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Symmetries of Integro-Differential Equations: A Survey of Methods Illustrated by the Benny Equations

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Abstract. Classical Lie group theory provides a universal tool for calculating symmetry groups for systems of differential equations. However Lie's method is not as much effective in the case of integral or integro-differential equations as well as in the case of infinite systems of differential equations.

This paper is aimed to survey the modern approaches to symmetries of integro-differential equations. As an illustration, an infinite symmetry Lie algebra is calculated for a system of integro-differential equations, namely the well-known Benny equations. The crucial idea is to look for symmetry generators in the form of canonical Lie–Bäcklund operators.

Keywords: Integro-differential equation, symmetry, Benny equations.

1. Introduction

The major obstacle for the application of Lie's infinitesimal techniques to integro-differential equations or infinite systems of differential equations is that the *frames* (see, e.g. [1] or [2]) of these equations are not locally defined in the space of differential functions. In consequence, the crucial idea of *s*plitting of determining equations into over-determined systems, commonly used in the classical Lie group analysis, fails.

1.1. Different Forms of the Benny Equations

The Benny equations referred to by the name of the author of a pioneering work [3] appear in long wavelength hydrodynamics of an ideal incompressible fluid of a finite depth in a gravitational field. From the group theoretical point of view they are of particular interest due to the existence of an infinite set of conservation laws obtained in [3]. The latter property of the Benny equations emphasizes their significance that goes far beyond an interesting example of an integrable system of hydrodynamic equations.

In practice, the Benny equations are used in various representation. One of them is the kinetic Benny equation (a kinetic equation with a self-consistent field):

$$f_t + v f_x - A_x^0 f_v = 0, \quad A^0(t,x) = \int_{-\infty}^{+\infty} f(t,x,v)\,\mathrm{d}v. \tag{1}$$

This equation appears as a unique representative of a set of hierarchy of kinetic equations of Vlasov-type [4]. A detailed study of its group properties will lead to better understanding of the symmetry properties of kinetic equations of collisionless plasma, viz. the Vlasov–Maxwell equations that have both theoretical and practical interest, e.g. while dealing with controlled nuclear fusion programme.

Another form of the Benny equations is an infinite set of coupled equations

$$A_t^i + A_x^{i+1} + i A_x^0 A^{i-1} = 0, \quad i \geq 0 \tag{2}$$

for a countable set of functions A^i of two independent variables, time t and the spatial coordinate x. In terms of hydrodynamics these functions appear as averaged values (with respect to the depth) of integer powers $i \geq 0$ of the horizontal component of the liquid flow velocity. The corresponding integrals that describe this averaging are taken over the vertical coordinate in the limits from the flat bottom up to the free liquid surface. Solutions, Hamiltonian structure and conservation laws for Equations (2) were discussed in detail in [5, 6].

From the kinetic point of view the system (2) can be treated as a system of equations for moments of the distribution function f that obeys the kinetic Benny equation (1)

$$A^i(t,x) = \int_{-\infty}^{+\infty} v^i f \,\mathrm{d}v, \quad i \geq 0. \tag{3}$$

This fact with the explicit formulation of the Benny equation (1) was first stated independently in [7, 8]. The Lagrangian change of the Euler velocity v,

$$v = V(t,x,u) \tag{4}$$

yields one more representation for Benny equations (1):

$$f_t + V f_x = 0, \quad V_t + V V_x = -A_x^0, \quad A^0(t,x) = \int V_u f(t,x,u)\,\mathrm{d}u. \tag{5}$$

Equations (5) are readily converted into the hydrodynamic-type form

$$n_t + (nV)_x = 0, \quad V_t + V V_x = -A_x^0, \quad A^0 = \int n(t,x,u)\,\mathrm{d}u, \tag{6}$$

if one employs the 'density' n depending on the Lagrangian velocity u:

$$n = f(t,x,u) V_u. \tag{7}$$

Using the form (6) of the Benny equations an infinite set of conservation laws were constructed in [7] with the densities regarded as functions of the Lagrangian velocity u.

Furthermore, we will rewrite the integro-differential Benny equations (1) in the form of differential equations by introducing the following nonlocal variables:

$$g = \int_{-\infty}^{v} f\,\mathrm{d}v, \quad h = \int_{v}^{+\infty} f\,\mathrm{d}v. \tag{8}$$

In terms of the latter variables Equations (1) are written as

$$f_t + v f_x - (g_x + h_x) f_v = 0, \quad g_v = f, \quad h_v = -f. \tag{9}$$

The knowledge of the complete Lie–Bäcklund symmetry for the Benny equations in different representations (1), (2), (5), (6) and (9) can clarify the question of structure of solutions and conservation laws for these equations. This statement is partially confirmed by the fact that one of the main results of the works [5, 6], namely the higher order Benny equations, can be re-formulated in terms of the first order Lie–Bäcklund group, admitted by the system (2). Unfortunately, the complete description of the Lie–Bäcklund symmetry for Equations (2) is not available in the literature. The goal of this paper is to contribute to this problem by calculating an infinite (countable) part of the Lie point symmetries of the moment equations (2).

2. Generalities

Here, we briefly discuss different known approaches to calculating symmetry groups for integro-differential equations. Loosely speaking, these approaches can be divided into two large groups: indirect and direct methods.

Algorithms of the first group rest on the possibility to replace in any way input nonlocal (integro-differential) equations by a system of differential equations. Then the resulting system of differential equations is analyzed using standard methods of a classical Lie group analysis. Here we point on two different ways of reducing nonlocal equations to differential ones.

2.1. Indirect Methods

2.1.1. *Method of Moments*

In this approach, the system of basic *integro-differential* equations for a function f (e.g., the kinetic equation (1)), that usually contains nonlocal terms depending on moments (3) of this function, is reduced to an infinite system of *differential* equations for these moments (in our case this is the system (2)). The admitted symmetry group is then calculated using the traditional methods of Lie group analysis for any *finite* subsystem of k equations of this general *infinite* system. Then an intersection of all admitted groups is defined and a transition to the limiting case $k \to \infty$ is fulfilled. The resultant algebra of group generators thus obtained is used to reconstruct the algebra with the original function f directly involved. This procedure usually employs the explicit form of finite group transformations for moments (3) and the relations between A^i and the function f. The last step is not trivial in any case as there may exist different representation for the equations for the moments A^i and the resultant group depends on the form of this representation. Hence, the transition from the algebra of group generators for the moments representation back to the symmetry of original equations formulated in terms of input functions form a special problem which we are going to discuss in Section 3.2. The above described algorithm was realized to calculate Lie point symmetry group for Vlasov–Maxwell equations in plasma theory [9–11] and for Benny, Vlasov-type and Boltzmann-type kinetic equations [4, 12, 13].

