplying A by $-c^2$, one obtains the following formula determining the energy of the free movement of a particle with the mass m in the space V_4 of with the constant curvature K :

$$\mathcal{E}_K = \left[1 + \frac{K}{2\Phi}\vec{x}\cdot(\vec{x} - \vec{v}t)\right]\frac{mc^2}{\sqrt{1-\beta}} \tag{26.5}$$

or

$$\mathcal{E}_K = \left[1 + \frac{K}{2\Phi}\vec{x}\cdot(\vec{x} - \vec{v}t)\right]\mathcal{E}_o.$$

(c) **Angular momentum**

The conserved quantity (25.4) for the rotation generator X_{12} has the form

$$A = \frac{m}{\Phi\sqrt{1-\beta}}(v^1x^2 - v^2x^1) = \frac{1}{\Phi}(p_o^1x^2 - p_o^2x^1).$$

Using the other rotation generators $X_{\mu\nu}$ we see that the invariance with respect to the rotations leads to conservation of the angular momentum defined by

$$\overrightarrow{M}_K = \frac{1}{\Phi}\vec{M}_o,$$

where the vector $\vec{M}_o$ is given by (25.10). Using Eqs. (26.4) and the equation $\vec{x}\times\vec{x} = 0$, we can also rewrite

$$\overrightarrow{M}_K = \frac{1}{2-\Phi}(\vec{x}\times\vec{p}_K). \tag{26.6}$$

(d) **Motion of the center of mass**

It was mentioned in the previous section that the Lorentz transformations with the generators $X_{\mu 4}$ lead to the relativistic center-of-mass theorem. Let us compute the similar conservation laws in the space V_4 of constant curvature. The operator X_{14} has the coordinates

$$\eta^1 = x^4, \quad \eta^2 = \eta^3 = 0, \quad \eta^4 = \frac{1}{c^2}\, x^1.$$

Substituting them in (25.4), we obtain the conserved quantity

$$A = \frac{m}{\Phi\sqrt{1-\beta}}(v^1 t - x^1).$$

Hence, we have the following conserved vector:

$$\overrightarrow{Q}_K = \frac{m}{\Phi\sqrt{1-\beta}}\,(\vec{x} - \vec{v}t) \equiv \frac{1}{\Phi}\overrightarrow{Q}_o. \tag{26.7}$$

§ 27 Nonlinear wave equation

27.1 Symmetries, Lagrangian and general form of conserved quantities

Let us apply Noether's theorem to the nonlinear wave equation

$$u_{tt} - \Delta u + \lambda u^3 = 0, \quad \lambda = \text{const.}, \tag{27.1}$$

where

$$\Delta u = u_{xx} + u_{yy} + u_{zz}.$$

In this case the number n of independent variables equals 4, and the number m of dependent variables equals 1.

If $\lambda = 0$, one has the usual linear wave equation. Sometimes, when it is convenient, the space variables are denoted by $x^\mu (\mu = 1, 2, 3)$ and time t by x^4 as we did before. The three-dimensional vector with the components x^μ is denoted by $\vec{x}$.

Eq. (27.1) can be represented as the Euler-Lagrange equation (21.2) for the functional (21.1) with the Lagrangian

$$\mathcal{L} = |\nabla u|^2 - u_t^2 + \frac{\lambda}{2} u^4, \tag{27.2}$$

where ∇u is the gradient of the function u :

$$\nabla u = (u_x, u_y, u_z).$$

According to Section 13.6, Eq. (27.1) with an arbitrary constant λ is invariant with respect to the 15-parameter group of conformal transformations in the flat space V_4 of normal hyperbolic type (Minkowski space-time). The generators (12.11) of this group are written

$$X_i = \frac{\partial}{\partial x^i} \quad (i = 1, \ldots, 4), \quad Z = x^i \frac{\partial}{\partial x^i} - u\frac{\partial}{\partial u}\,,$$
$$X_{\mu\nu} = x^\nu \frac{\partial}{\partial x^\mu} - x^\mu \frac{\partial}{\partial x^\nu} \quad (\mu < \nu),$$
$$X_{\mu 4} = x^4 \frac{\partial}{\partial x^\mu} + x^\mu \frac{\partial}{\partial x^4} \quad (\mu = 1, 2, 3), \tag{27.3}$$
$$Y_\mu = [2x^\mu x^i - (\rho^2 - t^2)\delta^{\mu i}]\frac{\partial}{\partial x^i} - 2x^\mu u \frac{\partial}{\partial u} \quad (\mu, \nu = 1, 2, 3),$$
$$Y_4 = 2x^4 x^\mu \frac{\partial}{\partial x^\mu} + (\rho^2 + t^2)\frac{\partial}{\partial x^4} - 2x^4 u \frac{\partial}{\partial u}\,,$$

where $\rho^2 = x^2 + y^2 + z^2$.

The vectors (21.9) satisfying the conservation equation (21.12) are four-dimensional and have the coordinates

$$\begin{aligned} A^\mu &= 2(\eta - u_i\xi^i)u_\mu + \xi^\mu \mathcal{L} \quad (\mu = 1, 2, 3), \\ A^4 &= -2(\eta - u_i\xi^i)u_4 + \xi^4 \mathcal{L}. \end{aligned} \tag{27.4}$$

27.2 Computation of several conserved vectors

One can readily evaluate the vectors (27.4) for all generators (27.3) of a conformal group. I will illustrate the computation only for the operators

$$X_4, \quad X_{\mu 4}, \quad Z, \quad Y_4.$$

Writing the operator X_4 in the form of (21.4) one obtains

$$\xi^\mu = 0 \ \ (\mu = 1, 2, 3), \quad \xi^4 = 1, \quad \eta = 0.$$

Therefore, introducing the three-dimensional vector $\vec{A} = (A^1, A^2, A^3)$, we obtain from Eqs. (27.4):

$$\vec{A} = -2u_t \nabla u, \quad A^4 = |\nabla u|^2 + u_t^2 + \frac{\lambda}{2}u^4.$$

Let us verify that the four-vector $A = (\vec{A}, A^4)$ with the above components satisfies the conservation equation (21.12). We have:

$$D_i(A^i) = \mathrm{div}\vec{A} + \frac{\partial A^4}{\partial t}\,\cdot$$

Using the usual notation $\vec{a}\cdot\vec{b}$ for the scalar product of three-dimensional vectors and invoking that

$$\mathrm{div}\vec{A} = \nabla\cdot\vec{A}, \quad \nabla\cdot\nabla = \Delta,$$

we obtain for our vector:

$$\mathrm{div}\vec{A} = -2u_t\Delta u - 2\nabla u\cdot\nabla u_t,$$

$$\frac{\partial A^4}{\partial t} = 2\nabla u\cdot\nabla u_t + 2u_t u_{tt} + 2\lambda u^3 u_t.$$

Therefore,

$$D_i(A^i)\big|_{(27.1)} = 2u_t(u_{tt} - \Delta u + \lambda u^3)\big|_{(27.1)} = 0,$$

i.e., the conservation equation (21.12) is satisfied.

According to Section 20.1, Eq. (20.4), the quantity

$$E_1 = \int\limits_{R^3}\Big(|\nabla u|^2 + u_t^2 + \frac{\lambda}{2}\,u^4\Big)dxdydz \tag{27.5}$$

does not depend on time t for solutions of Eq. (27.1) provided that these solutions decrease rapidly enough at infinity.

For the operators $X_{\mu 4},$ Z and Y_4 I will present only the quantities (20.4) independent of time.

These quantities provided by the operators $X_{\mu 4}(\mu = 1,2,3)$ compose the vector

$$\vec{E} = \int\limits_{R^3}[\vec{x}(|\nabla u|^2 + u_t^2 + \frac{\lambda}{2}u^4) + 2tu_t\nabla u]dxdydz. \tag{27.6}$$

Substituting the coefficients of the operator Z in the formula for A^4 given in (27.4) and using the equation $\vec{x}\cdot\nabla u = \rho u_\rho$, we obtain:

$$\begin{aligned}A^4 &= 2(u + x^i u_i)u_t + t(|\nabla u|^2 - u_t^2 + \frac{\lambda}{2}u^4)\\ &= 2u_t(u + \vec{x}\cdot\nabla u) + t(|\nabla u|^2 + u_t^2 + \frac{\lambda}{2}u^4)\\ &= 2u_t(\rho u)_\rho + t(|\nabla u|^2 + u_t^2 + \frac{\lambda}{2}u^4).\end{aligned}$$

The corresponding quantity (20.4), using E_1 defined by (27.5), is written

$$E_2 = tE_1 + 2\int\limits_{R^3}(\rho u)_\rho u_t dxdydz. \tag{27.7}$$

Similar calculations for the operator Y_4 yield:

$$A^4 = 2u_t[2t(\rho u)_\rho + (\rho^2+t^2)u_t] +$$
$$+(\rho^2+t^2)(|\nabla u|^2 - u_t^2 + \frac{\lambda}{2}u^4) - 2u^2 =$$
$$4tu_t(\rho u)_\rho + (\rho^2+t^2)(|\nabla u|^2 + u_t^2 + \frac{\lambda}{2}u^4) - 2u^2$$

Using the expressions (27.5) and (27.7), one can write the corresponding invariant (20.4) in the form

$$E_3 = t(2E_2 - tE_1) + \int\limits_{R^3} \left[\rho^2(|\nabla u|^2 + u_t^2 + \frac{\lambda}{2}u^4) - 2u^2\right]dxdydz. \tag{27.8}$$

§ 28 Transonic gas flow

28.1 Symmetries

The following equation derived in [98] is widely used in investigating the non-steady-state potential gas flow with transonic velocities:

$$-\varphi_x\varphi_{xx} - 2\varphi_{xt} + \varphi_{yy} = 0. \tag{28.1}$$

The group admitted by Eq. (28.1) is infinite and has the generators [101]

$$\begin{aligned}
X_f &= 3f(t)\frac{\partial}{\partial t} + (f'(t)x + f''(t)y^2)\frac{\partial}{\partial x} + 2f'(t)y\frac{\partial}{\partial y}\\
&\quad + \left[f''(t)x^2 + 2f'''(t)xy^2 + \frac{1}{3}f^{(4)}(t)y^4 - f'(t)\varphi\right]\frac{\partial}{\partial\varphi}\,,\\
X_g &= g'(t)y\frac{\partial}{\partial x} + g(t)\frac{\partial}{\partial y} + \left[2g''(t)xy + \frac{2}{3}g'''(t)y^3\right]\frac{\partial}{\partial\varphi}\,,\\
X_h &= h(t)\frac{\partial}{\partial x} + \left[2h'(t)x + 2h''(t)y^2\right]\frac{\partial}{\partial\varphi}\,, \qquad (28.2)\\
X_\sigma &= \sigma(t)y\frac{\partial}{\partial\varphi}\,,\\
X_\tau &= \tau(t)\frac{\partial}{\partial\varphi}\,,\\
X_0 &= 2t\frac{\partial}{\partial t} + y\frac{\partial}{\partial y} - 2\varphi\frac{\partial}{\partial\varphi}\cdot
\end{aligned}$$

Here $f(t), g(t), h(t), \sigma(t), \tau(t)$ are arbitrary functions, and $f'(t), f''(t)$, etc. are the first, second, etc. derivatives.

28.2 Conserved vectors

Using the results of § 21, one can verify that all operators (28.2), except X_0, lead to conservation laws. Application of Eqs. (21.9) and (21.15) to the operators $X_f, X_g, X_h, X_\sigma, X_\tau$ provides the following vectors $C = A - B$ (see Remark 1.6) satisfying the conservation equation

$$\text{div } C|_{(28.1)} = 0. \tag{28.3}$$

The components of the vectors C along the t, x and y axes are denoted by C^1, C^2 and C^3, respectively.

$$C_f \begin{cases} C^1 = & -\frac{1}{2}f\varphi_x^3 + (f'x + f''y^2)\varphi_x^2 + \frac{3}{2}f\varphi_y^2 + 2f'y\varphi_x\varphi_y \\ & +(f'\varphi - f''x^2 - 2f'''xy^2 - \frac{1}{3}f^{(4)}y^4)\varphi_x + 2(f''x + f'''y^2)\varphi, \\ C^2 = & \frac{1}{3}(f'x + f''y^2)\varphi_x^3 + (f'\varphi + 3f\varphi_t + 2f'y\varphi_y - f''x^2 \\ & -2f'''xy^2 - \frac{1}{3}f^{(4)}y^4)(\varphi_t + \frac{1}{2}\varphi_x^2) + \frac{1}{2}(f'x + f''y^2)\varphi_y^2 \\ & -\frac{1}{2}f''\varphi^2 + (f'''x^2 + 2f^{(4)}xy^2 + \frac{1}{3}f^{(5)}y^4)\varphi, \\ C^3 = & -f'y\varphi_y^2 - \frac{1}{3}f'y\varphi_x^3 - (f'x + f''y^2)\varphi_x\varphi_y - 2f'y\varphi_x\varphi_t \\ & -3f\varphi_y\varphi_t + (f''x^2 + 2f'''xy^2 + \frac{1}{3}f^{(4)}y^4 - f'\varphi)\varphi_y \\ & -4(f'''xy + \frac{1}{3}f^{(4)}y^3)\varphi. \end{cases}$$

$$C_g \begin{cases} C^1 = & g'y\varphi_x^2 - 2(g''xy + \frac{1}{3}g'''y^3)\varphi_x + g\varphi_x\varphi_y + 2g''y\varphi, \\ C^2 = & -(g''xy + \frac{1}{3}g'''y^3)\varphi_x^2 + \frac{1}{3}g'y\varphi_x^3 + \frac{1}{2}g\varphi_x^2\varphi_y + \frac{1}{2}g'y\varphi_y^2 \\ & +g\varphi_t\varphi_y - 2(g''xy + \frac{1}{3}g'''y^3)\varphi_t + 2(g'''xy + \frac{1}{3}g^{(4)}y^3)\varphi, \\ C^3 = & -\frac{1}{6}g\varphi_x^3 - g'y\varphi_x\varphi_y - g\varphi_t\varphi_x - \frac{1}{2}g\varphi_y^2 \\ & +2(g''xy + \frac{1}{3}g'''y^3)\varphi_y - 2(g''x + g'''y^2)\varphi. \end{cases}$$

$$C_h \begin{cases} C^1 = & h\varphi_x^2 - 2(h'x + h''y^2)\varphi_x + 2h'\varphi, \\ C^2 = & -2(h'x + h''y^2)\varphi_t + \frac{1}{3}h\varphi_x^3 - (h'x + h''y^2)\varphi_x^2 \\ & +\frac{1}{2}h\varphi_y^2 + 2(h''x + h'''y^2)\varphi, \\ C^3 = & 2(h'x + h''y^2)\varphi_y - h\varphi_x\varphi_y - 4h''y\varphi. \end{cases}$$

$$C_\sigma \begin{cases} C^1 = & -\sigma y\varphi_x, \\ C^2 = & -\sigma y\varphi_t - \frac{1}{2}\sigma y\varphi_x^2 + \sigma' y\varphi, \\ C^3 = & \sigma y\varphi_y - \sigma\varphi. \end{cases}$$

$$C_\tau \begin{cases} C^1 = & -\tau\varphi_x, \\ C^2 = & -\tau\varphi_t - \frac{1}{2}\tau\varphi_x^2 + \tau'\varphi, \\ C^3 = & \tau\varphi_y. \end{cases}$$

The extracted family of vectors C satisfying the conservation equation (28.3) depends on five arbitrary functions $f(t),\ g(t),\ h(t),\ \sigma(t),\ \tau(t)$. Thus, the nonlinear equation (28.1) has the remarkable property of possessing an infinite set of independent conservation laws.

28.3 Computation of one of the conserved vectors

In this section I provide details of computing for one of the above conserved vectors, namely C_σ. Other vectors are calculated likewise.

The Lagrangian for Eq. (28.1) has the form the function

$$\mathcal{L} = -\frac{1}{6}\varphi_x^3 - \varphi_t\varphi_x + \frac{1}{2}\varphi_y^2. \tag{28.4}$$

Substituting in (21.9) the coordinates of the operator X_σ and the Lagrangian (28.4) we obtain

$$A^1 = -\sigma y\varphi_x, \quad A^2 = -\sigma y\varphi_t - \frac{1}{2}\sigma y\varphi_x^2, \quad A^3 = \sigma y\varphi_y. \tag{28.5}$$

The first prolongation of the operator X_σ has the form

$$\widetilde{X}_\sigma = \sigma(t)y\frac{\partial}{\partial\varphi} + \sigma'(t)y\frac{\partial}{\partial\varphi_t} + \sigma(t)\frac{\partial}{\partial\varphi_y}\,\cdot$$

Noting that $D_i(\xi^i) = 0$ for the operator X_σ, we see that the left-hand side of Eq. (21.15) equals

$$\widetilde{X}_\sigma\mathcal{L} = -\sigma' y\varphi_x + \sigma\varphi_y = \frac{\partial}{\partial x}(-\sigma' y\varphi) + \frac{\partial}{\partial y}(\sigma\varphi).$$

Hence, Eq. (21.15) is satisfied if we take the vector B with the components

$$B' = 0, \quad B^2 = -\sigma' y\varphi, \quad B^3 = \sigma\varphi. \tag{28.6}$$

The difference of the vectors with the components (28.5) and (28.6) provides the vector C_σ from Section 28.2.

28.4 Three-dimensional case

Proceeding as above, one can obtain an infinite family of conservation laws for the three-dimensional transonic gas flow described by the equation

$$-\varphi_x\varphi_{xx} - 2\varphi_{xt} + \varphi_{yy} + \varphi_{zz} = 0. \tag{28.7}$$

The Lagrangian of this equation has the form

$$\mathcal{L} = -\frac{1}{6}\varphi_x^3 - \varphi_t\varphi_x + \frac{1}{2}\varphi_y^2 + \frac{1}{2}\varphi_z^2. \tag{28.8}$$

Eq. (28.7) admits an infinite group [86]. Consequently, it has an infinite number of conservation laws.

I will derive here only the family of conservation laws corresponding to the operator

$$X_\psi = \psi(y,z)\frac{\partial}{\partial\varphi},$$

which is admitted by Eq. (28.7) for an arbitrary solution $\psi(y,z)$ of the Laplace equation

$$\psi_{yy} + \psi_{zz} = 0.$$

Acting as in Section 28.3, we obtain the vector C_ψ with the components

$$C^1 = -\psi\varphi_x, \quad C^2 = -\psi\Big(\varphi_t + \frac{1}{2}\varphi_x^2\Big),$$

$$C^3 = \psi\varphi_y - \psi_y\varphi, \quad C^4 = \psi\varphi_z - \psi_z\varphi.$$

The validity of the conservation equation (21.12) follows from the equation

$$\frac{\partial C^1}{\partial t} + \frac{\partial C^2}{\partial x} + \frac{\partial C^3}{\partial y} + \frac{\partial C^4}{\partial z}$$

$$= (-\varphi_x\varphi_{xx} - 2\varphi_{xt} + \varphi_{yy} + \varphi_{zz})\,\psi - (\psi_{yy} + \psi_{zz})\,\varphi = 0.$$

§ 29 Dirac equations

29.1 Lagrangian

All examples considered in the previous sections deal with non-degenerate functionals. We will consider now an example where the corresponding functional is degenerate. The example is provided by the Dirac equations

$$\gamma^k\frac{\partial\psi}{\partial x^k} + m\psi = 0, \quad m = \text{const}. \tag{29.1}$$

Here the dependent variables are the components ψ^k $(k = 1, \ldots, 4)$ of the four-dimensional complex vector ψ. The independent variables are

$$x^1 = x, \quad x^2 = y, \quad x^3 = z, \quad x^4 = ict.$$

The coefficients γ^k $(k = 1, \ldots, 4)$ are the following complex 4×4 matrices:

$$\gamma' = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma^3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Splitting Eqs. (29.1) into real and imaginary parts, one can deal with eight real equations instead of four complex differential equations (29.1). However, it is more convenient to save the complex notation and consider Eq. (29.1) together with the conjugate equation

$$\frac{\partial \widetilde{\psi}}{\partial x^k}\gamma^k - m\widetilde{\psi} = 0, \tag{29.2}$$

where

$$\widetilde{\psi} = \psi^{*T}\gamma^4. \tag{29.3}$$

The indices $*$ and T indicate the transition to complex conjugate and transposed values, respectively. It is convenient to assume that ψ is a column vector and ψ^T (and consequently $\widetilde{\psi}$ too) is a row vector. Hence, the lower and upper indices are used accordingly: components of the column vector and the row vector are written with the upper and lower indices, respectively.

Equations (29.1), (29.2) can be derived as the Euler-Lagrange equations (21.2) of the functional (21.1) with the Lagrange function

$$\mathcal{L} = \frac{1}{2}\left\{\widetilde{\psi}\left(\gamma^k\frac{\partial\psi}{\partial x^k} + m\psi\right) - \left(\frac{\partial\widetilde{\psi}}{\partial x^k}\gamma^k - m\widetilde{\psi}\right)\psi\right\} \tag{29.4}$$

The order of derivatives involved in the Dirac equations (29.1), (29.2) coincides with the order of derivatives contained in the Lagrangian (29.4). Moreover, it is important that *all* equations of the system of equations (29.1), (29.2) are equations of the first order. Therefore, according to Definition 1.17, we are dealing with the degenerate functional (21.1).

29.2 Symmetries

Let us discuss the symmetry properties of the Dirac equations. We note first of all that since these equations linear and homogeneous, they admit the group G^+ with the generators

$$X_\varphi = \varphi^k(x)\frac{\partial}{\partial\psi^k} + \widetilde{\varphi}_k(x)\frac{\partial}{\partial\widetilde{\psi}_k} \tag{29.5}$$

and

$$X = \psi^k\frac{\partial}{\partial\psi^k} + \tilde{\psi}_k\frac{\partial}{\partial\widetilde{\psi}_k}\,\cdot \tag{29.6}$$

The column vector $\varphi(x) = (\varphi^1(x), \ldots, \varphi^4(x))$ in the operator (29.5) is any solution of Eqs. (29.1). The row vector $\widetilde{\varphi}(x)$ is obtained from $\varphi(x)$ according to the formula (29.3) and solves the conjugate equation (29.2). Since Eqs. (29.1) have infinitely many linearly independent solutions, the group G^+ is infinite. This group, like in the case of linear equations of the second order (see Section 12.2), is a normal subgroup of the whole group G admitted by the Dirac equations. In what follows, instead of the group G we consider its quotient group with respect to the normal subgroup G^+.

Below we shall write only transformations of the independent variables x^k and functions ψ, keeping in mind that the transformed value of the function $\widetilde{\psi}$ is derived from the transformed function ψ by the formula (29.3):

$$\widetilde{\overline{\psi}} = \overline{\psi}^{\,*T}\gamma^4. \tag{29.7}$$

Theorem 1.24. Let G be the widest group G admitted by Eqs. (29.1), (29.2) with the mass $m = 0$. Then its quotient group G/G^+ with respect to the normal subgroup G^+ is a 22-parameter group. Namely, G/G^+ is composed by the following seven one-parameter groups of transformations:

$$\overline{\psi} = \psi e^{-ia}, \tag{29.8}$$

$$\overline{\psi} = \psi\cosh a + \gamma^4\gamma^2\widetilde{\psi}^T\sinh a, \tag{29.9}$$

$$\overline{\psi} = \psi\cosh a + i\gamma^4\gamma^2\widetilde{\psi}^T\sinh a, \tag{29.10}$$

$$\overline{\psi} = \psi e^{ia\gamma^5}, \quad \gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4, \tag{29.11}$$

$$\overline{\psi} = \psi e^{-a\gamma^5}, \tag{29.12}$$

$$\overline{\psi} = \psi\cos a + \gamma^3\gamma^1\widetilde{\psi}^T\sin a, \tag{29.13}$$

$$\overline{\psi} = \psi\cos a + i\gamma^3\gamma^1\widetilde{\psi}^T\sin a, \tag{29.14}$$

and by the 15-parameter group with the generators

$$X = X^0 + (S\psi)^k \frac{\partial}{\partial \psi^k} + (\widetilde{\psi}\widetilde{S})_k \frac{\partial}{\partial \widetilde{\psi}_k}, \tag{29.15}$$

where the operator

$$X^0 = \xi^k(x) \frac{\partial}{\partial x^k}$$

runs over the set of the generators (9.8) of the group of conformal transformations in the flat space V_4, and

$$S = \frac{1}{8} \sum_{k,l=4}^{4} \frac{\partial \xi^k}{\partial x^l} (\gamma^k \gamma^l - \gamma^l \gamma^k - 3\delta^{kl}), \quad \widetilde{S} = \gamma^4 S^{*T} \gamma^4. \tag{29.16}$$

If the mass $m \neq 0$, then the admitted group G/G^+ is composed by the transformations (29.8)-(29.10) and by the 10-parameter group with the generators (29.15), where X^o runs the set of the generators of the group of isometric motions in the flat space V_4.

Proof. All statements of the theorem were obtained in [55] by solving the *determining equations* (Section 4.2) for the Dirac equations (29.1), (29.2).

Remark 1.9. The conformal invariance of Eqs. (29.1) with $m = 0$, i.e. the invariance under the 15-parameter group with the generators (29.15) was established by Dirac [26]. Later Pauli [115] discovered, by considering the case $m = 0$, that the Dirac equations (29.1), considered together with the conjugate equations (29.2) admit, along with the conformal group, three more symmetries, namely the transformations (29.12)-(29.14). The transformations (29.12)-(29.14) together with the transformations (29.11) are often called the 4-parameter *Pauli group.*

29.3 Invariance of extremal values of the functional

Straightforward verification shows that the extremal values of the functional (21.1) with the Lagrange function (29.4) are invariant with respect to the whole group G admitted by Eqs. (29.1), (29.2). I will illustrate this property by the one-parameter group of transformations (29.12).

Substituting the expressions of the matrices γ^k given in Section 29.1 in the definition of the matrix γ^5 (formula (29.11)), one obtains

$$\gamma^5 \cdot \gamma^5 = I,$$

where I is the unit matrix. Therefore,

$$e^{-a\gamma^5} = \Big(I - a\gamma^5 + \frac{a^2}{2!}\gamma^5\cdot\gamma^5 - \ldots\Big) = \Big(1 + \frac{a^2}{2!} + \frac{a^4}{4!} + \ldots\Big)I$$
$$-\Big(a + \frac{a^3}{3!} + \frac{a^5}{5!} + \ldots\Big)\gamma^5 = I\cosh a - \gamma^5\sinh a.$$

Substituting the resulting expression for $e^{-a\gamma^5}$ in (29.12) and using the formula (29.7) we obtain the one-parameter group of transformations

$$\overline{\psi} = \psi\cosh a - \gamma^5\psi\sinh a,$$
$$\overline{\widetilde{\psi}} = \widetilde{\psi}\cosh a - \widetilde{\psi}\gamma^5\sinh a$$

with the real valued parameter a. Since these transformations do not change the variables x^k, we have to find the transformation of the Lagrangian (29.4) only. Substituting the transformed functions $\overline{\psi}$ and $\overline{\widetilde{\psi}}$ in (29.4), we find the transformed value of the Lagrangian:

$$\overline{\mathcal{L}} = (\cosh^2 a + \sinh^2 a)\mathcal{L} + \cosh a\sinh a\Big(\widetilde{\psi}\gamma^5\gamma^k\frac{\partial\psi}{\partial x^k} + \frac{\partial\widetilde{\psi}}{\partial x^k}\gamma^k\gamma^5\psi\Big).$$

It is manifest that if ψ is an arbitrary function, then

$$\overline{\mathcal{L}} \neq \mathcal{L},$$

and hence the functional (21.1) is not invariant with respect to the transformations (29.12).

However, if ψ is the solution of the Dirac equations (29.1) (and consequently $\widetilde{\psi}$ solves Eqs. (29.2)) with $m = 0$, then

$$\gamma^k\frac{\partial\psi}{\partial x^k} = \frac{\partial\widetilde{\psi}}{\partial x^k}\gamma^k = 0,$$

and therefore

$$\overline{\mathcal{L}} = \mathcal{L} = 0.$$

It means that the extremal values of the functional are invariant with respect to the considered group.

Thus, according to Theorem 1.22, the group G admitted by the Dirac equations leads to the conservation laws (21.12). It is clear from the transformation of the Lagrangian under the transformation (29.12), that the Noether theorem (see Section 21.3) is not applicable to the whole symmetry group G. Specifically, it is not applicable to the subgroup $G^+ \subset G$ and to the Pauli transformations (29.12)-(29.14).

29.4 Conserved vectors

Let us obtain the vectors (21.9) satisfying the conservation equation (21.12). I will consider the case $m = 0$ and then indicate which conservation laws hold when $m \neq 0$.

The vectors (21.9) provided by the dilation generator (29.6) and by the Pauli transformations (29.12)-(29.14) are equal to zero, in accordance with Remark 1.7 from § 22.

The operator (29.5) furnishes the conserved vector with the components

$$A^k_\varphi = \widetilde{\psi}\gamma^k\varphi(x) - \widetilde{\varphi}(x)\gamma^k\psi \quad (k = 1, \ldots, 4).$$

Taking all possible solutions $\varphi(x)$ of the Dirac equations (29.1), one obtains an infinite set of conservation laws.

Let $m = 0$. Then, substituting in (21.9) the operators (29.15), where X^0 successively equals to the translation generators X_l, rotation generators X_{jl}, dilation generator Z and the operators Y_l from (9.8), we obtain the twelve conserved vectors with the following components A^k $(k = 1, \ldots, 4)$:

$$A^k_l = \frac{1}{2}\Big[\frac{\partial\widetilde{\psi}}{\partial x^l}\gamma^k\psi - \widetilde{\psi}\gamma^k\frac{\partial\psi}{\partial x^l} + \delta^k_l\Big(\widetilde{\psi}\gamma^j\frac{\partial\psi}{\partial x^j} - \frac{\partial\widetilde{\psi}}{\partial x^j}\gamma^j\psi\Big)\Big], \tag{29.17}$$

$$A^k_{jl} = \frac{1}{4}\Big[\widetilde{\psi}(\gamma^k\gamma^j\gamma^l + \gamma^j\gamma^l\gamma^k)\psi\Big] + x^l A^k_j - x^j A^k_l, \quad (j < l), \tag{29.18}$$

$$A^k = x^l A^k_l, \tag{29.19}$$

$$B^k_l = 2x^j A^k_{jl} + |x|^2 A^k_l, \tag{29.20}$$

where $j, l = 1, \ldots, 4$. In addition, using the one-parameter groups (29.8)-(29.11), we obtain the following four conserved vectors:

$$C^k_1 = -i\widetilde{\psi}\gamma^k\psi, \tag{29.21}$$

$$C^k_2 = \frac{1}{2}(\widetilde{\psi}\gamma^k\gamma^4\gamma^2\widetilde{\psi}^T - \psi^\tau\gamma^4\gamma^2\gamma^k\psi), \tag{29.22}$$

$$C^k_3 = \frac{i}{2}(\widetilde{\psi}\gamma^k\gamma^4\gamma^2\widetilde{\psi}^T + \psi^T\gamma^4\gamma^2\gamma^k\psi), \tag{29.23}$$

$$C^k_4 = i\widetilde{\psi}\gamma^k\gamma^5\psi. \tag{29.24}$$

If $m \neq 0$, we obtain the conserved vectors with the components

$$A^k_\varphi, \quad A^k_l, \quad A^k_{jl}, \quad C^k_1, \quad C^k_2, \quad C^k_3, \tag{29.25}$$

where the vectors (29.17) are replaced by

$$A^k_l = \frac{1}{2}\Big[\frac{\partial\widetilde{\psi}}{\partial x^l}\gamma^k\psi - \widetilde{\psi}\gamma^k\frac{\partial\psi}{\partial x^l} + \delta^k_l\Big(\widetilde{\psi}\gamma^j\frac{\partial\psi}{\partial x^j} - \frac{\partial\widetilde{\psi}}{\partial x^j}\gamma^j\psi\Big)\Big] + m\widetilde{\psi}\psi. \tag{29.26}$$

Open problem: Physical significance of new conservation laws

The above conservation laws, except those that correspond to the vectors A_φ, C_2, C_3, are well known in physics and have a definite physical meaning (see, e.g. [19]). Conservation laws with the vectors A^k_φ, C^k_2 and C^k_3 have not received the corresponding physical interpretation yet.

Problem 1.5. Find a physical meaning of the conservation laws corresponding to the conserved vectors A_φ, C_2, C_3 from Section 29.4.

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Paper 2

Approximate symmetries of equations with small parameter

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Abstarct

The paper develops the theory of approximate group analysis of differential equations that enables one to construct symmetries that are stable with respect to small perturbations. Approximate symmetries of nonlinear wave equations are calculated as well as those of a wide variety of evolutionary equations, including, e.g. the Korteweg-de Vries and Burgers-Korteweg-de Vries equations.

Introduction

Methods of classical group analysis allow to single out symmetries with remarkable properties (see e.g. [97], [114], [61]) among all equations of mathematical physics. Unfortunately, any small perturbation of an equation breaks the admissible group and reduces the applied value of these "refined" equations and group theoretical methods in general. Therefore, *development of methods of group analysis stable with respect to small perturbations of differential equations has become vital.* The present work develops such method based on notions of an approximate transformation group and approximate symmetries.

Let us introduce the following notation: $z = z^1, \dots, z^N$ is the independent variable, ε is a small parameter. All functions are considered to be analytic with respect to the set of their arguments. Together with the notation $\xi^k \frac{\partial}{\partial z^k}$ for expressions of the form $\sum\limits_{k=1}^{N} \xi^k \frac{\partial}{\partial z^k}$, the vector notation $\xi \frac{\partial}{\partial z}$ is also used. In what follows $\theta_p(z, \varepsilon)$ indicates the infinitesimal function of the order ε^{p+1}, $p \geq 0$, i.e.

$$\theta_p(z, \varepsilon) = o(\varepsilon^p),$$

and this equality (in case if functions are analytic in the vicinity $\varepsilon = 0$) is equivalent to satisfaction of any of the following conditions:

$$\lim_{\varepsilon \to 0} \frac{\theta_p(z, \varepsilon)}{\varepsilon^p} = 0;$$

there exists a constant $C > 0$ such that

$$|\theta_p(z, \varepsilon)| \leq C|\varepsilon|^{p+1}; \tag{0.1}$$

there exists a function $\varphi(z, \varepsilon)$ analytic in the vicinity of $\varepsilon = 0$ such that

$$\theta_p(z, \varepsilon) = \varepsilon|^{p+1} \varphi(z, \varepsilon). \tag{0.2}$$

In what follows, the approximate equality $f \approx g$ indicates that the equality $f(z, \varepsilon) = g(z, \varepsilon) + o(\varepsilon^p)$ with a fixed value $p \geq 0$ holds.

The following notation is used in section § 6: t, x are independent variables, u is the differential variable with successive derivatives (with respect to x) $u_1, u_2, \dots,$: $u_{\alpha+1} = D(u_\alpha)$, $u_o = u$, $D = \frac{\partial}{\partial x} + \sum\limits_{\alpha \geq 0} u_{\alpha+1} \frac{\partial}{\partial u_\alpha}$; $\mathcal{A}$ is the space of differential functions, i.e. of analytic functions of an arbitrary finite number of variables

$$t, x, u, u_1, \dots; \ f_t = \frac{\partial f}{\partial t}, \ f_x = \frac{\partial f}{\partial x}, \ f_\alpha = \frac{\partial f}{\partial u_\alpha}, \ f_* = \sum_{\alpha \geq 0} f_\alpha D^\alpha.$$

§ 1 The approximate Cauchy problem

The following interpretation of the theorem on continuous dependence of solution of the Cauchy problem on parameters is used below.

Theorem 2.1. Let the functions $f(z, \varepsilon)$, $g(z, \varepsilon)$, analytic in the neighborhood of the point (z_o, ε), satisfy the condition

$$g(z, \varepsilon) = f(z, \varepsilon) + o(\varepsilon^p), \tag{1.1}$$

and let $z = x(t, \varepsilon)$ and $\tilde{z} = \tilde{z}(t, \varepsilon)$ be solutions of the problems

$$\frac{dz}{dt} = f(z, \varepsilon), \ \ z|_{t=0} = \alpha(\varepsilon)$$

and

$$\frac{d\tilde{z}}{dt} = g(\tilde{z}, \varepsilon), \ \ \tilde{z}|_{t=0} = \beta(\varepsilon),$$

respectively. Here $\alpha(0) = \beta(0) = z_o, \beta(\varepsilon) = \alpha(\varepsilon) + o(\varepsilon^p)$. Then,

$$\tilde{z}(t, \varepsilon) = z(t, \varepsilon) + o(\varepsilon^p). \tag{1.2}$$

Proof. Let us introduce the function $u(t, \varepsilon) = z(t, \varepsilon) - \tilde{z}(t, \varepsilon)$. It satisfies the conditions

$$u(0, \varepsilon) = o(\varepsilon^p), \tag{1.3}$$

$$\left|\frac{du}{dt}\right| \leq |f(z, \varepsilon) - g(\tilde{z}, \varepsilon)| \leq |f(z, \varepsilon) - g(z, \varepsilon)| + |g(z, \varepsilon) - g(\tilde{z}, \varepsilon)|. \tag{1.4}$$

Using (1.1) written in the form of inequality (0.1)

$$|g(z, \varepsilon) - f(z, \varepsilon)| \leq C\varepsilon^{p+1}, \ \ C = \text{const},$$

and the Lipschiz condition $|g(z, \varepsilon) - g(\tilde{z}, \varepsilon)| \leq K|z - \tilde{z}|, K =$const. from (1.4), one obtains

$$\left|\frac{du}{dt}\right| \leq K|u| + C\varepsilon^{p+1}. \tag{1.5}$$

For every fixed ε there exists such t_ε, that the function $u(t, \varepsilon)$ has a constant sign on the interval form 0 to t_ε. One has

$$\frac{d}{dt}|u| \leq K|u| + C\varepsilon^{p+1}$$

on this interval from (1.5). Dividing the latter inequality by $|u| + \frac{C}{K}\varepsilon^{p+1}$ and integrating from 0 to t when $|t| \leq |t_\varepsilon|$, one obtains

$$|u(t, \varepsilon)| \leq \frac{C}{K}(e^{Kt_\varepsilon} - 1)\varepsilon^{p+1} + u(0, \varepsilon)e^{Kt_\varepsilon},$$

whence, with regard to (1.3), one obtains the condition (1.2). The theorem is proved.

Let us consider the approximate Cauchy problem

$$\frac{dz}{dt} \approx f(z, \varepsilon), \tag{1.6}$$

$$z|_{t=0} \approx \alpha(\varepsilon), \tag{1.7}$$

determined as follows. The approximate differential equation (1.6) is interpreted as a family of differential equations

$$\frac{dz}{dt} = g(z, \varepsilon) \quad \text{with} \quad g(z, \varepsilon) \approx f(z, \varepsilon). \tag{1.8}$$

The approximate initial condition (1.7) is interpreted likewise:

$$z|_{t=0} = \beta(\varepsilon) \quad \text{with} \quad \beta(\varepsilon) \approx \alpha(\varepsilon). \tag{1.9}$$

The approximate equality in (1.8), (1.9) has the same accuracy p as in (1.6), (1.9). According to the theorem 2.1, solutions of all problems of the form (1.8), (1.9) coincide with the accuracy to $o(\varepsilon^p)$. Therefore, *solutions of any problem (1.8), (1.9), considered with the accuracy up to $o(\varepsilon^p)$ are called solutions of the approximate Cauchy problem (1.6), (1.7).* The uniqueness of this solution (with the mentioned accuracy) follows from Theorem 2.1.

§ 2 One-parameter approximate groups

Given (local) transformations

$$z' = g(z, \varepsilon, a),$$

generating a one-parameter group with respect to a, so that

$$g(z, \varepsilon, 0) = z, \;\; g(g(z, \varepsilon, a), \varepsilon, b) = g(z, \varepsilon, a + b), \tag{2.1}$$

and depending on the small parameter ε. Let $f \approx g$, i.e.

$$f(z, \varepsilon, a) = g(z, \varepsilon, a) + o(\varepsilon^p). \tag{2.2}$$

Together with the points z', the "close" points $\tilde{z}$, determined by the formula

$$\tilde{z} = f(z, \varepsilon, a), \tag{2.3}$$

are introduced. Substituting (2.2) into (2.1), one can easily demonstrate that the formula (2.3) sets the approximate group in the sense of the following definition.

Definition 2.1. Transformations (2.3), or

$$z' \approx f(z, \varepsilon, a), \tag{2.4}$$

generate an approximate one-parameter group with respect to the parameter a if

$$f(z, \varepsilon, 0) \approx z, \tag{2.5}$$

$$f(f(z, \varepsilon, a), \varepsilon, b) \approx f(z, \varepsilon, a + b), \tag{2.6}$$

and satisfaction of the condition $f(z, \varepsilon, a) \approx z$ for any z entails $a = 0$.

The basic statements about infinitesimal description of local Lie groups hold when one starts to consider approximate groups changing the exact equalities to approximate ones.

Theorem 2.2. (*The approximate Lie theorem.*) Let us assume that transformations (2.4) generate an approximate group with a tangent vector field

$$\xi(z, \varepsilon) \approx \left. \frac{\partial f(z, \varepsilon, a)}{\partial a} \right|_{a=0}. \tag{2.7}$$

Then, the function $f(z, \varepsilon, a)$ satisfies the equation

$$\frac{\partial f(z, \varepsilon, a)}{\partial a} \approx \xi(f(z, \varepsilon, a), \varepsilon). \tag{2.8}$$

Conversely, solution (2.4) of the approximate Cauchy problem

$$\frac{dz'}{da} \approx \xi(z', a), \tag{2.9}$$

$$\left. z' \right|_{a=0} \approx z \tag{2.10}$$

for any (smooth) function $\xi(z, \varepsilon)$ determines the approximate one-parameter group with the group parameter a.

Remark 2.1. We will refer to equation (2.9) as the approximate Lie equation.

Proof. Let us assume that the function $f(z, \varepsilon, a)$ sets the approximate transformation group (2.4). Upon extraction of the dominant terms with respect to b, the correlation (2.6) takes the form

$$f(f(z, \varepsilon, a), \varepsilon, 0) + \left. \frac{\partial f(f(z, \varepsilon, a), \varepsilon, b)}{\partial b} \right|_{b=0} \cdot b + o(b) \approx$$

$$f(z, \varepsilon, a) + \frac{\partial f(z, \varepsilon, a)}{\partial a} \cdot b + o(b).$$

Whence, the approximate equation (2.8) is obtained by means of transforming the left-hand side by (2.5) and (2.7), dividing by b and limit transition $b \to 0$.

On the contrary, let the function (2.4) be the solution of the approximate problem (2.9), (2.10). In order to prove that $f(z, \varepsilon, a)$ sets the approximate group, it is sufficient to verify that the approximate equality (2.6) holds:

$$f(f(z, \varepsilon, a), \varepsilon, b) \approx f(z, \varepsilon, a + b).$$

Let us designate the left-hand and the right-hand sides of (2.6), considered (with fixed z, a) as functions of (b, ε), by $x(b, \varepsilon)$ and $y(b, \varepsilon)$, respectively. By virtue of (2.9) they satisfy one and the same approximate Cauchy problem

$$\frac{\partial x}{\partial b} \approx \xi(x, \varepsilon), \quad x|_{b=0} \approx g(z, \varepsilon, a),$$

$$\frac{\partial y}{\partial b} \approx \xi(y, \varepsilon), \quad y|_{b=0} \approx g(z, \varepsilon, a).$$

Therefore, according to Theorem 2.2, the approximate equality $x(b, \varepsilon,) \approx y(b, \varepsilon)$, i.e. the group property (2.6), holds.

§ 3 The algorithm for constructing an approximate group

The construction of an approximate group by means of a given infinitesimal operator is carried out on the basis of the approximate Lie theorem. In order to illustrate how to solve the approximate Lie equation (2.9), let us consider the case $p = 0$ first.

We seek the approximate transformation group

$$z' = f_o(z, a) + \varepsilon f_1(z, a), \tag{3.1}$$

determined by the infinitesimal operator

$$X = (\xi_o(z) + \varepsilon \xi_1(z)) \frac{\partial}{\partial z} \cdot \tag{3.2}$$

Upon extracting the dominant terms with respect to ε, the corresponding approximate Lie equation

$$\frac{d(f_o + \varepsilon f_1)}{da} \approx \xi_o(f_o + \varepsilon f_1) + \varepsilon \xi_1(f_o + \varepsilon f_1)$$

can be rewritten in the form of the system

$$\frac{df_o}{da} \approx \xi_o(f_o), \quad \frac{df_1}{da} \approx \xi_o'(f_o)f_1 + \xi_1(f_o),$$

where $\xi_o'(\lambda) = \|\frac{\partial \xi_o^i}{\partial \lambda^j}\|$ is the derivative of ξ_o. The initial condition $z'|_{a=0} \approx z$ yields

$$f_o\Big|_{a=0} \approx z, \quad f_1\Big|_{a=0} \approx 0.$$

Thus, according to the definition of solution of the approximate Cauchy problem (see § 1), in order to construct the approximate (with the accuracy up to $o(\varepsilon)$) group (3.1) by the given infinitesimal operator (3.2), it is sufficient to solve the following (exact) Cauchy problem:

$$\frac{df_o}{da} = \xi_o(f_o), \quad \frac{df_1}{da} = \xi_o'(f_o)f_1 + \xi_1(f_o),$$

$$f_o|_{a=0} = z, \quad f_1|_{a=0} = 0. \tag{3.3}$$

Example 2.1. Let $N = 1$, $X = (1 + \varepsilon x)\frac{\partial}{\partial x}$. The corresponding Cauchy problem (3.3)

$$\frac{df_o}{da} = 1, \quad \frac{df_1}{da} = f_o,$$

$$f_o|_{a=0} = x, \quad f_1|_{a=0} = 0$$

is readily solved and provides $f_o = x + a$, $f_1 = xa + \frac{a^2}{2}$. Hence, the approximate group is determined by the formula

$$x' \approx x + a + (xa + \frac{a^2}{2})\varepsilon.$$

This formula is obviously the dominant term of the Taylor expansion of the exact group

$$x' = xe^{a\varepsilon} + \frac{e^{a\varepsilon} - 1}{\varepsilon} =$$

$$(x + a) + a(x + \frac{a}{2})\varepsilon + \frac{a^2}{2}(x + \frac{a^2}{3})\varepsilon^2 + \ldots,$$

with respect to ε. The latter group is generated by the operator under consideration

$$X = (1 + \varepsilon x)\frac{\partial}{\partial x} \cdot$$

Example 2.2. Let us construct the approximate transformation group

$$x' = f_o^1(x, y, a) + \varepsilon f_1^1(x, y, a), \quad y' = f_o^2(x, y, a) + \varepsilon f_1^2(x, y, a)$$

on the plane (x, y) determined by the operator

$$X = (1 + \varepsilon x^2)\frac{\partial}{\partial x} + \varepsilon xy\frac{\partial}{\partial y}\cdot$$

Upon solving the problem (3.3)

$$\frac{df_o^1}{da} = 1, \ \frac{df_o^2}{da} = 0, \ \frac{df_1^1}{da} = (f_o^1)^2, \ \frac{df_1^2}{da} = f_o^1 f_o^2,$$

$$f_o^1\|_{a=0} = x, \ f_o^2\|_{a=0} = y, \ f_1^1\|_{a=0} = 0, \ f_1^2\|_{a=0} = 0,$$

one obtains

$$x' \approx x + a + (x^2 a + xa^2 + \frac{a^3}{3})\varepsilon, \quad y' \approx y + (xya + \frac{ya^2}{2})\varepsilon.$$

Transformations of the corresponding exact group in the given case have the form

$$x' = \frac{\delta x \cos\delta a + \sin\delta a}{\delta(\cos\delta a - \delta x \sin\delta a)}, \quad y' = \frac{y}{\cos\delta a - \delta x\sin\delta a}, \quad \delta = \sqrt{\varepsilon}.$$

In order to construct the approximate group (with the accuracy up to $o(\varepsilon^p)$) with the arbitrary p we will need the formula for the dominant (with respect to ε) part of the function of the form $F(y_o + \varepsilon y_1 + \ldots + \varepsilon^p y_p)$. According to the Taylor formula, one has

$$F(y_o{+}\varepsilon y_1{+}\ldots{+}\varepsilon^p y_p) = F(y_o){+}\sum_{|\delta|=1}^{p}\frac{1}{\delta^1}F^{(\delta)}(y_o)(\varepsilon y_1{+}\ldots\varepsilon^p y_p)^\delta{+}o(\varepsilon^p), \quad (3.4)$$

where

$$F^{(\delta)} = \frac{\partial^{|\delta|}F}{(\partial z^1)^{\delta_1}\ldots(\partial z^N)^{\delta_N}}, \quad (\varepsilon y_1 + \ldots + \varepsilon^p y_p)^\delta =$$

$$\prod_{k=1}^{N}(\varepsilon y_1^k + \ldots + \varepsilon^p y_p^k)^{\delta_k}. \quad (3.5)$$

Here $\delta = (\delta_1, \ldots, \delta_N)$ is the multiindex, $|\delta| = \delta_1 + \ldots + \delta_N$, $\delta! = \delta_1! \ldots \delta_N!$, indices $\delta_1, \ldots, \delta_N$ run the values from 0 to p. Let us extract terms up to the order ε^p from the latter expression:

$$\prod_{k=1}^{N}(\varepsilon y_1^k + \ldots + \varepsilon^p y_p^k)^{\delta k} = \prod_{k=1}^{N}\left(\sum_{i_1,\ldots,i_{\delta_k}=1}^{p} y_{i_1}^k \ldots y_{i_{\delta_k}}^k \varepsilon^{i_1+\ldots+i_{\delta_k}}\right) \approx$$

$$\prod_{k=1}^{N}\left(\sum_{\nu_k=\delta_k}^{p}\varepsilon^{\nu_k}\sum_{i_1+\ldots+i_{\delta_k}=\nu_k} y_{i_1}^k \ldots y_{i_{\delta_k}}^k\right) \equiv \prod_{k=1}^{N}\sum_{\nu_k=\delta_k}^{p}\varepsilon^{\nu_k} y_{(\nu_k)}^k \approx$$

$$\sum_{j=|\delta|}^{p}\varepsilon^j\left(\sum_{\nu_1+\ldots+\nu_N=j} y_{(\nu_1)}^1 \ldots y_{(\nu_N)}^N\right) \equiv \sum_{j=|\delta|}^{p}\varepsilon^j \sum_{|\nu|=j} y_{(\nu)}. \tag{3.6}$$

Here, the following notation is used:

$$y_{(\nu_k)}^k \equiv \sum_{i_1+\ldots+i_{\delta_k}=\nu_k} y_{i_1}^k \ldots y_{i_{\delta_k}}^k, \quad y_{(\nu)} = y_{(\nu_1)}^1 \ldots y_{(\nu_N)}^N, \tag{3.7}$$

where the indices $i_1, \ldots, i_{\delta_k}$ run the values $1, \ldots, p$, and $\nu = \nu(\delta) = (\nu_1, \ldots, \nu_N)$ is the multiindex associated with the multiindex δ so that if the index δ_1 in δ is equal to zero, then the corresponding index ν_1 is absent in ν, while every remaining index ν_k takes the values from δ_k to p; e.g. for $\delta = (0, \delta_2, \delta_3, 0, \ldots, 0)$ with $\delta_2, \delta_3 \neq 0$ one has $\nu = (\nu_2, \nu_3)$, so that $y_{(\nu)} = y_{(\nu_2)}^2 y_{(\nu_3)}^3$.

Substituting (3.6) into (3.4) and changing the order of summation with respect to δ and j, one obtains the following formula for the principal part:

$$F(y_o+\varepsilon y_1+\ldots+\varepsilon^p y_p) = F(y_o)+\sum_{j=1}^{p}\varepsilon^j\sum_{|\delta|=1}^{j}\frac{1}{\delta!}F^{(\delta)}(y_o)\sum_{|\nu|=j}y_{(\nu)}+o(\varepsilon^p), \tag{3.8}$$

where the notation (3.5), (3.7) is used. For instance,

$$F(y_o + \varepsilon y_1 + \varepsilon^2 y_2 + \varepsilon^3 y_3) = F(y_o) + \varepsilon\sum_{k=1}^{N}\frac{\partial F(y_o)}{\partial z^k}y_{y_1}^k + \varepsilon^2\left(\sum_{k=1}^{N}\frac{\partial F(y_o)}{\partial z^k}y_2^k + \right.$$

$$\left.\frac{1}{2}\sum_{k=1}^{N}\sum_{l=1}^{N}\frac{\partial^2 F(y_o)}{\partial z^k \partial z^l}y_1^k y_1^l\right) + \varepsilon^3\left(\sum_{k=1}^{N}\frac{\partial F(y_o)}{\partial z^k}y_3^k + \frac{1}{2}\sum_{k=1}^{N}\sum_{l=1}^{N}\frac{\partial^2 F(y_o)}{\partial z^k \partial z^l}\cdot\right.$$

$$\left.\cdot(y_1^k y_2^l + y_1^l y_2^k) + \frac{1}{3}\sum_{k=1}^{N}\sum_{l=1}^{N}\sum_{m=1}^{N}\frac{\partial^3 F(y_o)}{\partial z^k \partial z^l \partial z^m}y_1^k y_1^l y_1^m\right) + o(\varepsilon^3).$$

We will also need generalization of the formula (3.8) for the expression

$$\sum_{i=0}^{p}\varepsilon^i F_i(y_o + \varepsilon y_1 + \ldots + \varepsilon^p y_p).$$

Applying formula (3.8) to every function F_i and introducing the notation

$$\tau_{j,i} = \sum_{|\delta|=1}^{j} \frac{1}{\delta!} F_i^{(\delta)}(y_o) \sum_{|\nu|=j} y_{(\nu)}$$

for the sake of brevity, one obtains

$$\sum_{i=0}^{p} \varepsilon^i F_i(y_o + \varepsilon y_1 + \ldots + \varepsilon^p y_p) \approx \sum_{i=0}^{p} \varepsilon^i [F_i(y_o) + \sum_{j=1}^{p} \varepsilon^j \tau_{j,i}] \approx$$

$$\sum_{i=0}^{p} \varepsilon^i F_i(y_o) + \sum_{i=0}^{p-1} \sum_{j=1}^{p-i} \varepsilon^{i+j} \tau_{j,i}$$

with the accuracy up to $o(\varepsilon^p)$. In order to sort the latter summand with respect to degrees of ε let us make the standard transformations

$$\sum_{i=0}^{p-1} \sum_{j=1}^{p-i} \varepsilon^{i+j} \tau_{j,i} = \sum_{i=0}^{p-1} \sum_{l=i+1}^{p} \varepsilon^l \tau_{l-i,i} =$$

$$\sum_{l=1}^{p} \varepsilon^l \sum_{i=0}^{l-j} \tau_{l-i,i} = \sum_{l=1}^{p} \varepsilon^l \sum_{j=1}^{l} \tau_{j,l-j}.$$

As a result one arrives to the following generalization of the formula (3.8):

$$\sum_{i=0}^{p} \varepsilon^i F_i(y_o + \varepsilon y_1 + \ldots + \varepsilon^p y_p) \approx$$

$$F_o(y_o) + \sum_{i=1}^{p} \varepsilon^i [F_i(y_o) + \sum_{j=1}^{i} \sum_{|\delta|=1}^{j} \frac{1}{\delta!} F_{i-j}^{(\delta)}(y_o) \sum_{|\nu|=j} y_{(\nu)}] \tag{3.9}$$

with the same notation (3.5), (3.7).

