Notes on Topology

These are links to PostScript files containing notes for various topics in topology.

I learned topology at M.I.T. from *Topology: A First Course* by James Munkres. At the time, the first edition was just coming out; I still have the photocopies we were given before the printed version was ready!

Hence, I'm a bit biased: I still think Munkres' book is the best book to learn from. The writing is clear and lively, the choice of topics is still pretty good, and the exercises are wonderful. Munkres also has a gift for naming things in useful ways (The Pasting Lemma, the Sequence Lemma, the Tube Lemma).

(I like Paul Halmos's suggestion that things be named in descriptive ways. On the other hand, some names in topology are terrible --- "first countable" and "second countable" come to mind. They are almost as bad as "regions of type I" and "regions of type II" which you still sometimes encounter in books on multivariable calculus.)

I've used Munkres both of the times I've taught topology, the most recent occasion being this summer (1999). The lecture notes below follow the order of the topics in the book, with a few minor variations.

Here are some areas in which I decided to do things differently from Munkres:

- I prefer to motivate continuity by recalling the epsilon-delta definition which students see in analysis (or calculus); therefore, I took the pointwise definition as a starting point, and derived the inverse image version later.
- I stated the results on quotient maps so as to emphasize the universal property.
- I prefer Zorn's lemma to the Maximal Principle; for example, in constructing connected components, I use Zorn's lemma to construct maximal connected sets.
- I did Tychonoff's theorem at the same time as the other stuff on compactness. The proof for a finite product is long enough that I decided to save some time by omitting it and just doing the general case.
- It seems simpler to me to do Urysohn's lemma by indexing the level sets with dyadic rationals in [0,1] rather than rationals.

- How do I view/print the notes?
- How may I use/distribute the notes? How did you write them?

- funct.ps (86,912 bytes; 5 pages): Review of topics in set theory.
- top.ps (168,720 bytes; 6 pages): Topological spaces, bases.
- closed.ps (55,509 bytes; 4 pages): Closed sets and limit points.
- haus.ps (132,025 bytes; 2 pages): Hausdorff spaces.
- contin.ps (100,021 bytes; 3 pages): Continuous functions.
- homeo.ps (83,891 bytes; 3 pages): Homeomorphisms.
- prod.ps (51,962 bytes; 3 pages): The product topology.
- metric.ps (140,993 bytes; 8 pages): Metric spaces.
- quot.ps (116,395 bytes; 5 pages): Quotient spaces.
- conn.ps (82,400 bytes; 4 pages): Connectedness.
- realconn.ps (55,934 bytes; 3 pages): Connected subsets of the reals.
- pathco.ps (45,136 bytes; 3 pages): Path connectedness, local connectedness.
- compact.ps (240,878 bytes; 8 pages): Compactness (including Tychonoff's theorem).
- compreal.ps (69,020 bytes; 3 pages): Compact subsets of the reals.
- limcomp.ps (107,691 bytes; 4 pages): Limit point compactness and sequential compactness.
- loccpt.ps (70,399 bytes; 3 pages): Local compactness; the one-point compactification.
- countab.ps (40,734 bytes; 2 pages): The countability axioms.
- sep.ps (139,569 bytes; 5 pages): The separation axioms: regularity and normality.
- urysohn.ps (69,043 bytes; 2 pages): Urysohn's lemma.
- tietze.ps (45,448 bytes; 3 pages): The Tietze Extension Theorem.

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Review: Set Theory, Functions, and Relations

This is a brief review of material on set theory, functions, and relations. I'll concentrate on things which are often not covered (or given cursory treatment) in courses on mathematical proof.

Definition. Let A and B be sets, and let $f : A \to B$ be a function from A to B.

1. f is **injective** if for all $a_1, a_2 \in A$, $f(a_1) = f(a_2)$ implies $a_1 = a_2$.
2. f is **surjective** if for all $b \in B$, there is an element $a \in A$ such that $f(a) = b$.
3. f is **bijective** if f is injective and surjective.

Definition. Let $f : A \to B$ be a function. Let $B_0 \subset B$.

1. The **image** of f is the set
$$f(A) = \{f(a) \mid a \in A\}.$$
2. The **inverse image** of B_0 is the set
$$f^{-1}(B_0) = \{a \in A \mid f(a) \in B_0\}.$$

Remark. f is surjective if and only if $f(A) = B$.

Example. (a) Let $f : A \to B$ and $g, h : B \to C$ be functions. Suppose that f is surjective and $g \cdot f = h \cdot f$. Prove that $g = h$.

Let $b \in B$. I must show that $g(b) = h(b)$.
Since f is surjective, I may find an element $a \in A$ such that $f(a) = b$. Then $g \cdot f = h \cdot f$ implies that

$$g(f(a)) = h(f(a)), \quad \text{so} \quad g(b) = h(b).$$

Therefore, $g = h$. $\square$

(b) Let $f : A \to B$. Suppose that for all $g, h : B \to C$, if $g \cdot f = h \cdot f$, then $g = h$. Prove that f is surjective.

First, I'll take care of the cases where $A = \emptyset$ or A contains a single element.
If $A = \emptyset$ and $B = \emptyset$, then f is vacuously surjective: Every element of B (there are none) is an image of an element of A.
If $A = \emptyset$ and $B \neq \emptyset$, then B contains at least one element b. Define $g : B \to \{0, 1\}$ by $g(b) = 0$ and $h : B \to \{0, 1\}$ by $h(b) = 1$. It is vacuously true that $g(f(a)) = h(f(a))$ for all $a \in A$ (since there are no a's in A), but $g \neq h$. Therefore, the hypothesis isn't satisfied, and this case is ruled out.
Suppose A consists of a single element a. If $B = \emptyset$, then f is vacuously surjective. If B consists of a single element, then f is obviously surjective.
Finally, suppose that B contains more than one element. $f(a)$ is an element of B; suppose that $b_0 \in B$ and $b_0 \neq f(a)$. Consider the functions $g, h : B \to B$ given by

$$g(b) = b \quad \text{for all} \quad b \in B,$$

$$h(b) = \begin{cases} b & \text{if } b \neq b_0 \\ f(a) & \text{if } b = b_0 \end{cases}.$$

Then $g(f(a)) = f(a) = h(f(a))$, so $g \cdot f = h \cdot f$. However, $g \neq h$. Therefore, the hypothesis isn't satisfied, and this case is ruled out.

This concludes the proof for the trivial cases.

Assume that A contains more than one element.

Suppose, on the contrary, that $f(A) \neq B$. I'll show that the hypothesis is violated.

If $b \in f(A)$, *choose* an element $a_b \in A$ such that $f(a_b) = b$. Since A contains more than one element, I may find elements $a_1, a_2 \in A$ such that $a_1 \neq a_2$. Define $g, h : B \to A$ by

$$g(b) = \begin{cases} a_b & \text{if } b \in f(A) \\ a_1 & \text{if } b \notin f(A) \end{cases} \quad \text{and} \quad g(b) = \begin{cases} a_b & \text{if } b \in f(A) \\ a_2 & \text{if } b \notin f(A) \end{cases}.$$

If $a \in A$, then $f(a) \in f(A)$, so

$$g(f(a)) = a_{f(a)} = h(f(a)).$$

Thus, $g \cdot f = h \cdot f$.

However, since $f(A) \neq B$, there is an element $b_0 \in B - f(A)$. For this element, $g(b_0) = a_1$, but $h(b_0) = a_2$. Therefore, $g \neq h$.

This proves that the hypothesis is violated. This contradiction in turn establishes that $f(A) = B$, so f is surjective. □

Remark. This result is often expressed by saying that surjective functions are **right-cancellable**. The dual result is also true: Injective functions are **left-cancellable**.

Example. Let $f : A \to B$, and suppose that $B_1, B_2 \subset B$. Prove that $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.

Suppose $a \in f^{-1}(B_1 \cup B_2)$. Then $f(a) \in B_1 \cup B_2$, so $f(a) \in B_1$ or $f(a) \in B_2$.

If $f(a) \in B_1$, then $a \in f^{-1}(B_1)$, so $a \in f^{-1}(B_1) \cup f^{-1}(B_2)$. If $f(a) \in B_2$, then $a \in f^{-1}(B_2)$, so $a \in f^{-1}(B_1) \cup f^{-1}(B_2)$.

Therefore, $a \in f^{-1}(B_1) \cup f^{-1}(B_2)$, so $f^{-1}(B_1 \cup B_2) \subset f^{-1}(B_1) \cup f^{-1}(B_2)$.

Conversely, suppose that $a \in f^{-1}(B_1) \cup f^{-1}(B_2)$, so $a \in f^{-1}(B_1)$ or $a \in f^{-1}(B_2)$.

If $a \in f^{-1}(B_1)$, then $f(a) \in B_1$, so $f(a) \in B_1 \cup B_2$, and hence $a \in f^{-1}(B_1 \cup B_2)$. If $a \in f^{-1}(B_2)$, then $f(a) \in B_2$, so $f(a) \in B_1 \cup B_2$, and hence $a \in f^{-1}(B_1 \cup B_2)$.

Therefore, $a \in f^{-1}(B_1 \cup B_2)$, so $f^{-1}(B_1) \cup f^{-1}(B_2) \subset f^{-1}(B_1 \cup B_2)$.

This proves that $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$. □

Definition. Let A be a set. A relation $<$ on A is a **simple order** (or **linear order**) if:

1. (**Comparability**) For all $a, b \in A$, exactly one of the following is true: $a < b$, $b < a$, or $a = b$.
2. (**Nonreflexivity**) There is no element $a \in A$ such that $a < a$.
3. (**Transitivity**) For all $a, b, c \in A$, if $a < b$ and $b < c$,then $a < c$.

Example. Let $<$ be a simple order on a set A. The **dictionary order** (or **lexicographic order**) on $A \times A$ is given by: $(a, b) < (a', b')$ if and only if

1. $a < a'$; or
2. $a = a'$, and $b < b'$.

The words in a dictionary are ordered by their first letter. Among words with the same first letter, they are ordered by their second letter, and so on. This explains the name of this order relation.

To give a specific example, in $\mathbb{R}^2$ under the dictionary order, $(1, 4) < (2, 4)$ and $(1, 6) < (1, 8)$.

Definition. If A is an ordered set, $a, b \in A$, and $a < b$, then the **open interval** from a to b is the set

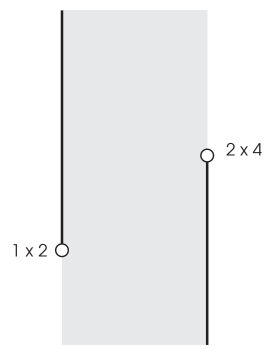
$$(a, b) = \{x \mid a < x < b\}.$$

It may happen that this set is empty. In that case, since there are no elements "between" a and b, it's natural to say that a is the **predecessor** of b and b is the successor of a.

Unfortunately, "(a, b)" can mean a *point* in a Cartesian product or an *open interval* in a simple order. To avoid confusion, I'll sometimes write $a \times b$ for the point when both concepts occur in the same discussion.

Example. Consider the dictionary order on $\mathbb{R}^2$. The open interval from the point 1×2 to the point 2×4 is the union of:

1. The open ray $\{1 \times y \mid y > 2\}$.
2. The lines $\{x \times y \mid 1 < x < 2\}$.
3. The open ray $\{2 \times y \mid y < 4\}$.



□

Definition. Suppose A and B are sets with simple orders $<_A$ and$<_B$, respectively. Let $f : A \to B$ be a function. f is **order-preserving** if for all $a_1, a_2 \in A$,

$$a_1 <_A a_2 \quad \text{implies} \quad f(a_1) <_B f(a_2).$$

Example. With the usual order relation on $\mathbb{R}$, define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3$. Then f is order-preserving.

In fact, a function $\mathbb{R} \to \mathbb{R}$ is order-preserving if it is an **increasing** function. □

Definition. Let A and B be ordered sets. A and B have the **same order type** if there's an order-preserving bijection from A to B.

Definition. A set is **well-ordered** if every nonempty subset has a smallest element.

The **Well-Ordering Axiom** says that every set can be well-ordered. This is an axiom of set theory; it's equivalent to other results which can be taken as axioms, such as the **Axiom of Choice** and **Zorn's lemma**.

Well-ordering means that for a given set, there is *some* order relation on the set relative to which the set is well-ordered. That well-ordering will usually have little to do with familiar order relations.

Definition. Let A be an ordered set, and let $B \subset A$.

1. $c \in B$ is the **largest element** of B if $b < c$ for all $b \in B$, $b \neq c$.

   Likewise, $c \in B$ is the **smallest element** of B if $c < b$ for all $b \in B$, $b \neq c$.

2. An element $a \in A$ is an **upper bound** for B if $b < a$ for all $b \in B$, $b \neq a$. If B has an upper bound, B is **bounded above**.

   Likewise, an element $a \in A$ is a **lower bound** for B if $a < b$ for all $b \in B$, $b \neq a$. If B has a lower bound, B is **bounded below**.

3. If the set of upper bounds of B is nonempty, then the smallest element of that set is the **least upper bound** for B. It is denoted $\operatorname{lub} B$ or $\sup B$.

   Likewise, if the set of lower bounds of B is nonempty, the largest element of that set is the **greatest lower bound** for B. It is denoted $\operatorname{glb} B$ or $\inf B$.

4. A set has the **least upper bound property** if every nonempty subset that is bounded above has a least upper bound.

   A set has the **greatest lower bound property** if every nonempty subset that is bounded below has a greatest lower bound.

Example. The real numbers with the usual ordering satisfy the least upper bound property. In fact, it is one of the axioms for the real numbers. □

Mathematics is based on a collection of axioms for set theory. They are called the **Zermelo-Fraenkel Axioms with the Axiom of Choice** (often abbreviated to **ZFC**).

While the Axiom of Choice is accepted and used by most mathematicians, the situation was different earlier in this century. Some mathematicians felt that the Choice Axiom was problematic and refused to use it; others took care to note when they used the Choice Axiom in a proof. Many of the objections stemmed from the often startling results which can be proved using the axiom.

Axiom of Choice. Let $\{X_a\}$ be a collection of disjoint nonempty sets. There exists a set X containing exactly one element from each X_a.

The Axiom of Choice is equivalent to the Well-Ordering Axiom, and also to an important result known as **Zorn's lemma**. The statement of Zorn's lemma requires some preliminaries.

Definition. A relation $\leq$ on a set X is a **partial order** if:

1. (**Reflexivity**) $x \leq x$ for all $x \in X$.

2. (**Antisymmetry**) If $x \leq y$ and $y \leq x$, then $x = y$.

3. (**Transitivity**) If $x \leq y$ and $y \leq z$, then $x \leq z$.

Example. The set of subsets of a set is partially ordered by inclusion. □

Example. The usual "less than or equal to" relation $\leq$ on $\mathbb{R}$ is a partial order. □

Definition. Let X be a partially ordered set.

1. A subset $Y \subset X$ is **totally ordered** (or **linearly ordered**) if for all $x, y \in Y$, either $x \le y$ or $y \le x$.

   A totally ordered subset is also said to be a **chain**.

2. If $Y \subset X$, then Y has an **upper bound** if there is an element $x \in X$ such that $y \le x$ for all $y \in Y$.

3. An element $x \in X$ is **maximal** if $y \in X$ and $x \le y$ implies $x = y$.

Zorn's Lemma. If X is a nonempty partially ordered set in which every chain has an upper bound, then X has maximal elements.

Zorn's lemma is an immensely useful form of the Axiom of Choice. For example, you may have seen a proof that every *finite-dimensional* vector space has a basis. Zorn's lemma can be used to show that *every* vector space has a basis.

Example. Let G be a group and let $g \in G$, $g \ne 1$. There is a subgroup $H < G$ such that $g \notin H$ and H is maximal among subgroups with this property.

Let $\mathcal{S}$ be the collection of subgroups of G which do not contain g. $\mathcal{S} \ne \emptyset$, since $\{1\} \in \mathcal{S}$. $\mathcal{S}$ is partially ordered by inclusion.

Let $\mathcal{T}$ be a chain in $\mathcal{S}$. Let

$$H_0 = \bigcup_{H \in \mathcal{T}} H.$$

Thus, H_0 is just the union of all the elements of the chain.

I claim that H_0 is a subgroup of G which does not contain g.

First, $1 \in H$ for all $H \in \mathcal{T}$, so certainly $1 \in H_0$.

If $h, h' \in H_0$, then $h \in H$ and $h' \in H'$ for some $H, H' \in \mathcal{T}$. But $\mathcal{T}$ is a chain, so either $H \subset H'$ or $H' \subset H$.

Suppose $H \subset H'$. Then $h, h' \in H'$, so (since H' is a subgroup) $hh'^{-1} \in H'$. Therefore, $hh'^{-1} \in H_0$, and hence H_0 is a subgroup.

If $g \in H_0$, then $g \in H$ for some $H \in \mathcal{T}$. But $H \in \mathcal{S}$, so $g \notin H$ by definition of $\mathcal{S}$. Therefore, $g \notin H_0$.

I've shown that H_0 is a subgroup of G which does not contain g, so $H_0 \in \mathcal{S}$. Since $H \subset H_0$ for all $H \in \mathcal{T}$, it follows that H_0 is an upper bound for $\mathcal{T}$ in $\mathcal{S}$.

Since every chain has an upper bound, Zorn's lemma implies that $\mathcal{S}$ has a maximal element H. Then H is a subgroup of G which is maximal among subgroups which do not contain g. □

Topological Spaces

A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at $x = a$ if and only if

For every $\epsilon > 0$, there is a $\delta > 0$ such that if $\delta > |x - a| > 0$, then $\epsilon > |f(x) - f(a)|$.

In words,

You can make $f(x)$ arbitrarily close to $f(a)$ by making x sufficiently close to a.

Continuity is important in analysis and calculus. The idea of "closeness" is an important part of the definition of continuity. What would "closeness" mean in an abstract setting — one in which there's no way to measure the *distance* between two points?

A **topology** for a space is, roughly speaking, a specification of all "closeness relations" in the space.

Definition. Let X be a set. A **topology** on X is a collection $\mathcal{T}$ of subsets of X satisfying the following conditions:

1. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.
2. The union of elements of $\mathcal{T}$ is an element of $\mathcal{T}$.
3. The *finite* intersection of elements of $\mathcal{T}$ is an element of $\mathcal{T}$.

The elements of $\mathcal{T}$ are called **open sets**. The set X together with the topology $\mathcal{T}$ is a **topological space**.

Example. If X is a set, take $\mathcal{D}$ to be $\mathcal{P}(X)$, the power set of X. (Recall that $\mathcal{P}(X)$ is the set of all subsets of X.)

$\mathcal{D}$ is clearly a topology on X; it is called the **discrete topology**. In the discrete topology, all subsets are open. □

The other extreme is the **indiscrete topology** on X. In this case, $\mathcal{I} = \{\emptyset, X\}$. Again, it is clear that this is a topology on X. □

Definition. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on a set X, then $\mathcal{T}_1$ is **finer** than $\mathcal{T}_2$ (or $\mathcal{T}_2$ is **coarser** than $\mathcal{T}_1$) if $\mathcal{T}_1 \supset \mathcal{T}_2$. In other words, finer topologies contain more open sets.

Thus, if $\mathcal{D}$ is the discrete topology on X, then $\mathcal{D}$ is finer than any other topology on X. If $\mathcal{I}$ is the indiscrete topology on X, then $\mathcal{I}$ is coarser than any other topology on X.

It is, of course, possible for two topologies on X to be incomparable, i.e. neither may be finer than the other.

Analysts often refer to "stronger" and "weaker" topologies, but their usage is often at variance with the usage of topologists.

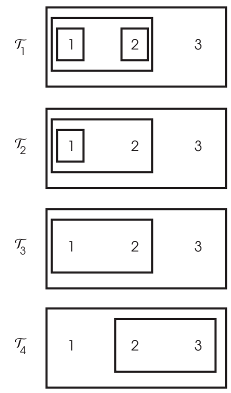
Example. Let $X = \{1, 2, 3\}$, and consider the following topologies on X:

$$\mathcal{T}_1 = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, X\},$$

$$\mathcal{T}_2 = \{\emptyset, \{1\}, \{1, 2\}, X\},$$

$$\mathcal{T}_3 = \{\emptyset, \{1, 2\}, X\}.$$

$$\mathcal{T}_4 = \{\emptyset, \{2, 3\}, X\}.$$



Then $\mathcal{T}_1$ is finer than $\mathcal{T}_2$, and $\mathcal{T}_2$ is finer than $\mathcal{T}_3$. On the other hand, $\mathcal{T}_4$ is not comparable with the other three.

As this example shows, topology does not necessarily concern itself with the concepts of measurement or distance. They might not be applicable to the space under consideration. □

Example. The standard topology on $\mathbb{R}$ is the topology whose open sets consist of arbitrary unions of open intervals.

The empty set is considered to be a degenerate open interval, and $\mathbb{R}$ can be written as a union of open intervals.

A union of unions of open intervals is a union of open intervals.

It remains to show that this topology is closed under finite intersections. It's fairly obvious that two open intervals either do not intersect, or intersect in an open interval; I won't write out the details. The general case for finite intersections reduces to this case by distributing intersections over unions. □

Example. The standard topology on $\mathbb{R}^n$ is obtained by starting with open balls

$$B(x; r) = \{y \in \mathbb{R}^n \mid |x - y| < r\}.$$

($B(x; r)$ is called the **open r-ball centered at x**.) The open sets are arbitrary unions of open balls.

I'll defer the verification that this gives a topology, since it will follow easily once I develop the notion of a **basis**. □

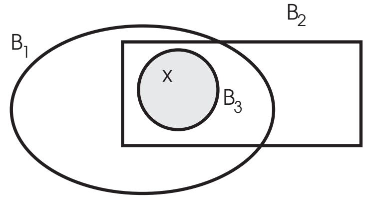
The examples above give a typical way of defining a topology: Start with a collection of subsets and then extend the collection to a topology by taking arbitrary unions. The next definition makes precise the conditions I need to be able to do this.

Definition. Let X be a set. A collection $\mathcal{B}$ of subsets of X is a **basis** if:

1. Every element of X is contained in some element of $\mathcal{B}$.

2. If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there is an element $B_3 \in \mathcal{B}$ such that

$$x \in B_3 \subset B_1 \cap B_2.$$



The elements of $\mathcal{B}$ are called **basic subsets**, or **basis elements**.

Remark. By induction, it follows that if $\mathcal{B}$ is a basis, $B_1, \ldots, B_n \in \mathcal{B}$, and $x \in B_1 \cap \cdots \cap B_n$, then there is an element $B \in \mathcal{B}$ such that

$$x \in B \subset B_1 \cap \cdots \cap B_n.$$

Example. The open intervals in $\mathbb{R}$ form a basis.

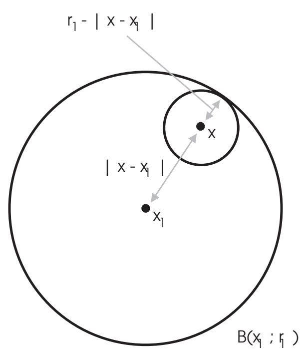
If $x \in \mathbb{R}$, then $x \in (x-1, x+1)$ (for instance).

If x is contained in the intersection of two open intervals, the intersection is itself an open interval which contains x and is contained in the intersection.

The open balls $B(x; r)$ in $\mathbb{R}^n$ form a basis. Obviously, every point in $\mathbb{R}^n$ is contained in some open ball.

Take open balls $B(x_1; r_1)$ and $B(x_2; r_2)$. Assume that they intersect, and let $x \in B(x_1; r_1) \cap B(x_2; r_2)$. I have to find an open ball centered at x which is contained in the intersection.

Let $s_1 = r_1 - |x - x_1|$. Then $B(x; s_1) \subset B(x_1, r_1)$.



Likewise, let $s_2 = r_2 - |x - x_2|$. Then $B(x; s_2) \subset B(x_2, r_2)$.

Therefore, if I let $r = \min(s_1, s_2)$, then

$$B(x; r) \subset B(x_1; r_1) \cap B(x_2; r_2).$$

This shows that the open balls in $\mathbb{R}^n$ form a basis. $\square$

Proposition. Let X be a set, and let $\mathcal{B}$ be a basis. Define $\mathcal{T}$ to be the collection of subsets $U \subset X$ satisfying the following property:

$U \in \mathcal{T}$ if and only if for all $x \in U$, there is an element $B \in \mathcal{B}$ such that $x \in B \subset U$.

Then $\mathcal{T}$ is a topology on X.

$\mathcal{T}$ is the **topology generated by the basis** $\mathcal{B}$.

Proof. The empty set is in $\mathcal{T}$; it satisfies the defining condition vacuously, since it has no elements.

$X \in \mathcal{T}$: If $x \in X$, the first axiom for a basis implies that there is an element $B \in \mathcal{B}$ such that $x \in B \subset X$.

Let $\{U_a\}$ be a collection of elements of $\mathcal{T}$. I need to show that the union $U = \bigcup_a U_a$ is in $\mathcal{T}$.

Let $x \in U$. Then $x \in U_a$ for some a. Since $U_a \in \mathcal{T}$, there is an element $B \in \mathcal{B}$ such that $x \in B \subset U_a$. Then

$$x \in B \subset U_a \subset U.$$

Hence, $U \in \mathcal{T}$.

Let $V_1, \ldots, V_n \in \mathcal{T}$. I must show that $V = \bigcap_{i=1}^n V_i$ is in $\mathcal{T}$. Let $x \in V$. Then $x \in V_i$ for all i. Since $V_i \in \mathcal{T}$, for each i I may find an element $B_i \in \mathcal{B}$ such that $x \in B_i \subset V_i$. By the remark above, I may find an element $B \in \mathcal{B}$ such that $x \in B \subset \bigcap_{i=1}^n B_i$. Then

$$x \in B \subset \bigcap_{i=1}^n B_i \subset \bigcap_{i=1}^n V_i = V.$$

Hence, $V \in \mathcal{T}$.

Therefore, $\mathcal{T}$ is a topology. $\square$

Notice that elements of a basis are automatically elements of the topology they generate: Basic sets are open.

Definition. If $\mathcal{T}$ is a topology and $\mathcal{B}$ is a basis, then $\mathcal{B}$ is a **basis for** $\mathcal{T}$ if the topology generated by $\mathcal{B}$ is the same as $\mathcal{T}$.

Lemma. Let X be a set, $\mathcal{B}$ a basis, and $\mathcal{T}$ the topology generated by $\mathcal{B}$. Then $\mathcal{T}$ is equal to the collection of all unions of elements of $\mathcal{B}$.

Proof. Since basis elements are open, unions of basis elements are open. Therefore, the collection of all unions of basis elements is contained in $\mathcal{T}$.

Conversely, let $U \in \mathcal{T}$. I must express U as a union of basis elements. For each $x \in U$, find a basis element B_x such that $x \in B_x \subset U$. Then $U = \bigcup_{x \in U} B_x$. $\square$

Here are two more facts about bases. The proofs are straightforward.

1. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X generated by bases $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. Then $\mathcal{T}_1$ is finer than $\mathcal{T}_2$ if and only if for every $U \in \mathcal{B}_2$ and $x \in U$, there is an element $V \in \mathcal{B}_1$ such that $x \in V \subset U$.

2. Given a topology $\mathcal{T}$ on X, a subcollection $\mathcal{B}$ of open sets is a basis for $\mathcal{T}$ if for all $U \in \mathcal{T}$ and $x \in U$, there is an element $B \in \mathcal{B}$ such that $x \in B \subset U$.

It's possible to go to a "lower" level and generate a topology by taking unions and finite intersections.

Definition. If X is a set, a **subbasis** is a collection of subsets of X whose union is X.

Proposition. If $\mathcal{S}$ is a subbasis in X, the collection of all unions of finite intersections of elements of $\mathcal{S}$ is a topology on X.

Proof. Let $\mathcal{T}$ denote the collection of all unions of finite intersections of elements of $\mathcal{S}$.

The union of the elements of the subbasis is X, so $X \in \mathcal{T}$. Taking the empty union gives the empty set, so $\emptyset \in \mathcal{T}$.

The union of unions of finite intersections of elements of $\mathcal{S}$ are unions of finite intersections of elements of $\mathcal{S}$, so such a union is an element of $\mathcal{T}$.

The verification that $\mathcal{T}$ is closed under finite intersections is routine, using the fact that intersections distribute over unions. I'll omit the details. $\square$

Example. (**Order topologies**) Let X be a set linearly ordered by a relation $<$. Consider the following collection of subsets of X:

1. All intervals $[s, b)$, where $b > s$ and s is the smallest element of X (if there is one).

2. All intervals $(a, l]$, where $a < l$ and l is the largest element of X (if there is one).

3. All intervals (a, b), where $a < b$.

I claim that this collection is a basis.

First, if $x \in X$, I must find an element of the basis containing x. If x is the largest element, take an interval of the form $(a, x]$; if x is the smallest element, take an interval of the form $[x, b)$. If x is neither largest nor smallest, I may find a, b such that $a < x < b$. Then $x \in (a, b)$.

To verify the second axiom for a basis, just observe that the intersection of two intervals is either empty, or an interval.

The topology generated by this basis is called the **order topology**. On $\mathbb{R}$, for instance, this topology is the standard topology. □

Example. (**Partition topologies**) Let X be a set, and let $\mathcal{P}$ be a partition of X. Then $\mathcal{P}$ is a basis: the union of the sets in the collection is X, and since the sets do not intersect, the second basis axiom is satisfied vacuously.

