THE GEOMETRY OF THE LINEAR PARTIAL DIFFERENTIAL EQUATION OF THE SECOND ORDER.*

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1. Introduction. The general linear partial differential equation of the second order in one unknown and n variables has an intrinsic geometry defined by its coefficients. This was investigated first by E. Cotton [1], to whom all the basic results are due. Further work introduced little that was new. Struik and Wiener [2], who were mainly interested in a certain physical application of the Cotton theory, recognized that the geometry of the quadratic and linear differential forms involved, under the groups allowed, could be unified in the concept of one geometry—the Weyl geometry—but otherwise added nothing new to the mathematical theory. Levi-Cività [3], using the Cotton theory, confined himself to the problem of finding normal forms, eliminating one of the most interesting groups by a normalization. Moreover, he paid most attention to the case $n=2$, an exceptional case to which the general theory does not apply.

The present paper aims first, by making consistent use of the intrinsic Weyl geometry, to cast the known theory in the form in which the powerful transformation calculus of modern differential geometry can be most directly applied to the equivalence problem (which yields a classification) and to the problems of simplifying the equation in the large by suitable transformations and of finding solutions. Second, making use of these methods, it gives several new results, of which the most important is the criterion for the equivalence of two such equations expressed in finite form in terms of complete sets of invariants of the corresponding Weyl geometries. As a corollary, the criterion that an equation be reducible to ordinary Laplacian form is immediate.

2. The intrinsic geometry. We treat only equations with vanishing undifferentiated term. Let the equation be written (cf. [4] for a concise summary of the notations used by Schouten and others)

$$(2.1)\qquad g^{rs}\partial^2{}_{rs}\phi + d^r\partial_r\phi = 0 \;\; [\partial_r = \partial/\partial x^r;\, r, s, \cdots, = 1, \cdots, n;\, \det g^{rs} \neq 0].$$

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The question of whether the left member might represent simply the Laplacian of ϕ in a curved space endowed with a suitable linear connection and metric is answered in the affirmative by the following theorem.

THEOREM 1. *Equations of the type (2. 1) can always be written as the generalized Laplacian of ϕ equals zero in terms of covariant differentiation with respect to a unique Weyl-type linear connection. The associated intrinsic geometry of the quation is a Weyl geometry W_n $(n \neq 2)$ and is uniquely determined.*

Proof. The first part of the theorem states that (2. 1) is identical with

$$\nabla_r(g^{rs}\partial_s\phi) = 0 \tag{2. 2}$$

for a unique Weyl-type connection, i. e., a symmetric linear connection $\Lambda_{pq}{}^t = \Lambda_{qp}{}^t$ for which there exist a symmetric tensor G_{rs}(det $G_{rs} \neq 0$) and a vector F_r such that

$$\Lambda_{pq}{}^t = C_{pq}{}^t - \tfrac{1}{2}(A^t{}_pF_q + A^t{}_qF_p - G_{pq}F^t) \tag{2. 3}$$

where $C_{pq}{}^t$ is the Christoffel symbol of G_{rs}, G^{rs} are the normalized cofactors, the unit tensor $A^p{}_q \equiv \delta^p{}_q$, and $F^t = G^{tr}F_r$. Expanding (2. 2) using (2. 3), and comparing with (2. 1), one obtains the unique solutions $G_{rs} = g_{rs}$, the normalized cofactors of g^{rs}, and

$$F^r = -2/n(d^r + g^{pq}C_{pq}{}^r), \tag{2. 4}$$

which proves the first part. For reasons which will emerge in a moment we define $(n \neq 2)$:

$$f^r \equiv (1-2/n)^{-1}F^r = (1-n/2)^{-1}(d^r + g^{pq}C_{pq}{}^r), \qquad f_s = g_{sr}f^r. \tag{2. 4$'$}$$

Then the connection in (2. 2) in terms of g_{rs} and f_s is

$$\Lambda_{pq}{}^t = C_{pq}{}^t - \tfrac{1}{2}(1-2/n)(A^t{}_pf_q + A^t{}_qf_p - g_{pq}f^t). \tag{2. 3$'$}$$

