Linearization of fourth-order ordinary differential equations by point transformations

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Abstract. We present here the solution of the problem on linearization of fourth-order equations by means of point transformations. We show that all fourth-order equations that are linearizable by point transformations are contained in the class of equations which is linear in the third-order derivative. We provide the linearization test and describe the procedure for obtaining the linearizing transformations as well as the linearized equation.

Keywords: Nonlinear ordinary differential equations, candidates for linearization, linearization test.

1. Introduction

The problem on linearization of second-order ordinary differential equations by means of point transformations was solved by Sophus Lie [1] in 1883. More specifically, he showed that the linearizable equations are at most cubic in the first-order derivative and gave the linearization test in terms of the coefficients of these equations.

In 1997, G. Grebot [2] studied the linearization of third-order equations by means of a restricted class of point transformations, namely $t = \varphi(x), u = \psi(x, y)$. However, the problem was not completely solved.

In 2004, N.H. Ibragimov and S.V. Meleshko [3] solved the problem of linearization of third-order equations by means of point transformations. They showed that all third-order equations that are linearizable by point transformations are contained either in the class of equations which is linear in the second-order derivative, or in the class of equations which is quadratic in the second-order derivative. They provided the linearization test for each of these classes and describe the procedure for obtaining the linearizing transformations as well as the linearized equation.

The present paper is devoted to obtain criteria for a fourth-order equation to be linearizable by change of the dependent and independent variables. In our calculations we used computer algebra packages. The final results were checked by comparing with theoretical results on invariants as well as by applying to numerous known and new examples of linearization. The paper is organized as follows.

2. Point transformations of fourth-order equations

We consider the fourth-order ordinary differential equation

$$y^{(4)} = f(x, y, y', y'', y''') . \tag{1}$$

We apply a point transformation

$$t = \varphi(x, y), \quad u = \psi(x, y) \tag{2}$$

to equation (1).

We begin with investigating the necessary conditions for linearization. The general form of (1) that can be obtained from linear equations by any point transformations (2) is found on this step. In consequence, we identify two candidates for linearization.

A linear fourth-order ordinary differential equation we use in the Laguerre form. In 1879, E. Laguerre showed that in linear ordinary differential equation of order $n \geq 3$ the two terms of orders next below the highest can be simultaneously removed by equivalence transformation (see [4], Section 10.2.1 and the references therein). Therefore, we write the general linear fourth-order equation in Laguerre's form

$$u^{(4)} + \alpha(t) u' + \beta(t) u = 0, \tag{3}$$

where t and u are the independent and dependent variables, respectively.

2.1. The candidates for linearization

Considering t and u as the new independent and dependent variables, respectively, one obtains the following transformation of the first-order derivative

$$u' = \frac{D_x(\psi)}{D_x(\varphi)} = \frac{\psi_x + y'\psi_y}{\varphi_x + y'\varphi_y}\,, \tag{4}$$

where $\varphi_x = \frac{\partial\varphi}{\partial x}$, $\varphi_y = \frac{\partial\varphi}{\partial y}$, etc., and

$$D_x = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + y'''\frac{\partial}{\partial y''} + y^{(4)}\frac{\partial}{\partial y'''} + \cdots$$

is the total derivative. Likewise, one obtains the transformation of derivatives of the second and higher order. Namely, denoting by $P(x,y,y')$ the right-hand side of (4),

$$P(x,y,y') = \frac{\psi_x + y'\psi_y}{\varphi_x + y'\varphi_y}$$

one has

$$u'' = \frac{D_x(P)}{D_x(\varphi)} = \frac{P_x + y'P_y + y''P_{y'}}{\varphi_x + y'\varphi_y} = \frac{\Delta}{(\varphi_x + y'\varphi_y)^3}y'' + \cdots. \tag{5}$$

Denoting by $Q(x,y,y',y'')$ the right-hand side of (5),

$$Q = \frac{\Delta}{(\varphi_x + y'\varphi_y)^3}\,y'' + \cdots$$

one has

$$\begin{aligned} u''' &= \tfrac{D_x(Q)}{D_x(\varphi)} = \tfrac{Q_x + y'Q_y + y''Q_{y'} + y'''Q_{y''}}{\varphi_x + y'\varphi_y} \\ &= \tfrac{\Delta}{(\varphi_x + y'\varphi_y)^5}\left[(\varphi_x + y'\varphi_y)\,y''' - 3\varphi_y\,(y'')^2\right] + \cdots. \end{aligned} \tag{6}$$

Denoting by $R(x,y,y',y'',y''')$ the right-hand side of (6),

$$R = \frac{\Delta}{(\varphi_x + y'\varphi_y)^5}\left[(\varphi_x + y'\varphi_y)\,y''' - 3\varphi_y\,(y'')^2\right] + \cdots$$

hence,

$$\begin{aligned} u^{(4)} &= \tfrac{D_x(R)}{D_x(\varphi)} = \tfrac{R_x + y'R_y + y''R_{y'} + y'''R_{y''} + y^{(4)}R_{y'''}}{\varphi_x + y'\varphi_y} \\ &= \tfrac{\Delta}{(\varphi_x + y'\varphi_y)^7}\left[(\varphi_x + y'\varphi_y)^2 y^{(4)}\right] + \cdots. \end{aligned} \tag{7}$$

Thus, (3) becomes

$$\begin{aligned} &\frac{1}{(\varphi_x + y'\varphi_y)^7}\Big[(\varphi_x + \varphi_y y')^2\Delta y^{(4)} + [-10\Delta(\varphi_x + \varphi_y y')\varphi_y y'' \\ &-2(2(5\varphi_{xy}\Delta - \varphi_y\Delta_x)\varphi_y + (5\varphi_{yy}\Delta - 4\varphi_y\Delta_y)\varphi_x)y'^2 \\ &-2(5(2\varphi_{xy}\varphi_x + \varphi_{xx}\varphi_y)\Delta - 2(\varphi_x\Delta_y + 2\varphi_y\Delta_x)\varphi_x)y' \\ &+\cdots]y''' + \cdots\Big] = 0. \end{aligned} \tag{8}$$

Here

$$\Delta = \varphi_x\psi_y - \varphi_y\psi_x \neq 0$$

is the Jacobian of the change of variables (2). It is manifest from (8) that the transformations (2) with $\varphi_y = 0$ and $\varphi_y \neq 0$, respectively, provide two distinctly different candidates for linearization.

If $\varphi_y = 0$ we work out the missing terms in (8), substitute the resulting expression in (3) and obtain the following equation

$$\begin{aligned} y^{(4)} \quad & +(A_1 y' + A_0)y''' + B_0 y''^2 + (C_2 y'^2 + C_1 y' + C_0)y'' \\ & +D_4 y'^4 + D_3 y'^3 + D_2 y'^2 + D_1 y' + D_0 = 0, \end{aligned} \tag{9}$$

where

$$A_1 = 4(\psi_y)^{-1}\psi_{yy}, \tag{10}$$

$$A_0 = -2(\varphi_x\psi_y)^{-1}(3\varphi_{xx}\psi_y - 2\varphi_x\psi_{xy}), \tag{11}$$

$$B_0 = 3(\psi_y)^{-1}\psi_{yy}, \tag{12}$$

$$C_2 = 6(\psi_y)^{-1}\psi_{yyy}, \tag{13}$$

$$C_1 = -6(\varphi_x\psi_y)^{-1}(3\varphi_{xx}\psi_{yy} - 2\varphi_x\psi_{xyy}), \tag{14}$$

$$C_0 = -(\varphi_x^2\psi_y)^{-1}\Big[(4\varphi_{xxx}\varphi_x - 15\varphi_{xx}^2)\psi_y + 6(3\varphi_{xx}\psi_{xy} - \varphi_x\psi_{xxy})\varphi_x\Big], \tag{15}$$

$$D_4 = (\psi_y)^{-1}\psi_{yyyy}, \tag{16}$$

$$D_3 = -2(\varphi_x\psi_y)^{-1}(3\varphi_{xx}\psi_{yyy} - 2\varphi_x\psi_{xyyy}), \tag{17}$$

$$D_2 = -(\varphi_x^2\psi_y)^{-1}(4\varphi_{xxx}\varphi_x\psi_{yy} - 15\varphi_{xx}^2\psi_{yy} + 18\varphi_{xx}\varphi_x\psi_{xyy} - 6\varphi_x^2\psi_{xxyy}), \tag{18}$$

$$\begin{aligned} D_1 = -(\varphi_x^3\psi_y)^{-1}\Big[&3(5\varphi_{xx}^2\psi_y - 10\varphi_{xx}\varphi_x\psi_{xy} + 6\varphi_x^2\psi_{xxy})\varphi_{xx} - (\varphi_x^3\psi_y\alpha + 4\psi_{xxxy})\varphi_x^3 \\ &-2(5\varphi_{xx}\psi_y - 4\varphi_x\psi_{xy})\varphi_{xxx}\varphi_x + \varphi_{xxxx}\varphi_x^2\psi_y\Big], \end{aligned} \tag{19}$$

$$\begin{aligned} D_0 = -(\varphi_x^3\psi_y)^{-1}\Big[&(15\varphi_{xx}^3 - \varphi_x^6\alpha + \varphi_{xxxx}\varphi_x^2)\psi_x - (10\varphi_{xxx}\varphi_{xx}\psi_x - 4\varphi_{xxx}\varphi_x\psi_{xx} \\ &+15\varphi_{xx}^2\psi_{xx} - 6\varphi_{xx}\varphi_x\psi_{xxx} + \varphi_x^6\beta\psi + \varphi_x^2\psi_{xxxx})\varphi_x\Big]. \end{aligned} \tag{20}$$

Definition 1. We call (9) with arbitrary coefficients $A_0 = A_0(x,y)$, $A_1 = A_1(x,y)$, $B_0 = B_0(x,y)$, $C_0 = C_0(x,y)$, $C_1 = C_1(x,y)$, $C_2 = C_2(x,y)$, and $D_i = D_i(x,y), (i = 0,\ldots,4)$, the first candidate for linearization.

