LECTURE NOTES ON JET SPACES, SYMMETRIES OF PDES, AND SYMMETRY-INVARIANT SOLUTIONS

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1. Introduction

In these notes we try to describe the theory of (generalized or higher) symmetries of PDEs in the most general, compact, and coordinate-free form. In particular, the initial bundle is allowed to be nontrivial and non-vector, the subset of a jet space determined by a system of PDEs is not required to be a submanifold and is not required to be formally integrable in the classical sense. All necessary coordinate formulas are also presented.

It seems that in the literature the theory of symmetries is developed rigorously only for formally integrable PDEs. However, as we show in Example 4 in Section 5, there are many PDEs (and some of them are very simple) that are not formally integrable. Therefore, the standard approach to symmetries must be generalized in order to include this kind of PDEs, and we hope that the present notes fill this gap.

The theory is illustrated with the examples of the KdV and sine-Gordon equations. More examples can be found in [2, 5, 7]. Our framework is close to that of [2, 4, 5], but is more general. Also, we announce without proof a new result from [3] on the existence of symmetry-invariant solutions.

All manifolds, mappings, and functions are supposed to be smooth.

Remark 1. In the analytic situation the theory is essentially the same, but one must consider sheaves instead of globally defined functions and vector fields.

For a bundle τ the space of sections is denoted by $\Gamma(\tau)$. For a smooth map $f\colon M_1 \to M_2$ the differential is denoted by f_* and the pullback map is $f^*\colon C^\infty(M_2) \to C^\infty(M_1)$. For a manifold N the space of vector fields is denoted by $D(N)$. The letter D here reflects the fact that vector fields on N are in one-to-one correspondence with derivations of the algebra $C^\infty(N)$.

2. An instructive example: solutions of the KdV equation

To motivate the general theory, consider first an example.

For a function $u = u(x,t)$ introduce the following notation for partial derivatives

$$u_x = \frac{\partial u}{\partial x}, \quad u_t = \frac{\partial u}{\partial t}, \quad u_{xt} = \frac{\partial^2 u}{\partial x \partial t}, \quad u_{xxx} = \frac{\partial^3 u}{\partial x^3}.$$

The famous Korteweg-de Vries equation reads

$$u_t = u_{xxx} + u_x u. \tag{1}$$

In what follows we call it the KdV equation.

By the Cauchy-Kovalevskaya theorem, for any analytic function $f(x)$ there is a unique solution $u(x,t)$ of (1) such that $u(x,0) = f(x)$. However, this solution is usually hard to describe explicitly. We want to find explicit solutions of (1), for example, solutions that can be expressed in terms of elementary or special functions. To do this, we add to (1) another PDE

$$\varphi(x,t,u,u_x,u_{xx},\dots) = 0. \tag{2}$$

It turns out that a system of two equations (1), (2) is easier to solve than the initial equation (1). Indeed, one can solve first equation (2) as an ordinary differential equation with respect to the variable x, treating t as a parameter. Then one solves (1) as a first-order ordinary differential equation with respect to t.

However, for a random equation (2) this system is inconsistent and has no solutions. The theory that is described in these notes allows to construct many functions φ such that system (1), (2) possesses solutions. It turns out that these solutions are in some sense the most interesting of all KdV solutions and have a lot of applications in physics.

For example, one can take

$$\varphi = u_{xxx} + u_x u - c u_x \tag{3}$$

for arbitrary constant $c \in \mathbb{R}$. Let us solve system (1), (2) for this φ. From

$$u_{xxx} + u_x u - c u_x = 0 \tag{4}$$

and (1) one obtains

$$u_t = c u_x \tag{5}$$

Exercise 1. Deduce from equations (5) and (4) that $u(x,t)$ is of the form $u = P(x+ct)$, where P is a function of one variable s and satisfies the ordinary differential equation

$$P''' + P'P - cP' = 0. \tag{6}$$

Integrating equation (6), we obtain

$$P'' + \frac{1}{2}P^2 - cP = a$$

for some constant a. Multiplying this by P' and integrating once again, one gets

$$\frac{1}{2}(P')^2 + \frac{1}{6}P^3 - \frac{c}{2}P^2 = aP + b \tag{7}$$

for some constant b. This equation can be solved in terms of so-called elliptic functions.

If $a = b = 0$ and $c > 0$ then the general solution of (7) is $P(s) = 3c\,\mathrm{sech}^2(\frac{1}{2}\sqrt{c}s + d)$, where $d \in \mathbb{R}$ is an arbitrary constant. Here

$$\mathrm{sech}(x) = \frac{2}{\mathrm{e}^x + \mathrm{e}^{-x}}$$

is the hyperbolic secant function. The corresponding solution $u(x,t) = P(x+ct)$ of the KdV equation is called the *one-soliton* solution.

Function (3) corresponds to a symmetry of the KdV equation, and the obtained solution is an example of a symmetry-invariant solution.

3. Symmetries of finite jets

3.1. **Jet spaces.** Let $\pi\colon E \to M$ be a fiber bundle. For a section s of π denote by $\Gamma_s \subset E$ its graph. Let k be a non-negative integer. Two sections s_1, s_2 defined on a neighborhood of a point $x \in M$ are said to be *tangent of order k at x* if $s_1(x) = s_2(x)$ and the submanifolds Γ_{s_1}, Γ_{s_2} are tangent of order k at the point $s_1(x) = s_2(x) \in E$. This determines an equivalence relation on the set of germs of local sections at x. The set of equivalence classes is denoted $J^k_x(\pi)$, and the equivalence class $[s]^k_x$ of a section s is called the *k-th order jet of s at x*.

The set

$$J^k(\pi) = \bigcup_{x\in M} J^k_x(\pi)$$

is said to be the *k-th order jet space* of the bundle π. We have a natural map

$$\pi_k\colon J^k(\pi) \to M, \quad [s]^k_x \mapsto x. \tag{8}$$

Obviously, $J^0(\pi)$ can be identified with E, then $\pi_0 = \pi$.

Set $n = \dim M$ and $m = \dim \pi$. Let $x \in M$, $a \in \pi^{-1}(x) \subset E$, and $U \subset M$ be a neighborhood of x diffeomorphic to an open subset of $\mathbb{R}^n$ with coordinates $x_1, \dots, x_n$. One can choose U such that there is an open subset $V \subset \mathbb{R}^m$ with coordinates $u^1, \dots, u^m$ and an open subset $W \subset E$ containing a and diffeomorphic to $U \times V$ in such a way that

$$\pi\big|_W\colon (x_1, \dots, x_n, u^1, \dots, u^m) \mapsto (x_1, \dots, x_n).$$

Such subset W with coordinates $x_1, \dots, x_n, u^1, \dots, u^m$ is called an *adapted coordinate chart* on E.

Remark 2. In what follows, speaking about local coordinates in jet spaces we always mean coordinates on the preimage $\pi_k^{-1}(W)$ of an adapted coordinate chart $W \subset E$. As we show below, the set $\pi_k^{-1}(W)$ has coordinates (9) and $\pi_\infty^{-1}(W) \subset J^\infty(\pi)$ has coordinates (17).

Let $S(U,W)$ be the set of local sections on open subsets of U whose graphs lie in $W \cong U \times V$. Such a section s is determined by smooth functions

$$\big(f^1(x_1,\dots,x_n),\dots,f^m(x_1,\dots,x_n)\big)$$

on an open subset of U. Denote by $J(U,W) \subset J^k(\pi)$ the set of k-th order jets of sections from $S(U,W)$. For any $i_1,\dots,i_p \in \{1,\dots,n\}$ and $j=1,\dots,m$ consider the functions

$$u^j_\sigma\colon J(U,W)\to\mathbb{R},\quad u^j_\sigma\big([s]^k_x\big)=\frac{\partial^p f^j}{\partial x_{i_1}\dots\partial x_{i_p}}.$$

Here $\sigma=(i_1,\dots,i_p)$ is a *symmetric multi-index*, i.e., a non-ordered collection of numbers from $\{1,\dots,n\}$. Set $|\sigma|=p$.

The multi-index σ is also allowed to be empty, and we can identify

$$u^j = u^j_\emptyset\colon J(U,W)\to\mathbb{R}.$$

Exercise 2. Prove that the number of symmetric multi-indices $\sigma=(i_1,\dots,i_p)$ with $1\le i_l\le n$, $0\le p\le k$ is equal to $s(n,k)=\dfrac{(n+k)!}{n!\,k!}$.

We have also the functions $x_i\colon J(U,W)\to\mathbb{R}$, $i=1,\dots,n$, determined by

$$\Big(x_1\big([s]^k_x\big),\dots,x_n\big([s]^k_x\big)\Big)=x\in U.$$

The functions x_i, u^j_σ provide a map from $J(U,W)$ onto an open subset of $\mathbb{R}^{n+m\cdot s(n,k)}$.

It is easily seen that $J^k(\pi)$ can be covered by subsets of the form $J(U,W)$ and this determines on $J^k(\pi)$ a structure of a smooth manifold of dimension $n+m\cdot s(n,k)$. Mappings (8) become smooth bundles. Thus a system of local coordinates for $J^k(\pi)$ consists of

$$x_i,\quad u^j_\sigma,\quad i=1,\dots,n,\quad j=1,\dots,m,\quad |\sigma|\le k. \tag{9}$$

Remark 3. In the classical language the symbol u^j_σ corresponds to the partial derivative

$$u^j_\sigma=\frac{\partial^{|\sigma|}u^j}{\partial x_{i_1}\dots\partial x_{i_p}}. \tag{10}$$

3.2. **The Cartan distribution and symmetries.** For a section s of π denote by $j_k(s)$ the section of π_k given by

$$j_k(s)(x)=[s]^k_x,\quad x\in M.$$

Let s be a section of π on a neighborhood U of $x\in M$ and $a=[s]^k_x\in J^k(\pi)$. The n-dimensional vector subspace $R(s,x)\subset T_aJ^k(\pi)$ equal to the tangent space at a of the submanifold $j_k(s)(U)\subset J^k(\pi)$ is called an *R-plane.* The *Cartan subspace* $C(a)$ of $T_aJ^k(\pi)$ is the linear span of all R-planes $R(s',x)$ such that $[s']^k_x=a$. It can be shown that Cartan subspaces form a smooth distribution on $J^k(\pi)$ called the *Cartan distribution.* The following propositions can be proved straightforwardly.

Proposition 1. *In a coordinate neighborhood one has the following basis of vector fields for the Cartan distribution*

$$D_{k,i} = \frac{\partial}{\partial x_i} + \sum_{\substack{j=1,\dots,m,\\ |\sigma|\le k-1}} u^j_{\sigma i}\frac{\partial}{\partial u^j_\sigma}, \quad i=1,\dots,n,$$

$$\frac{\partial}{\partial u^{j'}_{\sigma'}}, \quad j'=1,\dots,m, \quad |\sigma'|=k.$$

The Cartan distribution is also given by the 1-forms

$$w^j_\sigma = du^j_\sigma - \sum_{i=1}^{n} u^j_{\sigma i}dx_i, \quad |\sigma|\le k-1, \quad j=1,\dots,m,$$

which are called Cartan forms. *Here an below for $\sigma=(i_1,\dots,i_p)$ the symmetric multi-index σi is $(i_1,\dots,i_p,i)$.*

In particular, for $k\ge 1$ the fibers of the natural bundle

$$\pi_{k,k-1}\colon J^k(\pi)\to J^{k-1}(\pi), \quad [s]^k_x \mapsto [s]^{k-1}_x,$$

are integral submanifolds of this distribution.