The topology generated by $\mathcal{P}$ is called the **partition topology** on X. The open sets are unions of partition elements.

Note that if the partition consists of X itself, the partition topology is the indiscrete topology on X. And if the partition elements are the points of X, the partition topology is the discrete topology. □

Example. (**Product Topology**) Suppose $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ are topological spaces. (That is, $\mathcal{S}$ is the topology on X and $\mathcal{T}$ is the topology on Y.)

Define a basis on the product $X \times Y$ by

$$\mathcal{B} = \{U \times V \mid U \in \mathcal{S}, V \in \mathcal{T}\}.$$

I need to verify that this is a basis.

First, if $(x, y) \in X \times Y$, find open sets $U \subset X$, $V \subset Y$ such that $x \in U \subset X$ and $y \in V \subset Y$. Then $(x, y) \in U \times V \in \mathcal{B}$.

Now let U_1 and U_2 be open in X, and let V_1 and V_2 be open in Y. Suppose that

$$(x, y) \in (U_1 \times V_1) \cap (U_2 \times V_2).$$

Note that

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2).$$

Find open sets $U \subset X$ and $V \subset Y$ such that $x \in U \subset U_1 \cap U_2$ and $y \in V \subset V_1 \cap V_2$. Then $U \times V \in \mathcal{B}$, and

$$(x, y) \in U \times V \subset (U_1 \cap U_2) \times (V_1 \cap V_2).$$

Therefore, $\mathcal{B}$ is a basis.

The topology generated by $\mathcal{B}$ is the **product topology** on $X \times Y$.

In fact, if $\mathcal{C}$ is a basis for the topology on X and $\mathcal{D}$ is a basis for the topology on Y, then

$$\mathcal{E} = \{U \times V \mid U \in \mathcal{C}, V \in \mathcal{D}\}$$

is a basis for the product topology on $X \times Y$.

For instance, give $\mathbb{R}$ the standard topology generated by the open intervals. Then the product topology on $\mathbb{R}^2$ is generated by products of open intervals, i.e. open rectangles. In fact, this topology is the same as the topology generated by open balls. This will follow from the next lemma, which gives an easy way of telling when two topologies are the same. □

Lemma. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X. Then $\mathcal{T}_1 \subset \mathcal{T}_2$ if and only if for every $U \in \mathcal{T}_1$ and $x \in U$, there is an element $V \in \mathcal{T}_2$ such that $x \in V \subset U$.

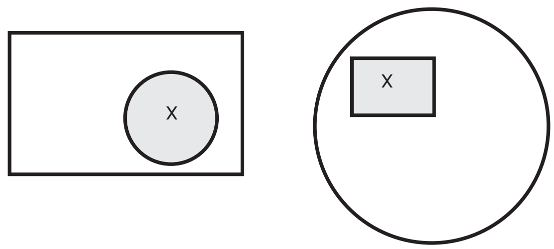
Proof. Suppose $\mathcal{T}_1 \subset \mathcal{T}_2$. If $U \in \mathcal{T}_1$ and $x \in U$, then $U \in \mathcal{T}_2$, so $x \in U \subset U$, and the condition holds.

Conversely, suppose that for every $U \in \mathcal{T}_1$ and $x \in U$, there is an element $V \in \mathcal{T}_2$ such that $x \in V \subset U$.

Let U be open in $\mathcal{T}_1$. I want to show that $U \in \mathcal{T}_2$. For each $x \in U$, find an element $V_x \in \mathcal{T}_2$ such that $x \in V_x \subset U$. Then $\bigcup_{x \in U} V_x \in \mathcal{T}_2$, but clearly $\bigcup_{x \in U} = U$. Therefore, $U \in \mathcal{T}_2$. Hence, $\mathcal{T}_1 \subset \mathcal{T}_2$. □

Corollary. The product topology on $\mathbb{R}^2$ is the same as the topology generated by the open balls.

Proof. The picture below indicates the idea:



□

Example. (**Subspace topology**) Let X be a topological space, and let $A \subset X$. Define a topology on A by stipulating that the open sets be the intersections of open sets of X with A.

It is easy to check that this is a topology. It is called the **subspace topology** on A. When a subset A of X is given the subspace topology, A is said to be a **subspace** of X.

It is also easy to check that if $\mathcal{B}$ is a basis for the topology on X, then

$$\{U \cap A \mid U \in \mathcal{B}\}$$

is a basis for the subspace topology on A. □

Remark. If $A \subset X$, a subset of A that is open in the subspace topology need not be open in X.

For example, consider the subset $\mathbb{Z} \subset \mathbb{R}$, where $\mathbb{R}$ has the standard topology. Then $\{0\}$ is open in the subspace topology on $\mathbb{Z}$, since $(-1, 1)$ is open in $\mathbb{R}$ and $\{0\} = \mathbb{Z} \cap (-1, 1)$.

However, $\{0\}$ is not open in $\mathbb{R}$. Take $0 \in \{0\}$. There is no open interval (a, b) in $\mathbb{R}$ such that $0 \in (a, b) \subset \{0\}$.

In fact, the subspace topology on $\mathbb{Z}$ is just the discrete topology. □

Theorem. Let X and Y be topological spaces. Let A be a subspace of X, and let B be a subspace of Y. The product topology on $A \times B$ agrees with topology it inherits as a subset of $X \times Y$.

Proof. Let U be open in A and let V be open in B. Thus,

$$U = A \cap U' \quad \text{and} \quad V = B \cap V',$$

where U' is open in X and V' is open in Y. $U \times V$ is a basic open set in the product topology.

Now

$$U \times V = (A \cap U') \times (B \cap V') = (A \times B) \cap (U' \times V').$$

Therefore, $U \times V$ is open in the subspace topology on $A \times B$. Thus, the product topology is contained in the subspace topology.

Going the other way, consider a basic open set $(A \times B) \cap (U' \times V')$ in the subspace topology on $A \times B$. Then

$$(A \times B) \cap (U' \times V') = (A \cap U') \times (B \cap V').$$

This is a product of an open set in A and an open set in B, so it's open in the product topology on $A \times B$.

Since an arbitrary open set in the subspace topology on $A \times B$ is a union of basic open sets, it follows that the subspace topology is contained in the product topology. □

Closed Sets and Limit Points

Definition. If X is a topological space, a subset $A \subset X$ is **closed** if $X - A$ is open.

Example. In the standard topology on $\mathbb{R}$, closed intervals are closed and points are closed.

Note, however, than points need not be closed in an arbitrary topological space.

In $\mathbb{R}^n$, the **closed r-ball centered at x**

$$\overline{B(x;r)} = \{y \in \mathbb{R}^n \mid |x - y| \le r\}$$

is closed. □

Closed sets satisfy complementary properties relative to open sets.

Proposition. Let X be a topological space.

1. X and $\emptyset$ are closed.
2. Finite unions of closed sets are closed.
3. Arbitrary intersections of closed sets are closed.

Proof. 1. X is closed, since $X - X = \emptyset$ is open. $\emptyset$ is closed, since $X - \emptyset = X$ is open.

2. Let $C_1, \ldots, C_n$ be closed. For each i, write $C_i = X - U_i$, where U_i is open. Then

$$X - \bigcup_{i=1}^{n} C_i = X - \bigcup_{i=1}^{n} (X - U_i) = \bigcap_{i=1}^{n} (X - (X - U_i)) = \bigcap_{i=1}^{n} U_i.$$

$\bigcap_{i=1}^{n} U_i$ is open, because it's a finite intersection of open sets. Therefore, $\bigcup_{i=1}^{n} C_i$ is closed.

3. Let $\{C_a\}_{a \in A}$ be a family of closed sets. For each $a \in A$, write $C_a = X - U_a$, where U_a is open. Then

$$X - \bigcap_{a \in A} C_a = X - \bigcap_{a \in A} (X - U_a) = X - \left(X - \bigcup_{a \in A} U_a\right) = \bigcup_{a \in A} U_a.$$

$\bigcup_{a \in A} U_a$ is open, because it's a union of open sets. Therefore, $\bigcap_{a \in A} C_a$ is closed. □

It's possible to construct a topology by specifying the closed sets rather than the open sets. Then you can define the open sets to be the complements of closed sets.

Example. Give $\mathbb{R}$ the standard topology. The intervals $\left(-\dfrac{1}{n}, \dfrac{1}{n}\right)$ for $n = 1, 2, \ldots$ are open sets. However, their intersection is

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\},$$

which is a closed subset of $\mathbb{R}$.

Likewise, the intervals $\left[\frac{1}{n}, \frac{n-1}{n}\right]$ are closed sets. However, their union is

$$\bigcup_{n=1}^{\infty} \left[\frac{1}{n}, \frac{n-1}{n}\right] = (0,1),$$

which is open in $\mathbb{R}$. $\square$

Open sets in the subspace topology are intersections of the subspace with open sets in the big space. The next result says that the same is true of closed sets.

Proposition. Let X be a topological space, let Y be a subspace of X, and let $C \subset Y$. C is closed in Y if and only if $C = Y \cap D$, where D is closed in X.

Note that "closed in Y" means "closed in the *subspace topology* on Y".

Proof. Let D be closed in X. Write $D = X - U$, where U is open in X. Then

$$D \cap Y = (X - U) \cap Y = Y - (Y \cap U).$$

(If $x \in (X - U) \cap Y$, then $x \in Y$ and $x \notin U$, so $x \notin Y \cap U$. This means $x \in Y - (Y \cap U)$. Conversely, if $x \in Y - (Y \cap U)$, then $x \in Y$, and $x \notin U$ since $x \notin Y \cap U$. Therefore, $x \in X - U$, so $x \in (X - U) \cap Y$.)

$Y \cap U$ is open in Y, so $Y - (Y \cap U)$ is closed in Y.

Conversely, suppose C is closed in Y. Then $Y - C$ is open in Y, so $Y - C = Y \cap U$, where U is open in X. Hence,

$$C = Y - (Y - C) = Y - (Y \cap U) = (X - U) \cap Y.$$

$X - U$ is closed in X, so C is the intersection of a closed set in X with Y. $\square$

Proposition. Let X be a topological space, let Y be a subspace of X and let $Z \subset Y$.

1. If Z is open in Y and Y is open in X, then Z is open in X.
2. If Z is closed in Y and Y is closed in X, then Z is closed in X.

Proof. I'll prove the first assertion; the proof of the second is similar.

Suppose Z is open in Y and Y is open in X. $Z = Y \cap U$, where U is open in X. But Y is open in X, so Z is an intersection of two open sets of X — hence, it is open in X. $\square$

Definition. Let X be a topological space, and let $Y \subset X$.

1. The **closure** of Y (denoted $\overline{Y}$ or $\text{cl}(Y)$) is the intersection of all closed sets containing Y.
2. The **interior** of Y (denoted $\dot{Y}$ or $\text{int}(Y)$) is the union of all open sets contained in Y.

Notice that the closed set X contains every subset Y, so the intersection formed in taking the closure is not taken over an empty collection.

Remarks.

1. The closure of a set is closed, and the interior of a set is open.
2. The closure of a set is the smallest closed set containing the set; likewise, the interior of a set is the largest open set contained in the set.
3. A set Y is closed if and only if $Y = \overline{Y}$; a set Y is open if and only if $Y = \dot{Y}$.

Example. In the standard topology on $\mathbb{R}$, the closure of (a, b) is $[a, b]$ and the interior of $[a, b]$ is (a, b).

More interestingly, $\overline{\mathbb{Q}} = \mathbb{R}$. (A set whose closure is the whole space is said to be **dense**; thus, the rationals are dense in the reals.) □

Example. The closure of even a single point can be quite large. For example, give $\mathbb{Z}$ the topology whose open sets are the open intervals $(-n, n)$ for $n \in \mathbb{Z}^+$, together with $\emptyset$ and $\mathbb{Z}$.

What is the closure of $\{1\}$? The only closed sets which contain 1 are $\mathbb{Z}$ and $\mathbb{Z} - (-1, 1)$. Now $(-1, 1) = \{0, \}$, so the intersection of these two sets is $\mathbb{Z} - \{0\}$. Thus, the closure of a single point turns out to be everything in $\mathbb{Z}$ but 0! □

Lemma. Let X be a topological space, let Y be a subspace, and let $Z \subset Y$. Then

$$\text{cl}_Y Z = Y \cap \overline{Z}.$$

Here $\text{cl}_Y Z$ is the closure of Z in Y (with the subspace topology), whereas $\overline{Z}$ is the closure of Z in X.

Proof. $\text{cl}_Y Z$ is closed in Y, so $\text{cl}_Y Z = C \cap Y$, where C is closed in X. Now

$$Z \subset \text{cl}_Y Z = C \cap Y \subset C,$$

so C is a closed set in X containing Z.

Therefore, $\overline{Z} \subset C$, so

$$\overline{Z} \cap Y \subset C \cap Y = \text{cl}_Y Z.$$

Conversely, $\overline{Z}$ is closed in X and contains Z, so $\overline{Z} \cap Y$ is closed set in Y containing Z. Therefore, $\text{cl}_Y Z \subset \overline{Z} \cap Y$.

Hence, $\text{cl}_Y Z = Y \cap \overline{Z}$. □

Terminology.

1. If X is a topological space and $x \in X$, a **neighborhood** of x is any open set containing x.

   Some authors use "neighborhood" to mean a set containing x in its interior.

2. If X and Y are sets, I'll say that X **intersects** Y or X **meets** Y if $X \cap Y \neq \emptyset$.

Lemma. Let X be a topological space, and let $Y \subset X$. $x \in \overline{Y}$ if and only if every neighborhood of x intersects Y.

Proof. Suppose $x \in \overline{Y}$. Let U be a neighborhood of x. Suppose on the contrary that $U \cap Y = \emptyset$. Then $U \subset X - Y$, so $X - U \supset X - (X - Y) = Y$.

Now $X - U$ is a closed set containing Y, so $X - U \supset \overline{Y}$. But then $x \in \overline{Y} \subset X - U$ implies $x \notin U$, contrary to the assumption that U was a neighborhood of x.

Conversely, suppose every neighborhood of x intersects y. Let C be a closed set containing Y. I must show that $x \in C$; since C is arbitrary, this will prove that $x \in \overline{Y}$.

Suppose on the contrary that $x \notin C$. $Y \subset C$ implies $X - Y \supset X - C$ — that is, $X - C$ does not intersect Y. Further, $x \in X - C$, and $X - C$ is open. Thus, $X - C$ is a neighborhood of x which does not meet Y, contrary to assumption. Therefore, $x \in C$, which is what I wanted to show. □

Remark. If $\mathcal{B}$ is a basis for the topology on X, then $x \in \overline{Y}$ if and only if every $B \in \mathcal{B}$ intersects Y.

In a sense, the closure of a set consists of the original set together with other points that are "close to" the original set — you might picture these additional points as "boundary points". I want to make the notion of a "boundary point" precise.

Definition. Let X be a topological space, $Y \subset X$. A point $x \in X$ is a **limit point** of Y is every neighborhood of x intersects Y *in some point other than* x.

Example. In the standard topology on $\mathbb{R}$, every point of $[a, b]$ is a limit point of (a, b).

On the other hand, 0 is not a limit point of $(1, 2)$. For example, the open set $(-0.5, 0.5)$ contains 0, but does not intersect $(1, 2)$.

The set $\left\{ \dfrac{1}{n} \,\middle|\, n \in \mathbb{Z}^+ \right\}$ has 0 as its only limit point.

$\mathbb{Z}$ has no limit points in $\mathbb{R}$. About every point in $\mathbb{R} - \mathbb{Z}$, you can construct an open interval which does not contain any integers. And if $n \in \mathbb{Z}$, the open interval $(n - 0.5, n + 0.5)$ intersects $\mathbb{Z}$ in the point n.

Every real number is a limit point of $\mathbb{Q}$, since every open interval about a real number contains rational numbers. □

Definition. If Y is a subset of a topological space X, the **derived set** Y' is the set of limit points of Y.

Proposition. If Y is a subset of a topological space X, then

$$\overline{Y} = A \cup A'.$$

Proof. Suppose $x \in Y \cup Y'$.

If $x \in Y$, then $x \in \overline{Y}$.

If $x \notin Y$, then $x \in Y'$, so every neighborhood of x meets Y — in a point other than x, but this fact isn't relevant. Since every neighborhood of x meets Y, an earlier result implies that $x \in \overline{Y}$.

Conversely, suppose that $x \in \overline{Y}$. If $x \in Y$, then $x \in Y \cup Y'$, and I'm done.

Suppose then that $x \notin Y$. Since $x \in \overline{Y}$, every neighborhood of x intersects Y. But the intersection can't consist of x, since $x \notin Y$. Therefore, every neighborhood of x intersects Y in a point other than x. Hence, $x \in Y'$, and certainly $x \in Y \cup Y'$. □

Corollary. Let X be a topological space, and let $Y \subset X$. Then Y is closed if and only if $Y' \subset Y$.

Proof. If Y is closed, then $Y = \overline{Y} = Y \cup Y'$, so $y' \subset Y$.

Conversely, if $Y' \subset Y$, then $Y \cup Y' = Y$, so $\overline{Y} = Y$, and Y is closed. □

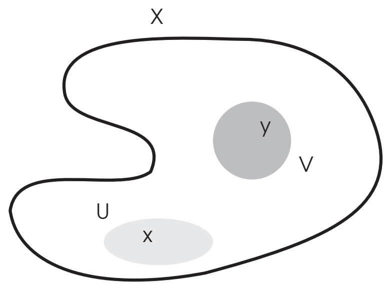
The last result is often expressed by saying that *a closed set contains all its limit points.* Thus, you often show a set is closed by taking a limit point and showing that it lies in the set.

6-17-1999

Hausdorff Spaces

If you try to study topological spaces without any conditions, the variety becomes overwhelming — there is little you can say about topological spaces *in general*. In order to say more, you have to assume more. The **Hausdorff** condition is a common assumption mathematicians make about topological spaces.

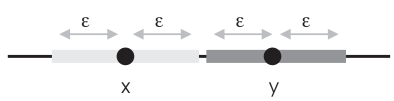
Definition. A topological space X is **Hausdorff** (or T_2)if for all $x, y \in X$, $x \neq y$, there are neighborhoods U of x and V of y such that $U \cap V = \emptyset$.



Remark. An easy induction shows that if X is Hausdorff and $x_1, \ldots, x_n \in X$, then there are neighborhoods U_1 of x_1, ..., U_n of x_n, such that the U's are pairwise disjoint.

Example. $\mathbb{R}$ is Hausdorff in the standard topology.

Let x and y be distinct points in $\mathbb{R}$. Let $\epsilon = \dfrac{1}{2}|x - y|$. Then $(x - \epsilon, x + \epsilon)$ and $(y - \epsilon, y + \epsilon)$ are disjoint open sets containing x and y, respectively.



□

Example. Consider the topology on $X = \{1, 2, 3\}$ in which the open sets are $\emptyset$, $\{1, 2\}$, and $\{1, 2, 3\}$.

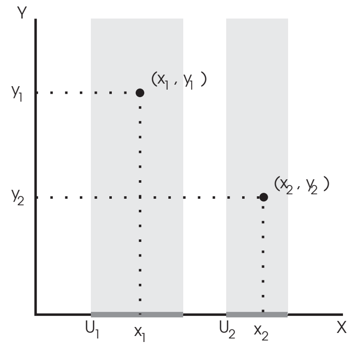
X is not Hausdorff, since it isn't possible to find disjoint neighborhoods containing 1 and 2. □

Proposition. The product of Hausdorff spaces is Hausdorff.

Proof. Let X and Y be Hausdorff spaces. Let $(x_1, y_1), (x_2, y_2) \in X \times Y$, and assume that $(x_1, y_1) \neq (x_2, y_2)$. I want to find disjoint neighborhoods in $X \times Y$ containing the points.

Since the points aren't the same, they must differ in at least one coordinate. Assume $x_1 \neq x_2$ (the argument is similar if instead $y_1 \neq y_2$.

Since X is Hausdorff, I can find disjoint neighborhoods U_1 of x_1 and U_2 of x_2. Then $U_1 \times Y$ and $U_2 \times Y$ are disjoint neighborhoods in $X \times Y$, $(x_1, y_1) \in U_1 \times Y$, and $(x_2, y_2) \in U_2 \times Y$.



Therefore, $X \times Y$ is Hausdorff. □

Proposition. A subspace of a Hausdorff space is Hausdorff.

Proof. Let X be Hausdorff, and let Y be a subspace of X. Let $y_1, y_2 \in Y$, where $y_1 \neq y_2$.

Since X is Hausdorff, there are disjoint neighborhoods U_1 of y_1 and U_2 of y_2 in X. Then $U_1 \cap Y$ is a neighborhood of y_1 in Y and $U_2 \cap Y$ is a neighborhood of y_2 in Y, and $U_1 \cap Y$ and $U_2 \cap Y$ are disjoint.

Therefore, Y is Hausdorff. □

Proposition. If X is Hausdorff and $x_1, \ldots x_n \in X$, then $\{x_1, \ldots, x_n\}$ is closed.

Proof. First, I'll show that a single point is closed. Let $x \in X$. I claim that $X - \{x\}$ is open.

Take $y \in X - \{x\}$. Since $x \neq y$, I may find disjoint neighborhoods U of x and V of y. Since V doesn't intersect U and $x \in U$, it follows that $V \subset X - \{x\}$. Thus, V is a neighborhood of y contained in $X - \{x\}$. Since y was arbitrary, $X - \{x\}$ is open.

Since a finite union of closed sets is closed, if $x_1, \ldots x_n \in X$, then $\{x_1, \ldots, x_n\}$ is closed. □

If y is a limit point of a set Y, then every neighborhood of y meets Y in a point other than y. In a Hausdorff space, every neighborhood of a limit point meets the set in *infinitely many points*.

Proposition. Let X be Hausdorff, and let $Y \subset X$. x is a limit point of Y if and only if every neighborhood of x meets Y in infinitely many points.

Proof. If every neighborhood of x meets Y in infinitely many points, then surely every neighborhood meets Y in a point other than Y. Therefore, x is a limit point of Y.

Conversely, suppose x is a limit point of Y. Let U be a neighborhood of x. I must show that U meets Y in infinitely many points.

Suppose U meets Y in only finitely many points. Then U meets $Y - \{x\}$ in only finitely many points $x_1, \ldots, x_n$. (Note that U must meet $Y - \{x\}$ in *at least one point*, since x is a limit point of Y.)

Now $X - \{x_1, \ldots, x_n\}$ is open and contains x, so $U \cap (X - \{x_1, \ldots, x_n\})$ is open and contains x. But in forming this set I've thrown out the points $x_1, \ldots, x_n$ where U meets Y, so it follows that $U \cap (X - \{x_1, \ldots, x_n\})$ does not meet Y at all! This contradicts the fact that every neighborhood of x must intersect Y.

Hence, U intersects Y in infinitely many points. □

6-21-1999

Continuous Functions

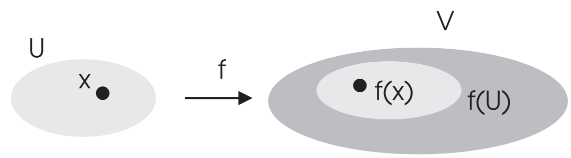
For a function $f : \mathbb{R} \to \mathbb{R}$, f is **continuous** at $x = a$ if for every $\epsilon > 0$, there is a $\delta > 0$ such that if

$$\delta > |x - a| > 0, \quad \text{then} \quad \epsilon > |f(x) - f(a)|.$$

Interpreted in terms of intervals — i.e. basic open sets in $\mathbb{R}$ — this says that for any interval I containing $f(a)$, there is an interval J containing a, such that f maps J into I.

This motivates the following definition.

Definition. Let $f : X \to Y$ be a function, where X and Y are topological spaces. f is **continuous** at $x \in X$ if for every neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.



f is **continuous** (on X) if it is continuous at every point of X.

The definition I've given is often called the *pointwise definition* of continuity. I'll give the equivalent *global form* shortly.

Example. Any function $f : \mathbb{R}^n \to \mathbb{R}^m$ that is continuous in the ϵ-δ sense is continuous according to the definition above. □

Example. Let X be a topological space. The identity function $\text{id} : X \to X$ is continuous.

To prove this, let $x \in X$ and let V be a neighborhood of $\text{id}(x) = x$. Then V is a neighborhood of x which satisfies $\text{id}(V) \subset V$. □

Example. Let X and Y be topological spaces, and let $y \in Y$. The constant function $c_y : X \to Y$ defined by $c_y(x) = y$ for all $x \in X$ is continuous.

To see this, suppose $x \in X$ and V is a neighborhood of $c_y(x) = y$. X is a neighborhood of x, and $c_y(X) = \{y\} \subset V$. Therefore, c_y is continuous at x. □

Example. Let Y be a subspace of the topological space X. The inclusion map $i : Y \hookrightarrow X$ given by $i(y) = y$ for all $y \in Y$ is continuous.

To prove this, let $y \in Y$ and let V be a neighborhood of $i(y) = y$ in X. Since V is open in X, $V \cap Y$ is open in Y, and $y \in V \cap Y$. Clearly, $i(V \cap Y) \subset V$, so $V \cap Y$ is a neighborhood of y in Y which i maps into V. Therefore, $\subset$ is continuous at y. □

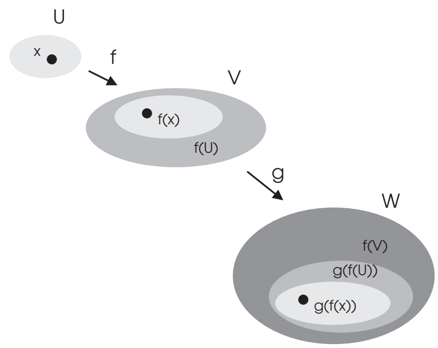
Example. The composite of continuous functions is continuous.

To be specific, suppose that X, Y, and Z are topological spaces, $f : X \to Y$ is continuous at x, and $g : Y \to Z$ is continuous at $f(x)$. Then $g \cdot f : X \to Z$ is continuous at x.

Thus, if f is continuous on X and g is continuous on Y, then $g \cdot f$ is continuous on X.

To prove the result at a point, assume that $f : X \to Y$ is continuous at x and $g : Y \to Z$ is continuous at $f(x)$. Let W be a neighborhood of $g(f(x))$ in Z. By continuity of g at $f(x)$, there is a neighborhood V of $f(x)$ such that $g(V) \subset W$.

By continuity of f at x, there is a neighborhood U of x such that $f(U) \subset V$.



Then $g(f(U)) \subset g(V) \subset W$. so $g \cdot f$ is continuous at x. □

Example. The restriction of a continuous function is continuous.

Specifically, let $f : X \to Y$ be a continuous function between topological spaces. Let U be a subspace of X. Then $f|U : U \to Y$ defined by $f|U(x) = f(x)$ for all $x \in U$ is continuous.

The result is immediate from the last two examples, since $f|U = f \cdot i$, where $i : U \hookrightarrow X$ is the inclusion map, and the composite of continuous functions is continuous. □

Example. If $f : X \to Y$ is a continuous function between topological spaces and Y is a subspace of Z, then the function $\tilde{f} : X \to Z$ defined by $\tilde{f}(x) = f(x)$ for all $x \in X$ is continuous. ($\tilde{f}$ is said to be obtained by **expanding the range**.)

This result is also easy, since $\tilde{f} = i \cdot f$, where $i : Y \hookrightarrow Z$ is the inclusion map and the composite of continuous functions is continuous. □

Next, I'll show that the pointwise definition of continuity is equivalent to the following global definition.

Proposition. Let $f : X \to Y$ be a function between topological spaces. f is continuous if and only if for every open set V in Y, $f^{-1}(V)$ is open in X.

This is expressed more concisely by saying that *the inverse image of an open set is open*.

Proof. Suppose f is continuous, and let V be open in Y. I want to show that $f^{-1}(V)$ is open in X.

Let $x \in f^{-1}(V)$. I want to find a neighborhood of x contained in $f^{-1}(V)$. Now $f(x) \in V$, so by continuity, I can find a neighborhood U of x such that $f(U) \subset V$. Then $U \subset f^{-1}(V)$, and U is the desired neighborhood of x. This proves that $f^{-1}(V)$ is open.

Conversely, suppose that the inverse image of an open set is open. I want to show that f is continuous. Let $x \in X$, and let V be a neighborhood of $f(x)$. I want to find a neighborhood U of x such that $f(U) \subset V$.

By assumption, $f^{-1}(V)$ is open. Since $f(x) \in V$, it follows that $x \in f^{-1}(V)$. Moreover, $f(f^{-1}(V)) \subset V$. Thus, $f^{-1}(V)$ is a neighborhood of x that f maps into V. Hence, f is continuous. □

I believe that the name of the next result is due to James R. Munkres; it's an enormously useful tool for constructing continuous functions.

Lemma. (**The Pasting Lemma**) Let X and Y be topological spaces, and let U and V be open sets in X

such that $X = U \cup V$. Suppose that $f : U \to Y$ and $g : V \to Y$ are continuous, and

$$f(x) = g(x) \quad \text{for all} \quad x \in U \cap V.$$

Then the function $h : X \to Y$ defined by

$$h(x) = \begin{cases} f(x) & \text{if } x \in U \\ g(x) & \text{if } x \in V \end{cases}$$

is continuous.

