Nonlinear Dynamics **22:** 61–71, 2000.

Contact Transformation Group Classification of Nonlinear Wave Equations

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(Received: 15 March 1999; accepted: 15 April 1999)

Abstract. Interest in nonlinear wave equations has been stimulated by numerous physical applications, such as telecommunication (e.g. nonlinear telegrapher equation), gasdynamics, anisotropic plasticity and nonlinear elasticity, etc. Mathematical models of these phenomena can often be reduced to particular types of the equation $u_{tt} = f(x, u_x)u_{xx} + g(x, u_x)$. In this paper, the problem of classification of the latter equation with respect to admitted contact transformation groups is reduced to the investigation of point transformation groups of the equivalent system of first-order quasi-linear equations $v_t = a(x, v)w_x$, $w_t = b(x, v)v_x$.

Keywords: Nonlinear wave equation, group classification, hodograph transform.

1. Introduction

The problem of group analysis of nonlinear wave phenomena has been discussed in numerous papers since the 1980s (see [1] and [2] and the references therein). In these papers, several particular types of so-called nonlinear wave equations were classified from the symmetry point of view. Recently, an attempt was undertaken in [2] to encapsulate these particular cases in a general group classification of nonlinear equations of the form

$$u_{tt} = f(x, u_x)u_{xx} + g(x, u_x). \tag{1}$$

Numerous particular types of Equations (1) were identified when the symmetry group is wider than that of the general equation (1). The approach employed in [2] is based on the so-called *method of preliminary group classification* and does not solve the problem of *complete group classification* of Equations (1).

We propose here to tackle the problem of group classification with respect to point as well as contact transformations by reducing the second-order equation (1) to an equivalent system of first-order quasi-linear equations of the form

$$v_t = a(x, v)w_x, \quad w_t = b(x, v)v_x. \tag{2}$$

The system (2) admits a large group of equivalence transformations. In particular, the system (2), unlike Equation (1), admits an equivalence transformation given by the hodograph transform. This fact is crucial for the group classification.

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To rewrite Equation (1) in the form (2), we first introduce two dependent variables, $p = u_x$ and $w = u_t$. Then the integrability condition $p_t = w_x$ and Equation (1) yield the following system of first-order partial differential equations:

$$p_t = w_x, \quad w_t = f(x, p)p_x + g(x, p).$$

It can be reduced to a homogenous system by a change of variables $p = h(x, v)$. Indeed, the above system is written

$$h_v v_t = w_x, \quad w_t = f(x, h)h_v v_x + f(x, h)h_x + g(x, h).$$

Now we choose the function $h(x, v)$ as a solution of an ordinary differential equation of the first order with the independent variable x:

$$f(x, h)h_x + g(x, h) = 0.$$

We assume that h depends on v being regarded as a parameter and that $h_v \neq 0$. Ultimately, we arrive at the homogeneous system of the form (2) with $a = 1/h_v$ and $b = f h_v$.

Equations (2) considered together with

$$u_t = w, \tag{3}$$

are equivalent to the original differential equation (1). Furthermore, any infinitesimal contact (in particular, Lie point) symmetry of Equation (1) (see, e.g., [3])

$$X = -\phi_{u_t}\frac{\partial}{\partial t} - \phi_{u_x}\frac{\partial}{\partial x} + \left(\phi - u_t\phi_{u_t} - u_x\phi_{u_x}\right)\frac{\partial}{\partial u} + (\phi_t + u_t\phi_u)\frac{\partial}{\partial u_t} + (\phi_x + u_x\phi_u)\frac{\partial}{\partial u_x},$$

where $\phi = \phi(x, t, u, u_x, u_t)$, provides a Lie point symmetry of the system (2–3). Indeed, we first transform X into a generator of point transformations of the variables x, t, u, p, w by setting $u_x = p, u_t = w$, then use the change of variables $p = h(x, v)$ and obtain the operator

$$Z = \xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x} + \varphi\frac{\partial}{\partial u} + \zeta\frac{\partial}{\partial v} + \chi\frac{\partial}{\partial w} \tag{4}$$

with the coordinates depending on x, t, u, v, w. This operator is obviously admitted by the system (2–3).