Let us turn back to constructing an approximate group with the accuracy up to $o(\varepsilon^p)$ with the arbitrary p. The approximate transformation group

$$z' \approx f_o(z, a) + \varepsilon f_1(z, a) + \ldots + \varepsilon^p f_p(z, a) \tag{3.10}$$

for the infinitesimal operator

$$X = [\xi_o(z) + \varepsilon \xi_1(z) + \ldots + \varepsilon^p \xi_p(z)] \frac{\partial}{\partial z}$$

is determined by the following approximate Lie equation

$$\frac{d}{da}(f_o + \varepsilon f_1 + \ldots + \varepsilon^p f_p) \approx \sum_{i=0}^{p} \varepsilon^i \xi_i(f_o + \varepsilon f_1 + \ldots + \varepsilon^p f_p). \tag{3.11}$$

Transforming the right-hand side of the equation by the formula (3.9) and equating coefficients of ε with the same powers, one obtains the system of equations (with notation (3.5),(3.7))

$$\frac{df_o}{da} = \xi_o(f_o), \tag{3.12}$$

$$\frac{df_1}{da} = \xi_i(f_o) + \sum_{j=1}^{i} \sum_{|\delta|=1}^{j} \frac{1}{\delta!} \xi_{i-j}^{(\delta)}(f_o) \sum_{|\nu|=j} f_{(\nu)}, i = 1, \ldots, p \tag{3.13}$$

equivalent to the approximate equation (3.11).

Thus, the problem of constructing the approximate group (3.10) is reduced to solution of the system (3.12), (3.13) with the initial conditions

$$f_o\Big|_{a=0} = z, \quad f_i\Big|_{a=0} = 0, \quad i = 1, \ldots, p. \tag{3.14}$$

Let us write out the first several equations of the system (3.12), (3.13) for the sake of illustration:

$$\frac{df_o}{da} = \xi_o(f_o),$$

$$\frac{df_1}{da} = \sum_{k=1}^{N} \frac{\partial \xi_o(f_o)}{\partial z^k} f_1^k + \xi_1(f_o),$$

$$\frac{df_2}{da} = \sum_{k=1}^{N} \frac{\partial \xi_o(f_o)}{\partial z^k} f_2^k + \frac{1}{2} \sum_{k=1}^{N} \sum_{l=1}^{N} \frac{\partial^2 \xi_o(f_o)}{\partial z^k \partial z^l} f_1^k f_1^l + \sum_{k=1}^{N} \frac{\partial \xi_1(f_o)}{\partial z^k} f_1^k + \xi_2(f_o),$$

$$\begin{aligned} \frac{df_3}{da} = {} & \sum_{k=1}^{N} \frac{\partial \xi_o(f_o)}{\partial z^k} f_3^k + \frac{1}{2} \sum_{k=1}^{N} \sum_{l=1}^{N} \frac{\partial^2 \xi_o(f_o)}{\partial z^k \partial z^l} (f_1^k f_2^l + f_1^l f_2^k) \\ & + \frac{1}{3!} \sum_{k=1}^{N} \sum_{l=1}^{N} \sum_{m=1}^{N} \frac{\partial^3 \xi_o(f_o)}{\partial z^k \partial z^l \partial z^m} f_1^k f_1^l f_1^m + \sum_{k=1}^{N} \frac{\partial \xi_1(f_o)}{\partial z^k} f_2^k \\ & + \frac{1}{2} \sum_{k=1}^{N} \sum_{l=1}^{N} \frac{\partial^2 \xi_1(f_o)}{\partial z^k \partial z^l} f_1^k f_1^l + \sum_{k=1}^{N} \frac{\partial \xi_2(f_o)}{\partial z^k} f_1^k + \xi_3(f_o). \end{aligned} \tag{3.15}$$

Example 2.3. Let us write out the system (3.12), (3.13) for the operator

$$X = (1+\varepsilon x^2)\frac{\partial}{\partial x} + \varepsilon xy\frac{\partial}{\partial y}$$

from Example 2.2. In the given case $N = 2$, $z = (x, y)$, $f_k = (f_k^1, f_k^2)$, $k = 0, 1, \ldots, p$; $\xi_o = (1, 0)$, $\xi_1 = (x^2, xy)$, $\xi_l = 0$ for $l \geq 2$. Equations (3.15) provide

$$\frac{df_o^1}{da} = 1, \quad \frac{df_o^2}{da} = 0; \quad \frac{df_1^1}{da} = (f_o^1)^2, \quad \frac{df_1^2}{da} = f_o^1 f_o^2;$$

$$\frac{df_2^1}{da} = 2f_o^1 f_1^1, \quad \frac{df_2^2}{da} = f_o^2 f_1^1 + f_o^1 f_1^2;$$

$$\frac{df_3^1}{da} = 2f_o^1 f_2^1 + (f_1^1)^2; \quad \frac{df_3^2}{da} = f_o^2 f_2^1 + f_o^1 f_2^2 + f_1^1 f_1^2.$$

Equation (3.13) is simplified for $i > 3$ due to the special form of the vector ξ. Namely, since ξ_o =const., $\xi_l = 0$ for $l \geq 2$, one has only terms with $j = i-1$ in the right-hand side of (3.13), and equation (3.13) is rewritten in the form

$$\frac{df_i}{da} = \sum_{|\delta|=1}^{i-1} \frac{1}{\delta!}\xi_1^{(\delta)}(f_o) \sum_{|\nu|=i-1} f_{(\nu)}.$$

Further reduction of these equations is connected with the form of the vector ξ_1. Since $\xi_1^1 = x^2, \xi_1^2 = xy$, writing the first component of the considered equations one uses only $\delta = (1, 0)$ and $\delta = (2, 0)$; and writing the second component one takes δ equal to $(1, 0)$, $(0, 1)$, and $(1, 1)$. As a result one obtains the following recurrent system:

$$\frac{df_i^1}{da} = 2f_o^1 f_{i-1}^1 + \sum_{i_1+i_2=i-1} f_{i_1}^1 f_{i_2}^1$$

$$\frac{df_i^2}{da} = f_o^2 f_{i-1}^1 + f_o^1 f_{i-1}^2 + \sum_{i_1+i_2=i-1} f_{i_1}^1 f_{i_2}^2.$$

These equations hold for every $i \geq 3$.

Example 2.4. Let us calculate the approximate transformation group of the order ε^p, dilated by the operator

$$X = (1+\varepsilon x)\frac{\partial}{\partial x}$$

from Example 2.1. The system (3.12), (3.13) in the given case takes the form

$$\frac{df_o}{da} = 1, \quad \frac{df_i}{da} = f_{i-1}, \quad i = 1, \ldots, p,$$

and with regard to initial conditions (3.14) it provides

$$f_i = \frac{xa^i}{i!} + \frac{a^{i+1}}{(i+1)!}, \quad i = 0, \ldots, p.$$

The corresponding approximate transformation group is determined by the formula

$$x' \approx \sum_{i=0}^{p} \frac{a^i}{i!}(x + \frac{a}{i+1})\varepsilon^i.$$

Let us verify conditions of Definition 2.1 of the approximate group for the latter transformation. Condition (2.5)

$$x'|_{a=0} = x,$$

obviously holds. The correlation (2.6), written in the given case as

$$\sum_{i=0}^{p} \frac{b^i}{i!} \left[\sum_{j=0}^{p} \frac{a^j}{j!}(x + \frac{a}{j+1})\varepsilon^j + \frac{b^i}{i+1} \right] \varepsilon^i \approx \sum_{i=0}^{p} \frac{(a+b)^i}{i!}(x + \frac{a+b}{i+1})\varepsilon^i,$$

follows from the chain of approximate equalities (with the accuracy up to $o(\varepsilon^p)$):

$$\sum_{i=0}^{p} \frac{b^i}{i!} \left[\sum_{j=0}^{p} \frac{a^j}{j!}(x + \frac{a}{j+1})\varepsilon^j + \frac{b^i}{i+1} \right] \approx$$

$$\sum_{i=0}^{p} \sum_{j=0}^{p-i} \left(\frac{b^i a^j}{i!j!} x + \frac{b^i a^{j+1}}{i!(j+1)!} \varepsilon^{i+j} + \sum_{i=0}^{p} \frac{b^{i+1}}{(i+1)!} \varepsilon^i \right) =$$

$$\sum_{i=0}^{p} \sum_{k=i}^{p} \left(\frac{b^i a^{k-i}}{i!(k-i)!} x + \frac{b^i a^{k+1-i}}{i!(k+1-i)!} \right) \varepsilon^k + \sum_{i=0}^{p} \frac{b^{i+1}}{(i+1)!} \varepsilon^i =$$

$$\sum_{k=0}^{p} \varepsilon^k \left[\sum_{i=0}^{k} \left(\frac{b^i a^{k-i}}{i!(k-i)!} x + \frac{b^i a^{k+1-i}}{i!(k+1-i)!} \right) + \frac{b^{k+1}}{(k+1)!} \right] =$$

$$\sum_{k=0}^{p} \varepsilon^k \left[\frac{(a+b)^k}{k!} x + \frac{(a+b)^{k+1}}{(k+1)!} \right].$$

§ 4 Criterion of approximate invariance

Definition 2.2. The approximate equation

$$F(z,\varepsilon)\approx 0 \tag{4.1}$$

is said to be invariant with respect to the approximate transformation group $z'\approx f(z,\varepsilon,a)$ if

$$F(f(z,\varepsilon,a),\varepsilon)\approx 0 \tag{4.2}$$

for all $z=(z^1,\dots,z^N)$, satisfying (4.1).

Theorem 2.3. Let the function $F(z,\varepsilon)=(F^1(z,\varepsilon),\dots,F^n(z,\varepsilon))$, analytic with respect to the set of variables z,ε, satisfy the condition

$$\text{rank } F'(z,0)|_{F(z,0)=0}=n, \tag{4.3}$$

where $F'(z,\varepsilon)=\|\frac{\partial F^\nu(z,\varepsilon)}{\partial z^i}\|$, $\nu=1,\dots,n;\ i=1,\dots,N$. The invariance of the approximate equation (4.1)

$$F(z,\varepsilon)=o(\varepsilon^p)$$

with respect to the approximate transformation group (2.4)

$$z'=f(z,\varepsilon,a)+o(\varepsilon^p)$$

with the infinitesimal operator

$$X=\xi^i(z,\varepsilon)\frac{\partial}{\partial z^i},\quad \xi^i=\frac{\partial f}{\partial a}\bigg|_{a=0}+o(\varepsilon^p), \tag{4.4}$$

is provided by the necessary and sufficient condition

$$XF(z,\varepsilon)|_{(4.1)}=o(\varepsilon^p). \tag{4.5}$$

Proof. Necessity. Let us assume that the condition (4.2) of invariance of the approximate equation (4.1)

$$F(f(z,\varepsilon,a),\varepsilon)\big|_{(4.1)}=o(\varepsilon^p)$$

is satisfied. Whence, one obtains the equality (4.5) by means of differentiation with respect to a when $a=0$.

Sufficiency. Let the equality (4.5) be satisfied for the function $F(z,\varepsilon)$, satisfying the condition (4.3). Let us prove the invariance of the approximate equation (4.1). With this purpose we introduce new variables

$$y^1=F^1(z,\varepsilon),\dots,\ y^n=F^n(z,\varepsilon),\ y^{n+1}=H^1(z,\varepsilon),\dots,\ y^N=H^{N-n}(z,\varepsilon)$$

instead of $z^1, \dots, z^N$, choosing such $H^1(z, \varepsilon), \dots, H^{N-n}(z, \varepsilon)$ that the functions $F^1, \dots, F^n, H^1, \dots, H^{N-n}$ are functionally independent (this is possible by virtue of the condition (4.3) when ε is sufficiently small). The original approximate equation (4.1), operator (4.4) and the condition (4.5) take the following form in the new variables

$$y^\nu = \theta_p^\nu(y, \varepsilon), \quad \nu = 1, \dots, n, \tag{4.6}$$

$$X = \eta^i(y, \varepsilon)\frac{\partial}{\partial y^i}, \text{ where } \eta^i \approx \xi^i(x, \varepsilon)\frac{\partial y^i(x, \varepsilon)}{\partial x^j} \tag{4.7}$$

$$\eta^\nu(\theta_p^1, \dots, \theta_p^n, y^{n+1}, \dots, y^N) = o(\varepsilon^p), \quad \nu = 1, \dots, n, \tag{4.8}$$

where $\theta_p^\nu = o(\varepsilon^p)$ (see 0.2)). According to Theorem 2.2, transformations of variables y are determined by the approximate Cauchy problem

$$\frac{dy'^\nu}{da} \approx \eta^\nu(y'^1, \dots, y'^n, y'^{n-1}, \dots, y'^N, \varepsilon), \; y'^\nu|_{a=0} = \theta_p^\nu(y, \varepsilon),$$

$$\frac{dy'^k}{da} \approx \eta^k(y'^1, \dots, y'^n, y'^{n+1}, \dots, y'^N, \varepsilon), y'^k|_{a=0} = y^k, \; k = n+1, \dots, N,$$

where the initial conditions for the first subsystem are written with regard to (4.1). According to Theorem 2.1, the solution of the problem is unique (with the accuracy under consideration), and by virtue of (4.8), it has the form $y' = \theta_p^1, \dots, \theta_p^n, y^{n+1}, y^N$. Turning back to the initial variables, one obtains $F^\nu(z', \varepsilon) = o(\varepsilon^p), \nu = 1, \dots, n$, i.e., the approximate equation (4.2). The theorem is proved.

Example 2.5. Let $N = 2, z = (x, y), p = 1$. Consider the approximate transformation group (see Example 2.2 of § 3)

$$x' = x + a + (x^2 a + x a^2 + \frac{a^3}{3})\varepsilon, \quad y' = y + (xya + \frac{ya^2}{2})\varepsilon \tag{4.9}$$

with the infinitesimal operator

$$X = (1 + \varepsilon x^2)\frac{\partial}{\partial x} + \varepsilon xy\frac{\partial}{\partial y}\cdot \tag{4.10}$$

Let us demonstrate that the approximate equation

$$F(x, y, \varepsilon) \equiv y^{2+\varepsilon} - \varepsilon x^2 - 1 \approx o(\varepsilon) \tag{4.11}$$

is invariant with respect to transformations (4.10).

First let us verify the invariance of (4.11) abiding by Definition 2.2. To this end, it is convenient to rewrite equation (4.11) in the following form:

$$\widetilde{F}(x,y,\varepsilon) \equiv y^2 - \varepsilon(x^2 - y^2 \ln y) - 1 \approx 0 \tag{4.12}$$

with the necessary accuracy preserved. Transformation (4.9) provides

$$\widetilde{F}(x',y',\varepsilon) = y'^2 - \varepsilon(x'^2 - y'^2 \ln y') - 1 \approx y^2 - \varepsilon(x^2 - y^2 \ln y) - 1 +$$
$$\varepsilon(2xa + a^2)(y^2 - 1) = \widetilde{F}(x,y,\varepsilon) + \varepsilon(2xa + a^2)[\widetilde{F}(x,y,\varepsilon) +$$
$$\varepsilon(x^2 - y^2 \ln y)] = [1 + \varepsilon(2ax + a^2)]\widetilde{F}(x,y,\varepsilon) + o(\varepsilon)\Big|_{\widetilde{F}=0} \approx 0,$$

whence, the necessary equality (4.2): $\widetilde{F}(x',y',\varepsilon)$ follows.

The function $F(x,y,\varepsilon)$ satisfies the condition 4.3 of Theorem 2.3. Therefore, the invariance can be manifested also by means of the infinitesimal criterion (4.5). One has

$$XF = (2+\varepsilon)\varepsilon x y^{2+\varepsilon} - 2\varepsilon x(1+\varepsilon x^2) = 2\varepsilon x(y^{2+\varepsilon} - 1) + o(\varepsilon) = 2\varepsilon x F + o(\varepsilon)$$

for the operator (4.10), so that the invariance criterion (4.5) obviously holds.

According to Theorem 2.3, the construction of the approximate group, leaving the equation $F(z,\varepsilon) \approx 0$ invariant, is reduced to solution of the determining equation

$$XF(z,\varepsilon)\Big|_{F\approx 0} \approx 0 \tag{4.13}$$

for the coordinates $\xi^k(z,\varepsilon)$ of the infinitesimal operator

$$X = \xi \frac{\partial}{\partial z}\cdot$$

In order to solve the determining equation (4.13) with the accuracy up to $o(\varepsilon)$ one has to represent the values z, F, ξ^k in the form

$$z = y_o + \varepsilon y_1 + \ldots + \varepsilon^p y_p,\ F(z,\varepsilon) \approx \sum_{i=0}^{p} \varepsilon^i F_i(z),\ \xi^k(z,\varepsilon) \approx \sum_{i=0}^{p} \varepsilon^i \xi_i^k(z), \tag{4.14}$$

substitute them into XF, and single out the principal terms there. One has

$$XF = \xi^k \frac{\partial F}{\partial z^k} = \left[\sum_{i=0}^{p} \varepsilon^i \xi^k (y_o + \varepsilon y_1 + \ldots + \varepsilon^p y_p)\right] -$$

$$\left[\sum_{j=0}^{p} \varepsilon^j \frac{\partial}{\partial z^k} F_j(y_o + \varepsilon y_1 + \ldots + \varepsilon^p y_p)\right].$$

Using the formula (3.9) and the notation

$$A_i^k = \xi_i^k(y_o) + \sum_{j=1}^{i} \sum_{|\delta|=1}^{j} \frac{1}{\delta!} (\xi_{i-j}^k)^{(\delta)}(y_o) \sum_{|\nu|=j} y_{(\nu)}, \tag{4.15}$$

$$B_{j,k} = \frac{\partial F_j(y_o)}{\partial z^k} + \sum_{i=1}^{j} \sum_{|\omega|=1}^{i} \frac{1}{\omega!} \left(\frac{\partial F_{j-i}}{\partial z^k}\right)^{(\omega)} (y_o) \sum_{|\mu|=i} y_{(\mu)}, \tag{4.16}$$

one obtains

$$XF = \left[\xi_o^k(y_o) + \sum_{i=1}^{p} \varepsilon^i A_i^k\right] \cdot \left[\frac{\partial F_o(y_o)}{\partial z^k} + \sum_{j=1}^{p} \varepsilon^j B_{j,k}\right],$$

whence,

$$XF = \xi_o^k(y_o) \frac{\partial F_o(y_o)}{\partial z^k} + \varepsilon \left[\xi_o^k(y_o) B_{1,k} + A_1^k \frac{\partial F_o(y_o)}{\partial z^k}\right] +$$

$$\sum_{s=2}^{p} \varepsilon^s \left[\xi_o^k(y_o) B_{s,k} + A_s^k \frac{\partial F_o(y_o)}{\partial z^k} + \sum_{i+j=s} A_i^k B_{j,k}\right]. \tag{4.17}$$

Combination of the formulae (4.13), (4.14), (4.15), (4.16), (4.17), and (3.9) provides the following form of the determining equation

$$\xi_o^k(y_o) \frac{\partial F_o(y_o)}{\partial z^k} = 0, \quad \xi_o^k(y_o) B_{1,k} + A_1^k \frac{\partial F_o(y_o)}{\partial z^k} = 0, \tag{4.18}$$

$$\xi_o^k(y_o) B_{l,k} + A_l^k \frac{\partial F_o(y_o)}{\partial z^k} + \sum_{i+j=l} A_i^k B_{j,k} = 0, \quad l = 2, \ldots, p.$$

Equations (4.18) hold for the set of all $y_o, y_1, \ldots, y_p$, satisfying the system

$$F_o(y_o) = 0, \; F_i(y_o) + \sum_{j=1}^{i} \sum_{|\delta|=1}^{j} \frac{1}{\delta!} F_{i-j}^{(\delta)}(y_o) \sum_{|\nu|=j} y_{(\nu)}, \quad i = 1, \ldots, p, \tag{4.19}$$

equivalent to the approximate equation (4.1). Thus, the problem of solving the approximate determining equation (4.13) is reduced to solving the system of exact equations (4.18), (4.19).

Let us write out the determining equations when $p = 1$. Equations (4.18), (4.19) provide

$$\xi_o^k(y_o)\frac{\partial F_o(y_o)}{\partial z^k} = 0, \tag{4.20}$$

$$\xi_1^k(y_o)\frac{\partial F_o(y_o)}{\partial z^k} + \xi_o^k(y_o)\frac{\partial F_1(y_o)}{\partial z^k} + y_1^l\frac{\partial}{\partial z^l}(\xi_o^k(y_o)\frac{\partial F_o(y_o)}{\partial z^k}) = 0 \tag{4.21}$$

with the conditions

$$F_o(y_o) = 0, \ F_1(y_o) + y_1^l\frac{\partial F_o(y_o)}{\partial z^l} = 0. \tag{4.22}$$

In the latter equations, as well as everywhere in this section, the following notation is used for the sake of brevity:

$$y_1^l\frac{\partial}{\partial z^l}(\xi_o^k(y_o)\frac{\partial F_o(y_o)}{\partial z^k}) \equiv \sum_{l=1}^N\sum_{k=1}^N y_1^l\frac{\partial}{\partial z^l}(\xi_o^k(z)\frac{\partial F_o(z)}{\partial z^k})\Big|_{z=y_o}.$$

In case if $p = 2$, equation

$$\begin{gathered}\xi_2^k(y_o)\frac{\partial F_o(y_o)}{\partial z^k} + \xi_1^k(y_o)\frac{\partial F_1(y_o)}{\partial z^k} + \xi_o^k(y_o)\frac{\partial F_2(y_o)}{\partial z^k} + \\ y_2^l\frac{\partial}{\partial z^l}\left(\xi_o^k(y_o)\frac{\partial F_o(y_o)}{\partial z^k}\right) + \frac{1}{2}y_1^l y_1^m\frac{\partial^2}{\partial z^l\partial z^m}\left(\xi_o^k(y_o)\frac{\partial F_o(y_o)}{\partial z^k}\right) + \\ y_1^l\frac{\partial}{\partial z^l}\left(\xi_o^k(y_o)\frac{\partial F_1(y_o)}{\partial z^k} + \xi_1^k(y_o)\frac{\partial F_o(y_o)}{\partial z^k}\right) = 0\end{gathered} \tag{4.23}$$

is added to (4.20), (4.21), and equation

$$F_2(y_o) + y_1^k\frac{\partial F_1(y_o)}{\partial z^k} + y_2^k\frac{\partial F_o(y_o)}{\partial z^k} + \frac{1}{2}y_1^k y_1^l\frac{\partial^2 F_o(y_o)}{\partial z^k\partial z^l} = 0 \tag{4.24}$$

is added to conditions (4.22).

Remark 2.2. Equations (4.18) and (4.19) (see also the particular cases (4.20)-(4.24)) manifest the necessity of the condition (4.3) imposed on the function $F(z, \varepsilon)$.

Example 2.6. Let us consider the approximate equation (4.11) of Example 2.5 again:

$$F(x, y, \varepsilon) \equiv y^{2+\varepsilon} - \varepsilon x^2 - 1 = o(\varepsilon).$$

In the notation (4.14) (se also (4.12)) one has

$$F_o(x, y) = y^2 - 1, \ F_1(x, y) = y^2 \ln y - x^2.$$

Taking into account the condition $y > 0$, one obtains

$$y_o = 1,\ y_1 = \frac{x_o^2}{2}$$

from equations (4.22); and the determining equations (4.20), (4.21), upon splitting with respect to the free variable x_1 and substituting $y_1 = x_o^2/2$, is written in the form

$$\xi_o^2(x_o, y_o) = 0,\ \frac{\partial \xi_o^2(x_o, y_o)}{\partial x} = 0,$$

$$y_o \xi_1^2(x_o, y_o) - x_o \xi_o^1(x_o, y_o) + \frac{x_o^2}{2}\frac{\partial \xi_o^2(x_o, y_o)}{\partial y} = 0. \tag{4.25}$$

Any operator

$$X = [\xi_o^1(x,y) + \varepsilon \xi_1^1(x,y)]\frac{\partial}{\partial x} + [\xi_o^2(x,y) + \varepsilon \xi_1^2(x,y)]\frac{\partial}{\partial y}$$

with the coordinates satisfying the equations (4.25) when $y_o = 1$ and x_o is arbitrary, dilates the approximate group that leaves equation (4.11) invariant with the accuracy up to $o(\varepsilon)$. For example, operators

$$X = x\frac{\partial}{\partial x} + 2(y-1)\frac{\partial}{\partial y},\quad X = xy\frac{\partial}{\partial x} + (y^2 - 1)\frac{\partial}{\partial y}\cdot$$

refer to such operators as well as (4.10).

Remark 2.3. If some variables z^k are not included into equation $F(z,\varepsilon) \approx 0$, then it is not necessary to represent the variable z^k in the form $\sum\limits_{i\geq 0} \varepsilon^i y_i^k$ in the determining equation (4.18).

§ 5 Equation $u_{tt} + \varepsilon u_t = [\varphi(u)u_x]_x$

Approximate symmetries (by which we mean either admissible approximate groups or their infinitesimal operators) of differential equations are calculated according to the algorithm given in § 4 using the ad hoc technic of dilation of infinitesimal operators to necessary derivatives. In what follows we consider approximate symmetries of the first order ($p = 1$) and classify equations of the second order with the small parameter

$$u_{tt} + \varepsilon u_t = (\varphi(u)u_x)_x,\quad \varphi \neq \text{ const}, \tag{5.1}$$

arising in various applied problems (see, e.g., [4]) according to such symmetries. The infinitesimal operator of the approximate symmetry is deduced in the form

$$X = (\xi_o^1 + \varepsilon\xi_1^1)\frac{\partial}{\partial t} + (\xi_o^2 + \varepsilon\xi_1^2)\frac{\partial}{\partial x} + (\eta_o + \varepsilon\eta_1)\frac{\partial}{\partial u}\cdot \tag{5.2}$$

Coordinates ξ and η of the operator (5.2) depend on t, x, u and are defined by the determining equations (4.20), (4.21), where

$$z = (t, x, u_t, u_x, u_{tt}, u_{tx}, u_{xx}), \quad F_o = u_{tt} - (\varphi(u)u_x)_x, F_i = u_t.$$

According to Remark (2.3), it is sufficient to dilate $z = y_o + \varepsilon(y_1)$ only for the differential variable (for t, x are not included into equation (4.1) explicitly):

$$u = u_o + \varepsilon u_1, u_x = (u_o)_x + \varepsilon(u_1)_x \text{ etc.}$$

Equation (4.20) is the determining equation for the operator

$$X^o = \xi_o^1\frac{\partial}{\partial t} + \xi_o^2\frac{\partial}{\partial x} + \eta_o\frac{\partial}{\partial u}, \tag{5.3}$$

admitted by the zero-order approximation of equation (5.1), i.e. by equation

$$u_{tt} = (\varphi(u)u_x)_x, \quad \varphi \neq \text{ const.} \tag{5.4}$$

Hence, the first stage of classification of equations (5.1) with respect to approximate symmetries is classification of equations (5.4) with respect to exact symmetries. The second stage is solution of the determining equation (4.21) with the known F_o and the values of the coordinates ξ_o^1, ξ_o^2, η_o of the operator (5.3).

Group classification of equations (5.4) (with respect to point symmetries) is carried out in [4] and the result with dilations and shifts applied can be written in the form of Table1.

Now let us turn to the second stage of constructing approximate symmetries. First, let us consider the case of an arbitrary function $\varphi(u)$. Substituting the values $\xi_o^1 = C_1 t + C_2$, $\xi_o^2 = C_1 x + C_3$, $\eta_o = 0$ into equation (4.21), one obtains $C_1 = 0, \xi_1^1 = K_1 t + K_2$, $\xi_1^2 = K_1 x + K_3$, $\eta_1 = 0$, where K_1 =const. Note that equation (5.1), as well as any admissible (exactly or approximately) operator X, admits the operator εX. These operators will be omitted from our consideration as unessential. In particular, the operators

$$\varepsilon\frac{\partial}{\partial t}, \quad \varepsilon\frac{\partial}{\partial x}$$

Table 1: Group classification of Eq. (5.4)

	$\varphi(u)$	ξ^1_o	ξ^2_o	η_o
	Arbitrary function	C_1t+C_2	C_1x+C_3	0
1	ku^{σ}	C_1t+C_2	C_3x+C_4	$\frac{2}{\sigma}(C_3-C_1)u$
2	$ku^{-\frac{1}{3}}$	C_1t+C_2	$C_3x^2+C_4x+C_5$	$-\frac{3}{2}(2C_3+C_4-C_1)u$
3	ku^{-4}	$C_1t^2+C_2t+C_3$	C_4x+C_5	$\frac{1}{2}(2C_1t+C_2-C_4)u$
4	ke^{u}	C_1t+C_2	C_3x+C_4	$2(C_3-C_1)$
$k=\pm 1$, σ is an arbitrary parameter, $C_1,\ldots,C_5=$const.				

ar unessential, and the constants K_2, K_3 can be considered to be equal to zero while solving the determining equation (4.21). Thus, when the function $\varphi(u)$ is arbitrary, equation (5.1) admits three essential operators of the approximate symmetry meeting the conditions of the constants C_2, C_3, K_1. Likewise, one can analyze the remaining cases of Table 1. The result is summarized in Table 2, where operators admitted by equations (5.4), and (5.1) exactly, and exactly and approximately, respectively are given for the sake of convenient comparison of approximate symmetries with the exact ones.

Remark 2.4. Table 2 provides bases of the admissible algebras for exact symmetries and their generators for approximate symmetries. The basis of the corresponding algebra is obtained by multiplying the generators by ε and eliminating the terms of the order ε^2. For example, when $\varphi(u)=ku^{-\frac{4}{3}}$, equations (5.4), and (5.1) admit a five-dimensional algebra, and a four-dimensional algebra of exact symmetries with a ten-dimensional algebra of approximate symmetries with the following basis, respectively:

$$X_1=\frac{\partial}{\partial t},\quad X_2=\frac{\partial}{\partial x},\quad X_3=(t-\frac{1}{4}\varepsilon t^2)\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}-\frac{3}{4}\varepsilon tu\frac{\partial}{\partial u},$$

$$X_4=2x\frac{\partial}{\partial x}-3u\frac{\partial}{\partial u},\quad X_5=x^2\frac{\partial}{\partial x}-3xu\frac{\partial}{\partial u},\quad X_6=\varepsilon X_1,$$

$$X_7=\varepsilon X_2,\quad X_8=\varepsilon(t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}),\quad X_9=\varepsilon X_4,\quad X_{10}=\varepsilon X_5.$$

Table 2: Comparative table of exact and approximate symmetries

$\varphi(u)$	Symmetries for Eq. (5.4)	Symmetries for Eq. (5.1) Exact	Approximate
Any	$X_1^o = \frac{\partial}{\partial t},\ X_2^o = \frac{\partial}{\partial x}$ $X_3^o = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$	$Y_1 = X_1^o$ $Y_2 = X_2^o$	$X_1 = X_1^o,\ X_2 = X_2^o$ $X_3 = \varepsilon X_3^o$
ku^σ	$X_4^o = \sigma x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}$	$Y_3 = X_4^o$	$\widetilde{X}_3 = X_3^o + \frac{\varepsilon}{\sigma+4}\left(\frac{\sigma}{2}t^2\frac{\partial}{\partial t} - 2tu\frac{\partial}{\partial u}\right)$ $X_4 = X_4^o$
$ku^{-\frac{4}{3}}$	$X_4^o = 2x\frac{\partial}{\partial x} - 3u\frac{\partial}{\partial u}$ $X_5^o = x^2\frac{\partial}{\partial x} - 3xu\frac{\partial}{\partial u}$	$Y_3 = X_4^o$ $Y_4 = X_5^o$	$\widetilde{X}_3 = X_3^o - \frac{\varepsilon}{4}\left(t^2\frac{\partial}{\partial t} + 3tu\frac{\partial}{\partial u}\right)$ $X_4 = X_4^o,\ X_5 = X_5^o$
ku^{-4}	$X_4^o = 2x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}$ $X_5^o = t^2\frac{\partial}{\partial t} + tu\frac{\partial}{\partial u}$	$Y_3 = X_4^o$ $Y_4 = X_5^o$	$X_4 = X_4^o$ $X_5 = \varepsilon X_5^o$
ke^u	$X_4^o = x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial u}$	$Y_3 = X_4^o$	$\widetilde{X}_3 = X_3^o + \varepsilon\left(\frac{t^2}{2}\frac{\partial}{\partial t} - t\frac{\partial}{\partial u}\right)$ $X_4 = X_4^o$

§ 6 Equation $u_t = h(u)u_x + \varepsilon H$

The present section considers the class of evolutionary equations of the form

$$u_t = h(u)u_1 + \varepsilon H, \quad H \in \mathcal{A} \tag{6.1}$$

containing, in particular, Korteweg-de Vries and Burgers Korteweg-de Vries equations, etc.

Theorem 2.4. Equation (6.1) inherits approximately (with any degree of accuracy) all symmetries of equation

$$u_t = h(u)u_1. \tag{6.2}$$

Namely, any canonical Lie-Bäcklund operator [61]

$$X^o = f^o\frac{\partial}{\partial u} + \ldots,$$

admitted by equation (6.2), generates an approximate symmetry (of an arbitrary order p) for (6.1) determined by the coordinate

$$f = \sum_{i=0}^{p} \varepsilon^i f^i, \quad f^i \in \mathcal{A} \tag{6.3}$$

of the canonical operator

$$X = f\frac{\partial}{\partial u} + \ldots.$$

Proof. The approximate symmetries (6.3) of the equation (6.1) are derived from the determining equation (4.18), which in the given case takes the form

$$f_t^o - h(u) f_x^o + \sum_{\alpha \geq 1} [D^\alpha (hu_1) - h_{\alpha+1}] f_\alpha^o - h'(u) u_1 f^o = 0, \tag{6.4}$$

$$f_t^i - h(u) f_x^i + \sum_{\alpha \geq 1} [D^\alpha (hu_1) - hu_{\alpha+1}] f_\alpha^i - h'(u) u_1 f^i =$$
$$\sum_{\alpha \geq 0} [D^\alpha (f^{i-1}) H_\alpha - f_\alpha^{i-1} D^\alpha (H)], \quad i = 1, \ldots, p. \tag{6.5}$$

Equation (6.4) in f^o is the determining equation for deriving the exact group of transformations admitted by (6.2). Let us assume that f^o is an arbitrary solution of equation (6.4) and that it is a differential function of the order $k \geq 0$, and let H be a differential function of the order $n \geq 1$, i.e.

$$f^o = f^o(t, x, u, \ldots, u_{k_o}), \quad H = H(t, x, u, \ldots, u_n).$$

We will be looking for solution f^1 of equation (6.5) in the form of the differential function of the order $k_1 = n + k_o - 1$. Then, (6.5) is a linear partial differential equation of the first order with respect to the function f^1 of $k_1 + 3$ arguments $t, x, u, u_1, \ldots, u_{k_1}$, and therefore, it is solvable. Substitution of any solution $f^1(t, x, u, u_1, \ldots, u_{k_1})$ into the right-hand side of equation (6.5) with $i = 2$ demonstrates that f^2 can be sought in the form of a differential function of the order $k_2 = n + k_1 - 1$, and the corresponding equation in f^2 is solvable. The remaining coefficients f^i, $i = 3, \ldots, p$ of the series (6.3) are determined recurrently from equation (6.5). The theorem is proved.

Theorem 2.4 entails, in particular, that *any* point *symmetry of equation (6.2)* determined by the infinitesimal operator

$$Y = \theta(t,x,u)\frac{\partial}{\partial t} + [\varphi(x+tu,u) - t\psi(x+tu,u) - u\theta(t,x,u)]\frac{\partial}{\partial x} + \psi(x+tu,u)\frac{\partial}{\partial u}$$

with the arbitrary functions φ, ψ, θ or by the corresponding canonical Lie-Bäcklund operator with the coordinate

$$f^o = [\varphi(x + tu, u) - t\psi(x + tu, u)] u_1 - \psi(x + tu, u), \tag{6.6}$$

is approximately inherited by equation (6.1). Let us illustrate the algorithm for constructing these approximate symmetries by examples.

First, let us consider the Korteweg-de Vries equation

$$u_t = uu_1 + \varepsilon u_3 \tag{6.7}$$

and find the approximate symmetry (6.3) of the second order ($p = 2$) with $f^p = \varphi(u)u_1$. For this purpose one has to solve equations (6.5) (equation (6.4) holds identically) for $i = 1, 2$ with $h(u) = u$, $H = u_3$. When $i = 1$, equation (6.5) is written in the form

$$f^1_t - uf^1_x + f^1_1 u_1^2 + 3f^1_2 u_1 u_2 + f^1_3(4u_1u_3 + 3u_1^2) - f^1 u_1 =$$

$$3\varphi' u_1 u_3 + 3\varphi' u_2^2 + 6\varphi'' u_1^2 u_2 + \varphi''' u_1^4. \tag{6.8}$$

According to the proof of Theorem 2.4, its solution should be sought in the form of a differential function of the third order, i.e. $f^1 = f^1(t, x, u, u_1, u_2, u_3)$. Assuming that f^1 is independent of t, x, one finds the following particular solution of equation (6.8):

$$f^1 = \varphi' u_3 + 2\varphi'' u_1 u_2 + \frac{1}{2}\varphi''' u_1^3 + \alpha(u)u_1 + \beta(u)\frac{u_2}{u_1^2},$$

where $\alpha(u), \beta(u)$ are arbitrary functions chosen so that $\alpha = \beta = 0$ for the sake of simplicity. Substituting the resulting expression for f^1 into the right-hand side of equation (6.5) with $i = 2$, one obtains

$$f^2 = \frac{3}{5}\varphi'' u_5 + \frac{9}{5}\varphi''' u_1 u_4 + (3\varphi''' u_2 + \frac{23}{10}\varphi^{IV} u_1^2)u_3 +$$

$$\frac{31}{10}\varphi^{IV} u_1 u_2^2 + \frac{8}{5}\varphi^{V} u_1^3 u_2 + \frac{1}{8}\varphi^{VI} u_1^5.$$

Thus, equation (6.7) pertains the following approximate symmetry of the second order

$$f = \varphi(u)u_1 + \varepsilon[\varphi' u_3 + 2\varphi'' u_1 u_2 + \frac{1}{2}\varphi''' u_1^3] + \varepsilon^2[\frac{3}{5}\varphi'' u_5 + \frac{9}{5}\varphi''' u_1 u_4 +$$

$$(3\varphi''' u_2 + \frac{23}{10}\varphi^{IV} u_1^2)u_3 + \frac{31}{10}\varphi^{IV} u^1 u_2^2 + \frac{8}{5}\varphi^{V} u_1^3 u_2 + \frac{1}{8}\varphi^{VI} u_1^5] + o(\varepsilon^2). \tag{6.9}$$

Calculation of further coefficients of the approximate symmetry (6.3) with $f^o = \varphi(u)u_1$ demonstrates that the coefficient f^i is a differential function of the order $2i + 1$, and f^i contains derivatives of $\varphi(u)$ only of the order $\geq i$. Hence, *when $\varphi(u)$ is a polynomial in the power of n*, all coefficients f^i in (6.3) vanish when $i > n + 1$ and the *approximate symmetry of the order $p = n$ is the exact Lie-Bäcklund symmetry of the order $2n + 1$;* in this case one san assume that $\varepsilon = 1$ and obtain the exact symmetries of equation

$$u_t = uu_1 + u_3. \tag{6.10}$$

For example, when $\varphi(u) = u^2$ and $p = 2$, the formula (6.8) provides the first non-trivial Lie-Bäcklund symmetry

$$f = u^2 u_1 + 4u_1 u_2 + 2uu_3 + \frac{6}{5} u_5$$

of the Kortweg-de Vries equation (6.10) (cf. [61] p. 191).

Let us consider another particular case and find the approximate symmetry of the first order for the Korteweg-de Vries equation (6.7), generated by the function $f^o = \varphi(x+tu)u_1$. The determining equation (6.5) with $i = 1$ yields

$$f = \varphi(x+tu)u_1 + \varepsilon \left\{ \varphi' \frac{1+tu_1}{u_1} u_3 - \varphi' \frac{u_2^2}{u_1^2} + \right.$$

$$\left. \varphi'' \cdot [2(1+tu_1)^2 - \frac{3}{2}(1+tu_1)] \frac{u_2}{u_1} + \frac{\varphi'''}{2}(1+tu_1)^3 \right\} + o(\varepsilon). \qquad (6.11)$$

Approximate symmetries generated by $f^o = \varphi(x+tu)u_1$ *never become exact whatever the function* φ *is.*

Let us draw another example of an approximate symmetry generated by a non-point symmetry. Equation $u_t = uu_1$ manifestly pertains the Lie-Bäcklund symmetry $f^o = \frac{u_2}{u_1}$. Solving equation (6.5) with $i = 1$ when $h = u$, $H = u_3$ for the given function f^o, one obtains the following approximate symmetry of the first order of equation (6.7):

$$f \approx \frac{u_2}{u_1^2} - \varepsilon \left(2\frac{u_2 u_4}{u_1^4} + 2\frac{u_3^2}{u_1^4} - 17\frac{u_2^2 u_3}{u_1^5} + 15\frac{u_2^4}{u_1^6} \right). \qquad (6.12)$$

Finally, let us consider the Burgers Kortweg-de Vries equation

$$u_t = uu_1 + \varepsilon(au_3 + bu_2), \quad a, b = \text{ const},$$

and find its approximate symmetry of the second order generated by point symmetry $f^o = \varphi(u)u_1$ of equation $u_t = uu_1$. The system of determining equations (6.5) with $h = u$, $H = au_3 + bu_2$, $i = 1, 2$ provides the unknown approximate symmetry

$$f = \varphi(u)u_1 + \varepsilon(a\varphi' u_3 + 2a\varphi'' u_1 u_2 + \frac{1}{2} a\varphi''' u_1^3 + b\varphi' u_2 + b\varphi'' u_1^2) +$$

$$\varepsilon^2 \left(\frac{3}{5} a^2 \varphi'' u_5 + \frac{5}{4} ab\varphi'' u_4 + \frac{1}{10} ab\varphi'' \frac{u_2 u_3}{u_1} - \frac{1}{20} ab\varphi'' \frac{u_2^3}{u_1^2} + \frac{2}{3} b^2 \varphi'' u_3 + \right.$$

$$\frac{9}{5} a^2 \varphi''' u_1 u_4 + 3a^2 \varphi''' u_2 u_3 + \frac{7}{2} ab\varphi''' u_1 u_3 + \frac{23}{10} ab\varphi'' u_2^2 + \frac{5}{3} b^2 \varphi''' u_1 u_2 +$$

$$\frac{23}{10}a^2\varphi^{IV}u_1^2u_3+\frac{31}{10}a^2\varphi^{IV}u_1u_2^2+\frac{15}{4}ab\varphi^{IV}u_1^2u_2+\frac{1}{2}b^2\varphi^{IV}u_1^3+$$

$$\left.\frac{8}{5}a^2\varphi^{V}u_1^3u_2+\frac{1}{2}ab\varphi^{5}u_1^4+\frac{1}{8}a^2\varphi^{VI}u_1^5\right)+o(\varepsilon^2). \qquad (6.13)$$

Note that symmetries (6.11), (6.12) and (6.9) with a non-polynomial function $\varphi(u)$ provide new symmetries of the Korteweg-de Vries equation that cannot be obtained in the framework of Lie and Lie-Bäcklund group theory. Symmetry (6.13) is also new for the Burgers-Korteweg-de Vries equation.

Translated by E.D. Avdonina

Paper 3

Approximate transformation groups

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In calculating approximate symmetries of differential equations with small parameter, Baikov, Gazizov, and Ibragimov [7], [9], [10] found that equations of the form

$$F(z,\varepsilon) \equiv F_o(z) + \varepsilon F_1(z) + \ldots + \varepsilon^q F_q(z) \approx 0 \tag{0.1}$$

with small parameter ε, considered with an accuracy of $o(\varepsilon^q)$ $(q > 0)$ can approximately [with an accuracy of $o(\varepsilon^p)$ $(p \geq q)$] admit infinitesimal operators of two types, namely, *zero-order* operators (referred to as essential operators in [9])

$$X = X_o + \varepsilon X_1 + \ldots + \varepsilon^p X_p + o(\varepsilon^p), \tag{0.2}$$

which are obtained by "inheriting" some (not necessarily all) symmetries X_o of the nonperturbed equation

$$F_o(z) = 0, \tag{0.3}$$

and k *th-order* operators

$$X = \varepsilon^k (X_o + \varepsilon X_1 + \ldots + \varepsilon^{p-k} X_{p-k}), \quad k = 1, \ldots, p. \tag{0.4}$$

As shown in [7], any operator (0.2) or (0.4) generates an approximate one-parameter group, and the composition of these one-parameter transformations determines a new object, namely, an approximate multi-parameter transformation group (see Definition 3.2).

The set of all operators X_o admitted by equation (0.3) is a Lie algebra with respect to the commutator. The zero-order operators (0.2) admitted approximately by equation (0.1) do not form a Lie algebra by themselves in general. Even the operators (0.2) and (0.4) are considered together, they still do not form a Lie algebra from the viewpoint of classical Lie algebras, but they do form a so-called *approximate Lie algebra.*

In this paper we study approximate multiparameter transformation groups (analytic with respect to the small parameter) and the related approximate Lie algebras. We also prove analogs of the first and second main Lie theorems on transformation groups. The third Lie theorem is not considered here.

We use the following notation. We write $\theta(z,\varepsilon) = o(\varepsilon^p)$ to indicate that $\theta(z,\varepsilon)$ can be represented in the form $\varepsilon^{p+1}\varphi(z,\varepsilon)$ where $\varphi(z,\varepsilon)$ is a series in nonnegative powers of ε. The notation $f \approx g$ is used if $f(z,\varepsilon) = g(z,\varepsilon) + o(\varepsilon^p)$ for some fixed p if the order p is not fixed, then we assume that the last equation holds for all p. We assume summation with respect to the repeating indices in expressions like $\xi^i_\alpha(z,\varepsilon)\frac{\partial}{\partial z^i}$. All considered functions are assumed to be as smooth as desired.

§ 1 Illustrative Examples

Here we show that if Eq. (0.1) approximately admits operators $X_1, \ldots X_k$ of the form (0.2) and (0.4), then the approximate relation

$$[X_i, X_j] = c^k_{ij} X_k + o(\varepsilon^p)$$

holds, where c^k_{ij} are constants (independent of ε). In this case we say that the operators $X_1, \ldots, X_k$ generate an approximate Lie algebra [with an accuracy of $o(\varepsilon^p)$.]

Example 3.1. The ordinary differential equation $y'' = f(y')$ with an arbitrary function f admits the two-dimension Lie algebra with the basis

$$X^o_1 = \frac{\partial}{\partial x}, \quad X^o_2 = \frac{\partial}{\partial y} \cdot$$

The equation

$$y'' = f(y') + \varepsilon[yy'^2 f'(y') - 3yy' f(y') - x f'(y')]$$

with small parameter ε admits the zero-order infinitesimal operators

$$X_1 = \frac{\partial}{\partial x} + \varepsilon x \frac{\partial}{\partial y} \quad \text{and} \quad X_2 = \frac{\partial}{\partial y} + \varepsilon y \frac{\partial}{\partial x}$$

with an accuracy of $o(\varepsilon)$. Their commutator $[X_1, X_2]$ is equal to

$$X_3 = \varepsilon^2 \left(x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right),$$

i.e., the operators X_1 and X_2 do not form a precise Lie algebra but form approximate Lie algebra with an accuracy of $o(\varepsilon)$.

Example 3.2. The operators

$$X_1 = \frac{\partial}{\partial x} + \varepsilon x \frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial y} + \varepsilon y \frac{\partial}{\partial x}, \quad X_3 = \varepsilon \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right),$$
$$X_4 = \varepsilon \frac{\partial}{\partial x}, \quad X_5 = \varepsilon \frac{\partial}{\partial y} \qquad (1.1)$$

do not form a Lie algebra, but form an approximate Lie algebra [with an accuracy of $o(\varepsilon)$]. Thus, for example,

$$[X_1, X_2] = \varepsilon^2 \left(x \frac{\partial}{\partial x} - y \frac{\partial}{\partial x} \right).$$

If we add the operators

$$X_6 = \varepsilon^2 \left(x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right), \quad X_7 = \varepsilon^2 \frac{\partial}{\partial x}, \quad X_8 = \varepsilon^2 \frac{\partial}{\partial y} \qquad (1.2)$$

to (1.1), we obtain an eight-dimensional approximate algebra with an accuracy of $o(\varepsilon^2)$. Considering the commutators of the operators (1.1) and (1.2) together, we see that additional operators are necessary to obtain an approximate Lie algebra of higher accuracy. For example, an approximate [with an accuracy of $o(\varepsilon^3)$] Lie algebra is generated by the operators (1.1), (1.2), and

$$X_9 = \varepsilon^3 x \frac{\partial}{\partial y}, \quad X_{10} = \varepsilon^3 y \frac{\partial}{\partial x}, \quad X_{11} = \varepsilon^3 \frac{\partial}{\partial x}, \quad X_{12} = \varepsilon^3 \frac{\partial}{\partial y} \cdot \qquad (1.3)$$

Remark 3.1. The operators (1.1) and (1.2) form a precise Lie algebra for $\varepsilon = 1$.

A one-parameter Lie group is associated with each approximate operator in (1.1)-(1.3). By analogy with the Lie theory, we shall construct approximate multiparameter groups as compositions of one-parameter subgroups. By solving the corresponding Lie equations for one-parameter approximate groups, the following formulas we obtain for the operators $X_1, \ldots, X_{12}$:

$$
\begin{aligned}
&X_1: \quad x' = x + a^1, \quad y' = y + \varepsilon x a^1 + \frac{1}{2}\varepsilon (a^1)^2;\\
&X_2: \quad x' = x + \varepsilon y a^2 + \frac{1}{2}\varepsilon (a^2)^2, \quad y' = y + a^2;\\
&X_3: \quad x' = x \exp(\varepsilon a^3), \quad y' = y \exp(\varepsilon a^3);\\
&X_4: \quad x' = x + \varepsilon a^4, \quad y' = y;\\
&X_5: \quad x' = x, \quad y' = y + \varepsilon a^5;\\
&X_6: \quad x' = x \exp(\varepsilon^2 a^6), \quad y' = y \exp(-\varepsilon^2 a^6);\\
&X_7: \quad x' = x + \varepsilon^2 a^7, \quad y' = y;\\
&X_8: \quad x' = x, \quad y' = y + \varepsilon^2 a^8;\\
&X_9: \quad x' = x, \quad y' = y + \varepsilon^3 a^9 x;\\
&X_{10}: \quad x' = x + \varepsilon^3 a^{10} y, \quad y' = y;\\
&X_{11}: \quad x' = x + \varepsilon^3 a^{11}, \quad y' = y;\\
&X_{12}: \quad x' = x, \quad y' = \varepsilon^3 a^{12}.
\end{aligned}
$$

The composition of these transformations considered with an accuracy of $o(\varepsilon^3)$ yields the approximate transformation

$$
\begin{aligned}
x' =& x + a^1 + \varepsilon[x a^3 + y a^2 + a^4 + \frac{1}{2}(a^2)^2 + a^1 a^3]\\
&+ \varepsilon^2 \left[x\left(\frac{1}{2}(a^3)^2 + a^6 + a^1 a^2\right) + y a^2 a^3 + a^7 + \frac{1}{2}(a^2)^2 a^3 + \frac{1}{2}a^1 (a^3)^2 + a^1 a^6 + \frac{1}{2}(a^1)^2 a^2\right]\\
&+ \varepsilon^3 \Bigg[x\left(\frac{1}{6}(a^3)^3 + a^3 a^6 + a^1 a^2 a^3\right) + y\left(\frac{1}{2}(a^3)^2 a^2 + a^2 a^6 + a^{10}\right) + a^{11} + a^4 a^6 + a^2 a^{10}\\
&+ \frac{1}{4}(a^2)^2 (a^3)^2 + \frac{1}{2}(a^2)^2 a^6 + \frac{1}{6}a^1 (a^3)^3 + a^1 a^3 a^6 + \frac{1}{2}(a^1)^2 a^2 a^3\Bigg] + \cdots,\\
y' =& y + a^2 + \varepsilon\left[x a^1 + y a^3 + a^5 + a^2 a^3 + \frac{1}{2}(a^1)^2\right] \qquad (1.4)\\
&+ \varepsilon^2 \left[x a^1 a^3 + y\left(\frac{1}{2}(a^3)^2 - a^6\right) + a^8 + \frac{1}{2}a^2 (a^3)^2 - a^2 a^6 + \frac{1}{2}(a^1)^2 a^3\right]\\
&+ \varepsilon^3 \Bigg[x\left(a^9 + \frac{1}{2}a^1 (a^3)^2 - a^1 a^6\right) + y\left(\frac{1}{6}(a^3)^3 - a^3 a^6\right) + a^{12} - a^5 a^6\\
&+ \frac{1}{6}a^2 (a^3)^3 - a^2 a^3 a^6 + \frac{1}{4}(a^1)^2 (a^3)^2 - \frac{1}{2}(a^1)^2 a^6 + a^1 a^9\Bigg] + \cdots
\end{aligned}
$$

depending on 12 parameters.

§ 2 Essential parameters of a family of functions depending on a small parameter

Consider a family

$$f^i(z, a, \varepsilon) \equiv f_0^i(z, a) + \varepsilon f_1^i(z, a) + \cdots + \varepsilon^p f_p^i(z, a) + o(\varepsilon^p) \tag{2.1}$$

of N functions depending on N variables $z = (z^1, ..., z^N)$, r parameters $a = (a^1, \ldots, a^r)$, and a small parameter ε.

Definition 3.1. The parameters $a^1, \ldots, a^r$ are said to be essential if one cannot select functions $A^1(a), \ldots, A^{r-1}(a)$ such that the following approximate equation holds:

$$f^i(z, a^1, \ldots, a^r, \varepsilon) \approx F^i(z, A^1(a), \ldots, A^{r-1}(a), \varepsilon). \tag{2.2}$$

Theorem 3.1. For all r parameters a to be essential in the family of functions (2.1) it is necessary and sufficient that the functions f^i satisfy no approximate equation of the form

$$\frac{\partial f^i}{\partial a^\alpha} \varphi^\alpha(a) \approx 0. \tag{2.3}$$

Proof. Suppose that not all parameters $a^1, \ldots, a^r$ are essential, i.e., that (2.2) is satisfied. Since the functions $A^1, \ldots, A^{r-1}$ depend only on the parameters $a^1, \ldots, a^r$ in (2.2), it follows that

$$\text{rang} \quad \left\| \frac{\partial A}{\partial a} \right\| \leq r - 1.$$

Consequently, there exists a system of functions $\varphi^\alpha(a)$ that do not vanish identically and satisfy the system

$$\frac{\partial A^\sigma}{\partial a^\alpha} \varphi^\alpha(a) = 0, \quad \sigma = 1, \ldots, r-1, \quad \alpha = 1, \ldots, r. \tag{2.4}$$

Then any function f of $A^1(a), \ldots, A^{r-1}(a)$ satisfies some equation of the form (2.4), namely,

$$\frac{\partial f}{\partial a^\alpha} \varphi^\alpha(a) = 0. \tag{2.5}$$

In particular, the functions $F^i(z, A^1(a), \ldots, A^{r-1}(a), \varepsilon)$ satisfy this equation; the variables z and ε can be considered as parameters in these functions (since φ^α do not depend on these variables). Therefore, by virtue of (2.2), the functions f^i satisfy the approximate equation (2.3), namely,

$$\frac{\partial}{\partial a^\alpha} (f_0(z, a) + \varepsilon f_1(z, a) + \cdots + \varepsilon^p f_p(z, a)) \varphi^\alpha(a) \approx 0.$$

Equating to zero the coefficients of ε^s $s=0,\ldots,p$, we obtain $p+1$ first-order equations with the same coefficients. Their general solution is

$$f_k = F_k(z, A^1(a), \ldots, A^{r-1}(a)), \quad k=0,\ldots,p,$$

where the functions $A(a)$ are the same for all f_k. Hence, Eq. (2.2) holds.

