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# A solution to the problem of invariants for parabolic equations

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ABSTRACT

The article is devoted to the solution of the invariants problem for the one-dimensional parabolic equations written in the two-coefficient canonical form used recently by N.H. Ibragimov:

$$u_t - u_{xx} + a(t,x)u_x + c(t,x)u = 0.$$

A simple invariant condition is obtained for determining all equations that are reducible to the heat equation by the general group of equivalence transformations.

The solution to the problem of invariants is given also in the one-coefficient canonical

$$u_t - u_{xx} + c(t,x)u = 0.$$

One of the main differences between these two canonical forms is that the equivalence group for the two-coefficient form contains the *arbitrary* linear transformation of the dependent variable whereas this group for the one-coefficient form contains only a *special type* of the linear transformations of the dependent variable.
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## 1. Two-coefficient representation of parabolic equations

Any one-dimensional linear homogeneous parabolic equation can be reduced by an appropriate change of the independent variables to the following form:

$$u_t - u_{xx} + a(t,x)u_x + c(t,x)u = 0. \tag{1}$$

This form has been used in [4] (see also Preprint in [3]) for obtaining a simple criteria for identifying the parabolic equations reducible to the heat equation by the *linear transformation of the dependent variable*, without changing the independent variables. The present paper is a continuation of the work [4] and contains, *inter alia*, a criteria (Theorem 1) for reducibility to the heat equation by the general group of equivalence transformations including changes of the independent variables.

### 1.1. Equivalence transformations

The equivalence algebra for Eq. (1) is spanned by the generator

$$Y_\sigma = \sigma u\frac{\partial}{\partial u} + 2\sigma_x\frac{\partial}{\partial a} + (\sigma_{xx} - \sigma_t - a\sigma_x)\frac{\partial}{\partial c} \tag{2}$$

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of the usual linear transformation of the dependent variable and by the generators

$$Y_\alpha = \alpha\frac{\partial}{\partial x} + \alpha'\frac{\partial}{\partial a}, \quad Y_\gamma = 2\gamma\frac{\partial}{\partial t} + \gamma' x\frac{\partial}{\partial x} + (x\gamma'' - a\gamma')\frac{\partial}{\partial a} - c\gamma'\frac{\partial}{\partial c} \tag{3}$$

of the corresponding transformations of the independent variables. Here $\gamma = \gamma(t)$, $\alpha = \alpha(t)$ and $\sigma = \sigma(t,x)$ are arbitrary functions, the prime denotes the differentiation with respect to $t$.

### 1.2. Invariants

The semi-invariant obtained in [2] is written for Eq. (1) as follows (see [3]):

$$K = aa_x - a_{xx} + a_t + 2c_x. \tag{4}$$

The problem of invariants reduces to calculation of invariants of the form

$$J = J(K, K_x, K_t, K_{tt}, K_{xt}, K_{xx}, \ldots) \tag{5}$$

for the operators (3). The generators (2) and (3) become

$$Y_\sigma = \sigma u\frac{\partial}{\partial u}, \; Y_\alpha = \alpha\frac{\partial}{\partial x} + \alpha''\frac{\partial}{\partial K}, \quad Y_\gamma = 2\gamma\frac{\partial}{\partial t} + \gamma' x\frac{\partial}{\partial x} + (x\gamma''' - 3\gamma' K)\frac{\partial}{\partial K} \tag{6}$$

We use the prolongations of the generator

$$X = \xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x} + \zeta\frac{\partial}{\partial u} + \zeta^{K_t}\frac{\partial}{K_t} + \zeta^{K_x}\frac{\partial}{K_x} + \zeta^{K_{tt}}\frac{\partial}{K_{tt}} + \zeta^{K_{xt}}\frac{\partial}{K_{xt}} + \zeta^{K_{xx}}\frac{\partial}{K_{xx}} + \zeta^{K_{ttt}}\frac{\partial}{K_{ttt}} + \zeta^{K_{xtt}}\partial_{K_{xtt}} + \zeta^{K_{xxt}}\frac{\partial}{K_{xxt}} + \zeta^{K_{xxx}}\partial_{K_{xxx}} + \cdots$$

and consider invariants of order $N$ (the maximal order of the derivatives involved in the invariant). It is better to use the notation

$$K_{kl} = \frac{\partial^{k+l} K}{\partial x^k \partial t^l}.$$

Then the form of the generator becomes

$$X = \xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x} + \zeta\frac{\partial}{\partial u} + \zeta^{K}\frac{\partial}{\partial K} + \zeta^{K_t}\frac{\partial}{K_t} + \zeta^{K_x}\partial_{K_x} + \sum_{k,l}^{k+l\leqslant N}\zeta^{K_{kl}}\frac{\partial}{K_{kl}}$$