If $\varphi_y \neq 0$, we proceed likewise and setting $r(x,y) = \frac{\varphi_x}{\varphi_y}$, arrive at the following equation

$$\begin{aligned} & y^{(4)} + \tfrac{1}{y'+r}(-10y'' + F_2 y'^2 + F_1 y' + F_0)y''' \\ & +\tfrac{1}{(y'+r)^2}\,[15y''^3 + (H_2 y'^2 + H_1 y' + H_0)y''^2 \\ & +(J_4 y'^4 + J_3 y'^3 + J_2 y'^2 + J_1 y' + J_0)y'' \\ & +K_7 y'^7 + K_6 y'^6 + K_5 y'^5 + K_4 y'^4 \\ & +K_3 y'^3 + K_2 y'^2 + K_1 y' + K_0] = 0, \end{aligned} \tag{21}$$

where

$$F_2 = -2(\varphi_y\Delta)^{-1}(5\varphi_{yy}\Delta - 2\varphi_y\Delta_y), \tag{22}$$

$$F_1 = 4(\varphi_y\Delta)^{-1}\Big[(\Delta_x + \Delta_y r - 5r_y\Delta)\varphi_y - 5\varphi_{yy}r\Delta\Big], \tag{23}$$

$$F_0 = -2(\varphi_y\Delta)^{-1}\Big[((5r_y\Delta - 2\Delta_x)r + 5r_x\Delta)\varphi_y + 5\varphi_{yy}r^2\Delta\Big], \tag{24}$$

$$H_2 = 6(\varphi_y\Delta)^{-1}(5\varphi_{yy}\Delta - 2\varphi_y\Delta_y), \tag{25}$$

$$H_1 = -3(\varphi_y\Delta)^{-1}\Big[(5\Delta_x + 3\Delta_y r - 25r_y\Delta)\varphi_y - 20\varphi_{yy}r\Delta\Big], \tag{26}$$

$$H_0 = 3(\varphi_y\Delta)^{-1}\Big[(5(3r_x + 2r_y r)\Delta - (5\Delta_x - \Delta_y r)r)\varphi_y + 10\varphi_{yy}r^2\Delta\Big], \tag{27}$$

$$J_4 = -(\varphi_y^2\Delta)^{-1}(10\varphi_{yyy}\varphi_y\Delta - 45\varphi_{yy}^2\Delta + 30\varphi_{yy}\varphi_y\Delta_y - 6\varphi_y^2\Delta_{yy}), \tag{28}$$

$$\begin{aligned} J_3 = 2(\varphi_y^2\Delta)^{-1}\Big[&3((2(\Delta_{xy} + \Delta_{yy}r - 5r_y\Delta_y) - 5r_{yy}\Delta)\varphi_y^2 \\ &-5((\Delta_x + 3\Delta_y r - 4r_y\Delta)\varphi_y - 6\varphi_{yy}r\Delta)\varphi_{yy}) - 20\varphi_{yyy}\varphi_y r\Delta\Big], \end{aligned} \tag{29}$$

$$\begin{aligned} J_2 = 6(\varphi_y^2\Delta)^{-1}\Big[&(\Delta_{xx} + \Delta_{yy}r^2 + 4\Delta_{xy}r - 5(2\Delta_x + 3\Delta_y r - 5r_y\Delta)r_y \\ &-10r_{yy}r\Delta - 5r_x\Delta_y - 5r_{xy}\Delta)\varphi_y^2 - 5(((3(\Delta_x + \Delta_y r) - 10r_y\Delta)r \\ &-2r_x\Delta)\varphi_y - 9\varphi_{yy}r^2\Delta)\varphi_{yy} - 10\varphi_{yyy}\varphi_y r^2\Delta\Big], \end{aligned} \tag{30}$$

$$\begin{aligned} J_1 = -2(\varphi_y^2\Delta)^{-1}\Big[&((5(3(3\Delta_x + \Delta_y r) - 14r_y\Delta)r_y - 6(\Delta_{xy}r + \Delta_{xx}) \\ &+20r_{yy}r\Delta)r + 5(3(\Delta_x + \Delta_y r) - 16r_y\Delta)r_x + 5r_{xx}\Delta + 20r_{xy}r\Delta)\varphi_y^2 \\ &+15(((3\Delta_x + \Delta_y r - 8r_y\Delta)r - 4r_x\Delta)\varphi_y - 6\varphi_{yy}r^2\Delta)\varphi_{yy}r \\ &+20\varphi_{yyy}\varphi_y r^3\Delta\Big], \end{aligned} \tag{31}$$

$$\begin{aligned} J_0 = -(\varphi_y^2\Delta)^{-1}\Big[&((2((5r_{yy}r\Delta - 3\Delta_{xx})r + 5r_{xx}\Delta + 5r_{xy}r\Delta) \\ &-5(7r_y\Delta - 6\Delta_x)r_y r)r - 5(2(7r_y\Delta - 3\Delta_x)r + 9r_x\Delta)r_x)\varphi_y^2 \\ &-5(3(2((2r_y\Delta - \Delta_x)r + 2r_x\Delta)\varphi_y + 3\varphi_{yy}r^2\Delta)\varphi_{yy} \\ &-2\varphi_{yyy}\varphi_y r^2\Delta)r^2\Big], \end{aligned} \tag{32}$$

$$\begin{aligned} K_7 = -(\varphi_y^2\Delta)^{-1}\Big[&\varphi_{yyyy}\varphi_y^2\psi_y - 10\varphi_{yyy}\varphi_{yy}\varphi_y\psi_y + 4\varphi_{yyy}\varphi_y^2\psi_{yy} + 15\varphi_{yy}^3\psi_y \\ &-15\varphi_{yy}^2\varphi_y\psi_{yy} + 6\varphi_{yy}\varphi_y^2\psi_{yyy} - \varphi_y^7\beta\psi - \varphi_y^6\psi_y\alpha - \varphi_y^3\psi_{yyyy}\Big], \end{aligned} \tag{33}$$

$$\begin{aligned} K_6 = (\varphi_y^3\Delta)^{-1}\Big[&3(5((7\varphi_y\psi_{yy}r - 6\Delta_y)\varphi_y - 7(\varphi_y\psi_y r - \Delta)\varphi_{yy})\varphi_{yy} \\ &-2(7\varphi_y\psi_{yyy}r - 5\Delta_{yy})\varphi_y^2)\varphi_{yy} + (7\varphi_y^5\beta\psi r + 7\varphi_y^4\psi_y\alpha r - \varphi_y^3\alpha\Delta \\ &+7\varphi_y\psi_{yyyy}r - 4\Delta_{yyy})\varphi_y^3 + 2(35\varphi_{yy}\varphi_y\psi_y r - 30\varphi_{yy}\Delta - 14\varphi_y^2\psi_{yy}r \\ &+10\varphi_y\Delta_y)\varphi_{yyy}\varphi_y - (7\varphi_y\psi_y r - 5\Delta)\varphi_{yyyy}\varphi_y^2\Big], \end{aligned} \tag{34}$$

$$\begin{aligned} K_5 = -(\varphi_y^3\Delta)^{-1}\Big[&(2(3(\Delta_{xyy} + 3\Delta_{yyy}r - 5r_y\Delta_{yy} - 5r_{yy}\Delta_y) - 5r_{yyy}\Delta) \\ &-3(7\varphi_y^4\beta\psi r + 7\varphi_y^3\psi_y\alpha r - 2\varphi_y^2\alpha\Delta + 7\psi_{yyyy}r)\varphi_y r)\varphi_y^3 \\ &-3(2(5(\Delta_{xy} + 5\Delta_{yy}r - 4r_y\Delta_y - 2r_{yy}\Delta) - 21\varphi_y\psi_{yyy}r^2)\varphi_y^2 \end{aligned}$$

$$-15((\Delta_x+11\Delta_y r-3r_y\Delta-7\varphi_y\psi_{yy}r^2)\varphi_y$$
$$+7(\varphi_y\psi_y r-2\Delta)\varphi_{yy}r)\varphi_{yy})\varphi_{yy}-2((5(\Delta_x+11\Delta_y r-3r_y\Delta)$$
$$-42\varphi_y\psi_{yy}r^2)\varphi_y+15(7\varphi_y\psi_y r-12\Delta)\varphi_{yy}r)\varphi_{yyy}\varphi_y$$
$$+3(7\varphi_y\psi_y r-10\Delta)\varphi_{yyyy}\varphi_y^2 r\Big], \quad (35)$$