Proof. Let W be open in Y. Then

$$h^{-1}(W) = f^{-1}(W) \cup g^{-1}(W).$$

By continuity, $f^{-1}(W)$ is open in U; since U is open in X, $f^{-1}(W)$ is open in X.
By continuity, $g^{-1}(W)$ is open in V; since V is open in X, $g^{-1}(W)$ is open in X.
Therefore, $f^{-1}(W) \cup g^{-1}(W)$ is open in X, so $h^{-1}(W)$ is open in X. By the preceding result, h is continuous.

An analogous result holds if U and V are closed; the proof follows from the next result, which says that you can also express continuity in terms of closed sets.

Proposition. Let $f : X \to Y$ be a function between topological spaces. The following statements are equivalent:

1. f is continuous.
2. For all $U \subset X$, $f(\overline{U}) \subset \overline{f(U)}$.
3. The inverse image of a closed set is closed.

Proof. ($1 \Rightarrow 2$) Suppose f is continuous, and let $U \subset X$. Since $\overline{U} = U \cup U'$, I have

$$f(\overline{U}) = f(U \cup U') = f(U) \cup f(U').$$

Clearly, $f(U) \subset \overline{f(U)}$; I need to show that $f(U') \subset \overline{f(U)}$.
Let x be a limit point of U, and let V be a neighborhood of $f(x)$. I must show that V intersects $f(U)$. Since $f^{-1}(V)$ is a neighborhood of x and x is a limit point of U, there is a $y \in f^{-1}(V) \cap U$ such that $y \neq x$. Then

$$f(y) \in f\left(f^{-1}(V) \cap U\right) \subset f(f^{-1}(V)) \cap f(U) \subset V \cap f(U).$$

This proves that V intersects $f(U)$, so $f(x) \in \overline{f(U)}$. Thus, $f(U') \subset \overline{f(U)}$, and so $f(U) \cup f(U') \subset \overline{f(U)}$.
Therefore, $f(\overline{U}) \subset \overline{f(U)}$.

($2 \Rightarrow 3$) Suppose that for all $U \subset X$, $f(\overline{U}) \subset \overline{f(U)}$. I want to show that the inverse image of a closed set is closed.
Let C be closed in Y. I want to show that $f^{-1}(C)$ is closed in X. Now

$$f\left(\overline{f^{-1}(C)}\right) \subset \overline{f\left(f^{-1}(C)\right)} \subset \overline{C} = C,$$

where the first inclusion follows from statement 2, the second inclusion follows from $f\left(f^{-1}(C)\right) \subset C$, and the last equality follows from the fact that C is closed.
Thus, $\overline{f^{-1}(C)} \subset f^{-1}(C)$. But clearly, $f^{-1}(C) \subset \overline{f^{-1}(C)}$, so $\overline{f^{-1}(C)} = f^{-1}(C)$, and $f^{-1}(C)$ is closed.

($3 \Rightarrow 1$) Suppose the inverse image of a closed set is closed. I want to show that f is continuous.
Let V be an open subset of Y. Then $Y - V$ is closed, so by assumption $f^{-1}(Y - V)$ is closed. But

$$f^{-1}(Y - V) = f^{-1}(Y) - f^{-1}(V) = X - f^{-1}(V).$$

Therefore, $X - f^{-1}(V)$ is closed, so $f^{-1}(V)$ is open. Therefore, f is continuous. □

6-22-1999

Homeomorphisms

Definition. Let X and Y be topological spaces, and let $f : X \to Y$. f is a **homeomorphism** if f is bijective, and both f and f^{-1} are continuous.

If there is a homeomorphism between topological spaces X and Y, then X and Y are **homeomorphic**.

Homeomorphic spaces are "the same" as topological spaces, in the same way that isomorphic groups are "the same" as groups. The big problem in topology — which is much too difficult to deal with in this generality — is to classify all spaces up to homeomorphism.

The following properties are obvious:

1. The identity map of a topological space is a homeomorphism.
2. The composite of homeomorphisms is a homeomorphism.
3. The inverse of a homeomorphism is a homeomorphism.

Example. Define $f : \mathbb{R} \to \mathbb{R}$ by
$$f(x) = x + k,$$
where $k \in \mathbb{R}$. Then f is a homeomorphism.

It's clear that f is continuous, since the inverse image of an open interval is an open interval — it's just a translate of the original interval. Then the inverse function $f^{-1} : \mathbb{R} \to \mathbb{R}$ given by $f^{-1}(x) = x - k$ is also continuous, since it can be written as $f^{-1}(x) = x + (-k)$.

Define $g : \mathbb{R} \to \mathbb{R}$ by
$$g(x) = rx,$$
where $r \in \mathbb{R}$ and $r \neq 0$. Then g is a homeomorphism.

Again, it's clear that g is continuous, since the inverse image of an open interval is an open interval. Replacing r with $\dfrac{1}{r}$ shows that the inverse $g^{-1} : \mathbb{R} \to \mathbb{R}$ is continuous.

From these facts, it follows that any two nonempty open intervals in $\mathbb{R}$ (with the subspace topologies) are homeomorphic. For if (a, b) is a nonempty open interval, then:

1. $f(x) = x - a$ maps (a, b) to $(0, b - a)$.
2. $g(x) = \dfrac{x}{b - a}$ maps $(0, b - a)$ to $(0, 1)$.

f and g are homeomorphisms, by observations above. This proves that every open interval is homeomorphic to $(0, 1)$. But homeomorphism is transitive (since the composite of homeomorphisms is a homeomorphism), so any two nonempty open intervals are homeomorphic.

The function $\tan : \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) \to \mathbb{R}$ is continuous, and its inverse $\arctan : \mathbb{R} \to \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ is continuous. Therefore, $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ is homeomorphic to $\mathbb{R}$. But every nonempty open interval is homeomorphic to $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$, so it follows that every nonempty open interval is homeomorphic to $\mathbb{R}$. $\square$

Definition. Let $f : X \to Y$ be a function between topological spaces. f is **open** if f takes open sets to open sets. Likewise, f is **closed** if f takes closed sets to closed sets.

Lemma. Homeomorphisms are open maps and closed maps.

Proof. I'll do the open map case; the closed map case is similar.

Let $f : X \to Y$ be a homeomorphism, and let U be open in X. I want to show that $f(U)$ is open in Y.

Since $f : X \to Y$ is a homeomorphism, it has a continuous inverse f^{-1}. By continuity, $(f^{-1})^{-1}(U)$ is open in Y. But $(f^{-1})^{-1}(U) = f(U)$, so $f(U)$ is open in Y. □

Example. You can tell that two spaces are *not* homeomorphic if you can find a topological property that one has that the other does not. In a sense, this is a circular observation: A *topological property* is a property preserved by homeomorphisms. So this observation becomes useful as you develop a repertoire of properties that are preserved by homeomorphisms.

For example, the circle

$$S^1 = \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; x^2 + y^2 = 1\right\}$$

with the subspace topology is not homeomorphic to the interval $(0, 1)$ in $\mathbb{R}$ with the subspace topology. One reason: Removing a point from $(0, 1)$ leaves a disjoint union of two sets open in $(0, 1)$, but removing a point from S^1 does not leave a disjoint union of two sets open in S^1.

Take for granted that the assertion about S^1 is true. To show that this difference implies that the spaces aren't homeomorphic, suppose $f : (0, 1) \to S^1$ is a homeomorphism. Take $x \in (0, 1)$, so $(0, 1) - \{x\} = (0, x) \cup (x, 1)$, a disjoint union of open subsets of $(0, 1)$.

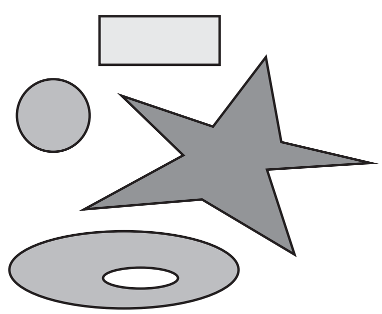
Since f is a homemorphism, $f[(0, x)]$ and $f[(x, 1)]$ are disjoint open subsets of S^1, and $S^1 - \{f(x)\} = f[(0, x)] \cup f[(x, 1)]$. This contradicts my claim about S^1, so S^1 and $(0, 1)$ aren't homeomorphic.

The assertion about S^1 will follow from results I'll prove later about **connected set**. □

Example. Removing a point from $\mathbb{R}$ leaves a disjoint union of two open sets. The same is not true when a point is removed from $\mathbb{R}^n$ for $n \geq 2$. Therefore, $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$ for $n \geq 2$.

The fact that $\mathbb{R}^m$ and $\mathbb{R}^n$ aren't homeomorphic for $m \neq n$ is a fact called **invariance of domain**; the best proofs use methods from **algebraic topology**. □

The standard informal example of homeomorphic spaces is that a coffee cup is homeomorphic to a doughnut. Popular accounts often say that two spaces are homeomorphic if you can imagine deforming one into the other with cutting, tearing, or breaking.

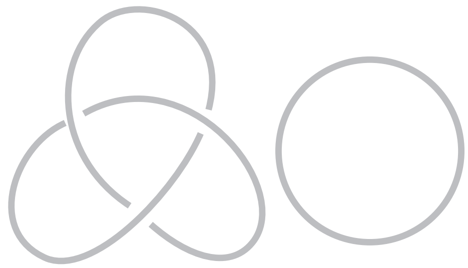


One of these things is not like the others.

In the picture above, the three solid objects are homeomorphic; the object shaped like a ring is not homeomorphic to the others. This kind of assertion (like the one about the coffee cup and the doughnut) must be taken in the right spirit. No one would set out to *prove* the assertion by writing down specific homeomorphisms!

The notion of homeomorphism does not include the idea of "deforming" one thing into another. In the first place, a deformation requires a "big space" inside of which the deformation takes place. In the second place, a deformation is something which takes places in time. Neither of these notions is part of the definition of a homeomorphism. The notion of **isotopy** comes closer to the idea of a continuous deformation of one space into another.

Homeomorphism is also independent of the **embedding** of a space in a larger space. For example, the trefoil knot on the left and the circle on the right are homeomorphic as topological spaces:



In order to get a hold of the obvious difference that exists between the two, you need to consider the *complements* of these sets in $\mathbb{R}^3$. In this case, the ambient space $\mathbb{R}^3$ is responsible for the "knottedness" of the trefoil.

The Product Topology

If X and Y are topological spaces, the **product topology** on $X \times Y$ is the topology generated by the basis consisting of all sets $U \times V$, where U is open in X and V is open in Y. This may be extended to a product of finitely many spaces in the obvious fashion. How do you define a topology on a product of infinitely many spaces?

Definition. Let $\{X_a\}_{a \in A}$ be a family of topological spaces. The **box topology** on $\prod_{a \in A} X_a$ is the topology generated by the basis consisting of sets $\prod_{a \in A} U_a$, where U_a is open in X_a.

Let $\mathcal{B}$ be the collection of sets $\prod_{a \in A} U_a$, where U_a is open in X_a. I want to show that $\mathcal{B}$ is a basis.

Let $(x_a)_{a \in A} \in \prod_{a \in A} X_a$. Since X_a is open in X_a, $\prod_{a \in A} X_a$ is an element of $\mathcal{B}$, and it obviously contains $(x_a)_{a \in A}$.

Let $(x_a)_{a \in A} \in \left(\prod_{a \in A} U_a \cap \prod_{a \in A} V_a\right)$, where U_a and V_a are open in X_a for each a. Then $U_a \times V_a$ is open in X_a, so $\prod_{a \in A}(U_a \cap V_a)$ is an element of $\mathcal{B}$. Moreover,

$$(x_a)_{a \in A} \in \prod_{a \in A}(U_a \cap V_a) \subset \left(\prod_{a \in A} U_a \cap \prod_{a \in A} V_a\right).$$

This proves that $\mathcal{B}$ is a basis.

There is another way to generate a topology on a product. I'll need some preliminary results.

Lemma. If $\{\mathcal{T}_a\}_{a \in A}$ is a family of topologies on a set X, then $\mathcal{T} = \bigcap_{a \in A} \mathcal{T}_a$ is a topology on X.

Proof. Since $\emptyset, X \in \mathcal{T}_a$ for all a, it follows that $\emptyset, X \in \mathcal{T}$.

If $\{U_i\}_{i \in I}$ is a collection of open sets in $\mathcal{T}$, then for each a, I have $U_i \in \mathcal{T}_a$ for all i. Since $\mathcal{T}_a$ is a topology, $\bigcup_{i \in I} U_i \in \mathcal{T}_a$ for all a. Hence, $\bigcup_{i \in I} U_i \in \mathcal{T}$. Thus, $\mathcal{T}$ is closed under arbitrary unions.

If $\{U_1, \ldots, U_n\}$ is a collection of open sets in $\mathcal{T}$, then for each a, I have $U_i \in \mathcal{T}_a$ for all $i = 1, \ldots, n$. Since $\mathcal{T}_a$ is a topology, $\bigcap_{i=1}^n U_i \in \mathcal{T}_a$ for all a. Hence, $\bigcap_{i=1}^n U_i \in \mathcal{T}$. Thus, $\mathcal{T}$ is closed under finite intersections.

Therefore, $\mathcal{T}$ is a topology. □

Definition. Let $\{X_a\}_{a \in A}$ be a family of topological spaces, let X be a set, and let $\{f_a : X \to X_a \mid a \in A\}$ be a family of functions. The **topology induced on X by the family $\{f_a\}_{a \in A}$** is the smallest topology on X which makes all the f_a's continuous.

There is at least one topology on X which makes all the f_a's continuous, namely the discrete topology. If I intersect *all* the topologies on X which make the f_a's continuous, I get a topology (by the last lemma) which makes all the f_a's continuous. It is clearly the smallest such topology, in the sense that it's contained in every topology which makes all the f_a's continuous.

While this construction accomplishes the goal of making the f_a's continuous, it would be nice to have a more concrete description of the open sets.

Lemma. Let $\{X_a\}_{a \in A}$ be a family of topological spaces, let X be a set, and let $\{f_a : X \to X_a \mid a \in A\}$ be a family of functions. The topology induced on X by the f_a's is the same as the topology generated by the subbasis

$$\left\{f_a^{-1}(U) \mid U \overset{\text{open}}{\subset} X_a, \quad a \in A\right\}.$$

Proof. First, this collection is a subbasis: If $x \in X$, fix $a \in A$ and consider $f_a(x) \in X_a$. Then X_a is open in X_a and $f_a^{-1}(X_a)$ is an element of the collection which contains x.

Now let $\mathcal{S}$ be the topology induced on X by the f_a's and let $\mathcal{T}$ be the topology generated by the subbasis.

If U is open in X_a and $f_a^{-1}(U)$ is an element of the subbasis, then $f_a^{-1}(U)$ is open in $\mathcal{T}$. Since U and f_a were arbitrary, this shows that $\mathcal{T}$ makes all the f_a's continuous. But $\mathcal{S}$ is the smallest topology which makes the f_a's continuous. Hence, $\mathcal{S} \subset \mathcal{T}$.

Conversely, let U be open in X_a and let $f_a^{-1}(U)$ be an element of the subbasis. I claim that $f_a^{-1}(U)$ is an open set in $\mathcal{S}$. For if not, then f_a is *not* continuous, since the inverse image under f_a of the open set U isn't open in X.

Now I know that all the elements of the subbasis are contained in $\mathcal{S}$. But $\mathcal{S}$ is a topology, so arbitrary unions of finite intersections of elements of the subbasis are also contained in $\mathcal{S}$. Therefore, $\mathcal{T} \subset \mathcal{S}$.

Hence, the two topologies are the same. □

Here's an important property of this topology.

Proposition. Let $\{X_a\}_{a \in A}$ be a family of topological spaces, let X be a set, and let $\{f_a : X \to X_a \mid a \in A\}$ be a family of functions. Give X the topology induced on X by the f_a's, and let W be a topological space. A function $g : W \to X$ is continuous if and only if $f_a \cdot g : W \to X_a$ is continuous for all a.

Proof. If g is continuous, then $f_a \cdot g : W \to X_a$ is continuous for all a, since $f_a \cdot g$ is a composite of continuous maps.

Conversely, suppose $f_a \cdot g : W \to X_a$ is continuous for all a. I want to show that g is continuous. Let U be open in X; I must show that $g^{-1}(U)$ is open in W.

To begin with, take U to be a subbasic open set. Thus, suppose $U = f_a^{-1}(V)$, where V is open in X_a. Then

$$g^{-1}(U) = g^{-1}(f_a^{-1}(V)) = (f_a \cdot g)^{-1}(V).$$

Since $f_a \cdot g$ is continuous, $(f_a \cdot g)^{-1}(V)$ is open in W.

For a general open set U, simply write U as a union of a finite intersections of subbasic open sets, then use the fact that $g^{-1}(-)$ commutes with unions and intersections. □

Now specialize to the case where $X = \prod_{a \in A} X_a$, and give X the topology induced by the family of projection maps

$$\left\{ \pi_b : \prod_{a \in A} X_a \to X_b \mid b \in A \right\}.$$

The topology generated in this way is called the **product topology**; *it is the smallest topology which makes all the projection maps continuous.*

Notice that I've already used the name "product topology" in the case of a product of finitely many spaces to mean the topology generated by the basis consisting of products of open sets from the factors. In fact, if there are finitely many factors, these two topologies are the same.

To see this, it's helpful to get an explicit description for the *basic* open sets in the product topology. A typical basis element has the form

$$\pi_{a_1}^{-1}(U_1) \cap \pi_{a_2}^{-1}(U_2) \cap \ldots \cap \pi_{a_n}^{-1}(U_n),$$

where U_i is open in X_{a_i}. Note that $\pi_a^{-1}(U) \cap \pi_a^{-1}(V) = \pi_a^{-1}(U \cap V)$, so I can combine terms with the same index. Thus, I may as well assume that a_1, a_2, ..., a_n are distinct.

In this case, the set above is

$$\prod_{a \in A} V_a, \quad \text{where} \quad V_a = \begin{cases} X_a & \text{if } a \neq a_1, a_2, \ldots, a_n \\ U_i & \text{if } a = a_i, \quad i = 1, 2, \ldots, n \end{cases}.$$

Thus, a typical basis element for the product topology is a product of sets open in the factors, where all but finitely many of these sets are the whole spaces. In the case where there are only finitely many factors, this is simply products of sets open in the factors – the same as the product topology I defined earlier.

The next result says that a function into a product is continuous if and only if its factors are continuous.

Corollary. Let $\{X_a\}_{a \in A}$ be a family of topological spaces, and give $\prod_{a \in A} X_a$ the product topology. Let W be a topological space, and let $g : W \to \prod_{a \in A} X_a$ be a function. g is continuous if and only if $\pi_a \cdot g$ is continuous for all a. □

Example. Define $f : \mathbb{R} \to \mathbb{R}^3$ by

$$f(t) = (\cos t, \sin t, t).$$

In this case, the projections are

$$(\pi_1 \cdot f)(t) = \cos t, \quad (\pi_2 \cdot f)(t) = \sin t, \quad (\pi_3 \cdot f)(t) = t.$$

Since these are continuous as functions $\mathbb{R} \to \mathbb{R}$, it follows from the corollary that f is continuous as well.

□

A final note: A basis element in the box topology is a product of sets open in the factors. Since this is *more general* than the condition for basis elements of the product topology, it follows that *the box topology is finer than the product topology.*

Metric Spaces

Definition. If X is a set, a **metric** on X is a function $d : X \times X \to \mathbb{R}$ such that:

1. $d(x, y) \geq 0$ for all $x, y \in X$; $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. (**Triangle Inequality**) For all $x, y, z \in X$,

$$d(x, y) + d(y, z) \leq d(x, z).$$

Lemma. If X is a set with a metric, the collection of open balls

$$B(x; \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$$

is a basis.

Proof. If $x \in X$, then $x \in B(x; 1)$.

Suppose $B(x_1; \epsilon_1)$ and $B(x_2; \epsilon_2)$ are open balls. Let $x \in B(x_1; \epsilon_1) \cap B(x_2; \epsilon_2)$.

Let

$$\epsilon = \min(\epsilon_1 - d(x, x_1), \epsilon_2 - d(x, x_2)).$$

Then $x \in B(x; \epsilon) \subset B(x_1; \epsilon_1) \cap B(x_2; \epsilon_2)$. Therefore, the collection of open balls forms a basis. □

Definition. If X is a set with a metric, the **metric topology** on X is the topology generated by the basis consisting of open balls $B(x; \epsilon)$, where $x \in X$ and $\epsilon > 0$. A **metric space** consists of a set X together with a metric d, where X is given the metric topology induced by d.

Remark. In generating a metric topology, it suffices to consider balls of rational radius.

Example. The usual metric on $\mathbb{R}^n$ is defined by

$$d(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2}.$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.

It is clear that $d(x, y) \geq 0$ and that $d(x, x) = 0$ for all $x, y \in \mathbb{R}^n$. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ and $d(x, y) = 0$, then

$$\left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2} = 0, \quad \text{so} \quad \sum_{i=1}^{n} (x_i - y_i)^2 = 0.$$

This is only possible if $(x_i - y_i)^2 = 0$ for all i, and this in turn implies that $x_i = y_i$ for all i. Therefore, $x = y$.

It is obvious that $d(x, y) = d(y, x)$ for all $x, y \in \mathbb{R}^n$.

Note that $d(x, y) = |x - y|$, where $|\cdot|$ is the standard norm which gives the length of a vector. Now $|u|^2 = u \cdot u$, where $\cdot$ denotes the dot product in $\mathbb{R}^n$. By standard properties of the dot product,

$$|u + v|^2 = (u + v) \cdot (u + v) = u \cdot u + 2u \cdot v + v \cdot v = |u|^2 + 2u \cdot v + |v|^2 \leq |u|^2 + 2|u||v| + |v|^2 = (|u| + |v|)^2.$$

(The inequality follows from the **Schwarz inequality** $|u \cdot v| \leq |u||v|$.) Then

$$|u + v| \leq |u| + |v|.$$

Now let $u = x - y$ and $v = y - z$. Then

$$|x - z| \le |x - y| + |y - z|, \quad \text{or} \quad d(x, z) \le d(x, y) + d(y, z).$$

Here is a proof of the Schwarz inequality in case you ahven't seen it. Given $u = (u_1, \ldots, u_n), v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, I want to show that $|u \cdot v| \le |u||v|$; I'll show that $|u \cdot v|^2 \le |u|^2|v|^2$, and the result will follow by taking square roots.

Set $A = |u|^2$, $C = |v|^2$, and $B = |u \cdot v|$. I want to show that $B^2 \le AC$.

If $A = 0$, then $u = 0$, and the result is obvious. Assume then that $A > 0$. For all $x \in \mathbb{R}$,

$$\sum_{i=1}^n (u_i x + v_i)^2 \ge 0$$
$$x^2 \sum_{i=1}^n u_i^2 + 2x \sum_{i=1}^n u_i v_i + \sum_{i=1}^n v_i^2 \ge 0$$
$$x^2 A + 2xB + C \ge 0$$

Take $x = -\dfrac{B}{A}$. The last inequality yields

$$\frac{B^2}{A^2} \cdot A - 2\frac{B}{A} \cdot B + C \ge 0$$
$$-\frac{B^2}{A} + C \ge 0$$
$$AC \ge C$$

This completes the proof of the Schwarz inequality.

Thus, the standard metric on $\mathbb{R}^n$ satisfies the axioms for a metric. Obviously, the metric topology is just the standard topology. □

Lemma. (**Comparison Lemma for Metric Topologies**) Let d and d' be metrics on X inducing topologies $\mathcal{T}$ and $\mathcal{T}'$. $\mathcal{T}$ is finer than $\mathcal{T}'$ if and only if for all $x \in X$ and $\epsilon > 0$, there is a $\delta > 0$ such that $B_d(x; \delta) \subset B_{d'}(x; \epsilon)$.

Proof. Suppose first that $\mathcal{T}' \subset \mathcal{T}$. Let $x \in X$, and let $\epsilon > 0$. $B_{d'}(x; \epsilon)$ is open in $\mathcal{T}'$, so it's open in $\mathcal{T}$. Since the open d-balls form a basis for $\mathcal{T}$, there is an open ball $B_d(x; \delta)$ such that

$$x \in B_d(x; \delta) \subset B_{d'}(x; \epsilon).$$

Conversely, suppose that for all $x \in X$ and $\epsilon > 0$, there is a $\delta > 0$ such that $B_d(x; \delta) \subset B_{d'}(x; \epsilon)$. I want to show that $\mathcal{T}' \subset \mathcal{T}$.

Let U be open in $\mathcal{T}'$. I want to show that it's open in $\mathcal{T}$. Let $x \in U$. Since the d'-balls form a basis for $\mathcal{T}'$, there is an $\epsilon > 0$ such that

$$x \in B_{d'}(x; \epsilon) \subset U.$$

By assumption, there is a $\delta > 0$ such that

$$x \in B_d(x; \delta) \subset B_{d'}(x; \epsilon).$$

Therefore, $x \in B_d(x; \delta) \subset U$.

Now $B_d(x; \delta)$ is a $\mathcal{T}$-open set containing x and contained in U. Since $x \in U$ was arbitrary, U is open in $\mathcal{T}$. Therefore, $\mathcal{T}' \subset \mathcal{T}$. □

The standard metric on $\mathbb{R}^n$ is unbounded, in the sense that you can find pairs of points which are arbitrarily far apart. However, you can always replace a metric with a *bounded* metric which gives the same topology.

Definition. If (X, d) is a metric space and $Y \subset X$, then Y is **bounded** if there is an $M \in \mathbb{R}$ such that

$$d(x, y) \le M \quad \text{for all} \quad x, y \in Y.$$

Lemma. Let X be a metric space with metric d. Define

$$\overline{d}(x, y) = \min(d(x, y), 1).$$

1. $\overline{d}$ is a metric.

2. d and $\overline{d}$ induce the same topology on X.

Proof. 1. Let $x, y \in X$. Since $d(x, y) \ge 0$, $\overline{d}(x, y) = \min(d(x, y), 1) \ge 0$, and

$$\overline{d}(x, x) = \min(d(x, x), 1) = \min(0, 1) = 0.$$

If $\overline{d}(x, y) = \min(d(x, y), 1) = 0$, then $d(x, y) = 0$, so $x = y$. This shows that the first metric axiom holds.
Since $\overline{d}(x, y) = \min(d(x, y), 1) = \min(d(y, x), 1) = \overline{d}(y, x)$, the second metric axiom holds.
To verify the third axiom, take $x, y, z \in X$. Begin by noting that if either $d(x, y) \ge 1$ or $d(y, z) \ge 1$, then $\overline{d}(x, y) = 1$ or $\overline{d}(y, z) = 1$. Therefore,

$$\overline{d}(x, y) + \overline{d}(y, z) \ge 1 \ge \overline{d}(x, z).$$

Assume that $d(x, y) < 1$ and $d(y, z) < 1$. Then

$$\overline{d}(x, y) + \overline{d}(y, z) = d(x, y) + d(y, z) \ge d(x, z) \ge \overline{d}(x, z).$$

This verifies the third axiom, so $\overline{d}$ is a metric.

2. Observe that for $0 < \epsilon < 1$, $B_d(x; \epsilon) = B_{\overline{d}}(x; \epsilon)$. The idea is to apply the Comparison Lemma, shrinking balls if necessary to make their radii less than 1.
Let $x \in X$ and let $\epsilon > 0$.
If $\epsilon < 1$, then $x \in B_d(x; \epsilon) = B_{\overline{d}}(x; \epsilon)$.
If $\epsilon \ge 1$, then

$$x \in B_d(x; 0.5) = B_{\overline{d'}}(x; 0.5) \subset B_{\overline{d}}(x; \epsilon).$$

Therefore, the d-topology is finer than the d'-topology. The other inclusion follows by simply swapping the d's and d''s. □

It follows that *boundedness* is not a topological notion, since every subset is bounded in the standard bounded metric.

Example. The **square metric** on $\mathbb{R}^n$ is given by

$$\rho(x, y) = \max(|x_1 - y_1|, \ldots, |x_n - y_n|).$$

Relative to this metric, $B(x; \epsilon)$ is an n-cube centered at x with sides of length 2ϵ.
First, I'll show that ρ is a metric. Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$.
Clearly, $\rho(x, y) \ge 0$ and $\rho(x, x) = 0$. If $\rho(x, y) = 0$, then $|x_i - y_i| = 0$ for all i, so $x = y$.
It's also obvious that $\rho(x, y) = \rho(y, x)$.

If $x, y, z \in \mathbb{R}^n$, then for each j,

$$\rho(x, y) + \rho(y, z) = \max_i\{|x_i - y_i|\} + \max_i\{|y_i - z_i|\} \geq |x_j - y_j| + |y_j - z_j| \geq |x_j - z_j|.$$

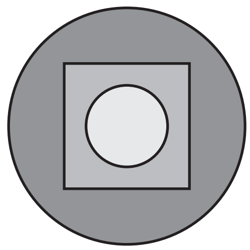
Therefore,

$$\rho(x, y) + \rho(y, z) \geq \max_j\{|x_j - z_j|\} = \rho(x, z).$$

Thus, ρ is a metric. □

Lemma. ρ induces the same topology on $\mathbb{R}^n$ as the standard metric.

Proof. The idea of the proof is depicted below.