Example 3.3. Consider the functions

$$\begin{aligned} f_1 =& x + a^1 + \varepsilon\left[xa^3 + ya^2 + a^4 + \frac{1}{2}(a^2)^2 + a^1a^3\right] \\ &+ \varepsilon^2\left[x\left(\frac{1}{2}(a^3)^2 + a^6 + a^1a^2\right) + ya^2a^3 + a^7\right. \\ &\left.+\frac{1}{2}(a^2)^2a^3 + \frac{1}{2}a^1(a^3)^2 + a^1a^6 + \frac{1}{2}(a^1)^2a^2\right], \\ f^2 =& y + a^2 + \varepsilon\left[xa^1 + ya^3 + a^5 + a^2a^3 + \frac{1}{2}(a^1)^2\right] \\ &+ \varepsilon^2\left[xa^1a^3 + y\left(\frac{1}{2}(a^3)^2 - a^6\right) + a^8 + \frac{1}{2}a^2(a^3)^2 - a^2a^6 + \frac{1}{2}(a^1)^2a^3\right], \end{aligned} \tag{2.6}$$

which are the right-hand sides of approximate [with an accuracy of $o(\varepsilon^2)$] transformations (1.4) from Section § 1 and depend on the variables x and y and on the eight parameters $a^1,\ldots,a^8$. In this case the equations in (2.3) have the form

$$\begin{aligned} &\left[1 + \varepsilon a^3 + \varepsilon^2\left[xa^2 + \left(\frac{1}{2}(a^3)^2 + a^6 + a^1a^2\right)\right]\right]\varphi^1(a) \\ &+\left[\varepsilon(y+a^2) + \varepsilon^2\left(xa^1 + ya^3 + a^2a^3 + \frac{1}{2}(a^1)^2\right)\right]\varphi^2(a) \\ &+\left[\varepsilon(x+a^1) + \varepsilon^2\left(xa^3 + ya^2 + \frac{1}{2}(a^2)^2 + a^1a^3\right)\right]\varphi^3(a) \\ &+\varepsilon\varphi^4(a) + \varepsilon^2(x+a^1)\varphi^6(a) + \varepsilon^2\varphi^7(a) \approx 0, \end{aligned} \tag{2.7}$$

$$\begin{aligned} &\left[\varepsilon(x+a^1) + \varepsilon^2(xa^3 + a^1a^3)\right]\varphi^1(a) + \left[1 + \varepsilon a^3 + \varepsilon^2\left(\frac{1}{2}(a^3)^2 - a^6\right)\right]\varphi^2(a) \\ &+\left[\varepsilon(y+a^2) + \varepsilon^2\left(xa^1 + ya^3 + a^2a^3 + \frac{1}{2}(a^1)^2\right)\right]\varphi^3(a) \\ &+\varepsilon\varphi^5(a) + \varepsilon^2(y-a^2)\varphi^6(a) + \varepsilon^2\varphi^8(a) \approx 0. \end{aligned}$$

Equating to zero the coefficients of $\varepsilon^0, \varepsilon^1$ and ε^2 we obtain

$$\varepsilon^0: \quad \varphi^1(a) = 0, \quad \varphi^2(a) = 0,$$

$$\varepsilon^1 : (x+a^1)\varphi^3(a)+\varphi^4(a)=0, \quad (y+a^2)\varphi^3(a)+\varphi^5(a)=0.$$

By separating x and y we find that

$$\varphi^3(a)=\varphi^4(a)=\varphi^5(a)=0.$$

Furthermore, we have

$$\varphi^2 : \quad (x+a^1)\varphi^6(a)+\varphi^7(a)=0, \quad (y+a^2)\varphi^6(a)+\varphi^8(a)=0,$$

whence it follows that

$$\varphi^6(a)=\varphi^7(a)=\varphi^8(a)=0.$$

Thus all coefficients $\varphi^\alpha(a)$ in (2.7) vanish identically, and hence, according to Theorem 3.1, all parameters $a^1,\ldots,a^8$ of the function family (2.6) are essential.

§ 3 Approximate transformation groups. The first direct Lie theorem

Consider approximate (local) transformations

$$z'^1 = f^1(z,a,\varepsilon) \equiv f_0^1(z,a)+\varepsilon f_1^i(z,a)+\cdots+\varepsilon^p f_p^i(z,a)+o(\varepsilon^p) \tag{3.1}$$

$$i=1,\ldots,N,$$

depending on r essential parameters $a=(a^1,\ldots,a^r)$ and on a small parameter ε.

Definition 3.2. The transformations (3.1) form a (local) approximate (of order p) r-parameter group, if there exists a system of functions $\varphi^\alpha(a,b)$, $\alpha=1,\ldots,r$, such that

$$f^i(f(z,a,\varepsilon),b,\varepsilon)\approx f^i(z,\varphi(a,b),\varepsilon) \tag{3.2}$$

for sufficiently small a and b, and

$$f(z,0,\varepsilon)\approx z. \tag{3.3}$$

It follows from the definition that

$$\varphi(a,0)=a, \quad \varphi(0,b)=b,$$

$$\left.\frac{\partial\varphi^\alpha}{\partial a^\beta}\right|_{b=0}=\delta^\alpha_\beta, \quad \left.\frac{\partial\varphi^\alpha}{\partial b^\beta}\right|_{a=0}=\delta^\alpha_\beta.$$

Let us introduce the auxiliary functions

$$A^\alpha_\beta(a) = \frac{\partial \varphi^\alpha}{\partial b^\beta}\bigg|_{b=0} \quad (\alpha, \beta = 1, \ldots, r). \tag{3.4}$$

Then

$$A^\alpha_\beta(0) = \delta^\alpha_\beta, \tag{3.5}$$

and

$$\varphi^\alpha(a, b) = a^\alpha + A^\alpha_\beta(a) b^\beta + o(|b|) \quad \text{as} \quad |b| \to 0. \tag{3.6}$$

By condition (3.5), the matrix $A(a)$ is invertible, namely, there exists a matrix $V^\beta_\gamma(a)$ such that

$$A^\alpha_\beta(a) \cdot V^\beta_\gamma(a) = \delta^\alpha_\gamma. \tag{3.7}$$

Just as in the theory of precise local Lie transformation groups, the study of approximate Lie transformation groups is closely related to the consideration of the tangent vector fields $\xi_\alpha(z, \varepsilon)$ with coordinates

$$\xi^i_\alpha(z, \varepsilon) \approx \frac{\partial f^i(z, a, \varepsilon)}{\partial a^\alpha}\bigg|_{a=0} \quad i = 1, \ldots, N, \quad \alpha = 1, \ldots, r, \tag{3.8}$$

or, which is the same, of the differential operators

$$X_\alpha = \xi^i_\alpha(z, \varepsilon) \frac{\partial}{\partial z^i} \cdot \tag{3.9}$$

In this case, the following first direct Lie theorem holds for approximate transformation groups.

Theorem 3.2. If the transformations (3.1) form an approximate r-parameter Lie transformation group, then the functions $z'^i = f^i(z, a, \varepsilon)$ satisfy the system of approximate equations

$$\frac{\partial z'^i}{\partial a^\alpha} \approx \xi^i_\beta(z', \varepsilon) \cdot V^\beta_\alpha(a), \quad \alpha = 1, \ldots, r, \tag{3.10}$$

which are called the approximate Lie equations. The corresponding differential operators (3.9) are linearly independent in the considered approximation.

Proof. By differentiating (3.2) with respect to b^β at the point $b = 0$, we obtain

$$\frac{\partial f^i(z', b, \varepsilon)}{\partial b^\beta}\bigg|_{b=0} \approx \frac{\partial f^i(z, \varphi, \varepsilon)}{\partial \varphi^\alpha} \frac{\partial \varphi^\alpha}{\partial b^\beta}\bigg|_{b=0}.$$

Hence, taking into account (3.8) and (3.4), we have

$$\xi^i_\beta(z', \varepsilon) \approx \frac{\partial z'^i}{\partial a^\alpha} A^\alpha_\beta(a). \tag{3.11}$$

Multiplying (3.11) by $V^\beta_\gamma(a)$ and taking into account (3.7), we obtain the system of approximate differential equations (3.10).

It remains to check that the differential operators (3.9) are linearly independent, namely, that if

$$C^\alpha X_\alpha \equiv C^\alpha \xi^i_\alpha(z, \varepsilon) \frac{\partial}{\partial z^i} \approx 0 \tag{3.12}$$

with constants $C^1, \ldots, C^r$ independent of ε, then all these constants are zero. Substituting (3.11) into (3.12), we obtain

$$C^\alpha X_\alpha \equiv C^\alpha \xi^i_\alpha(z', \varepsilon) \frac{\partial}{\partial z'^i} \approx C^\alpha \left[\frac{\partial z'^i}{\partial a^\beta} A^\beta_\alpha(a)\right] \frac{\partial}{\partial z'^i} \approx$$

$$\frac{\partial z'^i}{\partial a^\beta} [C^\alpha A^\beta_\alpha(a)] \frac{\partial}{\partial z'^i} \cdot \tag{3.13}$$

Therefore, Eq. (3.12) takes the form

$$\frac{\partial z'^i}{\partial a^\beta} C^\alpha A^\beta_\alpha(a) = 0, \quad i = 1, \ldots, N. \tag{3.14}$$

Since all parameters $a^1, \ldots, a^r$ in (3.1) are essential, it follows from (3.14), by virtue of Theorem 3.1. that $C^\alpha A^\beta_\alpha(a) = 0$ for all β. Hence, taking into account (3.5), we have $C^1 = \ldots = C^r = 0$. Thus the theorem is proved.

Remark 3.2. The approximate linear independence of the differential operators (3.9) is equivalent, to the approximate linear independence of the vector-valued functions $\xi_\alpha(z, \varepsilon) = (\xi^1_\alpha(z, \varepsilon), \ldots, \xi^N_\alpha(z, \varepsilon))$.

Example 3.4. Let us consider the approximate [with the accuracy $o(\varepsilon)$] transformations

$$x' = x + a^1 + \varepsilon \left[xa^3 + ya^2 + a^4 + \frac{1}{2}(a^2)^2 + a^1 a^3\right] + \ldots,$$

$$y' = y + a^2 + \varepsilon \left[xa^1 + ya^3 + a^5 + a^2 a^3 + \frac{1}{2}(a^1)^2\right] + \ldots \tag{3.15}$$

depending on five essential parameters $a^1, \ldots, a^5$ (see Example § 3). It is easy to show that these transformations form a local approximate transformation group, where

$$\varphi^1(a, b) = a^1 + b^1; \quad \varphi^2(a, b) = a^2 + b^2; \quad \varphi^3(a, b) = a^3 + b^3;$$

$$\varphi^4(a,b) = a^4 + b^4 + a^3 b^1; \quad \varphi^5(a,b) = a^5 + b^5 - a^3 b^2.$$

The auxiliary functions A_α^β and $V_\beta^\gamma(a)$ are given by the matrices

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -a^3 & 0 & 0 & 1 & 0 \\ 0 & -a^3 & 0 & 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ a^3 & 0 & 0 & 1 & 0 \\ 0 & a^3 & 0 & 0 & 1 \end{pmatrix}$$

and the vector field (3.8) by

$$\xi(z,\varepsilon) = \begin{pmatrix} 1 & \varepsilon y & \varepsilon x & \varepsilon & 0 \\ \varepsilon x & 1 & \varepsilon y & 0 & \varepsilon \end{pmatrix} + o(\varepsilon).$$

Therefore, in this case the approximate Lie equations in (3.10) read

$$\frac{\partial x'}{\partial a^1} = 1 + \varepsilon a^3, \quad \frac{\partial x'}{\partial a^2} = \varepsilon y', \quad \frac{\partial x'}{\partial a^3} = \varepsilon x', \quad \frac{\partial x'}{\partial a^4} = \varepsilon, \quad \frac{\partial x'}{\partial a^5} = 0; \qquad (3.16)$$

$$\frac{\partial y'}{\partial a^1} = \varepsilon x', \quad \frac{\partial y'}{\partial a^2} = 1 + \varepsilon a^3, \quad \frac{\partial y'}{\partial a^3} = \varepsilon y', \quad \frac{\partial y'}{\partial a^4} = 0, \quad \frac{\partial y'}{\partial a^5} = \varepsilon.$$

§ 4 The inverse first Lie theorem

Let us first prove some statements on solutions to systems of approximate first-order partial differential equations and next use them to analyze Lie equations.

Consider the system of approximate equations

$$\frac{\partial z^i}{\partial a^\alpha} \approx \psi_\alpha^i(z,a,\varepsilon), \quad i = 1,\ldots,N; \quad \alpha = 1,\ldots,r. \qquad (4.1)$$

[From now on we consider approximate equalities with an accuracy of $o(\varepsilon^p)$, $p \geq 1$. We seek the approximate [with an accuracy of $o(\varepsilon^p)$] solution to system (4.1) in the form

$$z^i = f_0^i(a) + \varepsilon f_1^i(a) + \ldots + \varepsilon^p f_p^i(a) + o(\varepsilon^p). \qquad (4.2)$$

For this purpose, we expand the functions $\psi_\alpha^i(z,a,\varepsilon)$ in powers of ε :

$$\psi_\alpha^i(z,a,\varepsilon) = \psi_{\alpha,0}^i(z,a) + \varepsilon\psi_{\alpha,1}^i(z,a) + \ldots + \varepsilon^p\psi_{\alpha,p}^i(z,a) + o(\varepsilon^p). \qquad (4.3)$$

Substituting (4.2) and (4.3) into (4.1) and equating the coefficients of the same powers of ε, we obtain the following system of equations for $f_0^i(a), \ldots, f_p^i(a)$, $i = 1, \ldots, N$, which is equivalent to (4.1) with an accuracy of $o(\varepsilon^p)$:

$$\frac{\partial f_0^i}{\partial a^\alpha} = \psi_{\alpha,0}^i(f_0, a), \tag{4.4}$$

$$\frac{\partial f_1^i}{\partial a^\alpha} = \psi_{\alpha,1}^i(f_0, a) + \sum_{j=1}^{N} \frac{\partial \psi_{\alpha,0}^i(f_0, a)}{\partial f_0^j} f_1^j,$$

....

$$\frac{\partial f_p^i}{\partial a^\alpha} = \psi_{\alpha,p}^i(f_0, a) + \sum_{\nu=1}^{p} \sum_{|\sigma|=1}^{\nu} \frac{1}{\sigma!} \frac{\partial^{|\sigma|} \psi_{\alpha,p-\nu}^i(f_0, a)}{(\partial f_0^1)^{\sigma_1} \cdots (\partial f_0^N)^{\sigma_N}} \sum_{|\omega|=\nu} f(\omega).$$

Here we use the following notation: $\sigma = (\sigma_1, \ldots, \sigma_N)$ is a multiindex, $|\sigma| = \sigma_1 + \ldots + \sigma_N$, $\sigma! = \sigma_1! \ldots \sigma_N!$, and the indices $\sigma_1, \ldots, \sigma_N$ run from 0 to p. Furthermore,

$$f_{(\omega)} = f_{(\omega_1)}^1 \cdots f_{(\omega_N)}^N,$$

$$f_{(\omega_k)}^k = \sum_{i_1+\ldots+i_{\sigma_k}=\omega_k} f_{i_1}^k \cdots f_{i_{\sigma_k}}^k,$$

where the indices $i_1, \ldots, i_{\sigma_k}$ run from 1 to p and $\omega = \omega(\sigma) = (\omega_1, \ldots, \omega_N)$ is the multiindex related to σ as follows: if the index σ_s is zero in σ, then the corresponding index ω_s is absent in ω, and the remaining indices ω_k run from σ_k to p; for example, for $\sigma = (0, \sigma_2, \sigma_3, 0, \ldots)$ with $\sigma_2, \sigma_3 \neq 0$, we have $\nu = (\nu_2, \nu_3)$, so that $f_{(\omega)} = f_{(\omega_2)}^2 f_{(\omega_3)}^3$.

Definition 3.3. The system of approximate equations (4.1) is said to be completely integrable if the approximate equations

$$\frac{\partial}{\partial a^\beta}\left(\frac{\partial z^i}{\partial a^\alpha}\right) \approx \frac{\partial}{\partial a^\alpha}\left(\frac{\partial z^i}{\partial a^\beta}\right) \tag{4.5}$$

hold by virtue of equations (4.1).

Let us show that system (4.4) obtained from a completely integrable system of approximate equations (4.1) is completely integrable. To this end, let us rewrite (4.5) grouping together the coefficients of the same powers of ε. Taking into account (4.1) and (4.3), we obtain

$$\frac{\partial}{\partial a^\beta}\left(\frac{\partial z^i}{\partial a^\alpha}\right) - \frac{\partial}{\partial a^\alpha}\left(\frac{\partial z^i}{\partial a^\beta}\right) \approx \frac{\partial}{\partial a^\beta}(\psi_{\alpha,0}^i(z, a) + \varepsilon \psi_{\alpha,1}^i(z, a) + \cdots + \varepsilon^p \psi_{\alpha,p}^i(z, a)) -$$

$$\frac{\partial}{\partial a^\beta}(\psi^i_{\beta,0}(z,a)+\varepsilon\psi^i_{\beta,1}(z,a)+\cdots+\varepsilon^p\psi^i_{\beta,p}(z,a))\approx$$

$$\Bigg\{\frac{\partial\psi^i_{\alpha,0}}{\partial a^\beta}+\frac{\partial\psi^i_{\alpha,0}(z,a)}{\partial z^j}[\psi^j_{\beta,0}+\varepsilon\psi^j_{\beta,1}+\cdots+\varepsilon^p\psi^j_{\beta,p}]+$$

$$\varepsilon\left[\frac{\partial\psi^i_{\alpha,1}}{\partial a^\beta}+\sum_{j=1}^N\frac{\partial\psi^i_{\alpha,1}}{\partial z^j}[\psi^j_{\beta,0}+\varepsilon\psi^j_{\beta,1}+\cdots+\varepsilon^{p-1}\psi^j_{\beta,p-1}]\right]+\cdots+$$

$$\varepsilon^p\left[\frac{\partial\psi^i_{\alpha,p}}{\partial a^\beta}+\sum_{j=1}^N\frac{\partial\psi^i_{\alpha,p}}{\partial z^j}\psi^j_{\beta,0}\right]\Bigg\}-\Bigg\{\frac{\partial\psi^i_{\beta,0}}{\partial a^\alpha}+\sum_{j=1}^N\frac{\partial\psi^i_{\beta,0}}{\partial z^j}[\psi^j_{\alpha,0}+\varepsilon\psi^j_{\alpha,1}+\cdots+\varepsilon^p\psi^j_{\alpha,p}]+$$

$$\varepsilon\left[\frac{\partial\psi^i_{\beta,1}}{\partial a^\alpha}+\sum_{j=1}^N\frac{\partial\psi^i_{\beta,1}}{\partial z^j}[\psi^j_{\alpha,0}+\varepsilon\psi^j_{\alpha,1}+\cdots+\varepsilon^{p-1}\psi^j_{\alpha,p-1}]\right]+\cdots+$$

$$\varepsilon^p\left[\frac{\partial\psi^i_{\beta,p}}{\partial a^\alpha}+\sum_{j=1}^N\frac{\partial\psi^i_{\beta,p}}{\partial z^j}\psi^j_{\alpha,0}\right]\Bigg\},$$

i.e., the functions $\psi^i_{\alpha,\nu}(z,a)$ satisfy the system

$$\frac{\partial\psi^i_{\alpha,0}}{\partial a^\beta}-\frac{\partial\psi^i_{\beta,0}}{\partial a^\alpha}+\sum_{j=1}^N\left[\frac{\partial\psi^i_{\alpha,0}}{\partial z^j}\psi^j_{\beta,0}-\frac{\partial\psi^i_{\beta,0}}{\partial z^j}\psi^j_{\alpha,0}\right]=0, \tag{4.6}$$

$$\frac{\partial\psi^i_{\alpha,1}}{\partial a^\beta}-\frac{\partial\psi^i_{\beta,1}}{\partial a^\alpha}+\sum_{j=1}^N\left[\frac{\partial\psi^i_{\alpha,0}}{\partial z^j}\psi^j_{\beta,1}+\frac{\partial\psi^i_{\alpha,1}}{\partial z^j}\psi^j_{\beta,0}-\frac{\partial\psi^i_{\beta,0}}{\partial z^j}\psi^j_{\alpha,1}-\frac{\partial\psi^i_{\beta,1}}{\partial z^j}\psi^j_{\alpha,0}\right]=0,\ldots.$$

Let us check the integrability condition for system (4.4). The equations for $f^i_0(a)$ yield

$$\frac{\partial}{\partial a^\beta}\left(\frac{\partial f^i_0}{\partial a^\alpha}\right)-\frac{\partial}{\partial a^\alpha}\left(\frac{\partial f^i_0}{\partial a^\beta}\right)=\frac{\partial}{\partial a^\beta}(\psi^i_{\alpha,0}(f_0,a))-\frac{\partial}{\partial a^\alpha}(\psi^i_{\beta,0}(f_0,a))=$$

$$\frac{\partial\psi^i_{\alpha,0}}{\partial a^\beta}-\frac{\partial\psi^i_{\beta,0}}{\partial a^\alpha}+\sum_{j=1}^N\left[\frac{\partial\psi^i_{\alpha,0}}{\partial f^j}\psi^j_{\beta,0}-\frac{\partial\psi^i_{\beta,0}}{\partial f^j_0}\psi^j_{\alpha,0}\right],$$

i.e., the integrability condition holds by virtue of (4.6). As to the equations for $f^i_1(a)$, we have

$$\frac{\partial}{\partial a^\beta}\left(\frac{\partial f^i_1}{\partial a^\alpha}\right)-\frac{\partial}{\partial a^\alpha}\left(\frac{\partial f^i_1}{\partial a^\beta}\right)=\frac{\partial}{\partial a^\beta}\left[\psi^i_{\alpha,1}(f_0,a)+\sum_{j=1}^N\frac{\partial\psi^i_{\alpha,0}(f_0,a)}{\partial f^j_0}f^j_1\right]-$$

$$\frac{\partial}{\partial a^\alpha}\left[\psi^i_{\beta,1}(f_0,a)+\sum_{j=1}^N \frac{\partial \psi^i_{\beta,0}(f_0,a)}{\partial f_0^j} f_1^j\right] =$$

$$\frac{\partial \psi^i_{\alpha,1}}{\partial a^\beta}+\sum_{j=1}^N \frac{\partial \psi^i_{\alpha,1}}{\partial f_0^j}\psi^j_{\beta,0}+\sum_{j=1}^N \frac{\partial^2 \psi^i_{\alpha,0}}{\partial f_0^j \partial a^\beta} f_1^j+\sum_{j=1}^N\sum_{k=1}^N \frac{\partial^2 \psi^i_{\alpha,0}}{\partial f_0^j \partial f_0^k}\psi^k_{\beta,0} f_1^j+$$

$$\sum_{j=1}^N \frac{\partial \psi^i_{\alpha,0}}{\partial f_0^j}\left[\psi^j_{\beta,1}+\sum_{k=1}^N \frac{\partial \psi^i_{\beta,0}}{\partial f_0^k}\right]-\frac{\partial \psi^i_{\beta,1}}{\partial a^\alpha}-\sum_{j=1}^N \frac{\partial \psi^i_{\beta,1}}{\partial f_0^j}\psi^j_{\alpha,0}-\sum_{j=1}^N \frac{\partial^2 \psi^i_{\beta,0}}{\partial f_0^j \partial a^\alpha} f_1^j -$$

$$\sum_{j=1}^N\sum_{k=1}^N \frac{\partial^2 \psi^i_{\beta,0}}{\partial f_0^j \partial f_0^k}\psi^k_{\alpha,0} f_1^j-\sum_{j=1}^N \frac{\partial \psi^i_{\beta,0}}{\partial f_0^j}\left[\psi^j_{\alpha,1}+\sum_{k=1}^N \frac{\partial \psi^i_{\alpha,0}}{\partial f_0^k} f_1^k\right] =$$

$$\frac{\partial \psi^i_{\alpha,1}}{\partial a^\beta}-\frac{\partial \psi^i_{\beta,1}}{\partial a^\alpha}+\sum_{j=1}^N\left[\frac{\partial \psi^i_{\alpha,0}}{\partial f_0^j}\psi^j_{\beta,1}+\frac{\partial \psi^i_{\alpha,1}}{\partial f_0^j}\psi^j_{\beta,0}-\frac{\partial \psi^i_{\beta,0}}{\partial f_0^j}\psi^j_{\alpha,1}-\frac{\partial \psi^i_{\beta,1}}{\partial f_0^j}\psi^j_{\alpha,0}\right]+$$

$$\sum_{j=1}^N \frac{\partial}{\partial f_0^j}\left[\frac{\partial \psi^i_{\alpha,0}}{\partial a^\beta}-\frac{\partial \psi^i_{\beta,0}}{\partial a^\alpha}+\sum_{k=1}^N\left[\frac{\partial \psi^i_{\alpha,0}}{\partial f_0^k}\psi^k_{\beta,0}-\frac{\partial \psi^i_{\beta,0}}{\partial f_0^k}\psi^k_{\alpha,0}\right]\right] f_1^j = 0$$

by virtue of (4.6).

The integrability condition for $f^i_\nu(a)$, $\nu = 2,\dots,p$, can be checked in a similar way.

Thus, system (4.4) is completely integrable. Hence, if the right-hand sides in the equations in (4.4) are continuously differentiable with respect to all variables, then this system has a unique solution for any initial data (e.g., [134], Vol. 2, Chap. VIII, Secs. 5 and 6 or [117], Chap. X, Sec. 55). Thus the following'statement holds.

Lemma 3.1. A completely integrable system of approximate equations (4.1) [i.e., system (4.1) satisfying the conditions (4.5)] has a unique solution of the form (4.2) for arbitrary initial data

$$z^i\Big|_{a=0} \approx z^i_0.$$

Multiplying the equations in (4.1) by da^α and summing with respect to α, we obtain the approximate Pfaff equations

$$dz^i \approx \psi^i_\alpha(z,a,\varepsilon)da^\alpha. \tag{4.7}$$

Lemma 3.2. A function F of the variables $(z^i, a^\alpha, \varepsilon)$ is constant on the solutions to the system, of approximate Pfaff equations (4.7) (i.e., is an integral of this system) if and only if it satisfies the system, of partial differential equations

$$\frac{\partial F}{\partial z^i}\varphi^i_\alpha + \frac{\partial F}{\partial a^\alpha} \approx 0, \tag{4.8}$$

Proof. *Necessity.* Let $F(z, a, \varepsilon)$ be constant on the solutions to system (4.7). Then its total differential is identically zero, that is,

$$0 = dF = \frac{\partial F}{\partial z^i}dz^i + \frac{\partial F}{\partial a^\alpha}da^\alpha \approx \frac{\partial F}{\partial z^i}\varphi^i_\alpha da^\alpha + \frac{\partial F}{\partial a^\alpha}da^\alpha = \left[\frac{\partial F}{\partial z^i}\varphi^i_\alpha + \frac{\partial F}{\partial a^\alpha}\right] da^\alpha.$$

Thus F satisfies (4.8) on the solutions to (4.7).

Let us now show that the equations in (4.8) hold identically with respect to z and a. Let

$$\frac{\partial F}{\partial z^i}\varphi^i_\alpha + \frac{\partial F}{\partial a^\alpha} = \tau_\alpha(z, a),$$

where $\tau_\alpha(z, a) = 0$ on the solutions to (4.7). Then any function $z = z(a)$ that satisfies the equation

$$\tau_\alpha(z, a) \approx C_\alpha \neq 0, \quad C_\alpha = \text{const.},$$

is also a solution to (4.7). But the differential dF of F has the form

$$dF \approx C_\alpha da^\alpha \neq 0$$

on this solution, which contradicts the condition that F is constant on the solutions to (4.7). Hence the equations in (4.8) are satisfied for any z and a.

Sufficiency. Let $F = F(z, a, \varepsilon)$ be a solution to (4.8). Then

$$dF = \frac{\partial F}{\partial a^\alpha}da^\alpha + \frac{\partial F}{\partial z^i}dz^i.$$

By virtue of (4.8), we have

$$\frac{\partial F}{\partial a^\alpha} \approx -\frac{\partial F}{\partial z^i}\varphi^i_\alpha,$$

and consequently,

$$dF = \frac{\partial F}{\partial z^i}(dz^i - \varphi^i_\alpha da^\alpha).$$

It follows that $dF \equiv 0$ if $z = z(a)$ satisfies system (4.7). Thus, Lemma 3.2 is proved. In what follows we consider the approximate Lie equations

$$\frac{\partial z'^i}{\partial a^\alpha} \approx \xi^i_\beta(z', \varepsilon) \cdot V^\beta_\alpha(a) \quad (i = 1, \ldots, N; \alpha = 1, \ldots, r) \tag{4.9}$$

satisfying the complete integrability conditions (4.5). These conditions can be rewritten for system (4.9) as

$$\frac{\partial \xi^i_\gamma}{\partial z'^j}\frac{\partial z'^j}{\partial a^\beta}V^\beta_\alpha(a)+\xi^i_\gamma\frac{\partial V^\gamma_\alpha}{\partial a^\beta}\approx\frac{\partial \xi'^i_\sigma}{\partial z'^k}\frac{\partial z'^k}{\partial a^\alpha}V^\sigma_\beta(a)+\xi^i_\sigma\frac{\partial V^\sigma_\beta}{\partial a^\alpha}$$

or, substituting the derivatives from (4.9), as

$$\frac{\partial \xi^i_\gamma}{\partial z'^j}\xi^j_\mu V^\mu_\beta V^\gamma_\alpha+\xi^i_\gamma\frac{\partial V^\gamma_\alpha}{\partial a^\beta}\approx\frac{\partial \xi^i_\sigma}{\partial z'^k}\xi^k_\nu V^\nu_\alpha V^\sigma_\beta+\xi^i_\sigma\frac{\partial V^\sigma_\beta}{\partial a^\alpha}\cdot$$

Hence

$$\left[\frac{\partial \xi'^i_\gamma}{\partial z'^j}\xi^j_\mu-\frac{\partial \xi^i_\mu}{\partial z'^j}\xi^j_\gamma\right]V^\mu_\beta V^\gamma_\alpha\approx\xi^i_\sigma\left(\frac{\partial V^\sigma_\beta}{\partial a^\alpha}-\frac{\partial V^\sigma_\alpha}{\partial a^\beta}\right). \tag{4.10}$$

Multiplying the latter relationship by $A^\alpha_q A^\beta_p$ and taking into account (3.7), we obtain

$$\left[\frac{\partial \xi'^i_\gamma}{\partial z'^j}\xi^j_\mu-\frac{\partial \xi^i_\mu}{\partial z'^j}\xi^j_\gamma\right]=\xi^i_\sigma\left(\frac{\partial V^\sigma_\beta}{\partial a^\alpha}-\frac{\partial V^\sigma_\alpha}{\partial a^\beta}\right)A^\alpha_\gamma A^\beta_\mu. \tag{4.11}$$

Denote

$$c^\sigma_{\gamma\mu}=-\left(\frac{\partial V^\sigma_\beta}{\partial a^\alpha}-\frac{\partial V^\sigma_\alpha}{\partial a^\beta}\right)A^\alpha_\gamma A^\beta_\mu. \tag{4.12}$$

Then the differentiation of both sides in (4.11) with respect to a yields

$$\xi^i_\sigma\frac{\partial c^\sigma_{\gamma\mu}}{\partial a^\nu}=0.$$

Since the vectors ξ_σ are linearly independent, it follows that

$$\frac{\partial c^\sigma_{\gamma\mu}}{\partial a^\nu}=0.$$

Therefore the $c^\sigma_{\gamma\mu}$ defined in (4.12) are constants.

In view of (3.7), Eq. (4.12) can be rewritten as

$$\frac{\partial V^\sigma_\beta}{\partial a^\alpha}-\frac{\partial V^\sigma_\alpha}{\partial a^\beta}=c^\sigma_{\gamma\mu}V^\mu_\beta V^\gamma_\alpha. \tag{4.13}$$

Thus we have proved the following statement.

Lemma 3.3. If the system of approximate Lie equations (4.9) is completely integrable, then the functions $V^\beta_\alpha(a)$ satisfy Eq. (4.13) with constant coefficients $c^\sigma_{\gamma\mu}$ defined in (4.12).

Equations (4.13) are known as the Maurer-Cartan equations. They can be rewritten for the functions $A^\beta_\alpha(a)$. By differentiating the relation $A^\alpha_\beta V^\beta_\gamma = \delta^\alpha_\gamma$ (3.7) with respect to a, we obtain

$$\frac{\partial A^\alpha_\beta}{\partial a^\sigma} V^\beta_\gamma = -A^\alpha_\beta \frac{\partial V^\beta_\gamma}{\partial a^\sigma} \cdot$$

Substituting these relations into the equations

$$c^\sigma_{\gamma\mu} A^\nu_\sigma = A^\nu_\sigma \left(\frac{\partial V^\sigma_\beta}{\partial a^\alpha} - \frac{\partial V^\sigma_\alpha}{\partial a^\beta} \right) A^\alpha_\gamma A^\beta_\mu,$$

which follow from (4.12), we obtain

$$c^\sigma_{\gamma\mu} A^\nu_\sigma = -\left(\frac{\partial A^\nu_l}{\partial a^\alpha} V^l_\beta - \frac{\partial A^\nu_l}{\partial a^\beta} V^l_\alpha \right) A^\alpha_\gamma A^\beta_\mu.$$

Hence, using (3.7), we obtain the Maurer-Cartan equations in the form

$$-c^\sigma_{\gamma\mu} A^\nu_\sigma = \left(\frac{\partial A^\nu_\mu}{\partial a^\alpha} A^\alpha_\gamma - \frac{\partial A^\nu_\gamma}{\partial a^\alpha} A^\alpha_\mu \right). \tag{4.14}$$

By Lemma 3.2, we can proceed from (4.9) to the equivalent system of linear homogeneous equations

$$\frac{\partial F}{\partial z'^i} \xi^i_\beta(z', \varepsilon) V^\beta_\alpha(a) + \frac{\partial F}{\partial a^\alpha} \approx 0, \quad \alpha = 1, \ldots, r. \tag{4.15}$$

Definition 3.4. The system of approximate linear homogeneous equations

$$X_\mu F \equiv \nu^i_\mu(z, \varepsilon) \frac{\partial}{\partial z^i} F \approx 0, \quad \mu = 1, \ldots, n,$$

is said to be complete, if

$$[X_\mu, X_\nu] \approx c^\sigma_{\mu\nu} X_\sigma,$$

i.e., if the commutator of any two operators X_μ can be approximately expressed as a linear combination of the operators $X_1, \ldots, X_n$. A complete system of approximate linear homogeneous equations is said to be Jacobian if the commutator of any two operators X_μ is approximately zero.

Let us show that system (4.15) is complete and even Jacobian. To this end, let us calculate the commutator of the operators

$$B_\alpha = \frac{\partial}{\partial a^\alpha} + \xi^i_\beta V^\beta_\alpha \frac{\partial}{\partial z'^i} \cdot$$

We have

$$B_\alpha B_\beta - B_\beta B_\alpha = \left(\frac{\partial}{\partial a^\alpha} + \xi^i_\gamma V^\gamma_\alpha \frac{\partial}{\partial z'^i}\right)\left(\frac{\partial}{\partial a^\beta} + \xi^j_\sigma V^\sigma_\beta \frac{\partial}{\partial z'^i}\right) -$$

$$\left(\frac{\partial}{\partial a^\beta} + \xi^j_\sigma V^\gamma_\beta \frac{\partial}{\partial z'^i}\right)\left(\frac{\partial}{\partial a^\alpha} + \xi^i_\gamma V^\gamma_\alpha \frac{\partial}{\partial z'^i}\right) =$$

$$\xi^j_\sigma \frac{\partial V^\sigma_\beta}{\partial a^\alpha}\frac{\partial}{\partial z'^j} + \xi^i_\gamma \frac{\partial \xi^j_\sigma}{\partial z'^i} V^\gamma_\alpha V^\sigma_\beta \frac{\partial}{\partial z'^j} - \xi^i_\gamma \frac{\partial V^\gamma_\alpha}{\partial a^\beta}\frac{\partial}{\partial z'^i} - \xi^j_\sigma \frac{\partial \xi^i_\gamma}{\partial z'^j} V^\sigma_\beta V^\gamma_\alpha \frac{\partial}{\partial z'^i} =$$

$$\left[\left(\xi^i_\gamma \frac{\partial \xi^i_\sigma}{\partial z'^j} - \xi^j_\sigma \frac{\partial \xi^i_\gamma}{\partial z'^j}\right) V^\gamma_\alpha V^\sigma_\beta - \xi^i_\gamma \left(\frac{\partial V^\gamma_\alpha}{\partial a^\beta} - \frac{\partial V^\gamma_\beta}{\partial a^\alpha}\right)\right]\frac{\partial}{\partial z'^i}\,.$$

The expression in the square brackets is zero by (4.10). Hence system (4.15) is Jacobian.

Let us construct the solution to system (4.9) under the initial conditions

$$z'^i\big|_{a=0} \approx z^i. \tag{4.16}$$

By Lemma 3.2, solving this problem is equivalent to finding the solution to (4.15) satisfying an analog of the initial data (4.16). Namely, if the solution to (4.15) has the form

$$F = F(z', a, \varepsilon),$$

then, by Lemma 3.2, this function is constant on the solutions to system (4.9), i.e.,

$$F = F(z', a, \varepsilon) \approx C.$$

The initial data in (4.16) yields

$$F(z', a, \varepsilon) \approx F(z, 0, \varepsilon). \tag{4.17}$$

Lemma 3.4. The system of approximate equations (4.15) with the initial data (4.17) has N functionally independent solutions, which determine a solution to problem (4.9), (4.16) of the form

$$z' = f_0(z, a) + \varepsilon f_1(z, a) + \cdots + \varepsilon^p f_p(z, a) + o(\varepsilon^p), \tag{4.18}$$

which depends on at most r parameters $a^1, \ldots, a^r$.

Proof. We seek the solution to (4.15) as the series

$$F(z', a, \varepsilon) = F_0(z', a) + \varepsilon F_1(z', a) + \cdots + \varepsilon^p F_p(z', a) + \cdots \tag{4.19}$$

in powers of ε. Let us substitute this expression and the representation

$$\xi^i_\beta(z',\varepsilon) = \xi^i_{\beta,0}(z') + \varepsilon\xi^i_{\beta,1}(z') + \cdots + \varepsilon^p\xi^i_{\beta,p}(z') + \cdots$$

into Eq. (4.15) and equate to zero the coefficients of the powers of ε. This results in the system of equations

$$\frac{\partial F_0}{\partial a^\alpha} + \frac{\partial F_0}{\partial z'^i}\xi^i_{\beta,0}(z')V^\beta_\alpha(a) = 0, \tag{4.20}$$

$$\frac{\partial F_1}{\partial a^\alpha} + \frac{\partial F_1}{\partial z'^i}\xi^i_{\beta,0}V^\beta_\alpha(a) = -\frac{\partial F_0}{\partial z'^i}\xi^i_{\beta,1}V^\beta_\alpha, \tag{4.21}$$

$$\frac{\partial F_2}{\partial a^\alpha} + \frac{\partial F_2}{\partial z'^i}\xi^i_{\beta,0}(z')V^\beta_\alpha = -\left[\frac{\partial F_0}{\partial z'^i}\xi^i_{\beta,2} + \frac{\partial F_1}{\partial z'^i}\xi^i_{\beta,1}\right]V^\beta_\alpha, \tag{4.22}$$

....

System (4.20) is Jacobian and has N functionally independent solutions (see Chebotarev [21], p. 71), namely,

$$F^1_0 = F^1_0(z',a), \ldots, \quad F^N_0 = F^N_0(z',a),$$

and any other solution is a function of these solutions. By virtue of Lemma 3.2 and of the initial data (4.16), for $\varepsilon = 0$ the solutions to (4.9) have the form

$$\begin{gathered} F^1_0(z',a) = F^1_0(z,0), \\ \cdots\cdots\cdots\cdots \\ F^N_0(z',a) = F^N_0(z,0). \end{gathered} \tag{4.23}$$

Since the determinant $|(\partial F^i_0)/(\partial z'^j)|$ is not zero [which follows from the form of system (4.20)], we see that the equations in (4.23) can be solved for z and we can assume that

$$F^i_0(z',a) = z^i, \quad i = 1,\ldots,N. \tag{4.24}$$

Moreover, the initial condition (4.17)

$$F(z',a,\varepsilon) = F(z,0,\varepsilon)$$

can also be solved for z and we can assume that

$$F(z',a,\varepsilon) \approx z. \tag{4.25}$$

The inversion of (4.24) yields the transformations

$$z'^i = f^i_0(z,a), \quad i = 1,\ldots,N,$$

which form a precise group with the number of essential parameters, equal to the number of zero-order operators.

Substituting each of the obtained functions $F_0^i(z', a)$ into (4.21), we obtain linear nonhomogeneous equations for $F_1^i(z', a)$; the corresponding homogeneous equations have the form (4.20). If $F_1^{i*}(z', a)$ is a particular solution to the nonhomogeneous equation, then the general solution has the form

$$F_1^i(z', a) = F_1^{i*}(z', a) + \varphi^i(F_0^1, \dots, F_0^N). \tag{4.26}$$

Substituting (4.26) into (4.19), we obtain an approximate [with the accuracy of $o(\varepsilon)$] solution to system (4.15):

$$F^i(z', a, \varepsilon) = F_0^i(z', a) + \varepsilon[F_1^{i*}(z', a) + \varphi^i(F_0^1(z', a), \dots, F_0^N(z', a))] + \cdots$$

By virtue of Lemma 3.2 and of the initial data (4.24), the corresponding solutions to system (4.9) can be determined from the relations

$$F_0^i(z', a) + \varepsilon[F_1^{i*}(z', a) + \varphi^i(F_0^1(z', a), \dots, F_0^N(z', a))] \approx z^i. \tag{4.27}$$

Then for $a = 0$ we have

$$F_1^{i*}(z, 0) + \varphi^i(z^i, \dots, Z^N) = 0,$$

i.e.,

$$\varphi^i(z^i, \dots, z^N) = -F_1^{i*}(z, 0).$$

Thus we have obtained N functionally independent approximate [with an accuracy of $o(\varepsilon)$] solutions to problem (4.15), (4.17):

$$F^i(z', a, \varepsilon) \approx F_0^i(z', a) + \varepsilon F_1^i(z', a) \equiv \tag{4.28}$$

$$F_0^i(z', a) + \varepsilon[F_1^{i*}(z', a) - F_1^{i*}(F_0^1(z', a), \dots, F_0^N(z', a), 0)], \quad i = 1, \dots, N.$$

Since $|(\partial F_0^i)/(\partial z'^j)| \neq 0$, it follows that this approximate equation can be solved [with an accuracy of $o(\varepsilon)$ for z' :

$$z'^i = f_0^i(z, a) + \varepsilon f_1^i(z, a). \tag{4.29}$$

To obtain explicit expressions for the functions $f_1^i(z, a)$, we substitute (4.29) into (4.27). We obtain, with an accuracy of $o(\varepsilon)$,

$$F_0^i(f_0(z, a) + \varepsilon f_1(z, a) + \dots, a) +$$

$$\varepsilon[F_1^{i*}(f_0(z, a), a) - F_1^{i*}(F_0^1(f_0, a), \dots, F_0^N(f_0, a), 0)] \approx z';$$

hence, by virtue of the initial data (4.24), we have

$$\varepsilon\left[\frac{\partial F_0^i}{\partial z'^j}(f_0,a)f_1^j(z,a)+F_1^{i*}(f_0,a)-F_1^{i*}(z,0)\right]=0.$$

Since $|(\partial F_0^i)/(\partial z'^j)|\neq 0$, we see that this system can be solved for $f_1^i(z,a)$, namely,

$$f_1^i(z,a)=\|\frac{\partial F_0^j}{\partial z'^i}\|\ [F_1^{i*}(z,0)-F_1^{i*}(f_0(z,a),a)]. \tag{4.30}$$

[Note that the functions $f_1^i(z,a)$ satisfy the initial data (4.16), i.e., $f_1^i(z,0)=0$.]

To obtain the approximate [with an accuracy of $o(\varepsilon^2)$] solution, we substitute N independent solutions of (4.28) into (4.22) and obtain a system of linear nonhomogeneous equations for the functions $F_2^i(z',a)$. By analogy with the first-order accuracy, we have

$$F_1^i(z',a,\varepsilon)=F_0^i(z',a)+ \tag{4.31}$$

$$\varepsilon F_1^i(z',a)+\varepsilon^2[F_2^{i*}(z',a)-F_2^{i*}(F_0^1(z',a),\ldots,F_0^N(z',a),0)].$$

The inversion of this formula for z' with an accuracy of $o(\varepsilon^2)$, we obtain

$$z'=f_0(z,a)+\varepsilon f_1(z,a)+\varepsilon^2 f_2(z,a).$$

The functions f_2 are determined by substituting z' into the initial condition (4.25), namely,

$$\varepsilon^2\frac{\partial F_0^i}{\partial z'^j}(f_0,a)f_2^j+\varepsilon^2\frac{1}{2}\frac{\partial^2 F_0^i}{\partial z'^j\partial z'^k}(f_0,a)f_1^jf_1^k$$

$$+\varepsilon^2\frac{\partial F_1^i}{\partial z'^j}(f_0,a)f_1+\varepsilon^2 f_2^{i*}(f_0,a)\approx\varepsilon^2F_2^{i*}(z,0).$$

These equations can be solved for f_2^i by virtue of the condition $|(\partial F_0^i)/(\partial z'^j)|\neq 0$.

Similarly, we can construct N independent approximate [with an accuracy of $o(\varepsilon^p)$ solutions to problem (4.15), (4.17) and the solution to (4.9), (4.16) of the form (4.18). Thus Lemma 3.4 is proved.

Let us now state and prove the inverse first Lie theorem for approximate transformation groups.

Theorem 3.3. Let the system of approximate Lie equations

$$\frac{\partial z'^i}{\partial a^\alpha}\approx\xi^i_\beta(z',\varepsilon)V^\beta_\alpha(a)\quad(i=1,\ldots,N,\ \alpha=1,\ldots,r) \tag{4.32}$$

be completely integrable, let the functions $\xi_\alpha(z', \varepsilon)$ be (approximately) linearly independent, and let the rank of the matrix $V(a) = (V_\alpha^\beta(a))$ be equal to r. Then the solution to equations (4.32) that satisfies the conditions

$$z'|_{a=0} \approx z \tag{4.33}$$

has the form

$$z' = f_0(z, a) + \varepsilon f_1(z, a) + \ldots + \varepsilon^p f_p(z, a) + o(\varepsilon^p) \tag{4.34}$$

and determines an approximate [with an accuracy of $o(\varepsilon^p)$] r -parameter transformation group.

Proof. By Lemmas 3.1 and 3.4, the solution to problem (4.32), (4.33) exists, is unique, and can be represented in the form (4.34). Let us prove that the transformations (4.34) form a local approximate transformation group.

First, let us show that the parameters a are essential in the family of N independent approximate solutions $F^i = F^i(z', a, \varepsilon)$ to problem (4.15), (4.17). Indeed, let the following equations of the form (2.3) hold:

$$\frac{\partial F^i}{\partial a^\alpha}\psi^\alpha(a) \approx 0, \quad i = 1, \ldots, N. \tag{4.35}$$

Then from (4.15) and (4.35) we obtain the system of linear homogeneous equations

$$\frac{\partial F^i}{\partial z'^j}\xi^j_\beta V_\alpha^\beta \psi^\alpha(a) \approx 0$$

with the matrix $(dF^i)/(dz'^j)$, whose determinant is not zero. Therefore, the expressions $\xi^j_\beta V_\alpha^\beta \psi^\alpha(a)$ are approximately zero. Since the vectors $\xi^i_\beta(z', \varepsilon)$ are linearly independent and the matrix $V_\alpha^\beta(a)$ is nondegenerate, we have

$$\psi^\alpha(a) \equiv 0.$$

This precisely means that the parameters a are essential.

Now let us show that the transformations (4.34) satisfy the group property (3.2). Rewriting (3.2) as

$$z'' \approx f(z', b, \varepsilon) \approx f(z, \varphi(a, b), \varepsilon), \tag{4.36}$$

we see that the approximate Lie equations (4.32) imply

$$\frac{\partial z''^i}{\partial b^\alpha} \approx \xi^i_\beta(z'', \varepsilon) V_\alpha^\beta(b) \quad \text{and} \quad \frac{\partial z''^i}{\partial \varphi^\alpha} \approx \xi^i_\beta(z'', \varepsilon) V_\alpha^\beta(\varphi).$$

Substituting the right-hand side of (4.36) into the first of these two systems, we obtain

$$\frac{\partial z''^i}{\partial \varphi^\gamma}\frac{\partial \varphi^\gamma}{\partial b^\alpha} \approx \xi^i_\beta(z'',\varepsilon)V^\beta_\alpha(b),$$

and from the second system we derive the equation

$$\xi_\mu(z'',\varepsilon)V^\mu_\gamma(\varphi)\frac{\partial \varphi^\gamma}{\partial b^\alpha} \approx \xi^i_\beta(z'',\varepsilon)V^\beta_\alpha(b).$$

Since the vectors $\xi_\beta(z'',\varepsilon)$ are linearly independent, we have

$$V^\beta_\gamma(\varphi)\frac{\partial \varphi^\gamma}{\partial b^\alpha} = V^\beta_\alpha(b)$$

and hence

$$\frac{\partial \varphi^\alpha}{\partial b^\beta} = A^\alpha_\sigma(\varphi)V^\sigma_\beta(b). \tag{4.37}$$

Thus if the composition $\varphi(a,b)$ is defined for transformations (4.34), then it necessarily satisfies system (4.37).

System (4.37) is completely integrable, i.e

$$\frac{\partial}{\partial b^\gamma}\left(\frac{\partial \varphi^\alpha}{\partial b^\beta}\right) = \frac{\partial}{\partial b^\beta}\left(\frac{\partial \varphi^\alpha}{\partial b^\gamma}\right)$$

Indeed, by virtue of (4.37), the last equation can be rewritten as follows:

$$\frac{\partial A^\alpha_\sigma}{\partial \varphi^\nu}\frac{\partial \varphi^\nu}{\partial b^\gamma}V^\sigma_\beta + A^\alpha_\sigma\frac{\partial V^\sigma_\beta}{\partial b^\gamma} = \frac{\partial A^\alpha_\mu}{\partial \varphi^l}\frac{\partial \varphi^l}{\partial b^\beta}V^\mu_\gamma + A^\alpha_\mu\frac{\partial V^\mu_\gamma}{\partial b^\beta}$$

or

$$\left(\frac{\partial A^\alpha_\sigma}{\partial \varphi^\nu}A^\nu_n - \frac{\partial A^\alpha_n}{\partial \varphi^\nu}A^\nu_\sigma\right)V^n_\gamma V^\sigma_\beta = -A^\alpha_\sigma\left(\frac{\partial V^\sigma_\beta}{\partial b^\gamma} - \frac{\partial V^\sigma_\gamma}{\partial b^\beta}\right).$$

Multiplying this equation by $A^\gamma_k A^\beta_l$ and using (4.11), we obtain equations (4.14) for the function $A^\alpha_\sigma(\varphi)$, namely,

$$\left(\frac{\partial A^\nu_\mu}{\partial \varphi^\alpha}A^\alpha_\gamma - \frac{\partial A^\nu_\gamma}{\partial \varphi^\alpha}A^\alpha_\mu\right) = -c^\sigma_{\gamma\mu}A^\nu_\sigma.$$

These equations follow from the Maurer-Cartan equation, which is equivalent to the integrability of the Lie equations (4.32) and holds by the condition of the theorem. Thus, system (4.37) is completely integrable.

It follows that the preceding argument concerning the approximate Lie equations (4.32) is valid for equations (4.37) as well (this has also been proved by Eisenkhart [31] and Chebotarev [21]). We see that the solution to (4.37) with the initial conditions

$$\varphi|_{b=0} = a, \tag{4.38}$$

where the components of $a = (a^1, \ldots, a^r)$ are sufficiently small, gives r independent integrals, since $|A^\alpha_\sigma(0)| = 1$.

Let us check that the transformations (4.34) satisfy property (3.2) with the functions $\varphi(a, b)$ obtained from problem (4.37), (4.38). Let

$$z' = f(z, a, \varepsilon), \quad z'' = f(z', b, \varepsilon), \quad w = f(z, \varphi(a, b), \varepsilon).$$

Clearly, $z'' = z'$ and $w = z'$ for $b = 0$. By the construction of f we have

$$\frac{\partial z''^i}{\partial b^\alpha} \approx \xi^i_\beta(z'', \varepsilon) V^\beta_\alpha(b),$$

$$\frac{\partial w^i}{\partial b^\alpha} = \frac{\partial w^i}{\partial \varphi^\beta}\frac{\partial \varphi^\beta}{\partial b^\alpha} \approx \xi^i_\gamma(w, \varepsilon) V^\gamma_\beta(\varphi)\frac{\partial \varphi^\beta}{\partial b^\alpha} =$$
$$\xi^i_\gamma V^\gamma_\beta(\varphi) A^\beta_\sigma(\sigma) V^\sigma_\alpha(b) = \xi^i_\gamma(w, \varepsilon) V^\gamma_\alpha(b),$$

where (4.38) was used to derive the last equalities. Thus the functions z'' and w satisfy the same Cauchy problem with the same initial conditions at $b = 0$. It follows from Lemma 3.1 that z'' and w coincide, and consequently, the group property (3.2) is satisfied.

The existence of the unit element follows from the construction of the transformations.

The inverse transformations also exist since (4.18) is invertible in the vicinity of $a = 0$. They can be obtained from (4.34) by choosing appropriate values of the parameters. Namely, these values can be determined by solving the equation

$$\varphi(a, b) = 0$$

for b. Thus the theorem is proved.