Remark 4. Let $\sigma' = (\underbrace{i,i,\dots,i}_{k})$, then

$$\left[\frac{\partial}{\partial u^{j'}_{\sigma'}}\,,\, D_{k,i}\right] = \frac{\partial}{\partial u^{j'}_{\sigma''}}, \quad \sigma'' = (\underbrace{i,i,\dots,i}_{k-1}).$$

Therefore, the Cartan distribution on $J^k(\pi)$ is not involutive, that is, not closed with respect to the commutator of vector fields.

Proposition 2. *Let $N\subset J^k(\pi)$ be a submanifold such that $\pi_k\big|_N\colon N\to M$ is a diffeomorphism onto an open subset $U\subset M$. Then N is an integral submanifold of the Cartan distribution if and only if $N=j_k(s)(U)$ for some section $s\colon U\to E$ of the bundle π.*

Proposition 3. *For an R-plane $R(s,x)\subset T_aJ^k(\pi)$ consider the point $[s]^{k+1}_x\in J^{k+1}(\pi)$. This correspondence provides a bijection between the set of R-planes at a and the set $\pi^{-1}_{k+1,k}(a)\subset J^{k+1}(\pi)$. The differential of the bundle $\pi_{k+1,k}\colon J^{k+1}(\pi)\to J^k(\pi)$ projects the Cartan subspace $\mathcal{C}([s]^{k+1}_x)$ onto $R(s,x)$.*

A vector field $X\in D(J^k(\pi))$ is said to be a *symmetry* if for any vector field X' from the Cartan distribution the commutator $[X,X']$ also belongs to the Cartan distribution.

Remark 5. Clearly, $X\in D(J^k(\pi))$ is a symmetry if and only if the corresponding one-parametric group of local diffeomorphisms preserves the Cartan distribution.

For $k_1\ge k_2$ we have the natural bundles

$$\pi_{k_1,k_2}\colon J^{k_1}(\pi)\to J^{k_2}(\pi), \quad [s]^{k_1}_x \mapsto [s]^{k_2}_x.$$

The next proposition is called the Lie-Bäcklund theorem in the infinitesimal form, its proof can be found in [2].

Proposition 4. *Let $X \in D(J^k(\pi))$ be a symmetry. For each $p \geq k$ there is a unique symmetry $X_p \in D(J^p(\pi))$ such that*

$$(\pi_{p,k})_*(X_p) = X. \tag{11}$$

If $\dim \pi > 1$ then for each $0 \leq p \leq k$ there is a unique symmetry $X_p \in D(J^p(\pi))$ such that

$$(\pi_{k,p})_*(X) = X_p.$$

If $\dim \pi = 1$ then the same statement holds for each $1 \leq p \leq k$.

For $p_1 \geq p_2$ we have also

$$(\pi_{p_1,p_2})_*(X_{p_1}) = X_{p_2}.$$

Thus for $\dim \pi > 1$ every symmetry of $J^k(\pi)$ is the lifting of some symmetry of $J^0(\pi) = E$.

If $\dim \pi = 1$ then every symmetry of $J^k(\pi)$ for $k \geq 1$ is the lifting of some symmetry of $J^1(\pi)$.

Remark 6. Let $X \in D(J^k(\pi))$ be a symmetry and $p \geq k$. The unique symmetry $X_p \in D(J^p(\pi))$ satisfying (11) is constructed as follows.

Let $a \in J^p(\pi)$. We have $a = [s]^p_{x_0}$ for $x_0 = \pi_p(a)$ and some section $s \in \Gamma(\pi)$. Consider the point

$$a_k = \pi_{p,k}(a) = [s]^k_{x_0} \in J^k(\pi).$$

Let A_t be the one-parametric group of local diffeomorphisms of $J^k(\pi)$ corresponding to the vector field $X \in D(J^k(\pi))$. Consider the submanifold

$$N = j_k(s)(M) = \{[s]^k_x \mid x \in M\} \subset J^k(\pi),$$

which is an integral submanifold of the Cartan distribution. Since $A_t(N)$ is also an integral submanifold of the Cartan distribution, according to Proposition 2 for small enough t and some neighborhood U of a_k we have $A_t(N) \cap U = j_k(s_t)(U')$ where $U' \subset M$ is some neighborhood of x_0 and s_t is a section of π over $U' \subset M$.

Set $x(t) = \pi_k(A_t(a_k)) \in M$. We have $x(0) = x_0$. Note that s_t depends smoothly on t and $s_0 = s\big|_{U'}$. Therefore, $a(t) = [s_t]^p_{x(t)}$ is a smooth curve in $J^p(\pi)$ and $a(0) = a$. Then the value of the vector field $X_p \in D(J^p(\pi))$ at the point $a \in J^p(\pi)$ is defined to be

$$\left.\frac{\mathrm{d}\, a(t)}{\mathrm{d}\, t}\right|_{t=0} \in T_a J^p(\pi).$$

Let us present explicit coordinate formulas for symmetries. Consider the *total derivative operators*

$$D_{x_i} = \frac{\partial}{\partial x_i} + \sum_{\substack{j=1,\dots,m,\\ |\sigma|\geq 0}} u^j_{\sigma i} \frac{\partial}{\partial u^j_\sigma}, \qquad i = 1, \dots, n. \tag{12}$$

These operators commute and are vector fields on the infinite jet space, see Section 4.1 below. In particular, $u^j_i = D_{x_i}(u^j)$.

Remark 7. The meaning of D_{x_i} is the following. Consider a function $f(x_s, u^d, u^j_\sigma)$ of coordinates (9). Set $g = D_{x_i}(f)$. Suppose that u^j are functions of $x_1, \dots, x_n$ and substitute (10) to f and g. Then f, g also become functions of $x_1, \dots, x_n$ and we have $g = \partial f / \partial x_i$.

For $\sigma = (i_1, \ldots, i_p)$ set

$$D_\sigma = D_{x_{i_1}} \circ \cdots \circ D_{x_{i_p}}.$$

The simplest way to prove the next two propositions is to use infinite jet spaces, we will explain this in Section 4.2.

Proposition 5. *For any collection of functions*

$$f^1, \ldots, f^n, g^1 \ldots, g^m \in C^\infty(E) \tag{13}$$

the vector field

$$X = \sum_{i=1}^n f^i \frac{\partial}{\partial x_i} + \sum_{\substack{j=1,\ldots,m,\\ |\sigma|\le k}} \left(D_\sigma \Big(g^j - \sum_{i=1}^n f^i u_i^j \Big) + \Big(\sum_{i=1}^n f^i u_{\sigma i}^j \Big) \right) \frac{\partial}{\partial u_\sigma^j} \tag{14}$$

is a symmetry of $D(J^k(\pi))$.

If $m > 1$ then for any symmetry $X \in D(J^k(\pi))$ there is a unique collection of functions (13) *such that* (14) *holds.*

Proposition 6. *Suppose that $m = 1$, $k \ge 1$ and denote*

$$u = u^1, \quad u_\sigma = u_\sigma^1.$$

For any symmetry $X \in D(J^k(\pi))$ there is a unique function

$$f \in C^\infty(J^1(\pi)) \tag{15}$$

such that

$$X = -\sum_{i=1}^n \frac{\partial f}{\partial u_i} \frac{\partial}{\partial x_i} + \sum_{|\sigma|\le k} \left(D_\sigma(f) - \Big(\sum_{i=1}^n \frac{\partial f}{\partial u_i} u_{\sigma i} \Big) \right) \frac{\partial}{\partial u_\sigma}. \tag{16}$$

And vice versa, for any function (15) *vector field* (16) *is a symmetry of $J^k(\pi)$.*

Remark 8. Note that, although (12) involves u_σ^j for arbitrary σ, vector fields (14), (16) involve only coordinates (9).

4. Symmetries of infinite jets

4.1. The infinite jet space and its Cartan distribution.

The infinite jet space $J^\infty(\pi)$ is the inverse (projective) limit of the sequence of bundles

$$\cdots \to J^{k+1}(\pi) \to J^k(\pi) \to \cdots \to J^1(\pi) \to E \to M.$$

Using the pullback of functions, we obtain natural embeddings

$$C^\infty(M) \hookrightarrow C^\infty(E) \hookrightarrow \ldots \hookrightarrow C^\infty(J^k(\pi)) \hookrightarrow C^\infty(J^{k+1}(\pi)) \hookrightarrow \ldots$$

Using these embeddings, set

$$C^\infty(J^\infty(\pi)) = \bigcup_k C^\infty(J^k(\pi)).$$

Below we identify the algebras $C^\infty(M)$, $C^\infty(E)$, and $C^\infty(J^k(\pi))$ with the corresponding subalgebras of $C^\infty(J^\infty(\pi))$. Local coordinates for $J^\infty(\pi)$ are

$$x_i, \quad u_\sigma^j, \quad i = 1, \ldots, n, \quad j = 1, \ldots, m, \quad |\sigma| \ge 0. \tag{17}$$

A smooth function on $J^\infty(\pi)$ depends on a finite number of coordinates (17).

For a point

$$a = (a_0, a_1, \dots, a_k, \dots) \in J^\infty(\pi), \;\; a_k \in J^k(\pi), \;\; \pi_{k_1,k_2}(a_{k_1}) = a_{k_2} \;\; \forall\, k_1 \geq k_2$$

a *tangent vector* $v \in T_a J^\infty(\pi)$ is a sequence

$$v_0, v_1, \dots, v_k, \dots, \quad v_k \in T_{a_k} J^k(\pi), \quad (\pi_{k_1,k_2})_*(v_{k_1}) = v_{k_2} \quad \forall\, k_1 \geq k_2. \tag{18}$$

Vector fields on $J^\infty(\pi)$ are derivations of the algebra $C^\infty(J^\infty(\pi))$. Each vector field $V \in D(J^\infty(\pi))$ determines naturally the tangent vector $V_a \in T_a J^\infty$ in the usual way. In coordinates V is determined by $V(x_i)$, $V(u^j_\sigma) \in C^\infty(J^\infty(\pi))$

$$V = \sum_i V(x_i) \frac{\partial}{\partial x_i} + \sum_{j,\sigma} V(u^j_\sigma) \frac{\partial}{\partial u^j_\sigma}.$$

Although this sum is infinite, $V(f) \in C^\infty(J^\infty(\pi))$ is well defined for any $f \in C^\infty(J^\infty(\pi))$.

A tangent vector (18) belongs to the *Cartan subspace* $\mathcal{C}(a)$ of $T_a J^\infty(\pi)$ if and only if v_k belongs to the Cartan distribution on $J^k(\pi)$ for all k. The spaces $\mathcal{C}(a)$ form the *Cartan distribution on* $J^\infty(\pi)$. From Proposition 3 it is easily seen that $\dim \mathcal{C}(a) = n$ and the differential of the natural map $\pi_\infty \colon J^\infty(\pi) \to M$ projects $\mathcal{C}(a)$ isomorphically onto the vector space $T_{\pi_\infty(a)} M$. This allows to define the $C^\infty(M)$-linear embedding

$$\mathbf{C} \colon D(M) \hookrightarrow D(J^\infty(\pi))$$

such that for any $X \in D(M)$ and $a \in J^\infty(\pi)$ one has $(\pi_\infty)_*\big(\mathbf{C}(X)\big) = X$ and the tangent vector at a corresponding to the vector field $\mathbf{C}(X)$ lies in $\mathcal{C}(a)$. The map $\mathbf{C}$ is called the *Cartan connection*.