Note that

$$\rho(x, y) = \max_i\{|x_i - y_i|\} = |x_j - y_j| = \left((x_j - y_j)^2\right)^{1/2} \leq \left(\sum_{i=1}^n (x_j - y_j)^2\right)^{1/2} = d(x, y),$$

$$d(x, y) = \left(\sum_{i=1}^n (x_j - y_j)^2\right)^{1/2} \leq \left(n \cdot \max_i\{(x_i - y_i)^2\}\right)^{1/2} = \sqrt{n}\left(\rho(x, y)^2\right)^{1/2} = \sqrt{n}\rho(x, y).$$

These inequalities may be used to get ρ-balls contained in d-balls and d-balls contained in ρ-balls; by the Comparison Lemma, this shows that the topologies are the same. □

Lemma. The square metric induces the product topology on $\mathbb{R}^n$. □

Proof. If $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, then

$$B(x; \epsilon) = (x_1 - \epsilon, x_1 + \epsilon) \times (x_2 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_n - \epsilon, x_n + \epsilon).$$

The set on the right is open in the product topology. Since the square metric-basic sets are open in the product topology, any square metric-open set is open in the product topology.

Conversely, let

$$U = (a_1, b_1) \times (a_2, b_2) \times \ldots \times (a_n, b_n).$$

It is easy to check that sets of this form comprise a basis for the product topology.

Let $x = (x_1, \ldots, x_n) \in U$, so $a_i < x_i < b_i$ for all i. Define

$$\epsilon = \min_i\{x_i - a_i, b_i - x_i\}.$$

Then

$$(x_i - \epsilon, x_i + \epsilon) \subset (a_i, b_i) \quad \text{for all} \quad i.$$

So

$$B(x; \epsilon) = \prod_i (x_i - \epsilon, x_i + \epsilon) \subset \prod_i (a_i, b_i) = U.$$

It follows that U is open in the square metric topology. Since the product topology basic sets are open in the square metric topology, any product topology open set is open in the square metric topology. $\square$

Lemma. Metric topologies are Hausdorff.

Proof. Let (X, d) be a metric space, and let x and y be distinct points of X. Let $\epsilon = d(x, y)$. Then $B\left(x; \frac{\epsilon}{2}\right)$ and $B\left(y; \frac{\epsilon}{2}\right)$ are disjoint open sets in the metric topology which contain x and y, respectively. $\square$

Lemma. If (X, d), (Y, d') are metric spaces, the ϵ-δ definition of continuity is valid. That is, a map $f : X \to Y$ is continuous at $x \in X$ if and only if for every $\epsilon > 0$, there is a $\delta > 0$ such that if $\delta > d(x, y)$ implies that $\epsilon > d'(f(x), f(y))$.

Proof. First, suppose that $f : X \to Y$ is continuous at $x \in X$. Let $\epsilon > 0$, and consider the ball $B(f(x); \epsilon)$. Since this is an open set containing $f(x)$, continuity implies that there is a $\delta > 0$ such that

$$f\left(B(x; \delta)\right) \subset B(f(x); \epsilon).$$

Now consider the conclusion to be established. Suppose $y \in X$ satisfies $\delta > d(x, y)$. Then $y \in B(x; \delta)$, so $f(y) \in f\left(B(x; \delta)\right)$. Therefore, $f(y) \in B(f(x); \epsilon)$, so $\epsilon > d'(f(x), f(y))$.

Conversely, suppose that for every $\epsilon > 0$, there is a $\delta > 0$ such that if $\delta > d(x, y)$ implies that $\epsilon > d'(f(x), f(y))$. I want to show that f is continuous.

Let $x \in X$, and let V be an open set in Y containing $f(x)$. I want to find a neighborhood U of x such that $f(U) \subset V$.

Since the ϵ-balls form a basis for the metric topology, I may find an $\epsilon > 0$ such that $f(x) \in B(f(x), \epsilon \subset V$. By assumption, there is a $\delta > 0$ such that if $\delta > d(x, y)$ implies that $\epsilon > d'(f(x), f(y))$.

Now consider the ball $B(x; \delta)$. This is an open set containing x. If $y \in B(x; \delta)$, then $\delta > d(x, y) > 0$. Therefore, $\epsilon > d'(f(x), f(y))$, so $f(y) \in B(f(x); \epsilon)$. This shows that $f(B(x; \delta)) \subset B(f(x); \epsilon) \subset V$, so f is continuous. $\square$

Definition. If X is a set, a **sequence** in X is a function $x : \mathbb{Z}^+ \to X$.

It's customary to write x_n for $x(n)$ in this situation, and to abuse terminology by referring to the collection $\{x_n\}$ as "the sequence".

Definition. Let X be a topological space. A sequence $\{x_n\}$ **converges** to a point $x \in X$ if for every neighborhood U of x, there is an integer N such that $x_n \in U$ for all $n \geq N$.

$x_n \to x$ means that $\{x_n\}$ converges to x.

Lemma. Let X be a Hausdorff space. Convergent sequences converge to *unique* points.

Proof. Let $x_n \to x$ and $x_n \to y$. I want to show that $x = y$.

Suppose $x \neq y$. Since X is Hausdorff, I can find disjoint neighborhoods U of x and V of y. Since $x_n \to x$, I can find an integer M such that $n \geq M$ implies $x_n \in U$. Since $x_n \to y$, I can find an integer N such that $n \geq N$ implies $x_n \in V$. Therefore, for $n \geq \max(M, N)$, I have $x_n \in U \cap V = \emptyset$. This is nonsense, so $x = y$. $\square$

In particular, limits of sequences are unique in metric spaces.

Lemma. (**The Sequence Lemma**) Let X be a topological space, let $Y \subset X$, and let $x \in X$. If there is a sequence $\{x_n\}$ with $x_n \in Y$ for all n and $x_n \to x$, then $x \in \overline{Y}$. The converse is true if X is a metric space.

Proof. Suppose that there is a sequence $\{x_n\}$ with $x_n \in Y$ for all n and $x_n \to x$. Let U be a neighborhood of x. Find an integer N such that $x_n \in U$ for all $n \geq N$. Obviously, U meets Y. This proves that $x \in \overline{Y}$.

Conversely, suppose that X is a metric space and $x \in \overline{Y}$. For each $n \in \mathbb{Z}^+$, the ball $B\left(x; \frac{1}{n}\right)$ meets Y, so I may choose $x_n \in B\left(x; \frac{1}{n}\right) \cap Y$. I claim that $x_n \to x$.

Let U be a neighborhood of x. Since the open balls form a basis for the metric topology, I may find $\epsilon > 0$ such that $B(x;\epsilon) \subset U$; then I may find $N \in \mathbb{Z}^+$ such that $\dfrac{1}{n} < \epsilon$, so $B\left(x; \dfrac{1}{N}\right) \subset B(x;\epsilon)$.

For all $n \geq N$, I have $\dfrac{1}{n} < \dfrac{1}{N}$, so $x_n \in B\left(x; \dfrac{1}{N}\right)$. Since $B\left(x; \dfrac{1}{N}\right) \subset U$, I have $x_n \in U$ for all $n \geq N$.

This proves that $x_n \to x$. □

Theorem. Let X be a metric space, let Y be a topological space, and let $f : X \to Y$. f is continuous if and only if $x_n \to x$ in X implies that $f(x_n) \to f(x)$ in Y.

More succinctly, continuous functions carry convergent sequences to convergent sequences.

Proof. Suppose f is continuous, and suppose $x_n \to x$ in X. Let V be a neighborhood of $f(x)$ in Y. By continuity, there is a neighborhood U of x such that $f(U) \subset V$.

Since $x_n \to x$, there is an integer N such that $x_n \in U$ for all $n \geq N$. Then $f(x_n) \in f(U) \subset V$ for all $n \geq N$. This proves that $f(x_n) \to f(x)$.

Conversely, suppose that $x_n \to x$ in X implies that $f(x_n) \to f(x)$ in Y. To show f is continuous, it will suffice to show that for all $A \subset X$, I have $f(\overline{A}) \subset \overline{f(A)}$.

Thus, take $x \in clA$. I want to show that $f(x) \in \overline{f(A)}$.

Now X is a metric space and $x \in clA$, so by the Sequence Lemma, there is a sequence of points $\{x_n\} \subset A$ with $x_n \to x$. By hypothesis, this implies that $f(x_n) \to f(x)$. Since $\{f(x_n)\}$ is a sequence in $f(A)$, the Sequence Lemma implies that $f(x) \in \overline{f(A)}$. Therefore, f is continuous. □

Definition. Let $\{f_n : X \to Y\}$ be a sequence of functions from X to Y, where Y is a metric space. $\{f_n\}$ **converges uniformly** to a function $f : X \to Y$ if for every $\epsilon > 0$, there is an integer N such that

$$d(f_n(x), f(x)) < \epsilon \quad \text{for} \quad n \geq N \quad \text{and} \quad x \in X.$$

Theorem. Let $\{f_n : X \to Y\}$ be a sequence of continuous functions from X to Y, where Y is a metric space. If $\{f_n\}$ converges uniformly to $f : X \to Y$, then f is continuous. □

This is often expressed by saying that *a uniform limit of continuous functions is continuous.*

Proof. Let $f(a) \in Y$ and let $B(f(a);\epsilon)$ be a neighborhood of $f(a)$. I want to find a neighborhood U of a such that $f(U) \subset B(f(a);\epsilon)$.

First, uniform continuity implies that there is an integer N such that

$$d(f_n(x), f(x)) < \frac{\epsilon}{3} \quad \text{for} \quad n \geq N, \quad x \in X. \qquad (*)$$

In particular,

$$d(f_N(a), f(a)) < \frac{\epsilon}{3}.$$

f_N is continuous, so there is a neighborhood U of a such that $f_N(U) \subset B\left(f_N(a), \dfrac{\epsilon}{3}\right)$. Thus,

$$d\left(f_N(x), f_N(a)\right) < \frac{\epsilon}{3} \quad \text{for all} \quad x \in U.$$

Moreover, restricting (*) to $n = N$ and $x \in U$, I have

$$d(f_N(x), f(x)) < \frac{\epsilon}{3} \quad \text{for all} \quad x \in U.$$

Therefore, the triangle inequality implies that

$$d(f(x), f(a)) \leq d(f_N(x), f(x)) + d\left(f_N(x), f_N(a)\right) + d(f_N(a), f(a)) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

for all $x \in U$.

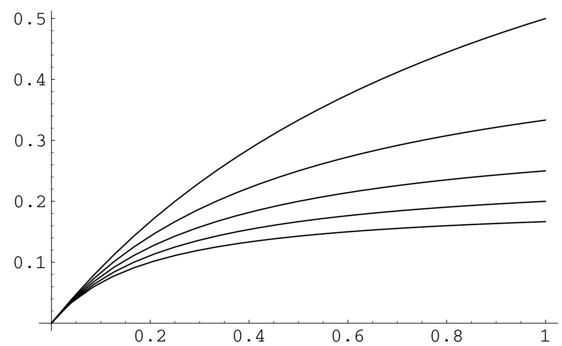
This proves that f is continuous. $\square$

Example. For $n \in \mathbb{Z}^+$, let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_n(x) = \frac{x}{nx + 1}.$$

For fixed x, $\lim_{n\to\infty} \frac{x}{nx+1} = 0$. Thus, this sequence of functions converges pointwise to the constant function 0.

The picture below shows the graphs of f_n for $n = 1, 2, 3, 4, 5$ on the interval $0 \le x \le 1$.



I will show that the convergence is uniform on the interval $0 \le x \le 1$. Thus, choose $\epsilon > 0$; I must find an integer N such that if $n \ge N$, then

$$|f_n(x)| < \epsilon.$$

Since $f_n'(x) = \frac{1}{(nx+1)^2}$, f_n is an increasing function; $f_n(1) = \frac{1}{n+1}$, so it follows that

$$|f_n(x)| < \frac{1}{n+1} \quad \text{for} \quad 0 \le x \le 1.$$

Now choose N such that $\frac{1}{N+1} < \epsilon$. Then if $n \ge N$,

$$|f_n(x)| < \frac{1}{n+1} \le \frac{1}{N+1} < \epsilon.$$

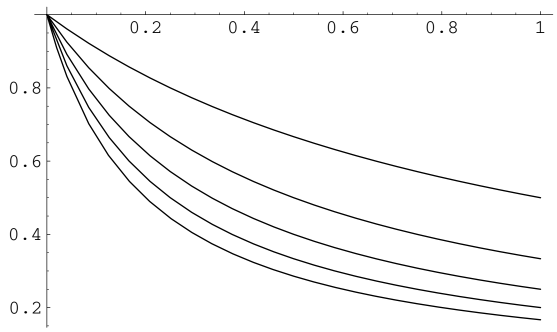
This proves that f_n converges uniformly to 0 on $0 \le x \le 1$. $\square$

Example. For $n \in \mathbb{Z}^+$, let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_n(x) = \frac{1}{nx + 1}.$$

For fixed $x \ne 0$, $\lim_{n\to\infty} \frac{1}{nx+1} = 0$. Thus, this sequence of functions converges pointwise to the constant function 0 for $x \ne 0$. It converges pointwise to 1 for $x = 0$.

The picture below shows the graphs of f_n for $n = 1, 2, 3, 4, 5$ on the interval $0 \le x \le 1$.



I will show that the convergence is *not* uniform on the interval $0 < x < 1$. In fact, I will show that there is no integer N such that if $n \ge N$, then

$$|f_n(x)| < \frac{1}{4}.$$

To see this, it suffices to note that $f_n\left(\frac{1}{n}\right) = \frac{1}{2}$, so there will always be a point in $0 < x < 1$ where the function exceeds $\frac{1}{4}$.

Therefore, $\{f_n\}$ converges pointwise, but not uniformly, to the zero function. $\square$

Quotient Spaces

You may have encountered **quotient groups** or **quotient rings** in an algebra course. In those situations, the idea was to turn a set of cosets of a subgroup or an ideal into a group or a ring.

A set of cosets is a **partition** of a group or a ring. In the topological setting, a **quotient space** arises as a topology on a set of partition elements. You can use this to construct new topological spaces by "gluing" and "pasting".

If X and Y are topological spaces, a function $f : X \to Y$ is continuous if and only if when V is open in Y, f^{-1} is open in X.

Definition. Let $f : X \to Y$, where X and Y are topological spaces. f is a **quotient map** if:

1. f is surjective.
2. V is open in Y if and only if $f^{-1}(V)$ is open in X.

Remarks.

1. Quotient maps are continuous.
2. If $f : X \to Y$ is a homeomorphism, then f is a quotient map.

   Since f is a homeomorphism, it's continuous; hence, if V is open in Y, then $f^{-1}(V)$ is open in X. Conversely, suppose $V \subset Y$ and $f^{-1}(V)$ is open in X. Since f^{-1} is continuous,

   $$V = f\left(f^{-1}(V)\right) = \left(f^{-1}\right)^{-1}\left(f^{-1}(V)\right) \quad \text{is open.}$$

3. The composite of two quotient maps is a quotient map.

In the definition of a quotient map, there are topologies on both X and Y. More often, you start with a topological space and a partition of the space, then try to topologize the set of partition elements so that the associated function (carrying an element of the space to the partition element containing it) is a quotient map.

Lemma. Let X be a topological space, let Y be a set, and let $f : X \to Y$ be a surjection. Define a collection of subsets of Y by requiring that $V \in \mathcal{T}$ if and only if $f^{-1}(V)$ is open in X. Then $\mathcal{T}$ is a topology on Y.

$\mathcal{T}$ is called the **quotient topology** on Y.

Proof. $f^{-1}(\emptyset) = \emptyset$ is open in X, so $\emptyset$ is open in Y. $f^{-1}(Y) = X$ is open in X, so Y is open in Y.

Suppose $\{V_i\}_{i \in I}$ is a collection of subsets of Y such that $f^{-1}(V_i)$ is open in X for all $i \in I$. Now

$$f^{-1}\left(\bigcup_{i \in I} V_i\right) = \bigcup_{i \in I} f^{-1}(V_i),$$

and $\bigcup_{i \in I} f^{-1}(V_i)$ is open in X. Therefore, $\bigcup_{i \in I} V_i$ is open in Y.

Finally, suppose $V_1, \ldots V_n$ is a collection of subsets of Y such that $f^{-1}(V_i)$ is open in X for $i = 1, \ldots, n$.

$$f^{-1}\left(\bigcap_{i=1}^{n} V_i\right) = \bigcap_{i=1}^{n} f^{-1}(V_i),$$

and $\bigcap_{i=1}^{n} f^{-1}(V_i)$ is open in X. Therefore, $\bigcap_{i=1}^{n} V_i$ is open in Y.

Therefore, $\mathcal{T}$ is a topology. $\square$

Example. (**Partition Topologies**) Let X be a topological space, and let $X^* = \{X_a\}_{a \in A}$ be a partition of X. Define $f : X \to X^*$ by

$$f(x) = X_a \quad \text{if} \quad x \in X_a.$$

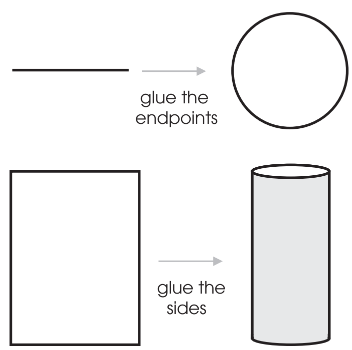
This makes sense, because every element of X lies in exactly one X_a.

f is clearly surjective; the quotient topology on X^* induced by f is called the **partition topology**.

In fact, all quotient topologies are partition topologies. For if X is a topological space and $f : X \to Y$ is surjective, the collection of inverse images $\{f^{-1}(y) \mid y \in Y\}$ forms a partition of X. The partition topology induced by this partition is the same as the quotient topology induced by f.

In applications, you often want to "glue" parts of a space together to make a new space. In this case, the partition elements are as follow. A point that isn't being glued to anything else is in a partition element by itself. Otherwise, a group of points that are being glued together form a partition element.

For example, take the interval $[0, 1]$. Partition $[0, 1]$ by defining $\{0, 1\}$ to be one partition element and defining $\{x\}$ to be a partition element for $0 < x < 1$. This has the effect of gluing the endpoints together; the quotient space is homeomorphic to a circle.



Similarly, if you take a rectangle and glue two opposite sides together, you get a cylinder. In this case, the partition elements consist of individual points not on the glued sides and pairs of corresponding points on the sides being glued.

If you twist the rectangle before gluing the opposite sides, the resulting quotient space is a Möbius strip. If you glue both pairs of opposite sides together with no twisting, you get a torus. □

Lemma. Let X and Y be topological spaces. If $f : X \to Y$ is continuous, open, and surjective, then f is a quotient map.

Proof. If V is open in Y, then $f^{-1}(V)$ is open in X by continuity.

Suppose $V \subset Y$ and $f^{-1}(V)$ is open in X. Since f is open, $f(f^{-1}(V))$ is open; since f is surjective, $V = f(f^{-1}(V))$. Therefore, V is open.

Hence, f is a quotient map. □

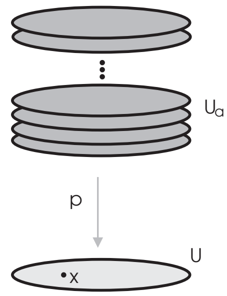
In general, quotient maps do not need to be open.

Covering spaces provide important examples of quotient maps; they are fundamental objects in **algebraic topology**.

Definition. Let $p : E \to B$ be a continuous surjection. p is a **covering map** if every point $x \in B$ has a neighborhood U such that $p^{-1}(U)$ is a disjoint union $\coprod_{a \in A} U_a$ of open sets and $p|_{U_a} : U_a \to U$ is a homeomorphism for all $a \in A$.

A neighborhood U which satisfies this condition is **evenly covered**.

Thus, $p^{-1}(U)$ looks like a stack of pancakes, where each pancake is carried homeomorphically onto U by p.



Lemma. Covering maps are open.

Proof. Let $p : E \to B$ be a covering map, and let $U \subset E$ be open. I want to show that $p(U)$ is open in B.

Take $x \in U$, so $p(x) \in p(U)$. I need to find a neighborhood V of $p(x)$ contained in $p(U)$. Let W be an evenly covered neighborhood of $p(x)$. Since $p^{-1}(W)$ is a disjoint union $\coprod_{a \in A} W_a$ of open sets homeomorphic to W and since $x \in p^{-1}(W)$, x must lie in one of the W_a's.

Thus, let W_a be a neighborhood of x mapped homeomorphically onto W by p. Then $W_a \cap U$ is open, p maps it homeomorphically onto $p(W_a \cap U)$, and $p(x) \in p(W_a \cap U)$. I claim that $p(W_a \cap U)$ is open in B.

To see this, observe that $p|_{W_a} : W_a \to p(W_a) = W$ is a homeomorphism, so the inverse $p|_{W_a}^{-1} : W \to W_a$ is continuous. Now $W_a \cap U$ is open in E, so it's open in W_a. Hence, $(p|_{W_a}^{-1})^{-1}(W_a \cap U) = p(W_a \cap U)$ is open in $p(W_a)$. But $p(W_a) = W$ is open in B, so $p(W_a \cap U)$ is open in B.

I've found a neighborhood $p(W_a \cap U)$ of x which is contained in $p(U)$. Therefore, $p(U)$ is open, and p is an open map. □

Corollary. Covering maps are quotient maps. □

Example. The map $e : \mathbb{R} \to S^1$ given by

$$e(t) = (\cos(2\pi t), \sin(2\pi t))$$

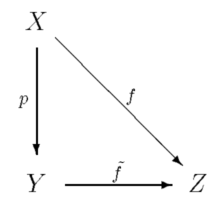
is a covering map; it wraps the real line around the circle. Each open interval of length 1 in $\mathbb{R}$ is mapped homeomorphically to the circle minus a point. □

Example. If $X \times Y$ is a product of topological spaces, the projection maps $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open, continuous, and surjective. Therefore, they are quotient maps. □

The next result tells you how to construct a map out of a quotient space; it is analogous to the results in algebra which tell you how to construct group maps out of quotient groups, or ring maps out of quotient rings.

Definition. Let $p : X \to Y$ be a quotient map. A **fiber** of p is a set $p^{-1}(y)$ for $y \in Y$.

Theorem. (**Universal Property for Quotients**) Let $p : X \to Y$ be a quotient map. Then any continuous map $f : X \to Z$ which is constant on each fiber of p factors uniquely through p; that is, there is a unique continuous map $\tilde{f} : Y \to Z$ such that $\tilde{f}p = f$.



Proof. Let $y \in Y$. Since f is constant on $p^{-1}(y)$, I may take any $x \in p^{-1}(y)$ and define

$$\tilde{f}(y) = f(x).$$

By construction, $\tilde{f}p = f$. I have to show that $\tilde{f}$ is continuous.

Let U be open in Z. I need to show $\tilde{f}^{-1}(U)$ is open in Y. By definition of the quotient topology, $\tilde{f}^{-1}(U)$ is open in Y if and only if $p^{-1}\left(\tilde{f}^{-1}(U)\right)$ is open in X. But since $\tilde{f}p = f$,

$$p^{-1}\left(\tilde{f}^{-1}(U)\right) = f^{-1}(U),$$

which is open in X by continuity of f.

This proves that $\tilde{f}$ is continuous.

To prove uniqueness, suppose that $\tilde{f}_1$ and $\tilde{f}_2$ are continuous maps $Y \to Z$ such that $\tilde{f}_1 p = f$ and $\tilde{f}_2 p = f$. Then $\tilde{f}_1 p = f = \tilde{f}_2 p$; since p is surjective, it is right-cancellable, so $\tilde{f}_1 = \tilde{f}_2$. □

If you think of the quotient topology as "gluing" or identifying the elements of each fiber, the universal property says that the quotient topology is the "smallest" way of accomplishing this "gluing", in the sense that any other map that accomplishes this "gluing" receives a map from the quotient space.

Corollary. Let $f : X \to Z$ be a continuous surjective map and suppose $p : X \to X^*$ is the quotient map obtained from the partition of X by the fibers $\{f^{-1}(z) \mid z \in Z\}$.

1. There is a continuous bijection $\tilde{f} : X^* \to Z$ such that $\tilde{f}p = f$.

2. $\tilde{f}$ is a homeomorphism if and only if f is a quotient map.

Proof. 1. p is the function which sends $x \in X$ to the partition element $f^{-1}(z) \in X^*$, where $z = f(x)$. f is trivially constant on each fiber, so the Universal Property yields a continuous map $\tilde{f} : X^* \to Z$ such that $\tilde{f}p = f$.

To be explicit, consider a fiber $f^{-1}(z) \in X^*$. Let $x \in f^{-1}(z)$. Then $p(x) = f^{-1}(z)$, so

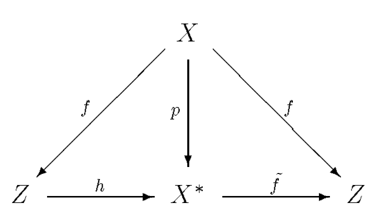
$$\tilde{f}\left(f^{-1}(z)\right) = \tilde{f}\left(p(x)\right) = f(x) = z.$$

The function $g : Z \to X^*$ given by $g(z) = f^{-1}(z)$ is obviously the inverse of $\tilde{f}$ *as a set map*, so $\tilde{f}$ is bijective. Thus, $\tilde{f}$ is a continuous bijection.

2. If $\tilde{f}$ is a homeomorphism, then $f = \tilde{f}p$ is a composite of quotient maps, so it's a quotient map.

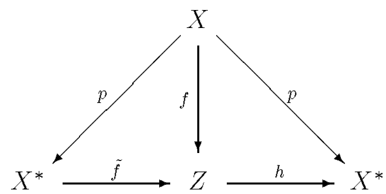
Conversely, suppose f is a quotient map. p is constant on each fiber of f, so the Universal Property applied to the quotient map f yields a continuous map $h : Z \to X^*$ such that $hf = p$.

Now $\tilde{f}hf = f$. That is, $\tilde{f}h$ makes the outer triangle below commute:



But the outer triangle clearly commutes if I replace $\tilde{f}h$ with id_Z. By the uniqueness part of the Universal Property applied to the quotient map f, it follows that $\tilde{f}h = \mathrm{id}_Z$.

Likewise, $h\tilde{f}p = p$. In other words, $h\tilde{f}$ makes the outer triangle below commute:



The outer triangle will commute if I replace $h\tilde{f}$ with id_{X^*}. By the uniqueness part of the Universal Property applied to the quotient map p, it follows that $h\tilde{f} = \mathrm{id}_{X^*}$.

Therefore, $h = \tilde{f}^{-1}$. (In fact, h is the map g defined in the first part of the proof.)

Since $\tilde{f}$ is a continuous bijection with a continuous inverse, it's a homeomorphism. □

Corollary. Let $f : X \to Z$ be a continuous surjective map and suppose $p : X \to X^*$ is the quotient map obtained from the partition of X by the fibers $\{f^{-1}(z) \mid z \in Z\}$. If Z is Hausdorff, then so is X^*.

Proof. Suppose $f^{-1}(z_1)$ and $f^{-1}(z_2)$ are distinct elements of X^*. Then $z_1 \neq z_2$, so I may find disjoint neighborhoods U_1 of z_1 and U_2 of z_2 in Z.

Look at $\tilde{f}^{-1}(U_1)$ and $\tilde{f}^{-1}(U_2)$. These are open in X^*, since $\tilde{f}$ is continuous. Since U_1 and U_2 are disjoint, $\tilde{f}^{-1}(U_1)$ and $\tilde{f}^{-1}(U_2)$ are disjoint.

Finally, $\tilde{f}\left(f^{-1}(z_1)\right) = z_1 \in U_1$ shows that $f^{-1}(z_1) \in \tilde{f}^{-1}(U_1)$, and likewise $f^{-1}(z_2) \in \tilde{f}^{-1}(U_2)$.

Thus, I have disjoint neighborhoods of $f^{-1}(z_1)$ and $f^{-1}(z_2)$ in X^*, so X^* is Hausdorff. □

Connected Spaces

Definition. A space is **connected** if it cannot be written as the disjoint union of two nonempty open sets.

A **separation** of a topological space X consists of two disjoint nonempty open sets U and V such that $X = U \cup V$.

Thus, a connected space is one that cannot be separated.

Example. If X is connected and Y is homeomorphic to X, then Y is connected. □

Example. A point is connected (considered as a subspace of an arbitrary topological space). □

Example. $\mathbb{R}$ is connected in the standard topology.

On the other hand, $\mathbb{Z}$ is not connected when considered as a subspace of $\mathbb{R}$. (The subspace topology is the discrete topology in this case.)

More generally, a set with more than one point is not connected in the discrete topology. □

Example. If $X = U \cup V$ is a separation of X, then U and V are both open and closed.

Conversely, if A is a subset of a topological space X, $A \neq \emptyset, X$, and A is both open and closed, then $X = A \cup (X - A)$ is a separation of X.

To put it another way, a space is connected if and only if there is no proper nontrivial subset which is both open and closed.

For example, any (nontrivial) partition topology is not connected. □

Proposition. Let $X = U \cup V$ be a separation of X. If $Y \subset X$ is connected, then $Y \subset U$ or $Y \subset V$.

Proof. Suppose on the contrary that $Y \cap U \neq \emptyset$ and $Y \cap V \neq \emptyset$. Then $Y \cap U$ and $Y \cap V$ are nonempty open subsets of V whose union is V, contradicting the fact that Y is connected. □

If a space X is not connected, what are the biggest connected subspaces of X?

Lemma. Let $\{U_a\}_{a \in A}$ be a collection of connected subsets of a topological space X. Suppose that $\bigcap_{a \in A} U_a \neq \emptyset$. Then $U = \bigcup_{a \in A} U_a$ is connected.