2.1.2. *Method of Boundary-Differential Equations*

The following method was developed in [14, 15] on basis of the concept of covering and applied to a coagulation kinetic equation. In this case each definite integral in the equation is replaced by the corresponding difference of values of the antiderivative on a boundary sets. The integro-differential equation takes the boundary (or functional) differential form. As shown in [14, 15] a geometric theory of boundary differential equations can be constructed in just the same way as the analogous theory of differential equations thus allowing to define and compute not only classical but higher symmetries as well. It is noted that the elimination of integrals by introducing potentials depends on the choice of potential variables and can be executed in different ways. Hence the result of group calculation and its dimension is influenced by the form of potential variables involved.

2.2. Direct Methods

Direct methods of finding symmetries were developed in [16–21] (see also [28]) and [22, 23] and applied to find symmetries of kinetic Boltzmann equation, the equations of motion of viscoelastic medium and Vlasov–Maxwell equations of plasma theory. To extend the classical Lie algorithm to integro-differential equations it appears necessary to resolve several problems. First, one should define the local one-parameter transformation group G for the nonlocal (integro-differential) equations and formulate the invariance criteria that lead to determining equations.

Let an integro-differential equation under consideration be expressed as a zero equality for some functional (here we indicate only one argument for a function f)

$$F[f(x)] = 0, \tag{10}$$

and let G be a local one-parameter group that transforms f to $\tilde{f}(x)$,

$$\tilde{f}(x) = f + a æ + o(a^2), \quad \tilde{x} = x, \tag{11}$$

Here we use the canonical group representation hence independent variables x do not vary. Then the local group G of point transformations (11) is called a symmetry group of integro-differential equations (10) iff for any a the function F does not vary [17],

$$F[\tilde{f}(x)] = 0. \tag{12}$$

Differentiating (12) with respect to group parameter a and assuming $a \to 0$ gives the determining equations. In contrast to the case of input differential equations these determining equations are in general also integro-differential.

The invariance criterion for F with respect to the admitted group can be expressed in an infinitesimal form using the canonical group operator Y,

$$YF\big|_{F=0} = 0, \quad \text{where} \quad Y \equiv \int \mathrm{d}y æ(y) \frac{\delta}{\delta f(y)} . \tag{13}$$

Here with the goal to generalize the action of a canonical group operator not only on differential functions but on *functionals* as well we use variational differentiation in the definition of Y [23, 24]. One can verify by direct calculation that the action of Y on any differential function and its derivatives, e.g., f and f_x, ... produces the usual result: $Yf = æ$, $Yf_x = D_x(æ)$ and so on. Hence, if F describe usual differential equations then formulas (13) lead to standard local determining equations, while for F having the form of integro-differential equations formulas (13) can be treated as *nonlocal* determining equations as they depend both on local and nonlocal variables.

In order to find solutions of determining equations one can use different approaches, e.g. expanding coordinates of group generator into formal power series and equating coefficients of various powers [16, 17]. However there exists a more traditional way. As we treat local and nonlocal variables in determining equations as independent it is possible to separate these equations into local and nonlocal. The procedure of solving local determining equations is fulfilled in a standard way using Lie algorithm based on splitting the system of over-determined equations with respect to local variables and their derivatives. As a result we get expressions for coordinates of group generator that define the so-called group of *intermediate* symmetry [23]. In the similar manner the solution of nonlocal determining equations is fulfilled using the information borrowed from an intermediate symmetry and by 'variational' splitting of nonlocal determining equations using the procedure of variational differentiation. Therefore, the algorithm of finding symmetries of nonlocal equations appears as an algorithmic procedure that consists of a sequence of several steps: (a) defining the set of local group variables, (b) constructing determining equations on basis of the infinitesimal criterion of invariance, that employs the generalization of the definition of the canonical operator, (c) separating determining equations into local and nonlocal, (d) solving local determining equations using a standard Lie algorithm, and (e) solving nonlocal determining equations using the procedure of variational differentiation.

In the next sections, the above methods are applied to the Benny equations.

3. Lie Subgroup and Lie–Bäcklund Group: Statement of the Problem

3.1. Lie Subgroup: Direct Method of Calculation

A Lie subgroup, admitted by the kinetic Benny equation (1) in the space of four group variables

$$t,\ x,\ v,\ f \tag{14}$$

is defined by five basic infinitesimal operators

$$\begin{aligned} X_1 &= \partial_t, \quad X_2 = \partial_x, \quad X_3 = t\partial_x + \partial_v, \\ X_4 &= t\partial_t - v\partial_v - f\partial_f, \quad X_5 = x\partial_x + v\partial_v + f\partial_f\,. \end{aligned} \tag{15}$$

With the less computation difficulties this group can be obtained using the approach developed in [22–24] in application to group analysis of Vlasov–Maxwell equations in plasma theory (see also chapter 16 in [25]).

Here we demonstrate the application of the general scheme to Benny equations (1). The merit of this direct approach is the possibility to present the criterion of invariance with respect to local one-parameter point group transformations in a standard infinitesimal form. For the Benny equations the point symmetry group generator has the form

$$X = \xi^1\partial_t + \xi^2\partial_x + \xi^3\partial_v + \eta^1\partial_f + \eta^2\partial_{A^0}, \tag{16}$$

where the coordinates ξ and η depend on t, x, v, f and A^0.

In the canonical form this operator is written:

$$Y = æ^1\partial_f + æ^2\partial_{A^0}, \tag{17}$$

where

$$æ^1 = \eta^1 - \xi^1 f_t - \xi^2 f_x - \xi^3 f_v, \quad æ^2 = \eta^2 - \xi^1 A^0_t - \xi^2 A^0_x,$$

and its action on any function or functional should be understood in generalized sense as in (13). Applying the canonical group operator to the joint system of basic equations (1) and one more 'evident' equation, that expresses the fact that the moment of f does not depend upon v

$$A^0_v = 0, \tag{18}$$

gives the system of determining equations

$$D_t(æ^1) + vD_x(æ^1) - A^0_x D_v(æ^1) - D_x(æ^2) f_v = 0, \quad D_v(æ^2) = 0; \tag{19}$$

$$æ^2 = \int æ^1 \, \mathrm{d}v, \tag{20}$$

which should be solved in view of the the complete set of basic equations (1), (18).