Using Proposition 1, one obtains that in local coordinates we have $\mathbf{C}\big(\frac{\partial}{\partial x_i}\big) = D_{x_i}$, where the vector field $D_{x_i} \in D(J^\infty(\pi))$ is given by formula (12). Since $[D_{x_i}, D_{x_j}] = 0$, the Cartan connection is flat, that is,

$$\mathbf{C}([X,Y]) = [\mathbf{C}(X), \mathbf{C}(Y)] \quad \forall\, X, Y \in D(M). \tag{19}$$

Denote by $\mathcal{C}(\pi) \subset D(J^\infty(\pi))$ the $C^\infty(J^\infty(\pi))$-submodule of vector fields that belong to Cartan distribution. In other words, the submodule $\mathcal{C}(\pi)$ is generated by vector fields of the form $\mathbf{C}(X)$ for $X \in D(M)$. In local coordinates $\mathcal{C}(\pi)$ is a free $C^\infty(J^\infty(\pi))$-module generated by $D_{x_1}, \dots, D_{x_n}$.

Property (19) implies $[\mathcal{C}(\pi), \mathcal{C}(\pi)] \subset \mathcal{C}(\pi)$. However, one cannot apply the Frobenius theorem to the Cartan distribution, because $\dim J^\infty(\pi) = \infty$. For a section $s \in \Gamma(\pi)$ consider the map

$$j_\infty(s) \colon M \to J^\infty(\pi), \quad x \mapsto [s]^\infty_x = ([s]^0_x, [s]^1_x, \dots, [s]^k_x, \dots) \in J^\infty(\pi).$$

The n-dimensional submanifold $j_\infty(s)(M) \subset J^\infty(\pi)$ is an integral submanifold of the Cartan distribution.

4.2. **The structure of the symmetry algebra.** Similarly to the finite-dimensional case, a vector field $X \in D(J^\infty(\pi))$ is said to be a *symmetry* if $[X, \mathcal{C}(\pi)] \subset \mathcal{C}(\pi)$.

Example 1. According to Proposition 4, for any symmetry X of $J^k(\pi)$ there is a unique symmetry $\tilde{X}$ of $J^\infty(\pi)$ such that

$$\tilde{X}\big|_{C^\infty(J^k(\pi))} = X, \quad \tilde{X}\big|_{C^\infty(J^p(\pi))} = X_p, \quad \forall\, p \geq k.$$

Such symmetries of $J^\infty(\pi)$ are said to be *classical*. Any classical symmetry determines a one-parametric group of local diffeomorphisms of $J^\infty(\pi)$, which preserve the Cartan distribution.

An arbitrary symmetry of $J^\infty(\pi)$ does not always determine a one-parametric group of local diffeomorphisms.

Denote by $\mathrm{Sym}(\pi) \subset D(J^\infty(\pi))$ the Lie algebra of symmetries. Obviously, $\mathcal{C}(\pi)$ is an ideal of $\mathrm{Sym}(\pi)$. Consider the following subalgebra

$$S_v(\pi) = \{X \in \mathrm{Sym}(\pi) \mid X\big|_{C^\infty(M)} = 0\}.$$

For any $X \in \mathrm{Sym}(\pi)$ there is a unique $X_h \in \mathcal{C}(\pi)$ such that $X - X_h \in S_v(\pi)$, that is,

$$X\big|_{C^\infty(M)} = X_h\big|_{C^\infty(M)}. \tag{20}$$

Indeed, in a coordinate neighborhood we set $X_h = \sum_i X(x_i) D_{x_i}$. Since for fixed X there is no more than one $X_h \in \mathcal{C}(\pi)$ satisfying (20), the formulas for X_h in coordinate neighborhoods provide a well-defined vector field on $J^\infty(\pi)$. Since $\mathcal{C}(\pi) \cap S_v(\pi) = 0$ and each $X \in \mathrm{Sym}(\pi)$ can be presented as a sum $X = X_h + (X - X_h)$, we obtain the decomposition

$$\mathrm{Sym}(\pi) = \mathcal{C}(\pi) \oplus S_v(\pi) \quad \text{(a direct sum of vector spaces)}. \tag{21}$$

The structure of $\mathcal{C}(\pi)$ is already known, so it remains to describe $S_v(\pi)$.

Proposition 7. *For any $X \in S_v(\pi)$ one has*

$$[X, \mathbf{C}(Y)] = 0 \quad \forall\, Y \in D(M).$$

In particular, in local coordinates

$$[X, D_{x_i}] = 0, \quad i = 1, \ldots, n. \tag{22}$$

Proof. Since $X\big|_{C^\infty(M)} = 0$, it is sufficient to prove (22). Since X is a symmetry, one has $[X, D_{x_i}] = \sum_{l=1}^n f_l D_{x_l}$ for some $f_l \in C^\infty(J^\infty(\pi))$. Using this and the fact that $X(x_a) = 0$, we obtain

$$f_a = \Big(\sum_{l=1}^n f_l D_{x_l}\Big)(x_a) = [X, D_{x_i}](x_a) = 0 \quad a = 1, \ldots, n.$$

□

Equation (22) implies $X(u^j_\sigma) = D_\sigma(X(u^j))$. Therefore, X is determined by

$$X(u^1), \ldots, X(u^m) \in C^\infty(J^\infty(\pi)).$$

In coordinate-free terms this means the following. Denote by $\varkappa(\pi)$ the vector space of $C^\infty(J^\infty(\pi))$-valued derivations $\varphi \colon C^\infty(E) \to C^\infty(J^\infty(\pi))$ of the algebra $C^\infty(E)$ such that $\varphi\big|_{C^\infty(M)} = 0$. That is, $\varphi \colon C^\infty(E) \to C^\infty(J^\infty(\pi))$ is an $\mathbb{R}$-linear map satisfying

$$\varphi(fg) = f\varphi(g) + g\varphi(f) \quad \forall\, f, g \in C^\infty(E), \quad \varphi(C^\infty(M)) = 0.$$

Proposition 8. *For any $\varphi \in \varkappa(\pi)$ there is a unique symmetry $E_\varphi \in S_v(\pi)$ such that*

$$E_\varphi\big|_{C^\infty(E)} = \varphi.$$

This is an isomorphism between the vector spaces $\varkappa(\pi)$ and $S_v(\pi)$.

Remark 9. For each point $a \in E$ consider the vector space

$$V(a) = \{v \in T_a E \mid \pi_*(v) = 0\}.$$

Consider the tangent bundle $TE \to E$ and the following submanifold of TE

$$V(E) = \bigcup_{a \in E} V(a) \subset TE.$$

Let $v(\pi)$ be the natural vector bundle $v(\pi)\colon V(E) \to E$, which is a subbundle of the tangent bundle of E.

Denote by

$$v^\infty(\pi)\colon V(\pi) \to J^\infty(\pi) \tag{23}$$

the m-dimensional vector bundle equal to the pullback of the bundle $v(\pi)$ by the natural map $\pi_{\infty,0}\colon J^\infty(\pi) \to E$. It is easily seen that the $C^\infty(J^\infty(\pi))$-module $\varkappa(\pi)$ is naturally isomorphic to the $C^\infty(J^\infty(\pi))$-module of sections of the bundle $v^\infty(\pi)$.

Remark 10. If π is a vector bundle then the bundle $v(\pi)$ is isomorphic to the pullback of π by π and the bundle $v^\infty(\pi)$ is isomorphic to the pullback of π by $\pi_\infty\colon J^\infty(\pi) \to M$.

Let us describe the space $S_v(\pi)$ in local coordinates. We have the isomorphism

$$\varkappa(\pi) \to \big(C^\infty(J^\infty)\big)^m, \quad \varphi \mapsto (\varphi(u^1), \dots, \varphi(u^m)).$$

For $\varphi = (\varphi^1, \dots, \varphi^m) \in \varkappa(\pi)$, $\varphi^j \in C^\infty(J^\infty)$, one has

$$E_\varphi = \sum_{j,\,\sigma} D_\sigma(\varphi^j) \frac{\partial}{\partial u^j_\sigma}. \tag{24}$$

Now we can prove Propositions 5, 6.

Proof of Proposition 5.

For any functions (13) consider the following vector field on E

$$X_0 = \sum_{i=1}^n f^i \frac{\partial}{\partial x_i} + \sum_{j=1}^m g^j \frac{\partial}{\partial u^j}.$$

Since the Cartan distribution on $J^0(\pi) = E$ coincides with the whole tangent bundle TE, the vector field X_0 is a symmetry of $J^0(\pi)$. By Example 1, there is a unique symmetry $\tilde X$ of $J^\infty(\pi)$ such that

$$\tilde X\big|_{C^\infty(E)} = X_0 \tag{25}$$

and $X_k = \tilde X\big|_{C^\infty(J^k(\pi))}$ is a symmetry of $J^k(\pi)$.

Remark 11. $\tilde X$ is unique, because, as it follows from the results of this section, any symmetry of $J^\infty(\pi)$ is uniquely determined by its restriction to $C^\infty(E)$.

By (21), we have

$$\tilde X = \sum_{i=1}^n h^i D_{x_i} + E_\varphi, \quad \varphi = (\varphi^1, \dots, \varphi^m), \quad h^i,\, \varphi^j \in C^\infty(J^\infty(\pi)). \tag{26}$$

Taking into account (25), one obtains

$$h^i = \tilde{X}(x_i) = X_0(x_i) = f^i,$$

$$\varphi^j = \tilde{X}(u^j) - \sum_{i=1}^{n} h^i D_{x_i}(u^j) = X_0(u^j) - \sum_{i=1}^{n} h^i u_i^j = g^j - \sum_{i=1}^{n} f^i u_i^j.$$

Combining this with (12) and (24), we get that the symmetry $X_k = \tilde{X}\big|_{C^\infty(J^k(\pi))}$ coincides with (14).

Now let X be an arbitrary symmetry of $J^k(\pi)$. Set $f^i = X(x_i)$, $g^j = X(u^j)$. By Proposition (4), one has f^i, $g^j \in C^\infty(E)$. According to Example 1, there is a classical symmetry $\tilde{X}$ of $J^\infty(\pi)$ such that $\tilde{X}\big|_{C^\infty(J^k(\pi))} = X$. Then

$$\tilde{X}\big|_{C^\infty(E)} = \sum_i f^i \frac{\partial}{\partial x_i} + \sum_j g^j \frac{\partial}{\partial u^j}.$$

As we have shown above, this implies that $X = \tilde{X}\big|_{C^\infty(J^k(\pi))}$ is given by (14).

Finally, if X is given by (14) then one can reconstruct f^i, g^j as $f^i = X(x_i)$, $g^j = X(u^j)$.

Proof of Proposition 6.

For any function (15) consider the symmetry

$$X' = E_f - \sum_{i=1}^{n} \frac{\partial f}{\partial u_i} D_{x_i} \tag{27}$$

of $J^\infty(\pi)$. It is easy to check that $X'\big(C^\infty(J^1(\pi))\big) \subset C^\infty(J^1(\pi))$ and $X_1 = X'\big|_{C^\infty(J^1(\pi))}$ is a symmetry of $J^1(\pi)$.

By Example 1, there is a classical symmetry $\tilde{X}$ of $J^\infty(\pi)$ such that $\tilde{X}\big|_{C^\infty(J^1(\pi))} = X_1$. According to Remark 11, since $\tilde{X}\big|_{C^\infty(J^1(\pi))} = X'\big|_{C^\infty(J^1(\pi))}$, one has $\tilde{X} = X'$. Since $\tilde{X}$ is a classical symmetry, the restriction $X_k = \tilde{X}\big|_{C^\infty(J^k(\pi))} = X'\big|_{C^\infty(J^k(\pi))}$ is a symmetry of $J^k(\pi)$. From (27) it follows that X_k is given by (16).