$$K_4=-(\varphi_y^3\Delta)^{-1}\Big[(2(45r_{yy}r_y\Delta-10r_{yy}\Delta_x-55r_{yy}\Delta_y r+50r_y^2\Delta_y$$
$$-20r_y\Delta_{xy}-50r_y\Delta_{yy}r+11\Delta_{xyy}r+2\Delta_{xxy}+17\Delta_{yyy}r^2$$
$$-20r_{yyy}r\Delta-5r_x\Delta_{yy}-10r_{xy}\Delta_y-5r_{xyy}\Delta)$$
$$-5(7\varphi_y^4\beta\psi r+7\varphi_y^3\psi_y\alpha r-3\varphi_y^2\alpha\Delta+7\psi_{yyyy}r)\varphi_y r^2)\varphi_y^3$$
$$+15((3((5(\Delta_x+5\Delta_y r)-14r_y\Delta)r-r_x\Delta)-35\varphi_y\psi_{yy}r^3)\varphi_y$$
$$+35(\varphi_y\psi_y r-3\Delta)\varphi_{yy}r^2)\varphi_{yy}^2-10(\Delta_{xx}+31\Delta_{yy}r^2+13\Delta_{xy}r$$
$$-8(\Delta_x+6\Delta_y r-2r_y\Delta)r_y-26r_{yy}r\Delta-4r_x\Delta_y-4r_{xy}\Delta$$
$$-21\varphi_y\psi_{yyy}r^3)\varphi_{yy}\varphi_y^2-10(((5(\Delta_x+5\Delta_y r)-14r_y\Delta)r-r_x\Delta$$
$$-14\varphi_y\psi_{yy}r^3)\varphi_y+5(7\varphi_y\psi_y r-18\Delta)\varphi_{yy}r^2)\varphi_{yyy}\varphi_y$$
$$+5(7\varphi_y\psi_y r-15\Delta)\varphi_{yyyy}\varphi_y^2 r^2\Big], \quad (36)$$

$$K_3=-(\varphi_y^3\Delta)^{-1}\Big[((13\Delta_{xxy}+35\Delta_{yyy}r^2)r+\Delta_{xxx}+31\Delta_{xyy}r^2$$
$$-5(3\Delta_{xx}+26\Delta_{yy}r^2+23\Delta_{xy}r-(15\Delta_x+49\Delta_y r-25r_y\Delta)r_y)r_y$$
$$-5(13\Delta_x+32\Delta_y r-50r_y\Delta)r_{yy}r-65r_{yyy}r^2\Delta-5(3\Delta_{xy}+5\Delta_{yy}r$$
$$-16r_y\Delta_y-7r_{yy}\Delta)r_x-5r_{xx}\Delta_y-5r_{xxy}\Delta$$
$$-5(3\Delta_x+11\Delta_y r-15r_y\Delta)r_{xy}-30r_{xyy}r\Delta-5(7\varphi_y^4\beta\psi r+7\varphi_y^3\psi_y\alpha r$$
$$-4\varphi_y^2\alpha\Delta+7\psi_{yyyy}r)\varphi_y r^3)\varphi_y^3-5(2((2(2\Delta_{xx}+17\Delta_{yy}r^2+11\Delta_{xy}r)$$
$$-(29\Delta_x+75\Delta_y r-51r_y\Delta)r_y-45r_{yy}r\Delta)r-(3\Delta_x+13\Delta_y r-13r_y\Delta)r_x$$
$$-r_{xx}\Delta-14r_{xy}r\Delta-21\varphi_y\psi_{yyy}r^4)\varphi_y^2-3((6((5(\Delta_x+3\Delta_y r)-13r_y\Delta)r$$
$$-2r_x\Delta)-35\varphi_y\psi_{yy}r^3)\varphi_y+35(\varphi_y\psi_y r-4\Delta)\varphi_{yy}r^2)\varphi_{yy}r)\varphi_{yy}$$
$$-10(2((5(\Delta_x+3\Delta_y r)-13r_y\Delta)r-2r_x\Delta-7\varphi_y\psi_{yy}r^3)\varphi_y$$
$$+5(7\varphi_y\psi_y r-24\Delta)\varphi_{yy}r^2)\varphi_{yyy}\varphi_y r+5(7\varphi_y\psi_y r-20\Delta)\varphi_{yyyy}\varphi_y^2 r^3\Big], \quad (37)$$

$$K_2=-(\varphi_y^3\Delta)^{-1}\Big[((3((5\Delta_{xxy}+7\Delta_{yyy}r^2)r+\Delta_{xxx}+7\Delta_{xyy}r^2)$$
$$-(3(13\Delta_{xx}+28\Delta_{yy}r^2+39\Delta_{xy}r)+(204r_y\Delta-161\Delta_x-217\Delta_y r)r_y)r_y$$
$$-(79\Delta_x+116\Delta_y r-264r_y\Delta)r_{yy}r-54r_{yyy}r^2\Delta)r$$
$$-(3(2\Delta_{xx}+7\Delta_{yy}r^2+11\Delta_{xy}r)+(171r_y\Delta-64\Delta_x-140\Delta_y r)r_y$$
$$-72r_{yy}r\Delta-18r_x\Delta_y)r_x-(4\Delta_x+11\Delta_y r-21r_y\Delta)r_{xx}-12r_{xxy}r\Delta$$
$$-r_{xxx}\Delta-((37\Delta_x+53\Delta_y r-150r_y\Delta)r-33r_x\Delta)r_{xy}-33r_{xyy}r^2\Delta$$
$$-3(7\varphi_y^4\beta\psi r+7\varphi_y^3\psi_y\alpha r-5\varphi_y^2\alpha\Delta+7\psi_{yyyy}r)\varphi_y r^4)\varphi_y^3$$
$$-3(2(5((2\Delta_{xx}+7\Delta_{yy}r^2+6\Delta_{xy}r-(13\Delta_x+19\Delta_y r-20r_y\Delta)r_y$$
$$-13r_{yy}r\Delta)r^2-((3\Delta_x+5\Delta_y r-11r_y\Delta)r-r_x\Delta)r_x-r_{xx}r\Delta$$

$$-6r_{xy}r^2\Delta) - 21\varphi_y\psi_{yyy}r^5)\varphi_y^2 - 15((2((5(\Delta_x + 2\Delta_y r) - 12r_y\Delta)r$$
$$-3r_x\Delta) - 7\varphi_y\psi_{yy}r^3)\varphi_y + 7(\varphi_y\psi_y r - 5\Delta)\varphi_{yy}r^2)\varphi_{yy}r^2)\varphi_{yy}$$
$$-2(2(5((5(\Delta_x + 2\Delta_y r) - 12r_y\Delta)r - 3r_x\Delta) - 21\varphi_y\psi_{yy}r^3)\varphi_y$$
$$+15(7\varphi_y\psi_y r - 30\Delta)\varphi_{yy}r^2)\varphi_{yyy}\varphi_y r^2 + 3(7\varphi_y\psi_y r - 25\Delta)\varphi_{yyyy}\varphi_y^2 r^4\Big], \quad (38)$$

$$K_1 = -(\varphi_y^3\Delta)^{-1}\Big[((7(\Delta_{xxy} + \Delta_{yyy}r^2)r + 3\Delta_{xxx} + 7\Delta_{xyy}r^2$$
$$-(33\Delta_{xx} + 28\Delta_{yy}r^2 + 49\Delta_{xy}r + 2(59r_y\Delta - 56\Delta_x - 42\Delta_y r)r_y)r_y$$
$$-(43\Delta_x + 42\Delta_y r - 128r_y\Delta)r_{yy}r - 23r_{yyy}r^2\Delta)r^2$$
$$-((12\Delta_{xx} + 7\Delta_{yy}r^2 + 21\Delta_{xy}r + 2(86r_y\Delta - 49\Delta_x - 35\Delta_y r)r_y$$
$$-49r_{yy}r\Delta)r + (85r_y\Delta - 15\Delta_x - 21\Delta_y r)r_x)r_x$$
$$-((8\Delta_x + 7\Delta_y r - 32r_y\Delta)r - 10r_x\Delta)r_{xx} - 9r_{xxy}r^2\Delta - 2r_{xxx}r\Delta$$
$$-((29\Delta_x + 21\Delta_y r - 95r_y\Delta)r - 46r_x\Delta)r_{xy}r - 16r_{xyy}r^3\Delta$$
$$-(7\varphi_y^4\beta\psi r + 7\varphi_y^3\psi_y\alpha r - 6\varphi_y^2\alpha\Delta + 7\psi_{yyyy}r)\varphi_y r^5)\varphi_y^3$$
$$-(2(5((4\Delta_{xx} + 7\Delta_{yy}r^2 + 7\Delta_{xy}r - (23\Delta_x + 21\Delta_y r - 31r_y\Delta)r_y$$
$$-17r_{yy}r\Delta)r^2 - ((9\Delta_x + 7\Delta_y r - 27r_y\Delta)r - 6r_x\Delta)r_x - 3r_{xx}r\Delta$$
$$-10r_{xy}r^2\Delta) - 21\varphi_y\psi_{yyy}r^5)\varphi_y^2 - 15((3((5\Delta_x + 7\Delta_y r - 11r_y\Delta)r$$
$$-4r_x\Delta) - 7\varphi_y\psi_{yy}r^3)\varphi_y + 7(\varphi_y\psi_y r - 6\Delta)\varphi_{yy}r^2)\varphi_{yy}r^2)\varphi_{yy}r$$
$$-2((5((5\Delta_x + 7\Delta_y r - 11r_y\Delta)r - 4r_x\Delta) - 14\varphi_y\psi_{yy}r^3)\varphi_y$$
$$+5(7\varphi_y\psi_y r - 36\Delta)\varphi_{yy}r^2)\varphi_{yyy}\varphi_y r^3 + (7\varphi_y\psi_y r - 30\Delta)\varphi_{yyyy}\varphi_y^2 r^5\Big], \quad (39)$$