Conversely, given an operator (4) admitted by the system (2–3), it is associated with an infinitesimal contact symmetry X of Equation (1) if and only if the following overdetermined system for an unknown ϕ is compatible:

$$\begin{aligned} &\xi = -\phi_{u_t}, \quad \eta = -\phi_{u_x}, \quad \varphi = \phi - u_t\phi_{u_t} - u_x\phi_{u_x}, \\ &\eta h_x + \zeta h_v = \phi_x + u_x\phi_u, \quad \chi = \phi_t + u_t\phi_u. \end{aligned} \tag{5}$$

Note that the system (2) does not involve the variable u. Consequently, in what follows we will discuss the symmetries

$$Y = \xi(x, t, v, w)\frac{\partial}{\partial t} + \eta(x, t, v, w)\frac{\partial}{\partial x} + \zeta(x, t, v, w)\frac{\partial}{\partial v} + \chi(x, t, v, w)\frac{\partial}{\partial w} \tag{6}$$

of the system (2) presupposing that the coordinates of Y do not explicitly depend upon u. One can associate with the operator (6) a symmetry (4) of the extended system (2–3) by first assuming that all arbitrary constants or arbitrary functions occuring in Y depend on u, and then extending the action of Y to the variable u in the form

$$Z = Y + \varphi(x, t, u, v, w)\frac{\partial}{\partial u} \tag{7}$$

with an unknown coefficient $\varphi(x, t, u, v, w)$. The function φ and the dependence on u of the coordinates of Y are determined from the usual infinitesimal invariance test of the differential equation (3) with respect to Z.

Thus, by the above procedure, one can obtain all point and contact symmetries of Equation (1) from point symmetries of the system (2). Furthermore, if the operator (7) does not satisfy the integrability condition of Equations (5), then one obtains 'non-local' symmetries of Equation (1).

2. Equivalence Transformations of the System (2)

In accordance with the general theory of equivalence transformations [4], Equations (2) are replaced by the following system:

$$v_t = aw_x, \quad w_t = bv_x, \quad a_t = a_w = b_t = b_w = 0. \tag{8}$$

We seek the generator of the continuous equivalence group in the general form:

$$E = \xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x} + \zeta\frac{\partial}{\partial v} + \chi\frac{\partial}{\partial w} + \alpha\frac{\partial}{\partial a} + \beta\frac{\partial}{\partial b}, \tag{9}$$

where $\xi, \eta, \zeta, \chi, \alpha$, and β are unknown functions of all variables involved in the system (8), i.e. of the variables t, x, v, w, a, and b. We extend the action of the operator (9) to the derivatives occurring in the system (8):

$$\begin{aligned} E \;=\; & \xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x} + \zeta\frac{\partial}{\partial v} + \chi\frac{\partial}{\partial w} + \alpha\frac{\partial}{\partial a} + \beta\frac{\partial}{\partial b} \\ & + (D_t\zeta - aw_x D_t\xi - v_x D_t\eta)\frac{\partial}{\partial v_t} + (D_x\zeta - aw_x D_x\xi - v_x D_x\eta)\frac{\partial}{\partial v_x} \\ & + (D_t\chi - bv_x D_t\xi - w_x D_t\eta)\frac{\partial}{\partial w_t} + (D_x\chi - bv_x D_x\xi - w_x D_x\eta)\frac{\partial}{\partial w_x} \\ & + (\widetilde{D}_t\alpha - a_x\widetilde{D}_t\eta - a_v\widetilde{D}_t\zeta)\frac{\partial}{\partial a_t} + (\widetilde{D}_w\alpha - a_x\widetilde{D}_w\eta - a_v\widetilde{D}_w\zeta)\frac{\partial}{\partial a_w} \\ & + (\widetilde{D}_t\beta - b_x\widetilde{D}_t\eta - b_v\widetilde{D}_t\zeta)\frac{\partial}{\partial b_t} + (\widetilde{D}_w\beta - b_x\widetilde{D}_w\eta - b_v\widetilde{D}_w\zeta)\frac{\partial}{\partial b_w}, \end{aligned}$$