Now let X be an arbitrary symmetry of $J^k(\pi)$ and $\tilde{X}$ be the classical symmetry of $J^\infty(\pi)$ such that $\tilde{X}\big|_{C^\infty(J^k(\pi))} = X$. By (21),

$$\tilde{X} = E_f + \sum_i h_i D_{x_i}, \quad \text{for some} \quad f,\, h_i \in C^\infty(J^\infty(\pi)).$$

Since $\tilde{X}$ is a classical symmetry, one has $\tilde{X}\big(C^\infty(J^1(\pi))\big) \subset C^\infty(J^1(\pi))$. This implies

$$h_i = \tilde{X}(x_i) \in C^\infty(J^1(\pi)), \qquad f = \tilde{X}(u) - \sum_i h_i u_i \in C^\infty(J^1(\pi)). \tag{28}$$

Then the condition $\tilde{X}(u_i) \in C^\infty(J^1(\pi))$ yields $h_i = -\partial f/\partial x_i$. Therefore, $\tilde{X}$ is given by formula (27), which implies (16) for $X = \tilde{X}\big|_{C^\infty(J^k(\pi))}$.

If X is given by (16) then one can reconstruct f as $f = X(u) - \sum_i X(x_i) u_i$.

Exercise 3. Let X be a symmetry of $J^\infty(\pi)$. Prove that

- if $\dim \pi > 1$ then X is a classical symmetry if and only if $X\big(C^\infty(E)\big) \subset C^\infty(E)$.
- if $\dim \pi = 1$ then X is a classical symmetry if and only if

$$X\big(C^\infty(J^1(\pi))\big) \subset C^\infty(J^1(\pi)).$$

5. Infinite prolongations of PDEs and symmetries

The described approach allows to associate with a system of PDEs a geometric object, a subset of the corresponding jet space.

Example 2. Consider the trivial bundle

$$\pi\colon \mathbb{R}^3 \to \mathbb{R}^2, \quad (x,t,u) \mapsto (x,t). \tag{29}$$

One has the following coordinates in $J^3(\pi)$

$$x,\ t,\ u,\ u_x,\ u_t,\ u_{xx},\ u_{xt},\ u_{tt},\ u_{xxx},\ u_{xxt},\ u_{xtt},\ u_{ttt}.$$

The KdV equation (1) determines a nonsingular algebraic hypersurface in $J^3(\pi)$.

Consider the *sine-Gordon equation* $u_{xt} = \sin u$. It determines a nonsingular analytic hypersurface in the space $J^2(\pi)$ with the coordinates

$$x,\ t,\ u,\ u_x,\ u_t,\ u_{xx},\ u_{xt},\ u_{tt}.$$

Return to the general case of an arbitrary fiber bundle π. Let $\mathcal{E}_k \subset J^k(\pi)$ be an arbitrary closed subset. Consider the ideal

$$I(\mathcal{E}_k) = \{ f \in C^\infty(J^k(\pi)) \mid f\big|_{\mathcal{E}_k} = 0 \}$$

of the algebra $C^\infty(J^k(\pi))$. Denote by $\mathcal{I}$ the ideal of $C^\infty(J^\infty(\pi))$ generated by all functions of the form

$$f, \quad V_1(V_2(\dots V_p(f)\dots))$$

where $f \in I(\mathcal{E}_k),\ V_i \in \mathcal{C}(\pi)$. The *infinite prolongation of* $\mathcal{E}_k$ is the subset

$$\mathcal{E} = \{ a \in J^\infty(\pi) \mid g(a) = 0 \ \ \forall\, g \in \mathcal{I} \} \subset J^\infty(\pi).$$

Example 3. In local coordinates, let $\mathcal{E}_k \subset J^k(\pi)$ be given by equations

$$F_s(x_i, u^l, u^j_\sigma, \dots) = 0, \quad s = 1, \dots, m'.$$

(We assume that $I(\mathcal{E}_k)$ is generated by $F_s \in C^\infty(J^k(\pi)),\ s = 1, \dots, m'$). Then $\mathcal{E} \subset J^\infty(\pi)$ is determined by the following infinite system of equations

$$F_s = 0, \quad D_\sigma(F_s) = 0, \quad s = 1, \dots, m', \quad |\sigma| \geq 0. \tag{30}$$

In other words, the infinite prolongation is given by the initial PDEs and all their differential consequences.

Clearly, points of $\mathcal{E}$ are in one-to-one correspondence with infinite formal Taylor series satisfying the initial system of PDEs.

We define the algebra $C^\infty(\mathcal{E})$ as follows. A function $f\colon \mathcal{E} \to \mathbb{R}$ belongs to $C^\infty(\mathcal{E})$ if and only if there is $\tilde{f} \in C^\infty(J^\infty(\pi))$ such that $f = \tilde{f}\big|_{\mathcal{E}}$. Consider the ideal

$$I(\mathcal{E}) = \{ f \in C^\infty(J^\infty(\pi)) \mid f\big|_{\mathcal{E}} = 0 \}$$

of the algebra $C^\infty(J^\infty(\pi))$. Then $C^\infty(\mathcal{E}) \cong C^\infty(J^\infty(\pi))/I(\mathcal{E})$. Clearly, $\mathcal{I} \subset I(\mathcal{E})$. We assume that

$$\mathcal{I} = I(\mathcal{E}). \tag{31}$$

The space of vector fields $D(\mathcal{E})$ by definition consists of derivations of the algebra $C^\infty(\mathcal{E})$. Let $X \in D(J^\infty(\pi))$ be such that $X(I(\mathcal{E})) \subset I(\mathcal{E})$, then X determines a derivation of $C^\infty(\mathcal{E})$ denoted by $X\big|_{\mathcal{E}} \in D(\mathcal{E})$. In this case X is said to be *tangent* to $\mathcal{E}$, and $X\big|_{\mathcal{E}}$ is called the *restriction* of X to $\mathcal{E}$.

Due to (31), for any $V \in \mathcal{C}(\pi) \subset D(J^\infty(\pi))$ one has $V(I(\mathcal{E})) \subset I(\mathcal{E})$. Therefore, the vector field $V\big|_{\mathcal{E}} \in D(\mathcal{E})$ is well defined. In other words, the Cartan distribution of $J^\infty(\pi)$ is tangent to $\mathcal{E}$. Denote by $\mathcal{C}(\mathcal{E}) \subset D(\mathcal{E})$ the $C^\infty(\mathcal{E})$-submodule of vector fields of the form $V\big|_{\mathcal{E}}$, where $V \in \mathcal{C}(\pi)$. One has $[\mathcal{C}(\mathcal{E}), \mathcal{C}(\mathcal{E})] \subset \mathcal{C}(\mathcal{E})$.

Remark 12. In local coordinates for each integer $p \geq k$ consider the subset

$$\mathcal{E}_p = \{a \in J^p(\pi) \mid f(a) = 0,\ \ D_\sigma(f)(a) = 0 \ \ \forall f \in I(\mathcal{E}_k),\ \ \forall \sigma,\ |\sigma| \leq p - k\}.$$

It is called the $(p-k)$*-th order prolongation of* $\mathcal{E}_k$ and can be defined also in a coordinate-free way [2, 4]. (We do not present the coordinate-free definition of $\mathcal{E}_p$, because we will not need this set).

The initial set $\mathcal{E}_k$ is said to be *formally integrable* if the following conditions hold

- $\mathcal{E}_p$ is a submanifold of $J^p(\pi)$ for all $p \geq k$,
- the map $\pi_{p_1,p_2}\big|_{\mathcal{E}_{p_1}} : \mathcal{E}_{p_1} \to \mathcal{E}_{p_2}$ is a bundle for all $p_1 \geq p_2 \geq k$.

It seems that in the literature one always requires the initial set $\mathcal{E}_k$ to be formally integrable. We do not make this assumption, for us $\mathcal{E}_k$ and $\mathcal{E}$ are closed subsets (not necessarily submanifolds) satisfying (31). It is not hard to show that condition (31) holds for any formally integrable $\mathcal{E}_k$. According to the next example, our class of sets $\mathcal{E}_k$ satisfying (31) is wider than the class of formally integrable sets. (In fact our class is much wider).

Example 4. Let $k = 2$, consider bundle (29) and the set $\mathcal{E}_2 \subset J^2(\pi)$ determined by the equation $u_t - uu_{xx} = 0$. Then $\mathcal{E}_3 \subset J^3(\pi)$ is given by the equations

$$u_t - uu_{xx} = 0, \quad D_x(u_t - uu_{xx}) = u_{xt} - u_x u_{xx} - uu_{xxx} = 0,$$
$$D_t(u_t - uu_{xx}) = u_{tt} - u_t u_{xx} - uu_{txx} = 0.$$

It is easily seen that any point of the form

$$(u = u_t = 0,\ u_x = u_{xx} = 1,\ u_{xt} = 2,\ x, t, u_{tt} \in \mathbb{R}) \in \mathcal{E}_2$$

does not belong to the image of the projection $\pi_{3,2} \colon \mathcal{E}_3 \to \mathcal{E}_2$. Therefore, $\mathcal{E}_2$ is not formally integrable.

The system

$$u^1_t = u^2_{xx}, \quad u^2_t = u^1_x \qquad \text{for} \quad u^1 = u^1(x,t), \quad u^2 = u^2(x,t),$$

is not formally integrable either. As we will show in Example 11, these examples satisfy condition (31), and, therefore, the presented approach is applicable to them.

A remark for specialists. The two described examples represent the two typical situations when a system of PDEs $\mathcal{E}_k$ is not formally integrable.

(1) If the symbol of $\mathcal{E}_k$ is not of constant rank.
(2) If the system consists of equations of different orders.

However, usually one can find a system of internal coordinates in the sense of Section 7.2, then, by Theorem 2, condition (31) holds and the presented theory of symmetries is applicable. (According to Remark 18, for any system of PDEs a system of internal coordinates exists on a neighborhood of any point from some open dense subset of the infinite prolongation).

Another situation that requires the presented approach is when $\mathcal{E}_k$ and $\mathcal{E}$ are algebraic varieties with singularities.

A vector field $X \in D(\mathcal{E})$ is said to be a *symmetry of* $\mathcal{E}$ if $[X, \mathcal{C}(\mathcal{E})] \subset \mathcal{C}(\mathcal{E})$.

Consider the projection $\pi_\infty\big|_{\mathcal{E}}\colon \mathcal{E} \to M$ and the pullback homomorphism

$$\big(\pi_\infty\big|_{\mathcal{E}}\big)^*\colon C^\infty(M) \to C^\infty(\mathcal{E}).$$

Set

$$\mathcal{A} = \big(\pi_\infty\big|_{\mathcal{E}}\big)^*\big(C^\infty(M)\big) \subset C^\infty(\mathcal{E}).$$

Similarly to (21), the space $\mathcal{C}(\mathcal{E})$ is an ideal of the Lie algebra $\mathrm{Sym}(\mathcal{E})$ of symmetries and

$$\mathrm{Sym}(\mathcal{E}) = \mathcal{C}(\mathcal{E}) \oplus S_v(\mathcal{E}) \quad \text{(a direct sum of vector spaces)}, \tag{32}$$

where

$$S_v(\mathcal{E}) = \{X \in \mathrm{Sym}(\mathcal{E}) \mid X\big|_{\mathcal{A}} = 0\}.$$

Theorem 1. *For any $X \in \mathrm{Sym}(\mathcal{E})$ there is $\tilde{X} \in \mathrm{Sym}(\pi)$ tangent to $\mathcal{E}$ such that $\tilde{X}\big|_{\mathcal{E}} = X$.*

If $X \in S_v(\mathcal{E})$ then there is $\varphi \in \varkappa(\pi)$ such that $E_\varphi \in S_v(\pi)$ is tangent to $\mathcal{E}$ and $E_\varphi\big|_{\mathcal{E}} = X$.

