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Differential invariants of nonlinear equations $v_{tt} = f(x, v_x)v_{xx} + g(x, v_x)$

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Abstract

Some 10 years ago, we considered the equations $v_{tt} = f(x, v_x)v_{xx} + g(x, v_x)$ with the intention of their group classification. We found, for the above equations, the equivalence group $\mathscr{E}$ generated by an infinite-dimensional Lie algebra $L_{\mathscr{E}}$ involving two arbitrary functions of the variable x. We utilized the method of preliminary group classification suggested earlier by one of the authors (NHI), and applied it to a finite-dimensional subalgebra of the equivalence Lie algebra $L_{\mathscr{E}}$. Consequently, we found 33 types of nonlinear wave equations admitting an extension by one of the principal Lie algebra, i.e. of the maximal Lie algebra admitted by our equation with arbitrary functions $f(x, v_x)$ and $g(x, v_x)$.

Recently, an infinitesimal technique was developed by NHI that allows one to find invariants of families of differential equations possessing finite or infinite equivalence groups. It is worth noting that the method does not depend on the assumption of linearity of equations. Here, we apply this method for calculation of invariants for the family of nonlinear equations formulated in the title. We show that the infinite-dimensional equivalence Lie algebra $L_{\mathscr{E}}$ has three functionally independent differential invariants of the second order. Knowledge of invariants of families of equations is essential for identifying distinctly different equations and therefore for the problem of group classification.

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1. Introduction

The paper of Ames et al. [1], dealing with the group properties of the nonlinear equation

$$u_{tt} = [f(u)u_x]_x$$

gave rise to numerous publications on symmetry analysis of nonlinear wave phenomena. In this framework Torrisi and Valenti, in [2] and [3], generalizing the above equation, investigated the symmetries of the equations

$$u_{tt} = [f(x,u)u_x]_x \quad \text{and} \quad u_{tt} = [f(u)u_x + g(x,u)]_x.$$

The above equations, after introducing the potential v defined by the equation $u = v_x$, can be written, respectively as follows:

$$v_{tt} = f(v_x)v_{xx}, \tag{1}$$

$$v_{tt} = f(x, v_x)v_{xx}, \tag{2}$$

$$v_{tt} = f(v_x)v_{xx} + g(x, v_x). \tag{3}$$

In [4] we attempted to perform a unified group classifications of a reasonably general class of nonlinear one-dimensional wave equations

$$v_{tt} = f(x, v_x)v_{xx} + g(x, v_x), \tag{4}$$

in order to encapsulate the previous results and possibly to find new classes of equations interesting from group point of view. We showed that the Lie algebra admitted by (4) for arbitrary f and g, i.e. the *principal Lie algebra*, is three-dimensional. Then by using the so-called *method of preliminary group classification* [5] we found 33 types of equations (4), admitting an extension by one of the principal Lie algebra.

Eq. (4) was studied successively in [6,7].

In [8] and later in [9] the following special type of equations (4)

$$v_{tt} = f(v_x)v_{xx} + g(v_x) \tag{5}$$

was investigated in some details.

In [10], Eq. (4) was discussed from point of view of classification with respect to contact transformation groups.

In this paper we calculate the differential invariants of equivalence transformations of equations (4) by using the infinitesimal method for calculation of invariants of families of equations developed in [11,12]. For the general theory of differential invariants of Lie groups, see, e.g. [13].

We recall that an equivalence transformation of (4) is an invertible transformation of the variables t, x and v,

$$t' = \psi_1(t,x,v), \quad x' = \psi_2(t,x,v), \quad v' = \psi_3(t,x,v),$$

mapping every equation of the form (4) into an equation of the same form,

$$v'_{t't'} = f'(x', v'_{x'})v_{x'x'} + g'(x', v'_{x'}),$$

where the transformed functions f' and g' can, in general, be different from the original functions f and g. The set of all equivalence transformations forms a group denoted by $\mathscr{E}$.