Let us use the operator $Y_\alpha$. Its first three prolongations have the coefficients

$$\begin{aligned}
&\zeta^{K_t} = \alpha^{(3)} - K_x\alpha', \quad \zeta^{K_x} = 0,\\
&\zeta^{K_{tt}} = \alpha^{(4)} - K_x\alpha'' - 2K_{xt}\alpha', \quad \zeta^{K_{xt}} = -K_{xx}\alpha', \quad \zeta^{K_{xx}} = 0,\\
&\zeta^{K_{ttt}} = \alpha^{(5)} - K_x\alpha^{(3)} - 3K_{xt}\alpha'' - 3K_{xtt}\alpha',\\
&\zeta^{K_{xtt}} = -K_{xx}\alpha'' - 2K_{xxt}\alpha', \quad \zeta^{K_{xxt}} = -K_{xxx}\alpha', \quad \zeta^{K_{xxx}} = 0,
\end{aligned}$$

and other prolongations of $k$th order give

$$\zeta^{K_{k0}} = 0, \; \zeta^{K_{0k}} = \alpha^{(k+2)} + a_k\alpha^{(k)} + \cdots, \; \zeta^{K_{ls}} = b_s\alpha^{(s)} + \cdots, \; (l \geqslant 1, \; l + s = k)$$

with some functions $a_k$ and $b_s$ which do not depend on $\alpha$ and its derivatives.

Let us use the operator $Y_\gamma$. Then the first and the second prolongations are defined by the coefficients:

$$\begin{aligned}
&Y_\gamma = 2\gamma\frac{\partial}{\partial t} + \gamma' \, x\frac{\partial}{\partial x} + (x\gamma''' - 3\gamma' K)\frac{\partial}{\partial K}\\
&\zeta^{K_t} = x\gamma^{(4)} - 3\gamma'' K - 5\gamma' K_t - \gamma'' x K_x,\\
&\zeta^{K_x} = \gamma^{(3)} - 4\gamma' K_x,\\
&\zeta^{K_{tt}} = x\gamma^{(5)} - 3\gamma''' K - 8\gamma'' K_t - \gamma''' x K_x - 7\gamma' K_{tt} - 2\gamma'' x K_{xt},\\
&\zeta^{K_{xt}} = \gamma^{(4)} - 4\gamma'' K_x - 6\gamma' K_{xt} - \gamma'' x K_{xx},\\
&\zeta^{K_{xx}} = -5\gamma' K_{xx},
\end{aligned}$$

Notice that other prolongations give that

$$\zeta^{K_{s0}} = -(s+3)\gamma', \quad (s \geqslant 3).$$

Let us assume that an invariant $J$ does not depend on the derivatives $K_{kl}$, where $l \geqslant 1$. Then the equation $Y_\alpha J = 0$ is identically satisfied, and the equation $Y_\gamma J = 0$ becomes

$$-\gamma^{(3)}\frac{\partial J}{\partial K_x} + \gamma'\left(4K_x\frac{\partial J}{\partial K_x} + 5K_{xx}\frac{\partial J}{\partial K_{xx}} + \sum_{l\geqslant 3}(l+3)K_{l0}\frac{\partial J}{\partial K_{l0}}\right) = 0.$$

This gives that $\frac{\partial J}{\partial K_x} = 0$, and any solution of the equation

$$5K_{xx}\frac{\partial J}{\partial K_{xx}} + \sum_{l\geqslant 3}(l+3)K_{l0}\frac{\partial J}{\partial K_{l0}} = 0$$

gives an invariant. The characteristic system of this equation is

$$\frac{d\ K_{xx}}{5K_{xx}} = \frac{dK_{xxx}}{6K_{xxx}} = \frac{d\ K_{xxxx}}{7K_{xxxx}} = \cdots$$

Further we use the following statements that can be easily proved by using the above prolongations of the generators (6).

