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Linearization of fourth-order ordinary differential equations by point transformations

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Abstract
The solution of the problem on linearization of fourth-order equations by means of point transformations is presented here. We show that all fourth-order equations that are linearizable by point transformations are contained in the class of equations which is linear in the third-order derivative. We provide the linearization test and describe the procedure for obtaining the linearizing transformations as well as the linearized equation. For ordinary differential equations of order greater than 4 we obtain necessary conditions, which separate all linearizable equations into two classes.

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1. Introduction

Almost all important governing equations in physics take the form of nonlinear differential equations, and, in general, are very difficult to solve explicitly. While solving problems related to nonlinear ordinary differential equations it is often expedient to simplify equations by a suitable change of variables. The simplest form of a differential equation is a linear form. The reduction of an ordinary differential equation to a linear ordinary differential equation besides simplification allows constructing an exact solution of the original equation. Analytical (explicit) solution has value, firstly, as an exact description of a real process in the framework of a given model; secondly, as a model to compare various numerical methods; thirdly, as a basis to improve the models used. Therefore, the linearization problem plays a significant role in the nonlinear problem. The linearization problem can be stated as follows: find a change of variables such that a transformed equation becomes a linear equation. If a change of variables includes derivatives, this change is called a tangent transformation. The present paper studies the linearization problem by using point transformations, where the change of variables only depends on the independent and dependent variables. Point transformations are the simplest type of transformations compared with tangent transformations.

The problem of linearization of ordinary differential equations has a long history. It attracted the attention of mathematicians like S Lie and E Cartan. The first solution of the linearization problem of a second-order ordinary differential equation was given by S Lie [1]. E Cartan tackled this problem by using differential forms. An application of symbolic computer calculations allows solving the linearization problem of order greater than 2. In a series of articles[3] [4–8] the linearization problem of a third-order ordinary differential equation with respect to point, contact and generalized Sundman transformations was considered and solved. The main difficulty in solving the linearization problem is related to voluminous and extremely complicated calculations. Because of this difficulty there are only a few attempts to solve this problem for orders higher than 3 even in a particular case, where a transformation can map a studied equation into the trivial equation $y^{(4)} = 0$. These attempts were made recently and they only considered contact transformations [9] and knowledge of an admitted Lie group [10]. It is worth noting that the application of contact transformations is more complicated than the application of point transformations. Moreover, as shown in [6, 7] for third-order ordinary differential equations, two sets (the set of equations linearizable by contact transformations and the set of equations linearizable by point transformations) complement each other. This is one of the motivations for studying the linearization problem by point transformations.

Another motivation of the study of fourth-order ordinary differential equations is as follows. Many systems of two second-order ordinary differential equations[4] can be reduced to a fourth-order ordinary differential equation. Hence, the linearization criteria obtained for fourth-order ordinary differential equations can also be applied to such type of systems.

One can continue studying the linearization problem increasing the order of equations. The case of fourth-order ordinary differential equations is on the boarder line of applications of point transformations for the linearization problem. The study of fourth-order equations allowed us to develop the method for obtaining necessary conditions of linearization of ordinary differential equations of any order greater than 4. These conditions separate all linearizable ordinary differential equations into two classes.

The present paper is devoted to obtaining complete criteria for fourth-order ordinary differential equations to be linearizable by a change of the dependent and independent variables. Intermediate calculations are obtained using computer algebra system Reduce [13]. For ordinary differential equations of order greater than 4 necessary conditions were obtained. The final results were checked by comparing with theoretical results on invariants as well as by applying to numerous known and new examples of linearization.

The paper is organized as follows. In section 2, the necessary conditions of linearization of an ordinary differential equation of order greater than 3 are presented. The next sections of the paper consider fourth-order ordinary differential equations. In section 3, we discuss the main results of the first class. We state the theorem that yields criteria for a fourth-order ordinary differential equation to be linearizable via a point transformation. Relations between coefficients of a linearizable equation and point transformations reducing this equation into a linear equation are presented in this section. These relations are necessary for proving the linearization theorem. The proof of this theorem, application of the linearization theorem

[3] A historical review can be found in [2], recent references can be seen in [3].

[4] A short review of results of solving the linearization problem for a system of two second-order ordinary differential equations can be found, for example, in [11, 12].

to one class of systems with two second-order ordinary differential equations and illustrative examples are provided in the subsequent subsections. The main results of the second class are studied in section 4 in a similar manner. For the sake of simplicity of reading, cumbersome formulae from this section are moved into the appendix.

2. Necessary conditions of linearization

Here we consider the *i*th-order ordinary differential equation

$$y^{(i)} = F(x, y, y', y'', \ldots, y^{(i-1)}). \tag{1}$$

We apply a point transformation

$$t = \varphi(x, y), \qquad u = \psi(x, y) \tag{2}$$

to equation (1).

We begin with investigating the necessary conditions for linearization. The general form of equation (1) that can be obtained from a linear ordinary differential equation by any point transformation (2) is found on this step. Necessary conditions for a linearizable fourth-order ordinary differential equation are studied here in more detail.

2.1. Necessary form of a linearizable ith-order ordinary differential equation

In 1879, E Laguerre showed that in the linear ordinary differential equation of order $i > 3$ the two terms of orders next below the highest can be simultaneously removed by an equivalence transformation. Therefore, the general linear *i*th-order ordinary differential equation in Laguerre's form is

$$u^{(i)} + \alpha_{i-3}(t)u^{(i-3)} + \cdots + \alpha_0(t)u = 0, \tag{3}$$

where t and u are the independent and dependent variables, respectively.

Applying a point transformation (2), the derivatives are changed as follows:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \psi_1 = \frac{D\psi}{D\varphi}, \qquad \frac{\mathrm{d}^2u}{\mathrm{d}t^2} = \psi_2 = \frac{D\psi_1}{D\varphi} = \frac{D^2\psi D\varphi - D^2\varphi D\psi}{(D\varphi)^3}, \ldots$$
$$\frac{\mathrm{d}^{k+1}u}{\mathrm{d}t^{k+1}} = \frac{D\psi_k}{D\varphi}, \qquad (k > 1),$$

where

$$D = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + y'''\frac{\partial}{\partial y''} + y^{(4)}\frac{\partial}{\partial y'''} + \cdots$$

is the operator of total derivative with respect to x. Note that

$$D^k f = y^{(k)} f_y + k y^{(k-1)} D f_y + h_f(x, y, y', \ldots, y^{(k-3)}, y^{(k-2)}), \qquad (k > 2),$$
$$\psi_i = \frac{1}{(D\varphi)^i}\left[D^i\psi - \frac{i(i-1)}{2}(D^2\varphi)(D\varphi)^{i-2}\psi_{i-1} - i(D^{i-1}\varphi)(D\varphi)\psi_2 - (D^i\varphi)\psi_1\right] + \cdots,$$
$$\psi_{i-1} = \frac{\Delta}{(D\varphi)^{i+1}} y^{(i-1)} + \cdots,$$

where $\Delta = \varphi_x\psi_y - \varphi_y\psi_x \neq 0$ is the Jacobian of the change of variables (2), $f = f(x, y)$ is an arbitrary function, and $i > 3$. Here ... means terms with derivatives of order less than

$i-1$. Hence,

$$(D\varphi)^i\psi_i = y^{(i)}\frac{\Delta}{D\varphi} + iy^{(i-1)}\left[D\psi_y - \frac{D\psi}{D\varphi}D\varphi_y - \frac{(i-1)}{2}(D^2\varphi)\frac{\Delta}{(D\varphi)^2} - \varphi_y\frac{D^2\psi D\varphi - D^2\varphi D\psi}{(D\varphi)^2}\right] + \cdots. \quad (4)$$

Calculations show that on the right-hand side of equation (4) the term with the derivative $y^{(i-1)}$ is

$$(D\varphi)^2\left[D\psi_y - \frac{D\psi}{D\varphi}D\varphi_y - \frac{(i-1)}{2}(D^2\varphi)\frac{\Delta}{(D\varphi)^2} - \varphi_y\frac{D^2\psi D\varphi - D^2\varphi D\psi}{(D\varphi)^2}\right]$$
$$= -y''\varphi_y\frac{(i+1)\Delta}{2} + y'^2\left(\varphi_{xy}\varphi_y\psi_y - \varphi_{yy}\frac{i\Delta}{2} - \varphi_{yy}\frac{(\varphi_x\psi_y + \varphi_y\psi_x)}{2} - \psi_{xy}\varphi_y^2 + \psi_{yy}\varphi_x\varphi_y\right) + y'\left(-\varphi_{xy}i\Delta + \varphi_{xx}\varphi_y\psi_y - \varphi_{yy}\varphi_x\psi_x - \psi_{xx}\varphi_y^2 + \psi_{yy}\varphi_x^2\right)$$
$$- \varphi_{xy}\varphi_x\psi_x - \varphi_{xx}\frac{i\Delta}{2} + \varphi_{xx}\frac{(\varphi_x\psi_y + \varphi_y\psi_x)}{2} + \psi_{xy}\varphi_x^2 - \psi_{xx}\varphi_x\varphi_y.$$

Thus, the necessary form of a linearizable ordinary differential equation of *i*th order is

$$y^{(i)} + iy^{(i-1)}\frac{1}{\Delta D\varphi}\left[-y''\varphi_y\frac{(i+1)\Delta}{2} + y'^2\left(\varphi_{xy}\varphi_y\psi_y - \varphi_{yy}\frac{i\Delta}{2} - \varphi_{yy}\frac{(\varphi_x\psi_y + \varphi_y\psi_x)}{2} - \psi_{xy}\varphi_y^2 + \psi_{yy}\varphi_x\varphi_y\right)\right.$$
$$+ y'\left(-\varphi_{xy}i\Delta + \varphi_{xx}\varphi_y\psi_y - \varphi_{yy}\varphi_x\psi_x - \psi_{xx}\varphi_y^2 + \psi_{yy}\varphi_x^2\right)$$
$$\left.- \varphi_{xy}\varphi_x\psi_x - \varphi_{xx}\frac{i\Delta}{2} + \varphi_{xx}\frac{(\varphi_x\psi_y + \varphi_y\psi_x)}{2} + \psi_{xy}\varphi_x^2 - \psi_{xx}\varphi_x\varphi_y\right] + \cdots = 0.$$

From this representation we can conclude that for the linearization problem one needs to study two cases: (a) $\varphi_y = 0$, and (b) $\varphi_y \neq 0$. This corresponds to the following two necessary forms of linearizable ordinary differential equations:

$$y^{(i)} + y^{(i-1)}[A_1y' + A_0] + \cdots = 0, \quad (5)$$

and

$$y^{(i)} + y^{(i-1)}\frac{1}{y'+r}\left[-y''\frac{i(i+1)}{2} + F_2y'^2 + F_1y' + F_0\right] + \cdots = 0, \quad (6)$$

where $F_j = F_j(x, y)$, $A_j = A_j(x, y)$. If $\varphi_y = 0$, in the literature this class of transformations is called a fiber-preserving transformation.