An *invariant of equations* (4) is a function

$$J(t, x, v, v_t, v_x, f, g) \tag{6}$$

that is invariant with respect to the equivalence group $\mathscr{E}$. Likewise, differential invariants of $\mathscr{E}$,

$$J(t, x, v, v_t, v_x, f, g, f_x, g_x, f_{v_x}, g_{v_x}, f_{xx}, g_{xx}, f_{xv_x}, g_{xv_x}, \ldots), \tag{7}$$

are termed *differential invariants of equations* (4). The invariant (6) is also called a *differential invariant of order zero*, whereas (7) is termed a *differential invariant of order s* if it involves derivatives of f and/or g whose maximal order is s.

Knowledge of invariants of families of equations is essential for their group classification.

2. Algebra $L_{\mathscr{E}}$. Absence of differential invariants of order zero

Our goal is to find the differential invariants for the family of equations

$$v_{tt} = f(x, v_x)v_{xx} + g(x, v_x), \quad f \neq 0. \tag{8}$$

In [4], in order to obtain the continuous group of equivalence transformations of equations (4) by means of the Lie infinitesimal invariance criterion [13], we search for the equivalence operator Y in the following form:

$$Y = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial v} + \zeta_1 \frac{\partial}{\partial v_t} + \zeta_2 \frac{\partial}{\partial v_x} + \mu^1 \frac{\partial}{\partial f} + \mu^2 \frac{\partial}{\partial g}, \tag{9}$$

where ξ^1, ξ^2 and η are depending on t, x and v, while μ^1 and μ^2 depend on t, x, v, v_t, v_x, f and g, while ζ_1 and ζ_2 are given by

$$\zeta_1 = D_t(\eta) - v_t D_t(\xi^1) - v_x D_t(\xi^2), \tag{10}$$

$$\zeta_2 = D_x(\eta) - v_t D_x(\xi^1) - v_x D_x(\xi^2). \tag{11}$$

The operators D_t and D_x denote total derivatives with respect to t and x:

$$D_t = \frac{\partial}{\partial t} + v_t \frac{\partial}{\partial v} + v_{tt} \frac{\partial J}{\partial v_t} + v_{tx} \frac{\partial}{\partial v_x} + \cdots, \tag{12}$$

$$D_x = \frac{\partial}{\partial x} + v_x \frac{\partial}{\partial v} + v_{tx} \frac{\partial}{\partial v_t} + v_{xx} \frac{\partial}{\partial v_x} + \cdots. \tag{13}$$

In [4], we found that the class of equations (4) has an infinite continuous group $\mathscr{E}$ of equivalence transformations generated by the Lie algebra $L_{\mathscr{E}}$ spanned by the operators:

$$Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial v}, \tag{14}$$

$$Y_3 = t\frac{\partial}{\partial v} + \frac{\partial}{\partial v_t}, \quad Y_4 = x\frac{\partial}{\partial v} + \frac{\partial}{\partial v_x}, \tag{15}$$

$$Y_5 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2v\frac{\partial}{\partial v} + v_t\frac{\partial}{\partial v_t} + v_x\frac{\partial}{\partial v_x}, \tag{16}$$

$$Y_6 = t\frac{\partial}{\partial t} - v_t\frac{\partial}{\partial v_t} - 2f\frac{\partial}{\partial f} - 2g\frac{\partial}{\partial g}, \tag{17}$$

$$Y_7 = t^2\frac{\partial}{\partial v} + 2t\frac{\partial}{\partial v_t} + 2\frac{\partial}{\partial g}, \tag{18}$$

$$Y_F = F\frac{\partial}{\partial v} + F'\frac{\partial}{\partial v_x} - F''f\frac{\partial}{\partial g}, \tag{19}$$

$$Y_\varphi = \varphi\frac{\partial}{\partial x} - \varphi' v_x\frac{\partial}{\partial v_x} + 2\varphi' f\frac{\partial}{\partial f} + \varphi'' v_x f\frac{\partial}{\partial g}, \tag{20}$$

where $\varphi = \varphi(x)$ and $F = F(x)$ are two arbitrary functions of x. Here, the prime denotes the differentiation with respect to x.

Now we seek for differential invariants of order zero, i.e. invariants of the form

$$J = J(t, x, v, v_t, v_x, f, g). \tag{21}$$

Applying the invariant test $Y(J) = 0$ to the operators Y_1, Y_2, Y_3, Y_4 and Y_φ with $\varphi = 1$, one can verify that the invariant (21) does not depend on t, x, v, v_t and v_x:

$$J = J(f, g). \tag{22}$$

Then, applying the invariant test to the operators Y_7 and Y_6, one obtains

$$\frac{\partial J}{\partial g} = 0, \qquad \frac{\partial J}{\partial f} = 0. \tag{23}$$

Hence, equations (8) do not have differential invariants of order zero.