where

$$\widetilde{D}_t = \frac{\partial}{\partial t}, \quad \widetilde{D}_w = \frac{\partial}{\partial w}, \quad D_t = \frac{\partial}{\partial t} + aw_x\frac{\partial}{\partial v} + bv_x\frac{\partial}{\partial w} + aa_v w_x\frac{\partial}{\partial a} + ab_v w_x\frac{\partial}{\partial b},$$

$$D_x = \frac{\partial}{\partial x} + v_x\frac{\partial}{\partial v} + w_x\frac{\partial}{\partial w} + (a_x + a_v v_x)\frac{\partial}{\partial a} + (b_x + b_v v_x)\frac{\partial}{\partial b}.$$

The infinitesimal invariance test of the system (8) leads to the following result.

THEOREM 1. *The Lie algebra of the most general continuous equivalence group of the system (2) is spanned by the following operators:*

$$E_1 = \frac{\partial}{\partial t}\,, \quad E_2 = \frac{\partial}{\partial w}\,, \quad E_3 = t\frac{\partial}{\partial t} - w\frac{\partial}{\partial w} - 2b\frac{\partial}{\partial b}\,, \quad E_4 = w\frac{\partial}{\partial w} - a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b}\,,$$

$$E_\eta = \eta(x)\frac{\partial}{\partial x} + \eta'(x)\left(a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b}\right), \quad E_\zeta = \zeta(v)\frac{\partial}{\partial v} - \zeta'(v)\left(a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b}\right),$$

where $\eta(x)$ and $\zeta(v)$ are arbitrary functions, and the prime denotes their derivatives. The group transformations generated by the basic operators have the form

$$E_1: \quad \bar{t} = t + \varepsilon_1;$$

$$E_2: \quad \overline{w} = w + \varepsilon_2;$$

$$E_3: \quad \bar{t} = t\varepsilon_3, \quad \overline{w} = w\varepsilon_3^{-1}, \quad \bar{b} = b\varepsilon_3^{-2};$$

$$E_4: \quad \overline{w} = w\varepsilon_4, \quad \bar{a} = a\varepsilon_4^{-1}, \quad \bar{b} = b\varepsilon_4;$$

$$E_\eta: \quad \bar{x} = F(x), \quad \bar{a}(F(x), v) = a(x, v)F'(x), \quad \bar{b}(F(x), v) = b(x, v)F'(x);$$

$$E_\zeta: \quad \bar{v} = H(v), \quad \bar{a}(x, H(v)) = a(x, v)H'(v), \quad \bar{b}(x, H(v)) = b(x, v)/H'(v), \tag{10}$$

where $\varepsilon_1, \ldots, \varepsilon_4$ are group parameters, and $F(x)$ and $H(v)$ are arbitrary functions.

The general equivalence group of the system (2) is discontinuous and contains, along with the continuous transformations (10), the *hodograph transform*

$$\bar{t} = w, \quad \overline{w} = t, \quad \bar{x} = v, \quad \bar{v} = x, \quad \bar{a}(\bar{x}, \bar{v}) = a(x, v), \quad \bar{b}(\bar{x}, \bar{v}) = 1/b(x, v) \tag{11}$$

and the following four independent reflections:

$$\bar{t} = -t, \quad \bar{a}(x, v) = -a(x, v), \quad \bar{b}(x, v) = -b(x, v); \tag{12}$$

$$\bar{x} = -x, \quad \bar{a}(\bar{x}, v) = -a(x, v), \quad \bar{b}(\bar{x}, v) = -b(x, v); \tag{13}$$

$$\bar{v} = -v, \quad \bar{a}(x, \bar{v}) = -a(x, v), \quad \bar{b}(x, \bar{v}) = -b(x, v); \tag{14}$$

$$\overline{w} = -w, \quad \bar{a}(x, v) = -a(x, v), \quad \bar{b}(x, v) = -b(x, v). \tag{15}$$