Theorem 1. *Any linearizable ith order* ($i \geqslant 4$) *ordinary differential equation has to be one of the forms either equation (5) or equation (6).*

2.2. The necessary form of a linearizable fourth-order ordinary differential equation

As was obtained in the previous subsection, the transformations (2) with $\varphi_y = 0$ and $\varphi_y \neq 0$, respectively, provide two distinctly different classes for the linearization.

If $\varphi_y = 0$ we work out the missing terms in equation (5), substitute the resulting expression into the linear equation

$$u^{(4)} + \alpha(t)u' + \beta(t)u = 0, \quad (7)$$

and obtain the following first class for linearization:

$$y^{(4)} + (A_1y' + A_0)y''' + B_0y''^2 + (C_2y'^2 + C_1y' + C_0)y'' + D_4y'^4 + D_3y'^3 + D_2y'^2 + D_1y' + D_0 = 0, \tag{8}$$

where $A_j = A_j(x, y)$, $B_j = B_j(x, y)$, $C_j = C_j(x, y)$ and $D_j = D_j(x, y)$ are arbitrary functions of x, y.

If $\varphi_y \neq 0$, we proceed likewise and setting $r = \frac{\varphi_x}{\varphi_y}$, arrive at the second class for linearization

$$y^{(4)} + \frac{1}{y'+r}(-10y'' + F_2y'^2 + F_1y' + F_0)y''' + \frac{1}{(y'+r)^2}[15y''^3 + (H_2y'^2 + H_1y' + H_0)y''^2 + (J_4y'^4 + J_3y'^3 + J_2y'^2 + J_1y' + J_0)y'' + K_7y'^7 + K_6y'^6 + K_5y'^5 + K_4y'^4 + K_3y'^3 + K_2y'^2 + K_1y' + K_0] = 0, \tag{9}$$

where $r = r(x, y)$, $F_j = F_j(x, y)$, $H_j = H_j(x, y)$, $J_j = J_j(x, y)$ and $K_j = K_j(x, y)$ are arbitrary functions of x, y.

Thus, we showed every linearizable fourth-order ordinary differential equation belongs either to the class of equations (8) or to the class of equations (9).

3. The first class of linearizable equations

3.1. The linearization test for equation (8)

In this case, the linearizing transformation (2) must be a fiber-preserving transformation, i.e., it has the form

$$t = \varphi(x), \qquad u = \psi(x, y). \tag{10}$$

Theorem 2. *Equation (8) is linearizable if and only if its coefficients obey the following equations:*

$$A_{0y} - A_{1x} = 0, \tag{11}$$

$$4B_0 - 3A_1 = 0, \qquad 12A_{1y} + 3A_1^2 - 8C_2 = 0, \tag{12}$$

$$12A_{1x} + 3A_0A_1 - 4C_1 = 0, \tag{13}$$

$$32C_{0y} + 12A_{0x}A_1 - 16C_{1x} + 3A_0^2A_1 - 4A_0C_1 = 0, \tag{14}$$

$$4C_{2y} + A_1C_2 - 24D_4 = 0, \qquad 4C_{1y} + A_1C_1 - 12D_3 = 0, \tag{15}$$

$$16C_{1x} - 12A_{0x}A_1 - 3A_0^2A_1 + 4A_0C_1 + 8A_1C_0 - 32D_2 = 0, \tag{16}$$

$$192D_{2x} + 36A_{0x}A_0A_1 - 48A_{0x}C_1 - 48C_{0x}A_1 - 288D_{1y} + 9A_0^3A_1 - 12A_0^2C_1 - 36A_0A_1C_0 + 48A_0D_2 + 32C_0C_1 = 0, \tag{17}$$

$$384D_{1xy} - \big[3\big((3A_0A_1 - 4C_1)A_0^2 + 16(2A_1D_1 + C_0C_1) - 16(A_1C_0 - D_2)A_0\big)A_0 - 32\big(4(C_1D_1 - 2C_2D_0 + C_0D_2) + \big(3A_1D_0 - C_0^2\big)A_1\big) - 96D_{1y}A_0 + 384D_{0y}A_1 + 1536D_{0yy} - 16(3A_0A_1 - 4C_1)C_{0x} + 12((3A_0A_1 - 4C_1)A_0 - 4(A_1C_0 - 4D_2))A_{0x}\big] = 0. \tag{18}$$

Corollary 1. *Provided that the conditions (11)–(18) are satisfied, the linearizing transformation (10) is defined by a fourth-order ordinary differential equation for the function* $\varphi(x)$*, namely by the Riccati equation*

$$40\frac{\mathrm{d}\chi}{\mathrm{d}x} - 20\chi^2 = 8C_0 - 3A_0^2 - 12A_{0x}, \tag{19}$$

for

$$\chi = \frac{\varphi_{xx}}{\varphi_x}, \tag{20}$$

and by the following integrable system of partial differential equations for the function $\psi(x, y)$*,*

$$4\psi_{yy} = \psi_y A_1, \qquad 4\psi_{xy} = \psi_y(A_0 + 6\chi), \tag{21}$$

and

$$\begin{aligned} 1600\psi_{xxxx} = {} & 9600\psi_{xxx}\chi + 160\psi_{xx}\left(-12A_{0x} - 3A_0^2 - 90\chi^2 + 8C_0\right) \\ & + 40\psi_x\left(12A_{0x}A_0 + 72A_{0x}\chi - 16C_{0x} + 3A_0^3 + 18A_0^2\chi - 12A_0C_0 \right. \\ & \left. + 120\chi^3 - 48\chi C_0 + 24D_1 - 8\Omega\right) + \psi\left(144A_{0x}^2 + 72A_{0x}A_0^2 - 352A_{0x}C_0 \right. \\ & - 160C_{0xx} - 80C_{0x}A_0 - 1600D_{0y} + 640D_{1x} - 80\Omega_x + 9A_0^4 - 88A_0^2C_0 \\ & \left. + 160A_0D_1 + 30A_0\Omega - 400A_1D_0 + 300\chi\Omega + 144C_0^2\right) + 1600\psi_y D_0, \end{aligned} \tag{22}$$

where χ *is given by equation (20) and* Ω *is the following expression:*

$$\Omega = A_0^3 - 4A_0C_0 + 8D_1 - 8C_{0x} + 6A_{0x}A_0 + 4A_{0xx}. \tag{23}$$

Finally, the coefficients α *and* β *of the resulting linear equation (7) are*

$$\alpha = \frac{\Omega}{8\varphi_x^3}, \tag{24}$$

$$\begin{aligned} \beta = {} & \left(1600\varphi_x^4\right)^{-1}\left(-144A_{0x}^2 - 72A_{0x}A_0^2 + 352A_{0x}C_0 + 160C_{0xx} + 80C_{0x}A_0 \right. \\ & + 1600D_{0y} - 640D_{1x} + 80\Omega_x - 9A_0^4 + 88A_0^2C_0 - 160A_0D_1 - 30A_0\Omega \\ & \left. + 400A_1D_0 - 300\chi\Omega - 144C_0^2\right). \end{aligned} \tag{25}$$

Remark 1. Since the system of equations (11)–(18) provides the necessary and sufficient conditions for linearization, it is invariant with respect to transformations (10). This means that the left-hand sides of equations (11)–(18) are relative invariants of second order for the equivalence transformations defined by (10).

3.2. Relations between coefficients and transformations

For proving the linearization theorem we need relations between $\varphi(x)$, $\psi(x, y)$ and the coefficients of equation (8). To obtain these relations, we complete equation (5) by adding the missing terms.

Lemma 1. *The coefficients of equation (8) and the functions* $\varphi(x)$ *and* $\psi(x, y)$ *in the transformation (10) are related by the following equations:*

$$A_1 = 4(\psi_y)^{-1}\psi_{yy}, \quad A_0 = -2(\varphi_x\psi_y)^{-1}(3\varphi_{xx}\psi_y - 2\varphi_x\psi_{xy}), \tag{26}$$

$$B_0 = 3(\psi_y)^{-1}\psi_{yy}, \quad C_2 = 6(\psi_y)^{-1}\psi_{yyy}, \quad C_1 = -6(\varphi_x\psi_y)^{-1}(3\varphi_{xx}\psi_{yy} - 2\varphi_x\psi_{xyy}), \tag{27}$$

$$C_0 = -\left(\varphi_x^2\psi_y\right)^{-1}\left[\left(4\varphi_{xxx}\varphi_x - 15\varphi_{xx}^2\right)\psi_y + 6(3\varphi_{xx}\psi_{xy} - \varphi_x\psi_{xxy})\varphi_x\right], \tag{28}$$

$$D_4 = (\psi_y)^{-1}\psi_{yyyy}, \tag{29}$$

$$D_3 = -2(\varphi_x\psi_y)^{-1}(3\varphi_{xx}\psi_{yyy} - 2\varphi_x\psi_{xyyy}), \tag{30}$$

$$D_2 = -\left(\varphi_x^2\psi_y\right)^{-1}\left(4\varphi_{xxx}\varphi_x\psi_{yy} - 15\varphi_{xx}^2\psi_{yy} + 18\varphi_{xx}\varphi_x\psi_{xyy} - 6\varphi_x^2\psi_{xxyy}\right), \tag{31}$$

$$\begin{aligned} D_1 = -\left(\varphi_x^3\psi_y\right)^{-1}\big[&3\left(5\varphi_{xx}^2\psi_y - 10\varphi_{xx}\varphi_x\psi_{xy} + 6\varphi_x^2\psi_{xxy}\right)\varphi_{xx} - \left(\varphi_x^3\psi_y\alpha + 4\psi_{xxxy}\right)\varphi_x^3 \\ &- 2(5\varphi_{xx}\psi_y - 4\varphi_x\psi_{xy})\varphi_{xxx}\varphi_x + \varphi_{xxxx}\varphi_x^2\psi_y\big], \end{aligned} \tag{32}$$

$$\begin{aligned} D_0 = -\left(\varphi_x^3\psi_y\right)^{-1}\big[&\left(15\varphi_{xx}^3 - \varphi_x^6\alpha + \varphi_{xxxx}\varphi_x^2\right)\psi_x - \big(10\varphi_{xxx}\varphi_{xx}\psi_x - 4\varphi_{xxx}\varphi_x\psi_{xx} \\ &+ 15\varphi_{xx}^2\psi_{xx} - 6\varphi_{xx}\varphi_x\psi_{xxx} + \varphi_x^6\beta\psi + \varphi_x^2\psi_{xxxx}\big)\varphi_x\big]. \end{aligned} \tag{33}$$