**Lemma.**

The equation

$$K_{xx} = 0 \tag{7}$$

is invariant under equivalence transformations with the generators (2) and (3).

**Proof.** Since the coefficients $\zeta^{K_{xx}}$ in the prolongations of the generators $Y_\sigma$ and $Y_\alpha$ vanish, the transformations corresponding to these operators do not change the derivative $K_{xx}$. Furthermore, the transformation of $K_{xx}$ corresponding to the generator $Y_\gamma$ is defined by two terms:

$$Y_\gamma = 2\gamma\frac{\partial}{\partial t}\zeta^{K_{xx}} - 5\gamma' K_{xx}\frac{\partial}{\partial K_{xx}}\cdots$$

Hence, the transformation corresponding to the generator $Y_\gamma$ just scales the derivative $K_{xx}$ with some function depending on $t$. This also does not change the property (7). This completes the proof.

**Theorem 1.** *Eq. (1) can be mapped to the heat equation by the general group of equivalence transformations with the generators (2) and (3) if and only if the semi-invariant $K$ of Eq. (1) satisfies the Eq. (7).*

**Proof.** According to (2) and (3), the equivalence transformations comprise the linear transformation of the dependent variable:

$$v = V(x,t)u \tag{8}$$

with an arbitrary function $V(x,t) \neq 0$ and the change of the independent variables

$$\tau = H(t), \quad y = x\ \varphi_1(t) + \varphi_0(t), \tag{9}$$

where $\varphi_1(t) \neq 0$ and $H(t)$ is defined by the equation

$$H'(t) = \varphi_1^2(t). \tag{10}$$

The requirement that the transformations (8) and (9) map the equation

$$v_\tau - v_{yy} + b_2(y,\tau)v_y + b_3(y,\tau)v = 0 \tag{11}$$

into the heat equation

$$u_t - u_{xx} = 0 \tag{12}$$

yields the following equations:

$$\begin{aligned} &2H'V_x\varphi_1 + (x\varphi_1' + \varphi_0' - H'b_2)\varphi_1^2 V = 0,\\ &H'V_{xx}\varphi_1 - H'V_x\varphi_1^2 b_2 - H'\varphi_1^3 b_3 V - V_t\varphi_1^3 + (x\varphi_1' + \varphi_0')V_x\varphi_1^2 = 0. \end{aligned} \tag{13}$$

The problem is to find the conditions for the coefficients $b_2(\tau,y)$, $b_3(\tau,y)$ that guarantee existence of the functions $V(t,x), H(t), \varphi_0(t)$ and $\varphi_1(t)$. To solve this problem, we have to investigate the compatibility of the overdetermined system of partial differential Eq. (13). To this end, we note that Eq. (13) yield

$$\begin{aligned} &V_x = \tfrac{\varphi_1 V}{2H'}(-\varphi_0' - \varphi_1' x + H'b_2),\\ &V_t = \tfrac{V}{4H'\varphi_1}[-{\varphi_1'}^2\varphi_1 x^2 + 2\varphi_1'\varphi_1 x(H'b_2 - \varphi_0') + \varphi_1({H'}^2(2b_{2y} - b_2^2 - 4b_3) - {\varphi_0'}^2 + 2\varphi_0' H' b_2) - 2\varphi_1' H']. \end{aligned}$$

Equating the mixed derivatives, $(V_t)_x = (V_x)_t$, we find

$${H'}^3 K = \varphi_0'' H' - \varphi_0' H'' + x(\varphi_1'' H' - \varphi_1' H'') = 0$$

or

$$K = \varphi_1^{-6}\left(y(\varphi_1''\varphi_1 - 2\varphi_1'^2) + \varphi_0''\varphi_1^2 - 2\varphi_0'\varphi_1'\varphi_1 - \varphi_1''\varphi_0\varphi_1 + 2{\varphi_1'}^2\varphi_0\right).$$

Thus, the semi-invariant $K$ of the equations that are equivalent to the heat equation has to be linear with respect to $y$, i.e. satisfy the equation $K_{yy} = 0$.