In solving determining equations (19), group variables

$$\{f,\ A^0,\ f_x,\ A^0_x,\ f_v,\ \ldots\} \tag{21}$$

are treated as independent ones. This assumption separates determining equations into local equations (19) and nonlocal equation (20). Local determining equations are solved in a standard way using the computational algorithm of Lie group analysis. Then the functions ξ and η thus obtained define the so-called *intermediate* symmetry [23]

$$\begin{aligned} \xi^1 &= \xi^1(t), \quad \xi^2 = \frac{x}{2}\xi^1_t(t) + \alpha x + \beta(t), \quad \xi^3 = \alpha v - \frac{v}{2}\xi^1_t + \frac{x}{2}\xi^1_{tt} + \beta_t, \\ \eta^1 &= \eta^1(f), \quad \eta^2 = \gamma(t) + (2\alpha - \xi^1_t)A^0 - x\beta_{tt} - \frac{x^2}{4}\xi^1_{ttt}. \end{aligned} \tag{22}$$

Here $\xi^1(t)$, $\beta(t)$, $\gamma(t)$ and $\eta^1(f)$ are arbitrary functions of their arguments and α and ν are constants.

Now let us turn to the solution of the nonlocal determining equation (20) that we rewrite in the following form

$$\eta^2 - \xi^1 A^0_t - \xi^2 A^0_x = \int (\eta^1 - \xi^1 f_t - \xi^2 f_x - \xi^3 f_v)\, \mathrm{d}v. \tag{23}$$

As in the case of the local determining equations (19), the latter should be solved in view of the original equations (1), (18). Hence, calculating the derivatives A^0_t, A^0_x and f_t from the basic equation (1) and inserting them into (23) and in view of the above expressions for coordinates ξ and η we obtain the following nonlocal determining equation

$$\int_{-\infty}^{+\infty} \left[\eta^1(f) - \left(\alpha - \frac{1}{2}\xi^1_t\right) f \right] \mathrm{d}v = \gamma - x\beta_{tt} - \frac{x^2}{4}\xi^1_{ttt}\,. \tag{24}$$

As any determining equation, (24) is the equality with respect to all group variables that appear in this equation. Therefore differentiating it with respect to any group variables also leads to equalities. Hence, nonlocal determining equation can be split with respect to independent group variable f using the variational differentiation. Since the right-hand part of (24) does not depend on f, it reduces after differentiation and the remaining terms are written as

$$\frac{\delta}{\delta f(v')} \int_{-\infty}^{+\infty} \left[\eta^1(f) - \left(\alpha - \frac{1}{2}\xi^1_t\right) f \right] \mathrm{d}v = 0. \tag{25}$$

It is essential that the nonlocal determining equation (23) should be solved simultaneously with its differential consequence, i.e. any solution of (25) must appear as the solution of (24). Introducing the variational derivative $(\delta/\delta f(v'))$ inside the integral over v

$$\int_{-\infty}^{+\infty} \left[\eta^1_f - \left(\alpha - \frac{1}{2}\xi^1_t\right)\right] \frac{\delta f(v)}{\delta f(v')} \, \mathrm{d}v = 0,$$

and eliminating integration over v with the help of the Dirac delta-function,

$$\frac{\delta f(v)}{\delta f(v')} = \delta(v - v'),$$

one comes to the first order differential equation for η^1

$$\eta^1_f - \left(\alpha - \frac{1}{2}\xi^1_t\right) = 0,$$

which gives the linear dependence of η^1 upon f,

$$\eta^1 = f\left(\alpha - \frac{1}{2}\xi^1_t\right) + C, \tag{26}$$

with some constant C. Substituting this result back into (24) yields the zero-value of this constant provided integral has finite value. As η^1 does not depend on t then differentiating (26) with respect to t gives $\xi^1_{tt} = 0$. Differentiating (24) with respect to x gives two more equations,

$$\gamma = \beta_{tt} = 0.$$

Solving these equations gives final expressions for coordinates ξ and η:

$$\begin{aligned} \xi^1 &= c^1 + c^4 t, \quad \xi^2 = c^2 + c^3 t + c^5 x, \quad \xi^3 = c^3 + (-c^4 + c^5)v, \\ \eta^1 &= (-c^4 + c^5) f, \quad \eta^2 = 2(-c^4 + c^5)A^0. \end{aligned} \tag{27}$$

These coordinates give rise to the five-dimensional Lie algebra with generators given by (15). One can see that terms proportional to A^0 which come from æ^2 are omitted in X_4, X_5 as they appear as the result of *prolongation* [26] of these operators *on a nonlocal variable* A^0. This procedure for any of operators X_4, X_5 is described in a concise form in the next section.

3.2. Extension of Lie Point Symmetry Generators to Nonlocal Variables

To fulfill the procedure of prolongation of the Lie point symmetry generators one should first rewrite the operator, say X_5, in a canonical form

$$Y_5 = æ^1 \partial_f, \quad æ^1 = (-x f_x - v f_v + f). \tag{28}$$

Then formally prolong this operator on the nonlocal variable A^0

$$Y_5 + æ^2 \equiv æ^1 \partial_f + æ^2 \partial_{A^0}. \tag{29}$$

The integral relation between æ^1 and æ^2 is obtained by applying the generator (29) to the second equation in (1) that is treated here as the *definition* of A^0. This relation here coincides with nonlocal determining equation (20). Substituting $æ^1$ from (28) in (20) and calculating integrals obtained (integrating by paths) gives the desired coordinate $æ^2$

$$æ^2 = (2A^0 - xA^0_x). \tag{30}$$

Inserting this into (29) and returning back to the noncanonical representation we get the following generator

$$X_5 = x\partial_x + v\partial_v + f\partial_f + 2A^0\partial_{A^0}, \tag{31}$$

that correlates with the result (27).