3. Determining Equations and the Principal Lie Algebra

The symmetries of the system (2) are obtained from the determining equations

$$Y\,[v_t - a(x, v)w_x] = 0, \quad Y\,[w_t - b(x, v)v_x] = 0, \tag{16}$$

where, in the left-hand sides, v_t and w_t are replaced by aw_x and bv_x, respectively. Here, Y is the operator (6) extended to the first derivatives of v and w. Since the left-hand sides of (16) involve the derivatives v_x, w_x regarded as independent variables, Equations (16) yield an overdetermined system comprising eight partial differential equations of the first order for four unknown coefficients of the operator (6):

$$\xi_t = \eta_x - \frac{(ab)_x}{2ab}\eta - \frac{(ab)_v}{2ab}\zeta, \quad \xi_x = \frac{1}{ab}\eta_t, \quad \xi_v = \frac{1}{a}\eta_w, \quad \xi_w = \frac{1}{b}\eta_v; \tag{17}$$

$$\chi_t = b\zeta_x, \quad \chi_x = \frac{1}{a}\zeta_t, \quad \chi_v = \frac{b}{a}\zeta_w, \quad \chi_w = \zeta_v + \frac{a}{2b}\left(\frac{b}{a}\right)_x \eta + \frac{a}{2b}\left(\frac{b}{a}\right)_v \zeta. \tag{18}$$

Recall that the maximal Lie algebra admitted by Equations (2) with arbitrary coefficients $a(x, v)$ and $b(x, v)$ is called the *principal Lie algebra* of the system (2). Assuming in the determining equations (17–18), that $a = a(x, v)$ and $b = b(x, v)$ are arbitrary functions, one readily obtains $\xi = C_1$, $\eta = 0$, $\zeta = 0$, $\chi = C_2$, where C_1 and C_2 are arbitrary constants. Hence, the principal Lie algebra $L_{\mathscr{P}}$ of the system (2) is a two-dimensional Lie algebra. It is spanned by

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial w}. \tag{19}$$

4. Integrability Conditions. The Classifying Relation

Integrability conditions of the system (17–18) yield four equations of the first order:

$$2\frac{(ab)_v}{ab}\eta_t + a\left(\frac{b}{a}\right)_x \eta_w + \frac{(ab)_v}{a}\zeta_w = 0, \tag{20}$$

$$\frac{(ab)_v}{ab}\zeta_t + 2a\left(\frac{b}{a}\right)_x \zeta_w + \frac{a}{b}\left(\frac{b}{a}\right)_x \eta_t = 0, \tag{21}$$

$$\frac{a}{b}\left(\frac{b}{a}\right)_x \eta_v - \left(\frac{(ab)_x}{ab}\right)_v \eta - \left[\frac{(ab)_v}{ab}\zeta\right]_v = 0, \tag{22}$$

$$\left[\frac{a}{b}\left(\frac{b}{a}\right)_x \eta\right]_x - \frac{(ab)_v}{ab}\zeta_x + \left(\frac{a}{b}\left(\frac{b}{a}\right)_v\right)_x \zeta = 0, \tag{23}$$

and eight equations of the second order:

$$2\left(\eta_{tt} - ab\,\eta_{xx}\right) = -\left(ab\right)_x \eta_x - \left(ab\right)_v \zeta_x - ab\left[\frac{(ab)_x}{ab}\right]_x \eta - ab\left[\frac{(ab)_v}{ab}\right]_x \zeta, \tag{24}$$

$$\eta_{tv} - b\,\eta_{xw} = \frac{(ab)_v}{ab}\eta_t - b\frac{a_x}{a}\eta_w, \tag{25}$$

$$\eta_{tw} - a\,\eta_{xv} = -a\frac{b_x}{b}\eta_v, \tag{26}$$

$$\eta_{vv} - \frac{b}{a}\eta_{ww} = \frac{b_v}{b}\eta_v, \tag{27}$$

$$\zeta_{tt} - ab\,\zeta_{xx} = ab_x \zeta_x, \tag{28}$$

$$\zeta_{tw} - a\,\zeta_{xv} = a\frac{b_v}{b}\zeta_x, \tag{29}$$

$$\zeta_{tv} - b\,\zeta_{xw} = \frac{a_v}{a}\zeta_t + a\left(\frac{b}{a}\right)_x \zeta_w, \tag{30}$$