Proof. Suppose on the contrary that $U = V \cup W$ is a separation of U. Let $x \in \bigcap_{a \in A} U_a$. Then $x \in V$ or $x \in W$; without loss of generality, suppose $x \in V$.

For each a, the connected set U_a is contained in either V or W, by the preceding result. But $x \in U_a$ for all a, so all the U_a's meet V. Therefore, $U_a \subset V$ for all a, and hence $U \subset V$.

This means that $W = \emptyset$, which contradicts the fact that V and W separated U.

Therefore, U is connected. □

Theorem. Let X be a topological space.

1. Every point $x \in X$ is contained in a unique maximal connected subset C_x of X.
2. $C_x = C_y$ or $C_x \cap C_y = \emptyset$.

To say that C_x is maximal means that C_x is not properly contained in any other connected subset of X.

Proof. 1. Let $x \in X$. Consider the collection $\mathcal{C}$ of connected subsets of X which contain x. $\mathcal{C}$ is nonempty, since $\{x\} \in C$. Order C by inclusion. I'll use Zorn's Lemma to show that $\mathcal{C}$ has maximal elements.

Let $\mathcal{D}$ be a chain in $\mathcal{C}$. Then every $U \in \mathcal{D}$ contains x, so $V = \bigcup_{U \in \mathcal{D}} U$ is a union of connected sets with a point in common. By the preceding result, V is connected; since it contains x and contains every $U \in \mathcal{D}$, it is an upper bound for $\mathcal{D}$ in $\mathcal{C}$.

By Zorn's Lemma, $\mathcal{C}$ has maximal elements.

Next, I'll show that $\mathcal{C}$ has a *unique* maximal element.

If U and V are maximal elements of C, then $U \cup V$ is a connected set, since U and V are connected and have x in common. Thus, $U \cup V$ is a connected set containing x, so it's an element of $\mathcal{C}$. Since U and V are maximal and are contained in $U \cup V$, this is only possible if $U = V$.

Note that if C_x is the unique maximal element in $\mathcal{C}$, any connected set containing C_x would be a connected set containing x — hence, an element of $\mathcal{C}$ containing the maximal element C_x. Therefore, there is no connected set which *properly* contains C_x.

2. Suppose $C_x \cap C_y \neq \emptyset$. Let $z \in C_x \cap C_y$. Then $C_x \cup C_y$ is connected, since it's the union of connected sets having a point in common. Moreover, $C_x \cup C_y$ contains the maximal sets C_x and C_y. By part 1, this is impossible unless $C_x = C_y$. □

The maximal connected sets described by the theorem form a partition of X. They are called the **connected components** of X. Here is another sense in which they are maximal.

Lemma. Let X be a topological space. Every connected subset of X is contained in a component of X.

Proof. Let U be a connected subset of X. U must intersect some component of X; suppose $U \cap C \neq \emptyset$, where U is a component. Then $U \cup C$ is a union of connected sets with nonempty intersection, so it's connected. Since $C \subset U \cup C$, and since C is maximal, I must have $C = U \cup C$. Therefore, $U \subset C$. □

Theorem. The continuous image of a connected set is connected.

Proof. Let $f : X \to Y$ be continuous and surjective, and suppose X is connected. I want to show that Y is connected.

Suppose on the contrary that $Y = U \coprod V$ is a separation of Y. Since U and V are open, $f^{-1}(U)$ and $f^{-1}(V)$ are open; since U and V are disjoint, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint. Therefore, $X = f^{-1}(U) \coprod f^{-1}(V)$ is a separation, contradicting the fact that X is connected. □

The theorem will be useful once I have some connected spaces to play with.

I'm working toward showing that the product of connected spaces is connected; the preliminaries are interesting in their own right.

Lemma. Let X be a topological space, let $Y \subset X$, and let $Y = A \coprod B$ be a partition of Y. A and B separate Y if and only if neither A nor B contains an X-limit point of the other.

Proof. Suppose A and B separate Y. Then A and B are closed in Y, so

$$\overline{A} \cap Y = \operatorname{cl}_Y A = A.$$

Suppose x is an X-limit point of A, so $x \in \overline{A}$. If $x \in B$, then $x \in Y$, so $x \in \overline{A} \cap Y = A$. But then $x \in A \cap B$, contradicting the fact that A and B are disjoint.

Therefore, B contains no X-limit points of A.

A similar argument shows that A contains to X-limit points of B.

Conversely, suppose neither A nor B contains an X-limit point of the other. Let A' denote the X-limit points of A. Then $A' \cap Y$ is the set of X-limit points of A contained in Y; by assumption, none of these are in B, so any such points must be in A. That is, $A' \cap Y \subset A$. Then

$$\overline{A} \cap Y = (A \cup A') \cap Y = (A \cap Y) \cup (A' \cap Y) \subset A \cup A = A \subset \overline{A} \cap Y.$$

Therefore, $A = \overline{A} \cap Y$, so A is closed in Y.

A similar argument shows that B is closed in Y.

Since A and B partition Y, they form a separation of Y. $\square$

Lemma. If A is connected and

$$A \subset B \subset \overline{A},$$

then B is connected.

Heuristically, if you add some limit points of a connected set to the set, you get a connected set.

Proof. Suppose $B = U \coprod V$ is a separation of B. Thus, U and V are disjoint nonempty open subsets of B. Since A is connected, A lies entirely in U or V. Without loss of generality, say $A \subset U$. Taking closures in X yields $\overline{A} \subset \overline{U}$.

U and V are disjoint, and by the preceding lemma V contains no limit points of U. Thus, $\overline{A} \cap V = \emptyset$. But $B \subset \overline{A}$, so B doesn't intersect V, either. This means that V is empty, which is a contradiction.

Therefore, B is connected. $\square$

Corollary. The closure of a connected set is connected. $\square$

Corollary. Connected components of a space are closed.

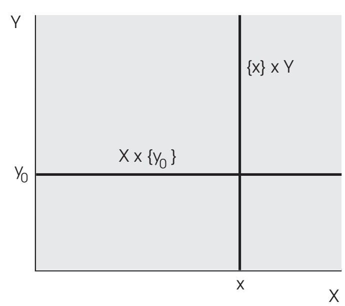
Proof. If C is a connected component of X, then $\overline{C}$ is a connected set containing C, by the previous corollary. By maximality, $C = \overline{C}$, so C is closed. $\square$

Lemma. The product of finitely many connected spaces is connected.

Proof. It suffices to prove this for two spaces, since the general result will follow by induction.

Thus, suppose X and Y are connected. I want to show that $X \times Y$ is connected. The idea is to use the fact that a union of connected sets with a point in common is connected.

To do this, picture $X \times Y$ as a plane. I'll decompose it as the union of "crosses": each "cross" consists of the union of a fixed "horizontal line" with an arbitrary "vertical line".



Fix $y_0 \in Y$. For each $x \in X$, consider the set

$$C_x = (X \times \{y_0\}) \cup (\{x\} \times Y).$$

$X \times \{y_0\}$ is connected, since it's a homeomorphic copy of X; $\{x\} \times Y$ is connected, since it's a homeomorphic copy of Y. The sets have the point (x, y_0) in common. Therefore, C_x is connected.

Next, the union of the C_x's is connected, because it's a union of connected sets having $X \times \{y_0\}$ in common. But the union of the C_x's is $X \times Y$, so $X \times Y$ is connected. $\square$

Theorem. An arbitrary product of connected spaces is connected.

Proof. First, I'll choose a point in the product. I'll construct a union of connected sets having this point in common; the earlier result on unions shows that the union is connected. Finally, I'll show that the closure of the union is the whole product; this will prove that the product is connected, by the earlier result on closures.

Let $X = \prod_{a\in A} X_a$ be a product of connected spaces. I want to show that X is connected. Let $y = (y_a)_{a\in A} \in X$.

For each finite collection $\{a_1, \ldots, a_n\}$ of indices from A, define

$$X(a_1, \ldots, a_n) = (x_a)_{a\in A} \mid x_a = y_a \quad \text{if} \quad a \neq a_1, \ldots, a_n\}.$$

Thus, a point of $X(a_1, \ldots, a_n)$ agrees with y except possibly in the $a_1, \ldots, a_n$ positions.

(Note that this doesn't mean that such an x *doesn't agree* with y in the $a_1, \ldots, a_n$ positions; it means only that such an x *does agree* with y in the other positions.)

I claim that $X(a_1, \ldots, a_n)$ is connected. I'll do this by showing that it's homeomorphic to a finite product of X_a's, which is connected by the preceding lemma.

Define $\phi : X_{a_1} \times \cdots \times X_{a_n} \to X(a_1, \ldots, a_n)$ by

$$[\phi(x_{a_1}, \ldots, x_{a_n})]_a = \begin{cases} x_{a_i} & \text{if } a = a_i \text{ for some } i \\ y_a & \text{if } a \neq a_i \text{ for any } i \end{cases}.$$

Thus, ϕ plugs $x_{a_1}, \ldots, x_{a_n}$ into the $a_1, \ldots, a_n$ components and uses y for the other components.

Define $\psi : X(a_1, \ldots, a_n) \to X_{a_1} \times \cdots \times X_{a_n}$ by

$$\psi[(x_a)] = (x_{a_1}, \ldots, x_{a_n}).$$

ϕ and ψ are easily seen to be inverses, so ϕ is bijective.

A basis element for the product topology on $X_{a_1} \times \cdots \times X_{a_n}$ is a product of open sets from the factors. A basis element for the subspace topology on $X(a_1, \ldots, a_n)$ consists of the intersection of $X(a_1, \ldots, a_n)$ with a basis element for the product $X = \prod_{a\in A} X_a$. A basis element for $X = \prod_{a\in A} X_a$ consists of a product of open sets from the factors, where at most finitely many are not the whole space.

With this description in mind, it is clear that ϕ carries a basis element for $X_{a_1} \times \cdots \times X_{a_n}$ onto a basis element for $X(a_1, \ldots, a_n)$, and ψ carries a basis element for $X(a_1, \ldots, a_n)$ onto a basis element for $X_{a_1} \times \cdots \times X_{a_n}$. Therefore, ϕ is a homeomorphism.

Since $X_{a_1} \times \cdots \times X_{a_n}$ is a finite product of connected spaces, it's connected. Therefore, $X(a_1, \ldots, a_n)$ is connected.

Since $y \in X(a_1, \ldots, a_n)$ for any finite subset $\{a_1, \ldots, a_n\} \subset A$, the union

$$X' = \bigcup \left\{ X(a_1, \ldots, a_n) \;\middle|\; \{a_1, \ldots, a_n\} \subset A \right\}$$

is a union of connected sets with a point in common, hence connected.

Finally, I claim that $\overline{X'} = X$. Let $x \in X$, and let $\prod_{a\in A} U_a$ be a neighborhood of x, where U_a is open in X_a and $U_a = X_a$ for all but finitely many a's. Let $a_1, \ldots, a_n$ be the indices for which $U_a \neq X_a$. Define $z = (z_a)_{a\in A}$ by

$$z_a = \begin{cases} x_{a_i} & \text{if } a = a_i \text{ for some } i \\ y_a & \text{if } a \neq a_i \text{ for any } i \end{cases}.$$

Then $z \in \prod_{a\in A} U_a$ and $z \in X(a_1, \ldots, a_n) \subset X'$.

Thus, every (basic) neighborhood of x intersects X'. Since $x \in X$ was arbitrary, it follows that $\overline{X'} = X$. Since X is the closure of a connected set, it's connected. $\square$

Connected Subsets of the Real Line

Proposition. $[0, 1]$ is connected in the standard topology.

Proof. Suppose $[0, 1] = U \cup V$ is a separation, so U and V are nonempty, disjoint, open sets.

The idea of the proof is to find a point on the "boundary" between U and V, then show that such a point can't be in either U or V.

Without loss of generality, suppose that $0 \in U$. Let

$$S = \{s \in [0, 1] \mid [0, s] \subset U\}.$$

Since $0 \in S$, S is nonempty. Moreover, since $0 \in U$ and U is open, there is an $\epsilon > 0$ such that $[0, \epsilon) \subset U$. Then $\left[0, \frac{\epsilon}{2}\right] \subset U$, so $\frac{\epsilon}{2} \in S$. Thus, $S \neq \{0\}$.

S is a nonempty set which is bounded above, so it has a least upper bound. Let

$$s_0 = \sup S.$$

Since $S \neq \{0\}$, $s_0 > 0$.

s_0 is the sort of "boundary point" I alluded to above. Before using it to obtain a contradiction, I'll prove two preliminary results.

Claim 1: If $s \in S$ and $0 \le r < s$, then $r \in S$.

$$[0, r] \subset [0, s] \subset U, \quad \text{since} \quad s \in S.$$

Therefore, $r \in S$.

Claim 2: If $0 \le s < s_0$, then $s \in S$ and $s \in U$.

Suppose $0 \le s < s_0$ but $s \notin S$. If $t > s$, then $t \notin S$, else $s \in S$ by Claim 1. Since no element of S is greater than s, it follows that s is an upper bound for S. But $s < s_0$, and s_0 is the *least* upper bound for S. This contradiction implies that $s \in S$; thus, $[0, s] \subset U$, so $s \in U$.

Finally, I'll obtain a contradiction by showing that s_0 can't be in S or its complement.

Suppose $s_0 \in S$, so $[0, s_0] \subset U$. Note that $s_0 < 1$ — for if $s_0 = 1$, then $[0, 1] = [0, s_0] \subset U$, contradicting the fact that V is nonempty.

Now $0 < s_0 < 1$, $s_0 \in U$, and U is open, so there is an $\epsilon > 0$ such that $(s_0 - \epsilon, s_0 + \epsilon) \subset U$. Hence,

$$\left[0, s_0 + \frac{\epsilon}{2}\right] = [0, s_0] \cup \left[s_0, s_0 + \frac{\epsilon}{2}\right] \subset U,$$

so $s_0 + \frac{\epsilon}{2} \in S$. This contradicts the fact that s_0 is the least upper bound for S.

It follows that $s_0 \notin S$. Therefore, $[0, s_0] \not\subset U$. By Claim 2, $[0, s_0) \subset U$, so $s_0 \notin U$. Therefore, $s_0 \in V$. Now $s_0 > 0$ and V is open, so there is an $\epsilon > 0$ such that $(s_0 - \epsilon, s_0] \subset V$. In particular, $s_0 - \frac{\epsilon}{2} \in V$, and this contradicts Claim 2.

It follows that there is no such separation, and hence $[0, 1]$ is connected. $\square$

Corollary. Any closed interval in $\mathbb{R}$ is connected.

Proof. Any closed interval in $\mathbb{R}$ is homeomorphic to $[0, 1]$. $\square$

Corollary. $\mathbb{R}$ is connected.

Proof.

$$\mathbb{R} = \bigcup_{n=1}^{\infty} [-n, n]$$

represents $\mathbb{R}$ as a union of connected sets having points in common (specifically, the points in $[-1, 1]$ are common to all of the intervals). By an earlier result, this proves that $\mathbb{R}$ is connected. □

Corollary. $\mathbb{R}^n$ is connected.

Proof. $\mathbb{R}^n$ is a product of connected spaces, so it's connected. □

Corollary. Any open interval in $\mathbb{R}$ is connected.

Proof. Any open interval in $\mathbb{R}$ is homeomorphic to $\mathbb{R}$. □

In fact, more is true. Let L be an ordered set with at least two elements. L is a **linear continuum** if L has the least upper bound property (every subset bounded above has a least upper bound) and if $x < y$ implies there exists z such that $x < z < y$. For example, $\mathbb{R}$ is a linear continuum. One may show that every linear continuum is connected.

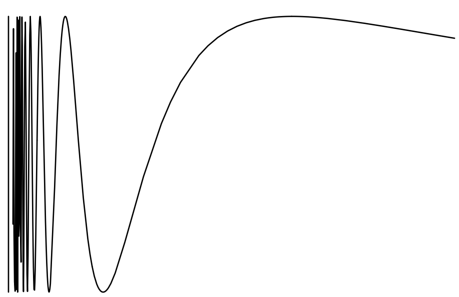
You can show that many sets in $\mathbb{R}^n$ are connected by using the results above and general stuff about connected sets.

Example. S^1 is a connected subset of $\mathbb{R}^2$.

To see this, note that the continuous function $f : \mathbb{R} \to S^1$ given by $f(t) = (\cos t, \sin t)$ maps the connected set $\mathbb{R}$ onto S^1.

More generally, if $f : \mathbb{R} \to \mathbb{R}^2$ is continuous, then the image is connected. For example, the graph of a function $y = f(x)$ defined for all $x \in \mathbb{R}$ is a connected set. □

Example. The **topologist's sine curve** consists of the graph of $y = \sin \dfrac{1}{x}$ for $0 < x \leq 1$. The **closed topologist's sine curve** consists of the union of the topologist's sine curve with the segment $\{(0, y) \mid -1 \leq y \leq 1\}$.



The topologist's sine curve is connected, because it's a continuous image of a connected set. The closed topologist's sine curve is the closure of the topologist's sine curve in $\mathbb{R}^2$, so it's connected as well. □

Here's a familiar result from calculus.

Theorem. (**Intermediate Value Theorem**) Let $f : X \to Y$ be a continuous function, where X is connected and Y is an ordered set with the order topology. Let $a, b \in X$, and let d be an element of Y between $f(a)$ and $f(b)$ (so $f(a) \leq d \leq f(b)$ or $f(b) \leq d \leq f(a)$). Then $d = f(c)$ for some $c \in X$.

Proof. If $d = f(a)$ or $d = f(b)$, then I'm done. So assume $d \neq f(a), f(b)$. Then $f(a) < d < f(b)$ or $f(b) < d < f(a)$.

$f(X)$ is connected, since X is connected. Consider the sets $f(X) \cap (-\infty, d)$ and $f(X) \cap (d, +\infty)$. These are disjoint open subsets of $f(X)$; they are nonempty, since one contains $f(a)$ and the other contains $f(b)$. Therefore, their union isn't all of $f(X)$. Hence, $f(X) \cap \{d\}$ is nonempty.

An element $f(X) \cap \{d\}$ is an element $f(c)$ for $c \in X$ such that $f(c) = d$. $\square$

7-5-1999

Path-Connectedness and Local Connectedness

Definition. Let X be a topological space. A **path** in X is a continuous function $\sigma : [0,1] \to X$. $\sigma(0)$ and $\sigma(1)$ are the **endpoints** of the path; σ is **a path from** $\sigma(0)$ **to** $\sigma(1)$.

Definition. Let X be a topological space. X is **path connected** if for all $x, y \in X$, there is a path in X from x to y.

Example. Points are path connected. □

Example. $\mathbb{R}^n$ is path connected. If $x, y \in \mathbb{R}^n$, then

$$\sigma(t) = (1-t)x + ty$$

is a path in $\mathbb{R}^n$ from x to y. □

Lemma. A path connected space is connected.

Proof. Suppose X is path connected, and suppose that $X = U \cup V$ is a separation of X. Let $x \in U$ and let $y \in V$. By assumption there is a path $\sigma : [0,1] \to X$ such that $\sigma(0) = x$ and $\sigma(1) = y$.

$\sigma^{-1}(U)$ and $\sigma^{-1}(V)$ are disjoint open sets in $[0,1]$, and $[0,1] = \sigma^{-1}(U) \cup \sigma^{-1}(V)$. This contradicts the fact that $[0,1]$ is connected.

Therefore, X is connected. □

Example. The **topologist's sine curve** S consists of the graph of $y = \sin \dfrac{1}{x}$ for $0 < x \le 1$.

The topologist's sine curve is connected, since it's a continuous image of the connected set $(0,1]$. The space $T = S \cup \{(0,0)\}$ is also connected, since it's the union of S with one of its limit points in $\mathbb{R}^2$. However, T is not path connected.

To prove this, I'll show that the point $(0,0)$ cannot be connected by a path to the point $(1, \sin 1)$.

Suppose on the contrary that $\sigma : [0,1] \to \overline{S}$ is a path from $(0,0)$ to $(1, \sin 1)$. Notice that $(0,0)$ is closed in T, so $\sigma^{-1}((0,0))$ is a closed subset of $[0,1]$. I will show that $\sigma^{-1}((0,0))$ is also open in $[0,1]$; this will prove that $\sigma^{-1}((0,0)) = [0,1]$.

Let $x \in \sigma^{-1}((0,0))$. Let V be the open rectangle $(-0,5.0.5) \times (-0.5, 0.5)$. The intersection of V with $\overline{S}$ is an open set in $\overline{S}$ containing $(0,0)$ — that is, a neighborhood of $(0,0)$ in $\overline{S}$. By continuity, I may find an open interval (a,b) in $\mathbb{R}$ containing x such that

$$\sigma\left([0,1] \cap (a,b)\right) \subset V \cap \overline{S}.$$

The intersection $[0,1] \cap (a,b)$ is an open interval or a half-open interval, so it's connected. Therefore, $\sigma\left([0,1] \cap (a,b)\right)$ is connected. $V \cap \overline{S}$ is a disjoint union of open arcs in S with the point $(0,0)$. Hence, $\sigma\left([0,1] \cap (a,b)\right)$ must map entirely into one of these components.

However, I know the point $x \in [0,1] \cap (a,b)$ maps to $(0,0)$. Therefore,

$$\sigma\left([0,1] \cap (a,b)\right) = (0,0).$$

Thus, $[0,1] \cap (a,b)$ is a neighborhood of x contained in $\sigma^{-1}((0,0))$. This proves that $\sigma^{-1}((0,0))$ is open.

Since $\sigma^{-1}((0,0))$ is a nonempty, open, and closed, and since it's a subset of the connected set $[0,1]$, it must be all of $[0,1]$. Thus, $\sigma([0,1]) = (0,0)$, contradicting the fact that σ is a path from $(0,0)$ to $(1, \sin 1)$.

It follows that there is no such path, and therefore T isn't path connected. □

Connected components were defined as the maximal connected subsets of a space; their existence followed from Zorn's Lemma. The **path components** of a space are defined in a similar fashion. I need a lemma which is analogous to one I prove for connected sets.

Lemma. Let $\{U_a\}_{a\in A}$ be a collection of path connected subsets of a topological space X. Suppose that $\bigcap_{a\in A} U_a \neq \emptyset$. Then $U = \bigcup_{a\in A} U_a$ is path connected.

Proof. Let $y, z \in U$. I must show that y and z can be joined by a path in U.

Suppose $y \in U_a$ and $z \in U_b$. Since $x \in U_a$ and U_a is path connected, x and y can be joined by a path in U_a. Since $x \in U_b$ and U_b is path connected, x and z can be joined by a path in U_b.

Concatenating the two paths produces a path in $U_a \cup U_b \subset U$ joining y and z. Therefore, U is path connected. □

Theorem. Let X be a topological space.

1. Every point $x \in X$ is contained in a unique maximal path connected subset C_x of X.

2. $C_x = C_y$ or $C_x \cap C_y = \emptyset$. □

To say that C_x is maximal means that C_x is not properly contained in any other path connected subset of X.

The proof will be omitted; it is essentially the same as the result for connected components. You can verify that the connected components proof required only the lemma on unions of connected sets with nonempty intersection; I just proved the analogous lemma for path connected sets above.

Definition. Let X be a topological space. The **path components** of X are the maximal path connected subsets of X.

Definition. Let X be a topological space.

1. X is **locally connected** if for every $x \in X$ and every neighborhood U of x, there is a connected neighborhood V of x such that $V \subset U$.

2. X is **locally path connected** if for every $x \in X$ and every neighborhood U of x, there is a path connected neighborhood V of x such that $V \subset U$.

Example. Since a path connected neighborhood is a connected neighborhood, every locally path connected space is locally connected. □

Example. $\mathbb{R}^n$ is locally path connected, since a ball $B(x;\epsilon)$ is path connected. □

Example. Let S denote the topologist's sine curve:

$$S = \left\{ \left(x, \sin \frac{1}{x}\right) \;\middle|\; 0 < x \leq 1 \right\}.$$

Let $T = S \cup \{(0,0)\}$. T is not locally connected; a neighborhood of $(0,0)$ consists of $(0,0)$ together with a disjoint union of open arcs, which is not a connected set. □

Theorem. A space is locally connected if and only if the components of any open subset are open.

In particular, the components of a locally connected space are open.

Proof. Suppose X is connected and $U \subset X$ is open. Let C be a component of U. I want to show C is open.

Let $x \in C$. By local connectedness, there is a connected neighborhood V of x such that $x \in V \subset U$. Now $V \cup C$ is a union of connected sets having the point x in common; since C is maximal, $V \cup C = C$, so $V \subset C$.

Since every point of C has a neighborhood contained in C, C is open.

Conversely, suppose that components of open subsets are open. Let $x \in X$, and let U be a neighborhood of x. I must find a connected neighborhood V of x such that $x \in V \subset U$. Take V to be the connected component of U which contains x. By assumption, V is open, and $x \in V \subset U$.

Hence, X is locally connected. $\square$

An essentially identical argument proves the following.

Theorem. A space is locally path connected if and only if the path components of any open set are open. $\square$

In particular, the path components of a locally path connected space are open.

The relationship between components and path components is described by the following result.

Proposition. Let X be a topological space.

1. Every path component is a subset of a component.

2. If X is locally path connected, then the path components and components coincide.

Proof. 1. A path component is path connected, and path connected sets are connected. Since every connected set is contained in a component (i.e. a maximal connected set), every path component is contained in a component.

2. Suppose X is locally path connected. Let C be a component. I have to show that C is a path component.

Let P be any path component contained in C. (For instance, let $x \in C$, and take P to be the path component containing x.) Let P' be the union of the path components of C other than P.

Since X is locally path connected, P is open. X is locally connected, so C is open; again, since X is locally path connected, the path components of the open set C are open. Hence, P' is open, since it's a union of open sets.

Now $C = P \cup P'$ is a disjoint union of open sets. Since C is connected, P and P' can't be nonempty; since $P \neq \emptyset$, I have $P' = \emptyset$. Thus, C has no path components other than P, which means that $C = P$. $\square$

7-11-1999

Compact Spaces

In the real line, compactness is equivalent to being closed and bounded. The Heine-Borel Theorem says that a closed and bounded subset B of $\mathbb{R}$ has the property that if $\{U_i\}_{i\in I}$ is a collection of open intervals whose union contains B (i.e., the collection **covers** B), then some finite subcollection of the U's also covers B. This is the appropriate way to generalize "closed and bounded" to arbitary spaces.

Definition.

1. A topological space X is **covered** by a collection $\{U_i\}_{i\in I}$ of subsets if $X = \bigcup_{i\in I} U_i$.
2. If X is a topological space, $Y \subset X$, and $\{U_i\}_{i\in I}$ is a collection of subsets of X, then $\{U_i\}_{i\in I}$ **covers** Y if $Y \subset \bigcup_{i\in I} U_i$.
3. If $\{U_i\}_{i\in I}$ covers X and each U_i is open, then $\{U_i\}_{i\in I}$ is an **open cover** of X.
4. If a subcollection of a covering also covers the space, the subcollection is a **subcovering**.

Example. The collection

$$\{(n, n+1) \mid n \in \mathbb{Z}\} \cup \left\{\left(n - \frac{1}{2}, n + \frac{1}{2}\right) \;\middle|\; n \in \mathbb{Z}\right\}$$

is an open cover of $\mathbb{R}$.

Every point of $\mathbb{R} - \mathbb{Z}$ is contained in an interval of the form $(n, n+1)$ for some $n \in \mathbb{Z}$. Each integer n is contained in $\left(n - \frac{1}{2}, n + \frac{1}{2}\right)$. Thus, the collection covers $\mathbb{R}$; since each element of the collection is open, it is an open cover.

The collection of open intervals with rational endpoints is also an open cover of $\mathbb{R}$.

More generally, the collection of open balls with rational centers and rational radii forms an open cover of $\mathbb{R}^n$. □

Definition. A space X is **compact** if for every open cover $\{U_i\}_{i\in I}$ of X, some finite subcollection $\{U_1, \ldots, U_n\}$ also covers X.

To say it another way, a space is compact if every covering has a finite subcovering.

It is clear that compactness is preserved by homeomorphisms: If X is compact and X is homeomorphic to Y, then Y is compact.

Example. Any finite set of points (with any topology) is compact, since any open cover can contain at most finitely many (distinct) open sets. □

Example. $\mathbb{R}$ is not compact.

Consider the following open cover of $\mathbb{R}$:

$$\{(n, n+1) \mid n \in \mathbb{Z}\} \cup \left\{\left(n - \frac{1}{2}, n + \frac{1}{2}\right) \;\middle|\; n \in \mathbb{Z}\right\}.$$

Each integer n is contained in exactly one element of the cover, namely $\left(n - \frac{1}{2}, n + \frac{1}{2}\right)$. Therefore, any finite subcollection contains at most finitely many integers, so no finite subcollection can cover $\mathbb{R}$.

Since every open interval (a, b) is homeomorphic to $\mathbb{R}$, open intervals aren't compact.

On the other hand, I'll show later that a closed interval $[a, b]$ is compact. □

The following condition is equivalent to compactness; it is extremely important in analysis.

Definition. Let X be a topological space. A collection of subsets $\mathcal{C}$ of X satisfies the **finite intersection condition** if the intersection of any finite subcollection of $\mathcal{C}$ is nonempty.