Example 3.5. Let us illustrate the solution of the Lie equations by the example of system (3.16). The equivalent system of linear homogeneous equations [see (4.15)] is

$$(1 + \varepsilon a^3)\frac{\partial F}{\partial x'} + \varepsilon x'\frac{\partial F}{\partial y'} + \frac{\partial F}{\partial a^1} \approx 0,$$

$$\varepsilon y'\frac{\partial F}{\partial x'} + (1 + \varepsilon a^3)\frac{\partial F}{\partial y'} + \frac{\partial F}{\partial a^2} \approx 0,$$

$$\varepsilon x'\frac{\partial F}{\partial x'}+\varepsilon y'\frac{\partial F}{\partial y'}+\frac{\partial F}{\partial a^3}\approx 0,$$

$$\varepsilon\frac{\partial F}{\partial x'}+\frac{\partial F}{\partial a^4}\approx 0,\quad \varepsilon\frac{\partial F}{\partial y'}+\frac{\partial F}{\partial a^5}\approx 0.$$

Substituting

$$F=F_0(x',y',a)+\varepsilon F_1(x',y',a)+\cdots$$

into these equations and equating the coefficients of ε^0 and ε^1 to zero we obtain

$$\frac{\partial F_0}{\partial x'}+\frac{\partial F_0}{\partial a^1}=0,\quad \frac{\partial F_0}{\partial y'}+\frac{\partial F_0}{\partial a^2}=0,\quad \frac{\partial F_0}{\partial a^i}=0,\quad i=3,4,5, \tag{4.39}$$

$$\begin{aligned}
&\frac{\partial F_1}{\partial x'}+\frac{\partial F_1}{\partial a^1}+a^3\frac{\partial F_0}{\partial x'}+x'\frac{\partial F_0}{\partial y'}=0,\\
&\frac{\partial F_1}{\partial y'}+\frac{\partial F_1}{\partial a^2}+y'\frac{\partial F_0}{\partial x'}+a^3\frac{\partial F_0}{\partial y'}=0,\\
&\frac{\partial F_1}{\partial a^3}+x'\frac{\partial F_0}{\partial x'}+y'\frac{\partial F_0}{\partial y'}=0,\\
&\frac{\partial F_1}{\partial a^4}+\frac{\partial F_0}{\partial x'}=0,\\
&\frac{\partial F_1}{\partial a^5}+\frac{\partial F_0}{\partial y'}=0.
\end{aligned} \tag{4.40}$$

System (4.39) has two functionally independent solutions, namely,

$$F_0^1=x'-a^1,\quad F_0^2=y'-a^2. \tag{4.41}$$

With the initial conditions

$$x'|_{a=0}=x,\quad y'|_{a=0}=y, \tag{4.42}$$

they determine the (precise) group

$$x'=x+a^1,\quad y'=y+a^2.$$

Substituting the functions (4.41) into (4.40), we obtain the systems for F_1^1,

$$\frac{\partial F_1^1}{\partial x'}+\frac{\partial F_1^1}{\partial a^1}=-a^3,\quad \frac{\partial F_1^1}{\partial y'}+\frac{\partial F_1^1}{\partial a^2}=-y',$$

$$\frac{\partial F_1^1}{\partial a^3}=-x',\quad \frac{\partial F_1^1}{\partial a^4}=-1,\quad \frac{\partial F_1^1}{\partial a^5}=0,$$

and for F_1^2

$$\frac{\partial F_1^2}{\partial x'} + \frac{\partial F_1^2}{\partial a^1} = -x', \quad \frac{\partial F_1^2}{\partial y'} + \frac{\partial F_1^2}{\partial a^2} = -a^3,$$

$$\frac{\partial F_1^2}{\partial a^3} = -y', \quad \frac{\partial F_1^2}{\partial a^4} = 0, \quad \frac{\partial F_1^2}{\partial a^5} = -1.$$

The general solutions to these systems have the form

$$F_1^1 = -a^4 - a^3 x' - \frac{1}{2}(y')^2 + \alpha^1(x' - a^1,\ y' - a^2)$$

and

$$F_1^2 = -a^5 - a^3 y' - \frac{1}{2}(x')^2 + \alpha^2(x' - a^1, y' - a^2),$$

respectively. Hence we have

$$F^1 = x' - a^1 + \varepsilon\left[-a^4 - a^3 x' - \frac{1}{2}(y')^2 + \alpha^1(x' - a^1,\ y' - a^2)\right],$$

$$F^2 = y' - a^2 + \varepsilon\left[-a^5 - a^3 y' - \frac{1}{2}(x')^2 + \alpha^2(x' - a^1,\ y' - a^2)\right].$$

By virtue of the initial conditions (4.25) and (4.33), which read

$$F(z', a, \varepsilon) \approx z, \quad z'|_{a=0} \approx z,$$

we have $\alpha^1 = y^2/2,\ \alpha^2 = x^2/2$. Therefore, by solving the relationships

$$x' - a^1 + \varepsilon\left[-a^4 - a^3 x' - a^2 y' + \frac{1}{2}(a^2)^2\right] \approx x,$$

$$y' - a^2 + \varepsilon\left[-a^5 - a^3 y' - a^1 x' + \frac{1}{2}(a^1)^2\right] \approx y$$

for z' and y' [with an accuracy of $o(\varepsilon)$], we obtain the approximate Lie transformation group

$$x' \approx x + a^1 + \varepsilon\left[a^4 + a^3 x + a^2 y + \frac{1}{2}(a^2)^2 + a^1 a^3\right],$$

$$y' \approx y + a^2 + \varepsilon\left[a^5 + a^3 y + a^1 x + \frac{1}{2}(a^1)^2 + a^2 a^3\right]$$

[cf. (1.4)].

§ 5 The second Lie theorem

Consider the approximate operators with an accuracy of $o(\varepsilon^p)$:

$$X = \xi^i(z,\varepsilon)\frac{\partial}{\partial z^i} \equiv [\xi^i_0(z) + \varepsilon\xi^i_1(z) + \cdots + \varepsilon^p\xi^i_p(z)]\frac{\partial}{\partial z^i},$$

$$Y = \eta^i(z,\varepsilon)\frac{\partial}{\partial z^i} \equiv [\eta^i_0(z) + \varepsilon\eta^i_1(z) + \cdots + \varepsilon^p\eta^i_p(z)]\frac{\partial}{\partial z^i}\cdot \tag{5.1}$$

It is clear that the addition of operators and multiplication by a number are well defined on the set of such operators, i.e., the structure of a linear space is defined.

Definition 3.5. The operator

$$[X,Y] \approx XY - YX \tag{5.2}$$

considered with an accuracy of $o(\varepsilon^p)$ is called the approximate [with an accuracy of $o(\varepsilon^p)$] commutator of the operators X and Y.

It is clear that the commutation introduced in such a way has all usual properties, namely,

1) linearity

$$[\alpha X + \beta Y, Z] \approx \alpha[X,Z] + \beta[Y,Z];$$

2) antisymmetry

$$[X,Y] \approx -[Y,X];$$

3) the Jacobi identity

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] \approx 0.$$

Definition 3.6. A vector space L of approximate operators (5.1) is called an approximate Lie algebra of operators if for any two operators in L their approximate commutator (5.2) is also in L.

Theorem 3.4. Let a local r-parameter approximate Lie transformation group (3.1) be given. Then the linear hull of the operators (3.9)

$$X_\alpha = \xi^i_\alpha(z,\varepsilon)\frac{\partial}{\partial z^i}, \quad i = 1,\ldots,N, \quad \alpha = 1,\ldots,r,$$

is an approximate Lie algebra of operators with structural constants given by (4.12) (i.e., coinciding with the structural constants of the approximate Lie transformation group), namely,

$$[X_\alpha, X_\beta] \approx c^\gamma_{\alpha\beta}X_\gamma. \tag{5.3}$$

Conversely, to any r linearly independent operators that satisfy conditions (5.3) with constant $c^\gamma_{\alpha\beta}$ there corresponds a local r-parameter approximate Lie transformations group.

Proof. Consider the approximate commutator

$$[X_\alpha, X_\beta] = \left[\xi^i_\alpha(z,\varepsilon)\frac{\partial}{\partial z^i},\ \xi^j_\beta(z,\varepsilon)\frac{\partial}{\partial z^j}\right] = \left[\xi^i_\alpha \frac{\partial \xi^j_\beta}{\partial z^i} - \xi^j_\beta \frac{\partial \xi^j_\alpha}{\partial z^j}\right]\frac{\partial}{\partial z^j}\,\cdot$$

By (4.11) and (4.12) this expression has the form (5.3), namely,

$$[X_\alpha, X_\beta] \approx c^\sigma_{\alpha\beta}\xi^j_\sigma \frac{\partial}{\partial z^j} \equiv c^\sigma_{\alpha\beta}X_\sigma.$$

Conversely, let the operators X_α form a basis of an approximate Lie algebra, and let (5.3) hold. Clearly, the structural constants $c^\sigma_{\alpha\beta}$ satisfy the usual conditions

$$c^\sigma_{\alpha\beta} = -c^\sigma_{\beta\alpha} \quad c^\sigma_{\alpha\beta}c^\tau_{\sigma\gamma} + c^\sigma_{\beta\gamma}c^\tau_{\sigma\alpha} + c^\sigma_{\gamma\alpha}c^\gamma_{\sigma\beta} = 0.$$

According to the third main Lie theorem (for precise groups) (see, for example, Eisenkhart [30], Chebotarev [21], and Ovsyannikov [111], [112]), such structural constants determine a system of functions $V^\alpha_\beta(b)$ satisfying the Maurer-Cartan equations (4.13). System (4.32) with these functions is completely integrable by (5.3). According to the first Lie theorem for approximate groups of transformations (see Theorem 3.3), we obtain an approximate r-parameter local Lie transformation group.

Example 3.6. Let us construct the auxiliary functions $V^\alpha_\beta(a)$ for the operators

$$X_1 = \frac{\partial}{\partial x} + \varepsilon x\frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial y} + \varepsilon y\frac{\partial}{\partial x},$$

$$X_3 = \varepsilon\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right), \quad X_4 = \varepsilon\frac{\partial}{\partial x}, \quad X_5 = \varepsilon\frac{\partial}{\partial y}\,\cdot$$

We have

$$[X_1, X_3] = -[X_3, X_1] = X_4; \quad [X_2, X_3] = -[X_3, X_2] = X_5;$$

all other commutators are zero. Therefore, only the following structural constants are not zero: $c^4_{13} = -c^4_{31} = 1$ and $c^5_{23} = -c^5_{32} = 1$. To find the functions $V^\beta_\alpha(b)$, we should find the integrals of the system of linear equations [[21], p. 63, Eq. (31)]

$$\frac{\partial \theta^\beta_\alpha}{\partial t} = \delta^\beta_\alpha + c^\beta_{\nu\mu}\lambda^\mu\theta^\nu_\alpha, \tag{5.4}$$

vanishing at $t = 0$. Then

$$b^\alpha = \lambda^\alpha t, \quad \alpha = 1,\ldots,r, \quad V^\beta_\alpha = \frac{1}{t}\theta^\beta_\alpha(\lambda^1,\ldots,\lambda^r, r). \tag{5.5}$$

In our case the solution to system (5.4) with the initial conditions taken into account has the form

$$\theta_\alpha^\alpha = t, \quad \theta_1^4 = \frac{1}{2}\lambda^3 t^2, \quad \theta_2^5 = \frac{1}{2}\lambda^3 t^2,$$

and $\theta_\alpha^\beta = 0$ otherwise. Then (5.5) gives

$$V_\alpha^\alpha = 1, \quad \alpha = 1, \ldots, 5; \quad V_1^4 = \frac{1}{2}b^3; \quad V_2^5 = \frac{1}{2}b^3,$$

and V_α^β otherwise. Replacing $b^3/2$ by a^3, we obtain the matrix $V(a)$ from Example 3.4.

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Paper 4

Approximate equivalence transformations

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The first stage of group classification of differential equations is to find equivalence transformations. A nonsingular change of dependent and independent variables which moves any equation of a given class into an equation of the same class is called an *equivalence transformation* of the given class of equations (the second-order classes of equations are determined by arbitrary parameters or functions). For example, for the class of differential equations with two independent variables of the form

$$A(x,y)u_{xx} + 2B(x,y)u_{xy} + C(x,y)u_{yy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u = 0,$$

the equivalence transformations are

$$\tilde{x} = f(x,y), \quad \tilde{y} = g(x,y), \quad \tilde{u} = h(x,y)u,$$

and for a narrower class of hyperbolic equations of the canonical form

$$u_{xy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u = 0,$$

the equivalence transformations are

$$\tilde{x} = f(x) \quad \tilde{y} = g(y), \quad \tilde{u} = h(x,y)u$$

(see [89] and its English translation in [65]).

In his works Lie calculated finite equivalence transformations. As far as we know, it was L. V. Ovsyannikov who first used infinitesimal approach to calculation of equivalence transformations (see [114] or, for details, [1]). In the present paper we use the infinitesimal approach to calculation of approximate equivalence transformations taking, for example, three classes of second-order ordinary differential equations with a small parameter.

§ 1 Equations $y'' = \varepsilon f(y)$

The infinitesimal approach to calculation of exact equivalence transformations can be applied to the equation

$$y'' = f(y) \tag{1.1}$$

$\varepsilon = 1$. For this purpose, we rewrite it as the system of equations

$$y'' = f, \quad f_x = 0, \tag{1.2}$$

where y and f are considered to be functions of the variables x and x, y, respectively.

We search for the generator of an equivalence group in the form

$$E = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \mu(x,y,f)\frac{\partial}{\partial f}$$

and found it from the invariance condition for system (1.2):

$$\widetilde{E}(y'' - f)|(1.2) = 0, \tag{1.3}$$

$$\widetilde{E}(f_x)|(1.2) = 0, \tag{1.4}$$

where $\widetilde{E} = E + \zeta_1\frac{\partial}{\partial y'} + \zeta_2\frac{\partial}{\partial y''} + \mu_1\frac{\partial}{\partial f_x}$ is the extended operator. The coefficients ζ_1 and ζ_2 are calculated by usual extension formulas while μ_1 is obtained by extending the function f, which is considered as a fuunction of the variables x and y, namely, $\mu_1 = \widetilde{D}_x(\mu) - f_x\widetilde{D}_x(\xi) - f_y\widetilde{D}_x(\eta)$, where $\widetilde{D}_x = \frac{\partial}{\partial x} + f_x\frac{\partial}{\partial f} + \cdots$. Taking into account that μ does not depend on f_y, we derive $\mu_x = 0$ and $\eta_x = 0$ from Eq. (1.4). Therefore the solution of Eq. (1.3) yields $\xi = C_1x + C_2, \eta = C_3y + C_4$, and $\mu = (C_3 - 2C_1)$. Setting in turn one of the constants C equal to 1 and the other constants equal to 0, we obtain the operators

$$E_1 = x\frac{\partial}{\partial x} - 2f\frac{\partial}{\partial f}, \quad E_2 = \frac{\partial}{\partial x}, \quad E_3 = y\frac{\partial}{\partial y} + f\frac{\partial}{\partial f}, \quad E_4 = \frac{\partial}{\partial y}, \tag{1.5}$$

which generate a 4−parametric group of equivalence transformations for Eq. (1.1)

A similar algorithm can be used for calculation of approximate (pointwise) equivalence transformations for the equation

$$y'' = \varepsilon f(y) \tag{1.6}$$

with the small parameter ε.

We call nonsingular in the zeroth order with respect to ε changes of variables of the form

$$\tilde{x} = \varphi_0(x, y) + \varepsilon\varphi_1(x, y), \quad \tilde{y} = \psi_0(x, y) + \varepsilon\psi_1(x, y),$$

which move (with the considered accuracy) Eq. (1.6) into an equation of the same form, the *approximate* [with the accuracy $o(\varepsilon)$] *equivalence transformations* for Eq. (1.6). Thus, taking into account the accuracy of approximation, it suffices to assume that ε does not depend on x. Then Eq. (1.6) is rewritten as the following system of equations

$$y'' = \varepsilon f, \quad \varepsilon f_x = 0. \tag{1.7}$$

The generator of the approximate group of equivalence transformations is sought in the form

$$E = (\xi^0(x, y) + \varepsilon\xi^1(x, y))\frac{\partial}{\partial x} + (\eta^0(x, y) + \varepsilon\eta^1(x, y))\frac{\partial}{\partial y} + \mu(x, y, f)\frac{\partial}{\partial f}$$

and is found from the approximate conditions of invariance for system (1.7) (see [8] and [9])

$$\widetilde{E}(y'' - \varepsilon f)|_{(1.7)} = o(\varepsilon), \tag{1.8}$$

$$\widetilde{E}(\varepsilon f_x)|_{(1.7)} = o(\varepsilon). \tag{1.9}$$

Here $\widetilde{E}$ is calculated in the same way as in the case of the exact group with the change of ξ and η by $\xi^0 + \varepsilon\xi^1$, and $\eta^0 + \varepsilon\eta^1$, respectively.

Separation of zeroth-order terms with respect to ε in (1.8) yields defining equations for the exact symmetries of the equation $y'' = 0$. Hence,

$$\begin{aligned} \xi^0 &= (C_1 x + C_2)y + C_3 x^2 + C_4 x + C_5, \\ \eta^0 &= C_1 y^2 + C_3 xy + C_6 y + C_7 x + C_8. \end{aligned} \tag{1.10}$$

The appropriate operators become

$$\begin{aligned} &E_1^0 = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} \quad E_2^0 = y\frac{\partial}{\partial x} \quad E_3^0 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, \\ &E_4^0 = x\frac{\partial}{\partial x}, \quad E_5^0 = \frac{\partial}{\partial x} \quad E_6^0 = y\frac{\partial}{\partial y}, \quad E_7^0 = x\frac{\partial}{\partial y}, \quad E_8^0 = \frac{\partial}{\partial y}\cdot \end{aligned} \tag{1.11}$$

The first-order terms with respect to ε in Eq. (1.9) yield $\mu^0_x = 0, \eta^0_x = 0$ and those in Eq. (1.8) yield ?????????, $\eta^1_{yy} - 2\xi^1_{xy} = 0, 2\eta^1_{xy} - \xi^1_{xx} - 3f\xi^0_y = 0, \mu = \eta^1_{xx} + f(\eta^0_y - 2\xi^0_x)$. The solution to these equations is

$$C_1 = 0, \quad C_2 = 0, \quad C_3 = 0, \quad C_7 = 0,$$

$$\xi^1 = (A_1x + A_2)y2B_1x^3 + A_3x^2 + A_4x + A_5,$$
$$\eta^1 = A_1y^2 + 3B_1x^2y + A_3xy + A_6y + B_2x^2 + A_7x + A_8,$$
$$\mu = 6B_1y + 2B_2 + (C_6 - 2C_4)f.$$

Thus, the approximate group of equivalence transformations is generated by

$$\begin{gathered} E_1 = x\frac{\partial}{\partial x} - 2f\frac{\partial}{\partial f} \quad E_2 = \frac{\partial}{\partial x}, \quad E_3 = y\frac{\partial}{\partial y} + f\frac{\partial}{\partial f}, \quad E_5 = \frac{\partial}{\partial y}, \\ E_5 = 2\varepsilon x^3\frac{\partial}{\partial x} + 3\varepsilon x^2\frac{\partial}{\partial y} + 6y\frac{\partial}{\partial f} \quad E_6 = \varepsilon x^2\frac{\partial}{\partial y} + 2\frac{\partial}{\partial f}, \end{gathered} \tag{1.12}$$

and by the eight generators obtained from (1.11) by multiplication by ε. The corresponding 14-parametric approximate group of equivalence transformations is written as

$$\tilde{x} = a_1x + a_2 + \varepsilon(2a_1a_5x^3 + a_1a_7xy + a_8y + a_1a_9x^2 + a_{10}x + a_{11}),$$

$$\tilde{y} = a_3y + a_4 + \varepsilon(3a_3a_5x^2y + a_6x^2 + a^3a_7y^2 + a_3a_9xy + a_{12}y + a_{13}x + a_{14}),$$
$$\tilde{f} = a_1^{-2}a_3f + 6a_1^{-2}a_3a_5y + 2a_1^{-2}a_6, \quad a_1a_3 \neq 0,$$

It should be mentioned that only the transformations generated by the operators E_1, E_3, E_4, E_5, E_6 in (1.12) are essentially used for group classification of the Eqs. (1.6) with respect to approximate symmetries.

§ 2 Equations $y'' = \varepsilon f(x, y)$

Like in Section § 1, we seek for a generator of the approximate group of equivalence transformations for the equation

$$y'' = \varepsilon f(x, y) \tag{2.13}$$

with the small parameter ε in the form

$$E = (\xi^0(x, y) + \varepsilon\xi^1(x, y))\frac{\partial}{\partial x} + (\eta^0(x, y) + \varepsilon\eta^1(x, y))\frac{\partial}{\partial y} + \mu(x, y, f)\frac{\partial}{\partial f}$$

and find it from the approximate condition of invariance for Eq. (2.13)

$$\widetilde{E}(y'' - \varepsilon f)|_{(2.13)} = o(\varepsilon) \tag{2.14}$$

Here ξ^0 and η^0 are defined by (1.10) and separation of first-order terms with respect to ε in Eq. (2.14) yields

$$\xi^1_{yy} = 0, \quad \eta^1_{yy} - 2\xi^1_{xy} = 0, \quad 2\eta^1_{xy} - \xi^1_{xx} - 3f(C_1 x + C_2) = 0,$$

$$\mu = \eta^1_{xx} + f(-3C_3 x + C_6 - 2C_4),$$

whence we derive

$$C_1 = 0, \quad C_2 = 0, \quad \xi^1 = A_1 xy + A_2 y + \beta(x),$$

$$\eta^1 = A_1 y^2 + 2^{-1}\beta'(x) y A_3 y + \sigma(x), \quad \mu 2^{-1}\beta'''(x) y \sigma''(x) + f(-3C_3 x + C_6 - 2C_4).$$

Thus, the approximate group of equivalence transformations for Eq. (2.13) is generated by

$$E_1 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} - 3xf\frac{\partial}{\partial f} \quad E_2 = x\frac{\partial}{\partial x} - 2f\frac{\partial}{\partial f}, \quad E_3 = \frac{\partial}{\partial x},$$
$$E_4 = x\frac{\partial}{\partial y}, \quad E_5 = y\frac{\partial}{\partial y} + f\frac{\partial}{\partial f}, \quad E_6 = \frac{\partial}{\partial y}, \quad E_\tau = \varepsilon xy\frac{\partial}{\partial x} + \varepsilon y^2\frac{\partial}{\partial y},$$
$$E_8 = \varepsilon y\frac{\partial}{\partial x}, \quad E_9 = \varepsilon y\frac{\partial}{\partial y}, \quad E_{10} = \varepsilon\beta(x)\frac{\partial}{\partial x} + \frac{\varepsilon}{2}\beta'(x)y\frac{\partial}{\partial y} + \frac{1}{2}\beta'''(x)y\frac{\partial}{\partial f},$$
$$E_{11} = \varepsilon\sigma(x)\frac{\partial}{\partial y} + \sigma''(x)\frac{\partial}{\partial f},$$

where $\beta(x)$ and $\sigma(x)$ are arbitrary functions. The approximate transformations have the form

$$\tilde{x} = \frac{a_2 x + a_3}{1 - a_1 a_3 - a_1 a_2 x} + \varepsilon\frac{a_2 a_\tau xy + a_2 a_8 y + a_2 a_{10}\beta}{(1 - a_1 a_3 - a_1 a_2 x)^2},$$

$$\tilde{y} = \frac{a_5 y + a_4 x + a_6}{1 - a_1 a_3 - a_1 a_2 x} + \frac{\varepsilon}{(1 - a_1 a_3 - a_1 a_2 x)^2}(a_1 a_2 a_5 a_8 + a_5 a_7 - a_1 a_3 a_5 a_7) y^2 +$$
$$+ a_1 a_2 a_5 a_{10}\beta(x) y + (2^{-1} a_5 a_{10} - 2^{-1} a_1 a_3 a_5 a_{10})\beta'(x) y -$$
$$-2^{-1} a_1 a_2 a_5 a_{10}\beta'(x) xy + (a_1 a_2 a_6 a_7 + a_4 a_7 - a_1 a_3 a_4 a_7 - a_1 a_2 a_5 a_9) xy +$$
$$+(a_1 a_2 a_6 a_8 + a_5 a_9 + a_4 a_8 - a_1 a_3 a_5 a_9 - a_1 a_3 a_4 a_8) y + (a_1 a_2 a_6 a_{10} + a_4 a_{10} -$$
$$-a_1 a_3 a_4 a_{10}\beta(x) - a_1 a_2 a_5 a_{11}\sigma(x) x + (a_5 a_{11} - a_1 a_3 a_5 a_{11})\sigma(x)),$$
$$\tilde{f} = a_2^{-2} a_5 (1 - a_1 a_3 - a_1 a_2 x)^3 (f + 2^{-1} a_{10}\beta'''(x) y + a_{11}\sigma''(x)).$$

For comparison, here we present the exact group of equivalence transformations for Eq. (2.13) with $\varepsilon = 1$, that is, of the equation $y'' = f(x, y)$. It admits the infinite-dimensional group equivalence transformations generated by the operators

$$E_1 = \alpha(x)\frac{\partial}{\partial x} + \frac{1}{2}\alpha'(x)y\frac{\partial}{\partial y} + \left(\frac{1}{2}\alpha'''(x)y - \frac{3}{2}\alpha'(x)f\right)\frac{\partial}{\partial f},$$

$$E_2 = \beta(x)\frac{\partial}{\partial y}\beta''(x)\frac{\partial}{\partial f}, E_3 = y\frac{\partial}{\partial y} + f\frac{\partial}{\partial f},$$

where $\alpha(x)$ and $\beta(x)$ are arbitrary functions. The corresponding transformations are

$$\tilde{x} = \gamma^{-1}(\gamma(x)+a_1), \quad \text{where} \quad \gamma(x) = \int \frac{dx}{\alpha(x)}, \quad \gamma^{-1} \text{ is the inverse to } \gamma,$$

$$\tilde{y} = (\alpha(\tilde{x})/\alpha(x))^{1/2}(a_3 y\beta(x)a_2),$$

$$\tilde{f} = (\alpha(x)/\alpha(\tilde{x}))^{3/2}(a_3 f + \beta''(x)a_2) - (1/2)(\alpha(\tilde{x}))^{-3/2}(\alpha(x))^{-1/2}$$

$$\times[\alpha(x)\alpha''(x) - (1/2)(\alpha'(x))^2 - \alpha(\tilde{x})\alpha''(\tilde{x}) + (1/2)(\alpha'(\tilde{x}))^2](a_3 y + \beta(x)a_2); \quad a_3 \neq 0.$$

§ 3 Equations $y'' = \varepsilon f(x, y, y')$

In the case of the equation

$$y'' = \varepsilon f(x, y, y') \tag{3.15}$$

(with the small parameter ε), a generator of the group of equivalence transformations becomes

$$E = (\xi^0(x, y) + \varepsilon\xi^1(x, y))\frac{\partial}{\partial x} + (\eta^0(x, y) + \varepsilon\eta^1(x, y))\frac{\partial}{\partial y} + \mu(x, y, y', f)\frac{\partial}{\partial f},$$

where the functions ξ^0 and η^0 are defined by (1.10). Unlike the two previous cases, the coefficient μ depends on y' and, therefore, the approximate condition of invariance for Eq. (3.15) implies that

$$\mu = \eta^1_{xx} + 2y'\eta^1_{xy} + (y')^2\eta^1_{yy} + f(-3C_1 xy' - 3C_2 y' - 3C_3 y' - 2C_4 + C_6) - y'\xi^1_{xx} -$$

$$-2(y')^2\xi^1_{xy} - (y')^3\xi^1_{yy},$$

where ξ^1 and η^1 are arbitrary functions.

Thus, the approximate group of equivalences transformations for Eq. (3.15) is infinite-dimensional and is generated by the operators

$$
\begin{aligned}
&E_1 = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} - 3xy'f\frac{\partial}{\partial f}, \quad E_2 = y\frac{\partial}{\partial x} - 3y'f\frac{\partial}{\partial f},\\
&E_3 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} - 3fx\frac{\partial}{\partial f}, \quad E_4 = x\frac{\partial}{\partial x} - 2f\frac{\partial}{\partial f},\\
&E_5 = \frac{\partial}{\partial x}, \quad E_6 = y\frac{\partial}{\partial y} + f\frac{\partial}{\partial f}, \qquad E_7 = x\frac{\partial}{\partial y}, \quad E_8 = \frac{\partial}{\partial y},\\
&E_\infty = \varepsilon\xi^1(x,y)\frac{\partial}{\partial x} + \varepsilon\eta^1(x,y)\frac{\partial}{\partial y} - ((y')^3\xi^1_{yy} + (y')^2(2\xi^1_{xy} - \eta^1_{yy}) +\\
&+y'\xi^1_{xx} - 2\eta^1_{xy}) - \eta^1_{xy})\frac{\partial}{\partial f},
\end{aligned}
$$

where $\xi^1(x,y)$ and $\eta^1(x,y)$ are arbitrary functions. The transformations corresponding to these generators have the form

$$
\begin{aligned}
&E_1 : \tilde{x} = \frac{x}{1-a_1y}, \quad \tilde{y} = \frac{y}{1-a_1y}, \quad \tilde{y}' = \frac{y'}{1-a_1y+a_1xy'},\\
&\qquad \tilde{f}\frac{fy^2(1-a_1y)^3}{(1-a_1y+a_1xy')^3};\\
&E_2 : \tilde{x} = x + a_2y, \quad \tilde{y} = y, \quad \tilde{y}' = y'/(1+a_2y');\\
&\qquad \tilde{f} = f/(1+a_2y')^3,\\
&E_3 : \tilde{x} = x/(1-a_3x), \quad \tilde{y} = y/(1-a_3x), \quad \tilde{y}' = y' + a_3(y - xy'),\\
&\qquad \tilde{f} = f(1-a_3x)^3;\\
&E_4 : \tilde{x} = a_4x, \tilde{y} = y, \quad \tilde{y}' = y'/a_4, \quad \tilde{f} = f/a_4^2;\\
&E_5 : \tilde{x} = x + a_5, \quad \tilde{y} = y, \quad \tilde{y}' = y', \quad \tilde{f} = f;\\
&E_6 : \tilde{x} = x, \quad \tilde{y} = a_6y, \quad \tilde{y}' = a_6y', \quad \tilde{f} = a_6f;\\
&E_7 : \tilde{x} = x, \quad \tilde{y} = y + a_7x, \quad \tilde{y}' = y' + a_7, \quad \tilde{f} = f;\\
&E_8 : \tilde{x} = x, \quad \tilde{y} = y + a_8, \quad \tilde{y}' = y', \quad \tilde{f} = f;\\
&E_\infty : \tilde{x} = x + \varepsilon a\xi^1(x,y), \quad \tilde{y} = y + \varepsilon a\eta^1(x,y),\\
&\qquad \tilde{y}' = y' + \varepsilon a(-(y')^2\xi^1_y + y'(\eta^1_y - \xi^1_x) + \eta^1_x),\\
&\qquad \tilde{f} = f - a((y')^3\xi^1_{yy} + (y')^2(2\xi^1_{xy} - \eta^1_{yy}) +\\
&\qquad\qquad +y'(\xi^1_{xx} - 2\eta^1_{xy}) - \eta^1_{xx}).
\end{aligned}
\tag{3.16}
$$

It should be noted that only in the case of Eq. (3.15), all the symmetries of the equation $y" = 0$ are stable with respect to the perturbations considered.

In the case of $\varepsilon = 1$, the obtained differential equation $y" = f(x, y, y')$ preserves its form under an arbitrary pointwise change of variables $\tilde{x} = \varphi(x, y), \tilde{y} = \psi(x, y)$. The generator of the corresponding equivalence transformations is:

$$E = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + (\eta_{xx} + y'(2\eta_{xy} - \xi_{xx}) + (y')^2(\eta_{yy} - 2\xi_{xy}) - (y')^3\xi_{yy})\frac{\partial}{\partial f}$$

[cf. the operator E_∞ in (3.16)].

Paper 5

Lie groups in turbulence

N.H. Ibragimov and G. Ünal [77]

Lie groups in turbulence: I. Kolmogorov's invariant and the algebra L_r.
Lie groups and their applications, Vol. 1, No. 2, 1994, pp. 98-103.

The usual application of the group theory to Navier-Stokes equations deals only with laminar flows, indeed point symmetry group G_φ of Navier-Stokes equations leaves unaltered the viscosity ν. In [133] an idea was initiated to use the equivalence group G_ε to adopt the Lie theory to turbulent flows. The equivalence transformation change the viscosity and therefore lead to a variety of liquids and scales of motions which depend on the viscosity. To identify those equivalence transformations connecting different scales of a given liquid Kolmogorov's invariant is used.

Here we employ Kolmogorov's invariant and determine the subgroup G_r of the equivalence group G_ε such that Kolmogorov's invariant is the first order differential invariant of G_r. It follows that the subgroup of G_r leaves invariant the energy balance equation. We believe that the group G_r plays a fundamental role in the turbulence theory similar to the heat representation of the Galilean group in heat conduction [66].

1 Equivalence and symmetry algebras for the Navier-Stokes equations

We consider the Navier-Stokes equations

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\Delta\mathbf{u}, \nabla\cdot\mathbf{u} = 0, \tag{1.1}$$

where $\mathbf{u} = (u^1, u^2, u^3)$,

$$\nabla = \left[\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right], \quad \Delta = \nabla \cdot \nabla,$$

and the density $\rho > 0$ is constant. Navier-Stokes equations admit the infinite dimensional equivalence Lie algebra L_ε consisting of the operators which are linear combination of the operators*

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, \quad X_2 = f(t)\frac{\partial}{\partial p}\\
X_3 &= x^i\frac{\partial}{\partial x^i} + 2t\frac{\partial}{\partial t} - u^i\frac{\partial}{\partial u^i} - 2p\frac{\partial}{\partial p}\\
X_{ij} &= x^j\frac{\partial}{\partial x^i} - x^i\frac{\partial}{\partial x^j} + u^j\frac{\partial}{\partial u^i} - u^i\frac{\partial}{\partial u^j}\\
X_4 &= h_1(t)\frac{\partial}{\partial x^1} + h_1'(t)\frac{\partial}{\partial u^1} - x^1 h_1''(t)\frac{\partial}{\partial p}\\
X_5 &= h_2(t)\frac{\partial}{\partial x^2} + h_2'(t)\frac{\partial}{\partial u^2} - x^2 h_2''(t)\frac{\partial}{\partial p}\\
X_6 &= h_3(t)\frac{\partial}{\partial x^3} + h_3'(t)\frac{\partial}{\partial u^3} - x^3 h_3''(t)\frac{\partial}{\partial p}\\
X_7 &= t\frac{\partial}{\partial t} + x^i\frac{\partial}{\partial x^i} + \nu\frac{\partial}{\partial \nu}
\end{aligned} \tag{1.2}$$

with arbitrary coefficients depending on ν [133], [78]. For all equations (1.1) with $\nu \neq 0$, the symmetry algebra is equivalent to the principal Lie algebra (in the sense defined in [76]) spanned by the $X_1, \ldots, X_6$ from (1.2) [118].

2 Kolmogorov's invariant and algebra L_r

Definition 5.1. The algebra L_r is the subalgebra of the equivalence algebra L_ε such that *Kolmogorov's invariant* [85], i.e. the energy dissipation rate

$$\varepsilon = \frac{\nu}{2}\sum_{r,s}(u_s^r + u_r^s)^2$$

is its differential invariant.

**Note to this 2007 edition:* Eqs. (1.1) allow the larger equivalence algebra where the functions $f(t), h_1(t), h_2(t)$, and $h_3(t)$ in the operators X_2, X_4, X_5 and X_6 are replaced by the arbitrary functions of two variables $f(t,\nu), h_1(t,\nu), h_2(t,\nu)$, and $h_3(t,\nu)$, respectively.

Theorem 5.1. The algebra L_r is infinite dimensional and it comprises the following operators:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = f(t)\frac{\partial}{\partial p}$$

$$X_\varepsilon = x^i\frac{\partial}{\partial x^i} + \frac{2}{3}t\frac{\partial}{\partial t} + \frac{1}{3}u^i\frac{\partial}{\partial u^i} + \frac{2}{3}p\frac{\partial}{\partial p} + \frac{4}{3}\nu\frac{\partial}{\partial \nu}$$

$$X_{ij} = x^j\frac{\partial}{\partial x^i} - x^i\frac{\partial}{\partial x^j} + u^j\frac{\partial}{\partial u^i} - u^i\frac{\partial}{\partial u^j}$$

$$X_4 = h_1(t)\frac{\partial}{\partial x^1} + h_1'(t)\frac{\partial}{\partial u^1} - x^1 h_1''(t)\frac{\partial}{\partial p}$$

$$X_5 = h_2(t)\frac{\partial}{\partial x^2} + h_2'(t)\frac{\partial}{\partial u^2} - x^2 h_2''(t)\frac{\partial}{\partial p}$$

$$X_6 = h_3(t)\frac{\partial}{\partial x^3} + h_3'(t)\frac{\partial}{\partial u^3} - x^3 h_3''(t)\frac{\partial}{\partial p} \tag{2.3}$$

Proof. Since the ε only involves the first order differentials, first prolongations of operators X_1 and X_2 do not comprise differential functions which appear in ε. These operators leave ε invariant:

$$X_1\varepsilon = 0,$$

$$\frac{\partial}{\partial t}[2(u_1^1)^2 + 2(u_2^2)^2 + 2(u_3^3)^2 + (u_2^1 + u_1^2)^2 + (u_3^1 + u_1^3)^2 + (u_3^2 + u_2^3)^2] = 0,$$

$$X_2\varepsilon = 0, \quad \left[f(t)\frac{\partial}{\partial p} + f(t)'\frac{\partial}{\partial p}\right]\varepsilon = 0.$$

Now we act by the first prolongation the linear combination $X_3^1 = \alpha X_3 + \beta X_7$ of the operators X_3 and X_7 on ε :

$$X_3^1\varepsilon = 0,$$

$$\left[\alpha x^i\frac{\partial}{\partial x^i} + (2\alpha+\beta)t\frac{\partial}{\partial t} + (\alpha+\beta)u^i\frac{\partial}{\partial u^i} - 2(\alpha+\beta)p\frac{\partial}{\partial p} - \beta\nu\frac{\partial}{\partial \nu}\right.$$
$$\left. -(2\alpha+\beta)u_j^i\frac{\partial}{\partial u_j^i} - (3\alpha+2\beta)u_t^i\frac{\partial}{\partial u_t^i} - (3\alpha+2\beta)p_m\frac{\partial}{\partial p_m}\right]\varepsilon = 0$$

to obtain

$$\nu(4\alpha+3\beta)[2(u_1^1)^2 + 2(u_2^2)^2 + 2(u_3^3)^2 + (u_2^1+u_1^2)^2 + (u_3^1+u_1^3)^2 + (u_3^2+u_2^3)^2] = 0.$$

Thus we have

$$\alpha = 1 \quad \text{and} \quad \beta = -\frac{4}{3}$$

which yields

$$X_3^1 = X_3 - \frac{4}{3} X_7$$

to be written. Next we act the first prolongation of operators X_{ij} on ε :

$$X_{ij}\varepsilon = 0,$$

$$\Big[x^j \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^j} + u^j \frac{\partial}{\partial u^i} - u^i \frac{\partial}{\partial u^j} + (u_m^j \delta_{i1} - u_i^1 \delta_{jm} + u_j^1 \delta_{im}) \frac{\partial}{\partial u_m^1} + (-u_m^i \delta_{j2}$$

$$+u_m^j \delta_{i2} - u_i^2 \delta_{jm} + u_j^2 \delta_{im}) \frac{\partial}{\partial u_m^2} + (-u_m^i \delta_{j3} - u_i^3 \delta_{jm} - u_j^3 \delta_{im}) \frac{\partial}{\partial u_m^3} +$$

$$(-p_i \delta_{jm} + p_j \delta_{im}) \frac{\partial}{\partial p_m}\Big] \, \varepsilon = 0,$$

to get

$$4u_1^1(u_1^j \delta_{i1} - u_i^1 \delta_{j1} + u_j^1 \delta_{i1}) + 4u_2^2(-u_2^i \delta_{j2} + u_2^j \delta_{i2} - u_i^2 \delta_{j2} + u_j^2 \delta_{i2}) + 4u_3^3(-u_3^i \delta_{j3}$$

$$-u_i^3 \delta_{j3} + u_j^3 \delta_{i3}) + 2(u_2^1 + u_1^2)(u_2^j \delta_{i1} - u_i^1 \delta_{j2} + u_j^1 \delta_{i2} - u_1^i \delta_{j2} + u_1^j \delta_{i2} - u_i^2 \delta_{j1} + u_j^2 \delta_{i1})$$

$$+2(u_3^1 + u_1^3)(u_3^j \delta_{i1} - u_i^1 \delta_{j3} + u_j^1 \delta_{i3} - u_1^i \delta_{j3} - u_i^3 \delta_{j1} + u_j^3 \delta_{i1}) + 2(u_3^2 + u_2^3)(-u_3^i \delta_{j2}$$

$$+u_3^j \delta_{i2} - u_i^2 \delta_{j3} + u_j^2 \delta_{i3} - u_i^2 \delta_{j3} - u_2^i \delta_{j3} - u_i^3 \delta_{j2} + u_j^3 \delta_{i2} = 0.$$

Latter vanishes when $i = 1, j = 2; i = 1, j = 3$ and $i = 2, j = 3$. Owing to the fact that the first prolongations of operators X_4, X_5 and X_6 do not involve the differential functions which appear in ε. These operators leave the ε. These operators leave the ε invariant.

$$X_4\varepsilon = 0,$$

$$\Big[h_1(t) \frac{\partial}{\partial x^1} + h_1'(t) \frac{\partial}{\partial u^1} - x^1 h_1''(t) \frac{\partial}{\partial p} + (h_1'' - u_1^1 h_1') \frac{\partial}{\partial u_t^1}$$

$$-u_1^2 h_1' \frac{\partial}{\partial u_t^2} - u_1^3 h_1' \frac{\partial}{\partial u_t^3} - h_1'' \frac{\partial}{\partial p_1} - (x^1 h_1'' + p_1 h_1') \frac{\partial}{\partial p_t}\Big] \, \varepsilon = 0,$$

$$X_5\varepsilon = 0,$$

$$\Big[h_2(t) \frac{\partial}{\partial x^2} + h_2'(t) \frac{\partial}{\partial u^2} - x^2 h_2''(t) \frac{\partial}{\partial p} + (h_2'' - u_2^2 h_2') \frac{\partial}{\partial u_t^2}$$

$$-u_2^1 h_2' \frac{\partial}{\partial u_t^1} - u_2^3 h_2' \frac{\partial}{\partial u_t^3} - h_2'' \frac{\partial}{\partial p_2} - (x^2 h_2'' + p_2 h_2') \frac{\partial}{\partial p_t}\Big] \, \varepsilon = 0,$$

$$X_6\varepsilon = 0,$$

$$\left[h_3(t)\frac{\partial}{\partial x^3} + h_3'(t)\frac{\partial}{\partial u^3} - x^3 h_3''(t)\frac{\partial}{\partial p} + (h_3'' - u_3^3 h_3')\frac{\partial}{\partial u_t^3}\right.$$
$$\left.-u_3^2 h_3'\frac{\partial}{\partial u_t^2} - u_3^1 h_3'\frac{\partial}{\partial u_t^1} - h_3''\frac{\partial}{\partial p_3} - (x^3 h_3'' + p_3 h_3')\frac{\partial}{\partial p_t}\right]\varepsilon = 0.$$

These calculation completes the proof.

3 The main property of algebra L_r

Theorem 5.2. The energy balance equation:

$$\Delta = \rho u^r u_t^r + \rho u^r u^3 u_3^r + u^r p_r - \mu u^r u_{33}^r = 0$$

is invariant with respect to a subgroup of L_r. Here μ is the dynamic viscosity ($\nu = \mu/\rho$).

Proof: Since the Δ involves second order differentials. The second prolongations of the operators should be acted on the Δ. The second prolongation of X_3^1 is given by:

$$X_\varepsilon = x^i\frac{\partial}{\partial x^i} + \frac{2}{3}t\frac{\partial}{\partial t} + \frac{1}{3}u^i\frac{\partial}{\partial u^i} + \frac{2}{3}p\frac{\partial}{\partial p} + \frac{4}{3}\nu\frac{\partial}{\partial \nu} - \frac{2}{3}u_j^i\frac{\partial}{\partial u_j^i} - \frac{1}{3}u_t^i\frac{\partial}{\partial u_t^i}$$
$$-\frac{1}{3}p_m\frac{\partial}{\partial p_m} - \left[\frac{2}{3}u_{nm}^\gamma - u_{mi}^\gamma\delta_{in} + \frac{2}{3}u_{mt}^\gamma\delta_{tn}\right]\frac{\partial}{\partial u_{nm}^\gamma}.$$

The second prolongation of X_{ij} is found to be:

$$X_{ij} = x^j\frac{\partial}{\partial x^i} - x^i\frac{\partial}{\partial x^j} + u^j\frac{\partial}{\partial u^i} - u^i\frac{\partial}{\partial u^j} + (u_m^i\delta_{i1} - u_i^1\delta_{jm} + u_j^1\delta_{im})\frac{\partial}{\partial u_m^1}$$
$$+(-u_m^i\delta_{j2} + u_m^j\delta_{i2} - u_i^2\delta_{jm} + u_j^2\delta_{im})\frac{\partial}{\partial u_m^2} + (-u_m^i\delta_{j3} - u_i^3\delta_{jm}$$
$$+u_j^3\delta_{im})\frac{\partial}{\partial u_m^3} + (-p_i\delta_{jm} + p_j\delta_{im})\frac{\partial}{\partial p_m} + (u_{nm}^2\delta_{i1}\delta_{j2} + u_{nm}^3\delta_{i1}\delta_{j3}$$
$$-u_{ni}^1\delta_{jm} + u_{nj}^1\delta_{im} - u_{mi}^1\delta_{jn} + u_{mj}^1\delta_{in})\frac{\partial}{\partial u_{mn}^1} + (-u_{nm}^1\delta_{i1}\delta_{j2} + u_{nm}^3\delta_{i2}\delta_{j3}$$
$$-u_{ni}^2\delta_{jm} + u_{nj}^2\delta_{im} - u_{mi}^2\delta_{jn} + u_{mj}^2\delta_{in})\frac{\partial}{\partial u_{mn}^2} + (-u_{nm}^1\delta_{i1}\delta_{j3} + u_{nm}^2\delta_{i2}\delta_{j3}$$
$$-u_{ni}^3\delta_{jm} + u_{nj}^3\delta_{im} - u_{mi}^3\delta_{jn} + u_{mj}^3\delta_{in})\frac{\partial}{\partial u_{mn}^3}$$

Now acting by the second prolongation of X_3^1 on δ one can obtain:

$$\frac{\rho}{3}u^r u_t^r - \frac{\rho}{3}u^r u_t^r + \frac{2\rho}{3}u^r u^3 u_3^r - \frac{2\rho}{3}u^r u^3 u_3^r + \frac{1}{3}u^r p_r - \frac{1}{3}u^r p_r + \frac{1}{3}\mu u_{33}^r + \frac{4}{3}\mu u^r u_{33}^r$$

$$-\frac{2}{3}\mu u^r u_{33}^r - \mu u^r u_{33}^r = 0$$

Finally we act the second prolongation of X_{ij} on Δ one can obtain:

$$T_1 + T_2 + T_3 + T_4 = 0$$

where

$$T_1 = \rho[u^j u_t^r \delta_{ri} - u^i u_t^r \delta_{rj} + u^1 u_t^j \delta_{i1} - u^2(-u_t^i \delta_{j2} + u_t^j \delta_{i2}) - u^3 u_t^i \delta_{j3}],$$

$$T_2 = \rho[u^j u^3 u_3^r \delta_{ir} - u^i u^3 u_3^r \delta_{j3} + u^1 u^m (u_m^j \delta_{i1} - u_i^1 \delta_{jm} + u_j^1 \delta_{im}) + u^2 u^m (-u_m^i \delta_{j2}$$
$$+ u_m^j \delta_{i2} - u_i^2 \delta_{jm} + u_j^2 \delta_{im}) + u^3 u^m (-u_m^i \delta_{j3} - u_i^3 \delta_{jm} + u_j^3 \delta_{im})],$$

$$T_3 = u^j p_r \delta_{ir} - u^i p_r \delta_{jr} - u^j p_i + u^i p_j,$$

$$T_4 = \mu[u^j u_{33}^i - u^i u_{33}^j + u^1(u_{33}^2 \delta_{i1}\delta_{j2} + u_{33}^3 \delta_{i1}\delta_{j3}) + u^2(-u_{33}^1 \delta_{i1}\delta_{j2} + u_{33}^3 \delta_{i2}\delta_{j3})$$
$$-u^3(-u_{33}^1 \delta_{i1}\delta_{j3} + u_{33}^2 \delta_{i2}\delta_{j3})].$$

$T_1 - T_4$ vanish when $\mathbf{i} = 1, j = 2; \mathbf{i} = 1, j = 3$ and $i = 2, j = 3$. It is clear that the groups generated by X_1 and X_2 leave the Δ invariant. Q.E.D.

4 How different scales of motion are generated by L_r

Let us consider the equivalence group generated by the operator X_ε

$$\bar{u}_i = e^a u_i,\ \bar{x}_i = e^{3a} x_i,\ \bar{t}e^{2a} t,\ \bar{\nu} = e^{4a}\nu \tag{4.4}$$

Since the Lie Groups are infinite groups, starting with constant viscosity we get different values of $\bar{\nu}$ by changing the group parameter a. What is physically happening here is the following: different scales of fluid motion (eddies) are affected differently by the viscosity. The effect of viscosity on small scale eddies are more severe than the larger ones (the faster the fluid motion it gets, higher the effect of viscosity becomes). Now we will illustrate this with an example. The equivalence group (4.4) allows one to write:

$$u_i = e^{-a} f_i(e^{2a} t, e^{3a} x_{i,} e^{4a} \nu).$$

For the sake of clarity let us take one dimensional periodic flow:

$$u = e^{-a} \sin(e^{2a} t)$$

increasing the value of parameter a increases the frequency of fluid motion but decreases its amplitude because of viscosity effects (damping). The correspondence between different incompressible fluid flows (each having different ν) can also be established through (4.4).

As it has already been shown in [78], the group of projective transformations:

$$\bar{\nu} = \frac{\nu}{1 - a\nu}, \quad \bar{t} = \frac{t}{1 - a\nu}, \quad \bar{x}^i = \frac{x^i}{1 - a\nu}, \tag{4.5}$$

are also admitted by the Navier-Stokes equations. In this case one can write the following:

$$u = \frac{1}{(1 - a\nu)^{1/4}} g(t(1 - a\nu)^{1/2},\ x(1 - a\nu)^{3/4},\ \frac{\nu}{1 - a\nu}).$$

In the limit where the parameter a approaches to the $1/\nu$ the amplitude and the period increases. This type of effect can be interpreted as negative viscosity effect.

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Paper 6

Integration of third-order equations admitting L_3 by Lie's method

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Abstract. We show how to integrate third order differential equations which admit a three-dimensional symmetry Lie algebra L_3. If L_3 is solvable, then we integrate the equation by quadrature, in accordance with Lie's theory. If L_3 is not solvable, then we can still integrate the given third order equation by reducing it to a first order equation, which can be transformed into a Riccati equation, thanks to the fact that L_3 transforms one of the symmetry of the third order equation into a non - local symmetry of the first order equation. Some examples are provided.

1 Introduction

Lie showed that an ordinary differential equation of order n with a known n-dimensional Lie symmetry algebra can be integrated by quadrature provided that its symmetry algebra is solvable. The general integrating procedure consists of n successive integrations and leads to quite lengthy calculations.

In the case of second order equations, Lie [93] simplified the integration procedure by using the canonical representation of two-dimensional algebras on the (x, y) plane. By this canonical representation, the corresponding differential equations assume a directly integrable form (e.g. see [64]).

The purpose of the present paper is to apply Lie's method to the integration of third order equations which admit a three-dimensional Lie symmetry algebra L_3.

If L_3 is solvable then we can reduce the given third order equation to a first order equation which is integrable by quadrature, and then obtain a second order equation which can be transformed into a directly integrable form.

If L_3 is not solvable then we can still reduce the given third order equation to a first order equation; this equation is not integrable by quadrature but can be easily reduced to a Riccati equation by using a non-local symmetry which comes from one of the symmetry of the original third order equation.

We have used the group classification of third order equations due to Mahomed [100] and Gat [35] in order to cover all the possible cases.

In the last section, we show some examples of third order differential equations which admit a three-dimensional Lie symmetry algebra.

We have found those Lie symmetry algebras by using an interactive REDUCE program developed by M.C.N. [108].

2 Equations admitting solvable Lie algebras

Let us consider a third order differential equation which admits a three-dimensional solvable Lie algebra L_3. First, we reduce it to a first order equation by using the differential invariants of an ideal $L_2 \subset L_3$. Then, the first order equation can be integrated by quadrature, because it admits the one-dimensional Lie algebra L_3/L_2. Its general solution becomes a second order equation in the original variables. This equation admits L_2, therefore it can be integrated by quadrature by using the canonical representation of L_2, as Lie showed [64].

2.1. The algebra L_3 with basis

$$\boxed{X_1 = \partial_x, \qquad X_2 = \partial_y, \qquad X_3 = x\partial_y} \tag{2.1}$$

and its most general invariant equation:

$$\boxed{y''' = f(y'')} \tag{2.2}$$

The commutators of (2.1) are:

$$[X_1, X_2] = 0, \qquad [X_1, X_3] = X_2, \qquad [X_2, X_3] = 0.$$

Therefore, X_2, X_3 span the ideal $L_2 = L_2 < X_2, X_3 >$. A basis of differential invariants of L_2 of order ≤ 2 is given by:

$$u = x, \qquad v = y''. \tag{2.3}$$

Then, equation (2.2) is easily reduced to the following first order equation:

$$\frac{\mathrm{d}v}{\mathrm{d}u} = f(v) \tag{2.4}$$

which admits the operator $X_1 = \partial_u$. Therefore, equation (2.4) can be integrated by quadrature, i.e.:

$$\int \frac{\mathrm{d}v}{f(v)} = u + c_1$$

which yields $v = F(u + c_1)$. By introducing the original variables and integrating twice, we obtain the general solution of (2.2), i.e.:

$$y = \int \left(\int F(x + c_1) dx \right) dx + c_2 x + c_3 \tag{2.5}$$

with c_1, c_2, c_3 arbitrary constants. The same result could be obtained by using the ideal L_2 spanned by X_1, X_2.

2.2. The algebra L_3 with basis

$$\boxed{X_1 = \partial_x, \qquad X_2 = \partial_y, \qquad X_3 = \lambda x \partial_x + y \partial_y \qquad (\lambda \neq \tfrac{1}{2})} \tag{2.6}$$

and its most general invariant equation:

$$\boxed{y''' = f\left(y''^{\lambda - 1} y'^{1 - 2\lambda} \right) y''^{\frac{3\lambda - 1}{2\lambda - 1}}} \tag{2.7}$$

The commutators of (2.6) are:

$$[X_1, X_2] = 0, \qquad [X_1, X_3] = \lambda X_1, \qquad [X_2, X_3] = X_2.$$

Therefore, X_1, X_2 span the ideal $L_2 = L_2 < X_1, X_2 >$. A basis of differential invariants of L_2 of order ≤ 2 is given by:

$$u = y', \qquad v = y''. \tag{2.8}$$

Then, equation (2.7) is reduced to the following first order equation:

$$\frac{\mathrm{d}v}{\mathrm{d}u} = f(v^{\lambda - 1} u^{1 - 2\lambda}) v^{\frac{\lambda}{2\lambda - 1}} \tag{2.9}$$

which admits the operator X_3 in the space of variables u, v, i.e.:

$$X_3 = (1 - \lambda) u \partial_u + (1 - 2\lambda) v \partial_v \,. \tag{2.10}$$

It should be noticed that if $\lambda = 1$ then equation (2.9) is integrable by quadrature. Therefore we assume $\lambda \neq 1$. We transform the operator (2.10) into its canonical form $X_3 = \partial_t$ by the change of variables:

$$t = \log u, \qquad z = v^{\lambda-1}u^{1-2\lambda}$$

which are found by solving the following equations:

$$u\frac{\partial t}{\partial u} + \frac{1-2\lambda}{1-\lambda}v\frac{\partial t}{\partial v} = 1,$$
$$u\frac{\partial z}{\partial u} + \frac{1-2\lambda}{1-\lambda}v\frac{\partial z}{\partial v} = 0.$$

In the variables t, z, equation (2.9) becomes:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = z\left[(\lambda-1)z^{1/(1-2\lambda)}f(z) + 1 - 2\lambda\right] \tag{2.11}$$

which is integrable by quadrature. Let $z = F(t, c_1)$ be its general integral. Then, by introducing the original variables, we obtain:

$$y''^{\lambda-1}y'^{1-2\lambda} = F(\log y', c_1)$$

which is a second order differential equations admitting the symmetry algebra $L_2 = L_2 < X_1, X_2 >$, and can easily be integrated by using Lie's approach [64].