Proof. By (32), $X = X_h + X_v$ for some $X_h \in \mathcal{C}(\mathcal{E})$ and $X_v \in S_v(\mathcal{E})$. By the definition of $\mathcal{C}(\mathcal{E})$, there is $\tilde{X}_h \in \mathcal{C}(\pi)$ such that $\tilde{X}_h\big|_{\mathcal{E}} = X_h$. Therefore, it remains to study the case when $X \in S_v(\mathcal{E})$.

Consider the map $\pi_{\infty,0}\big|_{\mathcal{E}}\colon \mathcal{E} \to E$ and the homomorphism $\big(\pi_{\infty,0}\big|_{\mathcal{E}}\big)^*\colon C^\infty(E) \to C^\infty(\mathcal{E})$. Set

$$\mathcal{A}' = \big(\pi_{\infty,0}\big|_{\mathcal{E}}\big)^*\big(C^\infty(E)\big) \subset C^\infty(\mathcal{E}).$$

Similarly to $\varkappa(\pi)$, denote by $\varkappa(\mathcal{E})$ the vector space of $C^\infty(\mathcal{E})$-valued derivations

$$\psi\colon \mathcal{A}' \to C^\infty(\mathcal{E})$$

of the algebra $\mathcal{A}'$ such that $\psi\big|_{\mathcal{A}} = 0$. An element $\varphi \in \varkappa(\pi)$ is said to be *tangent* to $\mathcal{E}$ if

$$\varphi\Big(\ker\big(\pi_{\infty,0}\big|_{\mathcal{E}}\big)^*\Big) \subset I(\mathcal{E})$$

Then φ determines a derivation $\mathcal{A}' \to C^\infty(\mathcal{E})$ denoted by $\varphi\big|_{\mathcal{E}} \in \varkappa(\mathcal{E})$.

Lemma 1. *For any $\psi \in \varkappa(\mathcal{E})$ there is $\varphi \in \varkappa(\pi)$ tangent to $\mathcal{E}$ such that $\varphi\big|_{\mathcal{E}} = \psi$.*

Proof. Suppose first that π is a trivial bundle with fiber diffeomorphic to an open subset of $\mathbb{R}^m$. Let $u^1, \dots, u^m$ be coordinates in fibers of π. By the definition of $C^\infty(\mathcal{E})$, there exist functions $\varphi^1, \dots, \varphi^m \in C^\infty(J^\infty(\pi))$ such that

$$\varphi^j\big|_{\mathcal{E}} = \psi(u^j\big|_{\mathcal{E}}), \quad j = 1, \dots, m.$$

Then the required derivation $\varphi\colon C^\infty(E) \to C^\infty(J^\infty(\pi))$ is defined by $\varphi(u^j) = \varphi^i$ and $\varphi(x_i) = 0$.

Now let π be an arbitrary bundle. By the above argument, the statement holds on each adapted coordinate chart of E. Then the proof is completed using a suitable partition of unity on the manifold E. □

Return to the proof of the theorem. For $X \in S_v(\mathcal{E})$ set $\psi = X\big|_{\mathcal{A}'} \in \varkappa(\mathcal{E})$. By the above lemma, there is $\varphi \in \varkappa(\pi)$ tangent to $\mathcal{E}$ such that $\varphi\big|_{\mathcal{E}} = \psi$. Let us show that $E_\varphi \in S_v(\pi)$ is tangent to $\mathcal{E}$ and $E_\varphi\big|_{\mathcal{E}} = X$. It is sufficient to check this in local coordinates x_i, u^j_σ. Similarly to Proposition 7 one has $[X, D_{x_i}\big|_{\mathcal{E}}] = 0$, and, therefore,

$$X\big(u^j_\sigma\big|_{\mathcal{E}}\big) = D_\sigma\big|_{\mathcal{E}}\Big(X\big(u^j\big|_{\mathcal{E}}\big)\Big) = D_\sigma\big|_{\mathcal{E}}\big(\varphi(u^j)\big|_{\mathcal{E}}\big) = E_\varphi(u^j_\sigma)\big|_{\mathcal{E}}.$$

Recall also that $X(x_i\big|_{\mathcal{E}}) = 0$ and $E_\varphi(x_i) = 0$.

Now the statement follows from the general fact that if $Y \in D(\mathcal{E})$ and $Y' \in D(J^\infty(\pi))$ satisfy

$$Y\big(u^j_\sigma\big|_{\mathcal{E}}\big) = Y'(u^j_\sigma)\big|_{\mathcal{E}}, \quad Y\big(x_i\big|_{\mathcal{E}}\big) = Y'(x_i)\big|_{\mathcal{E}}$$

then Y' is tangent to $\mathcal{E}$ and $Y'\big|_{\mathcal{E}} = Y$. □

Remark 13. In [2] this result is proved under the additional assumption that the map $\pi_{\infty,0}\big|_{\mathcal{E}} \colon \mathcal{E} \to E$ is surjective.

Remark 14. The partition of unity technique cannot be applied to symmetries directly, because for $X \in \mathrm{Sym}(\pi)$ and $f \in C^\infty(J^\infty(\pi))$ the vector field fX is not a symmetry in general. But this technique can be used for elements of $\varkappa(\pi)$, because the space $\varkappa(\pi)$ is a module over $C^\infty(J^\infty(\pi))$.

The structure of $\mathcal{C}(\mathcal{E})$ is clear, in local coordinates it is a free $C^\infty(\mathcal{E})$-module spanned by $D_{x_1}\big|_{\mathcal{E}}, \dots, D_{x_n}\big|_{\mathcal{E}}$. In order to study the algebra $S_v(\mathcal{E})$ we need to introduce nonlinear differential operators and their linearizations in the next section.

6. Nonlinear differential operators and the category of PDEs

6.1. **Nonlinear differential operators.** Let

$$\pi \colon E \to M, \quad \pi' \colon E' \to M$$

be fiber bundles over the same base M. A *nonlinear differential operator of order k* is a map $\Delta \colon \Gamma(\pi) \to \Gamma(\pi')$ of the form

$$\Delta \colon s \mapsto \delta \circ j_k(s), \tag{33}$$

where $\delta \colon J^k(\pi) \to E'$ is a smooth map satisfying $\pi' \circ \delta = \pi_k$. It is easily seen that for each $l = 0, 1, 2, \dots$ the smooth map

$$\delta_l \colon J^{k+l}(\pi) \to J^l(\pi'), \quad \delta_l([s]^{k+l}_x) = [\Delta(s)]^l_x, \quad s \in \Gamma(\pi),\ x \in M,$$

is well defined and $\delta_0 = \delta$. We have the commutative diagram

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & J^{k+l+1}(\pi) & \longrightarrow & J^{k+l}(\pi) & \longrightarrow & \dots & \longrightarrow & J^k(\pi) \\
 & & \Big\downarrow{\scriptstyle \delta_{l+1}} & & \Big\downarrow{\scriptstyle \delta_l} & & & & \Big\downarrow{\scriptstyle \delta} \\
\dots & \longrightarrow & J^{l+1}(\pi') & \longrightarrow & J^l(\pi') & \longrightarrow & \dots & \longrightarrow & E'
\end{array},$$

whose inverse limit determines a map

$$\Delta_\infty \colon J^\infty(\pi) \to J^\infty(\pi'). \tag{34}$$

In other words,

$$\Delta_\infty(a) = [\Delta(s)]^\infty_x, \quad a \in J^\infty(\pi),$$

where $x = \pi_\infty(a) \in M$ and $s \in \Gamma(\pi)$ is such that $a = [s]^\infty_x$. It is easy to check that the map Δ_∞ preserves the Cartan distribution.

6.2. **The category of PDEs.** One can define the *category of PDEs* as follows. An *object* of the category is a triple $(W, C^\infty(W), \mathcal{D})$, where

- W is a topological space,
- $C^\infty(W)$ is a subalgebra of the algebra of continuous functions on W (elements of $C^\infty(W)$ are called *smooth functions on* W),
- $\mathcal{D}$ is a $C^\infty(W)$-submodule of the module of derivations of the algebra $C^\infty(W)$ such that $[\mathcal{D}, \mathcal{D}] \subset \mathcal{D}$.

Using $C^\infty(W)$, one can define the tangent space T_aW for a point $a \in W$ in the standard way

$$T_aW = \left\{ v\colon C^\infty(W) \to \mathbb{R} \;\middle|\; \begin{array}{l} v \text{ is } \mathbb{R}\text{-linear,} \\ v(fg) = f(a)v(g) + g(a)v(f), \\ \forall f, g \in C^\infty(W) \end{array} \right\} \cong \mathrm{Hom}_{\mathbb{R}}(I_a/I_a^2, \mathbb{R}),$$

where $I_a = \{f \in C^\infty(W) \mid f(a) = 0\}$. We have the natural map

$$\mathrm{ev}_a\colon \mathcal{D} \to T_aW, \quad \mathrm{ev}_a(V)(f) = V(f)(a), \quad V \in \mathcal{D},\ \ f \in C^\infty(W).$$

Set $\mathcal{D}_a = \mathrm{ev}_a(\mathcal{D}) \subset T_aW$.

A *morphism* connecting two objects $(W_1, C^\infty(W_1), \mathcal{D}_1)$ and $(W_2, C^\infty(W_2), \mathcal{D}_2)$ is a continuous map $\tau\colon W_1 \to W_2$ such that $\tau^*\big(C^\infty(W_2)\big) \subset C^\infty(W_1)$ and for each $a \in W_1$ one has

$$\tau_*\big((\mathcal{D}_1)_a\big) \subset (\mathcal{D}_2)_{\tau(a)}.$$

Here $\tau_*\colon T_aW_1 \to T_{\tau(a)}W_2$ is the differential of τ defined in the standard way.

Example 5. The triples $(J^\infty(\pi), C^\infty(J^\infty(\pi)), \mathcal{C}(\pi))$ and $(J^\infty(\pi'), C^\infty(J^\infty(\pi')), \mathcal{C}(\pi'))$ are objects of this category, and Δ_∞ is a morphism. For $\mathcal{D} = \mathcal{C}(\pi)$ the subspace $\mathcal{D}_a \subset T_aJ^\infty(\pi)$ coincides with the Cartan subspace $\mathcal{C}(a)$.

Example 6. A triple $(\mathcal{E}, C^\infty(\mathcal{E}), \mathcal{C}(\mathcal{E}))$, where $\mathcal{E}$ is the infinite prolongation of a system of PDEs and $\mathcal{C}(\mathcal{E})$ corresponds to the Cartan distribution, is the main example of an object of the category of PDEs. So-called *Bäcklund transformations* in soliton theory are examples of morphisms of infinite prolongations of PDEs [2].

Exercise 4. For $a \in \mathcal{E}$ consider the vector space $\mathrm{ev}_a(\mathcal{C}(\mathcal{E})) \subset T_a\mathcal{E}$. Prove that

$$\dim \mathrm{ev}_a(\mathcal{C}(\mathcal{E})) = n = \dim M.$$

Remark 15. We have given the most general definition of the category of PDEs. Usually one imposes the following additional conditions on $(W, C^\infty(W), \mathcal{D})$.