Conversely, let us assume that the condition $K_{yy} = 0$ is satisfied for Eq. (11), i.e.

$$K = y\phi_1(\tau) + \phi_0(\tau). \tag{14}$$

Choosing $H(t), \varphi_0(t)$ and $\varphi_1(t)$ satisfying the Eq. (9) and

$$\varphi_1'' = 2\frac{\varphi_1'^2}{\varphi_1} + \varphi_1^5\phi_1, \quad \varphi_0'' = 2\varphi_0'\frac{\varphi_1'}{\varphi_1} + \varphi_1^4(\phi_1\varphi_0 + \phi_0) \tag{15}$$

one can transform Eq. (11) into the heat equation. This completes the proof.

**Example.** Consider the equation

$$v_\tau - v_{yy} + \frac{y}{3\tau}v_y = 0. \tag{16}$$

It has the form (11) with

$$b_2 = \frac{y}{3\tau}, \quad b_3 = 0.$$

The semi-invariant (4) for Eq. (16) is

$$K = -\frac{2y}{9\tau^2}\cdot$$

It has the form (14) with

$$\phi_1 = -\frac{2}{9\tau^2}, \quad \phi_2 = 0. \tag{17}$$

Let us investigate reducibility of Eq. (16) to the heat Eq. (12) by the change of the independent variables (9) without changing the dependent variable, i.e. by letting $V = 1$ in (8). Then Eq. (13) reduce to one equation

$$x\varphi_1' + \varphi_0' - H'b_2 = 0.$$

Substituting here the expression of $b_2$ and using Eqs. (9) and (10), we obtain:

$$x\varphi_1'(t) + \varphi_0'(t) - \varphi_1^2\frac{x\varphi_1(t) + \varphi_0(t)}{3H(t)} = 0.$$

Upon separating the variables, this equations splits into two equations:

$$\varphi_1' - \frac{\varphi_1^3}{3H} = 0$$

and

$$\varphi_0' - \frac{\varphi_0\varphi_1^2}{3H} = 0.$$

We rewrite the first equation in the form

$$H(t) = \frac{\varphi_1^3(t)}{3\varphi_1'(t)} \tag{18}$$

and satisfy the second equation by letting $\varphi_0 = 0$. Differentiating (18) and using (10), we obtain

$$\frac{\varphi_1^3}{3\varphi_1'}\varphi_1'' = 0,$$

whence $\varphi_1'' = 0$. Thus, $\varphi_1(t) = pt + q$ We take for the simplicity $q = 0$ and obtain

$$\varphi_1(t) = pt, \quad p = \text{const}.$$

Then Eqs. (18), (9) and (17) yield:

$$\tau = H(t) = \frac{p^2t^3}{3}, \quad \phi_1 = -\frac{2}{p^4t^6}\cdot$$

Now one can readily verify that Eq. (15) are satisfied for arbitrary $p$. We take $p = 1$ and obtain the following change of variables transformation Eq. (16) to the heat equation:

$$\tau = \frac{1}{3}t^3, \quad y = tx, \quad v = u. \tag{19}$$

**Remark.** In the case of three-coefficient representation of parabolic equations the test for reduction to the heat equation is more complicated, see, e.g. [5].

Further we will consider Eq. (1) that are not equivalent to the heat equation. In other words, taking into account Lemma 2, we will assume that $K_{xx} \neq 0$. This assumption allows us to obtain the invariants

$$K_{l0}K_{xx}^{-(l+3)/5}, \quad (l \geqslant 3). \tag{20}$$

The reckoning shows the following operator is an invariant differentiation:

$$\mathscr{D} = \frac{1}{K_{xx}^{1/5}} D_x \tag{21}$$

Moreover, we can easily see that applying the invariant differentiation (21) to the invariant corresponding to $l = 3$:

$$J_* = \frac{K_{xxx}}{K_{xx}^{6/5}}, \tag{22}$$

we obtain all invariants (20). Hence (22), provides a basis of these invariants.