2.3. The algebra L_3 with basis

$$\boxed{X_1 = \partial_x, \qquad X_2 = \partial_y, \qquad X_3 = x\partial_x + 2y\partial_y} \tag{2.12}$$

and its most general invariant equation:

$$\boxed{y''' = f(y'')y'^{-1}} \tag{2.13}$$

This is the previous case with $\lambda = \frac{1}{2}$. By using L_2, and the variables (2.8), we obtain:

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{f(v)}{uv}$$

which is integrable by quadrature.

2.4. The algebra L_3 with basis

$$\boxed{X_1 = \partial_y, \qquad X_2 = x\partial_y, \qquad X_3 = (\lambda-1)x\partial_x - y\partial_y \qquad (\lambda \neq \tfrac{1}{2}, 1)} \tag{2.14}$$

and its most general invariant equation:

$$\boxed{y''' = f\left(x^{1-2\lambda}y''^{1-\lambda}\right)y''^{\frac{3\lambda-2}{2\lambda-1}}} \tag{2.15}$$

The commutators of (2.14) are:

$$[X_1, X_2] = 0, \quad [X_1, X_3] = -X_1, \quad [X_2, X_3] = -\lambda X_2.$$

Therefore, X_1, X_2 span the ideal $L_2 = L_2 < X_1, X_2 >$. A basis of differential invariants of L_2 of order ≤ 2 is given by (2.3). Then, equation (2.15) is reduced to the following first order equation:

$$\frac{\mathrm{d}v}{\mathrm{d}u} = f(v^{1-\lambda}u^{1-2\lambda})v^{\frac{3\lambda-2}{2\lambda-1}} \tag{2.16}$$

which admits the operator X_3 in the space of variables u, v, i.e.:

$$X_3 = (\lambda - 1)u\partial_u + (1 - 2\lambda)v\partial_v \,. \tag{2.17}$$

We transform the operator (2.17) into its canonical form $X_3 = \partial_t$ by the change of variables:

$$t = \log u, \qquad z = v^{1-\lambda}u^{1-2\lambda}.$$

In these variables t, z, equation (2.16) becomes:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = (1 - 2\lambda)z + (1 - \lambda)z^{\frac{2(\lambda-1)}{2\lambda-1}} f(z) \tag{2.18}$$

which is integrable by quadrature.

2.5. The algebra L_3 with basis

$$\boxed{X_1 = \partial_x, \quad X_2 = x\partial_y, \quad X_3 = x\partial_x + 2y\partial_y} \tag{2.19}$$

and its most general invariant equation:

$$\boxed{y''' = f(y'')x^{-1}} \tag{2.20}$$

This is the previous case with $\lambda = \frac{1}{2}$. By using L_2, and the variables (2.3), we obtain:

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{f(v)}{u}$$

which is integrable by quadrature.

2.6. The algebra L_3 with basis

$$\boxed{X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = x\partial_x + (x + y)\partial_y} \tag{2.21}$$

and its most general invariant equation:

$$\boxed{y''' = e^{-y'}y''f\left(e^{y'}y''\right)} \tag{2.22}$$

The commutators of (2.21) are:

$$[X_1, X_2] = 0, \qquad [X_1, X_3] = X_1 + X_2, \qquad [X_2, X_3] = X_2.$$

Therefore, X_1, X_2 span the ideal $L_2 = L_2 < X_1, X_2 >$. A basis of differential invariants of L_2 of order ≤ 2 is given by (2.8). Then, equation (2.22) is reduced to the following first order equation:

$$\frac{\mathrm{d}v}{\mathrm{d}u} = e^{-u} f\left(e^u v\right) \tag{2.23}$$

which admits the operator X_3 in the space of variables u, v, i.e.:

$$X_3 = \partial_u - v\partial_v\,. \tag{2.24}$$

We transform the operator (2.24) into its canonical form $X_3 = \partial_t$ by the change of variables:

$$t = u, \qquad z = e^u v. \tag{2.25}$$

In these variables t, z, equation (2.23) becomes:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = z + f(z) \tag{2.26}$$

which is integrable by quadrature.

2.7. The algebra L_3 with basis

$$\boxed{X_1 = \partial_y, \qquad X_2 = x\partial_y, \qquad X_3 = \partial_x - y\partial_y} \tag{2.27}$$

and its most general invariant equation:

$$\boxed{y''' = e^{-x} f\left(e^x y''\right)} \tag{2.28}$$

The commutators of (2.27) are:

$$[X_1, X_2] = 0, \qquad [X_1, X_3] = -X_1, \qquad [X_2, X_3] = -X_1 - X_2.$$

Therefore, X_1, X_2 span the ideal $L_2 = L_2 < X_1, X_2 >$. A basis of differential invariants of L_2 of order ≤ 2 is given by (2.3). Then, equation (2.28) is reduced to the first order equation (2.23), and, of course, the operator X_3 becomes (2.24) in the space of variables u, v.

3 Equations admitting non-solvable Lie algebras

Let us consider a third order differential equation which admits a three-dimensional non-solvable Lie algebra L_3. First, we reduce it to a first order equation by using the differential invariants of a two-dimensional subalgebra $L_2 \subset L_3$, which always exists in the complex domain. Then, the third operator can be used to simplify the obtained first order equation, although it is non local. Indeed, a Riccati equation is always found. Its general solution becomes a second order equation in the original variables. This equation admits L_2, therefore it can be integrated by quadrature by using the canonical representation of L_2, as Lie showed [64].

We make use of the following:

Definition. For a given operator of the form

$$X = \xi\partial_x + \eta\partial_y \tag{3.1}$$

the variables z, u are called *semi-canonical* if they transform the operator (3.1) into the following *semi-canonical* form

$$X = F\partial_u \tag{3.2}$$

with F an arbitrary coefficient.

Semi-canonical variables exist for any operator (3.1), and they are determined by the equation:

$$X(z) = 0. \tag{3.3}$$

3.1. The algebra L_3 with basis

$$\boxed{X_1 = \partial_y, \quad X_2 = y\partial_y, \quad X_3 = y^2\partial_y} \tag{3.4}$$

and its most general invariant equation:

$$\boxed{y''' = \frac{3y''^2}{2y'} + f(x)y'} \tag{3.5}$$

The commutators of (3.4) are:

$$[X_1, X_2] = X_1, \quad [X_1, X_3] = 2X_2, \quad [X_2, X_3] = X_3.$$

Therefore, X_1, X_2 span a two-dimensional subalgebra L_2. A basis of differential invariants of L_2 of order ≤ 2 is given by:

$$u = x, \qquad v = y''/y'. \tag{3.6}$$

Then, equation (3.5) is reduced to the following first order equation:

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{v^2}{2} + f(u) \tag{3.7}$$

which admits the non-local operator X_3 in the space of variables u, v, i.e.:

$$X_3 = 2y'\partial_v \tag{3.8}$$

It is not a surprise that (3.7) is a Riccati equation. In fact, it is well known [48] that (3.5) can be transformed into the Riccati equation (3.7) by using the change of variables (3.6).

3.2. The algebra L_3 with basis

$$\boxed{X_1 = \partial_y, \qquad X_2 = x\partial_x + y\partial_y, \qquad X_3 = xy\partial_x + \frac{y^2}{2}\partial_y} \tag{3.9}$$

and its most general invariant equation:

$$\boxed{y''' = \frac{3y''^2}{y'} + f\left(\frac{2xy'' + y'}{2y'^3}\right)\frac{y'^4}{x^2}} \tag{3.10}$$

The commutators of (3.9) are:

$$[X_1, X_2] = X_1, \qquad [X_1, X_3] = X_2, \qquad [X_2, X_3] = X_3.$$

Therefore, X_1, X_2 span a two-dimensional subalgebra L_2. A basis of differential invariants of L_2 of order ≤ 2 is given by:

$$u = y', \qquad v = xy'' \tag{3.11}$$

Then, equation (3.10) is reduced to the following first order equation:

$$v\frac{\mathrm{d}v}{\mathrm{d}u} = v + \frac{3v^2}{u} + f\left(\frac{2v + u}{2u^3}\right) \tag{3.12}$$

which admits the non-local operator X_3 in the space of variables u, v, i.e.:

$$X_3 = -x\left(u^2\partial_u + (u^2 + 3uv)\partial_v\right). \tag{3.13}$$

This operator is non-local, due to the appearance of x. Yet, we can transform (3.13) into its semi-canonical form (3.2), i.e. $X_3 = -xu^2\partial_u$, by introducing the new variable

$$z = \frac{2v + u}{2u^3}$$

which is obtained by solving (refsemceq), i.e.:

$$u^2\frac{\partial z}{\partial u}+(u^2+3uv)\frac{\partial z}{\partial v}=0.$$

It turns out that equation (3.12) becomes a Riccati equation in the variables z, u, i.e.:

$$\frac{\mathrm{d}u}{\mathrm{d}z}=\frac{zu^2-1/2}{f(z)} \tag{3.14}$$

3.3. The algebra L_3 with basis

$$\boxed{X_1=\partial_y,\qquad X_2=x\partial_x+y\partial_y,\qquad X_3=xy\partial_x+\frac{x^2+y^2}{2}\partial_y} \tag{3.15}$$

and its most general invariant equation:

$$\boxed{y'''=\frac{3y''^2y'}{y'^2-1}+\frac{\left(y'^2-1\right)^2}{x^2}f\left(\frac{xy''}{\left(1-y'^2\right)^{3/2}}-\frac{y'}{\left(1-y'^2\right)^{1/2}}\right)} \tag{3.16}$$

The commutators of (3.15) are:

$$[X_1,X_2]=X_1,\qquad [X_1,X_3]=X_2,\qquad [X_2,X_3]=X_3.$$

Therefore, X_1, X_2 span a two-dimensional subalgebra L_2. A basis of differential invariants of L_2 of order ≤ 2 is given by (3.11). Then, equation (3.16) is reduced to the following first order equation:

$$v\frac{\mathrm{d}v}{\mathrm{d}u}=v-\frac{3uv^2}{1-u^2}+(1-u^2)^2f\left(\frac{v-u(1-u^2)}{(1-u^2)^{3/2}}\right) \tag{3.17}$$

which admits the non-local operator X_3 in the space of variables u, v, i.e.:

$$X_3=x\left((1-u^2)\partial_u+(1-u^2-3uv)\partial_v\right). \tag{3.18}$$

We put (3.18) into its semi-canonical form (3.2), i.e. $X_3=x(1-u^2)\partial_u$, by introducing the new variable

$$z=\frac{v-u(1-u^2)}{(1-u^2)^{3/2}}$$

which is obtained by solving (3.3), i.e.:

$$(1-u^2)\frac{\partial z}{\partial u}+(1-u^2-3uv)\frac{\partial z}{\partial v}=0.$$

It turns out that in the variables z, u, equation (3.17) becomes:

$$\frac{\mathrm{d}u}{\mathrm{d}z} = \frac{(1-u^2)z + u(1-u^2)^{1/2}}{f(z)} \,. \tag{3.19}$$

This equation satisfies the Vessiot-Guldberg-Lie theorem (see [64], Chapter IV), i.e. admits a fundamental system of solutions (also known as a nonlinear superposition principle). This means that equation (3.19) can be transformed into a Riccati equation by a change of the dependent variable u only. Let us follow the procedure given by the Vessiot-Guldberg-Lie theorem. First, we consider the following two operators:

$$\Gamma_1 = (1-u^2)\partial_u, \qquad \Gamma_2 = u(1-u^2)^{1/2}\partial_u \,. \tag{3.20}$$

Their commutator is equal to

$$[\Gamma_1, \Gamma_2] = (1-u^2)^{1/2}\partial_u$$

and is not a linear combination of the operators (3.20). Therefore, we have to consider a third operator

$$\Gamma_3 = (1-u^2)^{1/2}\partial_u \tag{3.21}$$

in order to obtain a Lie algebra. Operators (3.20)-(3.21) span a three-dimensional Lie algebra with the following commutators:

$$[\Gamma_1, \Gamma_2] = \Gamma_3 \,, \quad [\Gamma_1, \Gamma_3] = \Gamma_2 \,, \quad [\Gamma_2, \Gamma_3] = -\Gamma_1 \,. \tag{3.22}$$

Any Riccati equation of the form

$$\frac{\mathrm{d}\phi}{\mathrm{d}z} = P(z) + Q(z)\phi + R(z)\phi^2$$

admits the Vessiot-Guldberg-Lie algebra spanned by

$$Y_1 = \partial_\phi \,, \quad Y_2 = \phi\partial_\phi \,, \quad Y_3 = \phi^2\partial_\phi \tag{3.23}$$

with the following commutators:

$$[Y_1, Y_2] = Y_1 \,, \quad [Y_1, Y_3] = 2Y_2 \,, \quad [Y_2, Y_3] = Y_3.$$

Therefore, we have to transform the operators (3.20)-(3.21) into the form (3.23) by a change of the dependent variable

$$\phi = h(u) \tag{3.24}$$

in order for equation (3.19) to become a Riccati equation. If we consider

$$\overline{\Gamma}_1 = \Gamma_3 - \Gamma_2 \equiv Y_1\,, \quad \overline{\Gamma}_2 = \Gamma_1 \equiv Y_2\,, \quad \overline{\Gamma}_3 = \Gamma_3 + \Gamma_2 \equiv Y_3 \tag{3.25}$$

and introduce the new dependent variable

$$\phi = \left(\frac{1+u}{1-u}\right)^{1/2}$$

which is obtained by solving the following equation:

$$(1-u^2)\frac{\mathrm{d}\phi}{\mathrm{d}u} = \phi$$

then, equation (3.19) becomes a Riccati equation in the variables z, ϕ, i.e.:

$$\frac{\mathrm{d}\phi}{\mathrm{d}z} = \frac{2z\phi + \phi^2 - 1}{2f(z)}\,. \tag{3.26}$$

Recently, Clarkson and Olver [23] have shown that the Lie algebras in subsections 3.2. and 3.3. are connected by prolongation to the Lie algebra in subsection 3.1., which gives the theoretical explanation of the appearance of a Riccati equation in each case.

4 Examples

In this section, we give some examples of third order differential equations, which admit a three-dimensional Lie symmetry algebra L_3.

We remark that each of the following equation is reducible to a Riccati equation, even in the example with solvable L_3.

4.1. Equation

$$\boxed{w''' = 3w'' + \frac{w'w''}{w} - \frac{w'^2}{w} - 2w', \quad w = w(s)} \tag{4.1}$$

This equation was obtained by Whittaker in [139] by embedding the iterates

$$f^n(2\cos u) = 2\cos(2^n u)$$

into a continuously evolving system. He also found the general solution of (4.1) by “a lucky guess” method. Instead, we apply Lie's method to find that (4.1) admits a solvable Lie algebra L_3 with basis:

$$X_1 = \partial_s, \qquad X_2 = e^{-s}\partial_s, \qquad X_3 = w\partial_w\,. \tag{4.2}$$

Their commutators are:

$$[X_1, X_2] = -X_2, \qquad [X_1, X_3] = 0, \qquad [X_2, X_3] = 0.$$

It is easy to show that (4.1) is a particular case of (2.7) with $\lambda = 0$. In fact, by introducing the change of variables:

$$x = \log w, \qquad y = e^s$$

equation (4.1) transforms into:

$$y''' = \left(3\frac{y''}{y'} - 2\right) y'', \quad y = y(x) \tag{4.3}$$

which is a particular case of (2.7) with $\lambda = 0$ and $f(*) = 3/* - 2$. In order to obtain a Riccati equation, we consider the two-dimensional subalgebra spanned by X_1, X_2. A basis of its differential invariants of order ≤ 2 is given by:

$$u = w, \qquad v = (w'' - w')w'^{-2}. \tag{4.4}$$

Then, equation (4.1) is reduced to the following Riccati equation:

$$\frac{\mathrm{d}v}{\mathrm{d}u} = -2v^2 + \frac{v}{u} \tag{4.5}$$

which admits the operator X_3 in the space of variables u, v, i.e.:

$$X_3 = u\partial_u - v\partial_v\,. \tag{4.6}$$

The general solution of (4.5) is easily found to be:

$$v = -\frac{u}{c_1^2 - u^2} \tag{4.7}$$

which becomes a second order equation in the original variables, i.e.:

$$w'' = w' - \frac{ww'^2}{c_1^2 - w^2}$$

admitting the two-dimensional algebra spanned by X_1, X_2. Then, we easily find its general solution to be:

$$w = c_1 \cos\left(c_3 e^s + c_2\right).$$

We leave to the reader the application of the method delineated in subsection 2.2. to the equation (4.3).

4.2. Equation

$$\boxed{w''' = w^{-2}, \quad w = w(s)} \tag{4.8}$$

This equation represents the small-w limit of an equation which is relevant to fluid draining problems on a dry wall, and the large-w limit to draining over a wet wall [132]. Its general solution was given in [33] by "a lucky guess" method. Lie's method when applied to (4.8) leads to a non-solvable Lie algebra L_3 with basis:

$$X_1 = \partial_s, \qquad X_2 = s\partial_s + w\partial_w, \qquad X_3 = s^2\partial_s + 2sw\partial_w\,. \tag{4.9}$$

It is easy to show that (4.8) is a particular case of (3.10). In fact, by introducing the change of variables:

$$x = w, \qquad y = s$$

equation (4.8) transforms into:

$$y''' = \frac{3y''^2}{y'} - \frac{y'^4}{x^2}, \quad y = y(x) \tag{4.10}$$

which is a particular case of (3.10) with $f(*) = -1$. Let us consider the two-dimensional subalgebra spanned by X_1, X_2. A basis of its differential invariants of order ≤ 2 is given by:

$$u = w', \qquad v = ww''. \tag{4.11}$$

Then, equation (4.8) is reduced to the following first order equation:

$$v\frac{\mathrm{d}v}{\mathrm{d}u} = uv + 1 \tag{4.12}$$

which admits the non-local operator X_3 in the space of variables u, v, i.e.:

$$X_3 = w(\partial_u + u\partial_v). \tag{4.13}$$

We put (4.13) into its semi-canonical form (3.2), i.e. $X_3 = w\partial_u$, by introducing the new variable

$$z = v - u^2/2$$

which is obtained by solving (3.3), i.e.:

$$\frac{\partial z}{\partial u} + u\frac{\partial z}{\partial v} = 0.$$

Then, equation (4.12) becomes a Riccati equation in the variables z, u, i.e.:

$$\frac{\mathrm{d}u}{\mathrm{d}z} = \frac{u^2}{2} + z. \tag{4.14}$$

This equation was derived in [33], although it was not explained how the change of variables which reduces equation (4.8) to the Riccati equation (4.14) was found. Finally, the general solution of (4.14) is easily found t o be given in terms of Airy functions [33].

4.3. Equation

$$\boxed{w''' = -(4ww'' + 6w^2w' + 3w'^2 + w^4), \quad w = w(s)} \tag{4.15}$$

This equation is the third order member of the Riccati-chain [3]. Therefore, (4.15) can be transformed into the linear equation

$$W^{(iv)} = 0 \tag{4.16}$$

by the change of variable

$$w = W'/W \tag{4.17}$$

and its general solution is easily found to be

$$w = \frac{c_2 + 2c_3s + 3c_4s^2}{c_1 + c_2s + c_3s^2 + c_4s^3}\,.$$

Lie's method when applied to (4.15) leads to a non-solvable Lie algebra L_3 with basis:

$$X_1 = \partial_s, \qquad X_2 = s\partial_s - w\partial_w, \qquad X_3 = s^2\partial_s - (2sw - 3)\partial_w\,. \tag{4.18}$$

It is easy to show that (4.15) is a particular case of (3.16). In fact, by introducing the change of variables:

$$x = -\frac{3}{2w}, \qquad y = s - \frac{3}{2w}$$

equation (4.15) transforms into:

$$y''' = \frac{3y''^2}{y' - 1} + \frac{6y'y''}{x} + \frac{3(9y'^4 - 10y'^2 + 1)}{8x^2}, \quad y = y(x) \tag{4.19}$$

which is a particular case of (3.16) with $f(*) = -3*^2 + 3/8$. Let us consider the two-dimensional subalgebra spanned by X_1, X_2. A basis of its differential invariants of order ≤ 2 is given by:

$$u = w'w^{-2}, \qquad v = w''w^{-3}. \tag{4.20}$$

Then, equation (4.15) is reduced to the following first order equation:

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{3uv + 4v + 6u + 3u^2 + 1}{2u^2 - v} \tag{4.21}$$

which admits the non-local operator X_3 in the space of variables u, v, i.e.:

$$X_3 = -w^{-1}\left(2(3u+1)\partial_u + 3(3v+2u)\partial_v\right). \tag{4.22}$$

Operator (4.22) can be transformed into its semi-canonical form (3.2), i.e. $X_3 = -2w^{-1}(3u+1)\partial_u$, by introducing the new variable

$$z = \frac{9(3u+1)v + 54u^2 + 30u + 4}{9(3u+1)^{5/2}}$$

which is obtained by solving (3.3), i.e.:

$$2(3u+1)\frac{\partial z}{\partial u} + 3(3v+2u)\frac{\partial z}{\partial v} = 0.$$

Then, in the variables z, u, equation (4.21) becomes:

$$\frac{\mathrm{d}u}{\mathrm{d}z} = 2\frac{(3u+1)^{1/2}(6u+4) - 9z(3u+1)}{81z^2+2} \tag{4.23}$$

This equation satisfies the Vessiot-Guldberg-Lie theorem. In fact, operators

$$\Gamma_1 = u(3u+1)^{1/2}\partial_u\,, \quad \Gamma_2 = (3u+1)^{1/2}\partial_u\,, \quad \Gamma_3 = (3u+1)\partial_u \tag{4.24}$$

form a three-dimensional Lie algebra, and if we consider

$$\overline{\Gamma}_1 = \frac{2}{3}\Gamma_2 \equiv Y_1\,, \quad \overline{\Gamma}_2 = \frac{2}{3}\Gamma_3 \equiv Y_2\,, \quad \overline{\Gamma}_3 = 2\Gamma_1 + \frac{2}{3}\Gamma_2 \equiv Y_3 \tag{4.25}$$

and introduce the new dependent variable

$$\phi = (3u+1)^{1/2}$$

which is obtained by solving the following equation:

$$\frac{2}{3}(3u+1)\frac{\mathrm{d}\phi}{\mathrm{d}u} = \phi$$

then, equation (4.23) becomes a Riccati equation in the variables z, ϕ, i.e.:

$$\frac{\mathrm{d}\phi}{\mathrm{d}z} = 3\frac{-9z\phi + 2\phi^2 + 2}{81z^2+2}\,. \tag{4.26}$$

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Paper 7

Lie symmetry analysis of differential equations in finance

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Abstract. Lie group theory is applied to differential equations occurring as mathematical models in financial problems. We begin with the complete symmetry analysis of the one-dimensional Black-Scholes model and show that this equation is included in Sophus Lie's classification of linear second-order partial differential equations with two independent variables. Consequently, the Black-Scholes transformation of this model into the heat transfer equation follows directly from Lie's equivalence transformation formulas. Then we carry out the classification of the two-dimensional Jacobs-Jones model equations according to their symmetry groups. The classification provides a theoretical background for constructing exact (invariant) solutions, examples of which are presented.

1 Introduction

The works of R.C. Merton [105],[106] and F. Black and M. Scholes [18] opened a new era in mathematical modelling of problems in finance. Originally, their models are formulated in terms of stochastic differential equations. Under certain conditions, some of these models can be rewritten as linear evolutionary partial differential equations with variable coefficients.

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For example, the widely used one-dimensional model (one state variable plus time) known as the *Black-Scholes model*, is described by the equation

$$u_t + \frac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu = 0 \tag{1.1}$$

with constant coefficients A, B, C (parameters of the model). Black and Scholes reduced it to the classical heat equation and used this relation for solving Cauchy's problem with special initial data.

Along with (1.1), more complex models aimed at explaining additional effects are discussed in the current literature (see, e.g. [125]). We will consider here the two state variable model suggested by Jacobs and Jones [79]:

$$\begin{aligned} u_t = {} & \frac{1}{2}A^2x^2u_{xx} + ABCxyu_{xy} + \frac{1}{2}B^2y^2u_{yy} \\ & + \big(Dx\ln\frac{y}{x} - Ex^{\frac{3}{2}}\big)u_x + \big(Fy\ln\frac{G}{y} - Hyx^{\frac{1}{2}}\big)u_y - xu, \end{aligned} \tag{1.2}$$

where A, B, C, D, E, F, G, H are arbitrary constant coefficients. Jacobs and Jones [79] investigate the model numerically. An analytical study of solutions of this equation as well as of other complex financial mathematics models presents a challenge for mathematicians. This is due to the fact that, as a rule, these models unlike the Black-Scholes equation (1.1), can not be reduced to simple equations with known solutions. Here, we demonstrate this fact for the *Jacobs-Jones equation* (1.2) by using methods of the Lie group analysis.

The *Lie group analysis* is a mathematical theory that synthesizes symmetry of differential equations. This theory was originated by a great mathematician of 19th century, Sophus Lie (Norway, 1842–1899). One of Lie's striking achievements was the discovery that the known *ad hoc* methods of integration of differential equations could be derived by means of his theory of continuous groups. Further Lie gave a classification of differential equations in terms of their symmetry groups, thereby identifying the set of equations which could be integrated or reduced to lower-order equations by group theoretic algorithms. Moreover Lie [89] gave the group classification of linear second-order partial differential equations with two independent variables and developed methods of their integration. In particular, according to his classification all parabolic equations admitting the symmetry group of the highest order reduce to the heat conduction equation.

An extensive compilation and systematization of the results on symmetry analysis and group classification of differential equations obtained by S. Lie and his followers during the period of over one hundred years is presented in the Handbook [65], [68], [68]. The material in the Handbook is presented in a form convenient for immediate applications by applied scientists to their own problems.

This paper is aimed at Lie group analysis (symmetries, classification and invariant solutions) of the Black-Scholes (1.1) and the Jacobs-Jones (1.2) models.

The contents of the present paper is as follows. Section 2 is designed to meet the needs of beginners and contains a short account of methods of Lie group analysis.

The group analysis of the Black-Scholes model is presented in Section 3. It is shown (subsection 3.2) that symmetry group of this model equation is similar to that of the classical heat equation, and hence the Black-Scholes model is contained in the Lie classification [89]. However the practical utilization of Lie's classification is not trivial. Therefore, we discuss calculations for obtaining transformations of (1.1) into the heat equation (subsection 3.3), transformations of solutions (subsection 3.4) and invariant solutions (subsection 3.5). Moreover, the structure of the symmetry group of the equation (1.1) allows one to apply the recent method for constructing the fundamental solution based on the so called *invariance principle* [63], [69]. This application is discussed in subsection 3.6.

The Jacobs-Jones model is considered in Section 4. Subsection 4.2 contains the result of the Lie group classification of the equations (1.2) with the coefficients satisfying the restrictions $A, B \neq 0, C \neq 0, \pm 1$. It is shown that the dimension of the symmetry group depends essentially on the parameters $A, B, \ldots$, of the model and that the equations of the form (1.2) can not be reduced to the classical two-dimensional heat equation. The algorithm of construction of invariant solutions under two-parameter groups and an illustration are given in subsection 4.3.

2 Outline of methods from group analysis

2.1 Calculation of infinitesimal symmetries

Consider evolutionary partial differential equations of the second order,

$$u_t - F(t, x, u, u_{(1)}, u_{(2)}) = 0, \tag{2.1}$$

where u is a function of independent variables t and $x = (x^1, \ldots, x^n)$, and $u_{(1)}, u_{(2)}$ are the sets of its first and second order partial derivatives: $u_{(1)} = (u_{x^1}, \ldots, u_{x^n})$, $u_{(2)} = (u_{x^1x^1}, u_{x^1x^2}, \ldots, u_{x^nx^n})$.

Recall that invertible transformations of the variables t, x, u,

$$\bar{t} = f(t,x,u,a), \quad \bar{x}^i = g^i(t,x,u,a), \quad \bar{u} = h(t,x,u,a), \quad i = 1,\ldots,n, \tag{2.2}$$

depending on a continuous parameter a are said to be *symmetry transformations of the equation* (2.1), if the equation (2.1) has the same form in the new variables $\bar{t}, \bar{x}, \bar{u}$. The set G of all such transformations forms a *continuous group*, i.e. G contains the identity transformation

$$\bar{t} = t, \quad \bar{x}^i = x^i, \quad \bar{u} = u,$$

the inverse to any transformation from G and the composition of any two transformations from G. The symmetry group G is also known as the group *admitted* by the equation (2.1).

According to the Lie theory, the construction of the symmetry group G is equivalent to determination of its *infinitesimal transformations*:

$$\bar{t} \approx t + a\xi^0(t,x,u), \quad \bar{x}^i \approx x^i + a\xi^i(t,x,u), \quad \bar{u} \approx u + a\eta(t,x,u). \tag{2.3}$$

It is convenient to introduce the *symbol* (after S. Lie) of the infinitesimal transformation (2.3), i.e. the operator

$$X = \xi^0(t,x,u)\frac{\partial}{\partial t} + \xi^i(t,x,u)\frac{\partial}{\partial x^i} + \eta(t,x,u)\frac{\partial}{\partial u}. \tag{2.4}$$

The operator (2.4) also is known in the literature as the *infinitesimal operator* or *generator* of the group G. The symbol X of the group admitted by the equation (2.1) is called an *operator admitted by* (2.1).

The group transformations (2.2) corresponding to the infinitesimal transformations with the symbol (2.4) are found by solving the *Lie equations*

$$\frac{d\bar{t}}{da} = \xi^0(\bar{t},\bar{x},\bar{u}), \quad \frac{d\bar{x}^i}{da} = \xi^i(\bar{t},\bar{x},\bar{u}), \quad \frac{d\bar{u}}{da} = \eta(\bar{t},\bar{x},\bar{u}), \tag{2.5}$$

with the initial conditions:

$$\bar{t}\big|_{a=0} = t, \quad \bar{x}^i\big|_{a=0} = x^i, \quad \bar{u}\big|_{a=0} = u.$$

By definition, the transformations (2.2) form a symmetry group G of the equation (2.1) if the function $\bar{u} = \bar{u}(\bar{t}, \bar{x})$ satisfies the equation

$$\bar{u}_{\bar{t}} - F(\bar{t},\bar{x},\bar{u},\bar{u}_{(1)},\bar{u}_{(2)}) = 0, \tag{2.6}$$

whenever the function $u = u(t,x)$ satisfies the equation (2.1). Here $\bar{u}_{\bar{t}}$, $\bar{u}_{(1)}$, $\bar{u}_{(2)}$ are obtained from (2.2) according to the usual formulas of change of variables in derivatives. The infinitesimal form of these formulas are written:

$$\begin{gathered}\bar{u}_{\bar{t}} \approx u_t + a\ \zeta_0(t,x,u,u_t,u_{(1)})\,,\ \bar{u}_{\bar{x}^i} \approx u_{x^i} + a\ \zeta_i(t,x,u,u_t,u_{(1)})\,,\\ \bar{u}_{\bar{x}^i\bar{x}^j} \approx u_{x^ix^j} + a\ \zeta_{ij}(t,x,u,u_t,u_{(1)},u_{tx^k},u_{(2)})\,,\end{gathered} \tag{2.7}$$

where the functions $\zeta_0, \zeta_1, \zeta_{ij}$ are obtained by differentiation of ξ^0, ξ^i, η and are given by the *prolongation formulas*:

$$\zeta_0 = D_t(\eta) - u_t D_t(\xi^0) - u_{x^i} D_t(\xi^i), \quad \zeta_i = D_i(\eta) - u_t D_i(\xi^0) - u_{x^j} D_i(\xi^j),$$

$$\zeta_{ij} = D_j(\zeta_i) - u_{x^i x^k} D_j(\xi^k) - u_{t x^i} D_j(\xi^0). \tag{2.8}$$

Here D_t and D_i denote the total differentiations with respect to t and x^i:

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx^k} \frac{\partial}{\partial u_{x^k}} + \cdots ,$$

$$D_i = \frac{\partial}{\partial x^i} + u_{x^i} \frac{\partial}{\partial u} + u_{tx^i} \frac{\partial}{\partial u_t} + u_{x^i x^k} \frac{\partial}{\partial u_{x^k}} + \cdots .$$

Substitution of (2.3) and (2.7) into the left-hand side of the equation (2.6) yields:

$$\bar{u}_{\bar{t}} - F(\bar{t}, \bar{x}, \bar{u}, \bar{u}_{(1)}, \bar{u}_{(2)}) \approx u_t - F(t, x, u, u_{(1)}, u_{(2)})$$
$$+a\Big(\zeta_0 - \frac{\partial F}{\partial u_{x^i x^j}} \zeta_{ij} - \frac{\partial F}{\partial u_{x^i}} \zeta_i - \frac{\partial F}{\partial u} \eta - \frac{\partial F}{\partial x^i} \xi^i - \frac{\partial F}{\partial t} \xi^0\Big).$$

Therefore, by virtue of the equation (2.1), the equation (2.6) yields

$$\zeta_0 - \frac{\partial F}{\partial u_{x^i x^j}} \zeta_{ij} - \frac{\partial F}{\partial u_{x^i}} \zeta_i - \frac{\partial F}{\partial u} \eta - \frac{\partial F}{\partial x^i} \xi^i - \frac{\partial F}{\partial t} \xi^0 = 0, \tag{2.9}$$

where u_t is replaced by $F(t, x, u, u_{(1)}, u_{(2)})$ in $\zeta_0, \zeta_i, \zeta_{ij}$.

The equation (2.9) defines all infinitesimal symmetries of the equation (2.1) and therefore it is called the *determining equation.* Conventionally, it is written in the compact form

$$X\big(u_t - F(t, x, u, u_{(1)}, u_{(2)})\big)\Big|_{(2.1)} = 0. \tag{2.10}$$

Here X denotes the *prolongation* of the operator (2.4) to the first and second order derivatives:

$$X = \xi^0(t, x, u) \frac{\partial}{\partial t} + \xi^i(t, x, u) \frac{\partial}{\partial x^i} + \eta(t, x, u) \frac{\partial}{\partial u} + \zeta_0 \frac{\partial}{\partial u_t} + \zeta_i \frac{\partial}{\partial u_{x^i}} + \zeta_{ij} \frac{\partial}{\partial u_{x^i x^j}},$$

and the notation $\big|_{(2.1)}$ means evaluated on the equation (2.1).

The determining equation (2.9) (or its equivalent (2.10)) is a linear homogeneous partial differential equation of the second order for unknown functions $\xi^0(t, x, u), \xi^i(t, x, u), \eta(t, x, u)$ of the "independent variables" t, x, u.

At first glance, this equation seems to be more complicated than the original differential equation (2.1). However, this is an apparent complexity. Indeed, the left-hand side of the determining equation involves the derivatives u_{x^i}, $u_{x^ix^j}$, along with the variables t, x, u and functions ξ^0, ξ^i, η of these variables. Since the equation (2.9) is valid identically with respect to all the variables involved, the variables $t, x, u, u_{x^i}, u_{x^ix^j}$ are treated as "independent" ones. It follows that the determining equation decomposes into a system of several equations. As a rule, this is an overdetermined system (it contains more equations than a number $n+2$ of the unknown functions ξ^o, ξ^i, η). Therefore, in practical applications, the determining equation can be solved analytically, unlike the original differential equation (2.1). The solution of the determining equation can be carried out either "by hand" or, in simple cases, by using modern symbolic manipulation programs. Unfortunately, the existing software packages for symbolic manipulations do not provide solutions for complex determining equations, while a group theorist can solve the problem "by hand" (the disbelieving reader can try, for example, to obtain the result of the group classification of the Jacobs-Jones model (1.2) by computer). The reader interested in learning more about the calculation of symmetries by hand in complicated situations is referred to the classical book in this field, L.V. Ovsyannikov [111] containing the best presentation of the topic.

2.2 Exact solutions provided by symmetry groups

Group analysis provides two basic ways for construction of exact solutions: *group transformations* of known solutions and construction of *invariant solutions.*

Group transformations of known solutions. The first way is based on the fact that a symmetry group transforms any solutions of the equation in question into solution of the same equation. Namely, let (2.2) be a symmetry transformation group of the equation (2.1), and let a function

$$u = \phi(t, x) \tag{2.11}$$

solve the equation (2.1). Since (2.2) is a symmetry transformation, the solution (2.11) can be also written in the new variables:

$$\bar{u} = \phi(\bar{t}, \bar{x}). \tag{2.12}$$

Replacing here $\bar{u}, \bar{t}, \bar{x}$ from (2.2), we get

$$h(t, x, u, a) = \phi\big(f(t, x, u, a), g(t, x, u, a)\big).$$

Having solved this equation with respect to u, we arrive at the following one-parameter family (with the parameter a) of new solutions of the equation (2.1):

$$u = \psi_a(t, x). \tag{2.13}$$

Consequently, any known solution is a source of a multi-parameter class of new solutions provided that the differential equation considered admits a multi-parameter symmetry group. An example is given in subsection 3.4, where the procedure is applied to the Black-Scholes equation.

Invariant solutions. If a group transformation maps a solution into itself, we arrive at what is called a *self-similar* or *group invariant solution*. The search of this type of solutions reduces the number of independent variables of the equation in question. Namely, the invariance with respect to one-parameter group reduces the number the variables by one. The further reduction can be achieved by considering an invariance under symmetry groups with two or more parameters.

For example, the construction of these particular solutions is reduced, in the case of the equation (1.1), either to ordinary differential equations (if the solution is invariant under a one-parameter group, see subsection 3.5) or to an algebraic relation (if the solution is invariant with respect to a multi-parameter group, see subsection 3.6).

The construction of invariant solutions under one-parameter groups is widely known in the literature. Therefore, we briefly sketch the procedure in subsection 3.5 by considering one simple example only.

However, since the Jacobs-Jones equation involves three independent variables, its reduction to, e.g. ordinary differential equations requires an invariance under two-dimensional groups. Therefore, we discuss some details of the procedure in subsection 4.3 for the Jacobs-Jones equation.

2.3 Group classification of differential equations

Differential equations occurring in sciences as mathematical models, often involve undetermined parameters and/or arbitrary functions of certain variables. Usually, these arbitrary elements (parameters or functions) are found experimentally or chosen from a "simplicity criterion". Lie group theory provides a regular procedure for determining arbitrary elements from symmetry point of view. This direction of study is known today as *Lie group classification of differential equations.* For detailed presentations of methods used in Lie group classification of differential equations the reader is refereed to the first fundamental paper on this topic [89] dealing with the classification of linear second-order partial differential equations with two independent variables.

Lie group classification of differential equations provides a mathematical background for what can be called a *group theoretic modelling* (see [68], Ch. 6). In this approach, differential equations admitting more symmetries are considered to be "preferable". In this way, one often arrives at equations possessing remarkable physical properties.

Given a family of differential equations, the procedure of Lie group classification begins with determining the so-called *principal Lie group* of this family of equations. This is the group admitted by any equation of the family in question. The Lie algebra of the principal Lie group is called the *principal Lie algebra* of the equations and is denoted by $L_{\mathcal{P}}$ (see e.g. [67]). It may happen that for particular choice of arbitrary elements of the family the corresponding equation admits, along with the principal Lie group, additional symmetry transformations. Determination of all distinctly different particular cases when an extension of $L_{\mathcal{P}}$ occurs is the problem of the group classification.

3 The Black-Scholes model

3.1 The basic equation

For mathematical modelling stock option pricing, Black and Scholes [18] proposed the partial differential equation

$$u_t + \frac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu = 0 \tag{1.1}$$

with constant coefficients A, B, C (parameters of the model). It is shown in [18] that the equation (1.1) is transformable into the classical heat equation

$$v_\tau = v_{yy}\,, \tag{3.1}$$

provided that $A \neq 0, \quad \mathcal{D} \equiv B - A^2/2 \neq 0$. Using the connection between the equations (1.1) and (3.1), they give an explicit formula for the solution, defined in the interval $-\infty < t < t^*$, of the Cauchy problem with a special initial data at $t = t^*$.

3.2 Symmetries

For the Black-Scholes model (1.1), $n = 1, x^1 = x$ and the symbol of the infinitesimal symmetries has the form

$$X = \xi^0(t,x,u)\frac{\partial}{\partial t} + \xi^1(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}\,.$$

In this case, the determining equation (2.9) is written:

$$\zeta_0 + \frac{1}{2}A^2x^2\zeta_{11} + Bx\zeta_1 - C\eta + A^2xu_{xx} + Bu_x\xi^1\xi^1 = 0, \tag{3.2}$$

where according to the prolongation formulas (2.8), the functions $\zeta_0, \zeta_1, \zeta_{ij}$ are given by

$$\begin{aligned}
\zeta_0 &= \eta_t + u_t\eta_u - u_t\xi^0_t - u_t^2\xi^0_u - u_x\xi^1_t - u_tu_x\xi^1_u,\\
\zeta_1 &= \eta_x + u_x\eta_u - u_t\xi^0_x - u_tu_x\xi^0_u - u_x\xi^1_x - u_x^2\xi^1_u,\\
\zeta_{11} &= \eta_{xx} + 2u_x\eta_{xu} + u_{xx}\eta_u + u_x^2\eta_{uu}\\
&\quad -2u_{tx}\xi^0_x - u_t\xi^0_{xx} - 2u_tu_x\xi^0_{xu} - (u_tu_{xx} + 2u_xu_{tx})\xi^0_u - u_tu_x^2\xi^0_{uu}\\
&\quad -2u_{xx}\xi^1_x - u_x\xi^1_{xx} - 2u_x^2\xi^1_{xu} - 3u_xu_{xx}\xi^1_u - u_x^3\xi^1_{uu}.
\end{aligned}$$

The solution of the determining equation (3.2) provides the infinite dimensional vector space of the infinitesimal symmetries of the equation (1.1) spanned by following operators:

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t}\,, \qquad X_2 = x\frac{\partial}{\partial x}\,,\\
X_3 &= 2t\frac{\partial}{\partial t} + \big(\ln x + \mathcal{D}t\big)x\frac{\partial}{\partial x} + 2Ctu\frac{\partial}{\partial u}\,,\\
X_4 &= A^2tx\frac{\partial}{\partial x} + \big(\ln x - \mathcal{D}t\big)u\frac{\partial}{\partial u}\,,\\
X_5 &= 2A^2t^2\frac{\partial}{\partial t} + 2A^2tx\ln x\frac{\partial}{\partial x}\\
&\quad +\Big(\big(\ln x - \mathcal{D}t\big)^2 + 2A^2Ct^2 - A^2t\Big)u\frac{\partial}{\partial u}\,,
\end{aligned} \tag{3.3}$$

and

$$X_6 = u\frac{\partial}{\partial u}\,, \qquad X_\phi = \phi(t,x)\frac{\partial}{\partial u}\,\cdot \tag{3.4}$$

Here

$$\mathcal{D} = B - \frac{1}{2}A^2 \tag{3.5}$$

and $\phi(t,x)$ in (3.4) is an arbitrary solution of Eq. (1.1).

The finite symmetry transformations (see (2.2)),

$$\bar{t} = f(t,x,u,a), \quad \bar{x} = g(t,x,u,a), \quad \bar{u} = h(t,x,u,a),$$

corresponding to the basic generators (3.3) and (3.4), are obtained by solving the *Lie equations* (2.5). The result is as follows:

$$X_1: \quad \bar{t} = t + a_1, \ \bar{x} = x, \ \bar{u} = u;$$

$$X_2: \quad \bar{t} = t, \ \bar{x} = x\, a_2, \ \bar{u} = u, \ a_2 \neq 0;$$

$$X_3: \quad \bar{t} = t\, a_3^2, \ \bar{x} = x^{a_3} \mathrm{e}^{\mathcal{D}(a_3^2 - a_3)t}, \ \bar{u} = u\mathrm{e}^{C(a_3^2-1)t}, \ a_3 \neq 0;$$

$$X_4: \quad \bar{t} = t, \ \bar{x} = x\mathrm{e}^{A^2 t a_4}, \ \bar{u} = u x^{a_4} \mathrm{e}^{(\frac{1}{2}A^2 a_4^2 - \mathcal{D} a_4)t};$$

$$X_5: \quad \bar{t} = \frac{t}{1 - 2A^2 a_5 t}, \ \bar{x} = x^{\frac{t}{1-2A^2 a_5 t}},$$

$$\bar{u} = u\sqrt{1 - 2A^2 a_5 t}\ \exp\left(\frac{[(\ln x - \mathcal{D}t)^2 + 2A^2 C t^2] a_5}{1 - 2A^2 a_5 t}\right),$$

and

$$X_6: \quad \bar{t} = t, \ \bar{x} = x, \ \bar{u} = u\, a_6, \ a_6 \neq 0;$$

$$X_\phi: \quad \bar{t} = t, \ \bar{x} = x, \ \bar{u} = u + \phi(t, x).$$

Here $a_1, \ldots, a_6$ are the parameters of the one-parameter groups generated by $X_1, \ldots, X_6$, respectively, and $\phi(t, x)$ is an arbitrary solution of (1.1). Consequently, the operators $X_1, \ldots, X_6$ generate a six-parameter group and X_ϕ generates an infinite group. The general symmetry group is the composition of the above transformations.

Remark. The group of dilations generated by the operator X_6 reflects the homogeneity of the equation (1.1), while the infinite group with the operator X_ϕ represents the linear superposition principle for the equation (1.1). These transformations are common for all linear homogeneous differential equations. Hence, the specific (non-trivial) symmetries of (1.1) are given by the operators (3.3) that span a five-dimensional *Lie algebra*.

3.3 Transformation to the heat equation

Let us recall Lie's result of group classification of linear second-order partial differential equations with two independent variables. In the case of evolutionary parabolic equations this result is formulated as follows (see [89]):

Consider the family of linear parabolic equations

$$P(t, x)u_t + Q(t, x)u_x + R(t, x)u_{xx} + S(t, x)u = 0, \quad P \neq 0, \ R \neq 0. \quad (3.6)$$

The principal Lie algebra $L_{\mathcal{P}}$ (i.e. the Lie algebra of operators admitted by (3.6) with arbitrary coefficients $P(t,x), Q(t,x), R(t,x), S(t,x)$, see subsection 2.3) is spanned by the generators (3.4) of trivial symmetries. Any equation (3.6) can be reduced to the form

$$v_\tau = v_{yy} + Z(\tau, y)v \tag{3.7}$$

by a transformation, Lie's equivalence transformation:

$$y = \alpha(t,x), \quad \tau = \beta(t), \quad v = \gamma(t,x)u, \quad \alpha_x \neq 0,\ \beta_t \neq 0, \tag{3.8}$$

obtained with the help of two quadratures.

If the equation (3.6) admits an extension of the principal Lie algebra $L_{\mathcal{P}}$ by one additional symmetry operator then it is reduced to the form

$$v_\tau = v_{yy} + Z(y)v \tag{3.9}$$

for which the additional operator is

$$X = \frac{\partial}{\partial \tau}.$$

If $L_{\mathcal{P}}$ extends by three additional operators, the equation (3.6) is reduced to the form

$$v_\tau = v_{yy} + \frac{A}{y^2}v, \tag{3.10}$$

the three additional operators being:

$$X_1 = \frac{\partial}{\partial \tau}, \quad X_2 = 2\tau\frac{\partial}{\partial \tau} + y\frac{\partial}{\partial y}, \quad X_3 = \tau^2\frac{\partial}{\partial \tau} + \tau y\frac{\partial}{\partial y} - (\frac{1}{4}y^2 + \frac{1}{2}\tau)v\frac{\partial}{\partial v}.$$

If $L_{\mathcal{P}}$ extends by five additional operators, the equation (3.6) is reduced to the heat equation

$$v_\tau = v_{yy} \tag{3.11}$$

the five additional operators being:

$$X_1 = \frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial \tau}, \quad X_3 = 2\tau\frac{\partial}{\partial y} - yv\frac{\partial}{\partial v}, \quad X_4 = 2\tau\frac{\partial}{\partial \tau} + y\frac{\partial}{\partial y},$$

$$X_5 = \tau^2\frac{\partial}{\partial \tau} + \tau y\frac{\partial}{\partial y} - (\frac{1}{4}y^2 + \frac{1}{2}\tau)v\frac{\partial}{\partial v}.$$

The equations (3.9) to (3.11) provide the canonical forms of all linear parabolic second order equations (3.6) that admit non-trivial symmetries, i.e. extensions of the principal Lie algebra $L_{\mathcal{P}}$.

Thus, the Black-Scholes equation (1.1) belongs to the latter case and hence it reduces to the heat equation (3.11) by Lie's equivalence transformation. Let us find this transformation.

After the change of variables (3.8), the heat equation (3.11) becomes

$$u_{xx}+\Big[\frac{2\gamma_x}{\gamma}+\frac{\alpha_x\alpha_t}{\beta'}-\frac{\alpha_{xx}}{\alpha_x}\Big]u_x-\frac{\alpha_x^2}{\beta'}u_t+\Big[\frac{\gamma_{xx}}{\gamma}+\frac{\alpha_x\alpha_t\gamma_x}{\beta'\gamma}-\frac{\alpha_x^2\gamma_t}{\beta'\gamma}-\frac{\alpha_{xx}\gamma_x}{\alpha_x\gamma}\Big]u=0,$$

where the $'$ denotes the differentiation with respect to t. Comparing this equation with the Black-Scholes equation (1.1) rewritten in the form

$$u_{xx}+\frac{2B}{A^2x}u_x+\frac{2}{A^2x^2}u_t-\frac{2Cu}{A^2x^2}=0$$

and equating the respective coefficients, we arrive at the following system:

$$\frac{\alpha_x^2}{\beta'}=-\frac{2}{A^2x^2}\,, \tag{3.12}$$

$$\frac{2\gamma_x}{\gamma}+\frac{\alpha_x\alpha_t}{\beta'}-\frac{\alpha_{xx}}{\alpha_x}=\frac{2B}{A^2x}\,, \tag{3.13}$$

$$\frac{\gamma_{xx}}{\gamma}+\frac{\alpha_x\alpha_t\gamma_x}{\beta'\gamma}-\frac{\alpha_x^2\gamma_t}{\beta'\gamma}-\frac{\alpha_{xx}\gamma_x}{\alpha_x\gamma}=-\frac{2C}{A^2x^2}\,. \tag{3.14}$$

It follows from (3.12):

$$\alpha(t,x)=\frac{\varphi(t)}{A}\ln x+\psi(t),\quad \beta'(t)=-\frac{1}{2}\varphi^2(t),$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions. Using these formulas, one obtains from the equation (3.13):

$$\gamma(t,x)=\nu(t)x^{\frac{B}{A^2}-\frac{1}{2}+\frac{\psi'}{A\varphi}+\frac{\varphi'}{2A^2\varphi}\ln x}$$

with an arbitrary function $\nu(t)$. After substitution of the above expressions into the equation (3.14), one obtains two possibilities: either

$$\varphi=\frac{1}{L-Kt},\quad \psi=\frac{M}{L-Kt}+N,\quad K\neq 0,$$

and the function $\nu(t)$ satisfies the equation

$$\frac{\nu'}{\nu}=\frac{M^2K^2}{2(L-Kt)^2}-\frac{K}{2(L-Kt)}-\frac{A^2}{8}+\frac{B}{2}-\frac{B^2}{2A^2}-C,$$

or

$$\varphi = L, \quad \psi = Mt + N, \quad L \neq 0,$$

and

$$\frac{\nu'}{\nu} = \frac{M^2}{2L^2} - \frac{A^2}{8} + \frac{B}{2} - \frac{B^2}{2A^2} - C.$$

Here K, L, M, and N are arbitrary constants.

Thus, we arrive at the following two different transformations connecting the equations (1.1) and (3.11):

First transformation

$$y = \frac{\ln x}{A(L-Kt)} + \frac{M}{L-Kt} + N, \quad \tau = -\frac{1}{2K(L-Kt)} + P, \quad K \neq 0,$$

$$v = E\sqrt{L-Kt}\,\mathrm{e}^{\frac{M^2K}{2(L-Kt)} - \frac{1}{2}\left(\frac{B}{A} - \frac{A}{2}\right)^2 t - Ct}\, x^{\frac{B}{A^2} - \frac{1}{2} + \frac{MK}{A(L-Kt)} + \frac{K\ln x}{2A^2(L-Kt)}}\, u. \quad (3.15)$$

Second transformation

$$y = \frac{L}{A}\ln x + Mt + N, \quad \tau = -\frac{L^2}{2}t + P, \quad L \neq 0.$$

$$v = E\,\mathrm{e}^{\left[\frac{M^2}{2L^2} - \frac{1}{2}\left(\frac{B}{A} - \frac{A}{2}\right)^2 - C\right]t}\, x^{\frac{B}{A^2} - \frac{1}{2} + \frac{M}{AL}}\, u. \quad (3.16)$$

The Black-Scholes transformation (see [18], formula (9)) is a particular case of the second transformation (3.16) with

$$L = \frac{2}{A}\mathcal{D},\ M = -\frac{2}{A^2}\mathcal{D}^2,\ N = \frac{2}{A^2}\mathcal{D}(\mathcal{D}t^* - \ln c),\ P = \frac{2}{A^2}\mathcal{D}^2 t^*,\ E = \mathrm{e}^{Ct^*},$$

where t^*, c are constants involved in the initial value problem (8) of [18]. The transformation (3.15) is new and allows one to solve an initial value problem different from that given in [18].

3.4 Transformations of solutions

Let

$$u = \phi(t, x)$$

be a known solution of the equation (1.1). According to Section 2.2, one can use this solution to generate families of new solutions involving the group parameters. We apply here the procedure to the transformations generated by the basic operators (3.3), (3.4). Application of the formulas (2.11) to (2.13) yields:

$$X_1: \quad u = F(t - a_1, x);$$

$$X_2: \quad u = F(t, x\, a_2^{-1}),\ a_2 \neq 0;$$

$$X_3: \quad u = \mathrm{e}^{C(1-a_3^{-2})t} F\Big(t\, a_3^{-2}, x^{a_3^{-1}} \mathrm{e}^{\mathcal{D}(a_3^{-2}-a_3^{-1})t}\Big),\ a_3 \neq 0;$$

$$X_4: \quad u = x^{a_4} \mathrm{e}^{-(\frac{1}{2}A^2a_4^2 + \mathcal{D}a_4)t} F\big(t, x\mathrm{e}^{-A^2ta_4}\big);$$

$$X_5: \quad u = \frac{\exp\left(\frac{[(\ln x - \mathcal{D}t)^2 + 2A^2Ct^2]a_5}{1+2A^2a_5t}\right)}{\sqrt{1+2A^2a_5t}}\ F\Big(\frac{t}{1+2A^2a_5t}\, ,\ x^{\frac{t}{1+2A^2a_5t}}\Big);$$

and

$$X_6: \quad u = a_6F(t,x),\ a_6 \neq 0; \qquad X_\phi: \quad u = F(t,x) + \phi(t,x).$$

Example. Let us begin with the simple solution of the equation (1.1) depending only on t:

$$u = \mathrm{e}^{Ct}. \tag{3.17}$$

Using the transformation generated by X_4 we obtain the solution depending on the parameter a_4:

$$u = x^{a_4}\mathrm{e}^{-(\frac{1}{2}A^2a_4^2 + \mathcal{D}a_4 - C)t}.$$

Letting here, for the simplicity, $a_4 = 1$ we get

$$u = x\mathrm{e}^{(C-B)t}.$$

If we apply to this solution the transformation generated by X_5, we obtain the following solution of the equation (1.1):

$$u = \frac{x^{t/(1+2A^2a_5t)}}{\sqrt{1+2A^2a_5t}} \exp\left(\frac{[(\ln x - \mathcal{D}t)^2 + 2A^2Ct^2]a_5 + (C-B)t}{1+2A^2a_5t}\right). \tag{3.18}$$

Thus, beginning with the simplest solution (3.17) we arrive at the rather complicated solution (3.18). The iteration of this procedure yields more complex solutions.