3. Differential invariants of the first order

In order to obtain differential invariants of the first order,

$$J = J(f, g, f_x, f_{v_x}, g_x, g_{v_x}), \tag{24}$$

we consider the first prolongation of the operator Y (9):

$$Y^{(1)} = Y + \sigma_i^k\frac{\partial}{\partial f_i^k}. \tag{25}$$

Here we use the local notation $f^1 = f$, $f^2 = g$, $f_1^k = f_x^k$, $f_2^k = f_{v_x}^k$ and set

$$\sigma_i^k = \widetilde{D}_i(\mu^k) - f_1^k\widetilde{D}_i(\xi^2) - f_2^k\widetilde{D}_i(\zeta_2), \quad i, k = 1, 2, \tag{26}$$

where $\widetilde{D}_j$ $(j = 1, 2)$ denote the total derivatives with respect to x and v_x:

$$\widetilde{D}_1 = \frac{\partial}{\partial x} + f_1^k\frac{\partial}{\partial f^k} + f_{11}^k\frac{\partial}{\partial f_1^k} + f_{12}^k\frac{\partial}{\partial f_2^k} + \cdots, \tag{27}$$

$$\widetilde{D}_2 = \frac{\partial}{\partial v_x} + f_2^k\frac{\partial}{\partial f^k} + f_{12}^k\frac{\partial}{\partial f_1^k} + f_{22}^k\frac{\partial}{\partial f_2^k} + \cdots. \tag{28}$$

The equations $Y_1^{(1)}(J) = 0, \ldots, Y_4^{(1)}(J) = 0$ are satisfied identically. Moreover, if one keeps in the first prolongation $Y_7^{(1)}$ of the operator Y_7 only those terms needed in further calculations, one can readily see that this prolongation coincides with Y_7 itself, and hence

$$Y_7^{(1)}(J) = \frac{\partial J}{\partial g} = 0. \tag{29}$$

It follows that

$$J = J(f, f_x, f_{v_x}, g_x, g_{v_x}). \tag{30}$$

Likewise, keeping in the first prolongation of the operator Y_5 only the terms required for our purposes, we have

$$Y_5^{(1)} = -f_1 \frac{\partial}{\partial f_1} - g_1 \frac{\partial}{\partial g_1} - f_2 \frac{\partial}{\partial f_2} - g_2 \frac{\partial}{\partial g_2}, \tag{31}$$

where $f_1 = f_x$, $f_2 = f_{v_x}$, $g_1 = g_x$, $g_2 = g_{v_x}$.

Applying operator $Y_5^{(1)}$ to differential invariant given by (30), we have

$$Y_5^{(1)}(J) = -f_1 \frac{\partial J}{\partial f_1} - g_1 \frac{\partial J}{\partial g_1} - f_2 \frac{\partial J}{\partial f_2} - g_2 \frac{\partial J}{\partial g_2} = 0. \tag{32}$$

The characteristic equations

$$\frac{\mathrm{d}f_1}{f_1} = \frac{\mathrm{d}g_1}{g_1} = \frac{\mathrm{d}f_2}{f_2} = \frac{\mathrm{d}g_2}{g_2} \tag{33}$$

yield that $J = J(f, p_1, p_2, p_3)$, where

$$p_1 = \frac{g_1}{f_1}, \quad p_2 = \frac{f_2}{f_1}, \quad p_3 = \frac{g_2}{f_1}, \tag{34}$$

provided that $f_1 \neq 0$ (see the Note below).