**Theorem 2.** *The operator (21), $\mathscr{D} = K_{xx}^{-1/5}D_x$, is an invariant differentiation.*

**Proof.** Let $J$ be a differential invariant, i.e.

$$XJ = 0.$$

Recall that the coefficients of the generator $X$ are

$$\xi = \alpha + \gamma' x, \quad \eta = 2\gamma, \quad \zeta^{K_{xx}} = -5\gamma' K_{xx}.$$

Notice also that

$$X(K_{xx}^{-1/5}) - K_{xx}^{-1/5}D_x\xi = 0. \tag{23}$$

Using the identity (see [6])

$$D_x(XF) = X(D_xF) + D_x\xi D_xF + D_x\eta D_tF$$

valid for any function $F$ one has

$$X(K_{xx}^{-1/5}D_xJ) = (D_xJ)X(K_{xx}^{-1/5}) + K_{xx}^{-1/5}(D_x(XJ) - D_x\xi D_xJ - D_x\eta D_tJ). \tag{24}$$

Note that $XJ = 0$ because $J$ is an invariant, and $D_x\eta = 0$ because $\eta = 2\gamma(t)$. Therefore Eq. (24) becomes

$$X(K_{xx}^{-1/5}D_xJ) = (X(K_{xx}^{-1/5}) - K_{xx}^{-1/5}D_x\xi)D_xJ.$$

Now we use Eq. (23) and obtain the proof of the theorem:

$$X(K_{xx}^{-1/5}D_x(J)) = 0.$$

### 1.3. Fourth-order invariants

Splitting the equations $Y_\alpha J = 0$ and $Y_\gamma J = 0$ with respect to the functions $\alpha$, $\gamma$, and their derivatives, one obtains that[1]

$$J(K, K_x, K_{xx}, K_{xxx}, K_{xxt}, K_{xxxx}, K_{xxxt}, K_{xxtt})$$

and these equations are

$$\frac{\partial J}{\partial K} - K_{xxx}\frac{\partial J}{\partial K_{xxtt}} = 0, \quad \frac{\partial J}{\partial K_x} - 5K_{xx}\frac{\partial J}{\partial K_{xxtt}} = 0,$$
$$7K_{xxxx}\frac{\partial J}{\partial K_{xxxx}} + 6K_{xxx}\frac{\partial J}{\partial K_{xxx}} + 5K_{xx}\frac{\partial J}{\partial K_{xx}} + 4K_x\frac{\partial J}{\partial K_x} + 3K\frac{\partial J}{\partial K} + 8K_{xxxt}\frac{\partial J}{\partial K_{xxxt}} + 7K_{xxt}\frac{\partial J}{\partial K_{xxt}} + 9K_{xxtt}\frac{\partial J}{\partial K_{xxtt}} = 0,$$
$$6K_{xxx}\frac{\partial J}{\partial K_{xxxt}} + 12K_{xxt}\frac{\partial J}{\partial K_{xxtt}} + 5K_{xx}\frac{\partial J}{\partial K_{xxt}} = 0, \quad K_{xxxx}\frac{\partial J}{\partial K_{xxxt}} + 2K_{xxxt}\frac{\partial J}{\partial K_{xxtt}} + K_{xxx}\frac{\partial J}{\partial K_{xxt}} = 0.$$

This system of equations is a complete system. Solving the first four equations, this system becomes

$$(5J_2 - 6J_1^2)\frac{\partial J}{\partial z_2} + 10z_2\frac{\partial J}{\partial z_1} = 0,$$

[1] Here the order is the maximal order of derivatives of $K$ involved in the invariant.

where $J(J_1, J_2, z_1, z_2)$ and

$$J_1 = K_{xx}^{-6/5} K_{xxx}, \quad J_2 = K_{xx}^{-7/5} K_{xxxx}, \quad z_1 = K_{xx}^{-14/5}\left(K_{xx}(K_{xxtt} + KK_{xxx} + 5K_x K_{xx}) - \frac{6}{5}K_{xxt}^2\right), \quad z_2 = K_{xx}^{-13/5}\left(K_{xx}K_{xxxt} - \frac{6}{5}K_{xxt}K_{xxx}\right).$$

Notice that

$$\left(5J_2 - 6J_1^2\right) = K_{xx}^{-12/5}\left(5K_{xx}K_{xxxx} - 6K_{xxx}^2\right).$$

If it is assumed that

$$5K_{xxxx}K_{xx} - 6(K_{xxx})^2 \neq 0, \qquad (25)$$

then the invariants are

$$J_1, J_2, \left(5J_2 - 6J_1^2\right)z_1 - 5z_2^2.$$

If one assumes that

$$5K_{xxxx}K_{xx} - 6(K_{xxx})^2 = 0, \qquad (26)$$

then the invariants are defined by the equation

$$z_2 \frac{\partial J}{\partial z_1} = 0.$$

This means that: (a) if $z_2 = 0$, then the invariants are

$$J_1, J_2, z_1;$$

(b) if $z_2 \neq 0$, then the invariants are

$$J_1, J_2, z_2.$$

The invariants which do not depend on any conditions are only the invariants (20). Notice that the properties (25), (26) and $z_2 = 0$ or $z_2 \neq 0$ are invariant.