Note that the solution (3.17) is unalterable under the transformation generated by X_2. This is an example of so-called *invariant solutions* discussed in the next subsection.

3.5 Invariant solutions

An invariant solution with respect to a given subgroup of the symmetry group is a solution which is unalterable under the action of the transformations of the subgroup. Invariant solutions can be expressed via invariants of the subgroup (see, e.g., [67]). Here we illustrate the calculation of invariant solutions by considering the one-parameter subgroup with the generator

$$X = X_1 + X_2 + X_6 \equiv \frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}.$$

Invariants $I(t, x, u)$ of this group are found from the equation

$$XI = 0$$

and are given by

$$I = J(I_1, I_2),$$

where

$$I_1 = t - \ln x, \quad I_2 = \frac{u}{x}$$

are functionally independent invariants and hence form a basis of invariants. Therefore, the invariant solution can be taken in the form $I_2 = \phi(I_1)$, or

$$u = x\phi(z), \quad \text{where} \quad z = t - \ln x.$$

Substituting into the equation (1.1) we obtain the ordinary differential equation of the second order :

$$\frac{A^2}{2}\phi'' + (1 - B - \frac{A^2}{2})\phi' + (B - C)\phi = 0, \quad \text{where} \quad \phi' = \frac{d\phi}{dz}.$$

This equation with constant coefficients can be readily solved.

The above procedure can be applied to any linear combination (with constant coefficients) of the basic generators (3.3) - (3.4). Here we apply it to the generators (3.3). We have:

$$X_1 : \quad u = \phi(x), \quad \frac{1}{2}A^2x^2\phi'' + Bx\phi' - C\phi = 0,$$

this equation reduces to constant coefficients upon introducing the new independent variable $z = \ln x$;

$$X_2 : \quad u = \phi(t), \quad \phi' - C\phi = 0, \quad \text{whence} u = K\mathrm{e}^{Ct};$$

$$X_3 : \quad u = \mathrm{e}^{Ct}\phi\Big(\frac{\ln x}{\sqrt{t}} - \mathcal{D}\sqrt{t}\Big), \quad A^2\phi'' - z\phi' = 0, \; z = \frac{\ln x}{\sqrt{t}} - \mathcal{D}\sqrt{t},$$

whence

$$\phi(z) = K_1 \int_0^z \mathrm{e}^{\mu^2/(2A^2)}\, d\mu + K_2;$$

$$X_4: \quad u = \exp\left(\frac{(\ln x - \mathcal{D}t)^2}{2A^2 t}\right)\phi(t), \quad \phi' + \big(\frac{1}{2t} - C\big)\phi = 0,$$

whence

$$\phi = \frac{K}{\sqrt{t}}\mathrm{e}^{Ct},$$

and hence

$$u = \frac{K}{\sqrt{t}}\exp\left(\frac{(\ln x - \mathcal{D}t)^2}{2A^2 t} + Ct\right);$$

$$X_5: \quad u = \frac{1}{\sqrt{t}}\exp\left(\frac{(\ln x - \mathcal{D}t)^2}{2A^2 t} + Ct\right)\phi\Big(\frac{\ln x}{t}\Big), \quad \phi'' = 0,$$

hence

$$u = \Big(K_1 \frac{\ln x}{t^{3/2}} + \frac{K_2}{\sqrt{t}}\Big)\exp\left(\frac{(\ln x - \mathcal{D}t)^2}{2A^2 t} + Ct\right).$$

In the above solutions K, K_1, K_2 are constants of integration and $\mathcal{D}$ is given by Eq. (3.5), i.e.

$$\mathcal{D} = B - \frac{1}{2}A^2.$$

The operators $X_6\,, X_\phi$ do not provide invariant solutions.

3.6 The fundamental solution

Investigation of initial value problems for hyperbolic and parabolic linear partial differential equations can be reduced to the construction of a particular solution with specific singularities known in the literature as *elementary* or *fundamental* solutions (see, e.g. [41], [24] and [22]). Recently, it was shown [63] that for certain classes of equations, with constant and variable coefficients, admitting sufficiently wide symmetry groups, the fundamental solution is an invariant solution and it can be constructed by using the so-called *invariance principle.*

Here we find the fundamental solution for the equation (1.1) using the group theoretic approach presented in [69].

We can restrict ourselves by considering the fundamental solution

$$u = u(t, x; t_0, x_0)$$

of the Cauchy problem defined as follows:

$$u_t + \frac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu = 0, \quad t < t_0, \tag{3.19}$$

$$u\Big|_{t\to -t_0} = \delta(x - x_0). \tag{3.20}$$

Here $\delta(x - x_0)$ is the Dirac measure at x_0.

According to the invariance principle, we first find the subalgebra of the Lie algebra spanned by the operators (3.3) and the dilation generator

$$X_6 = u\frac{\partial}{\partial u}$$

(for our purposes it suffices to consider this finite-dimensional algebra obtained by omitting X_ϕ) such that this subalgebra leaves invariant the *initial manifold* (i.e. the line $t = t_0$) and its restriction on $t = t_0$ conserves the *initial conditions* given by $x = x_0$ and by the equation (3.20). This subalgebra is the three-dimensional algebra spanned by

$$Y_1 = 2(t - t_0)\tfrac{\partial}{\partial t} + \big(\ln x - \ln x_0 + \mathcal{D}(t - t_0)\big)x\tfrac{\partial}{\partial x} + \big(2C(t - t_0) - 1\big)u\tfrac{\partial}{\partial u}\,,$$

$$Y_2 = A^2(t - t_0)x\tfrac{\partial}{\partial x} + \big(\ln x - \ln x_0 - \mathcal{D}(t - t_0)\big)u\tfrac{\partial}{\partial u}\,,$$

$$Y_3 = 2A^2(t - t_0)^2\tfrac{\partial}{\partial t} + 2A^2(t - t_0)x\ln x\tfrac{\partial}{\partial x} +$$

$$+\Big(\big(\ln x - \mathcal{D}(t - t_0)\big)^2 - \ln^2 x_0 + 2A^2C(t - t_0)^2 - A^2(t - t_0)\Big)u\tfrac{\partial}{\partial u}\,.$$

Invariants are defined by the system

$$Y_1I = 0, \quad Y_2I = 0, \quad Y_3I = 0.$$

Since

$$Y_3 = A^2(t - t_0)Y_1 + \big(\frac{1}{2}A^2(t - t_0) - B(t - t_0) + \ln x + \ln x_0\big),$$

it suffices to solve only first two equations. Their solution is

$$I = ux^{\sigma(t)}\sqrt{t_0 - t}\,\mathrm{e}^{\omega(t,x)},$$

where

$$\sigma(t) = \frac{\mathcal{D}}{A^2} - \frac{\ln x_0}{A^2(t_0 - t)}\,, \tag{3.21}$$

$$\omega(t, x) = \frac{\ln^2 x + \ln^2 x_0}{2A^2(t_0 - t)} + \Big(\frac{\mathcal{D}^2}{2A^2} + C\Big)(t_0 - t).$$

The invariant solution is given by $I = K =$ const., and hence has the form

$$u = K\frac{x^{-\sigma(t)}}{\sqrt{t_0 - t}}\mathrm{e}^{-\omega(t,x)}, \tag{3.22}$$

where $\sigma(t), \omega(t,x)$ are defined by (3.21). One can readily verify that the function (3.22) satisfies the equation (3.19). The constant coefficient K is to be found from the initial condition (3.20).

We will use the well-known limit,

$$\lim_{s\to+0}\frac{1}{\sqrt{s}}\exp\Big(-\frac{(x-x_0)^2}{4s}\Big) = 2\sqrt{\pi}\delta(x-x_0), \tag{3.23}$$

and the formula of change of variables $z = z(x)$ in the Dirac measure (see, e.g. [24], p. 790):

$$\delta(x-x_0) = \left|\frac{\partial z(x)}{\partial x}\right|_{x=x_0}\delta(z-z_0). \tag{3.24}$$

For the function (3.22), we have

$$\lim_{t\to-t_0} u = \lim_{t\to-t_0}\frac{K}{\sqrt{t_0-t}}\mathrm{e}^{-\omega(t,x)-\sigma(t)\ln x}$$

$$= \lim_{t\to-t_0}\frac{K}{\sqrt{t_0-t}}\exp\left(-\frac{(\ln x-\ln x_0)^2}{2A^2(t_0-t)} - \frac{\mathcal{D}\ln x}{A^2}\right),$$

or, setting $s = t_0 - t, \quad z = \dfrac{\sqrt{2}}{A}\ln x,$

$$\lim_{t\to-t_0} u = K\exp\Big(-\frac{\mathcal{D}}{A^2}\ln x\Big)\lim_{s\to+0}\frac{1}{\sqrt{s}}\exp\Big(-\frac{(z-z_0)^2}{4s}\Big)$$

$$= 2\sqrt{\pi}K\exp\Big(-\frac{\mathcal{D}}{A^2}\ln x\Big)\delta(z-z_0).$$

By virtue of (3.24),

$$\delta(z-z_0) = \frac{Ax_0}{\sqrt{2}}\delta(x-x_0),$$

and hence

$$\lim_{t\to-t_0} u = \sqrt{2\pi}AKx_0\exp\Big(-\frac{\mathcal{D}}{A^2}\ln x_0\Big)\delta(x-x_0).$$

Therefore the initial condition (3.20) yields:

$$K = \frac{1}{\sqrt{2\pi}Ax_0}\exp\Big(\frac{\mathcal{D}}{A^2}\ln x_0\Big).$$

Thus, we arrive at the following fundamental solution to the Cauchy problem for the equation (1.1):

$$u = \frac{1}{Ax_0\sqrt{2\pi(t_0-t)}}\exp\Big[-\frac{(\ln x - \ln x_0)^2}{2A^2(t_0-t)} - \Big(\frac{\mathcal{D}^2}{2A^2}+C\Big)(t_0-t) - \frac{\mathcal{D}}{A^2}(\ln x - \ln x_0)\Big].$$

Remark. The fundamental solution can also be obtained from the fundamental solution

$$v = \frac{1}{2\sqrt{\pi\tau}}\exp\Big[-\frac{y^2}{4\tau}\Big]$$

of the heat equation (3.11) by the transformation of the form (3.16) with

$$M = -\frac{L}{A}\mathcal{D},\ N = \frac{L}{A}\mathcal{D}t_0 - \frac{L}{A}\ln x_0,\ P = \frac{L^2}{2}t_0,\ E = \frac{Ax_0}{L}\mathrm{e}^{Ct_0},$$

i.e. by the transformation

$$\tau = \frac{L^2}{2}(t_0-t),\ y = \frac{L}{A}\mathcal{D}(t_0-t) + \frac{L}{A}(\ln x - \ln x_0),\ v = \frac{Ax_0}{L}\mathrm{e}^{C(t_0-t)}u.$$

4 A two factor variable model

Methods of Lie group analysis can be successfully applied to other mathematical models used in mathematics of finance. Here we present results of calculation of symmetries for a two state variable model developed by Jacobs and Jones [79].

4.1 The Jacobs-Jones equation

The Jacobs-Jones model is described by the linear partial differential equation

$$u_t = \frac{1}{2}A^2x^2u_{xx} + ABCxyu_{xy} + \frac{1}{2}B^2y^2u_{yy} + \Big(Dx\ln\frac{y}{x} - Ex^{\frac{3}{2}}\Big)u_x + \Big(Fy\ln\frac{G}{y} - Hyx^{\frac{1}{2}}\Big)u_y - xu \tag{1.2}$$

with constant coefficients A, B, C, D, E, F, G, H.

4.2 The group classification

The equation (1.2) contains parameters $A, B, \ldots, H$. These parameters are "arbitrary elements" mentioned in subsection 2.3. According to subsection 2.3, it may happen that the Lie algebra of operators admitted by (1.2) with arbitrary coefficients (i.e. principal Lie algebra) extends for particular choices of the coefficients $A, B, \ldots, H$. It is shown here that the dimension of the symmetry Lie algebra for the model (1.2), unlike the Black-Scholes model, essentially depends on choice of the coefficients $A, B, C, \ldots, H$.

Result of the group classification

The principal Lie algebra $L_{\mathcal{P}}$ is infinite-dimensional and spanned by

$$X_1 = \frac{\partial}{\partial t}\,, \quad X_2 = u\frac{\partial}{\partial u}\,, \quad X_\omega = \omega(t,x,y)\frac{\partial}{\partial u}\,,$$

where $\omega(t,x,y)$ satisfies the equation (1.2).

We consider all possible extensions of $L_{\mathcal{P}}$ for *non-degenerate* equations (1.2), namely those satisfying the conditions

$$AB \neq 0, \quad C \neq \pm 1. \tag{4.1}$$

Moreover, we simplify calculations by imposing the additional restriction

$$C \neq 0. \tag{4.2}$$

Extensions of $L_{\mathcal{P}}$

The algebra $L_{\mathcal{P}}$ extends in the following cases:

1. $D = 0$,

$$X_3 = \mathrm{e}^{Ft} y \frac{\partial}{\partial y}.$$

Subcase: $AH - BCE = 0$ and $F = 0$. There is an additional extension

$$X_4 = 2AB^2(1 - C^2)ty\frac{\partial}{\partial y} + \big(2BC\ln x - 2A\ln y + (B - AC)ABt\big)u\frac{\partial}{\partial u}.$$

2. $D \neq 0$, $F = -\dfrac{BD}{2AC}$, $H = 0$,

$$X_3 = \exp\big(\frac{BD}{2AC}t\big)y\frac{\partial}{\partial y} + \Big(\frac{D}{ABC}\ln\frac{G}{y} + 1\Big)\exp\big(\frac{BD}{2AC}t\big)u\frac{\partial}{\partial u}.$$

3. $D \neq 0$, F is defined from the equation

$$A^2F^2 - A^2D^2 + 2ABCDF + B^2D^2 = 0,$$

and the constants E and H are connected by the relation

$$BE(ACF + ACD + BD) = AH(AF + AD + BCD),$$

$$X_3 = \mathrm{e}^{-Dt} y \frac{\partial}{\partial y} - \Big(\frac{ACF + ACD + BD}{A^2B(1-C^2)} \ln x - \frac{AF + AD + BCD}{AB^2(1-C^2)} \ln y + \frac{A^2CF + A^2CD - B^2CD - ABF}{2ABD(1-C^2)} + \frac{F \ln G(BCD + AF + AD)}{AB^2D(1-C^2)} \Big) \mathrm{e}^{-Dt} u \frac{\partial}{\partial u}.$$

Subcase: $B = 2AC,\ F = -D,\ H = 0$. There is an additional extension

$$X_4 = \mathrm{e}^{Dt} y \frac{\partial}{\partial y} + \Big(\frac{D}{2A^2C^2} \ln \frac{G}{y} + 1 \Big) \mathrm{e}^{Dt} u \frac{\partial}{\partial u}.$$

Remark 1. Most likely, the restriction (4.2) is not essential for the group classification. For example, one of the simplest equations of the form (4.1), namely

$$u_t = x^2 u_{xx} + y^2 u_{yy} - xu,$$

admits two additional operators to $L_{\mathcal{P}}$ and is included in the subcase of the case 1 of the classification.

Remark 2. The classification result shows that the equation (1.2) can not be transformed, for any choice of its coefficients, into the heat equation

$$v_\tau = v_{ss} + v_{zz}.$$

Indeed, the heat equation admits an extension of $L_{\mathcal{P}}$ by seven additional operators (see, e.g. [67], Section 7.2) while the equation (1.2) can admit an extension maximum by two operators.

4.3 Invariant solutions

The above results can be used for the construction of exact (invariant) solutions of Eq. (1.2). We consider here examples of solutions invariant under two-dimensional subalgebras of the symmetry Lie algebra. Then a solution of Eq. (1.2) is obtained from a linear second-order ordinary differential equations and hence the problem is reduced to a Riccati equation. The examples illustrate the general algorithm and can be easily adopted by the reader in other cases.

To construct a solution invariant under a two-dimensional symmetry algebra, one chooses two operators

$$Y_1 = \xi_1^0(t,x,y,u)\frac{\partial}{\partial t} + \xi_1^1(t,x,y,u)\frac{\partial}{\partial x} + \xi_1^2(t,x,y,u)\frac{\partial}{\partial y} + \eta_1(t,x,y,u)\frac{\partial}{\partial u},$$

$$Y_2 = \xi_2^0(t,x,y,u)\frac{\partial}{\partial t} + \xi_2^1(t,x,y,u)\frac{\partial}{\partial x} + \xi_2^2(t,x,y,u)\frac{\partial}{\partial y} + \eta_2(t,x,y,u)\frac{\partial}{\partial u}$$

that are admitted by the equation (1.2) and obey the Lie algebra relation:

$$[Y_1, Y_2] = \lambda_1 Y_1 + \lambda_2 Y_2,\ \ \lambda_1, \lambda_2 = \text{const}.$$

The two-dimensional Lie subalgebra spanned by Y_1, Y_2 will be denoted by

$$< Y_1, Y_2 > .$$

It has two functionally independent invariants, $I_1(t,x,z,u)$ and $I_2(t,x,z,u)$, provided that

$$\text{rank}\begin{pmatrix} \xi_1^0 & \xi_1^1 & \xi_1^2 & \eta_1 \\ \xi_2^0 & \xi_2^1 & \xi_2^2 & \eta_2 \end{pmatrix} = 2.$$

Under these conditions, the invariants are determined by the system of differential equations

$$Y_1 I = 0,\ \ Y_2 I = 0.$$

The invariants solution exists if

$$\text{rank}\left(\frac{\partial I_1}{\partial u}, \frac{\partial I_2}{\partial u}\right) = 1.$$

Then the invariant solution has the form

$$I_2 = \phi(I_1). \tag{4.3}$$

Substituting (4.3) into the equation (1.2), one arrives at an ordinary differential equation for the function ϕ.

Example. Consider the equation

$$u_t = \frac{1}{2}A^2x^2u_{xx} + ABCxyu_{xy} + \frac{1}{2}B^2y^2u_{yy} - Ex^{\frac{3}{2}}u_x - \frac{BCE}{A}yx^{\frac{1}{2}}u_y - xu. \tag{4.4}$$

According to the above group classification, the equation (4.4) admits the operators

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = u\frac{\partial}{\partial u}, \quad X_3 = y\frac{\partial}{\partial y}, \tag{4.5}$$

$$X_4 = 2AB^2(1-C^2)ty\frac{\partial}{\partial y} + \big[2BC\ln x - 2A\ln y + (B-AC)ABt\big]u\frac{\partial}{\partial u},$$

and

$$X_\omega = \omega(t,x,y)\frac{\partial}{\partial u}, \quad \text{where } \omega(t,x,y) \text{ solves Eq. (4.4).}$$

Here we consider invariants solutions with respect to three different two-dimensional subalgebras of the algebra (4.5).

1. The subalgebra $< X_1, X_3 >$ has the independent invariants $I_1 = x$ and $I_2 = u$. Hence, the invariant solution has the form

$$u = \phi(x), \tag{4.6}$$

and is determined by the equation

$$\frac{1}{2}A^2x^2\phi'' - Ex^{\frac{3}{2}}\phi' - x\phi = 0. \tag{4.7}$$

It reduces to the Riccati equation

$$\psi' + \psi^2 - \frac{2E}{A^2\sqrt{x}}\psi - \frac{2}{A^2x} = 0$$

by the standard substitution

$$\psi = \phi'/\phi. \tag{4.8}$$

2. The subalgebra $< X_1 + X_2, X_3 >$ has the invariants

$$I_1 = x, \quad I_2 = u\mathrm{e}^{-t}.$$

The corresponding invariant solution has the form

$$u = \mathrm{e}^t\phi(x). \tag{4.9}$$

The substitution into Eq. (4.4) yields:

$$\frac{1}{2}A^2x^2\phi'' - Ex^{\frac{3}{2}}\phi' - (x+1)\phi = 0. \tag{4.10}$$

It reduces to a Riccati equation by the substitution (4.8).

3. The subalgebra $< X_1, X_2 + X_3 >$ has the invariants

$$I_1 = x, \quad I_2 = \frac{u}{y} \cdot$$

The invariant solution has the form

$$u = y\phi(x) \tag{4.11}$$

with the function $\phi(x)$ defined by the equation

$$\frac{1}{2}A^2x^2\phi'' + \Big(ABCx - Ex^{\frac{3}{2}}\Big)\phi' - \Big(\frac{BCE}{A}x^{\frac{1}{2}} + x\Big)\phi = 0. \tag{4.12}$$

It reduces to a Riccati equation by the substitution (4.8).

4.4 Infinite ideal as a generator of new solutions

Recall that the infinite set of operators X_ω does not provide invariant solutions by the direct method (see the end of subsection 3.5). However, we can use it to generate new solutions from known ones as follows. Let

$$u = \omega(t, x, y)$$

be a known solution of Eq. (1.2) so that the operator X_ω is admitted by (1.2). Then, if X is any operator admitted by Eq. (1.2), one obtains that

$$[X_\omega, X] = X_{\overline{\omega}}, \tag{4.13}$$

where $\overline{\omega}(t, x, y)$ is a solution (in general, it is different from $\omega(t, x, y)$) of Eq. (1.2). The relation (4.13) means that the set L_ω of operators of the form X_ω is an *ideal* of the symmetry Lie algebra. Since the set of solutions $\omega(t, x, y)$ is infinite, L_ω is called an *infinite ideal*.

Thus, given a solution $\omega(t, x, y)$, the formula (4.13) provides a new solution $\overline{\omega}(t, x, y)$ to the equation (1.2). Let us apply this approach to the solutions given in the example of the previous subsection by letting $X = X_4$ from (4.5).

1. Starting with the solution (4.6), we have

$$\omega(t, x, y) = \phi(x),$$

where $\phi(x)$ is determined by the differential equation (4.7). Then

$$[X_\omega, X_4] = \big(2BC\ln x - 2A\ln y + (B - AC)ABt\big)\phi(x)\frac{\partial}{\partial u}\,.$$

Hence, the new solution $u = \overline{\omega}(t, x, y)$ is

$$\overline{\omega}(t, x, y) = \big(2BC\ln x - 2A\ln y + (B - AC)ABt\big)\phi(x) \tag{4.14}$$

with the function $\phi(x)$ determined by Eq. (4.7). Now we can repeat the procedure by taking the solution (4.14) as $\omega(t, x, y)$ in Eq. (4.13). Then

$$\begin{aligned}[][X_\omega, X_4] =& \Big[\big(2BC\ln x - 2A\ln y + (B - AC)ABt\big)^2\phi(x)\\ &+ 4A^2B^2(1 - C^2)t\phi(x)\Big]\frac{\partial}{\partial u}\cdot\end{aligned}$$

Hence, we arrive at the solution

$$u = \Big[\big(2BC\ln x - 2A\ln y + (B - AC)ABt\big)^2 + 4A^2B^2(1 - C^2)t\Big]\,\phi(x),$$

where $\phi(x)$ is again a solution of (4.7). By iterating this procedure, one obtains an infinite set of distinctly different solutions to Eq. (4.4). Further new solutions can be obtained by replacing X_4 by any linear combination of the operators (4.5).

2. For the solution (4.9), $\omega(t, x, y) = \mathrm{e}^t\phi(x)$, where $\phi(x)$ is determined by the equation (4.10). In this case,

$$[X_\omega, X_4] = \big(2BC\ln x - 2A\ln y + (B - AC)ABt\big)\mathrm{e}^t\phi(x)\frac{\partial}{\partial u}\,,$$

and the new solution $u = \overline{\omega}(t, x, y)$ has the form

$$\overline{\omega}(t, x, y) = \big(2BC\ln x - 2A\ln y + (B - AC)ABt\big)\mathrm{e}^t\phi(x) \tag{4.15}$$

with the function $\phi(x)$ determined by Eq. (4.10). One can iterate the procedure.

3. For the solution (4.11) we have

$$\omega(t, x, y) = y\phi(x),$$

where $\phi(x)$ is determined by the equation (4.12). In this case,

$$[X_\omega, X_4] = \big(2BC\ln x - 2A\ln y + (2BC^2 - B - AC)ABt\big)y\phi(x)\frac{\partial}{\partial u}\,,$$

and the new solution is

$$u = \big(2BC\ln x - 2A\ln y + (2BC^2 - B - AC)ABt\big)y\phi(x), \tag{4.16}$$

where the function $\phi(x)$ is determined by Eq. (4.12). The iteration of the procedure yields an infinite series of solutions.

5 Conclusion

The Lie group analysis is applied to the Black-Scholes and Jacobs-Jones models. The approach provides a wide class of analytic solutions of these equations. The fundamental solution to the Black-Scholes equation is constructed by means of the invariance principle. It can be used for general analysis of an arbitrary initial value problem.

For the Jacobs-Jones model, we present the group classification which shows that the dimension of the symmetry Lie algebra essentially depends on the parameters of the model. It also follows from this classification result that the Jacobs-Jones equation can not be transformed into the classical two-dimensional heat equation.

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Paper 8

Lie symmetry analysis of soil water motion equations

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Abstract. Exact solutions for a class of nonlinear partial differential equations modelling soil water infiltration and redistribution in irrigation systems are studied. These solutions are invariant under two-parameter symmetry groups obtained by the group classification of the governing equation. A general procedure for constructing invariant solutions is presented in a way convenient for investigating numerous new exact solutions.

1 Introduction

Differential equations occurring in the science and engineering, as mathematical models, often involve undetermined parameters and/or arbitrary functions of certain variables. Usually, these arbitrary elements (parameters or functions) are found experimentally or chosen from a "simplicity criterion". Lie group theory provides a regular procedure for determining arbitrary elements from symmetry point of view. This direction of study, originated by a great mathematician of 19th century Sophus Lie (Norway, 1842–1899), is known today as *Lie group classification of differential equations.* For detailed presentations of methods used in Lie group classification

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of differential equations the reader is refereed to the first fundamental paper on this topic [89] dealing with the classification of linear second-order partial differential equations with two independent variables. Modern approach to the problem is clearly presented in the classical book [111], Chapter 4, containing classification of several nonlinear equations and systems occurring in gasdynamics. See also [1], Sections 4 to 5, on the classification of heat conduction type equations, and [67], Chapter 2.

Lie group classification of differential equations provides a mathematical background for what can be called a *group theoretic modelling* (see [68], Chapter 6). In this approach, differential equations admitting more symmetries are considered to be “preferable”. In this way, one often arrives at equations possessing remarkable physical properties.

An extensive compilation and systematization of the results on symmetry analysis and group classification of differential equations obtained by S. Lie and his followers during the period of over one hundred years is presented in the Handbook [65], [67], [68]. The material in the Handbook is presented in a form convenient for immediate applications by applied scientists and engineers to their own problems.

This paper is aimed at Lie group analysis (symmetries and invariant solutions) of the mathematical model suggested in [136] (see also [135]) to simulate soil water infiltration and redistribution in a bedded soil profile irrigated by a drip irrigation system. The paper is closely related to the results of the authors on Lie group classification of nonlinear (2 + 1)-dimensional heat conduction type equations briefly presented in [67], Section 9.8.

The model discussed is described by the class of equations

$$C(\psi)\psi_t = (K(\psi)\psi_x)_x + (K(\psi)\,(\psi_z - 1))_z - S(\psi) \tag{1.1}$$

with three “arbitrary elements” represented by the functions $C(\psi) \neq 0$, $K(\psi) \neq 0$, and $S(\psi)$. Here ψ is a soil moisture pressure head, $C(\psi)$ is a specific water capacity, $K(\psi)$ is a unsaturated hydraulic conductivity, $S(\psi)$ is a sink or source term, t is a time, x is a horizontal and z is a vertical axis, which is considered positive downward. A discussion, based on analysis of numerical solutions, clarifying a validity of the model and its use in applied agricultural sciences is to be found in [135].

The contents of the present paper is as follows. Chapter 2 contains formulae for linking Eq. (1.1) with nonlinear heat conduction type equations. These formulae are necessary for a symmetry analysis of Eq. (1.1) using the group classification of heat conduction type equations given in Chapter 3 (see also [67], Section 9.8). Enumeration of all symmetries of equations (1.1) is given in Chapter 4. Following S. Lie [89] and L.V. Ovsyannikov [111], we present in Chapter 5 a simple algorithm for constructing invariant solutions in the case of the invariance under two-parameter groups. Examples of invariant solutions are given analytically and graphically.

2 Relation to heat conduction type equations

Theorem. There is a correspondence between the family of soil water motion equations (1.1) and the family of heat conduction type equations

$$u_t = (k(u)u_x)_x + (k(u)u_z)_z + l(u)u_z + p(u). \tag{2.1}$$

Namely, given an equation of the form (1.1), one can introduce a new variable $u = u(\psi)$ (depending on coefficients of the equation (1.1)) satisfying an equation of the form (2.1). Conversely, any equation of the form (2.1) is linked with an equation of the form (1.1).

Proof. Given an equation (1.1) with the coefficients $C(\psi)$, $K(\psi)$, $S(\psi)$, the new variable u is defined by

$$u = \int C(\psi)\, d\psi + A_1, \tag{2.2}$$

where A_1 is a constant of integration. The coefficients of the corresponding equation (2.1) are obtained by the formulae

$$k(u) = \frac{K(\psi)}{C(\psi)}\,, \quad l(u) = -\frac{K'(\psi)}{C(\psi)}\,, \quad p(u) = -S(\psi)\,, \tag{2.3}$$

where ψ is expressed, from (2.2), as a function $\psi = \psi(u)$.

Note that the mapping given by the equation (2.2) is not single valued because of its depends on the arbitrary constant A_1. Therefore, we let A_1 to be any fixed constant. However, the inverse to (2.2) may be not a single valued function even when the constant A_1 is fixed. In these cases, we assume in Eqs. (2.3) any fixed branch $\psi = \psi(u)$ of the inversion to the mapping (2.2).

To prove that any equation of the form (2.1) is linked with an equation of the form (1.1), let us consider the equations (2.2)-(2.3) as a system of functional-differential equations for $K(\psi), C(\psi), S(\psi)$ with known functions $k(u), l(u), p(u)$. Then, integrating the second equation in (2.3) with respect to ψ taking into account Eq. (2.2) and denoting

$$\int l(u)\, du = L(u), \tag{2.4}$$

we get

$$L(u) = A_2 - K(\psi), \quad A_2 = \text{const.} \tag{2.5}$$

The equation (2.2) yields

$$du = C(\psi)\, d\psi$$

or

$$\frac{d\psi}{du} = \frac{1}{C(\psi)}.$$

The first equation in (2.3) together with (2.5) yields

$$\frac{1}{C(\psi)} = \frac{k(u)}{K(\psi)} = \frac{k(u)}{A_2 - L(u)}.$$

Whence

$$d\psi = \frac{du}{C(\psi)} = \frac{k(u)\, du}{A_2 - L(u)}. \tag{2.6}$$

Hence, the dependence of ψ upon u is given by

$$\psi = \int \frac{k(u)}{A_2 - L(u)}\, du + A_3, \tag{2.7}$$

where A_3 is a constant of integration. Denoting by

$$u = U(\psi)$$

the inversion of (2.5), we obtain from Eqs. (2.2)-(2.3):

$$C(\psi) = \frac{\partial U}{\partial \psi}, \quad K(\psi) = k(U(\psi))\frac{\partial U}{\partial \psi}, \quad S(\psi) = -p(U(\psi)). \tag{2.8}$$

Thus, the equation (2.1) with the given coefficients $k(u), l(u), p(u)$ is linked with the equation of the form (1.1) with the coefficients $C(\psi)$, $K(\psi)$ and $S(\psi)$ by the formulae (2.7) and (2.8). This completes the proof.

Remark. Applying both of the described transformations to a single equation (1.1), one obtain a family of equations of the form (1.1) depending, in general, upon three parameters A_1, A_2, A_3. The family obtained contains the initial equation (1.1) for an appropriate choice of the parameters.

Example. Consider Eq. (1.1) with the coefficients

$$C(\psi) = 1,\ K(\psi) = \mathrm{e}^{\psi},\ S(\psi) = 0,$$

i.e. the equation

$$\psi_t = (\mathrm{e}^{\psi}\psi_x)_x + (\mathrm{e}^{\psi}\psi_z)_z - \mathrm{e}^{\psi}\psi_z\,. \tag{2.9}$$

The transformation (2.2) is written

$$u = \psi + A_1,$$

and the formulae (2.3) yield:

$$k(u) = a_1 \mathrm{e}^u,\ l(u) = -a_1 \mathrm{e}^u,\ p(u) = 0, \quad a_1 = \mathrm{e}^{-A_1}.$$

Hence, Eq. (2.9) is linked with the equation (2.1) of the form

$$u_t = (a_1 \mathrm{e}^u u_x)_x + (a_1 \mathrm{e}^u u_z)_z - a_1 \mathrm{e}^u u_z\,. \tag{2.10}$$

In this example the formula (2.7) yields

$$\psi = \ln|A_2 + a_1 \mathrm{e}^u| + A_3,$$

whence

$$u = U(\psi) \equiv \ln|\frac{a_3 \mathrm{e}^\psi A_2}{a_1}|, \quad a_3 = \mathrm{e}^{-A_3}.$$

Applying the formulae (2.8) we arrive at the following family of equations of the form (1.1):

$$\frac{\mathrm{e}^\psi}{a_3 \mathrm{e}^\psi - A_2} \psi_t = (\mathrm{e}^\psi \psi_x)_x + (\mathrm{e}^\psi(\psi_z - 1))_z\,,$$

depending, in this case, on two arbitrary parameters A_2, a_3 (cf. the above remark). The initial equation (2.9) is obtained by letting $A_2 = 0, a_3 = 1$.

3 Symmetries of equations (2.1)

One can consider transformations of the variables t, x, z, u leaving invariant every equation of the form (2.1) independently on a choice of the coefficients $k(u), l(u), p(u)$. These transformations form a Lie group known as the *principal Lie group* of the equations (2.1). The Lie algebra of the principal Lie group is called the *principal Lie algebra* of symmetries of Eq. (2.1) and is denoted by $L_{\mathcal{P}}$ (see, e.g., [67]).

It may happen that for particular choice of the coefficients Eq. (2.1) admits, along with the principal Lie group, additional symmetry transformations. Determination of all distinctly different particular cases of an extension of the principal Lie group is a problem of the group classification. This approach applied to problems of mathematical modelling is called *a group theoretic modelling* (see the reference in Introduction).

This section contains the result of the complete group classification of the equations (2.1). For the particular case $l(u) \equiv 0$, the symmetry of Eqs. (2.1) were investigated in [27] (see also [34] and [129]). Application of the transformations given in Section 2 provides equations of the form (1.1) with extended symmetries. The symmetries for Eq. (1.1) are denoted by $Y_1, Y_2, \ldots$. These symmetries, together with the coefficients of the corresponding equation (1.1), are presented in small font.

Classification result

Equivalence transformations

The result of group classification of the equations (2.1) is given up to the equivalence transformations

$$\bar{t} = \alpha_1 t + \alpha_2, \quad \bar{x} = \beta x + \gamma_1, \quad \bar{z} = \beta z + \gamma_2 t + \gamma_3,$$
$$\bar{u} = \delta_1 u + \delta_2, \quad \bar{k} = \frac{\beta^2}{\alpha_1} k, \quad \bar{l} = \frac{\beta}{\alpha_1} l - \frac{\gamma_2}{\alpha_1}, \quad \bar{p} = \frac{\delta_1}{\alpha_1} p \,, \tag{3.1}$$

where α_i, β, γ_i, and δ_i are arbitrary constants, $\alpha_1 \beta \delta_1 \neq 0$.

The principal Lie algebra

The principal Lie algebra $L_{\mathcal{P}}$ (i.e. the Lie algebra of the Lie transformation group admitted by Eq. (2.1) for arbitrary $k(u)$, $l(u)$ and $p(u)$) is the three-dimensional Lie algebra spanned by the following generators:

$$X_1 = \frac{\partial}{\partial t}\,, \quad X_2 = \frac{\partial}{\partial x}\,, \quad X_3 = \frac{\partial}{\partial z}\,.$$

The operators X_1, X_2, and X_3 generate groups of translations along the t, x, and z - axes, respectively. Hence the principal Lie group of the equation (2.1) is the three-parameter group of translations.

The principal Lie algebra $L_{\mathcal{P}}$ of (1.1) (i.e. the Lie algebra of the Lie transformation group admitted by Eq. (1.1) for arbitrary $C(\psi)$, $K(\psi)$ and $S(\psi)$) is the three-dimensional Lie algebra spanned by the following generators:

$$Y_1 = \frac{\partial}{\partial t}\,, \quad Y_2 = \frac{\partial}{\partial x}\,, \quad Y_3 = \frac{\partial}{\partial z}\,.$$

Extensions of the principal Lie algebra

The algebra $L_{\mathcal{P}}$ extends in the following cases:

I. $k(u)$ and $p(u)$ are arbitrary functions, $l(u) = 0$

$$X_4 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\,.$$

$C(\psi)$ and $S(\psi)$ are arbitrary functions, $K(\psi) = A$ (A is an arbitrary constant)

$$Y_4 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\,.$$

Subcase: For $p(u) = 0$, there is a further extension

$$X_5 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z}\,.$$

$S(\psi) = 0$

$$Y_5 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z}\,.$$

II. $k(u) = e^u$

1. $l(u) = Ae^u$, $p(u) = Be^u + D$ (A, B, and D are arbitrary constants, $A \neq 0$)

$$X_4 = \begin{cases} \exp(-Dt)\,\frac{\partial}{\partial t} + D\exp(-Dt)\,\frac{\partial}{\partial u} & \text{if } D \neq 0, \\ t\frac{\partial}{\partial t} - \frac{\partial}{\partial u} & \text{if } D = 0. \end{cases}$$

$C(\psi) = A\big(Me^{A\psi} - 1\big)^{-1}$, $K(\psi) = e^{-A\psi}$, $S(\psi) = -D + (B/A)\left(e^{-A\psi} - M\right)$, $\psi = -(1/A)\ln\left(M - Ae^u\right)$

$$Y_4 = \begin{cases} \exp(-Dt)\,\frac{\partial}{\partial t} + \frac{D}{A}\exp(-Dt)\left(Me^{A\psi} - 1\right)\frac{\partial}{\partial \psi} & \text{if } D \neq 0, \\ t\frac{\partial}{\partial t} - \frac{1}{A}\big(Me^{A\psi} - 1\big)\frac{\partial}{\partial \psi} & \text{if } D = 0. \end{cases}$$

Subcase: For $B = \dfrac{A^2}{4}$ there is a further extension:

$$X_5 = \sin\left(\frac{A}{4}x\right)\exp\left(-\frac{A}{4}z\right)\frac{\partial}{\partial x} - \cos\left(\frac{A}{4}x\right)\exp\left(-\frac{A}{4}z\right)\frac{\partial}{\partial z}$$
$$+\frac{A}{2}\cos\left(\frac{A}{4}x\right)\exp\left(-\frac{A}{4}z\right)\frac{\partial}{\partial u}\,,$$
$$X_6 = \cos\left(\frac{A}{4}x\right)\exp\left(-\frac{A}{4}z\right)\frac{\partial}{\partial x} + \sin\left(\frac{A}{4}x\right)\exp\left(-\frac{A}{4}z\right)\frac{\partial}{\partial z}$$
$$-\frac{A}{2}\sin\left(\frac{A}{4}x\right)\exp\left(-\frac{A}{4}z\right)\frac{\partial}{\partial u}\,,$$

$$Y_5 = \sin\left(\frac{A}{4}x\right)\exp\left(-\frac{A}{4}z\right)\frac{\partial}{\partial x} - \cos\left(\frac{A}{4}x\right)\exp\left(-\frac{A}{4}z\right)\frac{\partial}{\partial z} + \frac{1}{2}\cos\left(\frac{A}{4}x\right)\exp\left(-\frac{A}{4}z\right)\left(Me^{A\psi}-1\right)\frac{\partial}{\partial \psi},$$

$$Y_6 = \cos\left(\frac{A}{4}x\right)\exp\left(-\frac{A}{4}z\right)\frac{\partial}{\partial x} + \sin\left(\frac{A}{4}x\right)\exp\left(-\frac{A}{4}z\right)\frac{\partial}{\partial z} - \frac{1}{2}\sin\left(\frac{A}{4}x\right)\exp\left(-\frac{A}{4}z\right)\left(Me^{A\psi}-1\right)\frac{\partial}{\partial \psi}.$$

2. $l(u) = Ae^{\sigma u}$, $p(u) = Be^{(2\sigma-1)u}$
(A, B, and σ are arbitrary constants, $A \neq 0$, $\sigma \neq 0$).

$$X_4 = (1-2\sigma)t\frac{\partial}{\partial t} + (1-\sigma)x\frac{\partial}{\partial x} + (1-\sigma)z\frac{\partial}{\partial z} + \frac{\partial}{\partial u}.$$

$C(\psi) = e^{-u}(M - (A/\sigma)e^{\sigma u})$, $K(\psi) = M - (A/\sigma)e^{\sigma u}$, $S(\psi) = -Be^{(2\sigma-1)u}$,

$$\psi = \int^{\exp u} dw(M - (A/\sigma)w^{\sigma})^{-1}$$

$$Y_4 = (1-2\sigma)t\frac{\partial}{\partial t} + (1-\sigma)x\frac{\partial}{\partial x} + (1-\sigma)z\frac{\partial}{\partial z} + \frac{e^u}{M - (A/\sigma)e^{\sigma u}}\frac{\partial}{\partial \psi}.$$

3. $l(u) = Au$, $p(u) = Be^{-u}$ (A and B are arbitrary constants, $A \neq 0$)

$$X_4 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + (z - \mathrm{At})\frac{\partial}{\partial z} + \frac{\partial}{\partial u}.$$

$C(\psi) = e^{-u}\left(M - (A/2)u^2\right)$, $K(\psi) = M - (A/2)u^2$, $S(\psi) = -Be^{-u}$,

$$\psi = \frac{1}{\sqrt{2AM}}\left[e^{-\sqrt{2M/A}}\,\mathrm{li}\left(e^{u+\sqrt{2M/A}}\right) - e^{\sqrt{2M/A}}\,\mathrm{li}\left(e^{u-\sqrt{2M/A}}\right)\right],$$

where $\mathrm{li}(x) = \int_0^x \frac{dy}{\ln y}$ is a logarithm integral.

$$Y_4 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + (z - \mathrm{At})\frac{\partial}{\partial z} + e^u\left(M - \frac{A}{2}u^2\right)^{-1}\frac{\partial}{\partial \psi}.$$

4. $l(u) = 0$, $p(u)$ is an arbitrary function

$$X_4 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}.$$

$C(\psi) = (1/\psi)$, $K(\psi) = M$, $S(\psi)$ is an arbitrary function, $\psi = (1/M)e^u$

$$Y_4 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}.$$

For each of the following functions $p(u)$ (and $S(\psi)$) there is a further extension:

(i) $p(u) = \pm e^u + \lambda,\ \ \lambda = \pm 1.$

$$X_5 = \exp(-\lambda t)\frac{\partial}{\partial t} + \lambda \exp(-\lambda t)\frac{\partial}{\partial u}\,.$$

$S(\psi) = \mp M\psi + \lambda,\ \ \lambda = \pm 1\,.$

$$Y_5 = \exp(-\lambda t)\frac{\partial}{\partial t} + \lambda \exp(-\lambda t)\psi\frac{\partial}{\partial \psi}\,.$$

(ii) $p(u) = \pm e^{\sigma u},\ \ \sigma$ is an arbitrary constant, $\ \sigma \neq 0.$

$$X_5 = 2\sigma t\frac{\partial}{\partial t} + (\sigma - 1)x\frac{\partial}{\partial x} + (\sigma - 1)z\frac{\partial}{\partial z} - 2\frac{\partial}{\partial u}\,.$$

$S(\psi) = \mp(M\psi)^\sigma,\ \ \sigma$ is an arbitrary constant, $\ \sigma \neq 0.$

$$Y_5 = 2\sigma t\frac{\partial}{\partial t} + (\sigma - 1)x\frac{\partial}{\partial x} + (\sigma - 1)z\frac{\partial}{\partial z} - 2\psi\frac{\partial}{\partial \psi}\,.$$

(iii) $p(u) = \delta,\ \ \delta = 0, \pm 1.$

$$X_5 = x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} + 2\frac{\partial}{\partial u},$$

$$X_6 = \begin{cases} \exp(-\delta t)\frac{\partial}{\partial t} + \delta \exp(-\delta t)\frac{\partial}{\partial u} & \text{if } \delta \neq 0, \\ t\frac{\partial}{\partial t} - \frac{\partial}{\partial u} & \text{if } \delta = 0. \end{cases}$$

$S(\psi) = -\delta,\ \ \delta = 0, \pm 1.$

$$Y_5 = x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} + 2\psi\frac{\partial}{\partial \psi}\,,$$

$$Y_6 = \begin{cases} \exp(-\delta t)\frac{\partial}{\partial t} + \delta \exp(-\delta t)\psi\frac{\partial}{\partial \psi} & \text{if } \delta \neq 0, \\ t\frac{\partial}{\partial t} - \psi\frac{\partial}{\partial \psi} & \text{if } \delta = 0. \end{cases}$$

III. $k(u) = u^\sigma$, σ is an arbitrary constant, $\sigma \neq 0,\ -1$

1. $l(u) = Au^\sigma,\ p(u) = Bu^{\sigma+1} - Du$

(A, B, and D are arbitrary constants, $A \neq 0$)

$$X_4 = \begin{cases} \exp(D\sigma t)\,\frac{\partial}{\partial t} - Du\exp(D\sigma t)\,\frac{\partial}{\partial u} & \text{if } D \neq 0, \\ \sigma t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} & \text{if } D = 0. \end{cases}$$

$C(\psi) = (A/(\sigma+1))e^{-A\psi}\Big((M/A)(\sigma+1) - e^{-A\psi}\Big)^{-\sigma/(\sigma+1)}$, $K(\psi) = (A/(\sigma+1))e^{-A\psi}$,
$S(\psi) = D\big((M/A)(\sigma+1) - e^{-A\psi}\big)^{1/(\sigma+1)} - B\big((M/A)(\sigma+1) - e^{-A\psi}\big)$,
$\psi = -(1/A)\ln((\sigma+1)(M/A) - u^{\sigma+1})$.

$$Y_4 = \begin{cases} \exp(D\sigma t)\,\frac{\partial}{\partial t} - \frac{(\sigma+1)D}{A}\exp(D\sigma t)\left(\frac{M}{A}(\sigma+1)e^{A\psi} - 1\right)\frac{\partial}{\partial \psi} & \text{if } D \neq 0\,, \\ \sigma t\frac{\partial}{\partial t} - \frac{(\sigma+1)}{A}\left(\frac{M}{A}(\sigma+1)e^{A\psi} - 1\right)\frac{\partial}{\partial \psi} & \text{if } D = 0\,. \end{cases}$$

2. $l(u) = Au^{\mu}$, $p(u) = Bu^{1+2\mu-\sigma}$
(A, B, σ, and μ are arbitrary constants, $A \neq 0$, $\mu \neq 0$)

$$X_4 = (2\mu-\sigma)t\frac{\partial}{\partial t} + (\mu-\sigma)x\frac{\partial}{\partial x} + (\mu-\sigma)z\frac{\partial}{\partial z} - u\frac{\partial}{\partial u}\,.$$

$C(\psi) = \big(M - (A/(\mu+1))u^{\mu+1}\big)u^{-\sigma}$, $K(\psi) = \big(M - (A/(\mu+1))u^{\mu+1}\big)$, $S(\psi) = -Bu^{(1+2\mu-\sigma)}$,
$\psi = \dfrac{1}{\sigma+1}\int^{u^{\sigma+1}} dw\,\big(M - (A/(\mu+1))w^{(\mu+1)/(\sigma+1)}\big)^{-1}$.

$$Y_4 = (2\mu-\sigma)t\frac{\partial}{\partial t} + (\mu-\sigma)x\frac{\partial}{\partial x} + (\mu-\sigma)z\frac{\partial}{\partial z} - \frac{u^{\sigma+1}}{M - (A/(\mu+1))u^{\mu+1}}\frac{\partial}{\partial \psi}\,.$$

3. $l(u) = 0$, $p(u)$ is an arbitrary function

$$X_4 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\,.$$

$C(\psi) = M((\sigma+1)M\psi)^{-\sigma/(\sigma+1)}$, $K(\psi) = M$, $S(\psi)$ is an arbitrary function, $\psi = (1/((\sigma+1)M)u^{\sigma+1}$

$$Y_4 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\,.$$

For each of the following functions $p(u)$ (and $S(\psi)$) there is a further extension:

(i) $p(u) = \pm u^{\nu}$, ν is an arbitrary constant, $\nu \neq 0, 1$.

$$X_5 = 2(1-\nu)t\frac{\partial}{\partial t} + (\sigma-\nu+1)x\frac{\partial}{\partial x} + (\sigma-\nu+1)z\frac{\partial}{\partial z} + 2u\frac{\partial}{\partial u}\,.$$

$S(\psi) = \mp((\sigma+1)M\psi)^{\nu/(\sigma+1)}$, ν is an arbitrary constant, $\nu \neq 0, 1$

$$Y_5 = 2(1-\nu)t\frac{\partial}{\partial t} + (\sigma-\nu+1)x\frac{\partial}{\partial x} + (\sigma-\nu+1)z\frac{\partial}{\partial z} + 2(\sigma+1)\psi\frac{\partial}{\partial \psi}.$$

(ii) $p(u) = \pm u^{\sigma+1} + \lambda u, \quad \lambda = \pm 1.$

$$X_5 = \exp(-\lambda\sigma t)\frac{\partial}{\partial t} + \lambda\exp(-\lambda\sigma t)u\frac{\partial}{\partial u}.$$

$S(\psi) = \mp(\sigma+1)M\psi - \lambda((\sigma+1)M\psi)^{1/(\sigma+1)}$.

$$Y_5 = \exp(-\lambda\sigma t)\frac{\partial}{\partial t} + \lambda\exp(-\lambda\sigma t)(\sigma+1)\psi\frac{\partial}{\partial \psi}.$$

(iii) $p(u) = \delta u, \quad \delta = 0, \pm 1.$

$$X_5 = \sigma x\frac{\partial}{\partial x} + \sigma z\frac{\partial}{\partial z} + 2u\frac{\partial}{\partial u},$$

$$X_6 = \begin{cases} \exp(-\delta\sigma t)\,\frac{\partial}{\partial t} + \delta\exp(-\delta\sigma t)\,u\frac{\partial}{\partial u} & \text{if } \delta \neq 0, \\ \sigma t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} & \text{if } \delta = 0. \end{cases}$$

$S(\psi) = -\delta((\sigma+1)M\psi)^{1/(\sigma+1)}$

$$Y_5 = \sigma x\frac{\partial}{\partial x} + \sigma z\frac{\partial}{\partial z} + 2(\sigma+1)\psi\frac{\partial}{\partial \psi},$$

$$Y_6 = \begin{cases} \exp(-\delta\sigma t)\,\frac{\partial}{\partial t} + \delta\exp(-\delta\sigma t)\,(\sigma+1)\psi\frac{\partial}{\partial \psi} & \text{if } \delta \neq 0, \\ \sigma t\frac{\partial}{\partial t} - (\sigma+1)\psi\frac{\partial}{\partial \psi} & \text{if } \delta = 0. \end{cases}$$

IV. $k(u) = u^{-1},\ l(u) = 0,\ p(u)$ is an arbitrary function

$$X_4 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}.$$

$C(\psi) = Me^{M\psi},\ K(\psi) = M,\ S(\psi)$ is an arbitrary function, $\psi = (1/M)\ln u$

$$Y_4 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}.$$

For each of the following functions $p(u)$ (and $S(\psi)$) there is a further extension:

(i) $p(u) = \pm u^\nu$, ν is an arbitrary constant, $\nu \neq 1$.

$$X_5 = 2(\nu - 1)t\frac{\partial}{\partial t} + \nu x\frac{\partial}{\partial x} + \nu z\frac{\partial}{\partial z} - 2u\frac{\partial}{\partial u}\,.$$

$S(\psi) = \mp e^{M\nu\psi}$, ν is an arbitrary constant, $\nu \neq 1$.

$$Y_5 = 2(\nu - 1)t\frac{\partial}{\partial t} + \nu x\frac{\partial}{\partial x} + \nu z\frac{\partial}{\partial z} - \frac{2}{M}\frac{\partial}{\partial \psi}\,.$$

(ii) $p(u) = \lambda u \pm 1, \quad \lambda = \pm 1.$

$$X_5 = \exp(\lambda t)\frac{\partial}{\partial t} + \lambda \exp(\lambda t)\, u\frac{\partial}{\partial u}\,.$$

$S(\psi) = -\lambda e^{M\psi} \mp 1, \quad \lambda = \pm 1.$

$$Y_5 = \exp(\lambda t)\frac{\partial}{\partial t} + (\lambda/M)\exp(\lambda t)\,\frac{\partial}{\partial \psi}\,.$$

(iii) $p(u) = \delta u, \quad \delta = 0, \pm 1\,.$

$$X_5 = \begin{cases} \exp(\delta t)\,\frac{\partial}{\partial t} + \delta\exp(\delta t)\,u\frac{\partial}{\partial u} & \text{if } \delta \neq 0,\\[2ex] t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} & \text{if } \delta = 0,\end{cases}$$

$$X_\infty = \alpha(x,z)\,\frac{\partial}{\partial x} + \beta(x,z)\,\frac{\partial}{\partial z} - 2u\alpha_x(x,z)\frac{\partial}{\partial u}\,,$$

where the functions $\alpha(x,z)$ and $\beta(x,z)$ satisfy the system

$$\alpha_x = \beta_z, \quad \alpha_z = -\beta_x.$$

$S(\psi) = -\delta e^{M\psi}\,, \quad \delta = 0, \pm 1\,.$

$$Y_5 = \begin{cases} \exp(\delta t)\,\frac{\partial}{\partial t} + (\delta/M)\exp(\delta t)\,\frac{\partial}{\partial \psi} & \text{if } \delta \neq 0,\\[2ex] t\frac{\partial}{\partial t} + (1/M)\frac{\partial}{\partial \psi} & \text{if } \delta = 0,\end{cases}$$

$$Y_\infty = \alpha(x,z)\,\frac{\partial}{\partial x} + \beta(x,z)\,\frac{\partial}{\partial z} - \frac{2}{M}\alpha_x(x,z)\frac{\partial}{\partial \psi}\,,$$

where the functions $\alpha(x,z)$ and $\beta(x,z)$ satisfy the system

$$\alpha_x = \beta_z, \quad \alpha_z = -\beta_x.$$

V. $k(u) = 1$.

1. $l(u) = A\exp(\sigma u), \quad p(u) = B\exp(2\sigma u)$
(A, B, and σ are arbitrary constants, $A \neq 0, \quad \sigma \neq 0$).

$$X_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} - \frac{1}{\sigma}\frac{\partial}{\partial u}.$$

$C(\psi) = K(\psi) = M - (A/\sigma)\exp(\sigma u)$, $S(\psi) = -B\exp(2\sigma u)$, $\psi = (1/\sigma M)\left[-\sigma u + \ln(-\sigma(M/A) + \exp(\sigma u))\right]$.

$$Y_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} - \frac{1}{\sigma M - A\exp(\sigma u)}\frac{\partial}{\partial \psi}.$$

2. $l(u) = A\ln u, \quad p(u) = u(B\ln u + D)$
(A, B, and D are arbitrary constants, $A \neq 0$).

$$X_4 = \begin{cases} \exp(Bt)\,\frac{\partial}{\partial z} - \frac{B}{A}\exp(Bt)\,u\frac{\partial}{\partial u} & \text{if } B \neq 0, \\ t\frac{\partial}{\partial z} - \frac{u}{A}\frac{\partial}{\partial u} & \text{if } B = 0. \end{cases}$$