$$K_0 = (\varphi_y^3\Delta)^{-1}\Big[(((((2(r_{xxy} + 2r_{yyy}r^2)r + r_{xxx} + 3r_{xyy}r^2)\Delta$$
$$+3(3\Delta_x + 2\Delta_y r - 8r_y\Delta)r_{yy}r^2)r - ((10r_x + 11r_y r)\Delta$$
$$-(4\Delta_x + \Delta_y r)r)r_{xx} - ((13r_x + 20r_y r)\Delta - (7\Delta_x + 3\Delta_y r)r)r_{xy}r$$
$$+((\varphi_y^4\beta\psi + \varphi_y^3\psi_y\alpha + \psi_{yyyy})r - \varphi_y^2\alpha\Delta)\varphi_y r^5 + (9\Delta_{xx} + 4\Delta_{yy}r^2$$
$$+7\Delta_{xy}r - 2(13\Delta_x + 6\Delta_y r - 12r_y\Delta)r_y)r_y r^2 - ((\Delta_{xxy} + \Delta_{yyy}r^2)r$$
$$+\Delta_{xxx} + \Delta_{xyy}r^2)r^2)r - ((2((17\Delta_x + 5\Delta_y r - 23r_y\Delta)r_y + 6r_{yy}r\Delta)$$
$$-(6\Delta_{xx} + \Delta_{yy}r^2 + 3\Delta_{xy}r))r^2 - (5(3r_x + 8r_y r)\Delta$$
$$-3(5\Delta_x + \Delta_y r)r)r_x)r_x)\varphi_y^3 - ((2((5(r_{xx} + 3r_{yy}r^2 + 2r_{xy}r)\Delta$$
$$+3\varphi_y\psi_{yyy}r^4 + 5(5\Delta_x + 3\Delta_y r - 6r_y\Delta)r_y r - 5(\Delta_{xx} + \Delta_{yy}r^2 + \Delta_{xy}r)r)r$$
$$-5((3r_x + 7r_y r)\Delta - (3\Delta_x + \Delta_y r)r)r_x)\varphi_y^2 - 15((3(r_x + 2r_y r)\Delta$$
$$+\varphi_y\psi_{yy}r^3 - 3(\Delta_x + \Delta_y r)r)\varphi_y - (\varphi_y\psi_y r - 7\Delta)\varphi_{yy}r^2)\varphi_{yy}r^2)\varphi_{yy}$$
$$+(2((5(r_x + 2r_y r)\Delta + 2\varphi_y\psi_{yy}r^3 - 5(\Delta_x + \Delta_y r)r)\varphi_y$$
$$-5(\varphi_y\psi_y r - 6\Delta)\varphi_{yy}r^2)\varphi_{yyy} + (\varphi_y\psi_y r - 5\Delta)\varphi_{yyyy}\varphi_y r^2)\varphi_y r^2)r^2\Big]. \quad (40)$$

Definition 2. We call (21) with arbitrary coefficients $r = r(x,y)$, $F_0 = F_0(x,y)$, $F_1 = F_1(x,y)$, $F_2 = F_2(x,y)$, $H_0 = H_0(x,y)$, $H_1 = H_1(x,y)$, $H_2 = H_2(x,y)$, $J_0 = J_0(x,y)$, $J_1 = J_1(x,y)$, $J_2 = J_2(x,y)$, $J_3 = J_3(x,y)$, $J_4 = J_4(x,y)$, and $K_i = K_i(x,y)$, $(i = 0, \ldots, 7)$, the second candidate for linearization.

Thus, we showed that every linearizable fourth-order equations belong either to the class of (9) or to the class of (21). In Sections 2.2 and 2.3, we formulate the main theorems containing necessary and sufficient conditions for linearization as well as the methods for constructing the linearizing point transformations for each candidate. Proofs of the main theorems and illustrative examples are provided in the subsequent sections.

2.2. The linearization test for equation (9)

Consider the first canditate for linearization, i.e. equation (9). In this case, the linearizing transformations (2) have the form

$$t = \varphi(x), \quad u = \psi(x, y). \tag{41}$$

Theorem 1. Equation (9)

$$\begin{aligned} y^{(4)} & +(A_1 y' + A_0)y''' + B_0 y''^2 + (C_2 y'^2 + C_1 y' + C_0)y'' \\ & +D_4 y'^4 + D_3 y'^3 + D_2 y'^2 + D_1 y' + D_0 = 0, \end{aligned}$$

is linearizable if and only if its coefficients obey the following ten equations

$$A_{0y} - A_{1x} = 0, \tag{42}$$

$$4B_0 - 3A_1 = 0, \tag{43}$$

$$12A_{1y} + 3A_1^2 - 8C_2 = 0, \tag{44}$$

$$12A_{1x} + 3A_0 A_1 - 4C_1 = 0, \tag{45}$$

$$32C_{0y} + 12A_{0x}A_1 - 16C_{1x} + 3A_0^2 A_1 - 4A_0 C_1 = 0, \tag{46}$$

$$4C_{2y} + A_1 C_2 - 24D_4 = 0, \tag{47}$$

$$4C_{1y} + A_1 C_1 - 12D_3 = 0, \tag{48}$$

$$16C_{1x} - 12A_{0x}A_1 - 3A_0^2 A_1 + 4A_0 C_1 + 8A_1 C_0 - 32D_2 = 0, \tag{49}$$

$$\begin{aligned} & 192D_{2x} + 36A_{0x}A_0 A_1 - 48A_{0x}C_1 - 48C_{0x}A_1 - 288D_{1y} + 9A_0^3 A_1 - 12A_0^2 C_1 \\ & \quad -36A_0 A_1 C_0 + 48A_0 D_2 + 32C_0 C_1 = 0, \end{aligned} \tag{50}$$

$$\begin{aligned} & 384D_{1xy} - \Big[3((3A_0A_1 - 4C_1)A_0^2 + 16(2A_1D_1 + C_0C_1) - 16(A_1C_0 - D_2)A_0)A_0 \\ & \quad -32(4(C_1D_1 - 2C_2D_0 + C_0D_2) + (3A_1D_0 - C_0^2)A_1) - 96D_{1y}A_0 \\ & \quad +384D_{0y}A_1 + 1536D_{0yy} - 16(3A_0A_1 - 4C_1)C_{0x} \\ & \quad +12((3A_0A_1 - 4C_1)A_0 - 4(A_1C_0 - 4D_2))A_{0x}\Big] = 0. \end{aligned} \tag{51}$$

Provided that the conditions (42)-(51) are satisfied, the linearizing transformation (41) is defined by a fourth-order ordinary differential equation for the function $\varphi(x)$, namely by the Riccati equation

$$40\frac{d\chi}{dx} - 20\chi^2 = 8C_0 - 3A_0^2 - 12A_{0x}, \tag{52}$$

for

$$\chi = \frac{\varphi_{xx}}{\varphi_x}\,, \tag{53}$$

and by the following integrable system of partial differential equations for $\psi(x, y)$

$$4\psi_{yy} = \psi_y A_1, \tag{54}$$

$$4\psi_{xy} = \psi_y(A_0 + 6\chi), \tag{55}$$

and

$$\begin{aligned}1600\psi_{xxxx} = {} & 9600\psi_{xxx}\chi + 160\psi_{xx}(-12A_{0x} - 3A_0^2 - 90\chi^2 + 8C_0)\\ & +40\psi_x(12A_{0x}A_0 + 72A_{0x}\chi - 16C_{0x} + 3A_0^3 + 18A_0^2\chi - 12A_0C_0\\ & +120\chi^3 - 48\chi C_0 + 24D_1 - 8\Omega) + \psi(144A_{0x}^2 + 72A_{0x}A_0^2 - 352A_{0x}C_0\\ & -160C_{0xx} - 80C_{0x}A_0 - 1600D_{0y} + 640D_{1x} - 80\Omega_x + 9A_0^4 - 88A_0^2C_0\\ & +160A_0D_1 + 30A_0\Omega - 400A_1D_0 + 300\chi\Omega + 144C_0^2) + 1600\psi_y D_0,\end{aligned} \tag{56}$$

where χ is given by (53) and Ω is the following expression

$$\Omega = A_0^3 - 4A_0C_0 + 8D_1 - 8C_{0x} + 6A_{0x}A_0 + 4A_{0xx}. \tag{57}$$

Finally, the coefficients α and β of the resulting linear equation (3) is given by

$$\alpha = \frac{\Omega}{8\varphi_x^3}\,, \tag{58}$$

and

$$\begin{aligned}\beta = {} & (1600\varphi_x^4)^{-1}(-144A_{0x}^2 - 72A_{0x}A_0^2 + 352A_{0x}C_0 + 160C_{0xx} + 80C_{0x}A_0\\ & +1600D_{0y} - 640D_{1x} + 80\Omega_x - 9A_0^4 + 88A_0^2C_0 - 160A_0D_1 - 30A_0\Omega\\ & +400A_1D_0 - 300\chi\Omega - 144C_0^2).\end{aligned} \tag{59}$$

Remark 1. Since the system of equations (42)-(51) provides the necessary and sufficient conditions for linearization, it is invariant with respect to the transformations (41). It means that the left-hand sides of (42)-(51) are *relative invariants* (of the second-order) for the equivalence group (41).