3.3. Proof of the linearization theorems

The proof of the linearization theorem requires the study of integrability conditions for the unknown functions $\varphi(x)$ and $\psi(x, y)$. The functions $\varphi(x)$ and $\psi(x, y)$ satisfy equations (26)–(33) with given coefficients $A_i(x, y)$, $B_i(x, y)$, $C_i(x, y)$, $D_i(x, y)$.

We first rewrite expression (26) for A_1 and A_0 in the following form:

$$\psi_{yy} = \frac{\psi_y A_1}{4}, \qquad \psi_{xy} = \frac{(6\varphi_{xx} + \varphi_x A_0)}{4\varphi_x}\psi_y. \tag{34}$$

Comparing the mixed derivatives $(\psi_{yy})_x = (\psi_{xy})_y$, one arrives at equation (11): $A_{0y} = A_{1x}$. Then equations (27) become (12) and (13). Furthermore, equation (28) gives

$$\varphi_{xxx} = -\frac{\left(12A_{0x}\varphi_x^2 - 60\varphi_{xx}^2 + 3\varphi_x^2A_0^2 - 8\varphi_x^2C_0\right)}{40\varphi_x}. \tag{35}$$

Differentiation of equation (35) with respect to y yields equation (14). Equations (29), (30) and (31) become in the form of equations (15) and (16), respectively.

One can determine α from equation (32):

$$\alpha = \frac{4A_{0xx} + 6A_{0x}A_0 - 8C_{0x} + A_0^3 - 4A_0C_0 + 8D_1}{8\varphi_x^3}. \tag{36}$$

Since $\varphi = \varphi(x)$, then $\alpha_y = 0$, which yields equation (17). From equation (33) one finds

$$\begin{aligned} \psi_{xxxx} = -\frac{1}{40\varphi_x^3}\big[&32A_{0xx}\varphi_x^3\psi_x - 72A_{0x}\varphi_{xx}\varphi_x^2\psi_x + 48A_{0x}\varphi_x^3\psi_{xx} \\ &+ 36A_{0x}\varphi_x^3\psi_xA_0 - 48C_{0x}\varphi_x^3\psi_x - 120\varphi_{xx}^3\psi_x + 360\varphi_{xx}^2\varphi_x\psi_{xx} \\ &- 240\varphi_{xx}\varphi_x^2\psi_{xxx} - 18\varphi_{xx}\varphi_x^2\psi_xA_0^2 + 48\varphi_{xx}\varphi_x^2\psi_xC_0 + 40\varphi_x^7\beta\psi \\ &+ 12\varphi_x^3\psi_{xx}A_0^2 - 32\varphi_x^3\psi_{xx}C_0 + 5\varphi_x^3\psi_xA_0^3 - 20\varphi_x^3\psi_xA_0C_0 \\ &+ 40\varphi_x^3\psi_xD_1 - 40\varphi_x^3\psi_yD_0\big]. \end{aligned} \tag{37}$$

Forming the mixed derivative $(\psi_{xxxx})_y = (\psi_{xy})_{xxx}$ one obtains

$$\begin{aligned} \beta = \frac{1}{1600\varphi_x^5}\big[&\varphi_x\big(320A_{0xxx} + 360A_{0xx}A_0 + 336A_{0x}^2 - 12A_{0x}A_0^2 \\ &- 480C_{0xx} + 1600D_{0y} - 39A_0^4 + 32A_{0x}C_0 \\ &+ 208A_0^2C_0 - 400A_0D_1 + 400A_1D_0 - 144C_0^2\big) \\ &- 300\varphi_{xx}\left(4A_{0xx} + 6A_{0x}A_0 - 8C_{0x} + A_0^3 - 4A_0C_0 + 8D_1\right)\big]. \end{aligned} \tag{38}$$

Differentiation of β with respect to y (18). This completes the proof of theorem 2

3.4. Application of the first linearization theorem to a system of two second-order ordinary differential equations

In this subsection we give some sufficient conditions of linearization for a system of two second-order ordinary differential equations with two dependent variables y, z and one independent variable x

$$y'' = f_1(x, y, y', z), \qquad z'' = f_2(x, y, y', z). \tag{39}$$

Assuming that $f_{1z} \neq 0$, by virtue of the inverse function theorem the first equation of (39) can be solved with respect to $z = g(x, y, y', y'')$. Substituting this into the second equation of (39), one obtains that system (39) is equivalent to the fourth-order ordinary differential equation

$$y^{(4)} g_{y''} + y'''^2 g_{y''y''} + y'''(2g_{y'y''}y'' + g_{y'} + 2g_{y''x} + 2g_{y''y}y') + g_{y'y'}y''^2 + (2g_{y'x} + 2g_{y'y}y' + g_y)y'' + g_{yy}y'^2 + 2g_{xy}y' + g_{xx} - f_2 = 0. \tag{40}$$

Applying linearization theorems to equation (40) one can obtain conditions for the functions $f_2(x, y, y', z)$ and $g(x, y, y', z)$ which are necessary and sufficient for equation (40) to be linearizable. It is worth noting that, in general, these linearizing transformations, which are point transformations for equation (40), are not point transformations for system of equations (39).

Since one of the necessary conditions for the linearization of equation (40) requires that this equation has to be a linear equation with respect to the third-order derivative y', one obtains that $g_{y''y''} = 0$, i.e., $g = g_0 + g_1 y''$, where $g_i = g_i(x, y, y')$, $(i = 0, 1)$. Since $g_{y''} \neq 0$, the function $g_1 \neq 0$. Equation (40) becomes

$$y^{(4)} + \big[(g_{0y'} + 3g_{1y'}y'' + 2g_{1x} + 2g_{1y}y')y''' + g_{1y'y'}y''^3 + (2g_{1y'y}y' + g_{0y'y'} + 2g_{1y'x} + g_{1y})y''^2 + (2g_{0y'x} + g_{1xx} + g_{0y} + 2(g_{0y'y} + g_{1xy})y' + g_{1yy}y'^2)y'' + g_{0yy}y'^2 + 2g_{0xy}y' + g_{0xx} - f_2\big]/g_1 = 0. \tag{41}$$

Considering the coefficient related to the product $y''y'$, for a linearizable equation one obtains either $g_{1y'} = 0$ or $3(y' + r)g_{1y'} + 10g_1 = 0$, where $r = r(x, y)$. In the present paper we study the case $g_{1y'} = 0$. Since the coefficients with the derivative y' have to be linear with respect to the first-order derivative y', one obtains $g_{0y'y'y'} = 0$, that is

$$g_0 = g_{00} + g_{01}y' + g_{02}y'^2,$$

where $g_{0i} = g_{0i}(x, y)$, $(i = 0, 1, 2)$. Hence, the coefficients A_1 and A_0 in equation (8) are

$$A_1 = 2(g_{1y} + g_{02})/g_1, \qquad A_0 = (2g_{1x} + g_{01})/g_1.$$

Proceeding to compare coefficients of equation (40) with equation (8) we obtain that

$$f_2 = f_{22}z^2 + (f_{210} + f_{211}y' + f_{212}y'^2)z + f_{200} + f_{201}y' + f_{202}y'^2 + f_{203}y'^3 + f_{204}y'^4,$$

where $f_{22} = f_{22}(x, y)$, $f_{21i} = f_{21i}(x, y)$, $(i = 0, 1, 2)$, $f_{20i} = f_{20i}(x, y)$, $(i = 0, 1, 2, 3, 4)$ and

$$B_0 = \big(g_{1y} - f_{22}g_1^2 + 2g_{02}\big)/g_1,$$
$$C_2 = (5g_{02y} + g_{1yy} - f_{212}g_1 - 2f_{22}g_{02}g_1)/g_1,$$
$$C_1 = (3g_{01y} + 4g_{02x} + 2g_{1xy} - f_{211}g_1 - 2f_{22}g_{01}g_1)/g_1,$$
$$C_0 = (g_{00y} + 2g_{01x} + g_{1xx} - f_{210}g_1 - 2f_{22}g_{00}g_1)/g_1.$$
$$D_4 = \big(g_{02yy} - f_{204} - f_{212}g_{02} - f_{22}g_{02}^2\big)/g_1,$$
$$D_3 = (g_{01yy} + 2g_{02xy} - f_{203} - f_{211}g_{02} - f_{212}g_{01} - 2f_{22}g_{01}g_{02})/g_1,$$

$$D_2 = \big(g_{00yy} + 2g_{01xy} + g_{02xx} - f_{202} - f_{210}g_{02} - f_{211}g_{01} - f_{212}g_{00} - 2f_{22}g_{00}g_{02} - f_{22}g_{01}^2\big)/g_1,$$
$$D_1 = (2g_{00xy} + g_{01xx} - f_{201} - f_{210}g_{01} - f_{211}g_{00} - 2f_{22}g_{00}g_{01})/g_1,$$
$$D_0 = \big(g_{00xx} - f_{200} - f_{210}g_{00} - f_{22}g_{00}^2\big)/g_1.$$