$C(\psi) = K(\psi) = M - Au(\ln u - 1)$, $S(\psi) = -u(B\ln u + D)$, $\psi = \int \frac{du}{M - Au(\ln u - 1)}$.

$$Y_4 = \begin{cases} \exp(Bt)\,\frac{\partial}{\partial z} - \frac{B}{A}\exp(Bt)\,\frac{u}{M - Au(\ln u - 1)}\,\frac{\partial}{\partial \psi} & \text{if } B \neq 0, \\ t\frac{\partial}{\partial z} - \frac{u}{A(M - Au(\ln u - 1))}\,\frac{\partial}{\partial \psi} & \text{if } B = 0. \end{cases}$$

3. $l(u) = A\ln u, \quad p(u) = Bu + \frac{1}{4}A^2u\ln^2 u - \frac{1}{4}A^2u\ln u + Du\ln u$
(A, B, and D are arbitrary constants, $A \neq 0$).

$$X_4 = \exp\left(Dt - \frac{A}{2}z\right)u\frac{\partial}{\partial u}.$$

$C(\psi) = K(\psi) = M - Au(\ln u - 1)$, $S(\psi) = -Bu - \frac{1}{4}A^2u\ln^2 u + \frac{1}{4}A^2u\ln u - Du\ln u$,
$\psi = \int \frac{du}{M - Au(\ln u - 1)}$.

$$Y_4 = \exp\left(Dt - \frac{A}{2}z\right)\frac{u}{M - Au(\ln u - 1)}\frac{\partial}{\partial \psi}.$$

4. $l(u) = Au^{\sigma}, \quad p(u) = Bu^{2\sigma+1}$
(A, B, and σ are arbitrary constants, $A \neq 0, \quad \sigma \neq 0, 1$)

$$X_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} - \frac{u}{\sigma}\frac{\partial}{\partial u}\,.$$

$$C(\psi) = K(\psi) = M - (A/(\sigma+1))u^{\sigma+1},\, S(\psi) = -Bu^{2\sigma+1},\, \psi = (\sigma+1)\int du(M(\sigma+1) - Au^{\sigma+1})^{-1}$$

$$Y_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} - \frac{u}{\sigma M - (A\sigma/(\sigma+1))u^{\sigma+1}}\frac{\partial}{\partial \psi}\,.$$

5. $l(u) = Au,\ p(u) = Bu^3$ (A and B are arbitrary constants, $A \neq 0,\ B \neq 0$)

$$X_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} - u\frac{\partial}{\partial u}\,.$$

(i) $C(\psi) = K(\psi) = -M\sinh^{-2}\left(\sqrt{\frac{MA}{2}}\psi\right)$, $S(\psi) = -B(2M/A)^{3/2}\coth^3\left(\sqrt{\frac{MA}{2}}\psi\right)$,
$\psi = \frac{1}{\sqrt{2MA}}\ln\left(\frac{u+\sqrt{2M/A}}{u-\sqrt{2M/A}}\right)$

$$Y_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} + \frac{1}{\sqrt{2MA}}\sinh\left(\sqrt{2MA}\psi\right)\frac{\partial}{\partial \psi}$$

(ii) $C(\psi) = K(\psi) = M\cosh^{-2}\left(\sqrt{\frac{MA}{2}}\psi\right)$, $S(\psi) = -B(2M/A)^{3/2}\tanh^3\left(\sqrt{\frac{MA}{2}}\psi\right)$,
$\psi = \frac{1}{\sqrt{2MA}}\ln\left(\frac{u+\sqrt{2M/A}}{-u+\sqrt{2M/A}}\right)$.

$$Y_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} - \frac{1}{\sqrt{2MA}}\sinh\left(\sqrt{2MA}\psi\right)\frac{\partial}{\partial \psi}\,.$$

6. $l(u) = Au,\ p(u) = Bu$ (A and B are arbitrary constants, $A \neq 0,\ B \neq 0$)

$$X_4 = \exp(Bt)\,\frac{\partial}{\partial z} - \frac{B}{A}\exp(Bt)\,\frac{\partial}{\partial u}$$

(i) $C(\psi) = K(\psi) = -M\sinh^{-2}\left(\sqrt{\frac{MA}{2}}\psi\right)$, $S(\psi) = -B(2M/A)^{1/2}\coth\left(\sqrt{\frac{MA}{2}}\psi\right)$,
$\psi = \frac{1}{\sqrt{2MA}}\ln\left(\frac{u+\sqrt{2M/A}}{u-\sqrt{2M/A}}\right)$

$$Y_4 = \exp(Bt)\,\frac{\partial}{\partial z} + \frac{B}{MA}\exp(Bt)\sinh^2\left(\sqrt{\frac{MA}{2}}\psi\right)\frac{\partial}{\partial \psi}$$

(ii) $C(\psi) = K(\psi) = M\cosh^{-2}\left(\sqrt{\frac{MA}{2}}\psi\right)$, $S(\psi) = -B(2M/A)^{1/2}\tanh\left(\sqrt{\frac{MA}{2}}\psi\right)$,
$\psi = \frac{1}{\sqrt{2MA}}\ln\left(\frac{u+\sqrt{2M/A}}{-u+\sqrt{2M/A}}\right)$

$$Y_4 = \exp(Bt)\,\frac{\partial}{\partial z} - \frac{B}{MA}\exp(Bt)\cosh^2\left(\sqrt{\frac{MA}{2}}\psi\right)\frac{\partial}{\partial \psi}\,.$$

7. $l(u) = Au,\ p(u) = B$ (A and B are arbitrary constants, $A \neq 0$)

$$X_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + \left(z - \frac{3}{2}ABt^2\right)\frac{\partial}{\partial z} + (-u+3Bt)\frac{\partial}{\partial u}\,,$$

$$X_5 = t\frac{\partial}{\partial z} - \frac{1}{A}\frac{\partial}{\partial u}\,.$$

(i) $C(\psi) = K(\psi) = -M\sinh^{-2}\left(\sqrt{\frac{MA}{2}}\psi\right)$, $S(\psi) = -B$, $\psi = \frac{1}{\sqrt{2MA}}\ln\left(\frac{u+\sqrt{2M/A}}{u-\sqrt{2M/A}}\right)$

$$Y_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + \left(z - \frac{3}{2}ABt^2\right)\frac{\partial}{\partial z} + \left(\frac{1}{\sqrt{2MA}}\sinh\left(\sqrt{2MA}\psi\right) - \frac{3Bt}{M}\sinh^2\left(\sqrt{\frac{MA}{2}}\psi\right)\right)\frac{\partial}{\partial \psi}\,,$$

$$Y_5 = t\frac{\partial}{\partial z} + \frac{1}{MA}\sinh^2\left(\sqrt{\frac{MA}{2}}\psi\right)\frac{\partial}{\partial \psi}\,.$$

(ii) $C(\psi) = K(\psi) = M\cosh^{-2}\left(\sqrt{\frac{MA}{2}}\psi\right)$, $S(\psi) = -B$, $\psi = \frac{1}{\sqrt{2MA}}\ln\left(\frac{u+\sqrt{2M/A}}{-u+\sqrt{2M/A}}\right)$

$$Y_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + \left(z - \frac{3}{2}ABt^2\right)\frac{\partial}{\partial z} + \left(-\frac{1}{\sqrt{2MA}}\sinh\left(\sqrt{2MA}\psi\right) + \frac{3Bt}{M}\cosh^2\left(\sqrt{\frac{MA}{2}}\psi\right)\right)\frac{\partial}{\partial \psi}\,,$$

$$Y_5 = t\frac{\partial}{\partial z} - \frac{1}{MA}\cosh^2\left(\sqrt{\frac{MA}{2}}\psi\right)\frac{\partial}{\partial \psi}\,.$$

8. $l(u) = 0$, $p(u)$ is an arbitrary function ,

$$X_4 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\,.$$

$C(\psi) = K(\psi) = M$, $S(\psi)$ is an arbitrary function, $\psi = u/M$

$$Y_4 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\,.$$

For each of the following functions $p(u)$ (and $S(\psi)$) there is a further extension:

(i) $p(u) = \pm e^u$.

$$X_5 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} - 2\frac{\partial}{\partial u}\,.$$

$S(\psi) = \mp e^{M\psi}$.

$$Y_5 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} - \frac{2}{M}\frac{\partial}{\partial \psi}\,.$$

(ii) $p(u) = \pm u^\sigma$, σ is an arbitrary constant, $\sigma \neq 0, \pm 1$.

$$X_5 = 2(\sigma-1)t\frac{\partial}{\partial t} + (\sigma-1)x\frac{\partial}{\partial x} + (\sigma-1)z\frac{\partial}{\partial z} - 2u\frac{\partial}{\partial u}\,.$$

$S(\psi) = \mp (M\psi)^\sigma$, σ is an arbitrary constant, $\sigma \neq 0, \pm 1$.

$$Y_5 = 2(\sigma-1)t\frac{\partial}{\partial t} + (\sigma-1)x\frac{\partial}{\partial x} + (\sigma-1)z\frac{\partial}{\partial z} - 2\psi\frac{\partial}{\partial \psi}\,.$$

(iii) $p(u) = \delta u \ln u$, $\delta = \pm 1$.

$$X_5 = \exp(\delta t)u\frac{\partial}{\partial u}\,, \quad X_6 = \exp(\delta t)\frac{\partial}{\partial x} - \frac{\delta}{2}x\exp(\delta t)u\frac{\partial}{\partial u}\,,$$

$$X_7 = \exp(\delta t)\frac{\partial}{\partial z} - \frac{\delta}{2}z\exp(\delta t)u\frac{\partial}{\partial u}\,.$$

$S(\psi) = -\delta M\psi\ln(M\psi)$, $\delta = \pm 1$.

$$Y_5 = \exp(\delta t)\psi\frac{\partial}{\partial \psi}\,, \quad Y_6 = \exp(\delta t)\frac{\partial}{\partial x} - \frac{\delta}{2}x\exp(\delta t)\psi\frac{\partial}{\partial \psi}\,, \quad Y_7 = \exp(\delta t)\frac{\partial}{\partial z} - \frac{\delta}{2}z\exp(\delta t)\psi\frac{\partial}{\partial \psi}\,.$$

(iv) $p(u) = \delta u$, $\delta = 0, \pm 1$.

$$X_5 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} + 2\delta tu\frac{\partial}{\partial u}\,,$$

$$X_6 = 4t^2\frac{\partial}{\partial t} + 4tx\frac{\partial}{\partial x} + 4tz\frac{\partial}{\partial z} + (4\delta t^2 - 4t - x^2 - z^2)u\frac{\partial}{\partial u}\,,$$

$$X_7 = u\frac{\partial}{\partial u}\,, \quad X_8 = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}\,, \quad X_9 = 2t\frac{\partial}{\partial z} - zu\frac{\partial}{\partial u}\,,$$

$$X_\infty = \alpha(t,x,z)\frac{\partial}{\partial u}\,,$$

where α satisfies the equation

$$\alpha_t = \alpha_{xx} + \alpha_{zz} + \delta\alpha\,.$$

$S(\psi) = -\delta M\psi$, $\delta = 0, \pm 1$.

$$Y_5 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} + 2\delta t\psi\frac{\partial}{\partial \psi}\,,$$

$$Y_6 = 4t^2\frac{\partial}{\partial t} + 4tx\frac{\partial}{\partial x} + 4tz\frac{\partial}{\partial z} + (4\delta t^2 - 4t - x^2 - z^2)\psi\frac{\partial}{\partial \psi}\,,$$

$$Y_7 = u\frac{\partial}{\partial u}\,,\quad Y_8 = 2t\frac{\partial}{\partial x} - x\psi\frac{\partial}{\partial \psi}\,,\quad Y_9 = 2t\frac{\partial}{\partial z} - z\psi\frac{\partial}{\partial \psi}\,,$$

$$Y_\infty = \alpha(t,x,z)\frac{\partial}{\partial \psi}\,,$$

where α satisfies the equation

$$\alpha_t = \alpha_{xx} + \alpha_{zz} + \delta\alpha\,.$$

(v) $p(u) = \pm 1$.

$$X_5 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} + 2u\frac{\partial}{\partial u},$$

$$X_6 = 4t^2\frac{\partial}{\partial t} + 4tx\frac{\partial}{\partial x} + 4tz\frac{\partial}{\partial z} + \left[-(4t + x^2 + z^2)u + 8t^2 + tx^2 + tz^2\right]\frac{\partial}{\partial u},$$

$$X_7 = (u-t)\frac{\partial}{\partial u}\,,\quad X_8 = 2t\frac{\partial}{\partial x} + x(t-u)\frac{\partial}{\partial u}\,,\quad X_9 = 2t\frac{\partial}{\partial z} + z(t-u)\frac{\partial}{\partial u},$$

$$X_\infty = \alpha(t,x,z)\frac{\partial}{\partial u}\,,$$

where α satisfies the equation

$$\alpha_t = \alpha_{xx} + \alpha_{zz}\,.$$

$S(\psi) = \mp 1$.

$$Y_5 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} + 2\psi\frac{\partial}{\partial \psi},$$

$$Y_6 = 4t^2\frac{\partial}{\partial t} + 4tx\frac{\partial}{\partial x} + 4tz\frac{\partial}{\partial z} + \left[-(4t + x^2 + z^2)\psi + \frac{1}{M}(8t^2 + tx^2 + tz^2)\right]\frac{\partial}{\partial \psi},$$

$$Y_7 = (\psi - t/M)\frac{\partial}{\partial \psi}\,,\quad Y_8 = 2t\frac{\partial}{\partial x} - x(\psi - t/M)\frac{\partial}{\partial \psi}\,,\quad Y_9 = 2t\frac{\partial}{\partial z} - z(\psi - t/M)\frac{\partial}{\partial \psi},$$

$$Y_\infty = \alpha(t,x,z)\frac{\partial}{\partial \psi}\,,$$

where α satisfies the equation

$$\alpha_t = \alpha_{xx} + \alpha_{zz}\,.$$

4 Symmetries of equations (1.1)

The group classification of Eqs. (2.1) given in Section 3 does not provide, strictly speaking, the group classification of Eqs. (1.1). Indeed, the transformation (2.2) linking Eqs. (1.1) and (2.1) depends on the choice of coefficients (1.1) and therefore the transformation (2.2) does not map, in general, an equivalence transformation of Eqs. (2.1) into that of Eqs. (1.1). Let us illustrate this by the following example.

Example. Consider the particular case of the equivalence transformation (3.1)

$$\bar{t} = t, \quad \bar{x} = x, \quad \bar{z} = z + \gamma_2 t, \quad \bar{u} = u, \tag{4.1}$$

for which

$$\bar{k} = k, \quad \bar{l} = l - \gamma_2, \quad \bar{p} = p \tag{4.2}$$

with an arbitrary constant γ_2. Let us construct the image of this equaivalence transformation under the map (2.7), (2.8). Then, in accordance with (2.4),

$$\bar{L}(\bar{u}) = \int \bar{l}(\bar{u})\, d\bar{u},$$

whence, by using (4.1) and (2.5) we obtain

$$\bar{L}(\bar{u}) = \int (l(u) - \gamma_2) du = L(u) - \gamma_2 u.$$

The formula (2.5) is written in the new variables in the form

$$\bar{\psi} = \int \frac{\bar{k}(\bar{u})}{A_2 - \bar{L}(\bar{u})} d\bar{u} + A_3.$$

Since $\bar{k}(\bar{u})d\bar{u} = k(u)du$ (see (4.1)-(4.2)) and $k(u)du = (A_2 - L(u))d\psi$ (see (2.6)), the above formula is written

$$\bar{\psi} = \int \frac{A_2 - L(u)}{A_2 - L(u) + \gamma_2 u} d\psi + A_3,$$

or

$$\bar{\psi} = \psi - \gamma_2 \int \frac{u}{A_2 - L(u) + \gamma_2 u} d\psi + A_3.$$

Thus, after the map (2.2), the equivalence transformation (4.1) takes the form

$$\bar{t} = t, \quad \bar{x} = x, \quad \bar{z} = z + \gamma_2 t, \quad \bar{\psi} = \psi - \gamma_2 \int \frac{u}{A_2 - L(u) + \gamma_2 u} d\psi + A_3,$$

i.e. it is not an equivalence transformation in the usual sense. Such transformations can be called "quasi-equivalence transformations" for Eqs. (1.1).

Here, we systematize the equations (1.1) with extended symmetry obtained in Section 3.

The principal Lie algebra

The principal Lie algebra $L_{\mathcal{P}}$ (i.e. the Lie algebra of the Lie transformation group admitted by Eq. (1.1) for arbitrary $C(\psi)$, $K(\psi)$ and $S(\psi)$) is the three-dimensional Lie algebra spanned by the following generators:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial z}.$$

The operators X_1, X_2, and X_3 generate translations along the t, x, and z - axes, respectively. Hence the principal Lie group of the equation (1.1) is the three-parameter group of translations.

Extensions of the principal Lie algebra

The algebra $L_{\mathcal{P}}$ extends in the following cases.

I. $K(\psi) = 1,\ C(\psi)$ and $S(\psi)$ are arbitrary functions.

$$X_4 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}.$$

For the following functions $C(\psi)$ and $S(\psi)$ there is an additional extension:

1. $C(\psi)$ is an arbitrary function, $S(\psi) = 0$.

$$X_5 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z}.$$

2. $C(\psi) = \psi^\sigma,\ S(\psi)$ has one of the following forms:

(i) $S(\psi) = B\psi^\gamma$, where B and γ are arbitrary constants, $B \neq 0$, $\gamma \neq \sigma + 1$.

$$X_5 = 2(1+\sigma-\gamma)t\frac{\partial}{\partial t} + (1-\gamma)x\frac{\partial}{\partial x} + (1-\gamma)z\frac{\partial}{\partial z} + 2\psi\frac{\partial}{\partial \psi}.$$

(ii) $S(\psi) = B\psi^{\sigma+1} + D\psi$, where $B \neq 0$ and $D \neq 0$ are arbitrary constants.

$$X_5 = \exp(B\sigma t)\frac{\partial}{\partial t} + B\exp(B\sigma t)\psi\frac{\partial}{\partial \psi}.$$

If $D = 0, B \neq 0$ there is a further extension:

$$X_6 = \sigma x\frac{\partial}{\partial x} + \sigma z\frac{\partial}{\partial z} - 2\psi\frac{\partial}{\partial \psi}\,.$$

(iii) $S(\psi) = 0$.

$$X_5 = \sigma t\frac{\partial}{\partial t} + \psi\frac{\partial}{\partial \psi}\,,$$

$$X_6 = \sigma x\frac{\partial}{\partial x} + \sigma z\frac{\partial}{\partial z} - 2\psi\frac{\partial}{\partial \psi}\,.$$

3. $C(\psi) = \mathrm{e}^{\psi},\ S(\psi)$ has one of the following forms:

(i) $S(\psi) = B\mathrm{e}^{\nu\psi}$, ν is an arbitrary constant, $\nu \neq 1$.

$$X_5 = 2(\nu - 1)t\frac{\partial}{\partial t} + \nu x\frac{\partial}{\partial x} + \nu z\frac{\partial}{\partial z} - 2\frac{\partial}{\partial \psi}\,.$$

(ii) $S(\psi) = B\mathrm{e}^{\psi} + D$, where $B \neq 0$ and $D \neq 0$ are arbitrary constants.

$$X_5 = \exp(Bt)\frac{\partial}{\partial t} + B\exp(Bt)\,\frac{\partial}{\partial \psi}\,.$$

(iii) $S(\psi) = B\mathrm{e}^{\psi}$, where B is an arbitrary constant.

$$X_5 = \begin{cases} \exp(Bt)\,\frac{\partial}{\partial t} + B\exp(Bt)\,\frac{\partial}{\partial \psi} & \text{if } B \neq 0, \\ t\frac{\partial}{\partial t} + \frac{\partial}{\partial \psi} & \text{if } B = 0, \end{cases}$$

$$X_\infty = \alpha(x,z)\,\frac{\partial}{\partial x} + \beta(x,z)\,\frac{\partial}{\partial z} - 2\alpha_x(x,z)\frac{\partial}{\partial \psi}\,,$$

where the functions $\alpha(x,z)$ and $\beta(x,z)$ satisfy the system

$$\alpha_x = \beta_z, \quad \alpha_z = -\beta_x.$$

4. $C(\psi) = 1,\ S(\psi)$ has one of the following forms:

(i) $S(\psi) = \pm\mathrm{e}^{\psi}$.

$$X_5 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} - 2\frac{\partial}{\partial \psi}\,.$$

(ii) $S(\psi) = \pm\psi^{\sigma}$, where σ is an arbitrary constant, $\sigma \neq 0, 1$.

$$X_5 = 2(\sigma - 1)t\frac{\partial}{\partial t} + (\sigma - 1)x\frac{\partial}{\partial x} + (\sigma - 1)z\frac{\partial}{\partial z} - 2\psi\frac{\partial}{\partial \psi}\,.$$

(iii) $S(\psi) = -\delta\psi \ln(M\psi), \quad \delta = \pm 1.$

$$X_5 = \exp(\delta t)\psi \frac{\partial}{\partial \psi}\,, \quad X_6 = \exp(\delta t)\frac{\partial}{\partial x} - \frac{\delta}{2} x \exp(\delta t)\psi \frac{\partial}{\partial \psi}\,,$$

$$X_7 = \exp(\delta t)\frac{\partial}{\partial z} - \frac{\delta}{2} z \exp(\delta t)\psi \frac{\partial}{\partial \psi}\,.$$

(iv) $S(\psi) = -\delta\psi$, $\delta = 0, \pm 1.$

$$X_5 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} + 2\delta t\psi \frac{\partial}{\partial \psi}\,,$$

$$X_6 = 4t^2\frac{\partial}{\partial t} + 4tx\frac{\partial}{\partial x} + 4tz\frac{\partial}{\partial z} + (4\delta t^2 - 4t - x^2 - z^2)\psi \frac{\partial}{\partial \psi}\,,$$

$$X_7 = u\frac{\partial}{\partial u}\,, \quad X_8 = 2t\frac{\partial}{\partial x} - x\psi \frac{\partial}{\partial \psi}\,, \quad X_9 = 2t\frac{\partial}{\partial z} - z\psi \frac{\partial}{\partial \psi}\,,$$

$$X_\infty = \alpha(t, x, z)\frac{\partial}{\partial \psi}\,,$$

where α satisfies the equation

$$\alpha_t = \alpha_{xx} + \alpha_{zz} + \delta\alpha\,.$$

(v) $S(\psi) = \pm 1.$

$$X_5 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} + 2\psi \frac{\partial}{\partial \psi},$$

$$X_6 = 4t^2\frac{\partial}{\partial t} + 4tx\frac{\partial}{\partial x} + 4tz\frac{\partial}{\partial z} + \big[-(4t + x^2 + z^2)\psi + 8t^2 + tx^2 + tz^2\big]\frac{\partial}{\partial \psi},$$

$$X_7 = (\psi - t)\frac{\partial}{\partial \psi}\,,$$

$$X_8 = 2t\frac{\partial}{\partial x} - x(\psi - t)\frac{\partial}{\partial \psi}\,, \quad X_9 = 2t\frac{\partial}{\partial z} - z(\psi - t)\frac{\partial}{\partial \psi},$$

$$X_\infty = \alpha(t, x, z)\frac{\partial}{\partial \psi}\,,$$

where α satisfies the equation

$$\alpha_t = \alpha_{xx} + \alpha_{zz}\,.$$

II. $K(\psi) = C(\psi)$.

The principal Lie algebra extends for the following functions $C(\psi)$ and $S(\psi)$:

1. $C(\psi) = \sinh^{-2}(\psi)$, $S(\psi)$ has one of the following forms:

(i) $S(\psi) = B\coth^3(\psi)$, where $B \neq 0$ is an arbitrary constant.

$$X_4 = 4t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + 2z\frac{\partial}{\partial z} + \sinh(2\psi)\frac{\partial}{\partial \psi}\,.$$

(ii) $S(\psi) = B\coth(\psi)$, where $B \neq 0$ is an arbitrary constant.

$$X_4 = 2\exp(Bt)\,\frac{\partial}{\partial z} + B\exp(Bt)\sinh^2(\psi)\frac{\partial}{\partial \psi}\,.$$

(iii) $S(\psi) = B$, where B is an arbitrary constant.

$$X_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + (z - 3Bt^2)\frac{\partial}{\partial z} + \left(\frac{1}{2}\sinh(2\psi) - 3Bt\sinh^2(\psi)\right)\frac{\partial}{\partial \psi}\,,$$

$$X_5 = 2t\frac{\partial}{\partial z} + \sinh^2(\psi)\frac{\partial}{\partial \psi}\,.$$

2. $C(\psi) = \cosh^{-2}(\psi)$, $S(\psi)$ has one of the following forms:

(i) $S(\psi) = -B\tanh^3(\psi)$, where $B \neq 0$ is an arbitrary constant.

$$X_4 = 4t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + 2z\frac{\partial}{\partial z} - \sinh(\psi)\frac{\partial}{\partial \psi}\,.$$

(ii) $S(\psi) = -B\tanh(\psi)$, where $B \neq 0$ is an arbitrary constant.

$$X_4 = 2\exp(Bt)\,\frac{\partial}{\partial z} - B\exp(Bt)\cosh^2(\psi)\frac{\partial}{\partial \psi}\,.$$

(iii) $S(\psi) = -B$, where B is an arbitrary constant.

$$X_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + (z - 3Bt^2)\frac{\partial}{\partial z} + \left(-\frac{1}{2}\sinh(2\psi) + 3Bt\cosh^2(\psi)\right)\frac{\partial}{\partial \psi}\,,$$

$$X_5 = 2t\frac{\partial}{\partial z} - \cosh^2(\psi)\frac{\partial}{\partial \psi}\,.$$

3. $C(\psi) = M - Au(\ln u - 1)$, where $A \neq 0$, M are arbitrary constants, ψ is connected with u by the formula

$$\psi = \int \frac{du}{M - Au(\ln u - 1)},$$

and $S(\psi)$ has one of the following forms:

(i) $S(\psi) = -u(B\ln u + D)$, where B and D are arbitrary constants.

$$X_4 = \begin{cases} \exp(Bt)\,\frac{\partial}{\partial z} - \frac{B}{A}\exp(Bt)\,\frac{u}{M-Au(\ln u-1)}\frac{\partial}{\partial \psi} & \text{if } B \neq 0, \\ t\frac{\partial}{\partial z} - \frac{u}{A(M-Au(\ln u-1))}\frac{\partial}{\partial \psi} & \text{if } B = 0. \end{cases}$$

(ii) $S(\psi) = -Bu - \dfrac{1}{4}A^2 u \ln^2 u + \dfrac{1}{4}A^2 u\ln u - Du\ln u$,
where B and D are arbitrary constants.

$$X_4 = \exp\left(Dt - \frac{A}{2}z\right)\frac{u}{M - Au(\ln u - 1)}\frac{\partial}{\partial \psi}.$$

4. $C(\psi) = M - \dfrac{A}{\sigma}e^{\sigma u}$, $S(\psi) = -Be^{2\sigma u}$, where $A \neq 0$, B, $M \neq 0$, and $\sigma \neq 0$ are arbitrary constants, ψ is connected with u by the formula

$$\psi = \frac{1}{\sigma M}\left[-\sigma u + \ln(-\sigma(M/A) + e^{\sigma u})\right].$$

$$X_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} - \frac{1}{\sigma M - A\exp(\sigma u)}\frac{\partial}{\partial \psi}.$$

5. $C(\psi) = M - \dfrac{A}{\sigma+1}u^{\sigma+1}$, $S(\psi) = -Bu^{2\sigma+1}$, where $A \neq 0$, B, $M \neq 0$, and $\sigma \neq 0, -1$ are arbitrary constants, ψ is connected with u by the formula

$$\psi = (\sigma+1)\int (M(\sigma+1) - Au^{\sigma+1})^{-1}du.$$

$$X_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} - \frac{u}{\sigma M - (A\sigma/(\sigma+1))u^{\sigma+1}}\frac{\partial}{\partial \psi}.$$

III. $K(\psi) = e^{-\psi}$, $C(\psi) = e^{-\psi}(M - e^{-\psi})^{\sigma}$, $S(\psi) = -D(M - e^{-\psi})^{\sigma+1} + B(e^{-\psi} - M)$,
where B, D, and M are arbitrary constants.

$$X_4 = \begin{cases} \exp(D\sigma t)\,\frac{\partial}{\partial t} + D\exp(D\sigma t)\,(Me^{\psi} - 1)\frac{\partial}{\partial \psi} & \text{if } D \neq 0, \\ \sigma t\frac{\partial}{\partial t} + (Me^{\psi} - 1)\frac{\partial}{\partial \psi} & \text{if } D = 0. \end{cases}$$

If $\sigma=-1$ and $B=\dfrac{1}{4}$ there is an additional extension:

$$X_5=\sin\left(\frac{x}{4}\right)\,\exp\left(-\frac{z}{4}\right)\frac{\partial}{\partial x}-\cos\left(\frac{x}{4}\right)\,\exp\left(-\frac{z}{4}\right)\frac{\partial}{\partial z}$$
$$+\frac{1}{2}\cos\left(\frac{x}{4}\right)\,\exp\left(-\frac{z}{4}\right)\left(M\mathrm{e}^{\psi}-1\right)\frac{\partial}{\partial\psi}\,,$$
$$X_6=\cos\left(\frac{x}{4}\right)\,\exp\left(-\frac{z}{4}\right)\frac{\partial}{\partial x}+\sin\left(\frac{x}{4}\right)\,\exp\left(-\frac{z}{4}\right)\frac{\partial}{\partial z}$$
$$-\frac{1}{2}\sin\left(\frac{x}{4}\right)\,\exp\left(-\frac{z}{4}\right)\left(M\mathrm{e}^{\psi}-1\right)\frac{\partial}{\partial\psi}\,.$$

IV. $K(\psi)=M-A\mathrm{e}^{\sigma u},\ C(\psi)=(M-A\mathrm{e}^{\sigma u})\mathrm{e}^{-u},\ S(\psi)=B\mathrm{e}^{(2\sigma-1)u}$, where $A\neq 0, B, M, \sigma\neq 0$ are arbitrary constants, and ψ is connected with u by the formula

$$\psi=\int\limits_{u_0}^{\exp u}dw(M-Aw^{\sigma})^{-1}.$$

$$X_4=(1-2\sigma)t\frac{\partial}{\partial t}+(1-\sigma)x\frac{\partial}{\partial x}+(1-\sigma)z\frac{\partial}{\partial z}+\frac{e^u}{M-A\mathrm{e}^{\sigma u}}\frac{\partial}{\partial\psi}\,.$$

V. $K(\psi)=M-Au^2,\ C(\psi)=(M-Au^2)\,\mathrm{e}^{-u},\ S(\psi)=Be^{-u},$ where $A\neq 0, B, M\neq 0$ are arbitrary constants, and ψ is connected with u by the formula

$$\psi=\frac{1}{2\sqrt{AM}}\left[\mathrm{e}^{-\sqrt{M/A}}\,\mathrm{li}\left(\mathrm{e}^{u+\sqrt{M/A}}\right)-\mathrm{e}^{\sqrt{M/A}}\,\mathrm{li}\left(\mathrm{e}^{u-\sqrt{M/A}}\right)\right]$$

with $\mathrm{li}(x)$ the logarithm integral:

$$\mathrm{li}(x)=\int\limits_0^x\frac{dy}{\ln y}.$$

$$X_4=t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+(z-2At)\frac{\partial}{\partial z}+\frac{\mathrm{e}^u}{M-Au^2}\,\frac{\partial}{\partial\psi}\,.$$

VI. $K(\psi)=M-Au^{\mu+1},\ C(\psi)=(M-Au^{\mu+1})u^{-\sigma},\ S(\psi)=-Bu^{1+2\mu-\sigma},$ where

$A \neq 0, B, M, \sigma \neq 0, -1, \mu \neq -1$ are arbitrary constants, and ψ is connected with u by the formula

$$\psi = \frac{1}{\sigma+1} \int\limits_{u_0}^{u^{\sigma+1}} \left(M - Aw^{(\mu+1)/(\sigma+1)}\right)^{-1} dw \,.$$

$$X_4 = (2\mu - \sigma)t\frac{\partial}{\partial t} + (\mu - \sigma)x\frac{\partial}{\partial x} + (\mu - \sigma)z\frac{\partial}{\partial z} - \frac{u^{\sigma+1}}{M - Au^{\mu+1}}\frac{\partial}{\partial \psi}\,.$$

5 Invariant solutions

Here, the above results are used for the construction of exact (invariant) solutions of Eq. (1.1). Namely, we consider two particular equations of the form (1.1) when the principal Lie algebra $L_{\mathcal{P}}$ is extended by three and two operators, respectively. For each case, we look for solutions invariant under two-dimensional subalgebras of the symmetry Lie algebra. Then Eq. (1.1) reduces to second-order ordinary differential equations. The latter are solved in a closed form, and the solutions are also represented graphically. These examples illustrate the general algorithm due to Lie [89] and Ovsyannikov [111] for constructing invariant solutions and can be easily adopted by the reader in other cases.

Let the operators

$$X_1 = \xi_1^0(t,x,z,u)\frac{\partial}{\partial t} + \xi_1^1(t,x,z,u)\frac{\partial}{\partial x} + \xi_1^2(t,x,z,u)\frac{\partial}{\partial z} + \eta_1(t,x,z,u)\frac{\partial}{\partial u},$$

$$X_2 = \xi_2^0(t,x,z,u)\frac{\partial}{\partial t} + \xi_2^1(t,x,z,u)\frac{\partial}{\partial x} + \xi_2^2(t,x,z,u)\frac{\partial}{\partial z} + \eta_2(t,x,z,u)\frac{\partial}{\partial u}$$

be admitted by the equation (1.1). Let they span a two-dimensional Lie algebra, i.e.

$$[X_1, X_2] = \lambda_1 X_1 + \lambda_2 X_2, \ \ \lambda_1, \lambda_2 = \text{const},$$

and they satisfy the condition

$$\text{rank}\begin{pmatrix} \xi_1^0 & \xi_1^1 & \xi_1^2 & \eta_1 \\ \xi_2^0 & \xi_2^1 & \xi_2^2 & \eta_2 \end{pmatrix} = 2.$$

Then the system

$$X_1 I = 0, \ \ X_2 I = 0$$

has exactly two functionally independent solutions $I_1(t,x,z,u), I_2(t,x,z,u)$. The invariant solution has the form

$$I_2 = \phi(I_1). \tag{5.1}$$

After substitution of (5.1) into the corresponding equation (1.1), we obtain an ordinary differential equation for the function ϕ.

Note that all invariant solutions of the equations (1.1) can be obtained from the corresonding solutions of the equations (2.1) by transformations of the form (2.2). Numerous invariant solutions of Eqs. (2.1) with $l(u) \equiv 0$ are presented in [129]. According to the formula (2.3), $l(u) = 0$ corresponds to Eqs. (1.1) with a constant coefficient $K(\psi)$. Here we will consider examples of invariant solutions of Eqs. (1.1) with $K(\psi) \neq \text{const}$.

Example 1 (*Invariant solution under an Abelian subalgebra*). Consider Eq. (1.1) of the form

$$\frac{4}{Me^{4\psi}-1}\psi_t = \left(e^{-4\psi}\psi_x\right)_x + \left(e^{-4\psi}\psi_z\right)_z + 4e^{-4\psi}\psi_z + M - e^{-4\psi}. \tag{5.2}$$

This equation is obtained by a simple scaling of independent and dependent variables from the equation given in the above group classification in the case III with $\sigma = -1, B = 1/4$. According to the classification result, Eq. (5.2) admits an six-dimensional Lie algebra L_6 obtained by an extension of the principal Lie algebra $L_{\mathcal{P}}$ by the following three operators:

$$X_4 = t\frac{\partial}{\partial t} - \frac{1}{4}(M\mathrm{e}^{4\psi} - 1)\frac{\partial}{\partial \psi}\,,$$

$$X_5 = \sin x\ \mathrm{e}^{-z}\,\frac{\partial}{\partial x} - \cos x\ \mathrm{e}^{-z}\,\frac{\partial}{\partial z} + \frac{1}{2}\cos x\ \mathrm{e}^{-z}(M\mathrm{e}^{4\psi} - 1)\frac{\partial}{\partial \psi}\,,$$

$$X_6 = \cos x\ \mathrm{e}^{-z}\,\frac{\partial}{\partial x} + \sin x\ \mathrm{e}^{-z}\,\frac{\partial}{\partial z} - \frac{1}{2}\sin x\ \mathrm{e}^{-z}(M\mathrm{e}^{4\psi} - 1)\frac{\partial}{\partial \psi}\,.$$

Let us construct invariant solutions under the operators X_4, X_5. These operators span a two-dimensional subalgebra L_2 of the algebra L_6 and have two functionally independent invariants. First, we calculate a basis of invariants $I(t, x, z, \psi)$ by solving the following system of linear first-order partial differential equations:

$$X_4 I = 0,\ \ X_5 I = 0.$$

The subalgebra L_2 is Abelian ($[X_4, X_5] = 0$). Therefore, we can solve the equations of the above system successively in any order.

The first equation provides three functionally independent solutions

$$J_1 = x,\ \ J_2 = z,\ \ J_3 = t(\mathrm{e}^{-4\psi} - M).$$

Hence the common solution $I(t, x, z, \psi)$ of our system is defined as a function of J_1, J_2, J_3 only. Therefore we rewrite the action of X_5 on the space of

J_1, J_2, J_3 by the formula

$$X_5 = X_5(J_1)\frac{\partial}{\partial J_1} + X_5(J_2)\frac{\partial}{\partial J_2} + X_5(J_3)\frac{\partial}{\partial J_3}$$

to obtain

$$X_5 = \sin J_1 \ \mathrm{e}^{-J_2} \frac{\partial}{\partial J_1} - \cos J_1 \ \mathrm{e}^{-J_2} \frac{\partial}{\partial J_2} + 2\cos J_1 \ \mathrm{e}^{-J_2} J_3 \frac{\partial}{\partial J_3} .$$

Consequently, we easily obtain the following two functionally independent solutions (invariants) of the second equation of the system in discussion:

$$I_1 = \mathrm{e}^{J_2} \ \sin J_1 \equiv \mathrm{e}^{z} \ \sin x, \ \ I_2 = J_3 \ \mathrm{e}^{2J_2} \equiv (t\mathrm{e}^{-3\psi} - Mt) \ \mathrm{e}^{2z}.$$

Thus, according to (5.1), the invariant solution is given by

$$(t\mathrm{e}^{-3\psi} - Mt) \ \mathrm{e}^{2z} = \phi(\mathrm{e}^{z} \ \sin x).$$

Solving this equation with respect to ψ,

$$\psi = -\frac{1}{4} \ln \left| M + \frac{\mathrm{e}^{-2z}}{t}\phi(\xi)\right|, \ \text{where } \xi = \mathrm{e}^{z} \ \sin x, \tag{5.3}$$

and substituting into Eq. (5.2), we arrive at the following linear second-order ordinary differential equation:

$$\phi''(\xi) = 4.$$

Hence,

$$\phi(\xi) = 2\xi^2 + C_1\xi + C_2,$$

and Eq. (5.3) yields

$$\psi = -\frac{1}{4} \ln \left| M + \frac{\mathrm{e}^{-2z}}{t}\Big(2\mathrm{e}^{2z} \ \sin^2 x + \frac{C_1}{\mathrm{e}}^{z} \ \sin x + C_2\Big)\right|,$$

where C_1 and C_2 are arbitrary constants.

The same solution can be obtained from the solution of the corresponding equation (2.1) (see [67], Section 9.8).

The dynamics of the process described by this solution is illustrated graphically in the figures.

Example 2 (*Invariant solution under a non-Abelian subalgebra*). Consider Eq. (1.1) of the form

$$\sinh^{-2}\Big(\sqrt{\frac{M}{2}}\psi\Big)\psi_t = \left(\sinh^{-2}\Big(\sqrt{\frac{M}{2}}\psi\Big)\psi_x\right)_x + \left(\sinh^{-2}\Big(\sqrt{\frac{M}{2}}\psi\Big)\psi_z\right)_z$$

$$+\sqrt{2M}\coth\Big(\sqrt{\frac{M}{2}}\psi\Big)\sinh^{-2}\Big(\sqrt{\frac{M}{2}}\psi\Big)\psi_z - \frac{1}{M}\,. \tag{5.4}$$

This equation is obtained by a simple scaling of independent and dependent variables from the equation given in the above group classification in the case II.1(iii) with $B = 1$. According to the classification result, Eq. (5.4) admits a five-dimensional Lie algebra L_5 with the following two additional operators:

$$X_4 = 2t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+\big(z-\frac{3}{2}t^2\big)\frac{\partial}{\partial z}+\left(\frac{1}{\sqrt{2M}}\sinh\left(\sqrt{2M}\psi\right) - \frac{3t}{M}\sinh^2\Big(\sqrt{\frac{M}{2}}\psi\Big)\right)\frac{\partial}{\partial \psi}\,,$$

$$X_5 = t\frac{\partial}{\partial z} + \frac{1}{M}\sinh^2\Big(\sqrt{\frac{M}{2}}\psi\Big)\frac{\partial}{\partial \psi}\,.$$

The operators X_4, X_5 span a two-dimensional subalgebra L_2 of the algebra L_5. We have $[X_4, X_5] = X_5$. Hence, the subalgebra L_2 is non-Abelian and X_5 spans an ideal of L_2. Therefore, in this case, the system

$$X_4 I = 0, \;\; X_5 I = 0$$

for invariants can be solved successively begining with the equation $X_5 I = 0$. Then the first equation $X_4 I = 0$ will be represented in the space of three independent solutions of the equation $X_5 I = 0$. Repeating the calculations described in the previous example, we get the following invariant solution of the equation (5.4):

$$\psi = \frac{1}{\sqrt{2M}}\Big(\ln\Big|\sqrt{2M}+\frac{t}{2}-\frac{z}{t}+C_1\,\frac{x}{t}-C_2\Big[\frac{2}{\sqrt{t}}\exp\Big(-\frac{x^2}{4t}\Big)+\frac{\sqrt{\pi}x}{t}\,\mathrm{erf}\Big(\frac{x}{2\sqrt{t}}\Big)\Big]\Big|$$

$$-\ln\Big| - \sqrt{2M} + \frac{t}{2} - \frac{z}{t} + C_1\,\frac{x}{t} - C_2\,\Big[\frac{2}{\sqrt{t}}\exp\Big(- \frac{x^2}{4t}\Big) + \frac{\sqrt{\pi}x}{t}\,\mathrm{erf}\Big(\frac{x}{2\sqrt{t}}\Big)\Big]\Big|\Big).$$

Here C_1 and C_2 are arbitrary constants and

$$\mathrm{erf}(\xi) = \frac{2}{\sqrt{\pi}}\int_0^{\xi} e^{-\mu^2}d\mu.$$

The same solution can be obtained from the solution of the corresponding equation (2.1) (see [67], Section 9.8).

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Paper 9

A priori use of symmetry groups in nonlinear modelling

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Original unabridged version of the paper [73]

Abstract. Nonlinear mathematical models constructed on the basis of general physical concepts often have a rather complex form. Therefore, it is natural to simplify these models for further investigation. Unfortunately, however, a simplification often leads to the loss of certain symmetries of the model, and hence to the loss of physically important solutions.

The main idea of an *a priori* use of symmetries in mathematical modelling, suggested in this paper, is to augment the symmetry properties of the problem by means of a "reasonable" complication of the mathematical model. The method is illustrated by applying it to nonlinear acoustics and finding exact solutions that are of interest for the wave theory.

1 Motivation

The long term experience gained from solving problems of mathematical physics by means of group analysis furnishes ample evidence that numerous natural phenomena can be modelled directly in terms of symmetry groups. Then differential equations, conservation laws and often even solutions to initial value problems can be obtained as immediate consequences of group theoretic modelling.

On the other hand, mathematical models of physical processes that are constructed on the basis of general concepts from first principles) often have a rather complex form. Examples of such models are the systems of equations of mechanics of continua or electrodynamics, which are difficult to solve in the general form by common analytical and numerical methods, especially if the nonlinearity, inhomogeneity, hereditary and other properties of real media are taken into account. Therefore, it is natural to simplify the model equations as much as possible in accordance with the specific character of the problem under consideration. It is evident, that dealing with simpler models one can most probably find an appropriate method of solution. The simplification is achieved by neglecting the less important features of a given phenomenon and concentrating on its most important properties. A comprehensive description of ideas underlying this approach are given by A.A. Andronov, A.A. Vitt, and S.É. Khaĭkin in their classical monograph on the theory of oscillations [5]. A historically significant example of applying these ideas to the wave theory is the development of the method of a slowly varying profile by R.V. Khokhlov [81], [2]. This method considerably extended the possibilities for solving nonlinear wave problems analytically [121]. Unfortunately, a simplification of a mathematical model may lead, and often it does, to the loss of a number of symmetries possessed by the original general model, and hence to the loss of important exact solutions.

We suggest another approach, namely a complication of the model with the aim of augmenting symmetry properties of the mathematical model for finding desired solutions of a given simpler problem. This idea seems to be logically absurd. However, sometimes a complex model proves to be more simple to analyze. An example of such a useful complication in acoustics is the Burgers equation (see, e.g., [121] and the notation accepted there):

$$\frac{\partial V}{\partial z} - V\frac{\partial V}{\partial \theta} = \Gamma \frac{\partial^2 V}{\partial \theta^2}\,\cdot$$

The presence of the additional viscous term proportional to the higher (second-order) derivative on the right-hand side of this equation surprisingly does not complicate the initial first-order equation but, on the contrary, provides the possibility to solve the problem. With the transformation

$$V = 2\Gamma \frac{\partial}{\partial \theta} \ln U,$$

the Burgers equation can be linearized and reduced to the heat conduction equation for U :

$$\frac{\partial U}{\partial z} = \Gamma \frac{\partial^2 U}{\partial \theta^2}\,\cdot$$

Expressing the solution of the Burgers equation via the solution of the heat equations and then performing the limit $\Gamma \to 0$, one obtains a physically correct solution to the problem of interest for the non-viscous medium.

Evidently, this example is accidental and does not provide any general approach. However, the experience gained from the group classification of differential equations shows that in many cases a complication of the initial model supplies it with new symmetries, and hence allows one to find new physically important solutions.

A well-known regular and effective method of symmetry determination is the theory of continuous Lie groups [96], [93]. The conventional method of using Lie groups for finding analytical solutions to differential equations is as follows. Specific equations or systems of equations are considered. Symmetry groups are calculated for them. Then, these groups are used to construct exact particular solutions, conservation laws, and invariants. A more complicated and interesting problem is the group classification of equations involving unknown parameters or functions. The group classification allows one to find such parameters (or functions) at which the admitted group is wider than the symmetry group of the initial general equation. Many equations with physically interesting solutions have already been obtained in this way (see, e.g., [111], [114], [60], [65], [67], [68]). The approach described above can be called an *a posteriori* one, because it is based on analyzing given systems of equations.

In this paper, we propose a fundamentally new and simple approach providing new symmetric models. The approach is based on a reasonable complication of a given model without any loss of its physical content. A theoretical basis for our *principle of an a priori use of symmetries* is N.H. Ibragimovs theorem on projections of equivalence groups (see Section 4.2).

We begin with describing the standard methods of Lie group analysis of differential equations, adapting them to evolutionary equations appropriate to the models of nonlinear wave theory and nonlinear acoustics.

2 Outline of methods from group analysis

2.1 Symmetries of evolution equations

Consider evolutionary partial differential equations of the second order:

$$u_t = F(t, x, u, u_x, u_{xx}), \quad \partial F/\partial u_{xx} \neq 0. \tag{2.1}$$

Definition 9.1. A set G of invertible transformations of the variables t, x, u :

$$\bar{t} = f(t, x, u, a), \quad \bar{x} = g(t, x, u, a), \quad \bar{u} = h(t, x, u, a), \tag{2.2}$$

depending on a continuous parameter a is called a one-parameter group admitted by the equation (2.1), or a *symmetry group* of the equation (2.1), if the equation (2.1) has the same form in the new variables $\bar{t}, \bar{x}, \bar{u}$ and if G contains the inverse to any transformation from G, the identity transformation

$$\bar{t} = t, \quad \bar{x} = x, \quad \bar{u} = u,$$

and the composition of any two transformations, namely:

$$\bar{\bar{t}} \equiv f(\bar{t}, \bar{x}, \bar{u}, b) = f(t, x, u, a + b),$$

$$\bar{\bar{x}} \equiv g(\bar{t}, \bar{x}, \bar{u}, b) = g(t, x, u, a + b),$$

$$\bar{\bar{u}} \equiv h(\bar{t}, \bar{x}, \bar{u}, b) = h(t, x, u, a + b).$$

Thus, a group G is admitted by the equation (2.1) if the transformations (2.2) of the group G map every solution $u = u(t, x)$ of the equation (2.1) into a solution $\bar{u} = \bar{u}(\bar{t}, \bar{x})$ of the equation

$$\bar{u}_{\bar{t}} = F(\bar{t}, \bar{x}, \bar{u}, \bar{u}_{\bar{x}}, \bar{u}_{\bar{x}\bar{x}}), \tag{2.3}$$

where the function F has the same form in both equations (2.1) and (2.3).

According to the Lie theory, the construction of the symmetry group G is equivalent to determination of its *infinitesimal transformations*

$$\bar{t} \approx t + a\tau(t, x, u), \quad \bar{x} \approx x + a\xi(t, x, u), \quad \bar{u} \approx u + a\eta(t, x, u) \tag{2.4}$$

obtained from (2.2) by expanding into Taylor series with respect to the group parameter a and keeping only the terms linear in a. It is convenient to introduce the *symbol* (after S. Lie) of the infinitesimal transformation (2.4), i.e. the differential operator

$$X = \tau(t, x, u)\frac{\partial}{\partial t} + \xi(t, x, u)\frac{\partial}{\partial x} + \eta(t, x, u)\frac{\partial}{\partial u} \tag{2.5}$$

acting on any differentiable function $J(t, x, u)$ as follows:

$$X(J) = \tau(t, x, u)\frac{\partial J}{\partial t} + \xi(t, x, u)\frac{\partial J}{\partial x} + \eta(t, x, u)\frac{\partial J}{\partial u}\cdot$$

The operator (2.5) also is known in the literature as the *infinitesimal operator* or *generator* of the group G. The symbol X of the group admitted by the equation (2.1) is called an operator admitted by (2.1) or an *infinitesimal symmetry* for equation (2.1).

The group transformations (2.2) corresponding to the infinitesimal transformations with the symbol (2.5) are found by solving the *Lie equations*

$$\frac{d\bar{t}}{da} = \tau(\bar{t}, \bar{x}, \bar{u}), \quad \frac{d\bar{x}}{da} = \xi(\bar{t}, \bar{x}, \bar{u}), \quad \frac{d\bar{u}}{da} = \eta(\bar{t}, \bar{x}, \bar{u}), \tag{2.6}$$

with the initial conditions:

$$\bar{t}\big|_{a=0} = t, \quad \bar{x}\big|_{a=0} = x, \quad \bar{u}\big|_{a=0} = u.$$

Let us turn now to equation (2.3). The quantities $\bar{u}_{\bar{t}}$, $\bar{u}_{\bar{x}}$ and $\bar{u}_{\bar{x}\bar{x}}$ involved in (2.3) can be obtained by means of the usual rule of change of derivatives treating the equations (2.2) as a change of variables. Then, expanding the resulting expressions for $\bar{u}_{\bar{t}}$, $\bar{u}_{\bar{x}}$, $\bar{u}_{\bar{x}\bar{x}}$ into Taylor series series with respect to the parameter a, one can obtain the infinitesimal form of these expressions:

$$\begin{gathered}\bar{u}_{\bar{t}} \approx u_t + a\ \zeta_0(t,x,u,u_t,u_x)\,,\ \bar{u}_{\bar{x}} \approx u_x + a\ \zeta_1(t,x,u,u_t,u_x)\,,\\ \bar{u}_{\bar{x}\bar{x}} \approx u_{xx} + a\ \zeta_2(t,x,u,u_t,u_x,u_{tx},u_{xx})\,,\end{gathered} \tag{2.7}$$

where the functions $\zeta_0, \zeta_1, \zeta_2$ are given by the following *prolongation formulae*:

$$\begin{aligned}\zeta_0 &= D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi),\\ \zeta_1 &= D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi),\\ \zeta_2 &= D_x(\zeta_1) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi).\end{aligned} \tag{2.8}$$

Here D_t and D_x denote the total differentiations with respect to t and x:

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{tx}\frac{\partial}{\partial u_x}\,,$$

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{tx}\frac{\partial}{\partial u_t} + u_{xx}\frac{\partial}{\partial u_x}\,\cdot$$

Substitution of (2.4) and (2.7) in equation (2.3) yields:

$$\bar{u}_{\bar{t}} - F(\bar{t},\bar{x},\bar{u},\bar{u}_{\bar{x}},\bar{u}_{\bar{x}\bar{x}}) \approx u_t - F(t,x,u,u_x,u_{xx})$$
$$+a\Big(\zeta_0 - \frac{\partial F}{\partial u_{xx}}\zeta_2 - \frac{\partial F}{\partial u_x}\zeta_1 - \frac{\partial F}{\partial u}\eta - \frac{\partial F}{\partial x}\xi - \frac{\partial F}{\partial t}\tau\Big).$$

Therefore, by virtue of the equation (2.1), the equation (2.3) yields

$$\zeta_0 - \frac{\partial F}{\partial u_{xx}}\zeta_2 - \frac{\partial F}{\partial u_x}\zeta_1 - \frac{\partial F}{\partial u}\eta - \frac{\partial F}{\partial x}\xi - \frac{\partial F}{\partial t}\tau = 0, \tag{2.9}$$

where u_t is replaced by $F(t,x,u,u_x,u_{xx})$ in $\zeta_0, \zeta_1, \zeta_2$.

The equation (2.9) defines all infinitesimal symmetries of the equation (2.1) and therefore it is called the *determining equation.* Conventionally, it is written in the compact form

$$X\Big[u_t - F(t,x,u,u_x,u_{xx})\Big]_{u_t=F} = 0, \tag{2.10}$$

where X denotes the *prolongation* of the operator (2.5) to the first and second order derivatives:

$$X=\tau\frac{\partial}{\partial t}+\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial u}+\zeta_0\frac{\partial}{\partial u_t}+\zeta_1\frac{\partial}{\partial u_x}+\zeta_2\frac{\partial}{\partial u_{xx}}\cdot$$

The determining equation (2.9) (or its equivalent (2.10)) is a linear homogeneous partial differential equation of the second order for unknown functions $\tau(t,x,u)$, $\xi(t,x,u)$, $\eta(t,x,u)$. In consequence, the set of all solutions to the determining equation is a vector space L. Furthermore, the determining equation possesses the following significant and less evident property. The vector space L is a *Lie algebra,* i.e. it is closed with respect to the *commutator.* In other words, L contains, together with any operators X_1, X_2, their commutator $[X_1, X_2]$ defined by

$$[X_1,X_2]=X_1X_2-X_2X_1.$$

In particular, if $L=L_r$ is finite-dimensional and has a basis $X_1,\ldots,X_r$, then

$$[X_\alpha,X_\beta]=c^\gamma_{\alpha\beta}X_\gamma$$

with constant coefficients $c^\gamma_{\alpha\beta}$ known as the *structure constants* of L_r.

Note that the equation (2.9) should be satisfied identically with respect to all the variables involved, the variables t,x,u,u_x,u_{xx},u_{tx} are treated as five independent variables. Consequently, the determining equation decomposes into a system of several equations. As a rule, this is an overdetermined system since it contains more equations than three unknown functions τ,ξ and η. Therefore, in practical applications, the determining equation can be solved. The following preparatory lemma due to Lie (see [89], First part, Section III(10)) simplifies the calculations.