2.3. The linearization test for equation (21)

The following theorem provides the test for linearization of the second candidate. The necessary and sufficient conditions comprise *eighteen* differential equations (60)-(77) for *twenty one* coefficients of the (21). The linearizing change of variables (2) is determined by (78)-(81) for the functions $\varphi(x, y)$ and $\psi(x, y)$.

Theorem 2. Equation (21)

$$\begin{aligned} & y^{(4)} + \tfrac{1}{y'+r}(-10y'' + F_2y'^2 + F_1y' + F_0)y'''\\ & +\tfrac{1}{(y'+r)^2}\,[15y''^3 + (H_2y'^2 + H_1y' + H_0)y''^2\\ & +(J_4y'^4 + J_3y'^3 + J_2y'^2 + J_1y' + J_0)y''\\ & +K_7y'^7 + K_6y'^6 + K_5y'^5 + K_4y'^4\\ & +K_3y'^3 + K_2y'^2 + K_1y' + K_0] = 0,\end{aligned}$$

is linearizable if and only if its coefficients obey the following equations

$$10r_{yy} = -(F_{1y} + F_{2x} + F_{2y}r + r_yF_2), \tag{60}$$

$$10r_x = 10r_yr - F_0 + F_1r - F_2r^2, \tag{61}$$

$$H_2 = -3F_2, \tag{62}$$

$$4H_1 = -3(5F_1 - 2F_2r), \tag{63}$$

$$4H_0 = -3(6F_0 - F_1r), \tag{64}$$

$$\begin{aligned}10F_{1yy} = -(&F_{1y}F_2 - 40F_{2xy} - 16F_{2x}F_2 + 20F_{2yy}r + 40F_{2y}r_y + 14F_{2y}F_2r + 20J_{4x}\\ &- 20J_{4y}r + 14r_yF_2^2 - 40r_yJ_4),\end{aligned} \tag{65}$$

$$12F_{2x} = 12F_{2y}r - 3F_1F_2 + 6F_2^2r + 4J_3 - 16J_4r, \tag{66}$$

$$60F_{1x} = 60F_{1y}r - 36F_0F_2 - 15F_1^2 + 66F_1F_2r - 36F_2^2r^2 + 40J_2 - 80J_3r + 80J_4r^2, \tag{67}$$

$$\begin{aligned}60F_{0x} = 60F_{0y}r &- 51F_0F_1 + 66F_0F_2r + 36F_1^2r - 72F_1F_2r^2 + 36F_2^2r^3 + 60J_1\\ &- 80J_2r + 80J_3r^2 - 80J_4r^3,\end{aligned} \tag{68}$$

$$\begin{aligned}20J_0 = 9F_0^2 &- 18F_0F_1r + 18F_0F_2r^2 + 9F_1^2r^2 - 18F_1F_2r^3 + 9F_2^2r^4 + 20J_1r\\ &- 20J_2r^2 + 20J_3r^3 - 20J_4r^4,\end{aligned} \tag{69}$$

$$\begin{aligned}120J_{3yy} = 216F_{1y}F_{2y} &+ 54F_{1y}F_2^2 - 48F_{1y}J_4 + 360F_{2yy}r_y + 90F_{2yy}F_1 - 180F_{2yy}F_2r\\ &- 432F_{2y}^2r + 324F_{2y}r_yF_2 + 189F_{2y}F_1F_2 - 486F_{2y}F_2^2r - 192F_{2y}J_3\\ &+ 864F_{2y}J_4r - 60J_{3y}F_2 + 720J_{4xy} + 180J_{4x}F_2 - 240J_{4yy}r\\ &- 1200J_{4y}r_y + 60J_{4y}F_2r + 720K_{6x} - 720K_{6y}r - 5040K_{7x}r\\ &+ 5040K_{7y}r^2 + 36r_yF_2^3 - 432r_yF_2J_4 - 2160r_yK_6 + 15120r_yK_7r\\ &+ 504F_0K_7 + 36F_1F_2^3 - 102F_1F_2J_4 - 504F_1K_7r - 72F_2^4r\\ &- 48F_2^2J_3 + 396F_2^2J_4r + 504F_2K_7r^2 + 136J_3J_4 - 544J_4^2r,\end{aligned} \tag{70}$$

$$\begin{aligned}240J_{4xyy} = -(&36F_{1y}F_{2yy} + 162F_{1y}F_{2y}F_2 - 72F_{1y}J_{4y} + 36F_{1y}F_2^3 - 168F_{1y}F_2J_4\\ &- 72F_{1y}K_6 - 168F_{1y}K_7r - 72F_{2yy}F_{2y}r + 144F_{2yy}r_yF_2\\ &+ 54F_{2yy}F_1F_2 - 108F_{2yy}F_2^2r - 72F_{2yy}J_3 + 288F_{2yy}J_4r + 432F_{2y}^2r_y\\ &+ 108F_{2y}^2F_1 - 540F_{2y}^2F_2r - 144F_{2y}J_{3y} + 528F_{2y}J_{4x} + 192F_{2y}J_{4y}r\\ &+ 324F_{2y}r_yF_2^2 - 1008F_{2y}r_yJ_4 + 162F_{2y}F_1F_2^2 - 132F_{2y}F_1J_4\\ &- 396F_{2y}F_2^3r - 180F_{2y}F_2J_3 + 1320F_{2y}F_2J_4r + 144F_{2y}K_6r\\ &- 336F_{2y}K_7r^2 - 36J_{3y}F_2^2 + 176J_{3y}J_4 + 120J_{4xy}F_2 + 132J_{4x}F_2^2\\ &- 432J_{4x}J_4 - 240J_{4yyy}r - 960J_{4yy}r_y - 120J_{4yy}F_2r - 768J_{4y}r_yF_2\\ &- 138J_{4y}F_1F_2 + 288J_{4y}F_2^2r + 184J_{4y}J_3 - 1008J_{4y}J_4r + 960K_{6xy}\\ &+ 240K_{6x}F_2 - 960K_{6yy}r - 3840K_{6y}r_y - 240K_{6y}F_2r - 1920K_{7xy}r\end{aligned}$$

$$
\begin{aligned}
&- 2400K_{7xx} + 2880K_{7x}r_y - 600K_{7x}F_1 - 480K_{7x}F_2r + 4320K_{7yy}r^2 \\
&+ 24000K_{7y}r_yr + 432K_{7y}F_0 + 168K_{7y}F_1r + 912K_{7y}F_2r^2 \\
&+ 20160r_y^2K_7 + 1728r_yF_1K_7 + 36r_yF_2^4 - 264r_yF_2^2J_4 - 1248r_yF_2K_6 \\
&+ 5280r_yF_2K_7r + 160r_yJ_4^2 + 408F_0F_2K_7 + 150F_1^2K_7 + 27F_1F_2^4 \\
&- 120F_1F_2^2J_4 - 168F_1F_2K_6 + 168F_1F_2K_7r - 54F_2^5r - 36F_2^3J_3 \\
&+ 384F_2^3J_4r + 336F_2^2K_6r - 1344F_2^2K_7r^2 + 160F_2J_3J_4 - 640F_2J_4^2r \\
&- 400J_2K_7 + 224J_3K_6 - 368J_3K_7r - 896J_4K_6r + 3872J_4K_7r^2 \\
&+ 672F_{0y}K_7), \qquad (71)
\end{aligned}
$$

$$
4J_{4x} = 4J_{4y}r - F_1J_4 + 2F_2J_4r - 4K_5 + 24K_6r - 84K_7r^2, \qquad (72)
$$

$$
\begin{aligned}
60F_{0yy} = -(&30F_{0y}F_2 + 36F_{1y}F_1 - 36F_{1y}F_2r - 60F_{2yy}r^2 + 24F_{2y}F_0 - 36F_{2y}F_1r \\
&- 54F_{2y}F_2r^2 - 40J_{2y} + 40J_{3y}r + 80J_{4y}r^2 - 36r_yF_1F_2 + 36r_yF_2^2r \\
&+ 40r_yJ_3 - 80r_yJ_4r + 6F_0F_2^2 - 6F_0J_4 + 9F_1^2F_2 - 18F_1F_2^2r \\
&- 12F_1J_3 + 24F_1J_4r - 6F_2^3r^2 - 10F_2J_2 + 22F_2J_3r + 26F_2J_4r^2 \\
&- 60K_4 + 180K_5r - 180K_6r^2 - 420K_7r^3), \qquad (73)
\end{aligned}
$$

$$
\begin{aligned}
20J_{2x} = &20J_{2y}r + 20J_{3x}r - 20J_{3y}r^2 - 14F_0J_3 + 28F_0J_4r - 5F_1J_2 + 19F_1J_3r \\
&- 28F_1J_4r^2 + 10F_2J_2r - 24F_2J_3r^2 + 28F_2J_4r^3 - 120K_3 + 360K_4r \\
&- 640K_5r^2 + 840K_6r^3 - 840K_7r^4, \qquad (74)
\end{aligned}
$$

$$
\begin{aligned}
60J_{1x} = &60J_{1y}r - 40J_{3x}r^2 + 40J_{3y}r^3 - 42F_0J_2 + 42F_0J_3r - 70F_0J_4r^2 - 15F_1J_1 \\
&+ 42F_1J_2r - 52F_1J_3r^2 + 70F_1J_4r^3 + 30F_2J_1r - 42F_2J_2r^2 \\
&+ 62F_2J_3r^3 - 70F_2J_4r^4 - 600K_2 + 1080K_3r - 1380K_4r^2 \\
&+ 1700K_5r^3 - 2100K_6r^4 + 2100K_7r^5, \qquad (75)
\end{aligned}
$$