For the sake of simplicity we present here the linearization conditions for the case $f_{22} = 0$. One can verify that in the case $f_{22} = 0$ the found coefficients A_i, B_i, C_i and D_i satisfy the linearization conditions (11)–(18) if and only if

$$g_{01y} = \big(f_{210y}g_1^3 + g_{00y}g_{02}g_1 + g_{02xx}g_1^2 - 2g_{02x}g_{1x}g_1 + g_{02x}g_{01}g_1 - g_{1xx}g_{02}g_1 + 2g_{1x}^2g_{02} - g_{1x}g_{01}g_{02}\big)/g_1^2, \tag{42}$$
$$g_{1y} = f_{202}/g_1, \tag{43}$$
$$g_{00yy} = \big(f_{210y}g_1^3 + g_{00y}g_{02}g_1 + g_{02xx}g_1^2 - 2g_{02x}g_{1x}g_1 + g_{02x}g_{01}g_1 - g_{1xx}g_{02}g_1 + 2g_{1x}^2g_{02} - g_{1x}g_{01}g_{02}\big)/g_1^2, \tag{44}$$
$$f_{210y} = f_{202}/g_1, \tag{45}$$
$$f_{201y} = (2f_{202x}g_1 + f_{201}g_{02} - f_{202}g_{01})/g_1, \tag{46}$$
$$f_{200yy} = \big(f_{200y}g_{02}g_1 + f_{202xx}g_1^2 - f_{202y}g_{00}g_1 - 2g_{00y}f_{202}g_1 + g_{02x}f_{201}g_1 - g_{1x}f_{201}g_{02} - f_{202}f_{210}g_1^2 + 2f_{202}g_{00}g_{02}\big)/g_1^2, \tag{47}$$

and

$$f_{203} = f_{204} = f_{211} = f_{212} = 0.$$

One type of the functions $f_2(x, y, y', z)$ and $g(x, y, y', y'')$ satisfying conditions (42)–(47) is

$$f_2 = z\mu_2 + \mu_3 H + \mu_5, \qquad g = y''H_y + 2y'H_{xy} + y'^2H_{yy} + \mu_1 H + \mu_4,$$

where $\mu_i = \mu_i(x)$, $(i = 1, 2, 3, 4, 5)$ are arbitrary functions, and the function $H(x, y)$ satisfies the equation

$$\left(\left(\frac{H_{xy}}{H_y}\right)_x + \left(\frac{H_{xy}}{H_y}\right)^2\right)_y = 0.$$

System (39) corresponding to these functions is

$$y'' = z/H_y - (2y'H_{xy} + y'^2H_{yy} + \mu_4)/H - \mu_1, \qquad z'' = z\mu_2 + \mu_3 H + \mu_5.$$

3.5. Illustration of the linearization theorems

Example 1. Consider the nonlinear ordinary differential equation

$$x^2y(2y^{(4)} + y) + 8x^2y'y''' + 16xyy''' + 6x^2y''^2 + 48xy'y'' + 24yy'' + 24y'^2 = 0. \tag{48}$$

It is an equation of the form (8) with the coefficients

$$A_1 = \frac{4}{y}, \quad A_0 = \frac{8}{x}, \quad B_0 = \frac{3}{y}, \quad C_2 = 0, \quad C_1 = \frac{24}{xy}, \quad C_0 = \frac{12}{x^2},$$
$$D_4 = 0, \quad D_3 = 0, \quad D_2 = \frac{12}{x^2y}, \quad D_1 = 0, \quad D_0 = \frac{y}{2}. \tag{49}$$

One can check that the coefficients (49) obey the conditions (11)–(18). Thus, equation (48) is linearizable. We have

$$8C_0 - 3A_0^2 - 12A_{0x} = 0 \tag{50}$$

and equation (19) is written as

$$2\frac{\mathrm{d}\chi}{\mathrm{d}x} - \chi^2 = 0.$$

Let us take its simplest solution $\chi = 0$. Then invoking equation (20), we let

$$\varphi = x.$$

Now equations (21) are rewritten as

$$\frac{\psi_{yy}}{\psi_y} = \frac{1}{y}, \qquad \frac{\psi_{xy}}{\psi_y} = \frac{2}{x}$$

and yield

$$\psi_y = Kx^2y, \qquad K = \text{const}.$$

Hence

$$\psi = K\frac{x^2y^2}{2} + f(x).$$

Since one can use any particular solution, we set $K = 2$, $f(x) = 0$ and take

$$\psi = x^2y^2.$$

Invoking equation (50) and noting that equation (23) yields $\Omega = 0$, one can readily verify that the function $\psi = x^2y^2$ solves equation (22) as well. Hence, one obtains the following transformations:

$$t = x, \qquad u = x^2y^2. \tag{51}$$

Since $\Omega = 0$, equations (24) and (25) give

$$\alpha = 0, \qquad \beta = \frac{1}{\varphi_x^4} = 1.$$

Hence, equation (48) is mapped by the transformation (51) to the linear equation

$$u^{(4)} + u = 0.$$

Example 2. The third-order member of the Riccati hierarchy is given by Euler *et al* [14] as

$$y''' + 4yy'' + 3y'^2 + 6y^2y' + 4y^4 = 0. \tag{52}$$

Applying [7] and [8] one checks that the equation cannot be linearized by a point transformation or contact transformation or generalized Sundman transformation. Under the Riccati transformation $y = \frac{a\omega'}{\omega}$ equation (52) becomes [15]

$$\omega^3\omega^{(4)} + 4(a-1)\omega^2\omega'\omega''' + 3(a-1)\omega^2\omega''^2 + 6(a-1)(a-2)\omega\omega'^2\omega'' + (a-1)(a-2)(a-3)\omega'^4 = 0. \tag{53}$$

It is an equation of the form (8) with the coefficients

$$\begin{aligned} &A_1 = \frac{4(a-1)}{\omega}, \qquad A_0 = 0, \qquad B_0 = \frac{3(a-1)}{\omega},\\ &C_2 = \frac{6(a^2-3a+2)}{\omega^2}, \qquad C_1 = 0, \qquad C_0 = 0,\\ &D_4 = \frac{a^3-6a^2+11a-6}{\omega^3}, \qquad D_3 = 0, \quad D_2 = 0, \quad D_1 = 0, \quad D_0 = 0. \end{aligned} \tag{54}$$

One can verify that the coefficients (54) obey the linearization conditions (11)–(18). Furthermore,

$$8C_0 - 3A_0^2 - 12A_{0x} = 0 \tag{55}$$

and equation (19) is written as

$$2\frac{\mathrm{d}\chi}{\mathrm{d}x} - \chi^2 = 0.$$

We take its simplest solution $\chi = 0$ and obtain from equation (20) the equation $\varphi'' = 0$, whence

$$\varphi = x.$$

Equations (21) have the form

$$\frac{\psi_{\omega\omega}}{\psi_\omega} = \frac{a-1}{\omega}, \qquad \psi_{x\omega} = 0$$

and yield

$$\psi_\omega = K\omega^{(a-1)}, \qquad K = \text{const}.$$

Hence

$$\psi = K\frac{\omega^a}{a} + f(x).$$

Since one can use any particular solution, we set $K = a$, $f(x) = 0$ and take

$$\psi = \omega^a.$$

Invoking equation (55) and noting that equation (23) yields $\Omega = 0$, one can readily verify that the function $\psi = \omega^a$ solves equation (22) as well. One obtains the following transformation:

$$t = x, \qquad u = \omega^a. \tag{56}$$

Since $\Omega = 0$, equations (24) and (25) give

$$\alpha = 0, \qquad \beta = 0.$$

Hence, equation (53) is mapped by the transformations (56) to the linear equation

$$u^{(4)} = 0.$$

Example 3. Let us consider the Boussinesq equation

$$u_{tt} + uu_{xx} + u_x^2 + u_{xxxx} = 0. \tag{57}$$

Of particular interest among the solutions of the Boussinesq equation are traveling wave solutions:

$$u(x,t) = H(x - Dt).$$

Substituting the representation of a solution into equation (57), one finds

$$H^{(4)} + (H + D^2)H'' + H'^2 = 0. \tag{58}$$

It is an equation of the form (8) with the coefficients

$$\begin{array}{llllll} A_1 = 0, & A_0 = 0, & B_0 = 0, & C_2 = 0, & C_1 = 0, & C_0 = D^2 + H, \\ D_4 = 0, & D_3 = 0, & D_2 = 1, & D_1 = 0, & D_0 = 0. & \end{array} \tag{59}$$

Since the coefficients (59) do not satisfy the linearization conditions (14), (16) and (18), hence, the Boussinesq equation (58) is not linearizable.

4. The second class of linearizable equations

4.1. The linearization test for equation (9)

The following theorem provides the test for the linearization of the second class.

Theorem 3. *Equation (9) is linearizable if and only if its coefficients obey equations*[5] *(A.1)–(A.18).*

The necessary and sufficient conditions comprise 18 differential equations (A.1)–(A.18) for 21 coefficients of equation (9).

Corollary 2. *Provided that the conditions (A.1)–(A.18) are satisfied, the transformation (2) mapping equation (9) into a linear equation (7) is obtained by solving the compatible system of equations for the functions* $\varphi(x, y)$ *and* $\psi(x, y)$ *(A.19)–(A.22). The coefficients* α *and* β *are given by equations (A.23) and (A.24).*

Remark 2. Equations (A.1)–(A.18) define eighteen relative invariants of third order of point transformations (2).

4.2. Relations between coefficients and transformations

Lemma 2. *The coefficients of equation (9) and the functions* $\varphi(x, y)$ *and* $\psi(x, y)$ *in the transformation (2) are related by equations* [6] *(A.26)–(A.44).*

4.3. Proof of the linearization theorems

The problem is: for the given coefficients $F_i(x, y)$, $H_i(x, y)$, $J_i(x, y)$, $K_i(x, y)$ of equation (9) find the integrability conditions for the functions $\varphi(x, y)$ and $\psi(x, y)$.