The first prolongation of the operator Y_6, in the form which we need, is

$$Y_6^{(1)} = f \frac{\partial}{\partial f} + f_1 \frac{\partial}{\partial f_1} + g_1 \frac{\partial}{\partial g_1} + f_2 \frac{\partial}{\partial f_2} + g_2 \frac{\partial}{\partial g_2}. \tag{35}$$

Acting by this operator on the invariants (34), one obtains that

$$Y_6^{(1)}(p_1) = Y_6^{(1)}(p_2) = Y_6^{(1)}(p_3) = 0,$$

and hence

$$Y_6^{(1)}(J) \equiv f \frac{\partial J}{\partial f} = 0. \tag{36}$$

It follows that the quantities (34) provide a basis of invariants (24) for $Y_1, \ldots, Y_7$:

$$J = J(p_1, p_2, p_3). \tag{37}$$

Now we carry on the first prolongation of the operator Y_F and write it, as in the previous cases, in the following form:

$$Y_F^{(1)} = -F'' f_2 \frac{\partial}{\partial f_1} - (F''' f + F'' f_1 + F'' g_2) \frac{\partial}{\partial g_1} - F'' f_2 \frac{\partial}{\partial g_2}. \tag{38}$$

The invariant test $Y_F^{(1)}(J)=0$ is written:

$$F''f_2\frac{\partial J}{\partial f_1}+(F'''f+F''f_1+F''g_2)\frac{\partial J}{\partial g_1}+F''f_2\frac{\partial J}{\partial g_2}=0. \tag{39}$$

Since $F(x)$ is an arbitrary function, its derivatives F''' and F'' are functionally independent. In consequence, Eq. (39) splits into the following two equations:

$$f\frac{\partial J}{\partial g_1}=\frac{f}{f_1}\frac{\partial J}{\partial p_1}=0, \tag{40}$$

$$f_2\left(\frac{\partial J}{\partial f_1}+\frac{\partial J}{\partial g_2}\right)=-p_2\left[p_2\frac{\partial J}{\partial p_2}+(p_3-1)\frac{\partial J}{\partial p_3}\right]=0. \tag{41}$$

It follows that

$$J=J(q) \tag{42}$$

with

$$q=\frac{p_3-1}{p_2}\equiv\frac{g_2-f_1}{f_2}, \tag{43}$$

provided that $f_2\neq 0$.

Finally, we consider the first prolongation of the operator Y_φ keeping only the necessary terms:

$$Y_\varphi^{(1)}=(2\varphi''f+\varphi'f_1+\varphi''v_xf_2)\frac{\partial}{\partial f_1}+3\varphi'f_2\frac{\partial}{\partial f_2}+(\varphi''f+\varphi''v_xf_2+\varphi'g_2)\frac{\partial}{\partial g_2}.$$

Invoking the Eqs. (42) and (43), one has:

$$Y_\varphi^{(1)}(J)=-\frac{1}{f_2}[(\varphi''f-2\varphi'(f_1-g_2)]\frac{\partial J}{\partial q}=0.$$

Treating φ'' and φ' as independent functions and assuming that $f_2\neq 0$ one obtains:

$$\frac{\partial J}{\partial q}=0. \tag{44}$$

When the condition $f_2=0$ holds, the corresponding Eqs. (8), viz.

$$v_{tt}=l(x)v_{xx}+g(x,v_x),\quad l\neq 0, \tag{45}$$

should be considered separately because of the following lemma.

Lemma. *The equation $f_2=0$ is invariant with respect to the group $\mathscr{E}$.*

Proof. Indeed, it is evident that the invariance test $Y(f_2)|_{f_2=0}=0$ is satisfied for the first prolongation of Y_1, Y_2, Y_3, Y_4, and Y_7, since they do not contain differentiation with respect to f_2. One can readily verify, using the expressions for $Y_5^{(1)}$, $Y_6^{(1)}$, $Y_F^{(1)}$, and $Y_\varphi^{(1)}$, that the invariance test is satisfied for the remaining operators as well:

$$Y_5^{(1)}(f_2)\Big|_{f_2=0} = 0, \quad Y_6^{(1)}(f_2)\Big|_{f_2=0} = 0, \quad Y_F^{(1)}(f_2)\Big|_{f_2=0} = 0, \quad Y_\varphi^{(1)}(f_2)\Big|_{f_2=0} = 0. \qquad \square$$

Note. The equation $f_1 = 0$ is not invariant with respect to the group $\mathscr{E}$.

The above lemma and Eq. (44) prove the following statement.