- W is supposed to be a (possibly infinite-dimensional) manifold. An *infinite-dimensional manifold* here is a topological space N such that for any point $a \in N$ there is a neighborhood U_a endowed with a homeomorphism onto to the inverse (projective) limit of

$$\ldots \to M_a^{l+1} \xrightarrow{f_l} M_a^l \to \ldots \xrightarrow{f_2} M_a^2 \xrightarrow{f_1} M_a^1,$$

where M_a^l are finite-dimensional manifolds and f_l are bundles. We can identify $C^\infty(M_a^l)$ with the subalgebra $f_l^*(C^\infty(M_a^l))$ of $C^\infty(M_a^{l+1})$ and set

$$C^\infty(U_a) = \bigcup_{l \geq 1} C^\infty(M_a^l).$$

Then $C^\infty(N)$ is defined as follows. A continuous function $f\colon N \to \mathbb{R}$ belongs to $C^\infty(N)$ if and only if $f\big|_{U_a} \in C^\infty(U_a)$ for all $a \in N$. Rigorous theory of such infinite-dimensional manifolds is developed, for example, in [1].

- It is supposed that the spaces $\mathcal{D}_a \subset T_aW$ form a smooth distribution on W.

But for the general theory we do not need these conditions.

Example 7. The space $J^\infty(\pi)$ is an infinite-dimensional manifold in the above sense.

6.3. **Linearizations of nonlinear differential operators.** For a finite-dimensional manifold M' consider the triple $(M', C^\infty(M'), 0)$. This is an embedding of the category of finite-dimensional manifolds into the category of PDEs. Let f be a morphism in the category of finite-dimensional manifolds, that is, a smooth map $f\colon M_1 \to M_2$. Recall that the *differential* f_* of f is a morphism of vector bundles

$$\begin{array}{ccc} TM_1 & \xrightarrow{\;f_*\;} & f^*TM_2 \\ & {\scriptstyle \tau_1}\searrow \quad \swarrow{\scriptstyle f^*\tau_2} & \\ & M_1 & \end{array}$$

where $\tau_i\colon TM_i \to M_i$, $i = 1, 2$, are the tangent bundles and

$$f^*\tau_2\colon f^*TM_2 \to M_1$$

is the pullback of the bundle τ_2 by the map f.

We want to introduce a similar notion of 'differential' for the morphism

$$\Delta_\infty\colon J^\infty(\pi) \to J^\infty(\pi') \tag{35}$$

in the category of PDEs. This 'differential' is called the *linearization of* Δ, is denoted ℓ_Δ, and is a morphism of vector bundles

$$\begin{array}{ccc} V(\pi) & \xrightarrow{\;\ell_\Delta\;} & (\Delta_\infty)^*V(\pi') \\ & {\scriptstyle v^\infty(\pi)}\searrow \quad \swarrow{\scriptstyle (\Delta_\infty)^*v^\infty(\pi')} & \\ & J^\infty(\pi) & \end{array}$$

where

$$v^\infty(\pi)\colon V(\pi) \to J^\infty(\pi), \quad v^\infty(\pi')\colon V(\pi') \to J^\infty(\pi')$$

are the vector bundles introduced in Remark 9 and

$$(\Delta_\infty)^*v^\infty(\pi')\colon (\Delta_\infty)^*V(\pi') \to J^\infty(\pi)$$

is the pullback of $v^\infty(\pi')$ by (35).

The morphism ℓ_Δ is defined as follows. Recall that $\Gamma(v^\infty(\pi)) \cong \varkappa(\pi)$ consists of $C^\infty(J^\infty(\pi))$-valued derivations $\varphi\colon C^\infty(E) \to C^\infty(J^\infty(\pi))$ of the algebra $C^\infty(E)$ such that $\varphi\big|_{C^\infty(M)} = 0$.

For a section $\varphi \in \Gamma(v^\infty(\pi))$ we need to define a section $\ell_\Delta(\varphi) \in \Gamma\big((\Delta_\infty)^*v^\infty(\pi')\big)$. Note that $\Gamma\big((\Delta_\infty)^*v^\infty(\pi')\big)$ can be identified with the space of $\mathbb{R}$-linear maps

$$\gamma\colon C^\infty(E') \to C^\infty(J^\infty(\pi))$$

satisfying

$$\gamma(f_1f_2) = \gamma(f_1)\delta^*(f_2) + \delta^*(f_1)\gamma(f_2), \quad \forall\, f_1, f_2 \in C^\infty(E'), \quad \gamma\big|_{C^\infty(M)} = 0,$$

where $\delta\colon J^k(\pi) \to E'$ was introduced in (33) and $\delta^*\colon C^\infty(E') \to C^\infty(J^k(\pi))$ is the pullback homomorphism. Then for $f \in C^\infty(E')$ we set

$$\ell_\Delta(\varphi)(f) = E_\varphi(\delta^*(f)), \tag{36}$$

where E_φ was defined in Proposition 8.

Remark 16. Thus for the smooth map $\Delta_\infty\colon J^\infty(\pi) \to J^\infty(\pi')$ we have the usual differential $(\Delta_\infty)_*$ (which does not take into account the Cartan distribution) and the linearization ℓ_Δ, which is closely related to the Cartan distribution.

Let us describe ℓ_Δ in local coordinates. Let $m = \dim \pi$ and $m' = \dim \pi'$. Since ℓ_Δ is a morphism from some m-dimensional vector bundle to some m'-dimensional vector bundle over $J^\infty(\pi)$, locally ℓ_Δ is a $(m' \times m)$-matrix whose entries

$$[\ell_\Delta]_{ij}\colon C^\infty(J^\infty(\pi)) \to C^\infty(J^\infty(\pi)), \quad i = 1, \dots, m', \quad j = 1, \dots, m,$$

are linear differential operators. Let $v^1, \dots, v^{m'}$ be local coordinates in fibers of π'. Set $F_i = \delta^*(v^i) \in C^\infty(J^\infty(\pi))$. Then

$$[\ell_\Delta]_{ij} = \sum_\sigma \frac{\partial F_i}{\partial u^j_\sigma} D_\sigma. \tag{37}$$

The sum on the right-hand side is finite, because F_i depends only on a finite number of the coordinates u^j_σ.

7. Computation of symmetries of PDEs

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7.1. The defining equations for symmetries. Let us return to (32) and describe the space $S_v(\mathcal{E})$, where $\mathcal{E} \subset J^\infty(\pi)$ is the infinite prolongation of $\mathcal{E}_k \subset J^k(\pi)$. Suppose that there is a nonlinear differential operator (33) and a section $s_0\colon M \to E'$ of π' such that

$$\mathcal{E}_k = \delta^{-1}\big(s_0(M)\big). \tag{38}$$

Set $I(s_0) = \{f \in C^\infty(E') \mid f\big|_{s_0(M)} = 0\}$. Consider the pullback homomorphism

$$\delta^*\colon C^\infty(E') \to C^\infty(J^k(\pi)).$$

Due to (38) one has $\delta^*\big(I(s_0)\big) \subset I(\mathcal{E}_k)$. We assume that

(39) the ideal of $C^\infty(J^k(\pi))$ generated by $\delta^*\big(I(s_0)\big)$ coincides with $I(\mathcal{E}_k)$.

According to the following example, at least locally this can be achieved practically always.

Example 8. In local coordinates let $\mathcal{E}_k \subset J^k(\pi)$ be determined by a system of PDEs

$$F_s(x_i, u^l, u^j_\sigma, \dots) = 0, \quad s = 1, \dots, m'. \tag{40}$$

In other words, $\mathcal{E}_k = \{a \in J^k(\pi) \mid F_s(a) = 0,\ s = 1, \dots, m'\}$, where $F_s \in C^\infty(J^k(\pi))$. Consider the trivial bundle

$$\pi'\colon M \times \mathbb{R}^{m'} \to M, \quad (x_1, \dots, x_n, v^1, \dots, v^{m'}) \mapsto (x_1, \dots, x_n),$$

and the map

$$\delta\colon J^k(\pi) \to M \times \mathbb{R}^{m'}, \quad \delta^*(x_i) = x_i, \quad \delta^*(v^s) = F_s. \tag{41}$$

Then we have (38) for $s_0 = 0$. Condition (39) holds if and only if the functions F_s, $s = 1, \dots, m'$, generate the ideal $I(\mathcal{E}_k) \subset C^\infty(J^k(\pi))$.

If the ideal generated by the functions F_s does not coincide with $I(\mathcal{E}_k)$ then one should replace F_s by any finite collection of generators of the ideal $I(\mathcal{E}_k)$.

(A finite collection of generators exists unless the subset $\mathcal{E}_k \subset J^k(\pi)$ is extremely complicated. For example, it exists if $\mathcal{E}_k$ is a submanifold or an analytic subset).

Example 9. Let $n = k = 2$, $m = m' = 1$ and consider the PDE

$$F_1 = (u_t - u_{xx})^3 = 0.$$

In this case F_1 does not generated the ideal $I(\mathcal{E}_2)$, this ideal is generated by the function $u_t - u_{xx}$, so one should consider instead the equivalent PDE $u_t - u_{xx} = 0$.

It is easily seen that conditions (38) and (39) imply

$$\mathcal{E} = \big(\Delta_\infty\big)^{-1}\big(j_\infty(s_0)(M)\big), \tag{42}$$

where

$$j_\infty(s_0)\colon M \to J^\infty(\pi), \qquad x \mapsto [s_0]_x^\infty.$$

Recall that, by Theorem 1, the Lie algebra $S_v(\mathcal{E}) \subset D(\mathcal{E})$ consists of vector fields of the form $E_\varphi\big|_{\mathcal{E}}$, where $\varphi \in \varkappa(\pi)$ is such that $E_\varphi \in D(J^\infty(\pi))$ is tangent to $\mathcal{E} \subset J^\infty(\pi)$.

Proposition 9. *Suppose that* (38) *and* (39) *hold. For* $\varphi \in \varkappa(\pi)$ *the vector field* E_φ *is tangent to* $\mathcal{E}$ *if and only if*

$$\ell_\Delta(\varphi)\big|_{\mathcal{E}} = 0. \tag{43}$$

Proof. Recall that, according to Section 6.3, the element $\ell_\Delta(\varphi)$ is a section of the bundle $(\Delta_\infty)^* v^\infty(\pi')$, and at the same time $\ell_\Delta(\varphi)$ can be identified with a map

$$\ell_\Delta(\varphi)\colon C^\infty(E') \to C^\infty(J^\infty(\pi))$$

given by (36). In the latter interpretation of $\ell_\Delta(\varphi)$, condition (43) is means that

$$\ell_\Delta(\varphi)(f) = E_\varphi(\delta^*(f)) \in I(\mathcal{E}) \qquad \forall f \in C^\infty(E'). \tag{44}$$

Suppose that φ satisfies (44), then we must prove that E_φ is tangent to $\mathcal{E}$, that is,

$$E_\varphi\big(I(\mathcal{E})\big) \subset I(\mathcal{E}). \tag{45}$$

Combining (44) and (39), one obtains

$$E_\varphi\big(I(\mathcal{E}_k)\big) \subset I(\mathcal{E}). \tag{46}$$

Taking into account Proposition (7) and the definition of the ideal $\mathcal{I} \subset J^\infty(\pi)$ in Section 5, from (46) it follows that

$$E_\varphi\big(\mathcal{I}\big) \subset I(\mathcal{E}). \tag{47}$$

Finally, combining (47) with (31), we get (45).