Prolongation of infinitesimal operators (15) on nonlocal variables (3) extends the set of group variables (14) up to a countable set

$$t,\ x,\ v,\ f,\ A^0,\ \ldots,\ A^i,\ \ldots. \tag{32}$$

In the latter case infinitesimal operators (15) rewritten in the canonical form [1, sec. 8.4.2.] and restricted on the sub-manifold

$$t,\ x,\ A^0,\ \ldots,\ A^i,\ \ldots. \tag{33}$$

are given by the following expressions

$$\begin{aligned} X_1 &= \sum_{i=0}^{\infty}(A^{i+1}_x + iA^{i-1}A^0_x)\partial_{A^i}; \quad X_2 = \sum_{i=0}^{\infty} A^i_x\partial_{A^i}; \quad X_3 = \sum_{i=0}^{\infty}(iA^{i-1} - tA^i_x)\partial_{A^i}; \\ X_4 &= \sum_{i=0}^{\infty}[(i+2)A^i - t(A^{i+1}_x + iA^{i-1}A^0_x)]\partial_{A^i}; \quad X_5 = \sum_{i=0}^{\infty}[(i+2)A^i - xA^i_x]\partial_{A^i}. \end{aligned} \tag{34}$$

It can be easily checked that infinitesimal operators (34) are admitted by Benny equations (2) and it goes without saying that they directly result from the group analysis of Benny equations (2). Just in this way (i.e., using the method of moments) infinitesimal operators (15) were first obtained in [4] by using noncanonical form of infinitesimal operators (34) with the subsequent passage to the representation (15) in the space of variables (32).

3.3. Incompleteness of the Point Group: Statement of the Problem

It is evident, however, that the subgroup (34) does not exhaust the complete group symmetry of Benny equations (2). The incompleteness of the result (34) is obvious form many points of view. Here we shall only point on the nonconformity of finite dimension of the algebra (34) to the infinite set of conservation laws for Benny equations, and on the infinite extension of the point symmetry group for Benny equations in the form of (5), (6) with Lagrangian velocity (see also in the context chapter 16 in [25] where this extension was outlined for Vlasov kinetic equation in plasma theory). Here of principle significance for us is the following statement: *the group* (34) *is incomplete not only from the standpoint of Lie–Bäcklund symmetry for Benny equations but also from the standpoint of the Lie point symmetry*. The validity of the statement can be proved by direct solving of determining equations for the first order Lie–Bäcklund group (contact group, that is not reduced to point one)

$$D_t(æ^i) + D_x(æ^{i+1}) + iA^{i-1}D_x(æ^0) + iA^0_x æ^{i-1} = 0, \quad i \geq 0, \tag{35}$$

where coordinates $æ^i$ of canonical operator

$$X = \sum_{i=0}^{\infty} æ^i \partial_{A^i}, \tag{36}$$

depend upon the countered set of group variables

$$t,\ x;\ A^0,\ \ldots,\ A^j,\ \ldots;\ A^0_x,\ \ldots,\ A^j_x, \ldots; \quad j \geq 0. \tag{37}$$

To prove the above statement one can consider only partial solutions of determining equations (35)

$$æ^i = \eta^i(A^0, \ldots, A^j, \ldots); \quad i, j \geq 0, \tag{38}$$

that depend upon moments $A^j,\ j \geq 0$, and does not depend upon $t,\ x$. It appears that thanks to these infinitesimal operators (36), (38) an infinite extension of the group (34) takes place. Now the problem is to find these operators.

4. Determining Equations and Their Solution

4.1. General Form of the Determining Equations

Before proceeding further we write determining equations of first-order Lie–Bäcklund group, admitted by a more general (as compared to (2)) infinite system of coupling equations for functions $A^i(t, x)$ with the arbitrary element $\varphi(A^0)$

$$A^i_t + A^{i+1}_x + iA^{i-1}[\varphi(A^0)]_x = 0, \quad i \geq 0. \tag{39}$$

For the coordinates $æ^i$ of canonical infinitesimal operator (36) the following chains of determining equations are valid which result from splitting (39) with respect to second derivatives:

$$\begin{aligned}
&æ^{i+1}_{A^0_x} + i\varphi_1 A^{i-1} æ^0_{A^0_x} = \sum_{j=0}^{\infty} j\varphi_1 A^{j-1} æ^i_{A^j_x}, \quad i \geq 0;\\
&æ^{i+1}_{A^j_x} + i\varphi_1 A^{i-1} æ^0_{A^j_x} = æ^i_{A^{j-1}_x}, \quad i \geq 0,\ j \geq 1;\\
&æ^i_t + æ^{i+1}_x + i\varphi_1 A^{i-1} æ^0_x + A^0_x(i\varphi_1 æ^{i-1} + i\varphi_2 A^{i-1} æ^0)\\
&\quad + \sum_{j=0}^{\infty} [i\varphi_1 A^{i-1} A^j_x æ^0_{A^j} - (A^{j+1}_x + j\varphi_1 A^0_x A^{j-1}) æ^i_{A^j} + A^j_x æ^{i+1}_{A^j}]\\
&\quad - \sum_{j=0}^{\infty} jA^0_x(\varphi_1 A^{j-1}_x + \varphi_2 A^0_x A^{j-1}) æ^i_{A^j_x} = 0, \quad i \geq 0.
\end{aligned} \tag{40}$$

Here φ_1 and φ_2 are the first and the second derivatives of the function φ with respect to its argument. From the various standpoints at list three distinct values of the function φ are specified. In case $\varphi(A^0) = A^0$ we come to kinetic Benny equations (1), whereas for $\varphi = a(A^0)^2$

extension of the admitted point group takes place thanks to projective transformations in t, x-plane (see [4]). For $\varphi = a \ln A^0$ the corresponding kinetic equation

$$f_t + v f_x - a\frac{A_x^0}{A^0} f_v = 0, \quad A^0 = \int_{-\infty}^{+\infty} \mathrm{d}v\, f, \tag{41}$$

that gives rise to the discussed system of equations for moments, is of special interest in plasma theory. It appears as the equation for the distribution function of plasma ions, while electrons obey the Boltzmann distribution. More complicated dependencies of $\varphi(A^0)$ upon A^0 can also be of interest in plasma physics for non-Boltzmann distribution functions for hot electrons. Equation (41) was studied in detail in [27].