$$2\left(\zeta_{ww} - \frac{a}{b}\zeta_{vv}\right) = -\left(\frac{a}{b}\right)_x \eta_v - \left(\frac{a}{b}\right)_v \zeta_v - \frac{a}{b}\left[\frac{b}{a}\left(\frac{a}{b}\right)_x\right]_v \eta - \frac{a}{b}\left[\frac{b}{a}\left(\frac{a}{b}\right)_v\right]_v \zeta. \tag{31}$$

Comparing two expressions for η_{ttw} calculated from (24) and (26), respectively, and invoking (20), we arrive at the following basic *classifying relation*:

$$(ab)_{xv}\ \zeta_w = 0. \tag{32}$$

Consequently, we shall consider two alternative cases, $(ab)_{xv} \neq 0$ and $(ab)_{xv} = 0$.

5. The Case $(ab)_{xv} \neq 0$

It follows from (32) that $\zeta_w = 0$ and, hence, Equations (28–30) are written

$$\zeta_{tt} = a(b\,\zeta_x)_x = 0, \quad (b\,\zeta_x)_v = 0, \quad \left(\frac{1}{a}\zeta_t\right)_v = 0.$$

Thus, we have the following equations:

$$\zeta_w = 0, \quad \zeta_t = a\,A(t,x), \quad \zeta_x = \frac{1}{b}B(t,x), \quad A_t = B_x, \quad B_t = b\,(aA)_x. \tag{33}$$

We will distinguish two different cases, $A \neq 0$ (or $\zeta_t \neq 0$) and $A = 0$ (or $\zeta_t = 0$). Let $A \neq 0$. Then one can readily solve the last equation (32). Namely, writing the equation $B_t = b\,(aA)_x$ in the form

$$\frac{1}{b}\frac{B_t}{A} = a_x + a\frac{A_x}{A} \tag{34}$$

and invoking that a and b do not depend on t, one obtains by differentiating (34) with respect to t:

$$\frac{1}{b}\left(\frac{B_t}{A}\right)_t = a\left(\frac{A_x}{A}\right)_t.$$

If we assume that $(A_x/A)_t \neq 0$ and, hence, $(B_t/A)_t \neq 0$, we obtain from the above relation, invoking that A and B do not depend on v, that

$$(ab)_v = \left[\left(\frac{B_t}{A}\right)_t \Bigg/ \left(\frac{A_x}{A}\right)_t\right]_v = 0.$$

The latter relation contradicts the assumption $(ab)_{xv} \neq 0$. Hence, $(A_x/A)_t = 0$ and $(B_t/A)_t = 0$. It follows that $A_x/A = \mu(x)$ and $B_t/A)_t = \nu(x)$. Substituting these expressions in (34), we ultimately arrive at the following equations:

$$A_x = \mu(x)A, \quad B_t = \nu(x)A, \quad \frac{\nu(x)}{b} = a\mu(x) + a_x. \tag{35}$$

Furthermore, the functions $\mu(x)$ and $\nu(x)$ can be simplified by means of equivalence transformations. Namely, the transformation E_η from (10) reduces $\mu(x)$ to $\mu = 0$. Then the last relation (35) is written $ba_x = \nu(x)$. Substituting this relation in equations (33) and using the equivalence transformations, one can reduce the classification problem, when $A \neq 0$, to investigation of the three types of equations (2) whose coefficients obey the following distinctly different relations:

$$(1)\ a = 1; \qquad (2)\ ba_x = 1; \qquad (3)\ ba_x = \pm x. \tag{36}$$

However, we will not consider here these types of equations. Rather, we provide the complete investigation of the alternative case, i.e. $A = 0$, or $\zeta_t = 0$.