In other words, if $C_1, \ldots, C_n \in \mathcal{C}$, then $\bigcap_{i=1}^{n} C_i \neq \emptyset$.

Theorem. Let X be a topological space. X is compact if and only if every collection of closed sets satisfying the finite intersection condition has nonempty intersection.

Proof. Suppose X is compact, and let $\{C_i\}_{i \in I}$ be a collection of closed subsets of X such that every finite subcollection has nonempty intersection. I want to show that $\bigcap_{i \in I} C_i \neq \emptyset$.

Suppose on the contrary that $\bigcap_{i \in I} C_i = \emptyset$. Then

$$X = X - \bigcap_{i \in I} C_i = \bigcup_{i \in I} (X - C_i).$$

Therefore, $\{X - C_i\}_{i \in I}$ is an open cover of X. By compactness, some finite subollection $\{X - C_1, \ldots, X - C_n\}$ covers X. Thus,

$$X = \bigcup_{i=1}^{n} (X - C_i) = X - \bigcap_{i=1}^{n} C_i.$$

Hence, $\bigcap_{i=1}^{n} C_i = \emptyset$, contrary to assumption.

Therefore, $\bigcap_{i \in I} C_i \neq \emptyset$.

Conversely, suppose that every collection of closed subsets of X satisfying the finite intersection condition has nonempty intersection. Let $\{U_i\}_{i \in I}$ be an open cover of X. I want to show that $\{U_i\}_{i \in I}$ has a finite subcover.

Suppose on the contrary that no finite subcollection of $\{U_i\}_{i \in I}$ covers X. Consider the complements $\{X - U_i\}_{i \in I}$. This is a collection of closed sets. If $\{X - U_1, \ldots, X - U_n\}$ is a finite subcollection, then

$$\bigcap_{i=1}^{n} (X - U_i) = X - \bigcup_{i=1}^{n} U_i \neq \emptyset.$$

For $X \neq \bigcup_{i=1}^{n} U_i$, since no finite subcollection of $\{U_i\}_{i \in I}$ covers X.

Thus, every finite subcollection of $\{X - U_i\}_{i \in I}$ has nonempty intersection. By assumption, $\{X - U_i\}_{i \in I}$ has nonempty intersection. Therefore,

$$X - \bigcup_{i \in I} U_i = \bigcap_{i \in I} (X - U_i) \neq \emptyset.$$

This contradicts the fact that $\{U_i\}_{i \in I}$ covers X.

Hence, some finite subcollection of $\{U_i\}_{i \in I}$ covers X, and X is compact. □

Thus, the intersection condition is really just a translation of the definition of compactness from open to closed sets. In any situation, you can choose the version that is easier to apply.

The results that follow describe how compactness behaves in connection with other topological concepts.

Lemma. Let X be a topological space, and let $Y \subset X$. Y is compact if and only if every open cover of Y has a finite subcover.

Reminder: to say that $\{U_i\}_{i\in I}$ is an open cover of Y *as a subset of* X means that each U_i is open in X, and $Y \subset \bigcup_{i\in I} U_i$. That is, the U_i's are subsets of X, and not necessarily subsets of Y.

Proof. Suppose Y is compact. Let $\{U_i\}_{i\in I}$ be an open cover of Y by sets open in X. Then $\{U_i \cap Y\}_{i\in I}$ is an open cover of Y by sets open in Y. By compactness, some finite subcollection $\{U_1 \cap Y, \ldots, U_n \cap Y\}$ covers Y. Then $\{U_1, \ldots, U_n\}$ covers Y.

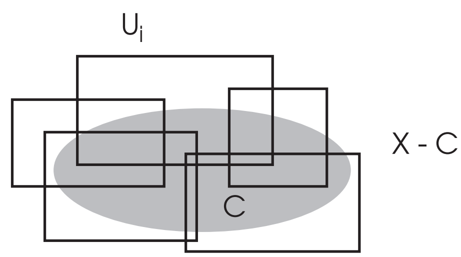
Conversely, suppose that every open cover of Y has a finite subcover. Let $\{U_i\}_{i\in I}$ be an open cover of Y by sets open in Y. For each i, $U_i = Y \cap V_i$, where V_i is open in X.

Now $\{V_i\}_{i\in I}$ is an open cover of Y by sets open in X, so by assumption a finite subcollection $\{V_1, \ldots, V_n\}$ covers Y. Then $\{Y \cap V_1, \ldots, Y \cap V_n\} = \{U_1, \ldots, U_n\}$ covers Y. Therefore, Y is compact. □

One way of putting this is: Compactness is intrinsic to the space. That is, to say Y is *compact* (in the sense of the "open cover" property) has the same meaning whether Y is considered as a subspace of another space or as a space in its own right. In contrast, whether a set is closed or not depends on the topology of the ambient space.

Theorem. A closed subset of a compact space is compact.

Proof. Let X be a compact topological space, and let C be a closed subset of X. Let $\{U_i\}_{i\in I}$ be an open cover of C, and consider the collection $\{U_i\}_{i\in I} \cup \{X - C\}$. This is an open cover of X, so by compactness it has a finite subcover.



If $X - C$ occurs in this subcover, throw it out. The remaining sets $\{U_1, \ldots, U_n\}$ cover C. Thus, I've found a finite subcover of C, so C is compact. □

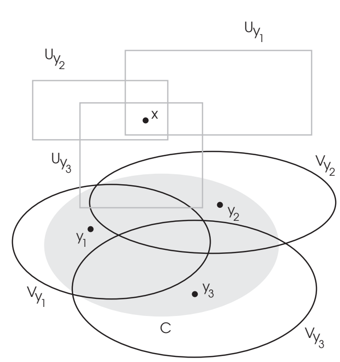
Theorem. A compact subset of a Hausdorff space is closed.

Proof. Let X be a Hausdorff space, and let C be a compact subset of X. I'll show that the complement of C is open.

Let $x \in X - C$. I want to find a neighborhood of x that is contained in $X - C$.

Since X is Hausdorff, for each $y \in C$, there are disjoint neighborhoods U_y of x and V_y of y. Now $\{V_y\}_{y\in Y}$ is an open cover of C, so by compactness there is a finite subcover $\{V_{y_1}, \ldots, V_{y_n}\}$. Let

$$U = \bigcap_{i=1}^{n} U_{y_i}.$$



If $z \in U$, then $z \in U_{y_i}$ for $i = 1, \ldots, n$. Hence, $z \notin V_{y_i}$ for $i = 1, \ldots, n$. But $\{V_{y_1}, \ldots, V_{y_n}\}$ cover C, so $z \notin C$, i.e. $z \in X - C$. Thus, U is a neighborhood of x contained in $X - C$. Hence, $X - C$ is open, so C is closed. □

Remark. In the course of the proof, I showed that in a Hausdorff space, a compact set (C) and a point in the complement (x) may be separated by disjoint open sets. □

Theorem. The continuous image of a compact space is compact.

Proof. Let X be a compact topological space, let Y be a topological space, and let $f : X \to Y$ be a surjective continuous function. I want to show that Y is compact.

Let $\{U_i\}_{i \in I}$ be an open cover of Y. Then $\{f^{-1}(U_i)\}_{i \in I}$ is an open cover of X, so by compactness I may find a finite subcover $\{f^{-1}(U_1), \ldots, f^{-1}(U_n)\}$.

Now if $y \in Y$, then $y = f(x)$ for some $x \in X$. Find $i \in \{1, \ldots, n\}$ such that $x \in f^{-1}(U_i)$. Then $y = f(x) \in U_i$. This shows that $\{U_1, \ldots, U_n\}$ covers Y.

Hence, Y is compact. □

The proof of the following theorem makes nice use of the last three results.

Theorem. Let X and Y be topological spaces, where X is compact and Y is Hausdorff. Let $f : X \to Y$ be a continuous bijection. Then f is a homeomorphism.

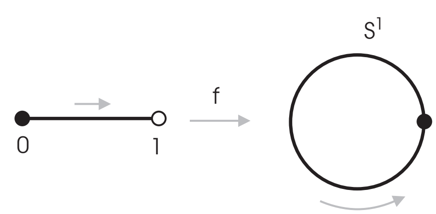
Proof. I need to show that f^{-1} is continuous. I will show that the inverse image of a closed set is closed.

Let C be closed in X; I want to show that $(f^{-1})^{-1}(C) = f(C)$ is closed in Y. Since X is compact and C is closed, C is compact. Since f is continuous and C is compact, $f(C)$ is compact. Since Y is Hausdorff and $f(C)$ is compact, $f(C)$ is closed.

Therefore, f^{-1} is continuous, and f is a homeomorphism. □

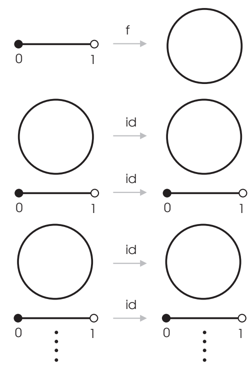
Example. Define $f : [0, 1) \to S^1$ by

$$f(t) = (\cos(2\pi t), \sin(2\pi t)).$$



f is a continuous bijection, and S^1 is surely Hausdorff. However, f is not a homeomorphism. Intuitively, f^{-1} "unwraps" the circle onto the interval; it fails to be continuous at the point $(1, 0)$, where the circle needs to be "cut" in order to do the unwrapping. This is not continuous, because points on either side of the cut point which start off close together wind up far apart, at opposite ends of $[0, 1)$.

Let X be the union of countably many copies of $[0, 1)$ and countably many copies of S^1. Construct a continuous bijection of X to itself by defining the map to be the "wrapping map" f from one of the $[0, 1)$'s to one of the S^1's and the identity maps $\mathrm{id} : [0, 1) \to [0, 1)$ and $\mathrm{id} : S^1 \to S^1$ on all the other pieces.

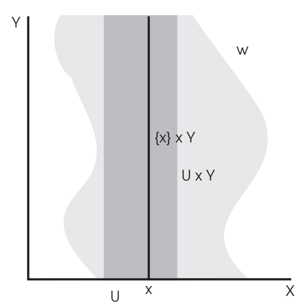


The inverse fails to be continuous for the same reason that f is not continuous.

This gives an example of a continuous bijection from a Hausdorff space to itself which is not a homeomorphism. □

The following result can be used to show that a *finite* product of compact spaces is compact; it's of interest in its own right.

Theorem. (**The Tube Lemma**) Let X and Y be topological spaces, and suppose Y is compact. Let $x \in X$, and let $W \subset X \times Y$ be an open set containing $\{x\} \times Y$. Then there is a neighborhood U of x such that $U \times Y \subset W$.



Proof. For each $y \in Y$, there is a neighborhood $U_y \times V_y$ of (x, y) such that U_y is a neighborhood of x in X, V_y is a neighborhood of y in Y, and $U_y \times V_y \subset W$. It follows that $\{U_y \times V_y\}_{y \in Y}$ is an open cover of $\{x\} \times Y$.

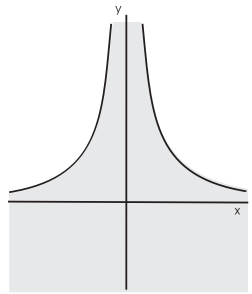
Now $\{x\} \times Y$ is compact, since it's homeomorphic to Y. Therefore, there is a finite subcollection $\{U_{u_1} \times V_{y_1}, \ldots U_{y_n} \times V_{y_n}\}$ which covers $\{x\} \times Y$.

Let $U = \bigcap_{i=1}^n U_{y_i}$. U is an open set containing x. If $(z, y) \in U \times Y$, then $y \in V_{y_i}$ for some $i \in \{1, \ldots, n\}$. So $(z, y) \in U \times V_{y_i} \subset W$, and hence $U \times Y \subset W$. □

Example.

$$W = \{(0, y) \mid y \in \mathbb{R}\} \cup \left\{(x, y) \mid y < \left|\frac{1}{x}\right|, \quad x \neq 0\right\}$$

is an open subset of $\mathbb{R}^2$ which contains the y-axis (i.e. $\{0\} \times \mathbb{R}$.



However, there is no open subset U of $\mathbb{R}$ such that

$$\{(0, y) \mid y \in \mathbb{R}\} \subset U \times \mathbb{R} \subset W.$$

The Tube Lemma does not apply because $\mathbb{R}$ is not compact. $\square$

Lemma. Let X be a topological space. Let $\mathcal{D}$ be a collection of subsets of X which is maximal with respect to satisfying the finite intersection condition. Then:

1. $\mathcal{D}$ is closed under finite intersections.

2. If $C \subset X$ and C meets every element of $\mathcal{D}$, then $C \in \mathcal{D}$.

Proof. 1. Suppose $D_1, \ldots, D_n \in \mathcal{D}$. I must show that $D = \bigcap_{i=1}^n D_i \in \mathcal{D}$.

Consider the set $\mathcal{D}' = \mathcal{D} \cup \{D\}$. I claim that $\mathcal{D}'$ satisfies the finite intersection condition.

Take $E_1, \ldots, E_m \in \mathcal{D}'$. If $E_1, \ldots, E_m \in \mathcal{D}$, then $\bigcap_{i=1}^m E_i \neq \emptyset$, since $\mathcal{D}$ satisfies the finite intersection condition.

Otherwise, one of the E's is D; without loss of generality, say $E_1 = D$ and $E_2, \ldots, E_m \in \mathcal{D}$. Then

$$\bigcap_{i=1}^m E_i = D \cap (E_2 \cap \cdots \cap E_m) = \bigcap_{i=1}^n D_i \cap (E_2 \cap \cdots \cap E_m) \neq \emptyset,$$

since this is a finite intersection of elements of $\mathcal{D}$.

Now $\mathcal{D}'$ satisfies the finite intersection condition and it contains $\mathcal{D}$. By maximality of $\mathcal{D}$, $\mathcal{D}' = \mathcal{D}$, which means that $D \in \mathcal{D}$.

2. Suppose $C \subset X$ and C meets every element of $\mathcal{D}$. I want to show that $C \in \mathcal{D}$.

Consider the set $\mathcal{D}' = \mathcal{D} \cup \{C\}$. I claim that $\mathcal{D}'$ satisfies the finite intersection condition.

Take $E_1, \ldots, E_m \in \mathcal{D}'$. If $E_1, \ldots, E_m \in \mathcal{D}$, then $\bigcap_{i=1}^m E_i \neq \emptyset$, since $\mathcal{D}$ satisfies the finite intersection condition.

Otherwise, one of the E's is C; without loss of generality, say $E_1 = C$ and $E_2, \ldots, E_m \in \mathcal{D}$. Now $E_2 \cap \cdots \cap E_m \in \mathcal{D}$ by part 1, so

$$C \cap (E_2 \cap \cdots \cap E_m) \neq \emptyset,$$

since C meets every element of $\mathcal{D}$.

Now $\mathcal{D}'$ satisfies the finite intersection condition and it contains $\mathcal{D}$. By maximality of $\mathcal{D}$, $\mathcal{D}' = \mathcal{D}$, which means that $C \in \mathcal{D}$. $\square$

All of the proofs of Tychonoff's theorem are sophisticated; it's a difficult and important result. Lang [1] credits the following proof to Nicholas Bourbaki, the *nom-de-plume* of a group of mathematicians who wrote an extremely influential series of expository monographs.

Theorem. (**Tychonoff**) Let $\{X_a\}_{a\in A}$ be a family of compact topological spaces. Then $X = \prod_{a\in A} X_a$ is compact.

Proof. Let $\mathcal{C}$ be a collection of closed subsets of X satisfying the finite intersection condition. I need to show that $\bigcap_{C\in\mathcal{C}} C \neq \emptyset$.

Step 1. Consider the set Γ of collections of subsets of X (closed or otherwise) which satisfy the finite intersection condition and contain $\mathcal{C}$. I claim that Γ has maximal elements.

Order Γ by inclusion.

Let Γ' be a chain in Γ. Let

$$\mathcal{F} = \bigcup_{\mathcal{G}\in\Gamma'} \mathcal{G}.$$

Since each $\mathcal{G} \in \Gamma'$ contains $\mathcal{C}$, $\mathcal{F}$ also contains $\mathcal{C}$. I claim that $\mathcal{F}$ satisfies the finite intersection condition.

Let $F_1, \ldots, F_n \in \mathcal{F}$. Suppose $F_1 \in \mathcal{G}_1, \ldots, F_n \in \mathcal{G}_n$, where $\mathcal{G}_1, \ldots, \mathcal{G}_n \in \Gamma'$. Since Γ' is a chain, there is an index $k \in \{1, \ldots, n\}$ such that $F_i \in \mathcal{G}_k$ for $1 \le i \neq n$. Since $\mathcal{G}_k$ satisfies the finite intersection condition, $F_1 \cap \cdots \cap F_n \neq \emptyset$. This proves that $\mathcal{F}$ satisfies the finite intersection condition.

Since $\mathcal{F}$ satisfies the finite intersection condition and contains $\mathcal{C}$, it's an element of Γ, and it's clearly an upper bound for Γ'. By Zorn's lemma, Γ has maximal elements.

Step 2. Let $\mathcal{D}$ be a maximal element of Γ. I'll construct an element $x \in \bigcap_{D\in\mathcal{D}} D$.

Let $\pi_a : X \to X_a$ be the a^{th} projection map. Fix $a \in A$, and consider the family of sets $\{\pi_a D\}_{D\in\mathcal{D}}$. For any finite subcollection $\{\pi_a(D_1), \ldots, \pi_a(D_n)\}$, the intersection $D_1 \cap \cdots \cap D_n$ is nonempty, because $\mathcal{D}$ satisfies the finite intersection condition. Since

$$\pi_a(D_1 \cap \cdots \cap D_n) \subset \pi_a(D_1) \cap \cdots \cap \pi_a(D_n),$$

it follows that $\pi_a(D_1) \cap \cdots \cap \pi_a(D_n) \neq \emptyset$.

Therefore, $\{\pi_a D\}_{D\in\mathcal{D}}$ satisfies the finite intersection condition.

Hence, $\{\overline{\pi_a D}\}_{D\in\mathcal{D}}$ satisfies the finite intersection condition.

Now $\{\overline{\pi_a D}\}_{D\in\mathcal{D}}$ is a family of closed subsets of the compact space X_a. Therefore, there is an element

$$x_a \in \bigcap_{D\in\mathcal{D}} \overline{\pi_a D}.$$

Define $x = (x_a)_{a\in A}$.

Step 3. Let U_a be a neighborhood of x_a in X_a. I claim that $\pi_a^{-1}(U_a) \in \mathcal{D}$.

Since U_a is a neighborhood of x_a and $x_a \in \overline{\pi_a D}$ for all $D \in \mathcal{D}$, U_a meets $\pi_a(D)$ for all $D \in \mathcal{D}$. But if $U_a \cap \pi_a(D) \neq \emptyset$, then $\pi_a^{-1}(U_a) \cap D \neq \emptyset$. To see this, note that if $z \in U_a \cap \pi_a(D)$, then $z = \pi_a(y)$ for some $y \in D$ and so $\pi_a(y) \in U_a$. Thus, $y \in \pi_a^{-1}(U_a)$, and hence $y \in \pi_a^{-1}(U_a) \cap D$.

Thus, $\pi_a^{-1}(U_a)$ meets every set $D \in \mathcal{D}$. By the second part of the lemma, $\pi_a^{-1}(U_a) \in \mathcal{D}$.

Step 4. Every basic open set containing x is contained in $\mathcal{D}$.

The sets $\pi_a^{-1}(U_a)$ for $a \in A$ and U_a a neighborhood of x_a in X_a form a subbasis for the open sets containing x. A basic open set containing x is a finite intersection of $\pi_a^{-1}(U_a)$-sets; such a finite intersection is in $\mathcal{D}$, by the first part of the lemma.

Step 5. Finally, I'll show that $x \in \bigcap_{C\in\mathcal{C}} C$.

Let $D \in \mathcal{D}$, and let U be a basic open set containing x. Since $\mathcal{D}$ satisfies the finite intersection condition, $U \cap D \neq \emptyset$. Since every basic open set containing x meets D, it follows that $x \in \overline{D}$.

In particular, if D is a *closed* set in $\mathcal{D}$, then $x \in D$. But all the sets in $\mathcal{C}$ are closed, and $\mathcal{C} \subset \mathcal{D}$, so $x \in C$ for all $C \in \mathcal{C}$. Therefore, $x \in \bigcap_{C\in\mathcal{C}} C$.

This completes the proof that X is compact. $\square$

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[1] Serge Lang, *Real analysis* (2^{nd} edition). Reading, Massachusetts: Addison-Wesley Publishing Company, 1983. [ISBN 0-201-14179-5]

7-11-1999

Compact Sets and the Real Numbers

Theorem. If X is an ordered set satisfying the least upper bound property, then any closed interval $[a, b]$ in X is compact.

Proof. The outline of the proof is as follows. Let $\{U_i\}_{i \in I}$ be an open cover of $[a, b]$. Construct the set C of points $y \in [a, b]$ such that $[a, y]$ can be covered by a finite subcollection of $\{U_i\}_{i \in I}$. Take the least upper bound c of C and show that $c \in C$. Finally, show that $c = b$.

Step 1. Suppose $a \leq x < b$. I claim that for some $y > x$, the interval $[x, y]$ can be covered by at most two elements of $\{U_i\}_{i \in I}$.

If x has an immediate successor $x + 1$, then $[x, x + 1]$ has only two elements, so it can be covered by at most two U's.

If x does not have an immediate successor, find U_i containing x. Pick $y' > x$ such that $[x, y') \subset U_i$; this is possible since U_i is open. Since x does not have an immediate successor, there is an element y such that $x < y < y'$. Then $[x, y] \subset U_i$, and $[x, y]$ is covered by a single element of $\{U_i\}_{i \in I}$.

Step 2. Now let

$$C = \{y \in (a, b] \mid [a, y] \text{ can be covered by finitely many } U_i\}.$$

By Step 1, there is an element $y > a$ such that $[a, y]$ can be covered by at most two elements of $\{U_i\}_{i \in I}$. Therefore, C is nonempty. Let c be the least upper bound of C in $[a, b]$.

Step 3. I claim that $c \in C$.

Find U_i containing c. U_i is open and $c > a$, so I may find an interval $(d, c] \subset U_i$. Since d can't be an upper bound for C, there is an element of C larger than d. Let $c' \in C$, where $d < c' < c$. Then $[a, c']$ can be covered by finitely many U's, and $[c', c] \subset U_i$. Therefore, $[a, c] = [a, c'] \cup [c', c]$ can be covered by finitely many U's. Hence, $c \in C$.

Step 4. I claim that $c = b$.

Suppose that $c < b$. By Step 1, there is a $y > c$ such that $[c, y]$ can be covered by at most two elements of $\{U_i\}_{i \in I}$. Since $c \in C$, $[a, c]$ can be covered by finitely many elements of $\{U_i\}_{i \in I}$. So $[a, y] = [a, c] \cup [c, y]$ can be covered by finitely many elements of $\{U_i\}_{i \in I}$, and therefore $y \in C$. This contradicts the fact that c was the least upper bound of C. Hence, $c = b$.

Since $b \in C$, $[a, b]$ can be covered by finitely many elements of $\{U_i\}_{i \in I}$. Therefore, $[a, b]$ is compact. □

Corollary. A closed interval $[a, b]$ in $\mathbb{R}$ is compact. □

Theorem. (**Heine-Borel**) A subset $C \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded relative to the standard metric or the square metric.

Remark. Recall that the standard metric d and the square metric ρ are related by

$$\rho(x, y) \leq d(x, y) \leq \sqrt{n} \cdot \rho(x, y).$$

Hence, a set bounded relative to one metric is bounded relative to the other.

Proof. Suppose that C is compact. $\mathbb{R}^n$ is Hausdorff, so C is closed. The collection of balls $\{B(0; n) \mid n \in \mathbb{Z}^+\}$ is an open cover of $\mathbb{R}^n$, so it is an open cover of C. By compactness, there is a finite subcover; the element of the subcover with the largest radius contains C, so C is bounded.

Conversely, suppose C is a closed and bounded subset of $\mathbb{R}^n$. Suppose that $\rho(x, y) \leq s$ for all $x, y \in C$. Fix $x \in C$ and let $t = \rho(x, 0)$. Then for all $y \in C$,

$$d(y, 0) \leq d(y, x) + d(x, 0) = s + t.$$

It follows that $C \subset [-(s+t), s+t]^n$. However, $[-(s+t), s+t]^n$ is a product of compact spaces, so it's compact. Thus, C is a closed subset of the compact set $[-(s+t), s+t]^n$, so it is compact. □

Theorem. Let X be a compact topological space, let Y be an ordered set with the order topology, and let $f : X \to Y$ be continuous. There are points $a, b \in X$ such that

$$f(a) \le f(x) \le f(b) \quad \text{for all} \quad x \in X.$$

Proof. $f(X)$ is a compact subset of Y. I'll show that $f(X)$ has a largest element and a smallest element.

Suppose that $f(X)$ does not have a largest element. Then the collection of open rays

$$\{(-\infty, f(x)) \mid x \in X\}$$

cover $f(X)$. By compactness, I may find $x_1, \ldots, x_n \in X$ such that

$$\{(\infty, f(x_1)), \ldots, (-\infty, f(x_n))\}$$

cover $f(X)$.

Find $m \in \{1, \ldots, n\}$ such that

$$f(x_m) = \max\{f(x_i) \mid 1 \le i \le n\}.$$

Then $f(X) \subset (-\infty, f(x_m))$, but $f(x_m) \notin (-\infty, f(x_m))$. This contradiction shows that $f(X)$ has a largest element; a similar argument shows that $f(X)$ has a smallest element. □

In the case where $X = [a, b]$ is a closed interval in $\mathbb{R}$ and $Y = \mathbb{R}$, this is the familiar result from calculus which says that a continuous function on a closed interval has a max and a min on the interval.

The next result and its corollary are more amusing than important, since you've probably seen the Cantor diagonalization proof of the uncountability of $[0, l]$. But this topological proof does not make any reference to *representations* of real numbers as infinite decimal.

Theorem. Let X be a (nonempty) compact Hausdorff space. Suppose every point of X is a limit point of X. Then X is uncountable.

Proof.

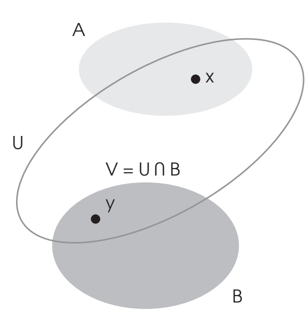
Step 1. Let U be a nonempty open subset of X, and let $x \in X$. I'll construct an open set $V \subset U$ such that $x \notin \overline{V}$.

If $U = \{x\}$, then U is a neighborhood of x which contains only x. This contradicts the assumption that every point of X is a limit point of X. Now x may not be in U in the first place, but this shows that in any event, U can't consist of x alone.

Thus, I may choose $y \in U$ such that $y \ne x$. Since X is Hausdorff, there are disjoint neighborhoods A of x and B of y. Let $V = B \cap U$. Now

$$V = B \cap U \subset B \subset X - A,$$

and $X - A$ is closed. Therefore, $\overline{V} \subset X - A$. In particular, since $x \in A$, $\overline{V}$ does not contain x.



Step 2. Next, I'll show that there is no surjection $f : \mathbb{Z}^+ \to X$.

Suppose that $f : \mathbb{Z}^+ \to X$. For the purposes of numbering, take $V_0 = X$. By Step 1, I may find a nonempty open set V_1 such that $f(1) \notin \overline{V_1}$.

Suppose inductively that $n > 1$, and I have found a nonempty open set $V_{n-1} \subset V_{n-2}$ such that $f(n-1) \notin \overline{V_{n-1}}$. Use Step 1 to find a nonempty open set $V_n \subset V_{n-1}$ such that $f(n) \notin \overline{V_n}$.

Consider the collection $\{\overline{V_n}\}_{n \in \mathbb{Z}^+}$. The collection has the finite intersection condition, since

$$\overline{V_1} \supset \overline{V_2} \supset \overline{V_3} \supset \ldots.$$

By compactness, there is a point $x \in \bigcap_{n \in \mathbb{Z}^+} \overline{V_n}$. Then $x \neq f(n)$ for any $n \in \mathbb{Z}^+$, so f is not surjective.

Since there is no surjective function from $\mathbb{Z}^+$ to X, X is uncountable. □

Corollary. The interval $[0, 1]$ in $\mathbb{R}$ is uncountable. □

Obviously, it follows that $\mathbb{R}$ and open intervals in $\mathbb{R}$ are uncountable.

Limit Point Compactness and Sequential Compactness

Definition. Let X be a topological space.

1. X is **limit point compact** if every infinite subset has a limit point.

2. X is **sequentially compact** if every sequence in X has a convergent subsequence.

If X is sequentially compact, then X is limit point compact. For every infinite subset of X contains an infinite sequence; the limit of a convergent subsequence is a limit point of the original subset. A partial converse is contained in the following theorem.

Theorem. Let X be a topological space.

1. If X is compact, then X is limit point compact.

2. If X is a metric space and X is limit point compact, then X is sequentially compact.

Proof. 1. Let X be compact, and let $S \subset X$ be an infinite subset. I must show that S has a limit point.

Suppose that S does not have a limit point. It is vacuously true that S contains all its limit points, so S is closed. Since X is compact, S is compact.

Let $x \in S$. Since x is not a limit point of S, there is a neighborhood U_x of x which does not intersect S in a point other than x. $\{U_x\}_{x \in S}$ is an open cover of S, so by compactness there is a finite subcover $\{U_{x_1}, \ldots, U_{x_n}\}$.