Recall that, according to our notations, the following equations hold $\alpha_x = r\alpha_y$, $\beta_x = r\beta_y$, and

$$\varphi_x = r\varphi_y, \qquad \psi_x = \frac{\psi_y\varphi_x - \Delta}{\varphi_y}. \tag{60}$$

From equations (A.26) and (A.27) one finds

$$\begin{aligned} &\varphi_{yy} = [(4\Delta_y - F_2\Delta)\varphi_y]/(10\Delta), \\ &\Delta_x = (20r_y\Delta + 4\Delta_y r + F_1\Delta - 2F_2 r\Delta)/4. \end{aligned} \tag{61}$$

Comparison of the mixed derivatives $(\varphi_x)_{yy} = (\varphi_{yy})_x$ gives equation (A.1). Then equations (A.28)–(A.32) become (A.2)–(A.5) and

$$\Delta_{yy} = -\left(20F_{2y}\Delta^2 - 48\Delta_y^2 + 4\Delta_y F_2\Delta + 7F_2^2\Delta^2 - 20J_4\Delta^2\right)\big/(40\Delta).$$

The equation $(\Delta_{yy})_x = (\Delta_x)_{yy}$ leads to equation (A.6). Equations (A.33)–(A.36) yield equations (A.7)–(A.10), and from equation (A.37) one finds

$$\psi_{yyyy} = \big[300\psi_{yyy}\varphi_y\Delta^2(4\Delta_y - F_2\Delta) + 5\psi_{yy}\varphi_y\Delta\big(-120F_{2y}\Delta^2 - 144\Delta_y^2$$
$$+ 72\Delta_y F_2\Delta - 39F_2^2\Delta^2 + 80J_4\Delta^2\big) + \psi_y\varphi_y\big(-500\varphi_y^3\alpha\Delta^3$$

[5] Since equations (A.1)–(A.18) and (A.19)–(A.22) are cumbersome, they are presented in the appendix.
[6] Equations (A.26)–(A.44) are presented in the appendix.

$$- 150F_{2yy}\Delta^3 + 360F_{2y}\Delta_y\Delta^2 - 165F_{2y}F_2\Delta^3 + 100J_{4y}\Delta^3 + 96\Delta_y^3 - 72\Delta_y^2F_2\Delta + 108\Delta_yF_2^2\Delta^2 - 240\Delta_yJ_4\Delta^2 - 24F_2^3\Delta^3 + 60F_2J_4\Delta^3\big) - 500\psi\varphi_y^5\beta\Delta^3 + 500K_7\Delta^4\big]\big/(500\varphi_y\Delta^3). \qquad (62)$$

Equation (A.38) defines α:

$$\alpha = \big(4F_{2yy} + 6F_{2y}F_2 - 8J_{4y} + F_2^3 - 4F_2J_4 - 8K_6 + 56K_7r\big)\big/\big(8\varphi_y^3\big). \qquad (63)$$

The equation $\alpha_x - r\alpha_y = 0$ leads to equation (A.11). Furthermore, considering $(\psi_x)_{yyyy} - (\psi_{yyyy})_x = 0$, one obtains

$$\beta = 120\Delta_y\big(-4F_{2yy} - 6F_{2y}F_2 + 8J_{4y} - F_2^3 + 4F_2J_4 + 8K_6 - 56K_7r\big) + \Delta\big(320F_{2yyy} + 480F_{2yy}F_2 + 336F_{2y}^2 + 168F_{2y}F_2^2 + 32F_{2y}J_4 - 480J_{4yy} - 240J_{4y}F_2 - 1600K_{7x} + 1600K_{7y}r - 400F_1K_7 - 9F_2^4 + 88F_2^2J_4 + 160F_2K_6 - 320F_2K_7r - 144J_4^2\big)\big/\big(1600\Delta\varphi_y^4\big). \qquad (64)$$

The equation $\beta_x - r\beta_y = 0$ leads to equation (A.12). Equations (A.39)–(A.44) become equations (A.13)–(A.18), respectively. Hence, we complete the proof of theorem 3.

4.4. Illustration of the linearization theorems

Example 4. Consider the nonlinear equation

$$y^{(4)} - \frac{10}{y'}y''y''' + \frac{1}{y'^2}(15y''^3 - xy'^7 - y'^6) = 0. \qquad (65)$$

It has the form of equation (9) with the following coefficients:

$$\begin{aligned} &r = 0, \quad F_2 = 0, \quad F_1 = 0, \quad F_0 = 0, \quad H_2 = 0, \quad H_1 = 0, \quad H_0 = 0,\\ &J_4 = 0, \quad J_3 = 0, \quad J_2 = 0, \quad J_1 = 0, \quad J_0 = 0, \quad K_7 = -x,\\ &K_6 = -1, \quad K_5 = 0, \quad K_4 = 0, \quad K_3 = 0, \quad K_2 = 0, \quad K_1 = 0, \quad K_0 = 0. \end{aligned} \qquad (66)$$

Let us test equation (65) for linearization by using theorem 3 It is manifest that equations (A.1)–(A.18) are satisfied by the coefficients (66). Thus, equation (65) is linearizable, and we can proceed further.

Let us take its simplest solution $\varphi = y$ and $\psi = x$ which satisfy the compatible system of equations (A.19)–(A.22). So that one obtains the following transformations:

$$t = y, \qquad u = x. \qquad (67)$$

Since $\Theta = 8$, equations (A.23) and (A.24) give

$$\alpha = 1, \qquad \beta = 1.$$

Hence, equation (65) is mapped by the transformations (67) into the linear equation

$$u^{(4)} + u' + u = 0.$$

This example shows that as for second-order ordinary differential equations [16][7] the Riccati substitution can map a third-order ordinary differential equation into a linearizable fourth-order ordinary differential equation. Using the obtained in this paper criteria of linearization, one can obtain complete criteria for third-order ordinary differential equations linearizable by the Riccati substitution.

[7] In [16] the complete study of second-order ordinary differential equations linearizable by the Riccati substitution is presented.

Appendix

In this section we present equations which were used in previous sections.

A.1. Equations for theorem 3 in section 4.1

$$10r_{yy} = -(F_{1y} + F_{2x} + F_{2y}r + r_yF_2), \quad \text{(A.1)}$$

$$10r_x = 10r_yr - F_0 + F_1r - F_2r^2, \quad \text{(A.2)}$$

$$H_2 = -3F_2, \quad \text{(A.3)}$$

$$4H_1 = -3(5F_1 - 2F_2r), \quad \text{(A.4)}$$

$$4H_0 = -3(6F_0 - F_1r), \quad \text{(A.5)}$$

$$10F_{1yy} = -\big(F_{1y}F_2 - 40F_{2xy} - 16F_{2x}F_2 + 20F_{2yy}r + 40F_{2y}r_y + 14F_{2y}F_2r + 20J_{4x} - 20J_{4y}r + 14r_yF_2^2 - 40r_yJ_4\big), \quad \text{(A.6)}$$

$$12F_{2x} = 12F_{2y}r - 3F_1F_2 + 6F_2^2r + 4J_3 - 16J_4r, \quad \text{(A.7)}$$

$$60F_{1x} = 60F_{1y}r - 36F_0F_2 - 15F_1^2 + 66F_1F_2r - 36F_2^2r^2 + 40J_2 - 80J_3r + 80J_4r^2, \quad \text{(A.8)}$$

$$60F_{0x} = 60F_{0y}r - 51F_0F_1 + 66F_0F_2r + 36F_1^2r - 72F_1F_2r^2 + 36F_2^2r^3 + 60J_1 - 80J_2r + 80J_3r^2 - 80J_4r^3, \quad \text{(A.9)}$$

$$20J_0 = 9F_0^2 - 18F_0F_1r + 18F_0F_2r^2 + 9F_1^2r^2 - 18F_1F_2r^3 + 9F_2^2r^4 + 20J_1r - 20J_2r^2 + 20J_3r^3 - 20J_4r^4, \quad \text{(A.10)}$$

$$\begin{aligned}120J_{3yy} = {} & 216F_{1y}F_{2y} + 54F_{1y}F_2^2 - 48F_{1y}J_4 + 360F_{2yy}r_y + 90F_{2yy}F_1 - 180F_{2yy}F_2r \\ & - 432F_{2y}^2r + 324F_{2y}r_yF_2 + 189F_{2y}F_1F_2 - 486F_{2y}F_2^2r - 192F_{2y}J_3 \\ & + 864F_{2y}J_4r - 60J_{3y}F_2 + 720J_{4xy} + 180J_{4x}F_2 - 240J_{4yy}r \\ & - 1200J_{4y}r_y + 60J_{4y}F_2r + 720K_{6x} - 720K_{6y}r - 5040K_{7x}r \\ & + 5040K_{7y}r^2 + 36r_yF_2^3 - 432r_yF_2J_4 - 2160r_yK_6 + 15120r_yK_7r \\ & + 504F_0K_7 + 36F_1F_2^3 - 102F_1F_2J_4 - 504F_1K_7r - 72F_2^4r \\ & - 48F_2^2J_3 + 396F_2^2J_4r + 504F_2K_7r^2 + 136J_3J_4 - 544J_4^2r, \end{aligned} \quad \text{(A.11)}$$