Theorem 1. *Eq.* (8) *have no differential invariants of the first order, but they have one invariant equation, namely*

$$f_2 \equiv f_{v_x} = 0. \tag{46}$$

4. Differential invariants of the second order

Here, we search for differential invariants involving second-order derivatives,

$$J = J(f^k, f_1^k, f_2^k, f_{11}^k, f_{12}^k, f_{22}^k), \quad k = 1, 2. \tag{47}$$

The second prolongation of the operator Y is written in the form

$$Y^{(2)} = Y + \sigma_i^k \frac{\partial}{\partial f_i^k} + \sigma_{ij}^k \frac{\partial}{\partial f_{ij}^k} \tag{48}$$

with

$$\sigma_{ij}^k = \widetilde{D}_j(\sigma_i^k) - f_{i1}^k \widetilde{D}_j(\xi^2) - f_{i2}^k \widetilde{D}_j(\zeta_2), \quad i, j, k = 1, 2. \tag{49}$$

The second prolongation $Y_7^{(2)}$ of the operator Y_7 coincides with Y_7. Therefore,

$$Y_7^{(2)}(J) = \frac{\partial J}{\partial g} = 0, \tag{50}$$

and hence

$$J = J(f, f_1^k, f_2^k, f_{11}^k, f_{12}^k, f_{22}^k), \quad k = 1, 2. \tag{51}$$

The action of the second prolongation of the operator Y_5 is defined by

$$\begin{aligned} Y_5^{(2)} = {} & -f_1 \frac{\partial}{\partial f_1} - g_1 \frac{\partial}{\partial g_1} - f_2 \frac{\partial}{\partial f_2} - g_2 \frac{\partial}{\partial g_2} - 2f_{11} \frac{\partial}{\partial f_{11}} - 2f_{12} \frac{\partial}{\partial f_{12}} \\ & - 2f_{22} \frac{\partial}{\partial f_{22}} - 2g_{11} \frac{\partial}{\partial g_{11}} - 2g_{12} \frac{\partial}{\partial g_{12}} - 2g_{22} \frac{\partial}{\partial g_{22}}, \end{aligned} \tag{52}$$

and the equation $Y_5^{(2)}(J) = 0$ yields

$$J = J(f, p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9), \tag{53}$$

where p_1, p_2, p_3 are defined by Eq. (34), and

$$p_4 = \frac{f_{11}}{(f_1)^2}, \quad p_5 = \frac{f_{12}}{(f_1)^2}, \quad p_6 = \frac{f_{22}}{(f_1)^2}, \quad p_7 = \frac{g_{11}}{(f_1)^2}, \quad p_8 = \frac{g_{12}}{(f_1)^2}, \quad p_9 = \frac{g_{22}}{(f_1)^2}. \tag{54}$$

Likewise, we have

$$\begin{aligned} Y_6^{(2)} &= f\frac{\partial}{\partial f} + f_1\frac{\partial}{\partial f_1} + g_1\frac{\partial}{\partial g_1} + f_2\frac{\partial}{\partial f_2} + g_2\frac{\partial}{\partial g_2} + f_{11}\frac{\partial}{\partial f_{11}} + f_{12}\frac{\partial}{\partial f_{12}} \\ &\quad + f_{22}\frac{\partial}{\partial f_{22}} + g_{11}\frac{\partial}{\partial g_{11}} + g_{12}\frac{\partial}{\partial g_{12}} + g_{22}\frac{\partial}{\partial g_{22}} \\ &= f\frac{\partial}{\partial f} - p_4\frac{\partial}{\partial p_4} - p_5\frac{\partial}{\partial p_5} - p_6\frac{\partial}{\partial p_6} - p_7\frac{\partial}{\partial p_7} - p_8\frac{\partial}{\partial p_8} - p_9\frac{\partial}{\partial p_9}, \end{aligned} \tag{55}$$

and the equation $Y_6^{(2)}(J) = 0$ yields $J = J(p_1, p_2, p_3, \bar{p}_h)$, where

$$\bar{p}_h = fp_h, \quad h = 4, 5, \ldots, 9, \tag{56}$$

with p_h defined in (54).