Lemma 9.1. The symmetry transformations (2.2) of Eqs. (2.1) have the form

$$\bar t=f(t,a),\quad \bar x=g(t,x,u,a),\quad \bar u=h(t,x,u,a). \tag{2.11}$$

It means that one can search the infinitesimal symmetries in the form

$$X=\tau(t)\frac{\partial}{\partial t}+\xi(t,x,u)\frac{\partial}{\partial x}+\eta(t,x,u)\frac{\partial}{\partial u}\cdot \tag{2.12}$$

Proof. Let us single out in the determining equation (2.9) the terms containing the variable u_{tx}. It is manifest from the prolongation formulae (2.8) that u_{tx} is contained only in ζ_2, namely, in its last term $u_{tx}D_x(\tau)$. Since the determining equation (2.9) holds identically in all variables t,x,u,u_x,u_{xx},u_{tx}, one concludes that $D_x(\tau) \equiv \tau_x + u_x\tau_u = 0$, whence $\tau_x = \tau_u = 0$. Thus, $\tau = \tau(t)$, and hence the generator (2.6) reduces to the form (2.12).

For the operators (2.12), the prolongation formulae (2.8) are written as follows:

$$\zeta_0 = D_t(\eta) - u_x D_t(\xi) - \tau'(t)u_t, \ \zeta_1 = D_x(\eta) - u_x D_x(\xi),$$

$$\zeta_2 = D_x(\zeta_1) - u_{xx}D_x(\xi) \equiv D_x^2(\eta) - u_x D_x^2(\xi) - 2u_{xx}D_x(\xi). \tag{2.13}$$

2.2 Invariant solutions

Group analysis provides two basic ways for construction of exact solutions: construction of *group invariant solutions* (or simply *invariant solutions*) and *group transformations* of known solutions.

If a group transformation maps a solution into itself, we arrive at what is called a *self-similar* or *group invariant solution.* Given an infinitesimal symmetry (2.5) of equation (2.1), the invariant solutions under the one-parameter group generated by X are obtained as follows. One calculates two independent *invariants* $J_1 = \lambda(t, x)$ and $J_2 = \mu(t, x, u)$ by solving the equation

$$X(J) \equiv \tau(t,x,u)\frac{\partial J}{\partial t} + \xi(t,x,u)\frac{\partial J}{\partial x} + \eta(t,x,u)\frac{\partial J}{\partial u} = 0,$$

or its characteristic system:

$$\frac{dt}{\tau(t,x,u)} = \frac{dx}{\xi(t,x,u)} = \frac{du}{\eta(t,x,u)}\,\cdot \tag{2.14}$$

Then one designates one of the invariants as a function of the other, e.g.

$$\mu = \phi(\lambda), \tag{2.15}$$

and solves equation (2.15) with respect to u. Finally, one substitutes the expression for u in equation (2.1) and obtains an ordinary differential equation for the unknown function $\phi(\lambda)$ of one variable. This procedure reduces the number of independent variables by one.

Further reductions, in the case of differential equations with many independent variables, can be achieved by considering invariant solutions under two or more infinitesimal symmetries.

2.3 Group transformations of known solutions

This application of symmetry groups is based on the fact that the transformations of this group map any solutions of the equation in question into solution of the same equation. Namely, let (2.2) be a symmetry transformation group of the equation (2.1), and let a function

$$u = \Phi(t, x)$$

solve the equation (2.1). Since (2.2) is a symmetry transformation, the above solution can be also written in the new variables:

$$\bar{u} = \Phi(\bar{t}, \bar{x}).$$

Replacing here $\bar{u}, \bar{t}, \bar{x}$ from (2.2), we get

$$h(t, x, u, a) = \Phi\big(f(t, x, u, a), g(t, x, u, a)\big). \tag{2.16}$$

Having solved equation (2.16) with respect to u, one obtains a one-parameter family (with the parameter a) of new solutions to the equation (2.1). Consequently, any known solution is a source of a multi-parameter class of new solutions provided that the differential equation considered admits a multi-parameter symmetry group. An example is given in Section 3, where the procedure is applied to the Burgers equation.

3 Group analysis of the Burgers equation

We illustrate the methods described in the previous section by considering the Burgers equation

$$u_t = u_{xx} + uu_x. \tag{3.1}$$

3.1 Calculation of symmetries

The determining equation (2.9) has the form

$$\zeta_0 - \zeta_2 - u\zeta_1 - \eta u_x = 0, \tag{3.2}$$

where ζ_0, ζ_1 and ζ_2 are given by (2.13). Let us single out and annul the terms with u_{xx}. Bearing in mind that u_t has to be replaced by $u_{xx} + uu_x$ and substituting in ζ_2 the expressions

$$D_x^2(\xi) = D_x(\xi_x + \xi_u\, u_x) = \xi_u\, u_{xx} + \xi_{uu}\, u_x^2 + 2\xi_{xu}\, u_x + \xi_{xx},$$

$$D_x^2(\eta) = D_x(\eta_x + \eta_u\, u_x) = \eta_u\, u_{xx} + \eta_{uu}\, u_x^2 + 2\eta_{xu}\, u_x + \eta_{xx} \tag{3.3}$$

we arrive at the following equation:

$$2\xi_u\, u_x + 2\xi_x - \tau'(t) = 0.$$

It splits into two equations, namely $\xi_u = 0$ and $2\xi_x - \tau'(t) = 0$. The first equation shows that ξ depends only on t, x, and integration of the second equation yields

$$\xi = \frac{1}{2}\,\tau'(t)\, x + p(t). \tag{3.4}$$

It follows from (3.4) that $D_x^2(\xi) = 0$. Now the determining equation (3.2) reduces to the form

$$u_x^2\eta_{uu} + \left[\frac{1}{2}\tau'(t)u + \frac{1}{2}\tau''(t)x + p'(t) + 2\eta_{xu} + \eta\right] u_x + u\eta_x + \eta_{xx} - \eta_t = 0$$

and splits into three equations:

$$\eta_{uu} = 0, \quad \frac{1}{2}\tau'(t)u + \frac{1}{2}\tau''(t)x + p'(t) + 2\eta_{xu} + \eta = 0, \quad u\eta_x + \eta_{xx} - \eta_t = 0. \tag{3.5}$$

The first equation (3.5) yields $\eta = \sigma(t,x)u + \mu(t,x)$, and the second equation (3.5) becomes:

$$\left(\frac{1}{2}\tau'(t) + \sigma\right) u + \frac{1}{2}\tau''(t)x + p'(t) + 2\sigma_x + \mu = 0,$$

whence

$$\sigma = -\frac{1}{2}\tau'(t), \quad \mu = -\frac{1}{2}\tau''(t)x - p'(t).$$

Thus, we have

$$\eta = -\frac{1}{2}\,\tau'(t)\, u - \frac{1}{2}\tau''(t)x - p'(t). \tag{3.6}$$

Finally, substitution of (3.6) in the third equation (3.5) yields

$$\frac{1}{2}\tau'''(t)x + p''(t) = 0,$$

whence $\tau'''(t) = 0$, $p''(t) = 0$, and hence

$$\tau(t) = C_1 t^2 + 2C_2 t + C_3, \quad p(t) = C_4 t + C_5.$$

Invoking (3.4) and (3.6), we ultimately arrive at the following general solution of the determining equation (3.2):

$$\tau(t) = C_1 t^2 + 2C_2 t + C_3,\ \xi = C_1 tx + C_2 x + C_4 t + C_5,\ \eta = -(C_1 t + C_2)u - C_1 x - C_4.$$

It contains five arbitrary constants C_i. It means that the infinitesimal symmetries of the Burgers equation (3.1) form the five-dimensional Lie algebra spanned by the following linearly independent operators:

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = t\frac{\partial}{\partial x} - \frac{\partial}{\partial u},$$
$$X_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}, \quad X_5 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} - (x+tu)\frac{\partial}{\partial u}\cdot \tag{3.7}$$

3.2 Invariant solutions

Example 9.1. One of the physically significant types of solutions is obtained by assuming the invariance under the time translation group generated by X_1. This assumption provides the stationary solutions

$$u = \Phi(x)$$

for which the Burgers equation yields

$$\Phi'' + \Phi\Phi' = 0. \tag{3.8}$$

Integrate it once:

$$\Phi' + \frac{\Phi^2}{2} = C_1,$$

and integrate again be setting $C_1 = 0, C_1 = \nu^2 > 0, C_1 = -\omega^2 < 0$ to obtain:

$$\Phi(x) = \frac{2}{x+C}, \quad \Phi(x) = \nu \,\mathrm{th}\Big(C + \frac{\nu}{2}\,x\Big), \quad \Phi(x) = \omega \,\mathrm{tg}\Big(C - \frac{\omega}{2}x\Big). \tag{3.9}$$

Note that the Galilean transformation $\bar{t} = t, \bar{x} = x + at, \bar{u} = u - a$ generated by X_3 maps X_1 to $X_1 + cX_2$. Consequently, it maps the stationary solutions to travelling waves $u = u(x - ct)$ which can also be obtained from the solutions (3.9).

Example 9.2. Let us find the invariant solutions under the projective group generated by X_5. The characteristic system (2.14) is written:

$$\frac{dt}{t^2} = \frac{dx}{tx} = -\frac{du}{x+tu}$$

and provides the invariants $\lambda = x/t$ and $\mu = x + tu$. Hence, the general expression (2.15) for invariant solution takes the form

$$u = -\frac{x}{t} + \frac{1}{t}\,\Phi(\lambda), \quad \lambda = \frac{x}{t}\cdot \tag{3.10}$$

Substituting this expression in the Burgers equation (3.1), one obtains for $\Phi(\lambda)$ precisely the equation (3.8). Hence, its general solution is obtained from (3.9) where x is replaced by λ. The corresponding invariant solutions are obtained by substituting in (3.10) the resulting expressions for $\Phi(\lambda)$. For example, using for $\Phi(\lambda)$ the second formula (3.9) by letting there $\nu = \pi$, one obtains the solution

$$u = -\frac{x}{t} + \frac{\pi}{t}\,\mathrm{th}\Big(C + \frac{\pi x}{2t}\Big) \tag{3.11}$$

derived by R.V. Khokhlov in 1961 by physical reasoning (see, e.g. [138], Chapter IX, §4, equation (4.15)).

Remark. The solution (3.11) can be obtained from the second solution (3.21) by setting $\nu = -\pi a$ and letting $a \to \infty$.

Example 9.3. The invariant solutions under the group of dilations with the generator X_4 lead to what is often called in the physical literature *similarity solutions* because of their connection with the dimensional analysis. In this case the characteristic system

$$\frac{dt}{2t} = \frac{dx}{x} = -\frac{du}{u}$$

provides the following invariants: $\lambda = x/\sqrt{t}, \mu = \sqrt{t}\,u$. Consequently, one seeks the invariant solutions in the form

$$u = \frac{1}{\sqrt{t}}\,\Phi(\lambda), \quad \lambda = \frac{x}{\sqrt{t}}\,,$$

and arrives at the following equation for the similarity solutions of the Burgers equation:

$$\Phi'' + \Phi\Phi' + \frac{1}{2}(\lambda\,\Phi' + \Phi) = 0. \tag{3.12}$$

Integrating once, one has:

$$\Phi' + \frac{1}{2}(\Phi^2 + \lambda\,\Phi) = C,$$

Letting $C = 0$, one obtains the following similarity solution (see [138], Chapter IX, §4, equation (4.11), or [65], Section 11.4):

$$u = \frac{2}{\sqrt{\pi t}}\,\frac{\mathrm{e}^{-x^2/(4t)}}{B + \mathrm{erf}(x/(2\sqrt{t})}\,,$$

where B is and arbitrary constant and

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z \mathrm{e}^{-s^2} ds$$

is the error function.

Example 9.4. Construction of exact solutions can be based not only on the basic infinitesimal symmetries (3.7), but also on their linear combinations. Consider an example based, e.g. on the operator

$$X_1 + X_5 = (1+t^2)\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} - (x+tu)\frac{\partial}{\partial u}\,\cdot \tag{3.13}$$

The characteristic system

$$\frac{dt}{1+t^2} = \frac{dx}{tx} = -\frac{du}{x+tu}$$

provides the invariants

$$\lambda = \frac{x}{\sqrt{1+t^2}}, \quad \mu = \frac{tx}{\sqrt{1+t^2}} + u\sqrt{1+t^2}.$$

Hence, invariant solution have the form

$$u = -\frac{tx}{1+t^2} + \frac{1}{\sqrt{1+t^2}}\,\Phi(\lambda), \quad \lambda = \frac{x}{\sqrt{1+t^2}}\,\cdot$$

Substituting this expression in the Burgers equation (3.1), one obtains for $\Phi(\lambda)$ the following equation:

$$\Phi'' + \Phi\Phi' + \lambda = 0, \tag{3.14}$$

or integrating once:

$$\Phi' + \frac{1}{2}(\Phi^2 + \lambda^2) = C.$$

3.3 Group transformations of solutions

One can find the group transformations (2.11) admitted by the Burgers equation by solving the Lie equations for the basic infinitesimal symmetries (3.7). In the case of the generators (3.7), the Lie equations (2.6) have the triangular form:

$$\frac{d\bar{t}}{da} = \tau(\bar{t}), \quad \frac{d\bar{x}}{da} = \xi(\bar{t}, \bar{x}), \quad \frac{d\bar{u}}{da} = \eta(\bar{t}, \bar{x}, \bar{u}). \tag{3.15}$$

Eqs. (3.15) and the initial conditions

$$\bar{t}\big|_{a=0} = t, \quad \bar{x}\big|_{a=0} = x, \quad \bar{u}\big|_{a=0} = u \tag{3.16}$$

are convenient for the consecutive integration. As an example, consider the generator X_5 from (3.7). The Lie equations (3.15) are written

$$\frac{d\bar{t}}{da} = \bar{t}^{\,2}, \quad \frac{d\bar{x}}{da} = \bar{t}\,\bar{x}, \quad \frac{d\bar{u}}{da} = -(\bar{x} + \bar{t}\,\bar{u}).$$

Integration of the first equation yields

$$\bar{t} = -\frac{1}{a + C_1} \cdot$$

We evaluate the constant of integration C_1 from the first initial condition (3.15) and obtain $C_1 = -1/t$. Hence,

$$\bar{t} = \frac{t}{1 - at} \cdot \tag{3.17}$$

Substituting (3.17) in the second Lie equation, we have

$$\frac{d\bar{x}}{\bar{x}} = \frac{t da}{1 - at} \equiv -\frac{d(1 - at)}{1 - at} \,,$$

whence

$$\bar{x} = \frac{C_2}{1 - at} \cdot$$

Using the second initial condition (3.15) we obtain $C_2 = x$, and hence

$$\bar{x} = \frac{x}{1 - at} \cdot \tag{3.18}$$

The equations (3.17) and (3.18) determine the *special projective transformation group* (the Möbius map) on the plane. Substituting (3.17) and (3.18) in the third Lie equation, one obtains a first-order non-homogeneous linear equation for $\bar{u}$:

$$\frac{d\bar{u}}{da} + \frac{t}{1 - at}\,\bar{u} + \frac{x}{1 - at} = 0.$$

Integrating this equation and evaluating the constant of integration from the third initial condition (3.15), one obtains:

$$\bar{u} = u(1 - at) - ax. \tag{3.19}$$

Using the projective transformations (3.17)-(3.19) and applying the equation (2.16) to any known solution $u = \Phi(t, x)$ of the Burgers equation, one can obtain the following one-parameter set of new solutions:

$$u = \frac{ax}{1 - at} + \frac{1}{1 - at}\,\Phi\Big(\frac{t}{1 - at}\,,\frac{x}{1 - at}\Big). \tag{3.20}$$

Example 9.5. One can obtain many examples, by choosing as an initial solution $u = \Phi(t, x)$, any invariant solution. Let us take, e.g. the invariant solution under the space translation generated by X_2 from (3.7). In this case the invariants are $\lambda = t$ and $\mu = u$, and hence Eq. (2.15) is written

$u = \phi(t)$. Substitution in the Burgers equation yields the obvious constant solution $u = k$. It is mapped by (3.20) into the following one-parameter set of solutions:

$$u = \frac{k + ax}{1 - at} \cdot$$

Example 9.6. If one applies the transformation (3.20) to the stationary solutions (3.9), one obtains the following new non-stationary solutions:

$$\begin{aligned} u &= \frac{ax}{1-at} + \frac{2}{x + C(1-at)}\,, \\ u &= \frac{1}{1-at}\left[ax + \nu\,\mathrm{th}\left(C + \frac{\nu x}{2(1-at)}\right)\right], \\ u &= \frac{1}{1-at}\left[ax + \omega\,\mathrm{tg}\left(C - \frac{\omega x}{2(1-at)}\right)\right]. \end{aligned} \tag{3.21}$$

4 Method of the a priori use of symmetries

In Section 3, we described examples of using symmetry groups. However, in many problems, a symmetry group is absent or is insufficiently wide to solve the given model equation. The method proposed below is aimed at constructing models with a higher symmetry without loss of the physical contents of the initial model. The essence of the method is as follows.

If the model contains arbitrary elements and has a sufficiently wide equivalence group (see further Definition 9.2 in Section 4.2), the symmetry group is determined as a suitable subgroup of the equivalence group. Otherwise the nonlinear model under consideration is generalized by means of its immersion into a wider model in a reasonable way, i.e. without losing the initial physical meaning but, at the same time, achieving the desired expansion of the equivalence group. In this case, the nonlinearity of the model is essential, because it provides the necessary flexibility in choosing the model and ensures the generality and efficiency of the method.

4.1 Immersion and application of Laplace's invariants

Consider Earnshow's equation (see, e.g., [40])

$$v_{tt} - c^2\,(1 + v_\xi)^{-(\gamma+1)}\,v_{\xi\xi} = 0 \tag{4.1}$$

describing the one-dimensional motion of a compressible gas in terms of Lagrange's variables. The physical meaning of the variables is as follows:

v is a shift of particles in the medium, γ is the adiabatic exponent in the equation of state, c is the acoustic velocity.

Equation (4.1) can be linearized by the "hodograph transform"

$$x = v_\xi, \quad y = v_t, \quad \xi = X(x,y), \quad t = u(x,y), \tag{4.2}$$

where Lagrange's coordinate ξ and time t are considered as functions of new independent variables x, y which are the first derivatives of the dependent variable v. The corresponding linearized equation has the form

$$u_{xx} - c^2\,(1+x)^{-(\gamma+1)}\,u_{yy} = 0. \tag{4.3}$$

If the acoustic Mach number $|v_\xi| = |x|$ is small, Eq. (4.3) can be approximated by a simpler equation

$$u_{xx} - c^2\,[1-(\gamma+1)x]\,u_{yy} = 0. \tag{4.4}$$

However, neither Eq. (4.3) nor its simplified version (4.4) are solvable because they do not possess sufficiently wide symmetry groups. Thus, the simplification (4.4) does not achieve the objective.

Therefore, we choose an approximating equation not for the simplicity of its form but for the presence of symmetry sufficient to make it solvable. Namely, let us consider a hyperbolic equation of the form

$$u_{xx} - c^2\,\psi^2(x)\,u_{yy} = 0, \tag{4.5}$$

generalizing the equations (4.3), (4.4), and undertake a search for such a function $\psi(x)$ that, first, coincides with the corresponding function in Eq. (4.3) (or, equivalently, in Eq. (4.4)) at small $|x|$ and, second, opens up a possibility to find a general solution to (4.5). We will satisfy the second property by choosing the function $\psi(x)$ so that Eq. (4.5) has the widest possible symmetry group.

It is well known from the theory of group analysis that a hyperbolic equation written in characteristic variables in the form

$$u_{\alpha\beta} + A(\alpha,\beta)u_\alpha + B(\alpha,\beta)u_\beta + P(\alpha,\beta)u = 0 \tag{4.6}$$

admits a widest possible symmetry group and therefore can be solved by reducing it to a wave equation if the Laplace invariants [88]

$$h = A_\alpha + AB - P, \quad k = B_\beta + AB - P \tag{4.7}$$

for Eq. (4.6) vanish. Therefore, let us calculate the Laplace invariants for Eq. (4.5) by rewriting it in the characteristic variables

$$\alpha = c\int \psi(x)dx - y, \quad \beta = c\int \psi(x)dx + y. \tag{4.8}$$

In these variables Eq. (4.5) takes the standard form (4.6):

$$u_{\alpha\beta} + \frac{\psi\,'(x)}{4c\psi^2(x)}\,(u_\alpha + u_\beta) = 0, \tag{4.9}$$

with the coefficients

$$A = B = \frac{\psi\,'(x)}{4c\psi^2(x)}\,, \quad P = 0, \tag{4.10}$$

where x is expressed through α and β by solving the equations (4.8):

$$x = \Psi^{-1}(z), \quad z = \frac{\alpha + \beta}{2c} = \Psi(x) \equiv \int \psi(x)dx. \tag{4.11}$$

Let us write $A_\alpha = A_x \cdot x_\alpha$ and invoke that

$$x_\alpha = x_z \cdot z_\alpha = \frac{z_\alpha}{z_x} = \frac{1}{2c\psi(x)}\,.$$

Then equations (4.7) and (4.8), yield

$$A_\alpha = \frac{\psi''}{8c^2\psi^3} - \frac{\psi'^2}{4c^2\psi^4}\,.$$

It is transparent that $B_\beta = A_\alpha$. Therefore, we get from (4.7) the following Laplace invariants for Eq. (4.9):

$$h = k = \frac{1}{8c^2\psi^4}\left(\psi\psi'' - \frac{3}{2}\psi'^2\right).$$

Hence, the conditions $h = 0,\ k = 0$ are reduced to the single equation

$$\psi\psi'' - \frac{3}{2}\psi'^2 = 0,$$

which is easily solved and yields:

$$\psi(x) = (l + sx)^{-2}, \quad l, s = \text{const.}$$

Thus, Eq. (4.5) with a widest possible symmetry group has the form

$$u_{xx} - c^2\,(l + sx)^{-4}\,u_{yy} = 0.$$

Comparing it with Eq. (4.4), we see that the constants l, s should be set as $l = 1,\ s = (\gamma + 1)/4$. As a result, we obtain the desired solvable equation

$$u_{xx} - c^2\left(1 + \frac{\varepsilon}{2}\,x\right)^{-4} u_{yy} = 0, \quad \varepsilon = \frac{\gamma + 1}{2}, \tag{4.12}$$

which approximates Eq. (4.3) with satisfactory accuracy when $|\varepsilon x| \ll 1$.

In order to solve Eq.(4.12), we rewrite it in the standard form (4.9) and obtain:

$$u_{\alpha\beta} + \frac{u_\alpha + u_\beta}{\alpha + \beta} = 0. \tag{4.13}$$

According to the general theory, Eq. (4.13) is reduced to the wave equation $w_{\alpha\beta} = 0$ by the substitution $w = (\alpha + \beta)u$. Therefore, the general solution of (4.12) is given by the formula

$$u(\alpha, \beta) = \frac{\Phi_1(\alpha) + \Phi_2(\beta)}{\alpha + \beta} \tag{4.14}$$

with two arbitrary functions Φ_1 and Φ_2. Let us return to the variables x, y in Eq. (4.14). Since $\psi(x) = [1 + (\varepsilon/2)x]^{-1}$, Eqs. (4.8) yield

$$\alpha = -y - \frac{c}{\varepsilon(2 + \varepsilon\, x)}\,, \quad \beta = y - \frac{c}{\varepsilon(2 + \varepsilon\, x)}\,, \quad \alpha + \beta = -\frac{2c}{\varepsilon(2 + \varepsilon\, x)}\,\cdot$$

Substituting these expressions in Eq. (4.14) and changing the inessential sign in the arbitrary functions Φ_1 and Φ_2, we obtain the following general solution to Eq. (4.12):

$$u(x, y) = \frac{\varepsilon(2 + \varepsilon x)}{2c}\left[\Phi_1\left(\frac{c}{\varepsilon(2 + \varepsilon\, x)} + y\right) + \Phi_2\left(\frac{c}{\varepsilon(2 + \varepsilon\, x)} - y\right)\right]. \tag{4.15}$$

Summarizing Eqs. (4.1), (4.3), (4.4), (4.5) and (4.15), we conclude that the solution (4.15) was found by way of "immersion" of the nonlinear model of our interest (4.1) in a more general model

$$v_{tt} - c^2\, \psi^2(v_\xi)\, v_{\xi\xi} = 0. \tag{4.16}$$

However, it should be emphasized that the possibility of constructing the solution (4.15) essentially depends on the incidental fact that Eq. (4.1) and its generalization (4.12) are linearized by the transformation (4.2). Therefore, although this example clearly illustrates the idea of immersion and shows how a generalization of a model can make it solvable, it still does not provide any practical and simple method for selecting the most symmetric equations from the generalized model. Such a method is discussed in the following section using a specific example.

4.2 Utilization of the theorem on projections

The theorem on projections was proved by N.H. Ibragimov in 1987 [62] and then used in the group classification problems [1] as the basis for the preliminary group classification (see also [76], [75], [74]).

We now proceed to the main examples illustrating the essence of our approach. These examples are of interest by themselves, because new non-linear equations are considered. Their physical content is discussed in the concluding Section 5.

We begin with the nonlinear equation

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} - u\frac{\partial u}{\partial t}\right) = -\beta\, u, \quad \beta = \text{const} \neq 0. \tag{4.17}$$

It admits only the three-dimensional Lie algebra L_3 with the basis

$$X_1 = \frac{\partial}{\partial t}\,, \quad X_2 = \frac{\partial}{\partial x}\,, \quad X_3 = t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}\,\cdot \tag{4.18}$$

Hence, Eq. (4.17) is not rich in group invariant solutions. They are limited to the travelling-wave solutions that are constructed using the translation generators X_1, X_2, and the self-similar solutions constructed using the dilation generator X_3.

Therefore, we use the immersion approach and consider two types of models generalizing Eq. (4.17). We will take the equation

$$\frac{\partial}{\partial t}\left[\frac{\partial u}{\partial x} - P(u)\frac{\partial u}{\partial t}\right] = F(x,u), \tag{4.19}$$

as the first generalization, and the equation

$$\frac{\partial}{\partial t}\left[\frac{\partial u}{\partial x} - Q(x,u)\frac{\partial u}{\partial t}\right] = F(x,u) \tag{4.20}$$

as the second generalized model.

Further calculations show that the second generalization is more appropriate and is the source of a symmetry group much richer than that of Eqs. (4.17) and (4.19). Therefore, we use the model (4.20) to illustrate the principle of the *a priori* use of symmetries.

Definition 9.2. A group of transformations

$$\bar{t} = f(t,x,u,a), \quad \bar{x} = g(t,x,u,a), \quad \bar{u} = h(t,x,u,a)$$

is called an equivalence transformation group for the family of equations (4.20), or briefly an *equivalence group* for Eqs. (4.20), if every equation of the given family (4.20) with any functions $Q(x,u)$ and $F(x,u)$ is transformed to an equation of the same family, i.e.

$$\frac{\partial}{\partial \bar{t}}\left[\frac{\partial \bar{u}}{\partial \bar{x}} - \overline{Q}(\bar{x},\bar{u})\frac{\partial \bar{u}}{\partial \bar{t}}\right] = \overline{F}(\bar{x},\bar{u}),$$

where the functions $\overline{Q}$, $\overline{F}$ may be different from Q, F. The generators of the equivalence group have the form

$$Y = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \mu^1 \frac{\partial}{\partial Q} + \mu^2 \frac{\partial}{\partial F}\,, \tag{4.21}$$

where

$$\xi^i = \xi^i(t,x,u), \quad \eta = \eta(t,x,u), \quad \mu^i = \mu^i(t,x,u,Q,F), \quad i = 1,2.$$

The generators (4.21) of the equivalence group form a Lie algebra which is called the *equivalence algebra* for Eqs.(4.20) and is denoted by $L_{\mathcal{E}}$. In the operator (4.21) and in its coordinates μ^i, the functions Q and F are considered as new variables together with the physical variables t, x, u.

As was shown by L.V. Ovsyannikov [114], the equivalence algebra can be found using Lie's infinitesimal technique (see Section 2.1). Namely, let us rewrite Eq. (4.20) in the equivalent form as the system

$$u_{tx} - Qu_{tt} - Q_u u_t^2 - F = 0, \quad Q_t = 0, \quad F_t = 0 \tag{4.22}$$

called the *extended system*. Now one can define the equivalence group for Eqs. (4.20) as the group admitted by the extended system (4.22). It is worth noting that despite the obvious similarity to the classical Lie theory, there are considerable technical differences between the calculation of infinitesimal symmetries and generators (4.21) for the equivalence group. Detailed calculations can be found in [1], [76].

Using this approach, one can show that the generators (4.21) of the equivalence group for Eq. (4.20) have the coordinates

$$\begin{aligned} &\xi^1 = C_1 t + \varphi(x), \quad \xi^2 = \psi(x), \quad \eta = (C_1 + C_2)u + \lambda(x),\\ &\mu^1 = -\varphi'(x) + [C_1 - \psi'(x)]Q, \quad \mu^2 = [C_2 - \psi'(x)]F, \end{aligned} \tag{4.23}$$

where $\varphi(x)$, $\psi(x)$ and $\lambda(x)$ are arbitrary functions. This means that Eq. (4.20) has an infinite-dimension equivalence algebra $L_{\mathcal{E}}$ with the basis

$$\begin{aligned} &Y_1 = \varphi(x)\frac{\partial}{\partial t} - \varphi'(x)\frac{\partial}{\partial Q}\,, \quad Y_2 = \psi(x)\frac{\partial}{\partial x} - \psi'(x)\Big[Q\frac{\partial}{\partial Q} + F\frac{\partial}{\partial F}\Big],\\ &Y_3 = \lambda(x)\frac{\partial}{\partial u}\,, \quad Y_4 = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + Q\frac{\partial}{\partial Q}\,, \quad Y_5 = u\frac{\partial}{\partial u} + F\frac{\partial}{\partial F}\,\cdot \end{aligned} \tag{4.24}$$

Remark 9.1. Likewise, it can be demonstrated that Eq. (4.19) has a seven-dimensional equivalence algebra spanned by the operators

$$Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial x}, \quad Y_3 = \frac{\partial}{\partial u}, \quad Y_4 = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + P\frac{\partial}{\partial P}, \tag{4.25}$$

$$Y_5 = u\frac{\partial}{\partial u} + F\frac{\partial}{\partial F}, \quad Y_6 = x\frac{\partial}{\partial t} - \frac{\partial}{\partial P}, \quad Y_7 = x\frac{\partial}{\partial x} - P\frac{\partial}{\partial P} - F\frac{\partial}{\partial F}\cdot$$

Further calculations are based on the theorem on projections, mentioned at the beginning of Section 4.2. Let us introduce the notation X and Z for the projections of the operator (4.21) to the physical variables t, x, u and the variables x, u, Q, F of arbitrary elements, respectively:

$$X = \text{pr}_{(t,x,u)}(Y) \equiv \xi^1\frac{\partial}{\partial t} + \xi^2\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u}, \tag{4.26}$$

$$Z = \text{pr}_{(x,u,Q,F)}(Y) \equiv \xi^2\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u} + \mu^1\frac{\partial}{\partial Q} + \mu^2\frac{\partial}{\partial F}\cdot \tag{4.27}$$

Substituting (4.23) into (4.26) and (4.27), one obtains the following projections:

$$X = \Big[C_1 t + \varphi(x)\Big]\frac{\partial}{\partial t} + \psi(x)\,\frac{\partial}{\partial x} + \Big[(C_1 + C_2)u + \lambda(x)\Big]\frac{\partial}{\partial u}, \tag{4.28}$$

$$\begin{aligned} Z = {} & \psi(x)\,\frac{\partial}{\partial x} + \Big[(C_1 + C_2)u + \lambda(x)\Big]\frac{\partial}{\partial u} \\ & + \Big[-\varphi'(x) + [C_1 - \psi'(x)]Q\Big]\frac{\partial}{\partial Q} + [C_2 - \psi'(x)]F\,\frac{\partial}{\partial F}\cdot \end{aligned} \tag{4.29}$$

The projections (4.28) and (4.29) are well defined in the sense that the coordinates of X depend on variables t, x, u alone, while the coordinates of Z depend on variables x, u, Q, F.

As applied to Eq. (4.20), the theorem on projections [62] is formulated as follows.

Theorem 9.1. (Theorem on projections). The operator X defined by (4.28) is an infinitesimal symmetry of Eq. (4.20) with the functions

$$Q = Q(x, u), \quad F = F(x, u) \tag{4.30}$$

if and only if the system of equations (4.30) is invariant with respect to the group with the generator Z defined by (4.29), i.e. if

$$Z[Q - Q(x, u)]\Big|_{(4.30)} = 0, \quad Z[F - F(x, u)]\Big|_{(4.30)} = 0. \tag{4.31}$$

Here the symbol $|_{(4.30)}$ means that the variables Q and F should be replaced by the functions $Q(x,u)$ and $F(x,u)$ according to equations (4.30). Substituting the expression (4.29) for Z in Eqs. (4.31), one obtains the followin system of linear first-order partial differential equations for the functions $Q(x,u)$ and $F(x,u)$:

$$\begin{aligned} &\psi(x)\frac{\partial Q}{\partial x} + \Big[(C_1+C_2)u+\lambda(x)\Big]\frac{\partial Q}{\partial u} + [\psi'(x) - C_1]Q + \varphi'(x) = 0,\\ &\psi(x)\frac{\partial F}{\partial x} + \Big[(C_1+C_2)u+\lambda(x)\Big]\frac{\partial F}{\partial u} + [\psi'(x) - C_2]F = 0. \end{aligned} \tag{4.32}$$

Remark 9.2. The similar formulation of this theorem for Eq. (4.19) is obtained by replacing the operators (4.24) by (4.25) and Eqs. (4.30) by

$$P = P(u), \quad F = F(x,u).$$

Example 9.7. Let us choose the constants and functions involved in (4.23) as follows:

$$C_1 = 1, \quad \varphi(x) = 0, \quad \psi(x) = x - k, \quad C_2 = 0, \quad \lambda(x) = 0$$

i.e. consider $Y' \in L_{\mathcal{E}}$ of a particular form

$$Y' = t\frac{\partial}{\partial t} + (x-k)\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} - F\frac{\partial}{\partial F}\cdot \tag{4.33}$$

Then, solving the system of equations (4.32), we obtain the functions

$$Q = \Phi\Big(\frac{u}{x-k}\Big), \quad F = \frac{1}{x-k}\Gamma\Big(\frac{u}{x-k}\Big), \tag{4.34}$$

where Φ and Γ are arbitrary functions of one and the same argument. Hence, according to Theorem 9.1, Eq. (4.20) of the form

$$\frac{\partial}{\partial t}\Big[\frac{\partial u}{\partial x} - \Phi\Big(\frac{u}{x-k}\Big)\frac{\partial u}{\partial t}\Big] = \frac{1}{x-k}\Gamma\Big(\frac{u}{x-k}\Big) \tag{4.35}$$

admits an additional operator

$$X' = t\frac{\partial}{\partial t} + (x-k)\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \tag{4.36}$$

together with the evident generator

$$X_0 = \frac{\partial}{\partial t}$$

of time transformation. One can use the additional symmetry X', e.g. for constructing invariant solutions. Solving the equation

$$X'\big(J(t,x,u)\big) = 0$$

we obtain the invariants

$$\lambda = \frac{t}{x-k}\,, \quad \mu = \frac{u}{x-k}\,\cdot$$

Hence we search the invariant solution in the form $\mu = G(\lambda$, i.e.

$$u = (x-k)G(\lambda).$$

The substitution in Eq. (4.35) shows that $G(\lambda)$ should satisfy the following second-order ordinary differential equation:

$$\lambda\, G'' + \big(\Phi(G)\, G'\big)' + \Gamma(G) = 0.$$

The particular case of interest is that of Eq. (4.35) with the linear functions Φ and Γ, when a quadratic nonlinear term appears in the left-hand side of Eq. (4.35), namely:

$$\frac{\partial}{\partial t}\left[\frac{\partial u}{\partial x} - \alpha \frac{u}{x-k}\frac{\partial u}{\partial t}\right] = -\beta \frac{u}{(x-k)^2}\,\cdot \tag{4.37}$$

This case will be considered in Section 5.

Example 9.8. Let us consider an equation with two additional symmetries. We will need two equivalence generators that span a two-dimensional Lie algebra. We again take (4.33) as the first generator and find the second generator Y'' by setting

$$C_1 = 0, \quad C_2 = 1, \quad \varphi(x) = \psi(x) = \lambda(x) = x - k$$

in (4.23), i.e. we choose

$$Y'' = (x-k)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) + \Big(u + x - k\Big)\frac{\partial}{\partial u} - \Big(1+Q\Big)\frac{\partial}{\partial Q}\,\cdot$$

The operators Y' and Y'' commute, and hence span an Abelian Lie algebra. Applying the theorem on projections to both operators Y' and Y'', i.e., in fact, solving Eqs. (4.32) with the coordinates of the operator Y'' and with the functions Q, F given by Eqs. (4.34), we obtain

$$Q = -1 + A\mathrm{e}^{-u/(x-k)}, \quad F = -\frac{B}{x-k}\,\mathrm{e}^{u/(x-k)}$$

where A and B are arbitrary constants.

Theorem 9.1 on projections guarantees that the equation

$$\frac{\partial}{\partial t}\left[\frac{\partial u}{\partial x} + \left(1 - A\mathrm{e}^{-u/(x-k)}\right)\frac{\partial u}{\partial t}\right] = -\frac{B}{x-k}\,\mathrm{e}^{u/(x-k)} \tag{4.38}$$

has the following two additional symmetries:

$$\begin{aligned} X' &= t\frac{\partial}{\partial t} + (x-k)\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}\,, \\ X'' &= (x-k)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) + \left(u + x - k\right)\frac{\partial}{\partial u}\cdot \end{aligned} \tag{4.39}$$

In particular, we can find the invariant solution with respect to the two-parameter symmetry group with two generators (4.39). For this purpose, it is necessary to solve the system of equations

$$X'(J) = 0, X''(J) = 0.$$

The first equation provides two invariants

$$\lambda = \frac{x-k}{t}\,, \quad \mu = \frac{u}{t}\cdot$$

Then, substituting $J = J(\lambda, \mu)$ in the second equation we obtain the relation

$$\mu = L\lambda + \lambda\,\ln\left|\frac{\lambda}{1-\lambda}\right|, \quad L = \mathrm{const.}$$

Substituting the values μ and λ, we obtain the following form for the invariant solution:

$$u = (x-k)\left[L + \ln\left|\frac{x-k}{t-x+k}\right|\right].$$

Substituting this expression in Eq. (4.38) we determine the the constant L, namely $L = \ln|1/B|$. Thus, the invariant solution of Eq. (4.38) constructed by means of its two infinitesimal symmetries (4.39) is given by

$$u = (x-k)\ln\left|\frac{x-k}{B(t-x+k)}\right|. \tag{4.40}$$

Example 9.9. Let us turn to Eq. (4.19) and apply the theorem on projections to the combination

$$Y = Y_3 + Y_5 + Y_6 + Y_7$$

of the operators (4.25), i.e. to the following equivalence generator:

$$Y = x\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + (1+u)\frac{\partial}{\partial u} - (1+P)\frac{\partial}{\partial P}\cdot$$

Using the projections similar to (4.26) and (4.27) we obtain:

$$X = x\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + (1+u)\frac{\partial}{\partial u}\,, \tag{4.41}$$

$$Z = x\frac{\partial}{\partial x} + (1+u)\frac{\partial}{\partial u} - (1+P)\frac{\partial}{\partial P}\cdot \tag{4.42}$$

Solving the equations (cf. Eqs. (4.31) and see Remark 9.2)

$$Z(P(u) - P) = 0, \quad Z(F(x,u) - F) = 0$$

which have the form

$$(1+u)\frac{dP}{du} + (1+P) = 0, \quad x\frac{\partial F}{\partial x} + (1+u)\frac{\partial F}{\partial u} = 0,$$

we obtain

$$F = \Gamma\left(\frac{x}{1+u}\right), \quad P = \frac{K}{1+u} - 1, \quad K = \text{const}.$$

Hence, the following articular equation of the form (4.19):

$$\frac{\partial}{\partial t}\left[\frac{\partial u}{\partial x} + \left(1 - \frac{K}{1+u}\right)\frac{\partial u}{\partial t}\right] = \Gamma\left(\frac{x}{1+u}\right) \tag{4.43}$$

admits, together with

$$X_0 = \frac{\partial}{\partial t}\,,$$

an additional operator X defined by (4.41).

Let us use the operator (4.41) for constructing an invariant solution. Two functionally independent invariants of the operator X are

$$\lambda = t - x, \quad \mu = \frac{u+1}{x}\,,$$

and hence we have the following form of the invariant solution:

$$u = x\phi(\lambda), \quad \lambda = t - x.$$

Substitution into (4.43) provides an ordinary differential equation

$$K\left(\frac{\phi'}{\phi}\right)' + \phi' = \Gamma(\phi).$$

Assuming that $K \neq 0$ we have

$$\phi'' - \frac{\phi'^2}{\phi} + \frac{\phi\,\phi'}{K} = \frac{\phi}{K}\,\Gamma(\phi). \tag{4.44}$$

Note that for small values of $|u|$, Eq. (4.43) provides a good approximation with an additional symmetry for the models described by equations of the type (4.17).

5 Physical discussion of the model (4.20)

The model (4.17),

$$\frac{\partial}{\partial t}\left[\frac{\partial u}{\partial x} - u\frac{\partial u}{\partial t}\right] = -\beta\,u, \tag{4.17}$$

appears in several problems. Let us first consider the oscillations of a compressible gas inside a cylinder with a cross section S. The cylinder is closed with a moving piston of mass m. The bottom of the cylinder is fixed at $(x = 0)$ (the x coordinate is measured along the generatrix, from the bottom upwards). The piston can oscillate with the displacement ζ relative to its mean position at $x = H$. The system of equations describing the motion of the piston, taking into account nonlinear gas movements, has the form

$$\rho\,c\,\frac{\partial\zeta}{\partial t} = p(t_-)\; - p(t_+)\;\; \frac{m}{S}\frac{\partial^2\zeta}{\partial t^2} = p(t_+) + p(t_-). \tag{5.1}$$

Here, ρ and c are the density of the gas and the sound velocity in it and $p(t)$ is the form of any of the two acoustic pressure waves propagating towards each other. The arguments contain the time shift determined by the length H of the resonator and by the nonlinear properties of the gas:

$$t_\pm = t \pm \frac{H}{c}\left(1 - \frac{\gamma+1}{2}\,\frac{p}{c^2\rho}\right).$$

Applying the method of transforming functional equations of the type (5.1) into differential evolution equations, described in [120], for the region near the acoustic resonance

$$\omega H/c = \pi + \Delta,$$

where Δ is a small frequency detuning, we obtain (see [119])

$$\frac{\partial}{\partial \xi}\left[\frac{\partial U}{\partial T} + \Delta\frac{\partial U}{\partial \xi} - \pi\frac{\gamma+1}{2}U\frac{\partial U}{\partial \xi}\right] = -\beta\,U. \tag{5.2}$$

Eq. (5.2) is written using the dimensionless quantities

$$U = \frac{p}{c^2\rho}\,, \quad \xi = \omega\, t + \pi\,, \quad T = \frac{\omega\, t_1}{\pi}\,, \quad \beta = \frac{1}{\pi}\frac{m_g}{m},$$

where t_1 is the slow" time, describing the setting of steady-state oscillations in the resonator and

$$m_g = \rho\, SH$$

is the mass of the gas in the cylinder. It is manifest that by changing the constants and variables according to

$$t = \xi - \Delta\, T, \quad T = x, \quad u = \pi\, \frac{\gamma+1}{2}\, U.$$

we can transform Eq. (5.2) to the form (4.17).

Approximate solutions to Eq. (5.2), related to problems of physical interest, are obtained in [119]. However, some exact solutions are also of important physical meaning. For example, let us consider the exact solution obtained by means of the dilation generator X_3 from (4.18). The invariants of X_3 and the form of the corresponding self-similar solution are

$$\lambda = xt, \quad \mu = ux^2, \quad u(x,t) = \frac{1}{x^2}\Phi(\lambda).$$

Substitution in Eq. (4.17) leads to the ordinary differential equation

$$\Phi\Phi'' + \Phi'^2 - \lambda\Phi'' + \Phi' - \beta\Phi = 0.$$

Its particular solution, which in the limit $\beta \to 0$ takes the form of the solution $u = x/t$ for the Riemann waves, is given by

$$\Phi(\lambda) = -\lambda + \frac{\beta}{6}\lambda^2, \quad u(x,t) = \frac{\beta}{6}\, t^2 - \frac{t}{x}\,\cdot \tag{5.3}$$

By analogy with the known procedure of constructing a sawtooth signal [121], in which the periodically continued solution $u = x/t$ is used to describe smooth linear portions of the profile (see the dashed curves in Fig. 1), we continue the solution (5.3) with the period of 2π. The shock fronts are localized at the points

$$t_n = (2n+1)\,\pi + \frac{3}{\beta x} - \sqrt{\left(\frac{3}{\beta x}\right)^2 - \frac{\pi^2}{3}},$$

$$n = 0, \pm 1, \pm 2, \ldots,$$

whose coordinates are calculated from the condition that the period-average value of the function $u(x,t)$ vanishes.

As shown in Fig. 1, the inclusion of a low-frequency dispersion ($\beta \neq 0$) leads to an asymmetric distortion of the waveform. The duration of the compression phase becomes shorter, and the duration of the rarefaction phase longer. The curves shown in Fig. 1 are plotted for $\beta = 3$ and two dimensionless distances $x = 0.2$ and 0.5 (curves $1, 1'$ and $2, 2'$, respectively). Manifestly, the construction is valid up to the distances $x \leq 3\sqrt{3}/\pi\beta$.

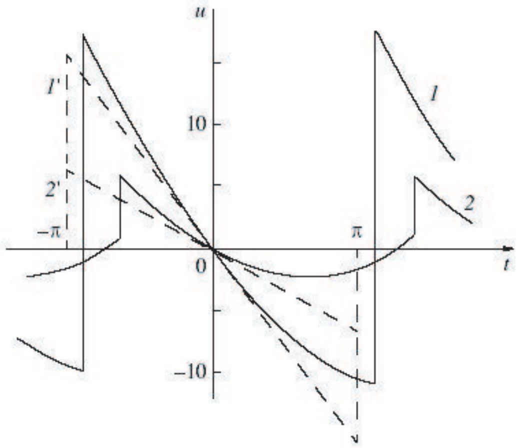

Figure 1: Effect of a low-frequency dispersion on the wave distortion process. The profiles are plotted for the distances $x = (1, 1')$ 0.2 and $(2, 2')$ 0.5. The solid curves are described by the invariant solution (5.3) for $\beta = 3$. The dashed curves are plotted taking into account only the nonlinearity ($\beta = 0$, the dispersion is absent) and are presented for the comparison with the profiles shown by the solid curves.

The waveform distortion shown in Fig. 1 is similar to that observed in experiments with high-intensity diffracted beams. The same behavior is predicted by the theory [122], [138]. Therefore the second physical problem associated with the model (4.17) belongs to the theory of nonlinear acoustic beams [122]. Let us consider the Khokhlov-Zabolotskaya equation in the form (cf. [138], p.346):

$$\frac{\partial}{\partial \tau}\left[\frac{\partial p}{\partial x} - \frac{\varepsilon}{c^3 \rho}\, p\, \frac{\partial p}{\partial \tau} - \frac{1}{c}\frac{\partial p}{\partial \tau}\left(\frac{\partial \Psi}{\partial x} + \frac{1}{2}\left(\frac{\partial \Psi}{\partial r}\right)^2\right)\right.$$
$$\left. + \frac{\partial p}{\partial r}\frac{\partial \Psi}{\partial r} + \frac{p}{2}\Delta_\perp \Psi\right] = \frac{c}{2}\,\Delta_\perp p. \tag{5.4}$$

Eq. (5.4) describes beams, circular in the cross-section, where

$$\Delta_\perp = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}, \qquad \tau = t - \frac{x}{c} - \frac{1}{c}\,\Psi(x, r),$$

$\varepsilon = (\gamma + 1)/2$ is a nonlinear parameter and Ψ is the eikonal. In a homogeneous medium, the distance x travelled by a wave is measured along a straight line. In a nonhomogeneous medium x can be measured along the ray, which is the axis of the beam [123]. Limiting our consideration to a vicinity of the axis, we set in Eq. (5.4)

$$\frac{\partial\Psi}{\partial x} + \frac{1}{2}\left(\frac{\partial\Psi}{\partial r}\right)^2 = \frac{r^2}{2} f(x) + q'(x), \tag{5.5}$$

where f and q' are known functions describing the behavior of the refraction index in the medium. Determining the eikonal from Eq. (5.5):

$$\Psi = \frac{r^2}{2}\Phi(x) + q(x), \quad \Phi' + \Phi^2 = f,$$

we rewrite Eq. (5.4) in the form

$$\frac{\partial}{\partial\tau}\left[\frac{\partial p}{\partial x} + \frac{f}{\Phi}\nu\frac{\partial p}{\partial\nu} - \frac{1}{c}\frac{\partial p}{\partial\tau}\left(\frac{\nu^2}{2}\frac{f}{\Phi^2} + q'\right) - \frac{\varepsilon}{c^3\rho}\, p\,\frac{\partial p}{\partial\tau} + \Phi p\right]$$

$$= \frac{c}{2}\Phi^2\left(\frac{\partial^2 p}{\partial\nu^2} + \frac{1}{\nu}\frac{\partial p}{\partial\nu}\right), \qquad \nu = r\Phi(x). \tag{5.6}$$

In the nonlinear geometric acoustics approximation, the right-hand side of Eq. (5.4) or Eq. (5.6) is assumed to be equal to zero. One can refine this approximation and take into account the diffraction corrections, if one uses the following model for the right-hand side of Eq. (5.6):

$$\left(\frac{\partial^2 p}{\partial\nu^2} + \frac{1}{\nu}\frac{\partial p}{\partial\nu}\right) \to -\frac{2}{a^2}\, p, \tag{5.7}$$

where a is the initial beam width. Note that, for a transverse structure described by the Bessel function $J_0(\sqrt{2}\nu/a)$, the representation (5.7) proves to be exact. Now, varying the right-hand side of Eq. (5.6) according to Eq. (5.7) and letting $\nu \to 0$, we obtain an equation describing the field of an acoustic beam near its axis:

$$\frac{\partial}{\partial\tau}\left[\frac{\partial p}{\partial x} - \frac{q'(x)}{c}\frac{\partial p}{\partial\tau} - \frac{\varepsilon}{c^3\rho}\, p\,\frac{\partial p}{\partial\tau} + \Phi(x)p\right] = -\frac{c}{a^2}\,\Phi^2(x)p. \tag{5.8}$$

In particular, for a focused wave, we set

$$\Phi = (x-k)^{-1}, \quad q' = 0,$$

where k is the focal distance, we reduce Eq. (5.8) to Eq. (4.37) with

$$u = p(x-k), \quad \alpha = \varepsilon\,(c^3\rho)^{-1}, \quad \beta = ca^{-2}.$$

Using the change of variables

$$t = c\Big[\tau + \frac{1}{c}q(x)\Big], \quad u = \frac{\varepsilon}{c^3\rho}\exp\left(\int\Phi(x)\,dx\right),$$

we reduce Eq. (5.8) to a simpler form

$$\frac{\partial}{\partial t}\left[\frac{\partial u}{\partial x} - Q(x)\,u\frac{\partial u}{\partial t}\right] = -\frac{\Phi^2(x)}{a^2}\,u, \tag{5.9}$$

where

$$Q(x) = \exp\left(-\int \Phi(x)\,dx\right).$$

In the general case, when all characteristics of the medium, including the nonlinearity parameter, may depend on the x coordinate, the generalized equation can be written in the form (4.20):

$$\frac{\partial}{\partial t}\left[\frac{\partial u}{\partial x} - Q(x,\ u)\frac{\partial u}{\partial t}\right] = F(x,\ u). \tag{5.10}$$

Equation (5.10) also takes into account the possibility of a more complex nonlinear response of the medium that cannot be described by a common quadratic nonlinearity.

If in Eq. (5.10) we set

$$Q = -\frac{\alpha}{k-x}, \quad F = -\frac{\beta}{(k-x)^2}\,u,$$

it takes the form of Eq. (4.37) and describes the focusing of a beam in a homogeneous medium. The invariant solution corresponding to operator (4.36) is

$$u = (k-x)G(\lambda), \quad \lambda = \frac{t}{k-x}\cdot$$

The function $G(\lambda)$ satisfies the equation

$$\lambda G'' - \alpha(GG')' + \beta G = 0. \tag{5.11}$$

When $\alpha = 0$, the solution to Eq. (5.11) is expressed in terms of the Bessel functions and has the form

$$G = \sqrt{\lambda}\big[C_1 J_1(2\sqrt{\beta\lambda}) + C_2 Y_1(2\sqrt{\beta\lambda})\big].$$

When $\beta = 0$, the solution is given implicitly by the equation

$$\lambda = -\alpha C_1 + C_2(G + C_1) - \alpha(G + C_1)\ln|G + C_1|.$$

It is clear that the models (4.17), (4.19), (4.20) are provided not only by the two above problems alone. Evidently, any linear distributed system with a low-frequency dispersion, described by the law

$$k(\omega) = \frac{\omega}{c} - \frac{\beta}{\omega},$$

can be associated with the differential equation

$$\frac{\partial^2 u}{\partial t\,\partial x} = -\beta\, u. \tag{5.12}$$

If the nonlinearity is weak, the corresponding term is added to the evolution equation, and Eq. (5.12) is transformed, e.g. to Eq. (4.17).

Note that the general form (5.10) is still specific enough. Its symmetry properties reflect the physics of the processes under study with a higher accuracy than, for example, the symmetries of the reduced model (4.17) or the most general model

$$\frac{\partial^2 u}{\partial t\,\partial x} = G(x,\, t,\, u,\, \frac{\partial u}{\partial t},\, \frac{\partial^2 u}{\partial t^2}),$$

which could be studied by the proposed method without any strong limitations on the class of problems under consideration.

6 Conclusions

The present study is based on the seemingly paradoxical statement that it is advisable to analyze nonlinear problems by way of their immersion into the class of more general and, hence, more complex models. The experience in studying the theory of nonlinear oscillations and waves on the basis of physically justified simplification of models seems to contradict the proposed approach based on the *a priori* use of symmetries. However, behind the external differences, one can discover the single nature of the two approaches. Evidently, the higher-symmetry model should contain more exact solutions. How one could obtain a higher symmetry? Of course, one can follow the conventional simplification method by "cutting off" the elements of the model that violate its symmetry (neglecting some of the terms in the equation or modifying them in some way). But one can make the model more symmetric by complication of the initial model. If the complex model allows a suitable exact solution, the necessary simplification can be performed at the last step of calculation, i.e., in the final formulae.

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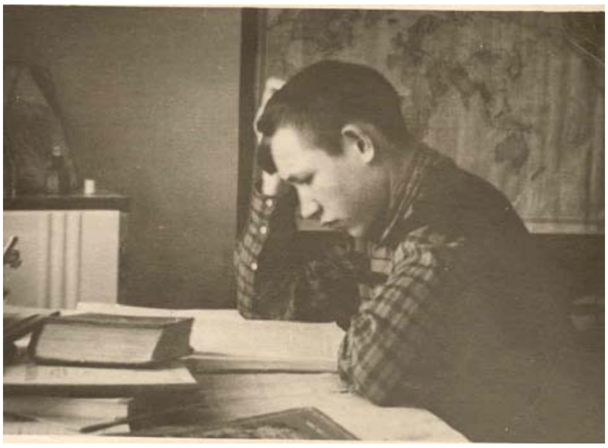

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