The obtained result confirms solutions obtained in [7] for invariants with three coefficients in a linear parabolic equation.

## 2. One-coefficient representation of parabolic equations

One can transform Eq. (1) to the equation (this representation was already known in the classical literature, see, e.g. [1])

$$u_t - u_{xx} + c(t,x)u = 0 \qquad (27)$$

with one-coefficient $c = c(x,t)$ by using the linear transformation

$$v = ue^{\rho(x,t)}$$

of the dependent variable, where $\rho(t,x)$ is defined by (see Eq. (3.7) in [3])

$$2\rho + a(x,t) = 0.$$

### 2.1. Equivalence transformations

The equivalence group for Eq. (27) is defined by the generator

$$8\gamma\frac{\partial}{\partial t} + 4(\gamma' x + 2\alpha)\frac{\partial}{\partial x} + u(2\beta - \gamma'' x^2 - 4\alpha' x)\frac{\partial}{\partial u} + (\gamma''' x^2 - 2\gamma'' + 4\alpha'' x - 2\beta' - 8\gamma' c)\frac{\partial}{\partial c}$$

spanned by the following three generators:

$$\begin{aligned} X_3 &= 8\gamma\tfrac{\partial}{\partial t} + 4\gamma' x\tfrac{\partial}{\partial x} - u\gamma'' x^2\tfrac{\partial}{\partial u} + (\gamma''' x^2 - 2\gamma'' - 8\gamma' c)\tfrac{\partial}{\partial c}, \\ X_2 &= 2\alpha\tfrac{\partial}{\partial x} - u\alpha' x\tfrac{\partial}{\partial u} + \alpha'' x\tfrac{\partial}{\partial c}, \\ X_1 &= \beta u\tfrac{\partial}{\partial u} - \beta'\tfrac{\partial}{\partial c}, \end{aligned} \qquad (28)$$

where $\gamma$, $\alpha$ and $\beta$ are arbitrary functions of $t$.

We use the prolongations of the generator

$$X = \xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x} + \zeta\frac{\partial}{\partial u} + \zeta^c\frac{\partial}{\partial c} + \zeta^{c_t}\frac{\partial}{\partial c_t} + \zeta^{c_x}\frac{\partial}{\partial c_x} + \zeta^{c_{tt}}\frac{\partial}{\partial c_{tt}} + \zeta^{c_{xt}}\frac{\partial}{\partial c_{xt}} + \zeta^{c_{xx}}\frac{\partial}{\partial c_{xx}} + \zeta^{c_{ttt}}\frac{\partial}{\partial c_{ttt}} + \zeta^{c_{xtt}}\frac{\partial}{\partial c_{xtt}} + \zeta^{c_{xxt}}\frac{\partial}{\partial c_{xxt}} + \zeta^{c_{xxx}}\frac{\partial}{\partial c_{xxx}} + \cdots$$

It is better to use the notation

$$c_{kl} = \frac{\partial^{k+l}c}{\partial x^k \partial t^l}.$$

Then the form of the generator becomes

$$X = \xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x} + \zeta\frac{\partial}{\partial u} + \zeta^c\frac{\partial}{\partial c} + \zeta^{c_t}\frac{\partial}{\partial c_t} + \zeta^{c_x}\frac{\partial}{\partial c_x} + \sum_{k,l}\zeta^{c_{kl}}\frac{\partial}{\partial c_{kl}}.$$

### *2.2. Invariants*

Let us use the operator $X_1$. Then the first prolongation is defined by the coefficients

$$\zeta^{c_t} = -\beta^{(2)}, \quad \zeta^{c_x} = 0,$$

and other prolongations of $k$th order give

$$\zeta^{c_{0k}} = -\beta^{(k+1)}, \quad \zeta^{c_{ls}} = 0, \quad (l \geqslant 1,\ l+s=k).$$