When $A = 0$, Equations (33) yield $B = C_1 = \text{const.}$, and hence

$$\zeta_w = 0, \quad \zeta_t = 0, \quad b\,\zeta_x = C_1. \tag{37}$$

Furthermore, invoking equations (20–31), we obtain

$$(b/a)_x \eta_t = 0, \quad 2\zeta_v + \eta[\ln(b/a)]_x + \zeta[\ln(b/a)]_v = \lambda(x),$$

$$\eta_w = 0, \quad \eta_v = C_2 b, \quad 2\eta_x - \eta[\ln(ab)]_x - \zeta[\ln(ab)]_v = \omega(v). \tag{38}$$

Since we are interested only in nonlinear equations (2), we can replace the first equation (38) by $\eta_t = 0$. Indeed, if $(b/a)_x = 0$, one can use the equivalence transformation to obtain $a(x, v) = \pm b(x, v)$. Then the corresponding equations (2) are linear provided that $\eta_t \neq 0$, since equation (20) yields $a_v = b_v = 0$.

Inspecting the integrability conditions of Equations (37–38) and using Equations (22) and (23), one arrives at the following equations:

$$\zeta_x = \frac{C_1}{b}, \quad 2\zeta_v = C_3 + \eta[\ln(a/b)]_x + \zeta[\ln(a/b)]_v,$$

$$2\eta_x = C_4 + \eta[\ln(ab)]_x + \zeta[\ln(ab)]_v, \quad \eta_v = C_2 b. \tag{39}$$

Their compatibility conditions yield:

$$\begin{aligned} \alpha\eta + \beta\zeta &= \frac{C_4}{2}[\ln(b/a)]_x - \frac{C_1}{b}[\ln(ab)]_v\,, \\ \gamma\eta + \delta\zeta &= C_2 b[\ln(b/a)]_x - \frac{C_3}{2}[\ln(ab)]_v\,, \end{aligned} \tag{40}$$

where

$$\alpha = \frac{a_{xx}}{a} - \frac{b_{xx}}{b} - \frac{1}{2}\frac{a_x^2}{a^2} + \frac{1}{2}\frac{b_x^2}{b^2}, \quad \beta = \frac{a_{xv}}{a} - \frac{b_{xv}}{b} + \frac{1}{2}\left(\frac{a_x}{a} + \frac{b_x}{b}\right)\left(\frac{b_v}{b} - \frac{a_v}{a}\right),$$

$$\gamma = \frac{a_{xv}}{a} + \frac{b_{xv}}{b} + \frac{1}{2}\left(\frac{a_x}{a} + \frac{b_x}{b}\right)\left(\frac{b_v}{b} - \frac{a_v}{a}\right) - 2\frac{b_x b_v}{b^2},$$

$$\delta = \frac{a_{vv}}{a} + \frac{b_{vv}}{b} - \frac{1}{2}\frac{a_v^2}{a^2} - \frac{3}{2}\frac{b_v^2}{b^2}.$$

The solution of the non-homogeneous linear system (40) for η and ζ has the form

$$\eta = C_1 I^1 + C_2 I^2 + C_3 I^3 + C_4 I^4, \quad \zeta = C_1 J^1 + C_2 J^2 + C_3 J^3 + C_4 J^4. \tag{41}$$

Here C_i are arbitrary constants and I_i and J_i are given by

$$\begin{aligned} I^1 &= -\frac{\delta}{b\Delta}[\ln(ab)]_v\,, \quad I^2 = -\frac{b\beta}{\Delta}[\ln(b/a)]_x\,, \\ I^3 &= \frac{\beta}{2\Delta}[\ln(ab)]_v\,, \quad I^4 = \frac{\delta}{2\Delta}[\ln(b/a)]_x \end{aligned}$$

and

$$\begin{aligned} J^1 &= \frac{\gamma}{b\Delta}[\ln(ab)]_v\,, \quad J^2 = \frac{b\alpha}{\Delta}[\ln(b/a)]_x\,, \\ J^3 &= -\frac{\alpha}{2\Delta}[\ln(ab)]_v\,, \quad J^4 = -\frac{\gamma}{2\Delta}[\ln(b/a)]_x\,, \end{aligned}$$

respectively, where $\Delta = \alpha\delta - \beta\gamma$.