$$\begin{aligned}240J_{4xyy} = -\big(& 36F_{1y}F_{2yy} + 162F_{1y}F_{2y}F_2 - 72F_{1y}J_{4y} + 36F_{1y}F_2^3 - 168F_{1y}F_2J_4 \\ & - 72F_{1y}K_6 - 168F_{1y}K_7r - 72F_{2yy}F_{2y}r + 144F_{2yy}r_yF_2 \\ & + 54F_{2yy}F_1F_2 - 108F_{2yy}F_2^2r - 72F_{2yy}J_3 + 288F_{2yy}J_4r + 432F_{2y}^2r_y \\ & + 108F_{2y}^2F_1 - 540F_{2y}^2F_2r - 144F_{2y}J_{3y} + 528F_{2y}J_{4x} + 192F_{2y}J_{4y}r \\ & + 324F_{2y}r_yF_2^2 - 1008F_{2y}r_yJ_4 + 162F_{2y}F_1F_2^2 - 132F_{2y}F_1J_4 \\ & - 396F_{2y}F_2^3r - 180F_{2y}F_2J_3 + 1320F_{2y}F_2J_4r + 144F_{2y}K_6r \\ & - 336F_{2y}K_7r^2 - 36J_{3y}F_2^2 + 176J_{3y}J_4 + 120J_{4xy}F_2 + 132J_{4x}F_2^2 \\ & - 432J_{4x}J_4 - 240J_{4yyy}r - 960J_{4yy}r_y - 120J_{4yy}F_2r - 768J_{4y}r_yF_2 \end{aligned}$$

$$
\begin{aligned}
&- 138J_{4y}F_1F_2 + 288J_{4y}F_2^2r + 184J_{4y}J_3 - 1008J_{4y}J_4r + 960K_{6xy} \\
&+ 240K_{6x}F_2 - 960K_{6yy}r - 3840K_{6y}r_y - 240K_{6y}F_2r - 1920K_{7xy}r \\
&- 2400K_{7xx} + 2880K_{7x}r_y - 600K_{7x}F_1 - 480K_{7x}F_2r + 4320K_{7yy}r^2 \\
&+ 24\,000K_{7y}r_yr + 432K_{7y}F_0 + 168K_{7y}F_1r + 912K_{7y}F_2r^2 \\
&+ 20160r_y^2K_7 + 1728r_yF_1K_7 + 36r_yF_2^4 - 264r_yF_2^2J_4 - 1248r_yF_2K_6 \\
&+ 5280r_yF_2K_7r + 160r_yJ_4^2 + 408F_0F_2K_7 + 150F_1^2K_7 + 27F_1F_2^4 \\
&- 120F_1F_2^2J_4 - 168F_1F_2K_6 + 168F_1F_2K_7r - 54F_2^5r - 36F_2^3J_3 \\
&+ 384F_2^3J_4r + 336F_2^2K_6r - 1344F_2^2K_7r^2 + 160F_2J_3J_4 - 640F_2J_4^2r \\
&- 400J_2K_7 + 224J_3K_6 - 368J_3K_7r - 896J_4K_6r + 3872J_4K_7r^2 \\
&+ 672F_{0y}K_7\big), \qquad \text{(A.12)}
\end{aligned}
$$

$$
4J_{4x} = 4J_{4y}r - F_1J_4 + 2F_2J_4r - 4K_5 + 24K_6r - 84K_7r^2, \qquad \text{(A.13)}
$$

$$
\begin{aligned}
60F_{0yy} = -\big(&30F_{0y}F_2 + 36F_{1y}F_1 - 36F_{1y}F_2r - 60F_{2yy}r^2 + 24F_{2y}F_0 - 36F_{2y}F_1r \\
&- 54F_{2y}F_2r^2 - 40J_{2y} + 40J_{3y}r + 80J_{4y}r^2 - 36r_yF_1F_2 + 36r_yF_2^2r \\
&+ 40r_yJ_3 - 80r_yJ_4r + 6F_0F_2^2 - 6F_0J_4 + 9F_1^2F_2 - 18F_1F_2^2r \\
&- 12F_1J_3 + 24F_1J_4r - 6F_2^3r^2 - 10F_2J_2 + 22F_2J_3r + 26F_2J_4r^2 \\
&- 60K_4 + 180K_5r - 180K_6r^2 - 420K_7r^3\big), \qquad \text{(A.14)}
\end{aligned}
$$

$$
\begin{aligned}
20J_{2x} = {}&20J_{2y}r + 20J_{3x}r - 20J_{3y}r^2 - 14F_0J_3 + 28F_0J_4r - 5F_1J_2 + 19F_1J_3r \\
&- 28F_1J_4r^2 + 10F_2J_2r - 24F_2J_3r^2 + 28F_2J_4r^3 - 120K_3 + 360K_4r \\
&- 640K_5r^2 + 840K_6r^3 - 840K_7r^4, \qquad \text{(A.15)}
\end{aligned}
$$

$$
\begin{aligned}
60J_{1x} = {}&60J_{1y}r - 40J_{3x}r^2 + 40J_{3y}r^3 - 42F_0J_2 + 42F_0J_3r - 70F_0J_4r^2 - 15F_1J_1 \\
&+ 42F_1J_2r - 52F_1J_3r^2 + 70F_1J_4r^3 + 30F_2J_1r - 42F_2J_2r^2 \\
&+ 62F_2J_3r^3 - 70F_2J_4r^4 - 600K_2 + 1080K_3r - 1380K_4r^2 \\
&+ 1700K_5r^3 - 2100K_6r^4 + 2100K_7r^5, \qquad \text{(A.16)}
\end{aligned}
$$

$$
\begin{aligned}
80K_1 = {}&3F_0^2F_1 - 6F_0^2F_2r - 6F_0F_1^2r + 18F_0F_1F_2r^2 - 12F_0F_2^2r^3 - 8F_0J_1 \\
&+ 16F_0J_2r - 24F_0J_3r^2 + 32F_0J_4r^3 + 3F_1^3r^2 - 12F_1^2F_2r^3 + 15F_1F_2^2r^4 \\
&+ 8F_1J_1r - 16F_1J_2r^2 + 24F_1J_3r^3 - 32F_1J_4r^4 - 6F_2^3r^5 - 8F_2J_1r^2 \\
&+ 16F_2J_2r^3 - 24F_2J_3r^4 + 32F_2J_4r^5 + 160K_2r - 240K_3r^2 + 320K_4r^3 \\
&- 400K_5r^4 + 480K_6r^5 - 560K_7r^6, \qquad \text{(A.17)}
\end{aligned}
$$

$$
\begin{aligned}
400K_0 = -\big(&6F_0^3 - 33F_0^2F_1r + 48F_0^2F_2r^2 + 48F_0F_1^2r^2 - 126F_0F_1F_2r^3 + 78F_0F_2^2r^4 \\
&+ 40F_0J_1r - 80F_0J_2r^2 + 120F_0J_3r^3 - 160F_0J_4r^4 - 21F_1^3r^3 \\
&+ 78F_1^2F_2r^4 - 93F_1F_2^2r^5 - 40F_1J_1r^2 + 80F_1J_2r^3 - 120F_1J_3r^4 \\
&+ 160F_1J_4r^5 + 36F_2^3r^6 + 40F_2J_1r^3 - 80F_2J_2r^4 + 120F_2J_3r^5 \\
&- 160F_2J_4r^6 - 400K_2r^2 + 800K_3r^3 - 1200K_4r^4 + 1600K_5r^5 \\
&- 2000K_6r^6 + 2400K_7r^7\big). \qquad \text{(A.18)}
\end{aligned}
$$

A.2. Equations for corollary 2 in section 4.1

$$\varphi_x = r\varphi_y, \tag{A.19}$$

$$\varphi_y\psi_x = r\varphi_y\psi_y - \Delta, \tag{A.20}$$

$$10\Delta\varphi_{yy} = \varphi_y(4\Delta_y - F_2\Delta), \tag{A.21}$$

$$\begin{aligned}500\varphi_y\psi_{yyyy}\Delta^3 &= 300\psi_{yyy}\varphi_y\Delta^2(4\Delta_y - F_2\Delta) + 5\psi_{yy}\varphi_y\Delta\big(-120F_{2y}\Delta^2 - 144\Delta_y^2\\ &\quad + 72\Delta_y F_2\Delta - 39F_2^2\Delta^2 + 80J_4\Delta^2\big) + \psi_y\varphi_y\big(-500\varphi_y^3\alpha\Delta^3\\ &\quad - 150F_{2yy}\Delta^3 + 360F_{2y}\Delta_y\Delta^2 - 165F_{2y}F_2\Delta^3 + 100J_{4y}\Delta^3 + 96\Delta_y^3\\ &\quad -72\Delta_y^2F_2\Delta + 108\Delta_yF_2^2\Delta^2 - 240\Delta_yJ_4\Delta^2 - 24F_2^3\Delta^3 + 60F_2J_4\Delta^3\big)\\ &\quad - 500\psi\varphi_y^5\beta\Delta^3 + 500K_7\Delta^4,\end{aligned} \tag{A.22}$$

$$\alpha = \frac{\Theta}{8\varphi_y^3}, \tag{A.23}$$

$$\begin{aligned}\beta &= \big(1600\Delta\varphi_y^4\big)^{-1}\big[\Delta\big(-144F_{2y}^2 - 72F_{2y}F_2^2 + 352F_{2y}J_4 + 160J_{4yy} + 80J_{4y}F_2\\ &\quad + 640K_{6y} - 1600K_{7x} - 2880K_{7y}r + 80\Theta_y - 4480r_yK_7 - 400F_1K_7\\ &\quad - 9F_2^4 + 88F_2^2J_4 + 160F_2K_6 - 320F_2K_7r - 144J_4^2\big) - 120\Delta_y\Theta\big],\end{aligned} \tag{A.24}$$

where Θ is the following expression:

$$\Theta = \big(F_2^2 - 4J_4\big)F_2 - 8(K_6 - 7K_7r) - 8J_{4y} + 6F_{2y}F_2 + 4F_{2yy}. \tag{A.25}$$