Now suppose that E_φ is tangent to $\mathcal{E}$. Then it is sufficient to check that

$$\ell_\Delta(\varphi)(f)\Big|_{\mathcal{E}} = 0 \qquad \forall f \in C^\infty(E').$$

in local coordinates. Coordinates in fibers of π' can be chosen such that $s_0 = 0$. Therefore, we can assume that $\mathcal{E}_k$ is given by (40) and δ is given by (41). Since $E_\varphi(F_s)\big|_{\mathcal{E}} = 0$ and $E_\varphi(x_i) = 0$, for any $f = f(x_1, \ldots, x_n, v^1, \ldots, v^{m'}) \in C^\infty(E')$ we have

$$\ell_\Delta(\varphi)(f)\Big|_{\mathcal{E}} = E_\varphi(\delta^*(f))\Big|_{\mathcal{E}} = E_\varphi\big(f(x_1, \ldots, x_n, F_1, \ldots, F_{m'})\big)\Big|_{\mathcal{E}} = \sum_{s=1}^{m'} \frac{\partial f}{\partial v^s}(x_1, \ldots, x_n, F_1, \ldots, F_{m'}) \cdot E_\varphi(F_s)\Big|_{\mathcal{E}} = 0.$$

□

Let us summarize the obtained structure of the algebra of symmetries $\mathrm{Sym}(\mathcal{E})$ is local coordinates. Suppose that $\mathcal{E}$ is the infinite prolongation of system (40). We have (32), where the ideal $\mathcal{C}(\mathcal{E})$ is spanned by $D_{x_1}\big|_{\mathcal{E}}, \ldots, D_{x_n}\big|_{\mathcal{E}}$ and the subalgebra $S_v(\mathcal{E})$ consists of vector fields of the form $E_\varphi\big|_{\mathcal{E}}$, where E_φ is given by (24) and $\varphi^1, \ldots, \varphi^m \in C^\infty(J^\infty(\pi))$ satisfy

$$\sum_{j,\,\sigma} \frac{\partial F_i}{\partial u^j_\sigma} D_\sigma(\varphi^j)\Bigg|_{\mathcal{E}} = 0, \quad i = 1, \ldots, m'. \tag{48}$$

Remark 17. Equations (48) appear from (43) and (37).

7.2. **Internal coordinates for infinite prolongations.** It is convenient to study equation (43) in terms of *internal coordinates* of the subset $\mathcal{E} \subset J^\infty(\pi)$. First recall the similar notion for submanifolds of $\mathbb{R}^n$.

Exercise 5. Consider the space $\mathbb{R}^3$ with coordinates (x, y, z). Let $f(x, y)$ be a smooth function and

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid z = f(x, y)\} \subset \mathbb{R}^3.$$

Prove that

- the ideal
$$I(S) = \{g \in C^\infty(\mathbb{R}^3) \mid g\big|_S = 0\}$$
of the algebra $C^\infty(\mathbb{R}^3)$ coincides with the ideal generated by the function
$$z - f(x, y),$$
- the algebra $C^\infty(S) = C^\infty(\mathbb{R}^3)/I(S)$ is isomorphic to the algebra of smooth functions in x, y. In other words, for any $h(x, y, z) \in C^\infty(\mathbb{R}^3)$ there is a unique smooth function $h_1(x, y)$ such that $h - h_1 \in I(S)$. Namely, one can set $h_1 = h(x, y, f(x, y))$. In particular, x, y form a system of coordinates for S.

Hint. Use the following property of smooth functions. Consider the space $\mathbb{R}^n$ with coordinates $x_1, \ldots, x_n$ and a fixed point $(a_1, \ldots, a_n) \in \mathbb{R}^n$. For any function $f \in C^\infty(\mathbb{R}^n)$ there are functions $g_1, \ldots, g_n \in C^\infty(\mathbb{R}^n)$ such that

$$f(x_1, \ldots, x_n) = f(a_1, \ldots, a_n) + \sum_{i=1}^{n} (x_i - a_i) g_i(x_1, \ldots, x_n) \qquad \forall\, x_1, \ldots, x_n.$$

Before giving the general definition of internal coordinates for $\mathcal{E}$, let us consider an example.

Example 10. Return to the KdV equation (1). It is easily seen that, using its differential consequences

$$D_\sigma(u_t - u_{xxx} - u_x u) = 0, \qquad |\sigma| \geq 0, \tag{49}$$

all derivatives of the form

$$\frac{\partial^{p+q} u}{\partial t^p \partial x^q}, \quad p > 0, \quad q \geq 0,$$

can be expressed in terms of

$$u_i = \frac{\partial^i u}{\partial x^i}, \quad i = 0, 1, 2, \dots$$

That is, one can obtain from (49) equations of the form

$$\frac{\partial^{p+q} u}{\partial t^p \partial x^q} = F_{p,q}(u, u_1, u_2, u_3, \dots) \quad \forall\, p > 0,\ q \geq 0. \tag{50}$$

For example, applying D_x to (1), one obtains

$$u_t = u_3 + u_1 u, \quad u_{tx} = u_4 + u_2 u + u_1^2, \quad u_{txx} = u_5 + u_3 u + 3u_2 u_1, \quad u_{txxx} = u_6 + u_4 u + 4u_3 u_1 + 3u_2^2, \tag{51}$$

applying D_t to (1) and using (51), we get

$$u_{tt} = u_{txxx} + u_{tx} u + u_x u_t = u_6 + u_4 u + 4u_3 u_1 + 3u_2^2 + (u_4 + u_2 u + u_1^2)u + u_1(u_3 + u_1 u),$$

and so on.

Moreover, it is easy to check that in the algebra $C^\infty(J^\infty(\pi))$ the ideal $\mathcal{I}$ generated by the functions

$$u_t - u_{xxx} - u_x u, \quad D_\sigma(u_t - u_{xxx} - u_x u), \quad |\sigma| \geq 0,$$

coincides with the ideal generated by the functions

$$\frac{\partial^{p+q} u}{\partial t^p \partial x^q} - F_{p,q}(u, u_1, u_2, u_3, \dots), \quad p > 0,\ q \geq 0. \tag{52}$$

where $\dfrac{\partial^{p+q} u}{\partial t^p \partial x^q}$ is considered as a coordinate of the jet space. Similarly to Exercise 5, one can show that the ideal generated by (52) coincides with the ideal $I(\mathcal{E})$. Therefore, condition (31) holds in this case. Besides, similarly to Exercise 5, we obtain that $C^\infty(\mathcal{E})$ is isomorphic to the algebra of smooth functions in the variables

$$x,\ t,\ u_i,\ i \geq 0. \tag{53}$$

This allows to call (53) *internal coordinates of* $\mathcal{E}$. One can also introduce on $\mathcal{E}$ the structure of an infinite-dimensional manifold in the sense of Remark 15 as follows. Set $M^l = \mathbb{R}^{n+l+1}$ with coordinates $x_1, \dots, x_n, u, \dots, u_l$, then $\mathcal{E}$ is isomorphic to the inverse limit of the sequence of bundles

$$\dots \to M^{l+1} \xrightarrow{f_l} M^l \to \dots \xrightarrow{f_2} M_a^2 \xrightarrow{f_1} M_a^1,$$
$$f_l \colon (x_1, \dots, x_n, u, \dots, u_{l+1}) \mapsto (x_1, \dots, x_n, u, \dots, u_l).$$

A vector field $V \in D(\mathcal{E})$ can be written in these coordinates as follows

$$V = V(x)\frac{\partial}{\partial x} + V(t)\frac{\partial}{\partial t} + \sum_{i \geq 0} V(u_i)\frac{\partial}{\partial u_i}.$$

In particular, the total derivative operators restricted to $\mathcal{E}$ are

$$D_x\big|_{\mathcal{E}} = \frac{\partial}{\partial x} + \sum_{i\geq 0} u_{i+1}\frac{\partial}{\partial u_i},$$

$$D_t\big|_{\mathcal{E}} = \frac{\partial}{\partial t} + \sum_{i\geq 0} \big(D_x\big|_{\mathcal{E}}\big)^i (u_3 + u_1 u)\frac{\partial}{\partial u_i}.$$

Equation (48) reads

$$D_t\big|_{\mathcal{E}}(\varphi) = \big(D_x\big|_{\mathcal{E}}\big)^3(\varphi) + uD_x\big|_{\mathcal{E}}(\varphi) + u_1\varphi. \tag{54}$$

Thus any symmetry $X \in S_v(\mathcal{E})$ is of the form

$$X = \sum_{i\geq 0} \big(D_x\big|_{\mathcal{E}}\big)^i(\varphi)\frac{\partial}{\partial u_i},$$

where $\varphi = \varphi(x, t, u, u_1, u_2, \dots, u_l) \in C^\infty(\mathcal{E})$ satisfies (54). For example, φ given by (3) is a solution of (54). In fact for any $p \geq 1$ there is a solution of the form

$$\varphi = u_{2p+1} + g(u, \dots, u_{2p}).$$

For any fixed integer $l \geq 0$ one can solve equation (54) for $\varphi = \varphi(x, t, u, \dots, u_l)$ straightforwardly and obtain a finite-dimensional space S_l of solutions. The space $S_v(\mathcal{E}) \cong \bigcup_l S_l$ is infinite-dimensional, and this fact is very important for soliton theory.

Now let us give a general definition motivated by the above example. Consider a system of PDEs

$$F_s(x_i, u^l, u^j_\sigma, \dots) = 0, \quad s = 1, \dots, m',$$

such that $I(\mathcal{E}_k)$ is generated by the functions F_s. Its infinite prolongation $\mathcal{E}$ is determined by equations (30), and the ideal $\mathcal{I} \subset C^\infty(J^\infty(\pi))$ is generated by the functions

$$F_s, \quad D_\sigma(F_s), \quad s = 1, \dots, m', \quad |\sigma| \geq 0. \tag{55}$$

Let Ω be the set of all symmetric multi-indices σ and $Z = \{1, \dots, m\}$. Suppose that there are a subset $\Omega' \subset Z \times \Omega$ and $G_{i,\sigma} \in C^\infty(J^\infty(\pi))$ for each $(i, \sigma) \in \Omega'$ such that

- the functions $G_{i,\sigma}$ do not depend on any of the coordinates $u^j_{\sigma'}$, $(j, \sigma') \in \Omega'$,
- the functions

$$u^i_\sigma - G_{i,\sigma}, \quad (i, \sigma) \in \Omega', \tag{56}$$

  generate the same ideal $\mathcal{I}$. In particular, $\mathcal{E}$ is determined by the equations

$$u^i_\sigma = G_{i,\sigma}, \quad (i, \sigma) \in \Omega'.$$

Then

$$x_1, \dots, x_n, \; u^j_\sigma, \quad (j, \sigma) \in \big(Z \times \Omega\big)\backslash\Omega', \tag{57}$$

are called *internal coordinates of* $\mathcal{E}$.

Similarly to Exercise 5 and Example 10, one gets that the ideal generated by (56) coincides with the ideal $I(\mathcal{E})$. Therefore, condition (31) holds true. Besides, the algebra $C^\infty(\mathcal{E})$ is isomorphic to the algebra of smooth functions in the variables (57). Thus we obtain the following result.

Theorem 2. *If a system of PDEs with infinite prolongation $\mathcal{E}$ has a set of internal coordinates then condition* (31) *holds, $C^\infty(\mathcal{E})$ is isomorphic to the algebra of smooth functions in the internal coordinates, and $\mathcal{E}$ has a structure of an infinite-dimensional manifold described in Remark 15.*

Remark 18. In [6] an algorithm is presented that for any system of PDEs $\mathcal{E}$ and any point a from some open dense subset of $\mathcal{E}$ constructs a set of internal coordinates for $\mathcal{E}$ on a neighborhood of a.