Substituting (41) in (39), one obtains four relations:

$$\begin{aligned} &\left(J_x^1 - \frac{1}{b}\right) C_1 + J_x^2 C_2 + J_x^3 C_3 + J_x^4 C_4 = 0, \\ &\left(J_v^1 - \frac{1}{2}[\ln(a/b)]_x I^1 - \frac{1}{2}[\ln(a/b)]_v J^1\right) C_1 \\ &\quad + \left(J_v^2 - \frac{1}{2}[\ln(a/b)]_x I^2 - \frac{1}{2}[\ln(a/b)]_v J^2\right) C_2 \\ &\quad + \left(J_v^3 - \frac{1}{2}[\ln(a/b)]_x I^3 - \frac{1}{2}[\ln(a/b)]_v J^3 - \frac{1}{2}\right) C_3 \\ &\quad + \left(J_v^4 - \frac{1}{2}[\ln(a/b)]_x I^4 - \frac{1}{2}[\ln(a/b)]_v J^4\right) C_4 = 0, \\ &\left(I_x^1 - \frac{1}{2}[\ln(ab)]_x I^1 - \frac{1}{2}[\ln(ab)]_v J^1\right) C_1 \\ &\quad + \left(I_x^2 - \frac{1}{2}[\ln(ab)]_x I^2 - \frac{1}{2}[\ln(ab)]_v J^2\right) C_2 \\ &\quad + \left(I_x^3 - \frac{1}{2}[\ln(ab)]_x I^3 - \frac{1}{2}[\ln(ab)]_v J^3\right) C_3 \\ &\quad + \left(I_x^4 - \frac{1}{2}[\ln(ab)]_x I^4 - \frac{1}{2}[\ln(ab)]_v J^4 - \frac{1}{2}\right) C_4 = 0, \\ &I_v^1 C_1 + (I_v^2 - b) C_2 + I_v^3 C_3 + I_v^4 C_4 = 0. \end{aligned} \tag{42}$$

Now we are in a position to prove the following classification result.

THEOREM 2. *Let* $(ab)_{xv} \neq 0$. *Then the principal Lie algebra (19) of the system (2) may be extended at most by four additional infintesimal symmetries (6) such that* $\zeta_t = 0$, *provided that Equations (2) are not linearizable*[1] *by the hodograph transform (11). The virtual number of additional symmetries is defined from Equations (42).*

Proof. Provided that the conditions (42) are satisfied, we substitute the expressions (41) of η and ζ in (17) and (18) to obtain completely integrable systems of first-order partial differential equations for ξ and χ, respectively. Whence, upon integration, we get the functions $\xi(t, x, v, w)$ and $\chi(t, x, v, w)$ containing one additive constant each. These two constants of integration furnish the basis (19) of the principal Lie algebra $L_{\mathcal{P}}$. Hence, the general solution of the determining equations (17–18) may depend at most upon four additional constants $C_1, \ldots, C_4$ involved in (41). This proves the first statement of the theorem.

In order to identify those equations (2) for which the principal Lie algebra $L_{\mathcal{P}}$ can be extended by four, three, two or one additional symmetry, we should find the restrictions on the coefficients a and b of the system (2) under which either all four parameters C_i in (42) may assume arbitrary values (then equations (2) have four additional symmetries), or this is possible for three, two, or one independent linear combinations of C_i. To enumerate all these possibilities, let us treat (42) as a homogeneous system of linear equations for $C_1, \ldots, C_4$.

Four parameters C_i can assume arbitrary values only if the functions $a(x, v)$ and $b(x, v)$ satisfy 16 equations obtained by equating to zero the coefficients of $C_1, \ldots, C_4$ in the left-hand sides of Equations (42).

The cases when three linear combinations of C_i may assume arbitrary values, are obtained by setting in (42) $C_i = k_{i1}s_1 + k_{i2}s_2 + k_{i3}s_3$, where $k_{ij} =$ const. Equating to zero the coefficients of s_1, s_2, s_3, one arrives at 12 equations for $a(x, v)$ and $b(x, v)$.