The second prolongation of the operator Y_F has the form

$$\begin{aligned} Y_F^{(2)} &= -F''f_2\frac{\partial}{\partial f_1} - (F'''f + F''f_1 + F''g_2)\frac{\partial}{\partial g_1} - F''f_2\frac{\partial}{\partial g_2} - (F'''f_2 + 2F''f_{12})\frac{\partial}{\partial f_{11}} \\ &\quad - F''f_{22}\frac{\partial}{\partial f_{12}} - (F^{\mathrm{IV}}f + 2F'''f_1 + F''f_{11} + F'''g_2 + 2F''g_{12})\frac{\partial}{\partial g_{11}} \\ &\quad - (F'''f_2 + F''f_{12} + F''g_{22})\frac{\partial}{\partial g_{12}} - F''f_{22}\frac{\partial}{\partial g_{22}}. \end{aligned} \tag{57}$$

Invoking that $Y_F^{(2)}(J) = 0$, because F^{IV}, F''' and F'' are independent functions, one obtains from the invariance test $Y_F^{(2)}(J) = 0$ the following three equations:

$$f\frac{\partial J}{\partial g_{11}} = \frac{f^2}{(f_1)^2}\frac{\partial J}{\partial \bar{p}_7} = 0, \tag{58}$$

$$f\frac{\partial J}{\partial g_1} + f_2\left(\frac{\partial J}{\partial f_{11}} + \frac{\partial J}{\partial g_{12}}\right) = \frac{f}{f_1}\left[\frac{\partial J}{\partial p_1} + p_2\left(\frac{\partial J}{\partial \bar{p}_4} + \frac{\partial J}{\partial \bar{p}_8}\right)\right] = 0, \tag{59}$$

$$f_2\frac{\partial J}{\partial f_1} + (f_1 + g_2)\frac{\partial J}{\partial g_1} + f_2\frac{\partial J}{\partial g_2} + 2f_{12}\frac{\partial J}{\partial f_{11}} + f_{22}\frac{\partial J}{\partial f_{12}} + (f_{12} + g_{22})\frac{\partial J}{\partial g_{12}} + f_{22}\frac{\partial J}{\partial g_{22}} = 0. \tag{60}$$

Eq. (58) and the characteristic equations for (59) yield:

$$J = J(p_2, p_3, \bar{p}_5, \bar{p}_6, \bar{p}_9, q_1, q_2), \tag{61}$$

with

$$q_1 = \bar{p}_4 - \bar{p}_8 \equiv \frac{f(f_{11} - g_{12})}{(f_1)^2}, \quad q_2 = \bar{p}_4 - p_1p_2 \equiv \frac{ff_{11} - g_1f_2}{(f_1)^2}. \tag{62}$$

Now Eq. (60) takes the form

$$\begin{aligned} &-p_2^2\frac{\partial J}{\partial p_2} - p_2(p_3 - 1)\frac{\partial J}{\partial p_3} - (2p_2\bar{p}_5 - \bar{p}_6)\frac{\partial J}{\partial \bar{p}_5} - 2p_2\bar{p}_6\frac{\partial J}{\partial \bar{p}_6} - (2p_2\bar{p}_9 - \bar{p}_6)\frac{\partial J}{\partial \bar{p}_9} \\ &\quad - (2p_2q_1 - \bar{p}_5 + \bar{p}_9)\frac{\partial J}{\partial q_1} - [p_2(2q_2 + p_3 + 1) - 2\bar{p}_5]\frac{\partial J}{\partial q_2} = 0, \end{aligned} \tag{63}$$

and its the characteristic equations yield:

$$J = J(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6), \tag{64}$$

where

$$\begin{aligned}
\lambda_1 &= \frac{p_3 - 1}{p_2} = \frac{g_2 - f_1}{f_2}, \quad \lambda_2 = \frac{\bar{p}_6}{p_2^2} = \frac{ff_{22}}{(f_2)^2},\\
\lambda_3 &= \frac{\bar{p}_6 - p_2\bar{p}_5}{(p_2)^3} = f\frac{f_1 f_{22} - f_2 f_{12}}{(f_2)^3},\\
\lambda_4 &= \frac{\bar{p}_9 - \bar{p}_5}{(p_2)^2} = f\frac{f_{12} - g_{22}}{(f_2)^2},\\
\lambda_5 &= \frac{\bar{p}_5 - \bar{p}_9 - q_1 p_2}{(p_2)^3} = f\frac{(f_{12} - g_{22})f_1 - (f_{11} - g_{12})f_2}{(f_2)^3},\\
\lambda_6 &= \frac{(p_2)^2(q_2 + p_3) - 2p_2\bar{p}_5 + \bar{p}_6}{(p_2)^4} = \frac{f(f_2)^2 f_{11} - (f_2)^3 g_1 + f_1(f_2)^2 g_2 - 2ff_1f_2f_{12} + f(f_1)^2 f_{22}}{(f_2)^4}.
\end{aligned} \tag{65}$$