Now $S \subset \{U_{x_1} \cup \ldots \cup U_{x_n}$, but each U_{x_i} meets S only in x_i. Therefore, $S = \{x_1, \ldots, x_n\}$, which contradicts the assumption that S is infinite.

Therefore, S has a limit point.

2. Suppose X is a metric space and X is limit point compact. Let $\{x_n\}$ be a sequence in X. I want to show that $\{x_n\}$ has a convergent subsequence.

First, I have to eliminate any repetitions in the sequence, so define

$$S = \{x_n \mid n \in \mathbb{Z}^+\}.$$

Thus, S is the *set of points* in $\{x_n\}$.

Suppose S is finite. Then for some $x \in X$, $x_n = x$ for infinitely many $n \in \mathbb{Z}^+$. Suppose this is true for $n_1, n_2, \ldots$. Then $\{x_{n_i}\}_{i \in \mathbb{Z}^+}$ is a convergent subsequence of $\{x_n\}$, because all the terms are equal to x.

The only other possibility is that S is infinite. Then by limit point compactness, S has a limit point x. Consider the nested sequence of neighborhoods $\left\{B\left(x; \frac{1}{k}\right) \mid k \in \mathbb{Z}^+\right\}$. Since metric spaces are Hausdorff and x is a limit point of S, each $B\left(x; \frac{1}{k}\right)$ contains infinitely many points of S.

Choose $n_1 \in \mathbb{Z}^+$ such that $x_{n_1} \in B(x; 1)$. Assuming that $k > 1$ and that n_{k_1} has been chosen, choose $n_k > n_{k-1}$ such that $x_{n_k} \in B\left(x; \frac{1}{k}\right)$.

I now have $x_{n_k} \to x$. For if $\epsilon > 0$, choose N so that $\frac{1}{n_k} < \epsilon$ for $k \geq N$. Then $x_{n_k} \in B(x; \epsilon)$ for $k \geq N$. $\square$

Example. The subsets $\{[2n-1, 2n] \mid n \in \mathbb{Z}^+\}$ form a partition of $\mathbb{Z}^+$. The corresponding partition topology is called the **odd-even topology**. Let X denote $\mathbb{Z}^+$ with this topology.

X is limit point compact. To see this, let A be an arbitrary nonempty subset of X, and let $a \in A$. For some $n \in \mathbb{Z}^+$, $a \in [2n-1, 2n]$. Notice that $[2n-1, 2n]$ is the smallest open set containing $2n-1$ or $2n$.

If $a = 2n - 1$, then $2n$ is a limit point of A, since every neighborhood of $2n$ contains a.

If $a = 2n$, then $2n - 1$ is a limit point of A, since every neighborhood of $2n - 1$ contains a.

In either case, A has a limit point. Since A was an arbitrary nonempty subset of X, surely every infinite subset of X has a limit point. Thus, X is limit point compact.

On the other hand, X is not sequentially compact. For example, the sequence $\{1, 2, 3, \ldots\}$ has no convergent subsequence.

This example shows that limit point compactness does not in general imply sequential compactness.

In fact, X is not compact: The open cover $\{[2n-1, 2n] \mid n \in \mathbb{Z}^+\}$ does not have a finite subcover. Thus, limit point compactness does not in general imply compactness. □

Recall that if Y is a subset of a metric space (X, d), the **diameter** of Y is

$$\operatorname{diam}(Y) = \sup\{d(x, y) \mid x, y \in Y\}.$$

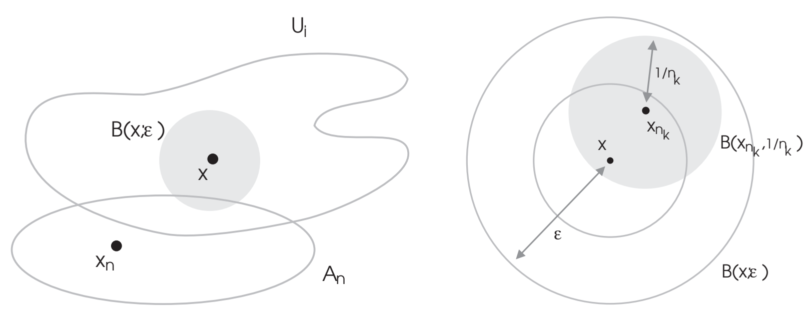
Theorem. (**Lebesgue Number Lemma**) Let X be a sequentially compact metric space, and let $\{U_i\}_{i \in I}$ be an open cover of X. There is an $\epsilon > 0$ such that every subset of X of diameter less than ϵ is contained in an element of $\{U_i\}_{i \in I}$.

ϵ is called a **Lebesgue number** for the cover.

Proof. Suppose on the contrary that no such ϵ exists. Then for every $n \in \mathbb{Z}^+$, there is a set A_n of diameter less than $\dfrac{1}{n}$ which is not contained in any U_i. Let $x_n \in A_n$. I'll show that $\{x_n\}$ has no convergent subsequence, which will contradict sequential compactness.

Suppose on the contrary that $\{x_{n_k}\}$ is a convergent subsequence converging to x. Suppose that $x \in U_i$. Find $\epsilon > 0$ such that $B(x; \epsilon) \subset U_i$. Then choose n_k sufficiently large so that

$$d(x, x_{n_k}) < \frac{\epsilon}{2} \quad \text{and} \quad \frac{1}{n_k} < \frac{\epsilon}{2}.$$



Since $\operatorname{diam}(A_{n_k}) < \dfrac{1}{n_k}$ and $x_{n_k} \in A_{n_k}$, it follows that $A_{n_k} \subset B\left(x_{n_k}; \dfrac{1}{n_k}\right)$. Suppose $y \in B\left(x_{n_k}; \dfrac{1}{n_k}\right)$. Then

$$d(x, y) \le d(x, x_{n_k}) + d(x_{n_k}, y) \le \frac{\epsilon}{2} + \frac{1}{n_k} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore, $B\left(x_{n_k}; \dfrac{1}{n_k}\right) \subset B(x; \epsilon)$. Hence,

$$A_{n_k} \subset B\left(x_{n_k}; \frac{1}{n_k}\right) \subset B(x; \epsilon) \subset U_i.$$

This contradicts the fact that A_{n_k} is not contained in any U_i. Therefore, there *is* a number ϵ satisfying the conclusion of the theorem. □

Theorem. Let X be a metric space. The following are equivalent:

1. X is compact.
2. X is limit point compact.
3. X is sequentially compact.

Proof. I've already proven $(1 \Rightarrow 2)$ and $(2 \Rightarrow 3)$. I need to prove $(3 \Rightarrow 1)$. Suppose then that X is sequentially compact. I need to show that X is compact.

Step 1. For every $\epsilon > 0$, there is a finite covering of X by ϵ-balls.

Let $x_1 \in X$ and construct $B(x_1; \epsilon)$. If $X = B(x_1; \epsilon)$, then I'm done. Otherwise, choose $x_2 \in X - B(x_1; \epsilon)$ and construct $B(x_2; \epsilon)$. Again, if $X = B(x_1; \epsilon) \cup B(x_2; \epsilon)$, I'm done; otherwise, choose $x_3 \in X - (B(x_1; \epsilon) \cup B(x_2; \epsilon))$. Keep going. Notice that each $B(x_n; \epsilon)$ contains only one x_k, namely x_n.

Suppose that the process does not terminate. Consider the sequence $\{x_n\}$. I claim that it has no convergent subsequence. Indeed, if $\{x_{n_k}\}$ is a subsequence converging to x, then $B\left(x; \dfrac{\epsilon}{2}\right)$ can contain at most one x_{n_k}: If it contains two x_{n_k}'s, then they are less than ϵ apart. Therefore, there is an ϵ-ball about an x_{n_k} containing another x_{n_k}

It follows that the process *must* terminate, so X is covered by a finite number of ϵ-balls.

Step 2. Every open cover of X contains a finite subcover.

Let $\{U_i\}_{i \in I}$ be an open cover of X. Let ϵ be a Lebesgue number for the cover. Cover X with a finite number of $\dfrac{\epsilon}{2}$ balls $\{B_1, \ldots, B_n\}$. Each B_j has diameter less than ϵ, so each B_j is contained in some U_{i_j}. Then $\{U_{i_j} \mid j = 1, \ldots, n\}$ is a finite subcollection of $\{U_i\}_{i \in I}$ which covers X. Therefore, X is compact. □

Definition. Let X and Y be metric spaces, and let $f : X \to Y$. f is **uniformly continuous** if for every $\epsilon > 0$, there is a $\delta > 0$ such that $d_X(a, b) < \delta$ implies $d_Y(f(a), f(b)) < \epsilon$ for all $a, b \in X$.

Theorem. Let X and Y be metric spaces, let $f : X \to Y$ be a continuous function, and suppose X is compact. Then f is uniformly continuous.

Proof. Given $\epsilon > 0$, the open balls $\left\{B\left(y; \dfrac{\epsilon}{2}\right) \;\middle|\; y \in Y\right\}$ cover Y. Therefore, $\left\{f^{-1}\left(B\left(y; \dfrac{\epsilon}{2}\right)\right) \;\middle|\; y \in Y\right\}$ is an open cover of X.

By the Lebesgue Number Lemma, there is a $\delta > 0$ such that every set of diameter less than δ is contained in a $f^{-1}\left(B\left(y; \dfrac{\epsilon}{2}\right)\right)$. In particular, if $d_X(a, b) < \delta$, then $\{a, b\}$ is a set of diameter less than δ, so $a, b \in f^{-1}\left(B\left(y; \dfrac{\epsilon}{2}\right)\right)$ for some y. This means that $f(a), f(b) \in B\left(y; \dfrac{\epsilon}{2}\right)$, so $d_Y(f(a), f(b)) < \epsilon$. □

Example. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. f is continuous, but f is not uniformly continuous.

To show that f is not uniformly continuous, I'll show that there is no $\delta > 0$ such that if $\delta > |x - y|$, then $1 > |f(x) - f(y)|$. Suppose on the contrary that such a δ exists. Choose $x > \dfrac{2}{\delta}$, and consider the points $x + \dfrac{\delta}{2}$ and x.

First,

$$\left(x + \frac{\delta}{2}\right) - x = \frac{\delta}{2} < \delta.$$

The points are less than δ units apart.

However,

$$\left(x + \frac{\delta}{2}\right) + x = 2x + \frac{\delta}{2} > 2 \cdot \frac{2}{\delta} + \frac{\delta}{2} = \frac{4}{\delta} + \frac{\delta}{2} > \frac{4}{\delta}.$$

So

$$f\left(x+\frac{\delta}{2}\right)-f(x)=\left(x+\frac{\delta}{2}\right)^2-x^2=\left(x+\frac{\delta}{2}-x\right)\left(x+\frac{\delta}{2}+x\right)=\frac{\delta}{2}\cdot\left(x+\frac{\delta}{2}+x\right)>\frac{\delta}{2}\cdot\frac{4}{\delta}=2.$$

The images are more than 1 unit apart.
This shows that f is not uniformly continuous. □

7-16-1999

Local Compactness

Definition. Let X be a topological space. X is **locally compact** if for all $x \in X$, there is a compact set C and a neighborhood U of x such that $x \in U \subset C$.

If X is compact, then X is locally compact: For any $x \in X$, X is a compact set containing the neighborhood X of x.

Example. If X is a space with the discrete topology, then X is locally compact. If $x \in X$, then $\{x\}$ is a compact set containing the neighborhood $\{x\}$ of x. □

Example. $\mathbb{R}^n$ is locally compact: If $x \in \mathbb{R}^n$, the closed ball $\overline{B(x;1)}$ is compact, and it contains the open ball $B(x;1)$. □

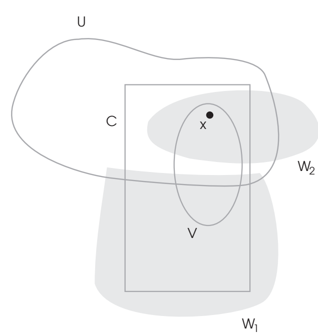
Theorem. Let X be a Hausdorff space. X is locally compact if and only if for every $x \in X$ and every neighborhood U of x, there is a neighborhood V of x such that $\overline{V}$ is compact and $\overline{V} \subset U$.

Proof. Suppose for every $x \in X$ and every neighborhood U of x, there is a neighborhood V of x such that $\overline{V}$ is compact and $\overline{V} \subset U$. Choose any neighborhood U of x — for example, X — and find a neighborhood V of x with compact closure such that $\overline{V} \subset U$. Then $\overline{V}$ is a compact set containing a neighborhood V of x, so X is locally compact.

Conversely, suppose X is locally compact. Let $x \in X$, and let U be a neighborhood of x. Let C be a compact set which contains a neighborhood V of x.

C is compact and X is Hausdorff, therefore C is closed. It follows that $C - U = C \cap (X - U)$ is closed; since it's a subset of the compact set C, $C - U$ is compact. Note that $x \in U$, so $x \notin C - U$.

I showed earlier that in a Hausdorff space, a compact set and a point not contained in it may be separated by disjoint open sets. Thus, I may find disjoint open sets W_1 and W_2 such that $C - U \subset W_1$ and $x \in W_2$.



Next, I'll show that $\overline{W_2 \cap V}$ is compact. C is a closed set containing V, so $\overline{V} \subset C$. But $W_2 \cap V \subset V$, so $\overline{W_2 \cap V} \subset \overline{V} \subset C$. Now $\overline{W_2 \cap V}$ is a closed subset of the compact set C, so $\overline{W_2 \cap V}$ is compact.

Moreover, $W_2 \cap V \subset W_2 \subset X - W_1$. $X - W_1$ is closed, so $\overline{W_2 \cap V} \subset X - W_1$. Since $\overline{W_2 \cap V}$ does not meet W_1, and since $C - U \subset W_1$, it follows that $\overline{W_2 \cap V}$ does not intersect $C - U$. But $\overline{W_2 \cap V}$ *is* a subset of C, so $\overline{W_2 \cap V} \subset U$.

Thus, $W_2 \cap V$ is a neighborhood of x with compact closure whose closure is contained in U. □

Corollary. Let X be a locally compact Hausdorff space, and let Y be a subspace of X. If Y is open or closed in X, then Y is locally compact.

Proof. Suppose Y is open in X. Let $x \in Y$. By the preceding result, I may find a neighborhood U of x such that $\overline{U}$ is compact and $\overline{U} \subset Y$. Note that U is automatically open in Y. Thus, x is contained in a compact set which contains a Y-neighborhood of x. Therefore, Y is locally compact.

Suppose Y is closed. Let $y \in Y$, and let C be a compact set containing a neighborhood U of y. C is closed, since X is Hausdorff; therefore, $C \cap Y$ is a closed subset of C. Since C is compact, $C \cap Y$ is compact. Moreover, $C \cap Y$ contains $U \cap Y$, which is a neighborhood of y. Therefore, Y is locally compact. □

Example. Let S denote the topologist's sine curve

$$S = \left\{ \left(x, \sin \frac{1}{x}\right) \;\Big|\; 0 < x \le 1 \right\}.$$

Let $X = S \cup \{(0,0)\}$. X is not locally compact.

Take a ball $B((0,0);\epsilon)$ about the origin. Consider the intersection of X with the line $y = \dfrac{\epsilon}{2}$. The intersection is an infinite sequence of points with all but finitely many terms lying inside $B((0,0);\epsilon)$. The sequence converges to $\left(0, \dfrac{\epsilon}{2}\right)$, which is not in X.

If there is a compact set C in X containing a neighborhood of $(0,0)$, then that neighborhood contains $B((0,0);\epsilon) \cap X$ for ϵ sufficiently small. By the argument above, such a neighborhood contains an infinite subset of X with no limit point. The same is true for C, which contradicts the fact that compact sets are limit point compact. □

It is often useful to embed a space in a compact Hausdorff space, because compact Hausdorff spaces are very nicely behaved. For example, I'll show later that compact Hausdorff spaces are **normal**: Any two disjoint closed sets can be separated by disjoint open sets. If a space is locally compact Hausdorff, it can be embedded in a compact Hausdorff space called the **one-point compactification**.

Lemma. Let X be a locally compact Hausdorff space. Define $Y = X \cup \{\infty\}$, where ∞ is a point not in X. Let $\mathcal{T}$ be the collection of subsets of Y consisting of:

1. Any open subset of X.
2. The complement in Y of a compact subset of X.

Then $\mathcal{T}$ is a topology on Y.

$(Y, \mathcal{T})$ is called the **one-point compactification** of X.

Proof. $\emptyset$ is open in X, so it's open in Y. $\emptyset$ is compact, so $Y = Y - \emptyset$ is open in Y.

To verify the axioms for unions and intersections, it's necessary to take cases. I'll show the work for unions; the proof for intersections is similar.

The union of open sets in X is open in X, so it's open in Y.

If $\{C_i\}$ is a family of compact subsets of X, then

$$\bigcup_{i \in I} (Y - C_i) = Y - \bigcap_{i \in I} C_i.$$

Now X is Hausdorff, so compact subsets are closed. Therefore, $\bigcap_{i \in I} C_i$ is closed. Since it's contained in any one of the compact sets C_i, it's also compact. Therefore, $\bigcup_{i \in I} (Y - C_i)$ is the complement in Y of a compact set in X, so it's open in Y.

The remaining possibility is a union of sets open in X with complements of compact subsets of X. Using the first two cases, this reduces to showing that if U is open in X and C is a compact subset of X, then $U \cup (Y - C)$ is open in Y. But

$$U \cup (Y - C) = Y - (C - U).$$

C is closed in X, U is open in X, so $C - U$ is closed in X. But $C - U \subset C$ and C is compact, therefore $C - U$ is compact. Hence, $U \cup (Y - C)$ is a complement in Y of a compact set in X, so it's open in Y. □

Here are some properties of the one-point compactification.

Lemma. Let X be a locally compact Hausdorff space, and let Y be its one-point compactification.

1. The subspace topology on X is the same as the original topology on X. (That is, the inclusion of X into Y is a homeomorphism onto its image.)
2. If X is not compact, then $Y = \overline{X}$.
3. Y is compact.
4. Y is Hausdorff.

Proof. 1. If U is open in X, then it's open in Y by definition. Conversely, an open set in Y is either an open set U in X (in which case $U \cap X = U$ is open in X) or $Y - C$, where C is compact in X. But $(Y - C) \cap X = X - C$, which is open in$X$ since C is closed in X.

2. Since the only point of Y not in X is ∞, this amounts to showing that ∞ is a limit point of X. Neighborhoods of ∞ are sets $Y - C$, where C is compact, and since X is not compact, $C \neq X$. Thus, $Y - C$ must intersect X, and so $\infty \in \overline{X}$.

3. Let $\{U_i\}_{i \in I}$ be an open cover of Y. One of the U's, say U_0, must contain ∞, so U_0 must be a set of the form $Y - C$, where C is compact in X. The sets $U_i \cap X$ for $U_i \neq U_0$ form an open cover of C; let $\{U_{i_1} \cap X, \ldots, U_{i_n} \cap X\}$ be a finite subcover. Then $\{U_{i_1}, \ldots, U_{i_n}, U_0\}$ covers Y.

4. Let x and y be distinct points of Y. If they're both in X, they may be separated by disjoint neighborhoods in X (since X is Hausdorff), and these neighborhoods are also open in Y.

Otherwise, I'm trying to separate $x \in X$ from ∞. Since X is locally compact, I may find a compact set C containing a neighborhood U of x. Then $Y - C$ and U are disjoint open sets in Y separating x and ∞. □

Corollary. A space X embeds as an open subset of a compact Hausdorff space if and only if X is locally compact Hausdorff. □

7-16-1999

The Countability Axioms

Definition. Let X be a topological space, and let $x \in X$.

1. X has a **countable basis at** x if there is a countable collection of neighborhoods $\{B_i\}_{i \in I}$ of x such that if U is a neighborhood of x, then $B_i \subset U$ for some i.

2. X is **first countable** if X has a countable basis at each point.

3. X is **second countable** if X has a countable basis for its topology.

Obviously, a second countable space is first countable.

Example. Every metric space is first countable. If X is a metric space and $x \in X$, then $\left\{ B\left(x; \frac{1}{n}\right) \mid n \in \mathbb{Z}^+ \right\}$ is a countable basis at x. □

Example. $\mathbb{R}^n$ is second countable, since the balls with rational centers and radii form a countable basis for the usual topology.

On the other hand, $\mathbb{R}$ with the discrete topology is not second countable. □

Example. Consider $\mathbb{R}$ with the finite complement topology. Thus, the open sets are $\emptyset$ and any set whose complement is finite.

I claim that $\mathbb{R}$ does not have a countable basis at any point. Suppose, for example, that there is a countable basis $\{B_1, B_2, \ldots\}$ at 0. I claim that $\bigcap_{i=1}^{\infty} B_i = \{0\}$. Clearly, 0 is in the intersection; I must show that no nonzero point is in the intersection.

Suppose that $x \neq 0$. The complement of $\mathbb{R} - \{x\}$ is finite, so it's an open set containing 0. Hence, $B_i \subset \mathbb{R} - \{x\}$ for some i. Then $x \notin B_i$, so $x \notin \bigcap_{i=1}^{\infty} B_i$.

Thus,

$$\mathbb{R} - \{0\} = \mathbb{R} - \bigcap_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} (\mathbb{R} - B_i).$$

Each set B_i is finite, so $\bigcup_{i=1}^{\infty} (\mathbb{R} - B_i)$ is countable. But $\mathbb{R}$ is uncountable, so $\mathbb{R} - \{0\}$ is uncountable. This contradiction shows that there is no countable basis at 0; obviously, the argument works for any $x \in \mathbb{R}$.

Thus, $\mathbb{R}$ is not first countable (or second countable) in the finite complement topology. □

The following results were proved for metric spaces. An examination of the proofs shows that they only depended on the fact that metric spaces are first countable.

Theorem. Let X and Y be topological spaces, and suppose X is first countable.

1. Let $A \subset X$. $x \in \overline{A}$ if and only if there is a sequence of points of A converging to x.

2. Let $f : X \to Y$. f is continuous if and only if whenever $\{x_n\}$ is a convergent sequence in X, $\{f(x_n)\}$ is a convergent sequence in Y. □

Proposition.

1. A subspace of a first countable space is first countable.

2. A countable product of first countable spaces is first countable.

3. A subspace of a second countable space is second countable.

4. A countable product of second countable spaces is second countable.

Proof. 1. Suppose X is first countable, and suppose Y is a subspace of X. Let $y \in Y$. Let $\{B_1, B_2, \ldots\}$ be a countable basis at y in X. Then $\{B_1 \cap Y, B_2 \cap Y, \ldots\}$ is a countable basis at y in Y.

2. Suppose $X_1, X_2, \ldots$ are first countable spaces. Let $(x_n) \in \prod_{n=1}^{\infty} X_n$. Let $\mathcal{B}_n$ be a countable basis for x_n in X_n. Consider the collection $\mathcal{U}$ of product neighborhoods $\prod_{n=1}^{\infty} U_n$, where $U_n = X_n$ for all but finitely many n, and if $U_n \neq X_n$, then $U_n \in \mathcal{B}_n$. Then $\mathcal{U}$ is a countable basis at (x_n) in $\prod_{n=1}^{\infty} X_n$, so $\prod_{n=1}^{\infty} X_n$ is first countable.

3. Suppose X is second countable, and suppose Y is a subspace of X. Let $\{B_1, B_2, \ldots\}$ be a countable basis for the topology on X. Then $\{B_1 \cap Y, B_2 \cap Y, \ldots\}$ is a countable basis for the subspace topology on Y.

4. Suppose $X_1, X_2, \ldots$ are second countable spaces. Let $\mathcal{B}_n$ be a countable basis for the topology on X_n.Consider the collection $\mathcal{U}$ of product neighborhoods $\prod_{n=1}^{\infty} U_n$, where $U_n = X_n$ for all but finitely many n, and if $U_n \neq X_n$, then $U_n \in \mathcal{B}_n$. Then $\mathcal{U}$ is a countable basis for the product topology on $\prod_{n=1}^{\infty} X_n$, so $\prod_{n=1}^{\infty} X_n$ is second countable. □

Definition. Let X be a topological space.

1. X is **Lindelöf** if every open cover of X has a countable subcover.

2. X is **separable** if X has a countable dense subset.

Example. Any compact space is Lindelöf, since every open cover has a *finite* subcover. □

Example. $\mathbb{R}$ is separable, since $\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$. □

Proposition. Let X be a second countable topological space.

1. X is Lindelöf.

2. X is separable.

Proof. 1. Let $\mathcal{U}$ be an open cover of X. Let $\{B_1, B_2, \ldots\}$ be a countable basis for X. For each $n \in \mathbb{Z}^+$, let

$$\mathcal{U}_n = \{U \in \mathcal{U} \mid B_n \subset U\}.$$

Next, for each nonempty $\mathcal{U}_n$, choose $V_n \in \mathcal{U}_n$.

For each $x \in X$, there is a $U \in \mathcal{U}$ such that $x \in U$. Moreover, there is a basis element B_n such that $x \in B_n \subset U$. Then $\mathcal{U}_n$ is nonempty, so V_n is defined, and $x \in B_n \subset V_n$.

Now x was an arbitrary point of X, so the collection $\{V_n\}$ covers X. $\{V_n\}$ is a countable subcover of $\mathcal{U}$.

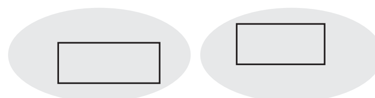
2. Let $\{B_1, B_2, \ldots\}$ be a countable basis for the topology of X. Choose $x_n \in B_n$ for each $n \in \mathbb{Z}^+$. Then $\{x_n\}$ is a countable dense subset of X. □

The Separation Axioms

Definition. Let X be a space in which singletons are closed.

1. X is **regular** if a closed set and a point outside it can be separated by disjoint open sets.
2. X is **normal** if disjoint closed sets can be separated by disjoint open sets.

regular



normal

Obviously, a normal space is regular, and a regular space is Hausdorff.

Terminology. Recall that a Hausdorff space is also known as a T_2 space.

X is a T_0 space if given distinct points x, y, there is a neighborhood U such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$.

X is a T_1 space if given distinct points x, y, there are neighborhoods U of x and V of y such that $y \notin U$ and $x \notin V$.

X is a $T_{2\frac{1}{2}}$ space if distinct points have neighborhoods whose closures are disjoint.

If you don't require that points be closed, then a space in which disjoint points and closed sets can be separated is T_3, and a space in which disjoint closed sets can be separated is T_4.

Lemma. Let X be a space in which singletons are closed.

1. X is regular if and only if for all $x \in X$ and every neighborhood U of x, there is a neighborhood V of x such that $\overline{V} \subset U$.
2. X is normal if and only if for every closed subset A of X and every open set U containing A, there is an open set V containing A such that $\overline{V} \subset U$.

Proof. 1. Suppose X is regular, $x \in X$, and U is a neighborhood of x. Then $X - U$ is a closed set disjoint from x, so by regularity I may find disjoint open sets V containing x and W containing $X - U$.

In particular, $x \in V \subset X - U$; since $X - U$ is closed, $\overline{V} \subset X - U$.

Conversely, suppose that for all $x \in X$ and every neighborhood U of x, there is a neighborhood V of x such that $\overline{V} \subset U$. Let $x \in X$ and let C be a closed set which does not contain x.

$X - C$ is an open set containing x, so there is a neighborhood V of x such that $\overline{V} \subset X - C$. Then $X - \overline{V}$ is an open set containing C, and it is disjoint from the open set V containing x. Therefore, X is regular.

2. The proof for normality is the same as the proof for regularity, with the point x replaced with the closed set A. $\square$

Corollary. If X is a regular space, then X is a $T_{2\frac{1}{2}}$ space.

Proof. Suppose X is regular. Let x and y be distinct points in X. Points in X are closed, so $X - \{y\}$ is a neighborhood of x. By the preceding result, there is a neighborhood U of x such that $\overline{U} \subset X - \{y\}$. Again by the preceding result, there is a neighborhood V of x such that $\overline{V} \subset U$.

$X - \overline{U}$ is a neighborhood of y, and $X - \overline{U} \subset X - U$. Since $X - U$ is closed, $\overline{X - \overline{U}} \subset X - U$. Since $\overline{V} \subset U$ and $\overline{X - \overline{U}} \subset X - U$, $\overline{V}$ and $\overline{X - \overline{U}}$ are disjoint.

Thus, V is a neighborhood of x, $X - \overline{U}$ is a neighborhood of y, and their closures are disjoint. Therefore, X is $T_{2\frac{1}{2}}$. □

Corollary. If X is a locally compact Hausdorff space, then X is regular. □

Example. (Irrational slope topology [1, Example 75]) Let

$$X = \mathbb{Q} \times \mathbb{Q}^{\geq 0},$$

the points with rational coordinates in the closed upper half-plane. Fix an irrational number θ.