A.3. Equations for lemma 2 in section 4.2

$$F_2 = -2(\varphi_y\Delta)^{-1}(5\varphi_{yy}\Delta - 2\varphi_y\Delta_y), \tag{A.26}$$

$$F_1 = 4(\varphi_y\Delta)^{-1}[(\Delta_x + \Delta_yr - 5r_y\Delta)\varphi_y - 5\varphi_{yy}r\Delta], \tag{A.27}$$

$$F_0 = -2(\varphi_y\Delta)^{-1}[((5r_y\Delta - 2\Delta_x)r + 5r_x\Delta)\varphi_y + 5\varphi_{yy}r^2\Delta], \tag{A.28}$$

$$H_2 = 6(\varphi_y\Delta)^{-1}(5\varphi_{yy}\Delta - 2\varphi_y\Delta_y), \tag{A.29}$$

$$H_1 = -3(\varphi_y\Delta)^{-1}[(5\Delta_x + 3\Delta_yr - 25r_y\Delta)\varphi_y - 20\varphi_{yy}r\Delta], \tag{A.30}$$

$$H_0 = 3(\varphi_y\Delta)^{-1}[(5(3r_x + 2r_yr)\Delta - (5\Delta_x - \Delta_yr)r)\varphi_y + 10\varphi_{yy}r^2\Delta], \tag{A.31}$$

$$J_4 = -\big(\varphi_y^2\Delta\big)^{-1}\big(10\varphi_{yyy}\varphi_y\Delta - 45\varphi_{yy}^2\Delta + 30\varphi_{yy}\varphi_y\Delta_y - 6\varphi_y^2\Delta_{yy}\big), \tag{A.32}$$

$$\begin{aligned}J_3 &= 2\big(\varphi_y^2\Delta\big)^{-1}\big[3((2(\Delta_{xy} + \Delta_{yy}r - 5r_y\Delta_y) - 5r_{yy}\Delta)\varphi_y^2\\ &\quad - 5((\Delta_x + 3\Delta_yr - 4r_y\Delta)\varphi_y - 6\varphi_{yy}r\Delta)\varphi_{yy}) - 20\varphi_{yyy}\varphi_yr\Delta\big],\end{aligned} \tag{A.33}$$

$$\begin{aligned}J_2 &= 6\big(\varphi_y^2\Delta\big)^{-1}\big[(\Delta_{xx} + \Delta_{yy}r^2 + 4\Delta_{xy}r - 5(2\Delta_x + 3\Delta_yr - 5r_y\Delta)r_y\\ &\quad - 10r_{yy}r\Delta - 5r_x\Delta_y - 5r_{xy}\Delta)\varphi_y^2 - 5(((3(\Delta_x + \Delta_yr) - 10r_y\Delta)r\\ &\quad - 2r_x\Delta)\varphi_y - 9\varphi_{yy}r^2\Delta)\varphi_{yy} - 10\varphi_{yyy}\varphi_yr^2\Delta\big],\end{aligned} \tag{A.34}$$

$$\begin{aligned}J_1 &= -2\big(\varphi_y^2\Delta\big)^{-1}\big[((5(3(3\Delta_x + \Delta_yr) - 14r_y\Delta)r_y - 6(\Delta_{xy}r + \Delta_{xx})\\ &\quad + 20r_{yy}r\Delta)r + 5(3(\Delta_x + \Delta_yr) - 16r_y\Delta)r_x + 5r_{xx}\Delta + 20r_{xy}r\Delta)\varphi_y^2\\ &\quad + 15(((3\Delta_x + \Delta_yr - 8r_y\Delta)r - 4r_x\Delta)\varphi_y - 6\varphi_{yy}r^2\Delta)\varphi_{yy}r\\ &\quad + 20\varphi_{yyy}\varphi_yr^3\Delta\big],\end{aligned} \tag{A.35}$$

$$J_0 = -\left(\varphi_y^2\Delta\right)^{-1}\big[((2((5r_{yy}r\Delta - 3\Delta_{xx})r + 5r_{xx}\Delta + 5r_{xy}r\Delta) - 5(7r_y\Delta - 6\Delta_x)r_yr)r - 5(2(7r_y\Delta - 3\Delta_x)r + 9r_x\Delta)r_x)\varphi_y^2 - 5(3(2((2r_y\Delta - \Delta_x)r + 2r_x\Delta)\varphi_y + 3\varphi_{yy}r^2\Delta)\varphi_{yy} - 2\varphi_{yyy}\varphi_yr^2\Delta)r^2\big], \tag{A.36}$$

$$K_7 = -\left(\varphi_y^2\Delta\right)^{-1}\big[\varphi_{yyyy}\varphi_y^2\psi_y - 10\varphi_{yyy}\varphi_{yy}\varphi_y\psi_y + 4\varphi_{yyy}\varphi_y^2\psi_{yy} + 15\varphi_{yy}^3\psi_y - 15\varphi_{yy}^2\varphi_y\psi_{yy} + 6\varphi_{yy}\varphi_y^2\psi_{yyy} - \varphi_y^7\beta\psi - \varphi_y^6\psi_y\alpha - \varphi_y^3\psi_{yyyy}\big], \tag{A.37}$$

$$K_6 = \left(\varphi_y^3\Delta\right)^{-1}\big[3\big(5((7\varphi_y\psi_{yy}r - 6\Delta_y)\varphi_y - 7(\varphi_y\psi_yr - \Delta)\varphi_{yy})\varphi_{yy} - 2(7\varphi_y\psi_{yyy}r - 5\Delta_{yy})\varphi_y^2\big)\varphi_{yy} + \big(7\varphi_y^5\beta\psi r + 7\varphi_y^4\psi_y\alpha r - \varphi_y^3\alpha\Delta + 7\varphi_y\psi_{yyyy}r - 4\Delta_{yyy}\big)\varphi_y^3 + 2\big(35\varphi_{yy}\varphi_y\psi_yr - 30\varphi_{yy}\Delta - 14\varphi_y^2\psi_{yy}r + 10\varphi_y\Delta_y\big)\varphi_{yyy}\varphi_y - (7\varphi_y\psi_yr - 5\Delta)\varphi_{yyyy}\varphi_y^2\big], \tag{A.38}$$

$$K_5 = -\left(\varphi_y^3\Delta\right)^{-1}\big[\big(2(3(\Delta_{xyy} + 3\Delta_{yyy}r - 5r_y\Delta_{yy} - 5r_{yy}\Delta_y) - 5r_{yyy}\Delta) - 3\big(7\varphi_y^4\beta\psi r + 7\varphi_y^3\psi_y\alpha r - 2\varphi_y^2\alpha\Delta + 7\psi_{yyyy}r\big)\varphi_yr\big)\varphi_y^3 - 3(2(5(\Delta_{xy} + 5\Delta_{yy}r - 4r_y\Delta_y - 2r_{yy}\Delta) - 21\varphi_y\psi_{yyy}r^2)\varphi_y^2 - 15((\Delta_x + 11\Delta_yr - 3r_y\Delta - 7\varphi_y\psi_{yy}r^2)\varphi_y + 7(\varphi_y\psi_yr - 2\Delta)\varphi_{yy}r)\varphi_{yy})\varphi_{yy} - 2((5(\Delta_x + 11\Delta_yr - 3r_y\Delta) - 42\varphi_y\psi_{yy}r^2)\varphi_y + 15(7\varphi_y\psi_yr - 12\Delta)\varphi_{yy}r)\varphi_{yyy}\varphi_y + 3(7\varphi_y\psi_yr - 10\Delta)\varphi_{yyyy}\varphi_y^2r\big], \tag{A.39}$$

$$K_4 = -\left(\varphi_y^3\Delta\right)^{-1}\big[\big(2\big(45r_{yy}r_y\Delta - 10r_{yy}\Delta_x - 55r_{yy}\Delta_yr + 50r_y^2\Delta_y - 20r_y\Delta_{xy} - 50r_y\Delta_{yy}r + 11\Delta_{xyy}r + 2\Delta_{xxy} + 17\Delta_{yyy}r^2 - 20r_{yyy}r\Delta - 5r_x\Delta_{yy} - 10r_{xy}\Delta_y - 5r_{xyy}\Delta\big) - 5\big(7\varphi_y^4\beta\psi r + 7\varphi_y^3\psi_y\alpha r - 3\varphi_y^2\alpha\Delta + 7\psi_{yyyy}r\big)\varphi_yr^2\big)\varphi_y^3 + 15((3((5(\Delta_x + 5\Delta_yr) - 14r_y\Delta)r - r_x\Delta) - 35\varphi_y\psi_{yy}r^3)\varphi_y + 35(\varphi_y\psi_yr - 3\Delta)\varphi_{yy}r^2)\varphi_{yy}^2 - 10(\Delta_{xx} + 31\Delta_{yy}r^2 + 13\Delta_{xy}r - 8(\Delta_x + 6\Delta_yr - 2r_y\Delta)r_y - 26r_{yy}r\Delta - 4r_x\Delta_y - 4r_{xy}\Delta - 21\varphi_y\psi_{yyy}r^3)\varphi_{yy}\varphi_y^2 - 10(((5(\Delta_x + 5\Delta_yr) - 14r_y\Delta)r - r_x\Delta - 14\varphi_y\psi_{yy}r^3)\varphi_y + 5(7\varphi_y\psi_yr - 18\Delta)\varphi_{yy}r^2)\varphi_{yyy}\varphi_y + 5(7\varphi_y\psi_yr - 15\Delta)\varphi_{yyyy}\varphi_y^2r^2\big], \tag{A.40}$$

$$K_3 = -\left(\varphi_y^3\Delta\right)^{-1}\big[((13\Delta_{xxy} + 35\Delta_{yyy}r^2)r + \Delta_{xxx} + 31\Delta_{xyy}r^2 - 5(3\Delta_{xx} + 26\Delta_{yy}r^2 + 23\Delta_{xy}r - (15\Delta_x + 49\Delta_yr - 25r_y\Delta)r_y)r_y - 5(13\Delta_x + 32\Delta_yr - 50r_y\Delta)r_{yy}r - 65r_{yyy}r^2\Delta - 5(3\Delta_{xy} + 5\Delta_{yy}r - 16r_y\Delta_y - 7r_{yy}\Delta)r_x - 5r_{xx}\Delta_y - 5r_{xxy}\Delta - 5(3\Delta_x + 11\Delta_yr - 15r_y\Delta)r_{xy} - 30r_{xyy}r\Delta - 5\big(7\varphi_y^4\beta\psi r + 7\varphi_y^3\psi_y\alpha r - 4\varphi_y^2\alpha\Delta + 7\psi_{yyyy}r\big)\varphi_yr^3)\varphi_y^3 - 5(2((2(2\Delta_{xx} + 17\Delta_{yy}r^2 + 11\Delta_{xy}r) - (29\Delta_x + 75\Delta_yr - 51r_y\Delta)r_y - 45r_{yy}r\Delta)r - (3\Delta_x + 13\Delta_yr - 13r_y\Delta)r_x$$