Let us consider the second prolongation of operator Y_φ, keeping only the necessary terms:

$$\begin{aligned}
Y_\varphi^{(2)} &= 2\varphi' f\frac{\partial}{\partial f} + (2\varphi'' f + \varphi' f_1 + \varphi'' v_x f_2)\frac{\partial}{\partial f_1} + 3\varphi' f_2\frac{\partial}{\partial f_2} + (\varphi''' v_x f + \varphi'' v_x f_1 - \varphi' g_1 + \varphi'' v_x g_2)\frac{\partial}{\partial g_1}\\
&\quad + (\varphi'' f + \varphi'' v_x f_2 + \varphi' g_2)\frac{\partial}{\partial g_2} + (2\varphi''' f + 3\varphi'' f_1 + \varphi''' v_x f_2 + 2\varphi'' v_x f_{12})\frac{\partial}{\partial f_{11}}\\
&\quad + (3\varphi'' f_2 + 2\varphi' f_{12} + \varphi'' v_x f_{22})\frac{\partial}{\partial f_{12}} + 4\varphi' f_{22}\frac{\partial}{\partial f_{22}} + (\varphi''' f + \varphi''' v_x f_2 + \varphi'' f_1 + \varphi'' v_x f_{12}\\
&\quad + \varphi'' g_2 + \varphi'' v_x g_{22})\frac{\partial}{\partial g_{12}} + (2\varphi'' f_2 + \varphi'' v_x f_{22} + 2\varphi' g_{22})\frac{\partial}{\partial g_{22}}.
\end{aligned} \tag{66}$$

Since φ', φ'' and φ''' are independent functions, the operator (66) furnishes the following three independent operators:

$$Y_{\varphi'}^{(2)} = 2f\frac{\partial}{\partial f} + f_1\frac{\partial}{\partial f_1} + 3f_2\frac{\partial}{\partial f_2} - g_1\frac{\partial}{\partial g_1} + g_2\frac{\partial}{\partial g_2} + 2f_{12}\frac{\partial}{\partial f_{12}} + 4f_{22}\frac{\partial}{\partial f_{22}} + 2g_{22}\frac{\partial}{\partial g_{22}}, \tag{67}$$

$$\begin{aligned}
Y_{\varphi''}^{(2)} &= (2f + v_x f_2)\frac{\partial}{\partial f_1} + v_x(f_1 + g_2)\frac{\partial}{\partial g_1} + (f + v_x f_2)\frac{\partial}{\partial g_2} + (3f_1 + 2v_x f_{12})\frac{\partial}{\partial f_{11}}\\
&\quad + (3f_2 + v_x f_{22})\frac{\partial}{\partial f_{12}} + [f_1 + g_2 + v_x(f_{12} + g_{22})]\frac{\partial}{\partial g_{12}} + (2f_2 + v_x f_{22})\frac{\partial}{\partial g_{22}},
\end{aligned} \tag{68}$$

$$Y_{\varphi'''}^{(2)} = v_x f\frac{\partial}{\partial g_1} + (2f + v_x f_2)\frac{\partial}{\partial f_{11}} + (f + v_x f_2)\frac{\partial}{\partial g_{12}}. \tag{69}$$

The invariant test

$$Y_{\varphi'''}^{(2)}(J) = 0, \tag{70}$$

yields

$$\frac{\partial J}{\partial \lambda_5} - 2\frac{\partial J}{\partial \lambda_6} = 0. \tag{71}$$

Thus,

$$J = J(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \mu_0) \tag{72}$$

with

$$\mu_0 = 2\lambda_5 + \lambda_6 = \frac{ff_1(f_1 f_{22} - 2f_2 g_{22})}{(f_2)^4} - \frac{f(f_{11} - 2g_{12}) + f_2 g_1 - f_1 g_2}{(f_2)^2}. \tag{73}$$

The reckoning shows that invariant test

$$Y^{(2)}_{\varphi''}(J) = 0, \tag{74}$$

yields, i.e.