Splitting with respect to the derivatives of the function $\beta(t)$ the equation for invariants

$$X_1^{(k)}J = 0,$$

we immediately come to the conditions that the invariant $J$ does not depend on the derivatives

$$c_{0k},\ (k \geqslant 1).$$

Hence, further the coefficients $\zeta^{c_{0k}}$ are not necessary to be calculated. Splitting with respect to $\beta'$ also excludes $c$ from the invariant $J$. Let us use the operator $X_2$. Then the first and the second prolongations are defined by the coefficients:

$$\zeta^{c_x} = \alpha'', \quad \zeta^{c_{xt}} = \alpha^{(3)} - 2\alpha' c_{xx}, \quad \zeta^{c_{xx}} = 0, \quad \zeta^{c_{xtt}} = \alpha^{(4)} - 2\alpha'' c_{xx} - 4\alpha' c_{xxt}, \quad \zeta^{c_{xxt}} = -2\alpha' c_{xxx}, \quad \zeta^{c_{xxx}} = 0.$$

In the $k$th prolongation the maximal order of the derivative $\alpha^{(k+1)}$ will be in the coefficient $\zeta^{c_{(k-1)1}}$. Hence, the invariant $J$ does not depend on the derivatives

$$c_{(k-2)1},\ c_{(k-1)1}.$$

If one assumes that the invariant $J$ does not depend on

$$c_{sl},\ (l \geqslant 1,\ s+l=k \geqslant 2),$$

then the term with $\alpha''$ in the equation $X_2 J = 0$ gives $\frac{\partial J}{\partial c_x} = 0$. Hence, the invariant can only depend on

$$c_{xx},\ c_{xxx},\ c_{xxxx}, \ldots, c_{k0}.$$

Let us use the operator $X_3$. The prolongations are defined by the coefficients:

$$\begin{aligned}
&\zeta^{c_x} = 2x\gamma''' - 12\gamma' c_x, \quad \zeta^{c_t} = x^2\gamma^{(4)} - 2\gamma^{(3)} - 4\gamma'' x c_x - 8\gamma'' c - 16\gamma' c_t,\\
&\zeta^{c_{xx}} = 2\gamma''' - 16\gamma' c_{xx}, \quad \zeta^{c_{xt}} = 2x\gamma^{(4)} - 4\gamma'' x c_{xx} - 12\gamma'' c_x - 20\gamma' c_{xt},\\
&\zeta^{c_{30}} = -20\gamma' c_{xxx}, \quad \zeta^{c_{40}} = -24\gamma' c_{40}, \quad \zeta^{c_{50}} = -28\gamma' c_{50}, \ldots
\end{aligned}$$

Splitting the equation $X_3 J = 0$ with respect to the derivative $\gamma'''$, one obtains $\frac{\partial J}{\partial c_{xx}} = 0$. The equation $X_3 J = 0$ becomes

$$5c_{xxx}\frac{\partial J}{\partial c_{xxx}} + \sum_{k \geqslant 4}(k+2)c_{k0}\frac{\partial J}{\partial c_{k0}} = 0.$$

For solving this equation one needs to solve the characteristic system of equations

$$\frac{dc_{xxx}}{5c_{xxx}} = \frac{dc_{xxxx}}{6c_{xxxx}} = \frac{dc_{xxxxx}}{7c_{xxxxx}} = \cdots$$

Note that the Eq. (27) with the coefficient $c(t,x)$ satisfying the condition $c_{xxx} = 0$ are equivalent to the heat equation. Hence, for the equations that are not equivalent to the heat equation all invariants not depending on $c_{kl}$ $(l \geqslant 1)$ are given by

$$c_{k0}c_{30}^{-(k+2)/5}, \quad k \geqslant 4. \tag{29}$$

Thus, the basis of differential invariants (29) consists of the invariant

$$J = \frac{c_{xxxx}}{(c_{xxx})^{6/5}} \tag{30}$$

and the operator of invariant differentiation is

$$\mathscr{D} = c_{xxx}^{-1/5} D_x. \tag{31}$$

Proceeding as in the case of two-coefficients, we arrive at the following statement.

**Theorem 3.** *The operator $\mathscr{D}$ given by (31) is an operator of invariant differentiation. Hence, all invariants (29) are obtained from the invariant (30) by the invariant differentiation (31).*

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