$$- r_{xx}\Delta - 14r_{xy}r\Delta - 21\varphi_y\psi_{yyy}r^4)\varphi_y^2 - 3((6((5(\Delta_x + 3\Delta_y r) - 13r_y\Delta)r$$
$$- 2r_x\Delta) - 35\varphi_y\psi_{yy}r^3)\varphi_y + 35(\varphi_y\psi_y r - 4\Delta)\varphi_{yy}r^2)\varphi_{yy}r)\varphi_{yy}$$
$$- 10(2((5(\Delta_x + 3\Delta_y r) - 13r_y\Delta)r - 2r_x\Delta - 7\varphi_y\psi_{yy}r^3)\varphi_y$$
$$+ 5(7\varphi_y\psi_y r - 24\Delta)\varphi_{yy}r^2)\varphi_{yyy}\varphi_y r + 5(7\varphi_y\psi_y r - 20\Delta)\varphi_{yyyy}\varphi_y^2 r^3\big], \qquad (A.41)$$

$$K_2 = -\big(\varphi_y^3\Delta\big)^{-1}\big[((3((5\Delta_{xxy} + 7\Delta_{yyy}r^2)r + \Delta_{xxx} + 7\Delta_{xyy}r^2)$$
$$- (3(13\Delta_{xx} + 28\Delta_{yy}r^2 + 39\Delta_{xy}r) + (204r_y\Delta - 161\Delta_x - 217\Delta_y r)r_y)r_y$$
$$- (79\Delta_x + 116\Delta_y r - 264r_y\Delta)r_{yy}r - 54r_{yyy}r^2\Delta)r$$
$$- (3(2\Delta_{xx} + 7\Delta_{yy}r^2 + 11\Delta_{xy}r) + (171r_y\Delta - 64\Delta_x - 140\Delta_y r)r_y$$
$$- 72r_{yy}r\Delta - 18r_x\Delta_y)r_x - (4\Delta_x + 11\Delta_y r - 21r_y\Delta)r_{xx} - 12r_{xxy}r\Delta$$
$$- r_{xxx}\Delta - ((37\Delta_x + 53\Delta_y r - 150r_y\Delta)r - 33r_x\Delta)r_{xy} - 33r_{xyy}r^2\Delta$$
$$- 3\big(7\varphi_y^4\beta\psi r + 7\varphi_y^3\psi_y\alpha r - 5\varphi_y^2\alpha\Delta + 7\psi_{yyyy}r\big)\varphi_y r^4)\varphi_y^3$$
$$- 3(2(5((2\Delta_{xx} + 7\Delta_{yy}r^2 + 6\Delta_{xy}r - (13\Delta_x + 19\Delta_y r - 20r_y\Delta)r_y$$
$$- 13r_{yy}r\Delta)r^2 - ((3\Delta_x + 5\Delta_y r - 11r_y\Delta)r - r_x\Delta)r_x - r_{xx}r\Delta$$
$$- 6r_{xy}r^2\Delta) - 21\varphi_y\psi_{yyy}r^5)\varphi_y^2 - 15((2((5(\Delta_x + 2\Delta_y r) - 12r_y\Delta)r$$
$$- 3r_x\Delta) - 7\varphi_y\psi_{yy}r^3)\varphi_y + 7(\varphi_y\psi_y r - 5\Delta)\varphi_{yy}r^2)\varphi_{yy}r^2)\varphi_{yy}$$
$$- 2(2(5((5(\Delta_x + 2\Delta_y r) - 12r_y\Delta)r - 3r_x\Delta) - 21\varphi_y\psi_{yy}r^3)\varphi_y$$
$$+ 15(7\varphi_y\psi_y r - 30\Delta)\varphi_{yy}r^2)\varphi_{yyy}\varphi_y r^2 + 3(7\varphi_y\psi_y r - 25\Delta)\varphi_{yyyy}\varphi_y^2 r^4\big], \qquad (A.42)$$

$$K_1 = -\big(\varphi_y^3\Delta\big)^{-1}\big[((7(\Delta_{xxy} + \Delta_{yyy}r^2)r + 3\Delta_{xxx} + 7\Delta_{xyy}r^2$$
$$- (33\Delta_{xx} + 28\Delta_{yy}r^2 + 49\Delta_{xy}r + 2(59r_y\Delta - 56\Delta_x - 42\Delta_y r)r_y)r_y$$
$$- (43\Delta_x + 42\Delta_y r - 128r_y\Delta)r_{yy}r - 23r_{yyy}r^2\Delta)r^2$$
$$- ((12\Delta_{xx} + 7\Delta_{yy}r^2 + 21\Delta_{xy}r + 2(86r_y\Delta - 49\Delta_x - 35\Delta_y r)r_y$$
$$- 49r_{yy}r\Delta)r + (85r_y\Delta - 15\Delta_x - 21\Delta_y r)r_x)r_x$$
$$- ((8\Delta_x + 7\Delta_y r - 32r_y\Delta)r - 10r_x\Delta)r_{xx} - 9r_{xxy}r^2\Delta - 2r_{xxx}r\Delta$$
$$- ((29\Delta_x + 21\Delta_y r - 95r_y\Delta)r - 46r_x\Delta)r_{xy}r - 16r_{xyy}r^3\Delta$$
$$- \big(7\varphi_y^4\beta\psi r + 7\varphi_y^3\psi_y\alpha r - 6\varphi_y^2\alpha\Delta + 7\psi_{yyyy}r\big)\varphi_y r^5)\varphi_y^3$$
$$- (2(5((4\Delta_{xx} + 7\Delta_{yy}r^2 + 7\Delta_{xy}r - (23\Delta_x + 21\Delta_y r - 31r_y\Delta)r_y$$
$$- 17r_{yy}r\Delta)r^2 - ((9\Delta_x + 7\Delta_y r - 27r_y\Delta)r - 6r_x\Delta)r_x - 3r_{xx}r\Delta$$
$$- 10r_{xy}r^2\Delta) - 21\varphi_y\psi_{yyy}r^5)\varphi_y^2 - 15((3((5\Delta_x + 7\Delta_y r - 11r_y\Delta)r$$
$$- 4r_x\Delta) - 7\varphi_y\psi_{yy}r^3)\varphi_y + 7(\varphi_y\psi_y r - 6\Delta)\varphi_{yy}r^2)\varphi_{yy}r^2)\varphi_{yy}r$$
$$- 2((5((5\Delta_x + 7\Delta_y r - 11r_y\Delta)r - 4r_x\Delta) - 14\varphi_y\psi_{yy}r^3)\varphi_y$$
$$+ 5(7\varphi_y\psi_y r - 36\Delta)\varphi_{yy}r^2)\varphi_{yyy}\varphi_y r^3 + (7\varphi_y\psi_y r - 30\Delta)\varphi_{yyyy}\varphi_y^2 r^5\big], \qquad (A.43)$$

$$K_0 = \big(\varphi_y^3\Delta\big)^{-1}\big[((((2(r_{xxy} + 2r_{yyy}r^2)r + r_{xxx} + 3r_{xyy}r^2)\Delta$$
$$+ 3(3\Delta_x + 2\Delta_y r - 8r_y\Delta)r_{yy}r^2)r - ((10r_x + 11r_y r)\Delta$$
$$- (4\Delta_x + \Delta_y r)r)r_{xx} - ((13r_x + 20r_y r)\Delta - (7\Delta_x + 3\Delta_y r)r)r_{xy}r$$
$$+ \big(\big(\varphi_y^4\beta\psi + \varphi_y^3\psi_y\alpha + \psi_{yyyy}\big)r - \varphi_y^2\alpha\Delta\big)\varphi_y r^5 + (9\Delta_{xx} + 4\Delta_{yy}r^2$$

$$
\begin{aligned}
&+7\Delta_{xy}r - 2(13\Delta_x + 6\Delta_y r - 12r_y\Delta)r_y)r_y r^2 - ((\Delta_{xxy} + \Delta_{yyy}r^2)r \\
&+ \Delta_{xxx} + \Delta_{xyy}r^2)r^2)r - ((2((17\Delta_x + 5\Delta_y r - 23r_y\Delta)r_y + 6r_{yy}r\Delta) \\
&- (6\Delta_{xx} + \Delta_{yy}r^2 + 3\Delta_{xy}r))r^2 - (5(3r_x + 8r_y r)\Delta \\
&- 3(5\Delta_x + \Delta_y r)r)r_x)r_x)\varphi_y^3 - ((2((5(r_{xx} + 3r_{yy}r^2 + 2r_{xy}r)\Delta \\
&+ 3\varphi_y\psi_{yyy}r^4 + 5(5\Delta_x + 3\Delta_y r - 6r_y\Delta)r_y r - 5(\Delta_{xx} + \Delta_{yy}r^2 + \Delta_{xy}r)r)r \\
&- 5((3r_x + 7r_y r)\Delta - (3\Delta_x + \Delta_y r)r)r_x)\varphi_y^2 - 15((3(r_x + 2r_y r)\Delta \\
&+ \varphi_y\psi_{yy}r^3 - 3(\Delta_x + \Delta_y r)r)\varphi_y - (\varphi_y\psi_y r - 7\Delta)\varphi_{yy}r^2)\varphi_{yy}r^2)\varphi_{yy} \\
&+ (2((5(r_x + 2r_y r)\Delta + 2\varphi_y\psi_{yy}r^3 - 5(\Delta_x + \Delta_y r)r)\varphi_y \\
&- 5(\varphi_y\psi_y r - 6\Delta)\varphi_{yy}r^2)\varphi_{yyy} + (\varphi_y\psi_y r - 5\Delta)\varphi_{yyyy}\varphi_y r^2)\varphi_y r^2)r^2\big]. \qquad \text{(A.44)}
\end{aligned}
$$

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