For the Benny equations (2) the determining equations (40) are rewritten in the following form

$$\begin{aligned}
&æ^{i+1}_{A_x^0} + iA^{i-1}æ^0_{A_x^0} - \sum_{j=0}^{\infty} jA^{j-1}æ^i_{A_x^j} = 0, \quad i \geq 0;\\
&æ^{i+1}_{A_x^{j+1}} - æ^i_{A_x^j} + iA^{i-1}æ^0_{A_x^{j+1}} = 0, \quad i \geq 0,\ j \geq 0;\\
&æ^i_t + æ^{i+1}_x + iA^{i-1}æ^0_x + A^0_x\left(iæ^{i-1} - \sum_{j=0}^{\infty} jA^{j-1}æ^i_{A^j} - \sum_{j=0}^{\infty}(j+1)A^j_x æ^i_{A_x^{j+1}}\right)\\
&\quad + iA^{i-1}\sum_{j=0}^{\infty} A^j_x æ^0_{A^j} + \sum_{j=0}^{\infty} A^j_x æ^{i+1}_{A^j} - \sum_{j=0}^{\infty} A^{j+1}_x æ^i_{A^j} = 0, \quad i \geq 0.
\end{aligned} \tag{42}$$

4.2. Solution of the Determining Equations

Under conditions (38) the determining equations (42) are splitted and reduced to two infinite chains of equalities, namely one-dimensional (vector) and two-dimensional (tensor):

$$\begin{aligned}
&\eta^{i+1}_{A^0} - \sum_{j=0}^{\infty} jA^{j-1}\eta^i_{A^j} + iA^{i-1}\eta^0_{A^0} + i\eta^{i-1} = 0, \quad i \geq 0;\\
&\eta^{i+1}_{A^{k+1}} - \eta^i_{A^k} + iA^{i-1}\eta^0_{A^{k+1}} = 0, \quad i \geq 0,\ k \geq 0.
\end{aligned} \tag{43}$$

The apparent difficulty in analytical solving of the given system of determining equations (43) is due to a ‘nonlocal’ nature of the second term in the vector chain in the form of an infinite sum with respect to index $j \geq 0$. The measure of this nonlocality is characterized by a number of nonzero components of tensor η^i_j. But in fact in case of an overdetermined system (43) we obtain a finite upper value of the summation index $j < \infty$, which depends upon the other index i of this tensor. In order to be sure that it is true we shall first consider the system of determining equations (43) in two particular cases, namely for $i = 0$ and $i = 1$. In case $i = 0$ we have two coupled determining equations just for two coordinates η^0 and η^1 of the desired infinitesimal operator (36), (38):

$$\eta^1_{A^0} - \sum_{j=0}^{\infty} jA^{j-1}\eta^0_{A^j} = 0, \quad \eta^1_{A^{k+1}} = \eta^0_{A^k}, \quad k \geq 0. \tag{44}$$

The second determining equation in (44) enables to eliminate the coordinate η^0 from the first determining equation and obtain as a consequence of the system (44) the following isolated scalar determining equation for the coordinate η^1 only

$$\eta^1_{A^0} - \sum_{j=0}^{\infty} jA^{j-1}\eta^1_{A^{j+1}} = 0. \tag{45}$$

For $i = 1$ the system (43) yields coupled equations for three coordinates η^0, η^1 and η^2

$$\eta^2_{A^0} = \sum_{j=0}^{\infty} jA^{j-1}\eta^1_{A^j} - (\eta^0 + A^0\eta^0_{A^0}), \quad \eta^2_{A^{k+1}} = \eta^1_{A^k} - A^0\eta^0_{A^{k+1}}, \quad k \geq 0. \tag{46}$$

Compatibility conditions for determining equations (46)

$$\eta^2_{A^0A^{k+1}} = \eta^2_{A^{k+1}A^0}, \quad k \geq 0, \tag{47}$$

enables to eliminate the coordinate η^2 from (46) and obtain one more closed vector determining equation for η^1 (more precisely, the determining equation that contains the first and the second derivatives of η^1 with respect to A^i):

$$\eta^1_{A^kA^0} - \sum_{j=0}^{\infty} jA^{j-1}\eta^1_{A^jA^{k+1}} - (k+2)\eta^1_{A^{k+2}} = 0, \quad k \geq 0. \tag{48}$$

Differentiating the scalar equality (45) by moments A^k yields one more vector corollary for the coordinate η^1:

$$\eta^1_{A^0A^k} - \sum_{j=0}^{\infty} jA^{j-1}\eta^1_{A^{j+1}A^k} - (k+1)\eta^1_{A^{k+2}} = 0, \quad k \geq 0. \tag{49}$$

From the compatibility conditions for two determining equations (48) and (49) it follows that the coordinate η^1 does not depend upon the moments A^i which are higher than A^1

$$\eta^1_{A^{i+2}} = 0, \quad i \geq 0. \tag{50}$$

This formula arises as a result of mutual subtraction of determining equations (48) and (49) in view of the equality of terms containing summation over index $j \geq 0$, that is the corollary of the second determining equation in (44)

$$\eta^1_{A^{j+1}A^k} = \eta^1_{A^jA^{k+1}}, \quad j,\ k \geq 0. \tag{51}$$

The symmetry of the second derivatives of η^1 given by (51) results from the tensor form of a compatibility condition for two vector determining equations from (44), which differ only in 'sounding' index

$$\eta^0_{A^k} = \eta^1_{A^{k+1}}, \quad \eta^0_{A^j} = \eta^1_{A^{j+1}}, \quad j,\ k \geq 0. \tag{52}$$

The result (50) is a milestone on the way of solving the system of determining equations (43). Indeed, in view of (50) the second determining equation of the system (44) yields the requirement of independence of the coordinate η^0 upon all higher moments, that differ from A^0

$$\eta^0_{A^{i+1}} = 0, \quad i \geq 0, \tag{53}$$

and with (53) in mind the tensor chain from (43) is simplified in such a way

$$\eta^{i+1}_{A^{k+1}} = \eta^{i}_{A^{k}}, \quad i \geq 0,\ k \geq 0, \tag{54}$$

that enables to modify the equality (50) in the sense that any coordinate η^i of the desired operator (36), (38) does not depend upon any moments A^j, higher than A^i

$$\eta^{i}_{A^{i+k+1}} = 0, \quad i \geq 0,\ k \geq 0. \tag{55}$$

The last equality sets a finite upper limit $i \geq j$ to the summation index $j \geq 0$ in the second (nonlocal) term of the left hand side of the vector determining equation of the system (43)

$$\eta^{i+1}_{A^0} - \sum_{j=0}^{i} jA^{j-1}\eta^{i}_{A^j} + iA^{i-1}\eta^{0}_{A^0} + i\eta^{i-1} = 0, \quad i \geq 0. \tag{56}$$