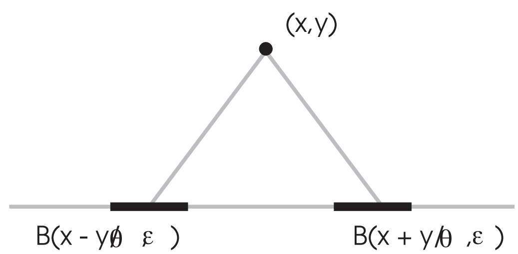
If $x, \epsilon \in \mathbb{R}$ and $\epsilon > 0$, define a subset of the x-axis $B(x; \epsilon)$ by

$$B(x; \epsilon) = \{y \in \mathbb{Q} \mid |x - y| < \epsilon\}.$$

Then for $(x, y) \in X$ and $\epsilon > 0$, set

$$N((x, y), \epsilon) = \{(x, y)\} \cup B\left(x - \frac{y}{\theta}, \epsilon\right) \cup B\left(x + \frac{y}{\theta}, \epsilon\right).$$

$N((x, y), \epsilon)$ consists of the point (x, y) together with two open ϵ-intervals of rationals on the x-axis. The intervals are centered at points on the x-axis which determine lines of slopes θ and $-\theta$ with the point (x, y).



Note that if $y = 0$, $N((x, y), \epsilon)$ reduces to an open interval of rationals about x in the x-axis. Moreover, if $x \in \mathbb{Q}$, then $N((x, 0), \epsilon)$ is a basic open set.

Define a topology by taking as a basis all the sets $N((x, y), \epsilon)$ for all real $\epsilon > 0$ and all $(x, y) \in X$. It's clear that the sets cover X.

If $(a, b) \in N((x, y), \epsilon_1) \cap N((x, y), \epsilon_2)$, then either $(a, b) = (x, y)$ or (a, b) lies in the intersection of the intervals in the x-axis.

In the first case, if $\epsilon = \min(\epsilon_1, \epsilon_2)$, then $N((x, y), \epsilon)$ is a basis element containing (a, b) and contained in $N((x, y), \epsilon_1) \cap N((x, y), \epsilon_2)$.

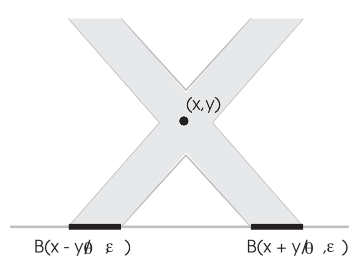
In the second case, $(a, b) = (a, 0)$, and $(a, 0) \in B\left(x \pm \frac{y}{\theta}, \epsilon_1\right) \cap B\left(x \pm \frac{y}{\theta}, \epsilon_2\right)$, and this is the intersection of open intervals in the x-axis. Choosing a sufficiently small $\epsilon > 0$, I may find an interval $N((a, 0), \epsilon)$ contained in this intersection.

If $(a, b) \in N((x_1, y_1), \epsilon_1) \cap N((x_2, y_2), \epsilon_2)$ where $(x_1, y_1) \neq (x_2, y_2)$, then again (a, b) must lie in the intersection of intervals in the x-axis. As before, I may find a basic interval containing x contained in the intersection of the original intervals.

Therefore, the collections of sets $N((x, y), \epsilon)$ for all real $\epsilon > 0$ and all $(x, y) \in X$ forms a basis.

Next, I'll show that the closures of any two basic neighborhoods must intersect.

What is $\overline{N((x, y), \epsilon)}$? It consists of two diagonal strips of slopes θ and $-\theta$ emanating from the intervals $B\left(x - \frac{y}{\theta}, \epsilon\right)$ and $B\left(x + \frac{y}{\theta}, \epsilon\right)$.



To see this, observe that a basic neighborhood of a point in these strips will contain intervals in the x-axis which intersect $B\left(x - \dfrac{y}{\theta}, \epsilon\right)$ or $B\left(x + \dfrac{y}{\theta}, \epsilon\right)$.

It is clear that any two such pairs of strips must intersect. Therefore, the closures of any two basic neighborhoods must intersect. By the result above, X is not regular. $\square$

I proved a special case of the first part of the following proposition; I'm repeating the proof of the second part for the sake of completeness.

Proposition.

1. A product of Hausdorff spaces is Hausdorff.
2. A subspace of a Hausdorff space is Hausdorff.

Proof. 1. Let $\{X_a\}_{a\in A}$ be a family of Hausdorff spaces. Let $(x_a), (y_a)$ be distinct points in $\prod_{a\in A} X_a$. For some index b, $x_b \neq y_b$. Choose disjoint open sets $U, V \subset X_b$ such that $x_b \in U$ and $y_b \in V$.

Define
$$U_a = \begin{cases} X_a & \text{if } a \neq b \\ U & \text{if } a = b \end{cases} \quad \text{and} \quad V_a = \begin{cases} X_a & \text{if } a \neq b \\ V & \text{if } a = b \end{cases}.$$

$\prod_{a\in A} U_a$ is a neighborhood of (x_a), $\prod_{a\in A} V_a$ is a neighborhood of (y_a), and $\prod_{a\in A} U_a$ and $\prod_{a\in A} V_a$ are disjoint. Therefore, $\{X_a\}_{a\in A}$ is Hausdorff.

2. Let X be a Hausdorff space, and let $Y \subset X$. Let $x, y \in Y$. Find disjoint neighborhoods U of x and V of y in X. Then $U \cap Y$ is a Y-neighborhood of x, $V \cap Y$ is a Y-neighborhood of y, and $U \cap Y$ and $V \cap y$ are disjoint. Therefore, Y is Hausdorff. $\square$

Proposition.

1. A product of regular spaces is regular.
2. A subspace of a regular space is regular.

Proof. 1. Let $\{X_a\}_{a\in A}$ be a family of regular topological spaces. For each a, X_a is Hausdorff. Therefore, $\prod_{a\in A} X_a$ is Hausdorff, so points in $\prod_{a\in A} X_a$ are closed.

Let (x_a) be a point in $\prod_{a\in A} X_a$. Take a neighborhood of (x_a) in $\prod_{a\in A} X_a$; replacing the neighborhood with a smaller one if necessary, I may assume that it has the form $\prod_{a\in A} U_a$, where U_a is open in X_a and $U_a = X_a$ for all but finitely many a's.

Since X_a is regular for each a, I may find a neighborhood V_a of x_a in X_a such that $\overline{V_a} \subset U_a$. If a is an index for which $U_a = X_a$, I will take $V_a = X_a$ as well. Then $\prod_{a\in A} V_a$ is a neighborhood of (x_a) in $\prod_{a\in A} X_a$, and
$$\overline{\prod_{a\in A} V_a} = \prod_{a\in A} \overline{V_a} \subset \prod_{a\in A} U_a.$$

It follows by an earlier result that $\prod_{a\in A} X_a$ is regular.

2. Let X be a regular topological space, and let $Y \subset X$. A point of Y is closed in X, so it is closed in Y.

Let $x \in Y$, and let C be a closed subset of Y which does not contain x. Write $C = Y \cap D$, where D is closed in X. Find disjoint open sets U and V in X such that $x \in U$ and $D \subset V$. Then $Y \cap U$ and $Y \cap V$ are disjoint open sets in Y, $x \in Y \cap U$, and $C \subset Y \cap V$. Therefore, Y is regular. $\square$

Remark. The preceding results are false for normal spaces: A subspace of a normal space need not be normal, and a product of normal spaces need not be normal.

The following results say that the class of normal spaces is large enough to contain many interesting spaces: specifically, metric spaces, compact Hausdorff spaces, and regular spaces having countable bases.

Theorem. Every metric space is normal.

Proof. Let (X, d) be a metric space. Since metric spaces are Hausdorff, points in X are closed.

Let C and D be disjoint closed sets in X. Since $C \subset X - D$ and $X - D$ is open, for every $x \in C$ I may find an open ball $B(x; \epsilon_x)$ contained in $X - D$.

By a similar argument, for every $y \in D$, there is a ball $B(y; \epsilon_y)$ contained in $X - C$.

Let

$$U = \bigcup_{x \in C} B\left(x; \frac{\epsilon_x}{2}\right) \quad \text{and} \quad V = \bigcup_{y \in D} B\left(y; \frac{\epsilon_y}{2}\right).$$

U and V are open sets, $C \subset U$, and $D \subset V$. I claim that U and V are disjoint.

Suppose on the contrary that $z \in U \cap V$ — say $z \in B\left(x; \frac{\epsilon_x}{2}\right)$ for $x \in C$ and $z \in B\left(y; \frac{\epsilon_y}{2}\right)$ for $y \in D$. Then

$$d(x, y) \le d(x, z) + d(y, z) < \frac{\epsilon_x}{2} + \frac{\epsilon_y}{2}.$$

If $\epsilon_x \le \epsilon_y$, then

$$d(x, y) < \frac{\epsilon_x}{2} + \frac{\epsilon_y}{2} \le \frac{\epsilon_y}{2} + \frac{\epsilon_y}{2} = \epsilon_y.$$

This means that $x \in B(y; \epsilon_y)$, so $x \notin C$, contradicting the fact that $x \in C$.

Likewise, $\epsilon_y \le \epsilon_x$ implies that $d(x, y) < \epsilon_x$, which in turns implies that $y \in B(x; \epsilon_x)$. This means that $y \notin D$, contradicting the fact that $y \in D$.

This proves my claim that U and V are disjoint. Since I've separated the closed sets C and D with disjoint open sets U and V, it follows that X is normal. □

Theorem. A compact Hausdorff space is normal.

Proof. In a Hausdorff space, points are closed.

I need to show that disjoint closed sets can be separated by disjoint open sets. First, recall that in a Hausdorff space, a point and a compact set that doesn't contain it can be separated by disjoint open sets.

Let C and D be disjoint closed subsets of the compact Hausdorff space X. Since D is closed and X is compact, D is compact. By the observation above, for each $x \in C$ I may find disjoint open sets U_x and V_x such that $x \in U_x$ and $D \subset V_x$.

$\{U_x\}_{x \in C}$ is an open cover of C; C is compact, since it's a closed subset of a compact space. Let $\{U_{x_1}, \ldots, U_{x_n}\}$ be a finite subcover of C. Define

$$U = \bigcup_{i=1}^{n} U_{x_i} \quad \text{and} \quad V = \bigcap_{i=1}^{n} V_{x_i}.$$

U and V are open sets, $C \subset U$, and $D \subset V$. If $y \in U \cap V$, then $y \in U_{x_i}$ for some i and $y \in V \subset V_{x_i}$, so $y \in U_{x_i} \cap V_{x_i} = \emptyset$. This contradiction shows that U and V are disjoint.

Therefore, X is normal. □

Theorem. A regular space with a countable basis is normal.

Proof. Let X be a regular space with a coutable basis $\{B_i\}_{i \in \mathbb{Z}^+}$. Since X is regular, points are closed.

Let C and D be disjoint closed subsets of X. Let $x \in C$. Since $X - D$ is an open set containing x, I may find a neighborhood U of x such that $x \in U \subset X - D$. Next, regularity implies that there is a neighborhood V of x such that $\overline{V} \subset U$. Finally, there is a basis element B_i containing x such that $B_i \subset V$. Now $\overline{B_i} \subset \overline{V} \subset U \subset X - D$, so $\overline{B_i}$ misses D.

Repeat this procedure for each $x \in C$. I wind up up with a countable subcollection $\{C_j\}$ of $\{B_i\}$ which covers C, and which satisfies $\overline{C_j} \cap D = \emptyset$ for all j.

Likewise, I may find a countable subcollection $\{D_k\}$ of $\{B_i\}$ which covers D, and which satisfies $\overline{D_k} \cap C = \emptyset$ for all k.

What I'd like to do is to take the unions of the two subcollections as my neighborhoods of C and D, but these unions may not be disjoint. However, since the subcollections are countable, I can inductively adjust the subcollections to produce new subcollections whose unions will be disjoint.

Thus, for each j and k, define

$$C'_j = C_j - \bigcup_{k=1}^{j} \overline{D_k} \quad \text{and} \quad D'_k = D_k - \bigcup_{j=1}^{k} \overline{C_j}.$$

Then let

$$A = \bigcup_{j=1}^{\infty} C'_j \quad \text{and} \quad B = \bigcup_{k=1}^{\infty} D'_k.$$

I claim that A and B are disjoint open sets, $C \subset A$, and $D \subset B$.

Each C'_j and each D'_k is open, since each is an open set minus a closed set. Therefore, A and B are unions of open sets, so they are open.

The union of the C_j's contains C. This is not changed by subtracting $\overline{D_k}$'s, because these sets missed C anyway. Therefore, $A = \bigcup_{j=1}^{\infty} C'_j$ contains C. Similarly, B contains D.

Finally, I'll show that A and B are disjoint. Suppose $y \in A \cap B$. Then $y \in C'_j \cap D'_k$ for some j and k. Without loss of generality, suppose that $j \le k$. Now $y \in C'_j$ implies $y \in C_j$, but $y \in D'_k$ implies $y \notin \overline{C_j}$ for $j \le k$. This contradiction proves that $A \cap B = \emptyset$.

Therefore, A and B are disjoint neighborhoods of C and D, and X is normal. □

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[1] Lynn A. Steen and J. Arthur Seebach, *Counterexamples in Topology.* New York: Holt, Rinehart, and Winston, Inc., 1970. [ISBN 0-03-079485-4]

7-22-1999

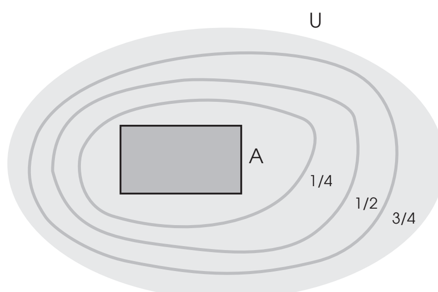
Urysohn's Lemma

Urysohn's lemma is often expressed by saying that disjoint closed sets in a normal space can be separated by a continuous function — that is, there is a continuous, real-valued function which is 0 on one of the closed sets and 1 on the other.

Note that if A and B are disjoint closed sets in a topological space X, then $X - B$ is an open set containing A. Conversely, if A is closed and U is an open set containing A, then A and $X - U$ are disjoint closed sets.

Thus, Urysohn's lemma can be expressed in another form: In a normal space, given a closed set and an open set containing it, there is a continuous, real-valued function which is 0 on the closed set and 1 outside the open set. It is this version that I'll prove; the discussion above shows that the other version is an easy corollary.

The idea is that, for such a function to exist, there should be "level sets" — a set where the function is equal to $\frac{1}{4}$, a set where the function is equal to $\frac{1}{2}$, and so on from 0 to 1. Going the other way, maybe I can define such a function by constructing the level sets.



There are various ways of constructing the level sets; they differ in the way they index the level sets. I will take the approach of [1] and [2], which index the level sets using the **dyadic rationals** in $[0,1]$ — the rationals which can be written in the form $\dfrac{k}{2^n}$. The first step is to show that, given an appropriate collection of such level sets, one may define a *continuous* function by using the level set indices in the obvious way.

Lemma. Let X and Y be topological spaces, let $f : X \to Y$, and let $\mathcal{S}$ be a subbasis for the topology on Y. f is continuous if and only if $f^{-1}(U)$ is open in X for every $U \in \mathcal{S}$.

Proof. Since subbasic sets are open, if f is continuous, then $f^{-1}(U)$ is open in X for every $U \in \mathcal{S}$.

Conversely, suppose that $f^{-1}(U)$ is open in X for every $U \in \mathcal{S}$. An arbitrary open set in Y is a union of finite intersections of elements of $\mathcal{S}$. Since f^{-1} preserves arbitrary unions and arbitrary intersections, the inverse image of an arbitrary open set in Y is open in X. Therefore, f is continuous. □

Lemma. Let X be a topological space, and let

$$S = \left\{ \frac{k}{2^n} \;\middle|\; n \geq 0, \quad 0 \leq k \leq 2^n \right\}.$$

Suppose that for each $s \in S$, there is an open set U_s in X such that if $r, s \in S$ and $r < s$, then $\overline{U_r} \subset U_s$. The function $f : X \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \inf_{\mathbb{R}}\{r \in S \mid x \in U_r\} & \text{if } x \in U_1 \\ 1 & \text{if } x \notin U_1 \end{cases}$$

is continuous.

Note that some elements of S are represented by more than one fraction of the form $\frac{k}{2^n}$ — but repetitions are eliminated by implication, since S is a *set*.

Proof. The open rays $(-\infty, a)$, $(b, +\infty)$ for $a, b \in \mathbb{R}$ form a subbasis for the standard topology on $\mathbb{R}$. It therefore suffices to show that $f^{-1}((-\infty a))$ and $f^{-1}((b, +\infty))$ are open for all $a, b \in \mathbb{R}$.

Consider the set $f^{-1}((-\infty, a))$. The range of f lies in $[0, 1]$. Thus, if $a > 1$, then $f^{-1}((-\infty, a)) = X$, and if $a \le 0$, then $f^{-1}((-\infty, a)) = \emptyset$. In either case, $f^{-1}((-\infty, a))$ is open.

Suppose then that $0 < a \le 1$. I will show that $f^{-1}((-\infty, a)) = \bigcup_{r<a} U_r$.

Let $x \in f^{-1}((-\infty, a))$, so $f(x) < a \le 1$. Since $f(x)$ is the greatest lower bound of the indices r such that $x \in U_r$, and since $f(x) < a$, there is an index $r < a$ such that $x \in U_r$. Hence, $x \in \bigcup_{r<a} U_r$.

Conversely, suppose $x \in \bigcup_{r<a} U_r$. Suppose $x \in U_r$, where $r < a$. Then $f(x) \le r < a$, so $x \in f^{-1}((-\infty, a))$.

Thus, $f^{-1}((-\infty, a)) = \bigcup_{r<a} U_r$, so $f^{-1}((-\infty, a))$ is open.

Now consider the set $f^{-1}((b, +\infty))$. The range of f lies in $(-\infty, 1]$. Thus, if $b \ge 1$, then $f^{-1}((b, +\infty)) = \emptyset$, and if $b < 0$, then $f^{-1}((b, +\infty)) = X$. In either case, $f^{-1}((b, +\infty))$ is open.

Suppose then that $0 \le b < 1$. I will show that $f^{-1}((b, +\infty)) = \bigcup_{r>b}(X - \overline{U_r})$.

Let $x \in f^{-1}((b, +\infty))$, so $f(x) > b$. Since in addition $b < 1$, I may find $r, s \in S$ such that $b < r < s < f(x)$; by construction, $\overline{U_r} \subset U_s$. Since $f(x)$ is the greatest lower bound of the indices t such that $x \in U_t$, and since $f(x) > s$, it follows that $x \notin U_s$. Hence, $x \notin \overline{U_r}$, so $x \in X - \overline{U_r}$, and $x \in \bigcup_{r>b}(X - \overline{U_r})$.

Conversely, suppose $x \in \bigcup_{r>b}(X - \overline{U_r})$. Suppose $x \in X - \overline{U_r}$, where $r > b$. Then $f(x) \notin U_r$, so the fact that the U's are nested implies that $f(x) \notin U_s$ for all $s \in S$ with $s \le r$. Therefore, $f(x) \ge r > b$, so $x \in f^{-1}((b, +\infty))$.

Thus, $f^{-1}((b, +\infty)) = \bigcup_{r>b}(X - \overline{U_r})$, so $f^{-1}((b, +\infty))$ is open.

Therefore, f is continuous. □

Theorem. (**Urysohn's Lemma**) Let X be a normal space, let C be a closed subset of X, and let U be an open set containing C. There is a continuous function $f : X \to [0, 1]$ such that $f(C) = 0$ and $f(X - U) = 1$.

Proof. I'll define a sequence of sets U_r indexed by

$$S = \left\{ \frac{k}{2^n} \;\middle|\; n \ge 0, \quad 0 \le k \le 2^n \right\}$$

and satisfying $\overline{U_r} \subset U_s$ for $r < s$.

Let $U_1 = U$. By normality, I may find an open set U_0 such that $C \subset U_0 \subset \overline{U_0} \subset U_1$.

By normality, I may find an open set $U_{1/2}$ such that

$$\overline{U_0} \subset U_{1/2} \subset \overline{U_{1/2}} \subset U_1.$$

By normality, I may find open sets $U_{1/4}$ and $U_{3/4}$ such that

$$\overline{U_0} \subset U_{1/4} \subset \overline{U_{1/4}} \subset U_{1/2}$$

and

$$\overline{U_{1/2}} \subset U_{3/4} \subset \overline{U_{3/4}} \subset U_1.$$

Keep going. By construction, the U's satisfy the hypotheses of the lemma. The function f constructed by the lemma satisfies the conclusion of the theorem. □

Corollary. Let X be a normal space, and let A and B be disjoint closed subset of X. There is a continuous function $f : X \to [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$. □

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[1] Glen Bredon, *Topology and Geometry*.New York: Springer-Verlag New York, Inc., 1993. [ISBN 0-387-97926-3].

[2] Serge Lang, *Real Analysis* (2^{nd} edition). Reading, Massachusetts: Addison-Wesley Publishing Company, Inc., 1983. [ISBN 0-201-14179-5].

The Tietze Extension Theorem

The **Tietze Extension Theorem** says that a continuous real-valued function on a closed subset of a normal space may be extended to the entire space. I'll show that this extension condition is essentially *equivalent* to normality.

The construction of the extension involves building a sequence of functions whose *sum* agrees with the given function on the subspace.

Lemma. Let X be a normal space, let A be a closed subset of X, and let $f : A \to \mathbb{R}$ be a continuous function satisfying $|f(x)| \le c$ for all $x \in A$ and some $c \in \mathbb{R}$. Then there is a continuous function $g : X \to \mathbb{R}$ such that

1. $|g(x)| \le \dfrac{1}{3}c$ for all $x \in X$.
2. $|f(x) - g(x)| \le \dfrac{2}{3}c$ for all $x \in A$.

Proof. The sets

$$f^{-1}\left(\left[\frac{1}{3}c, \infty\right)\right) \quad \text{and} \quad f^{-1}\left(\left(-\infty, -\frac{1}{3}c\right]\right)$$

are disjoint closed sets in A; since A is closed, these sets are also closed in X. By Urysohn's lemma, I may find a continuous function $g : X \to \left[-\frac{1}{3}c, \frac{1}{3}c\right]$ such that

$$g\left(f^{-1}\left(\left(-\infty, -\frac{1}{3}c\right]\right)\right) = -\frac{1}{3}c \quad \text{and} \quad g\left(f^{-1}\left(\left[\frac{1}{3}c, \infty\right)\right)\right) = \frac{1}{3}c.$$

By construction, $|g(x)| \le \dfrac{1}{3}c$ for all $x \in X$.

Now consider cases. If $-c \le f(x) \le -\dfrac{1}{3}c$, then $g(x) = -\dfrac{1}{3}c$, so $|f(x) - g(x)| \le \dfrac{2}{3}c$.

If $\dfrac{1}{3}c \le f(x) \le c$, then $g(x) = \dfrac{1}{3}c$, so $|f(x) - g(x)| \le \dfrac{2}{3}c$.

Finally, suppose $-\dfrac{1}{3}c < f(x) < \dfrac{1}{3}c$. Since $-\dfrac{1}{3}c \le g(x) \le \dfrac{1}{3}c$, I again have $|f(x) - g(x)| \le \dfrac{2}{3}c$. $\square$

Theorem. Let X be Hausdorff. The following are equivalent:

1. X is normal.
2. If A is a closed subset of X, any continuous function $f : A \to \mathbb{R}$ extends to a continuous function $F : X \to \mathbb{R}$.

   To say that F **extends** f means that $F|_A = f$.

Remark. $\mathbb{R}$ may be replaced by a closed interval $[a, b]$.

Proof. ($2 \Rightarrow 1$) Suppose that if A is a closed subset of X, any continuous function $f : A \to \mathbb{R}$ extends to a continuous function $F : X \to \mathbb{R}$. Let C and D be disjoint closed subsets of X. Define $f : C \cup D \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in C \\ 1 & \text{if } x \in D \end{cases}.$$

$C \cup D$ is closed, so by assumption, I may extend f to a function $F : X \to \mathbb{R}$. Pick disjoint open sets U and V in $\mathbb{R}$ such that $0 \in U$ and $1 \in V$. Then $F^{-1}(U)$ and $F^{-1}(V)$ are disjoint open sets in X, $C \subset F^{-1}(U)$, and $D \subset F^{-1}(V)$. Therefore, X is normal.

$(1 \Rightarrow 2)$ Suppose first that $|f(x)| \leq c$ for all $x \in A$ and some $c \in \mathbb{R}$. I'll extend f to $F : X \to \mathbb{R}$ such that $|F(x)| \leq c$.

By the Lemma, I may find a continuous function $g_0 : X \to \mathbb{R}$ such that

$$|g_0(x)| \leq \frac{1}{3}c \quad \text{for all} \quad x \in X,$$
$$|f(x) - g_0(x)| \leq \frac{2}{3}c \quad \text{for all} \quad x \in A.$$

Next, apply the Lemma to $f - g_0 : A \to \mathbb{R}$ to find a function $g_1 : X \to \mathbb{R}$ such that

$$|g_1(x)| \leq \frac{1}{3} \cdot \frac{2}{3}c \quad \text{for all} \quad x \in X,$$
$$|f(x) - g_0(x) - g_1(x)| \leq \frac{2}{3} \cdot \frac{2}{3}c \quad \text{for all} \quad x \in A.$$

Suppose $n > 1$ and $g_{n-1} : X \to \mathbb{R}$ has been defined and satisfies

$$|g_{n-1}(x)| \leq \frac{1}{3} \cdot \left(\frac{2}{3}\right)^{n-1} c \quad \text{for all} \quad x \in X,$$
$$|f(x) - g_0(x) - g_1(x) - \cdots - g_{n-1}(x)| \leq \left(\frac{2}{3}\right)^{n} c \quad \text{for all} \quad x \in A.$$

Apply the Lemma to find $g_n : X \to \mathbb{R}$ satisfying

$$|g_n(x)| \leq \frac{1}{3} \cdot \left(\frac{2}{3}\right)^{n} c \quad \text{for all} \quad x \in X,$$
$$|f(x) - g_0(x) - g_1(x) - \cdots - g_n(x)| \leq \left(\frac{2}{3}\right)^{n+1} c \quad \text{for all} \quad x \in A.$$

I want to define $F(x) = \displaystyle\sum_{n=0}^{\infty} g_n(x)$. I must show that this series converges. To see this, note that $|g_n(x)| \leq \dfrac{1}{3} \cdot \left(\dfrac{2}{3}\right)^n c$, and that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3} \cdot \left(\frac{2}{3}\right)^n c$ is a convergent geometric series. Therefore, $\sum_{n=0}^{\infty} g_n(x)$ converges absolutely by direct comparison.

Next, I want to show that $F(x)$ is continuous. Let $x \in X$, and let

$$s_k(x) = \sum_{n=0}^{k} g_n(x)$$

be the k^{th} partial sum.

If $k > j$, then

$$|s_k(x) - s_j(x)| = \left|\sum_{n=0}^{k} g_n(x) - \sum_{n=0}^{j} g_n(x)\right| = \left|\sum_{n=j+1}^{k} g_n(x)\right| \leq \sum_{n=j+1}^{k} |g_n(x)| \leq \sum_{n=j+1}^{k} \frac{1}{3} \cdot \left(\frac{2}{3}\right)^n c =$$

$$\frac{1}{3} \cdot \left(\frac{2}{3}\right)^{j+1} c \cdot \sum_{n=0}^{k-(j+1)} \left(\frac{2}{3}\right)^n = \frac{1}{3} \cdot \left(\frac{2}{3}\right)^{j+1} c \cdot \left(\frac{1 - \left(\frac{2}{3}\right)^{k-j}}{1 - \frac{2}{3}}\right) = \left(\left(\frac{2}{3}\right)^{j+1} - \left(\frac{2}{3}\right)^{k+1}\right) c.$$

Fix j and let $k \to \infty$:

$$|F(x) - s_j(x)| \le \left(\frac{2}{3}\right)^{j+1} c.$$

Since the right side goes to 0 as $j \to \infty$ *independent of* x, it follows that the partial sums converge uniformly to $F(x)$. Since the partial sums are continuous, it follows that $F(x)$ is continuous.

For $x \in A$, I have

$$\left|f(x) - \sum_{n=0}^{k} g_n(x)\right| \le \left(\frac{2}{3}\right)^{k+1} c.$$

As $k \to \infty$, the right side goes to 0, and the sum goes to $F(x)$. Thus, F agrees with f on A.

Before I consider the case where f is unbounded, I need to observe something about the preceding construction. I claim that if $|f(x)| < c$, I can construct an extension $F'(x)$ satisfying $|F'(x)| < c$.

To see this, first construct $F(x)$ as above. Note that

$$|F(x)| = \left|\sum_{n=0}^{\infty} g_n(x)\right| \le \sum_{n=0}^{\infty} |g_n(x)| \le \sum_{n=0}^{\infty} \frac{1}{3} \cdot \left(\frac{2}{3}\right)^n c = c.$$

Let $B = F^{-1}(\{-c\} \cup \{c\})$. This is a closed set disjoint from A, since if $x \in A$, $|F(x)| = |f(x)| < c$. Let ϕ be a continuous function such that $\phi(A) = 1$ and $\phi(B) = 0$. Define $F'(x) = \phi(x) \cdot F(x)$. F' still extends f, but now $|F'(x)| < c$.

Now I'll consider the unbounded case where $f : A \to \mathbb{R}$.

Let $\psi : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ be the arctangent function $\psi(x) = \arctan x$. Then $\psi \cdot f : A \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ satisfies $|\psi \cdot f(x)| < \frac{\pi}{2}$. Construct an extension $F : X \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then $\psi^{-1} \cdot F : X \to \mathbb{R}$ is a continuous function which extends f. $\square$