$$
\begin{aligned}
80K_1 = &3F_0^2F_1 - 6F_0^2F_2r - 6F_0F_1^2r + 18F_0F_1F_2r^2 - 12F_0F_2^2r^3 - 8F_0J_1 \\
&+ 16F_0J_2r - 24F_0J_3r^2 + 32F_0J_4r^3 + 3F_1^3r^2 - 12F_1^2F_2r^3 + 15F_1F_2^2r^4 \\
&+ 8F_1J_1r - 16F_1J_2r^2 + 24F_1J_3r^3 - 32F_1J_4r^4 - 6F_2^3r^5 - 8F_2J_1r^2 \\
&+ 16F_2J_2r^3 - 24F_2J_3r^4 + 32F_2J_4r^5 + 160K_2r - 240K_3r^2 + 320K_4r^3 \\
&- 400K_5r^4 + 480K_6r^5 - 560K_7r^6, \qquad (76)
\end{aligned}
$$

$$
\begin{aligned}
400K_0 = -(&6F_0^3 - 33F_0^2F_1r + 48F_0^2F_2r^2 + 48F_0F_1^2r^2 - 126F_0F_1F_2r^3 + 78F_0F_2^2r^4 \\
&+ 40F_0J_1r - 80F_0J_2r^2 + 120F_0J_3r^3 - 160F_0J_4r^4 - 21F_1^3r^3 \\
&+ 78F_1^2F_2r^4 - 93F_1F_2^2r^5 - 40F_1J_1r^2 + 80F_1J_2r^3 - 120F_1J_3r^4 \\
&+ 160F_1J_4r^5 + 36F_2^3r^6 + 40F_2J_1r^3 - 80F_2J_2r^4 + 120F_2J_3r^5 \\
&- 160F_2J_4r^6 - 400K_2r^2 + 800K_3r^3 - 1200K_4r^4 + 1600K_5r^5 \\
&- 2000K_6r^6 + 2400K_7r^7). \qquad (77)
\end{aligned}
$$

Provided that the conditions (60)-(77) are satisfied, the transformations (2) mapping equation (21) to a linear equation (3) is obtained by solving the following compatible system of equations for the functions $\varphi(x, y)$ and $\psi(x, y)$

$$\varphi_x = r\varphi_y, \tag{78}$$

$$\varphi_y\psi_x = r\varphi_y\psi_y - \Delta, \tag{79}$$

$$10\Delta\varphi_{yy} = \varphi_y(4\Delta_y - F_2\Delta), \tag{80}$$

and

$$\begin{aligned}500\varphi_y\psi_{yyyy}\Delta^3 = {} & 300\psi_{yyy}\varphi_y\Delta^2(4\Delta_y - F_2\Delta) + 5\psi_{yy}\varphi_y\Delta(-120F_{2y}\Delta^2 - 144\Delta_y^2 \\ & + 72\Delta_y F_2\Delta - 39F_2^2\Delta^2 + 80J_4\Delta^2) + \psi_y\varphi_y(-500\varphi_y^3\alpha\Delta^3 \\ & - 150F_{2yy}\Delta^3 + 360F_{2y}\Delta_y\Delta^2 - 165F_{2y}F_2\Delta^3 + 100J_{4y}\Delta^3 + 96\Delta_y^3 \\ & - 72\Delta_y^2F_2\Delta + 108\Delta_y F_2^2\Delta^2 - 240\Delta_y J_4\Delta^2 - 24F_2^3\Delta^3 + 60F_2J_4\Delta^3) \\ & - 500\psi\varphi_y^5\beta\Delta^3 + 500K_7\Delta^4.\end{aligned} \tag{81}$$

The coefficients α and β of the resulting linear equation (3) is given by

$$\alpha = \frac{\Theta}{8\varphi_y^3}\ , \tag{82}$$

and

$$\begin{aligned}\beta = (1600\Delta\varphi_y^4)^{-1}\Big[& \Delta(-144F_{2y}^2 - 72F_{2y}F_2^2 + 352F_{2y}J_4 + 160J_{4yy} + 80J_{4y}F_2 \\ & +640K_{6y} - 1600K_{7x} - 2880K_{7y}r + 80\Theta_y - 4480r_yK_7 - 400F_1K_7 \\ & -9F_2^4 + 88F_2^2J_4 + 160F_2K_6 - 320F_2K_7r - 144J_4^2) - 120\Delta_y\Theta\Big],\end{aligned} \tag{83}$$

where Θ is the following expression

$$\Theta = (F_2^2 - 4J_4)F_2 - 8(K_6 - 7K_7r) - 8J_{4y} + 6F_{2y}F_2 + 4F_{2yy}. \tag{84}$$

Remark 2. The equations (60)-(77) define eighteen *relative invariants* of the third-order for the general point transformation group (2).

3. Proof of the linearization theorems

The proof of the linearization theorems formulated above requires investigation of integrability conditions for the equations given in Section 2.1. We will consider the problem for the candidates (9) and (21) separately. The problem is formulated as follows. Given the coefficients $A_i(x, y), B_i(x, y), C_i(x, y), D_i(x, y)$ and $F_i(x, y), H_i(x, y), J_i(x, y), K_i(x, y)$ of the equations (9) and (21), respectively, find the integrability conditions of the respective equations for the functions φ and ψ.

3.1. Proof of Theorem 1

Let us turn to the proof of Theorem 1 on linearization of (9). Namely, given the coefficients $A_i(x,y), B_i(x,y), C_i(x,y), D_i(x,y)$ of (9), we have to find the necessary and sufficient conditions for integrability of the over-determined system (10)-(20) for the unknown functions $\varphi(x)$ and $\psi(x,y)$.

We first rewrite the expressions (10) and (11) for A_1 and A_0 in the following form

$$\psi_{yy} = \frac{\psi_y A_1}{4}, \tag{85}$$

$$\psi_{xy} = \frac{(6\varphi_{xx} + \varphi_x A_0)}{4\varphi_x}\psi_y. \tag{86}$$

Comparing the mixed derivative $(\psi_{yy})_x = (\psi_{xy})_y$, one arrives at (42)

$$A_{0y} = A_{1x}.$$

Then (12), (13) and (14) are written in the form

$$3A_1 - 4B_0 = 0,$$

$$3A_1^2 - 8C_2 + 12A_{1y} = 0,$$

and

$$12A_{1x} + 3A_0A_1 - 4C_1 = 0,$$

respectively. So that one obtains (43), (44) and (45) respectively. Furthermore, (15) for C_0 becomes

$$\varphi_{xxx} = -\frac{(12A_{0x}\varphi_x^2 - 60\varphi_{xx}^2 + 3\varphi_x^2A_0^2 - 8\varphi_x^2C_0)}{40\varphi_x}. \tag{87}$$

Differentiation of (87) with respect to y yields

$$12A_{0x}A_1 + 32C_{0y} - 16C_{1x} + 3A_0^2A_1 - 4A_0C_1 = 0.$$

Thus one gets (46). Therefore (16), (17) and (18) can be written in the form of (47), (48) and (49), respectively.

One can determine α from (19), as the following

$$\alpha = \frac{4A_{0xx} + 6A_{0x}A_0 - 8C_{0x} + A_0^3 - 4A_0C_0 + 8D_1}{8\varphi_x^3}. \tag{88}$$

Since $\varphi = \varphi(x)$, we have $\alpha_y = 0$ yields (50)

$$\begin{aligned} D_{2x} = \; & -\tfrac{1}{192}\Big[36A_{0x}A_0A_1 - 48A_{0x}C_1 - 48C_{0x}A_1 - 288D_{1y} + 9A_0^3A_1 \\ & -12A_0^2C_1 - 36A_0A_1C_0 + 48A_0D_2 + 32C_0C_1\Big]. \end{aligned}$$

From (20) one finds

$$\begin{aligned} \psi_{xxxx} = \; & -\frac{1}{40\varphi_x^3}\Big[32A_{0xx}\varphi_x^3\psi_x - 72A_{0x}\varphi_{xx}\varphi_x^2\psi_x + 48A_{0x}\varphi_x^3\psi_{xx} \\ & + 36A_{0x}\varphi_x^3\psi_xA_0 - 48C_{0x}\varphi_x^3\psi_x - 120\varphi_{xx}^3\psi_x + 360\varphi_{xx}^2\varphi_x\psi_{xx} \\ & - 240\varphi_{xx}\varphi_x^2\psi_{xxx} - 18\varphi_{xx}\varphi_x^2\psi_xA_0^2 + 48\varphi_{xx}\varphi_x^2\psi_xC_0 + 40\varphi_x^7\beta\psi \\ & + 12\varphi_x^3\psi_{xx}A_0^2 - 32\varphi_x^3\psi_{xx}C_0 + 5\varphi_x^3\psi_xA_0^3 - 20\varphi_x^3\psi_xA_0C_0 \\ & + 40\varphi_x^3\psi_xD_1 - 40\varphi_x^3\psi_yD_0\Big]. \end{aligned} \tag{89}$$