The use of one of the equalities (54)

$$\eta^{0}_{A^0} = \eta^{i}_{A^i}, \quad i \geq 0 \tag{57}$$

enables to rewrite the chain of determining equations (56) in even a more simple way

$$\eta^{i+1}_{A^0} - \sum_{j=0}^{i-1} jA^{j-1}\eta^{i}_{A^j} + i\eta^{i-1} = 0, \quad i \geq 0, \tag{58}$$

whereas one of the corollaries of determining equations (54), that results for $i = k+1$ in combination with (53)

$$\eta^{i+1}_{A^i} = 0, \quad i \geq 0, \tag{59}$$

lowers the upper value of the summation index $j \geq 0$ in (58) by unit

$$\eta^{i+1}_{A^0} - \sum_{j=0}^{i-2} jA^{j-1}\eta^{i}_{A^j} + i\eta^{i-1} = 0, \quad i \geq 0. \tag{60}$$

Collecting the arising determining equations (53), (54) and (59), (60) we arrive to a much more simplified (but equivalent) formulation of the system (43), which really is integrated below

$$\eta^{i+1}_{A^0} - \sum_{j=0}^{i-2} jA^{j-1}\eta^{i}_{A^j} + i\eta^{i-1} = 0, \quad \eta^{i+1}_{A^i} = 0, \quad i \geq 0;$$

$$\eta^{i+1}_{A^{k+1}} = \eta^{i}_{A^{k}}, \quad \eta^{i}_{A^{i+k}} = 0, \quad i \geq 0,\ k \geq 0. \tag{61}$$

The last of the four equalities in (61) is strengthened in comparison with (55) thanks to the condition

$$\eta^{i}_{A^i} = 0, \quad i \geq 0, \tag{62}$$

which can be obtained as follows. For example, the first two equalities (62) for $i = 0$ and $i = 1$ result as a corollary of (57) and compatibility conditions

$$\eta^{3}_{A^0 A^1} = \eta^{3}_{A^1 A^0} \tag{63}$$

for the first derivatives of the coordinate η^3 with respect to moments A^0 and A^1 (see (60) for $i = 1$ and $i = 2$)

$$\eta^3_{A^0} = -2\eta^1, \quad \eta^3_{A^1} = \eta^2_{A^0} = -\eta^0. \tag{64}$$

After that, the validity of the remaining equalities (62) becomes obvious thanks to (57).

Before proceeding to enumerating all solutions of the system of determining equations (61), we present here yet another form of the chain (60)

$$\eta^{i+1}_{A^0} - \sum_{j=0}^{i-2} jA^{j-1}\eta^{i-j}_{A^0} + i\eta^{i-1} = 0, \quad i \geq 0. \tag{65}$$

This form can be employed to clarify the general structure of these solutions on basis of the corresponding generating functions.

4.3. Discussion of the Solution of the Determining Equations

The integrability procedure in itself for determining equations (61) is of no difficulties. For example the first six coordinates η^i $(0 \leq i \leq 5)$ of the desired infinitesimal operator (36), (38) are given by the following formulas for the general solutions of determining equations (61) that depend upon six arbitrary constants C^j $(0 \leq j \leq 5)$ and are described by polynomials in moments A^l

$$\begin{aligned}
\eta^0 &= C^0, \quad \eta^1 = C^1, \quad \eta^2 = C^2 - C^0A^0, \quad \eta^3 = C^3 - 2C^1A^0 - C^0A^1, \\
\eta^4 &= C^4 - 3C^2A^0 - 2C^1A^1 + C^0[-A^2 + (A^0)^2], \\
\eta^5 &= C^5 - 4C^3A^0 - 3C^2A^1 + C^1[-2A^2 + 3(A^0)^2] + C^0(-A^3 + 2A^0A^1).
\end{aligned} \tag{66}$$

It appears that the polynomial dependence of any solution η^i of determining equations (61) upon moments A^j is a general property of components of the vector η^i for any $i \geq 0$. The example (66) demonstrates that the procedure of obtaining solutions of determining equations (61) is reduced to their enumeration. To be concrete, we assume the following scheme of indicating of the k-th basic solution η^i_k of determining equations (61) for the coordinate η^i:

$$\eta^i_k = \begin{cases} 0, & i < k; \\ 1, & i = k; \\ 0, & i = k+1; \end{cases} \qquad [\eta^i_k] = i - k, \ i \geq k+2; \quad i, \ k \geq 0. \tag{67}$$

In the solutions (66) this scheme demands quit definite choice of values of integration constants C^j in the form of Kronecker symbols

$$C^j = \delta_{jk}; \quad j, \ k \geq 0. \tag{68}$$

The last of the four equalities for η^i_k in (67) (in square brackets) indicates the homogeneity degree $(i - k)$ of the polynomial 'tail' of the solution η^i for $i \geq k + 2$ in accordance with the attributed to any of the moments A^i of the order i the homogeneity degree, which is equal to positive number $(i + 2)$ (see, e.g., [5])

$$[A^i] = i + 2, \quad i \geq 0. \tag{69}$$

For instance, the component η_1^5 of the basis solution η_1^i of determining equations (61) in accordance with (66), (67) and (69) has the homogeneity degree equal to four

$$\eta_1^5 = -2A^2 + 3(A^0)^2; \quad [\eta_1^5] = 4. \tag{70}$$

The indexing of the presented infinite (countable) vectors η^i by one more integral number $k \geq 0$ yields the desired representation of all linear independent solutions of determining equations (61) in the form of tensor of the second rank (matrix) η_k^i, in which the lower index $k \geq 0$ indicates the index of the basis infinitesimal operator in the general element of an infinite Lie algebra under consideration

$$X = \sum_{i,k=0}^{\infty} C^k \, \eta_k^i \, \partial_{A^i}, \tag{71}$$

Under the conditions (67) the integration of determining equations (61) for the given basis vector η_k^i for a fixed value $k \geq 0$ is carried out with boundary conditions, that are imposed by requirements (67) in a single way.

The representation of matrix η_k^i for different lines are as follows (i is the column number, k is the line number)

$$\eta_k^i = \{0, \dots, 0,\ 1,\ 0,\ -(k+1)A^0,\ -(k+1)A^1, \dots\}. \tag{72}$$

Here zeroes preceding unity describe matrix elements, which exist only for $i < k$, i.e. which are located below the principle diagonal $i = k$, that contains only units. The first nearest upper off-diagonal $i = k + 1$ also contains only zeroes. Expressions for elements from the second $i = k + 2$ and the third $i = k + 3$ upper off-diagonals are given in (72) explicitly: they contain monomials, the homogeneity degree of which is equal to 2 and 3 respectively, while the numerical coefficient $(k + 1)$ is defined by the line number.