Example 11. Consider a system of the form

$$u^i_t = F^i(x,t,u^1,\dots,u^d,u^1_x,\dots,u^d_x,u^1_{xx},\dots,u^d_{xx},\dots),\quad u^i=u^i(x,t),\quad i=1,\dots,d, \tag{58}$$

(on the right-hand side there are no derivatives with respect to t). Similarly to Example 10, one can show that

$$x,\ t,\ u^i,\ \frac{\partial^l u^i}{\partial x^l},\quad i=1,\dots,d,\quad l\ge 1,$$

form a system of internal coordinates for the infinite prolongation of (58). Therefore, by Theorem 2, such systems satisfy condition (31).

A remark for specialists. Note that if the order of the right-hand side of (58) is greater than 1 and the symbol of the right-hand side is not of constant rank then this system is not formally integrable.

Example 12. Consider the sine-Gordon equation $u_{xt}=\sin u$ and its infinite prolongation $\mathcal{E}$. It is easily seen that

$$x,\quad t,\quad u,\quad u_i=\frac{\partial^i u}{\partial x^i},\quad u'_i=\frac{\partial^i u}{\partial t^i},\quad i\ge 1,$$

form a system of internal coordinates for $\mathcal{E}$. We have

$$D_x\big|_{\mathcal{E}}=\frac{\partial}{\partial x}+\sum_{i\ge 0}u_{i+1}\frac{\partial}{\partial u_i}+\sin u\frac{\partial}{\partial u'_1}+u_1\cos u\frac{\partial}{\partial u'_2}+\big(u_2\cos u-u_1^2\sin u\big)\frac{\partial}{\partial u'_3}+\dots,$$

$$D_t\big|_{\mathcal{E}}=\frac{\partial}{\partial t}+\sum_{i\ge 0}u'_{i+1}\frac{\partial}{\partial u'_i}+\sin u\frac{\partial}{\partial u_1}+u'_1\cos u\frac{\partial}{\partial u_2}+\big(u'_2\cos u-{u'_1}^2\sin u\big)\frac{\partial}{\partial u_3}+\dots.$$

Equation (48) reads

$$D_x\big|_{\mathcal{E}}D_t\big|_{\mathcal{E}}(\varphi)=\cos u\cdot\varphi. \tag{59}$$

For example, $\varphi=u_3+\frac{1}{2}u_1^3$ and $\varphi=u'_3+\frac{1}{2}{u'_1}^3$ satisfy (59).

8. Applications to finding solutions of PDEs

8.1. **Symmetry-invariant solutions.** Let $\mathcal{E}_k$ be the closed subset of $J^k(\pi)$ corresponding to a system of PDEs and $\mathcal{E}\subset J^\infty(\pi)$ be its infinite prolongation. A section $s\colon U\to E$ of the bundle π over an open subset $U\subset M$ is said to be a *solution* of the system of PDEs if $j_k(s)(U)\subset\mathcal{E}_k$. Then $j_\infty(s)(U)\subset\mathcal{E}$.

It can be shown that any n-dimensional integral submanifold of the Cartan distribution on $J^\infty(\pi)$ is locally of the form $j_\infty(s)(U)$ for some U and s. Therefore, locally solutions of $\mathcal{E}$ are in one-to-one correspondence with n-dimensional integral submanifolds of the Cartan distribution on $\mathcal{E}$.

Consider a symmetry $X \in \operatorname{Sym}(\pi)$ tangent to $\mathcal{E}$ (that is, $X(I(\mathcal{E})) \subset I(\mathcal{E})$). A solution $s\colon U \to E$ of $\mathcal{E}$ is said to be *invariant with respect to the symmetry* X if the vector field X is tangent to the submanifold $j_\infty(s)(U) \subset J^\infty(\pi)$.

Remark 19. This means that for each $x \in U$ the tangent vector at $a(x) = [s]_x^\infty \in J^\infty(\pi)$ corresponding to the vector field X belongs to the subspace $\big(j_\infty(s)\big)_*(T_xU) \subset T_{a(x)}J^\infty(\pi)$, which is equal to the Cartan subspace $\mathcal{C}(a(x))$.

Proposition 10. *Suppose that $X \in \operatorname{Sym}(\pi)$ is tangent to $\mathcal{E}$. Consider the vertical part $E_\varphi \in S_v(\pi)$ of X with respect to the direct sum decomposition* (21)*, where*

$$\varphi \in \varkappa(\pi) \cong \Gamma(v^\infty(\pi)).$$

Then a solution $s\colon U \to E$ of $\mathcal{E}$ is invariant with respect to X if and only if

$$\forall\, x \in U \qquad \varphi([s]_x^\infty) = 0. \tag{60}$$

Here φ is regarded as a section of the bundle (23).

Proof. We have $X = X_h + E_\varphi$, where $X_h \in \mathcal{C}(\pi)$. Since X_h is tangent to the submanifold $j_\infty(s)(U)$ for any s, the symmetry X is tangent to $j_\infty(s)(U)$ if and only if E_φ is. By Remark 19, this is equivalent to the fact that for each $x \in U$ the tangent vector corresponding to E_φ at the point $[s]_x^\infty \in J^\infty(\pi)$ belongs to the Cartan subspace at this point. However, since $E_\varphi(C^\infty(M)) = 0$, this means that the vector field E_φ is equal to zero at the points $[s]_x^\infty$ for all $x \in U$. Finally, it is easy to check in local coordinates that the latter condition is equivalent to (60). □

For any reasonable system of PDEs the set $\pi_\infty(\mathcal{E}) \subset M$ contains an open subset U of M (otherwise there is no chance to find any solutions for $\mathcal{E}$). Restricting the bundle π to $\pi^{-1}(U) \subset E$ if necessary, we will assume in what follows that $\pi_\infty(\mathcal{E}) = M$.

Let us write down the statement of Proposition 10 in local coordinates. Suppose that the initial system of PDEs is

$$F_s(x_i, u^l, u^j_\sigma, \dots) = 0, \quad s = 1, \dots, m', \tag{61}$$

and $\varphi = (\varphi^1, \dots, \varphi^m) \in \varkappa(\pi) \cong \big(C^\infty(J^\infty(\pi))\big)^m$. Then functions

$$u^j(x_1, \dots, x_n), \quad j = 1, \dots, m,$$

form a solution invariant with respect to the symmetry X if and only if they satisfy the following extended system of PDEs

$$F_s(x_i, u^l, u^j_\sigma, \dots) = 0, \quad s = 1, \dots, m', \tag{62}$$

$$\varphi^j(x_i, u^l, u^j_\sigma, \dots) = 0, \quad j = 1, \dots, m. \tag{63}$$

This suggests the following method to obtain solutions for (61): find a symmetry E_φ for (61), and solve system (62), (63).

Note that system (62), (63) is overdetermined, because it consists of $m + m'$ equations for m unknown functions $u^1, \dots, u^m$. Nevertheless, according to the next theorem, this system almost always possesses solutions and is often easier to solve than the initial system (62).

Theorem 3 ([3])**.** *If E_φ is a symmetry of* (61) *and $\varphi = (\varphi^1, \dots, \varphi^m)$ satisfies some non-degeneracy conditions then system* (62), (63) *is consistent, that is, its infinite prolongation* *is non-empty. (Note that a system with non-empty infinite prolongation almost always possesses solutions).*

Moreover, system (62), (63) *is usually equivalent to a system with fewer than n independent variables. In particular, if* $n = 2$ *then this system is usually equivalent to a system of ordinary differential equations.*

Remark 20. For systems of the form (58) this result is well known.

We would like to stress that in Theorem 3 the initial system (61) is allowed to be overdetermined (but, as we said above, it must satisfy $\pi_\infty(\mathcal{E}) = M$, where $\mathcal{E}$ is the infinite prolongation of (61)).

Here we do not have a possibility to describe explicitly the non-degeneracy conditions mentioned in Theorem 3, but many studied examples suggest that practically all important symmetries satisfy these conditions.

Example 13. For a PDE of the form

$$u_t = f(x, t, u, u_x, u_{xx}, \dots) \tag{64}$$

the non-degeneracy condition for $\varphi = \varphi(x, t, u, u_1, u_2, \dots)$, where $u_l = \dfrac{\partial^l u}{\partial x^l}$, is

$$\exists\, l \geq 0 \qquad \frac{\partial \varphi}{\partial u_l} \neq 0.$$

Example 14. In Section 2 for the KdV equation (1) we studied solutions invariant with respect to the symmetry E_φ for φ equal to (3).

8.2. **Reproduction of solutions.** Let $\varphi \in \varkappa(\pi)$ be such that E_φ is tangent to $\mathcal{E}$. Suppose that there are $\psi \in \varkappa(\pi)$ and $V \in \mathcal{C}(\pi)$ such that

$$\psi(C^\infty(E)) \subset I(\mathcal{E}) \tag{65}$$

and the symmetry $Y = E_{\varphi+\psi} + V$ is classical. Condition (65) implies that E_ψ is tangent to $\mathcal{E}$ and $E_\psi\big|_{\mathcal{E}} = 0$, therefore, the symmetry Y is also tangent to $\mathcal{E}$. Recall that $\mathcal{E}$ is the infinite prolongation of $\mathcal{E}_k \subset J^k(\pi)$. We assume $k \geq 1$, because the case $k = 0$ is trivial.

Denote by

$$G_a \colon J^k(\pi) \to J^k(\pi), \quad a \in \mathbb{R}, \quad G_0 = \mathrm{Id},$$

the one-parametric group of local diffeomorphisms corresponding to the classical symmetry Y. Let $s \colon U \to E$ be a solution of $\mathcal{E}_k$. Since Y is tangent to $\mathcal{E}$, we have

$$G_a\big(j_k(s)(U)\big) \subset \mathcal{E}_k \qquad \forall\, a. \tag{66}$$

For a small enough open subset $U' \subset U$ and small enough a the projection $\pi_k\big|_{N_a} \colon N_a \to M$ is a diffeomorphism onto an open subset $U_a \subset U$, where $N_a = G_a\big(j_k(s)(U')\big) \subset \mathcal{E}_k$. Since Y is a symmetry, N_a is an integral submanifold of the Cartan distribution, and, by Proposition 2, there is a section $s_a \colon U_a \to E$ such that $N_a = j_k(s_a)(U_a)$. Since $N_a \subset \mathcal{E}_k$, the section s_a is a solution of $\mathcal{E}_k$.

Thus, using a classical symmetry of $\mathcal{E}_k$, for any solution s one obtains a one-parametric family of solutions s_a such that $s_0 = s$.

Remark 21. One has $s_a = s$ for all a if and only if the solution s is invariant with respect to the symmetry Y. Equivalently, s is invariant with respect to E_φ.

Example 15. For the KdV equation (1) consider the symmetry E_φ, where

$$\varphi = u_{xxx} + u_x u - cu_x$$

is equal to (3) and $c \in \mathbb{R}$ is a constant. For $\psi = u_t - u_{xxx} - u_x u$ and $V = cD_x - D_t$ set $Y = E_{\varphi+\psi} + V$. We have

$$Y(u) = 0, \quad Y(x) = c, \quad Y(t) = -1.$$

Therefore, by Exercise 3, the symmetry Y is classical and the scheme described above is applicable to it. For a solution $s(x,t)$ of (1) the one-parametric family of solutions s_a determined by Y is given by

$$s_a(x,t) = s(x + ac, t - a)$$

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