Likewise, by setting $C_i = k_{i1}s_1 + k_{i2}s_2$, one arrives at 8 equations for $a(x, v)$ and $b(x, v)$ providing conditions for extension of $L_{\mathcal{P}}$ by two symmetries.

Finally, the conditions for the extension of $L_{\mathcal{P}}$ by one symmetry are obtained by setting $C_i = k_i s$. This yields four equations for $a(x, v)$ and $b(x, v)$ obtained from (42) by formally replacing the parameters C_i by constant coefficients k_i.

6. The Case $(ab)_{xv} = 0$

Consider now the second possibility provided by the classifying relation (32), namely $(ab)_{xv} = 0$. Integrating this equation and employing the last two equivalence transformations (10) with appropriately chosen functions $F(x)$ and $H(v)$, one obtains

$$ab = \pm 1. \tag{43}$$

Now, assuming that (2) are nonlinear and using Equations (20–31), we obtain

$$\eta = C_1 x + C_2, \qquad C_1, C_2 = \text{const.}, \tag{44}$$

$\zeta_t = \zeta_w = 0$ and the following equations for $\zeta(x, v)$:

$$\zeta_x = Ca, \quad \zeta_v = \frac{a_v}{a}\zeta + \Big(C_1 x + C_2\Big)\frac{a_x}{a} + K, \tag{45}$$

$$C_1 \frac{a_x}{a} + (C_1 x + C_2)\left(\frac{a_x}{a}\right)_x + \zeta\left(\frac{a_x}{a}\right)_v = 0, \tag{46}$$

[1] Recall that linearizable equations admit an infinite-dimensional Lie algebra.

the latter being the consistency condition for Equations (45). Equation (46) yields:

$$\zeta = C_1 I + C_2 J, \tag{47}$$

where

$$I = -\left(x\frac{a_x}{a}\right)_x \left[\left(\frac{a_x}{a}\right)_v\right]^{-1}, \quad J = -\left(\frac{a_x}{a}\right)_x \left[\left(\frac{a_x}{a}\right)_v\right]^{-1}.$$

Substituting (47) in (45), one obtains two relations:

$$C_1 I_x + C_2 J_x - Ca = 0,$$
$$C_1(aI_v - Ia_v - xa_x) + C_2(aJ_v - Ja_v - a_x) - Ka = 0. \tag{48}$$

THEOREM 3. *Let $(ab)_{xv} = 0$. Then the principal Lie algebra (19) of the system (2) may be extended at most by two additional infinitesimal symmetries (6), provided that Equations (2) are not linearizable. The virtual number of additional symmetries is defined from Equations (48).*

Proof. Provided that the conditions (48) are satisfied, we substitute the expressions (44) and (47) of η and ζ, respectively, in (17) and (18) to obtain completely integrable systems of first-order partial differential equations for ξ and χ, respectively. After integration, we get the functions ξ and χ containing one additive constant each. These two constants of integration furnish the basis (19) of the principal Lie algebra $L_{\mathcal{P}}$. Hence, the general solution of the determining equations (17–18) may depend at most upon two additional constants C_1, C_2 involved in (44) and (47). This proves the first statement of the theorem.

In order to identify those equations (2) for which the principal Lie algebra $L_{\mathcal{P}}$ can be extended by two or one additional symmetry, we adopt the reasoning used in the proof of Theorem 2. Namely, setting $C_i = k_{i1}s_1 + k_{i2}s_2$ in (48) and equating to zero the coefficients of s_1 and s_2, we obtain four equations for $a(x, v)$. The latter equations provide the conditions for the existence of two additional symmetries. Likewise, the conditions for one additional symmetry are obtained by setting $C_i = k_i s$ and equating to zero the coefficients of s. This yields two equations for $a(x, v)$ formally obtained by replacing in (48) C_1 and C_2 by k_1 and k_2, respectively.

7. Conclusion

In this paper, the problem of classification of nonlinear wave equations (1) with respect to admitted contact transformation groups is reduced to the investigatiton of point transformation groups of the equivalent system of first-order quasi-linear equations (2). The main classification results are formulated in Theorems 2 and 3.

Acknowledgement

We acknowledge financial support from the Foundation for Research Development of South Africa.

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