In general, any one of the nonzero off-diagonals $i = k + s$ with the number $s \geq 2$ is presented by polynomials with the homogeneity degree equal to s. This 'line scheme' (72) is readily illustrated by a pictorial rendition of elements of the high left block of the discussed matrix ($0 \leq i \leq 5, 0 \leq k \leq 3$)

$$\eta_k^i = \begin{pmatrix} 1 & 0 & -A^0 & -A^1 & -A^2 + (A^0)^2 & -A^3 + 2A^0A^1 & \dots \\ 0 & 1 & 0 & -2A^0 & -2A^1 & -2A^2 + 3(A^0)^2 & \dots \\ 0 & 0 & 1 & 0 & -3A^0 & -3A^1 & \dots \\ 0 & 0 & 0 & 1 & 0 & -4A^0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \tag{73}$$

As a more illustrative example we present here the element η_1^i of the matrix (72) with sufficiently high column number $i = 10$ and the homogeneity degree 9, that is located in the line with $k = 1$ (the second from above)

$$\begin{aligned} \eta_1^{10} = & -2A^7 + 6A^5A^0 + 6A^4A^1 + 6A^3A^2 - 12A^3(A^0)^2 \\ & - 24A^2A^1A^0 - 4(A^1)^3 + 20A^1(A^0)^3. \end{aligned} \tag{74}$$

4.4. Illustrative Example for Matrix Elements

A much more comprehensive idea of definite expressions of matrix elements η^i_k is given by the following list of elements (with the previous result included) of the first 11 columns ($0 \le i \le 10$) and 4 lines ($0 \le k \le 3$) of matrix η^i_k, which define the k-th basic solution of determining equations (61) for vectors η^i_k of the canonical infinitesimal operator (36), (38). The lower index 'k' is omitted for simplicity.

(0) $k = 0;\ \eta^0 = 1,\ \eta^1 = 0,\ [\eta^i] = i,\ i \ge 2.$

$$\begin{aligned}
&\eta^2 = -A^0,\\
&\eta^3 = -A^1,\\
&\eta^4 = -A^2 + (A^0)^2,\\
&\eta^5 = -A^3 + 2A^0A^1,\\
&\eta^6 = -A^4 + 2A^0A^2 + (A^1)^2 - (A^0)^3,\\
&\eta^7 = -A^5 + 2A^0A^3 + 2A^2A^1 - 3A^1(A^0)^2,\\
&\eta^8 = -A^6 + 2A^0A^4 + 2A^3A^1 + (A^2)^2 - 3A^2(A^0)^2 - 3A^0(A^1)^2 + (A^0)^4,\\
&\eta^9 = -A^7 + 2A^0A^5 + 2A^4A^1 + 2A^3A^2 - 3A^3(A^0)^2\\
&\qquad - 6A^0A^1A^2 - (A^1)^3 + 4A^1(A^0)^3,\\
&\eta^{10} = -A^8 + 2A^0A^6 + 2A^5A^1 + A^4[2A^2 - 3(A^0)^2] + A^3[A^3 - 6A^0A^1]\\
&\qquad + A^2[-3(A^1)^2 - 3A^0A^2 + 4(A^0)^3] + 6(A^1)^2(A^0)^2 - (A^0)^5.
\end{aligned} \tag{75}$$

(1) $k = 1;\ \eta^0 = 0,\ \eta^1 = 1,\ \eta^2 = 0,\ [\eta^i] = i - 1,\ i \ge 3.$

$$\begin{aligned}
&\eta^3 = -2A^0,\\
&\eta^4 = -2A^1,\\
&\eta^5 = -2A^2 + 3(A^0)^2,\\
&\eta^6 = -2A^3 + 6A^0A^1,\\
&\eta^7 = -2A^4 + 6A^0A^2 + 3(A^1)^2 - 4(A^0)^3,\\
&\eta^8 = -2A^5 + 6A^0A^3 + 6A^2A^1 - 12A^1(A^0)^2,\\
&\eta^9 = -2A^6 + 6A^0A^4 + 6A^3A^1 + A^2[3A^2 - 12(A^0)^2] - 12A^0(A^1)^2 + 5(A^0)^4,\\
&\eta^{10} = -2A^7 + 6A^0A^5 + 6A^4A^1 + 6A^3[A^2 - 2(A^0)^2]\\
&\qquad - 24A^0A^1A^2 + A^1[-4(A^1)^2 + 20(A^0)^3].
\end{aligned} \tag{76}$$

(2) $k = 2;\ \eta^0 = 0,\ \eta^1 = 0,\ \eta^2 = 1, \eta^3 = 0,\ [\eta^i] = i - 2,\ i \ge 4.$

$$\begin{aligned}
&\eta^4 = -3A^0,\\
&\eta^5 = -3A^1,
\end{aligned}$$

$$\eta^6 = -3A^2 + 6(A^0)^2,$$
$$\eta^7 = -3A^3 + 12A^0A^1,$$
$$\eta^8 = -3A^4 + 12A^0A^2 + 6(A^1)^2 - 10(A^0)^3,$$
$$\eta^9 = -3A^5 + 12A^0A^3 + 12A^2A^1 - 30A^1(A^0)^2,$$
$$\eta^{10} = -3A^6 + 12A^0A^4 + 12A^3A^1 + 6(A^2)^2 - 30A^0(A^1)^2 + 15(A^0)^4 - 30A^2(A^0)^2. \quad (77)$$

(3) $k = 3;\ \eta^0 = 0,\ \eta^1 = 0,\ \eta^2 = 0,\ \eta^3 = 1,\ \eta^4 = 0,\ [\eta^i] = i - 3,\ i \geq 5.$

$$\eta^5 = -4A^0,$$
$$\eta^6 = -4A^1,$$
$$\eta^7 = -4A^2 + 10(A^0)^2,$$
$$\eta^8 = -4A^3 + 20A^0A^1,$$
$$\eta^9 = -4A^4 + 20A^0A^2 + 10(A^1)^2 - 20(A^0)^3,$$
$$\eta^{10} = -4A^5 + 20A^0A^3 + 20A^2A^1 - 60A^1(A^0)^2. \quad (78)$$