Forming the mixed derivative $(\psi_{xxxx})_y = (\psi_{xy})_{xxx}$ one obtains

$$\beta = \frac{1}{1600\varphi_x^5}\Big[320A_{0xxx}\varphi_x - 1200A_{0xx}\varphi_{xx} + 360A_{0xx}\varphi_x A_0 + 336A_{0x}^2\varphi_x - 1800A_{0x}\varphi_{xx}A_0 - 12A_{0x}\varphi_x A_0^2 + 32A_{0x}\varphi_x C_0 - 480C_{0xx}\varphi_x + 2400C_{0x}\varphi_{xx} + 1600D_{0y}\varphi_x - 300\varphi_{xx}A_0^3 + 1200\varphi_{xx}A_0C_0 - 2400\varphi_{xx}D_1 - 39\varphi_x A_0^4 + 208\varphi_x A_0^2C_0 - 400\varphi_x A_0 D_1 + 400\varphi_x A_1 D_0 - 144\varphi_x C_0^2\Big]. \quad (90)$$

Since $\varphi = \varphi(x)$, we have $\beta_y = 0$ yields (51)

$$D_{1xy} = \tfrac{3}{384}\Big[[(3A_0A_1 - 4C_1)A_0^2 + 16(2A_1D_1 + C_0C_1) - 16(A_1C_0 - D_2)A_0]A_0 - 32[4(C_1D_1 - 2C_2D_0 + C_0D_2) + (3A_1D_0 - C_0^2)A_1] - 96D_{1y}A_0 + 384D_{0y}A_1 + 1536D_{0yy} - 16(3A_0A_1 - 4C_1)C_{0x} + 12[(3A_0A_1 - 4C_1)A_0 - 4(A_1C_0 - 4D_2)]A_{0x}\Big].$$

From (87) one can rewrite the representation for C_0 upon denoting $\chi = \frac{\varphi_{xx}}{\varphi_x}$ leads to (52) and the representations for ψ_{yy} and ψ_{xy} in the equations (85) and (86) become (54) and (55). Rewriting the representation for α from (88) in the form

$$\alpha = \frac{\Omega}{8\varphi_x^3},$$

where

$$\Omega = A_0^3 - 4A_0C_0 + 8D_1 - 8C_{0x} + 6A_{0x}A_0 + 4A_{0xx},$$

and thus β of (90) becomes

$$\beta = (1600\varphi_x^4)^{-1}(-144A_{0x}^2 - 72A_{0x}A_0^2 + 352A_{0x}C_0 + 160C_{0xx} + 80C_{0x}A_0 + 1600D_{0y} - 640D_{1x} + 80\Omega_x - 9A_0^4 + 88A_0^2C_0 - 160A_0D_1 - 30A_0\Omega + 400A_1D_0 - 300\chi\Omega - 144C_0^2).$$

Finally, one obtains (89) in the form

$$1600\psi_{xxxx} = 9600\psi_{xxx}\chi + 160\psi_{xx}(-12A_{0x} - 3A_0^2 - 90\chi^2 + 8C_0) + 40\psi_x(12A_{0x}A_0 + 72A_{0x}\chi - 16C_{0x} + 3A_0^3 + 18A_0^2\chi - 12A_0C_0 + 120\chi^3 - 48\chi C_0 + 24D_1 - 8\Omega) + \psi(144A_{0x}^2 + 72A_{0x}A_0^2 - 352A_{0x}C_0 - 160C_{0xx} - 80C_{0x}A_0 - 1600D_{0y} + 640D_{1x} - 80\Omega_x + 9A_0^4 - 88A_0^2C_0 + 160A_0D_1 + 30A_0\Omega - 400A_1D_0 + 300\chi\Omega + 144C_0^2) + 1600\psi_yD_0.$$

Hence we complete the proof of Theorem 1.

3.2. Proof of Theorem 2

In the case of (21), the problem is formulated as follows. Given the coefficients $F_i(x,y), H_i(x,y), J_i(x,y), K_i(x,y)$ of (21), find the necessary and sufficient conditions for integrability of the over-determined system of equations (22)-(40) for the unknown functions $\varphi(x,y)$ and $\psi(x,y)$. Recall that, according to our notation, the following equations hold

$$\varphi_x = r\varphi_y, \quad \psi_x = \frac{\psi_y\varphi_x - \Delta}{\varphi_y}, \tag{91}$$

and

$$\alpha_x = \frac{\varphi_x}{\varphi_y}\,\alpha_y, \qquad \beta_x = \frac{\varphi_x}{\varphi_y}\,\beta_y.$$

Let us simplify the expression (22) as follow

$$\varphi_{yy} = \Big[(4\Delta_y - F_2\Delta)\varphi_y\Big]/(10\Delta). \tag{92}$$

Comparing the mixed derivative $(\varphi_x)_{yy} = (\varphi_{yy})_x$ one obtains

$$\begin{aligned}\Delta_{xy} = \Big[&F_{2x}\Delta^2 - F_{2y}r\Delta^2 + 10r_{yy}\Delta^2 + 4r_y\Delta_y\Delta - r_yF_2\Delta^2 \\ &+ 4\Delta_x\Delta_y + 4\Delta_{yy}r\Delta - 4\Delta_y^2 r\Big]/(4\Delta).\end{aligned} \tag{93}$$

Rewriting (23) in the form

$$\Delta_x = (20r_y\Delta + 4\Delta_y r + F_1\Delta - 2F_2r\Delta)/4.$$

Forming the mixed derivative $\Delta_{xy} = (\Delta_x)_y$ one arrives at (60)

$$r_{yy} = -(F_{1y} - F_{2x} - F_{2y}r - r_yF_2)/10.$$

Then (24)-(27) are written in the form of (61)-(64), respectively. Furthermore, (28) becomes

$$\Delta_{yy} = -(20F_{2y}\Delta^2 - 48\Delta_y^2 + 4\Delta_yF_2\Delta + 7F_2^2\Delta^2 - 20J_4\Delta^2)/(40\Delta).$$

Now, consider the equation $(\Delta_{yy})_x = (\Delta_x)_{yy}$, one gets (65)

$$\begin{aligned}F_{1yy} = \ & -(F_{1y}F_2 - 40F_{2xy} - 16F_{2x}F_2 + 20F_{2yy}r + 40F_{2y}r_y \\ & +14F_{2y}F_2r + 20J_{4x} - 20J_{4y}r + 14r_yF_2^2 - 40r_yJ_4)/10.\end{aligned}$$

Thus equations (29)-(32) yield (66)-(69), and from (33) one finds

$$\begin{aligned}\psi_{yyyy} = \Big[&300\psi_{yyy}\varphi_y\Delta^2(4\Delta_y - F_2\Delta) + 5\psi_{yy}\varphi_y\Delta(-120F_{2y}\Delta^2 - 144\Delta_y^2 \\ &+ 72\Delta_yF_2\Delta - 39F_2^2\Delta^2 + 80J_4\Delta^2) + \psi_y\varphi_y(-500\varphi_y^3\alpha\Delta^3 \\ &- 150F_{2yy}\Delta^3 + 360F_{2y}\Delta_y\Delta^2 - 165F_{2y}F_2\Delta^3 + 100J_{4y}\Delta^3 \\ &+ 96\Delta_y^3 - 72\Delta_y^2F_2\Delta + 108\Delta_yF_2^2\Delta^2 - 240\Delta_yJ_4\Delta^2 - 24F_2^3\Delta^3 \\ &+ 60F_2J_4\Delta^3) - 500\psi\varphi_y^5\beta\Delta^3 + 500K_7\Delta^4\Big]/(500\varphi_y\Delta^3).\end{aligned} \tag{94}$$

One can determine α from (34), as the following

$$\alpha = (4F_{2yy} + 6F_{2y}F_2 - 8J_{4y} + F_2^3 - 4F_2J_4 - 8K_6 + 56K_7r)/8\varphi_y^3. \tag{95}$$

Now the equation $\alpha_x - r\alpha_y = 0$ leads to (70). Furthermore, one considers $(\psi_x)_{yyyy} = (\psi_{yyyy})_x$, yields

$$\begin{aligned}\beta = \ & 120\Delta_y(-4F_{2yy} - 6F_{2y}F_2 + 8J_{4y} - F_2^3 + 4F_2J_4 + 8K_6 - 56K_7r) \\ & +\Delta(320F_{2yyy} + 480F_{2yy}F_2 + 336F_{2y}^2 + 168F_{2y}F_2^2 + 32F_{2y}J_4 \\ & -480J_{4yy} - 240J_{4y}F_2 - 1600K_{7x} + 1600K_{7y}r - 400F_1K_7 \\ & -9F_2^4 + 88F_2^2J_4 + 160F_2K_6 - 320F_2K_7r - 144J_4^2)/1600\Delta\varphi_y^4.\end{aligned} \tag{96}$$

The equation $\beta_x - r\beta_y = 0$ leads to (71). Therefore, (35)-(40) become (72)-(77), respectively.