$$\frac{\partial J}{\partial \lambda_1} + (3 - 2\lambda_2)\frac{\partial J}{\partial \lambda_3} - \frac{\partial J}{\partial \lambda_4} - 4(\lambda_1 + \lambda_3 + \lambda_4)\frac{\partial J}{\partial \mu_0} = 0. \tag{75}$$

Hence,

$$J = J(\lambda_2, \mu_1, \mu_2, \mu_3), \tag{76}$$

where the invariants μ_1, μ_2 and μ_3 are given by

$$\mu_1 = \lambda_1 + \lambda_4 = \frac{f_2(g_2 - f_1) + f(f_{12} - g_{22})}{(f_2)^2}, \tag{77}$$

$$\mu_2 = \lambda_3 - (3 - 2\lambda_2)\lambda_1 = \frac{ff_{22}(2g_2 - f_1) - ff_2 f_{12} - 3(f_2)^2(g_2 - f_1)}{(f_2)^3}, \tag{78}$$

$$\begin{aligned} \mu_3 &= \mu_0 + 2(2\lambda_2 - 1)(\lambda_1)^2 + 4(\lambda_3 + \lambda_4)\lambda_1 \\ &= f\frac{f_1(f_1 f_{22} + 2f_2 g_{22}) + 4g_2[f_{22}(g_2 - f_1) - f_2 g_{22}]}{(f_2)^4} \\ &\quad - \frac{2[(f_1)^2 + (g_2)^2] + f(f_{11} - 2g_{12}) + f_2 g_1 - 5f_1 g_2}{(f_2)^2}. \end{aligned} \tag{79}$$

Finally, the invariant test

$$Y^{(2)}_{\varphi'}(J) = 0, \tag{80}$$

after tedious calculations yields

$$\mu_1\frac{\partial J}{\partial \mu_1} + \mu_2\frac{\partial J}{\partial \mu_2} + 2\mu_3\frac{\partial J}{\partial \mu_3} = 0. \tag{81}$$

Upon solving this equation, we arrive at the following result.

Theorem 2. *The general form of the second-order differential invariants of Eqs.* (8) *is*

$$J = J(\lambda, \mu, \nu), \tag{82}$$

where λ, μ *and* ν *are three independent invariants defined by*

$$\lambda = \lambda_2 = \frac{ff_{22}}{(f_2)^2}, \tag{83}$$

$$\mu = \frac{\mu_2}{\mu_1} = \frac{ff_{22}(2g_2 - f_1) - ff_2f_{12} - 3(f_2)^2(g_2 - f_1)}{f_2[f_2(g_2 - f_1) + f(f_{12} - g_{22})]}, \tag{84}$$

$$\nu = \frac{\mu_3}{(\mu_1)^2} = f\frac{f_1(f_1f_{22} + 2f_2g_{22}) + 4g_2[f_{22}(g_2 - f_1) - f_2g_{22}]}{[f_2(g_2 - f_1) + f(f_{12} - g_{22})]^2} - (f_2)^2\frac{2[(f_1)^2 + (g_2)^2] + f(f_{11} - 2g_{12}) + f_2g_1 - 5f_1g_2}{[f_2(g_2 - f_1) + f(f_{12} - g_{22})]^2}. \tag{85}$$

5. Some applications

Previous invariants can be used, for example, in order to identify subsets of equations (8) that remain unaltered under the action of the equivalence group $\mathscr{E}$. The following examples clarify this application.

1. Eq. (8) with $f_2 \equiv f_{v_x} = 0$ are semilinear equations of the form (45):

   $$v_{tt} = l(x)v_{xx} + g(x, v_x).$$

   Since the equation $f_2 = 0$ is invariant with respect to $\mathscr{E}$, any semilinear equation (45) is transformed by the equivalence group $\mathscr{E}$ into an equation of the same form.
2. Likewise, the subset of Eqs. (8) defined by the conditions $f_2 \neq 0$ and

   $$J^1 \equiv \frac{ff_{22}}{(f_2)^2} = 0$$

   is also invariant under $\mathscr{E}$. The equations of this subset have the form:

   $$v_{tt} = [k(x)v_x + l(x)]v_{xx} + g(x, v_x), \quad k(x) \neq 0.$$

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