5. Conclusion

This paper presents a result of calculation of the infinite (countable) part of Lie point group admitted by the system of Benny equations – moment equations (2). In standard (noncanonical representation) the point Lie group of Benny equations (2) is described by the infinitesimal operator

$$X = \xi^1\partial_t + \xi^2\partial_x + \sum_{i=0}^{\infty}\eta^i\frac{\partial}{\partial A^i}, \quad (79)$$

where coordinates ξ and η obey the system of determining equations

$$\eta^{i+1}_{A^0} - \sum_{j=0}^{\infty} jA^{j-1}\eta^i_{A^j_x} + i\eta^{i-1} + iA^{i-1}(\eta^0_{A^0} + \xi^1_t - \xi^2_x) + (i+1)A^i\xi^1_x - \xi^2_t\delta_{i,0} = 0,$$
$$\eta^{i+1}_{A^{k+1}} - \eta^i_{A^k} + iA^{i-1}(\eta^0_{A^{k+1}} + \xi^1_x\delta_{0,k}) + (\xi^1_t - \xi^2_x)\delta_{i,k} + \xi^1_x\delta_{i+1,k} - \xi^2_t\delta_{i,k+1} = 0,$$
$$\eta^i_t + \eta^{i+1}_x + iA^{i-1}\eta^0_x = 0, \qquad i,\ k \geq 0. \quad (80)$$

Determining equations (80) result from (42) in account of relationships between coordinates of infinitesimal operators (79) and (36)

$$æ^i = \eta^i + \xi^1(A^{i+1}_x + iA^{i-1}A^0_x) - \xi^2A^i_x. \quad (81)$$

Infinitesimal operators (34), that were presented above, gives rise to the following coordinates

$$\xi^1 = K^4 + K^5t, \quad \xi^2 = K^1 + K^2t + K^3x,$$
$$\eta^i = iA^{i-1}K^2 + (i+2)A^i(K^3 - K^5). \quad (82)$$

The problem of finding coordinates of the operator (79) was first treated in [4], where only these solutions, namely (15), (34) and (82), were described. The main result of our paper is that point symmetries of Benny equations (2) are exhausted by formulas (34) and solutions of determining equations (61), i.e. determining equations (80) do not have any other solutions. Solutions of determining equations (61) which are responsible for the infinite part of the point group probably have not been known so far.

As a next step it seems intriguing to generalize the result (81), i.e. to find the first order Lie–Bäcklund group admitted by Benny equations (2) with coordinates $æ^i$ of the canonical infinitesimal operator (36), that has the linear form

$$æ^i = \eta^i + \sum_{j=0}^{\infty} \eta^{i,j} A^j_x, \quad i \geq 0. \tag{83}$$

Though the unique existence of the linear form (83) as well as the complete solution of determining equations[1] for the tensor $\eta^{i,j}$ has not yet been obtained, all known facts are in agreement with this linear form. In particular, results of [5, 6] mentioned above are consistent with the following expression for the tensor $\eta^{i,j}$ of the linear form

$$\eta^{i,j}_s = \sum_{k=0}^{\infty} k H^s_{A^k} \delta_{i+k,j+1} + s \sum_{k=0}^{s-j-2} (i+k) A^{i+k-1} H^{s-1}_{A^{j+k+1}}; \quad i,\ j,\ s \geq 0. \tag{84}$$

Here s is the number of the basis solution (similar to that used for η^i in (73)), H^s is a polynomial of the homogeneity degree $(s+2)$ in moments A^i. Compatibility conditions for determining equations for the tensor $\eta^{i,j}$ give rise to many relationships for H^s, for example

$$\sum_{j=0}^{\infty} j A^{j-1} H^s_{A^j} = s H^{s-1}, \quad s \geq 0. \tag{85}$$

An explicit form for the polynomial H^7 is presented below just to illustrate the aforesaid

$$\begin{aligned} H^7 \ =\ & A^7 + 7A^5A^0 + 7A^4A^1 + 7A^3A^2 + 21A^3(A^0)^2 + 42A^2A^1A^0 \\ & + 7(A^1)^3 + 35A^1(A^0)^3. \end{aligned} \tag{86}$$

Comparison between formulas (74) and (86) shows that they differ only in numerical values (and signs) of coefficients. The generating function for polynomials H^s is given in [5, 6]. So constructing of a recursion operator, which transforms the linear form (81) for point group to the linear form (83) is of principal interest.

To complete the conclusion we present formulas for the infinite dimensional algebra of a Lie point group admitted by the system of equations (9)

$$\begin{aligned} X_1 &= \xi(t)\partial_t + \frac{x}{2}\xi_t\partial_x + \frac{1}{2}(x\xi_{tt} - v\xi_t)\partial_v - \frac{g}{2}\xi_t\partial_g - \frac{1}{4}(2(g+2h)\xi_t + x^2\xi_{ttt})\partial_h, \\ X_2 &= \chi(t)\partial_x + \chi_t\partial_v - x\chi_{tt}\partial_h, \quad X_3 = x\partial_x + v\partial_v + g\partial_g + (g+2h)\partial_h, \\ X_4 &= f\partial_f + g(\partial_g - \partial_h), \quad X_5 = \partial_f + v(\partial_g - \partial_h), X_6 = \mu(t)\partial_h, \\ X_7 &= G(t, x, g+h)(\partial_g - \partial_h). \end{aligned} \tag{87}$$

[1] For simplicity these equations are omitted here.

Here in (87) $\xi(t)$, $\chi(t)$, $\mu(t)$ and $G(t, x, g + h)$ are arbitrary functions of their arguments. Using the procedure described in section 3.2 it is easily checked that prolongation of generators (15) on nonlocal variables g and h produce generators that directly follow from (87). Namely, the prolongation of X_1 from (15) gives X_1 provided $\xi = 1$, and X_2 from (15) gives X_2 provided $\chi = 1$. Next, prolongation of X_3 from (15) gives X_2 provided $\chi = t$, and X_4 from (15) gives $X_1 - X_3/2 - X_4$ provided $\xi = t$. At last, prolongation of X_5 from (15) gives $X_3 + X_4$.

It should be noted that on one hand the transition from nonlocal Benny equations (1) to their 'potential' analog does not introduce additional independent variables that relate to restriction onto the boundary sets (compare with [14, 15]). This fact reflects the property of the fast decay of the distribution function f in Benny equations at the infinity, $f(+\infty) = f(-\infty) = 0$. On the other hand the generators (87) give one more evidence in favor of constructing symmetry of nonlocal equations using various representations and different approaches.

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