Let us turn now to the integrability problem. One can find all fourth-order derivatives of the functions φ and ψ by using (91), (92) and (94). So that one obtains at (78)-(81). Finally, the coefficients α and β of the resulting linear equations (95) and (96) are given by

$$\alpha = \frac{\Theta}{8\varphi_y^3},$$

$$\begin{aligned}\beta = \;& (-144F_{2y}^2\Delta - 72F_{2y}F_2^2\Delta + 352F_{2y}J_4\Delta + 160J_{4yy}\Delta + 80J_{4y}F_2\Delta \\ & +640K_{6y}\Delta - 1600K_{7x}\Delta - 2880K_{7y}r\Delta - 4480r_yK_7\Delta + 80\Theta_y\Delta \\ & -120\Delta_y\Theta - 400F_1K_7\Delta - 9F_2^4\Delta + 88F_2^2J_4\Delta + 160F_2K_6\Delta \\ & -320F_2K_7r\Delta - 144J_4^2\Delta)/(1600\varphi_y^4\Delta),\end{aligned}$$

where

$$\Theta = (F_2^2 - 4J_4)F_2 - 8(K_6 - 7K_7r) - 8J_{4y} + 6F_{2y}F_2 + 4F_{2yy}.$$

Hence we complete the proof of Theorem 2.

4. Illustration of the linearization theorems

4.1. An example on Theorem 1

Example 1. Consider the nonlinear ordinary differential equation

$$x^2y(2y^{(4)}+y)+8x^2y'y'''+16xyy'''+6x^2y''^2+48xy'y''+24yy''+24y'^2 = 0.(97)$$

It is an equation of the form (9) with the coefficients

$$A_1 = \frac{4}{y}\,,\; A_0 = \frac{8}{x}\,,\; B_0 = \frac{3}{y}\,,\; C_2 = 0\,,\; C_1 = \frac{24}{xy}\,,\; C_0 = \frac{12}{x^2}\,,$$

$$D_4 = 0\,,\; D_3 = 0\,,\; D_2 = \frac{12}{x^2y}\,,\; D_1 = 0\,,\; D_0 = \frac{y}{2}\,\cdot \tag{98}$$

One can check that the coefficients (98) obey the conditions (42)-(51). Thus, the equation (97) is linearizable. We have

$$8C_0 - 3A_0^2 - 12A_{0x} = 0 \tag{99}$$

and the equation (52) is written as

$$2\frac{d\chi}{dx} - \chi^2 = 0.$$

Let us take its simplest solution $\chi = 0$. Then invoking (53), we let

$$\varphi = x.$$

Now the equations (54)-(55) are written

$$\frac{\psi_{yy}}{\psi_y} = \frac{1}{y}, \quad \frac{\psi_{xy}}{\psi_y} = \frac{2}{x}$$

and yield

$$\psi_y = Kx^2y, \qquad K = const.$$

Hence

$$\psi = K\frac{x^2y^2}{2} + f(x).$$

Since one can use any particular solution, we set $K = 2,\ \ f(x) = 0$ and take

$$\psi = x^2y^2.$$

Invoking (99) and noting that (57) yields $\Omega = 0$, one can readily verify that the function $\psi = x^2y^2$ solves equation (56) as well. Hence, one obtains the following transformations

$$t = x, \quad u = x^2y^2. \tag{100}$$

Since $\Omega = 0$, equations (58) and (59) give

$$\alpha = 0, \quad \beta = \frac{1}{\varphi_x^4} = 1$$

Hence, the equation (97) is mapped by the transformations (100) to the linear equation

$$u^{(4)} + u = 0.$$

Example 2. The third-order member of the Riccati Hierarchy is given by Euler et al. [5] as

$$y''' + 4yy'' + 3y'^2 + 6y^2y' + 4y^4 = 0. \tag{101}$$

Applying [6], and [7] one checks that equation cannot be linearized by a point transformation or contact transformation or generalized Sundman transformation. Under the Riccati transformation $y = \frac{a\omega'}{\omega}$ the equation (101) becomes [8]

$$\begin{aligned} &\omega^3\omega^{(4)} + 4(a-1)\,\omega^2\omega'\omega''' + 3(a-1)\,\omega^2\omega''^2 \\ &+ 6(a-1)(a-2)\,\omega\omega'^2\omega'' + (a-1)(a-2)(a-3)\,\omega'^4 = 0. \end{aligned} \tag{102}$$

It is an equation of the form (9) with the coefficients

$$\begin{array}{c} A_1 = \frac{4(a-1)}{\omega}\,,\ A_0 = 0\,,\ B_0 = \frac{3(a-1)}{\omega}\,, \\ C_2 = \frac{6(a^2-3a+2)}{\omega^2}\,,\ C_1 = 0\,,\ C_0 = 0\,, \\ D_4 = \frac{a^3-6a^2+11a-6}{\omega^3}\,,\ D_3 = 0\,,\ D_2 = 0\,,\ D_1 = 0\,,\ D_0 = 0. \end{array} \tag{103}$$

One can verify that the coefficients (103) obey the linearization conditions (42)-(51). Furthermore,

$$8C_0 - 3A_0^2 - 12A_{0x} = 0 \tag{104}$$

and the equation (52) is written as

$$2\frac{d\chi}{dx} - \chi^2 = 0.$$

We take its simplest solution $\chi = 0$ and obtain from (53) the equation $\varphi'' = 0$, whence

$$\varphi = x.$$

Equations (54) and (55) have the form

$$\frac{\psi_{\omega\omega}}{\psi_\omega} = \frac{a-1}{\omega}, \qquad \psi_{x\omega} = 0$$

and yield

$$\psi_\omega = K\omega^{(a-1)}, \qquad K = const.$$

Hence

$$\psi = K\frac{\omega^a}{a} + f(x).$$

Since one can use any particular solution, we set $K = a, \quad f(x) = 0$ and take

$$\psi = \omega^a.$$

Invoking (104) and noting that (57) yields $\Omega = 0$, one can readily verify that the function $\psi = \omega^a$ solves equation (56) as well. So that one obtains the following transformations

$$t = x, \quad u = \omega^a. \tag{105}$$

Since $\Omega = 0$, equations (58) and (59) gives

$$\alpha = 0, \quad \beta = 0.$$

Hence, the equation (102) is mapped by the transformations (105) to the linear equation

$$u^{(4)} = 0.$$

Example 3. Let us consider the Boussinesq equation

$$u_{tt} + uu_{xx} + u_x^2 + u_{xxxx} = 0. \tag{106}$$

Of particular interest among the solutions of the Boussinesq equation are travelling wave solutions:

$$u(x,t) = H(x - Dt).$$

Substituting the representation of a solution into (106), one finds

$$H^{(4)} + (H + D^2)H'' + H'^2 = 0. \tag{107}$$

It is an equation of the form (9) with the coefficients

$$\begin{gathered} A_1 = 0\,,\ A_0 = 0\,,\ B_0 = 0\,,\ C_2 = 0\,,\ C_1 = 0\,,\ C_0 = D^2 + H\,, \\ D_4 = 0\,,\ D_3 = 0\,,\ D_2 = 1\,,\ D_1 = 0\,,\ D_0 = 0. \end{gathered} \tag{108}$$

Since the coefficients (108) do not satisfy the linearization conditions (46), (49) and (51), hence, the equation (107) is not linearizable.

Example 4. Consider the non-linear equation

$$y^{(4)} - \frac{10}{y'}y''y''' + \frac{1}{y'^2}\left(15y''^3 - xy'^7 - y'^6\right) = 0. \tag{109}$$

It has the form (21) with the following coefficients:

$$r = 0\,,\ F_2 = 0\,,\ F_1 = 0\,,\ F_0 = 0\,,\ H_2 = 0\,,\ H_1 = 0\,,\ H_0 = 0\,,$$
$$J_4 = 0\,,\ J_3 = 0\,,\ J_2 = 0\,,\ J_1 = 0\,,\ J_0 = 0\,,\ K_7 = -x\,,$$
$$K_6 = -1\,,\ K_5 = 0\,,\ K_4 = 0\,,\ K_3 = 0\,,\ K_2 = 0\,,\ K_1 = 0\,,\ K_0 = 0. \quad (110)$$

Let us test the equation (109) for linearization by using Theorem 2. It is manifest that the equations (60)-(77) are satisfied by the coefficients (110). Thus, the equation (109) is linearizable, and we can proceed further.

Let us take its simplest solution $\varphi = y$ and $\psi = x$ which satisfy the compatible system of equations (78)-(81). So that one obtains the following transformations

$$t = y, \quad u = x. \quad (111)$$

Since $\Theta = 8$, equations (82) and (83) give

$$\alpha = 1, \quad \beta = 1.$$

Hence, the equation (109) is mapped by the transformations (111) to the linear equation

$$u^{(4)} + u' + u = 0.$$

References

[1] Lie S 1883 *Klassifikation und Integration von gewöhnlichen Differentialgleichungen zwischen x, y, die eine Gruppe von Transformationen gestatten III* (Archiv for Matematik og Naturvidenskab) vol 8 (Kristiania) p 371–458
[2] Grebot G 1997 it Journal of Mathematical Analysis and Applications **206** 364–388
[3] Ibragimov N H and Meleshko S V 2004 *Archives of ALGA* **1** 71–93
[4] Ibragimov N H 1999 *Elementary Lie group analysis and ordinary differential equations* (Chichester: Wiley)
[5] Euler M, Euler N and Leach P G L 2005 *preprint: Institut Mittag Leffler* (The Royal Swedish Academy of Sciences, Auravgen 17, SE-182 60 Djursholm, Sweden)
[6] Ibragimov N H and Meleshko S V 2005 *Journal of Mathematical Analysis and Applications* **308** 266–89
[7] Euler N, Wolf T, Leach P G L and Euler M 2003 *Acta Applicandae Mathematicae* **76** 89115
[8] Andriopoulos K and Leach P G L 2007 *Journal of Mathematical Analysis and Applications* **328** 625–639