The second part of the theorem asserts that we can really associate a *geometry* with the equation; that is, that the geometrical form in which we have cast (2. 1) persists unchanged throughout the group of allowable transformations which take (2. 1) into equivalent forms. Now this group[1] is the direct product of $\mathfrak{G}_n$: $x^{p'} = f^{p'}(x^q)$, det $\partial_q x^{p'} \neq 0$: non-singular change of coordinates; $\mathfrak{F}$: multiplication through of the equation by a factor $\tau(x^q) > 0$:

[1] Beside the two groups considered here, Cotton treats a third group $\phi \rightarrow \rho\phi$ transforming the unknown. If this further group is adjoined, the equivalence classes will be correspondingly larger.

gauge group. This second part then asserts that the intrinsic geometry is the Weyl gecmetry W_n (cf. [5], p. 81) defined by the symmetric and linear differential forms g_{rs} and f_s respectively in the precise sense that a) under a transformation of $\mathfrak{G}_n$, (2. 1) goes into $\nabla_{r'}(g^{r's'}\partial_{s'}\phi) = 0$, where the connection $\Lambda_{p'q'}{}^{t'}$ is given by (2. 3)′ with

$$\partial_{r'} = \partial/\partial x^{r'}; \qquad g_{r's'} = g_{rs}\partial_{r'}x^r\partial_{s'}x^s, \qquad f_{s'} = f_s\partial_{s'}x^s,$$

and that b) under a transformation of $\mathfrak{F}$ with τ it goes into $'\nabla_r('g^{rs}\partial_s\phi) = 0$, where the connection $'\Lambda_{pq}{}^t$ is given by (2. 3)′ with

$$'g_{rs} = \lambda g_{rs}, \qquad 'f_s = f_s + \partial_s \log \lambda; \qquad \lambda = \tau^{-1}.$$

For then g_{rs} and f_s transform under $\mathfrak{G}_n \times \mathfrak{F}$ as in W_n.

Proof. a) is immediate from the invariant form of (2. 2). b) is shown as follows: $g^{rs} \rightarrow {'g^{rs}} \equiv \tau g^{rs}$, hence $g_{rs} \rightarrow {'g_{rs}} = \lambda g_{rs}$ for $\lambda = \tau^{-1}$. By (2. 4)′,

$$f_s \rightarrow {'f_s} \equiv (1 - n/2)^{-1}('g_{sr}'d^r + 'g^{pq}'C_{pq}{}^{r\prime}g_{sr}), \qquad 'd^r \equiv \tau d^r = \lambda^{-1}d^r.$$

This equals

$$f_s + (2 - n)^{-1}\lambda^{-1}g^{pq}(2g_{ps}\partial_q\lambda - g_{pq}\partial_s\lambda) = f_s + \partial_s \log \lambda,$$

q. e. d. (It should be noted that the connection (2. 3)′ in terms of which (2. 2) is written is *not* the same as the Weyl connection belonging to W_n. In particular, $\Lambda_{pq}{}^r$ is not gauge-invariant.)

It is remarkable that this theory breaks down for $n = 2$. This case is treated briefly at the end of the article.

Of particular interest is the subclass of self-adjoint equations, those which, after multiplication by a suitable (positive) factor, can be written in the form

(2. 1)′ $\qquad \partial_r(P^{rs}\partial_s\phi) = 0 \qquad$ [for some set of functions P^{rs}].

This property has a very nice geometrical characterization (remark: this geometrical characterization of self-adjointness is independent of the boundary behavior of ϕ):

THEOREM 2. Equation (2. 1) is self-adjoint if and only if its intrinsic geometry is Riemannian, ($n \neq 2$).

We recall that W_n is Riemannian if and only if $\partial_{[s}f_{r]} = 0$; i. e., there exists a gauge in which $f_r = 0$.

Proof. If W_n is Riemannian, then in the gauge in which $f_r = 0$ by (2.3)′ ∇_r becomes covariant differentiation with respect to $C_{pq}{}^t$. Hence (2.2) reads

$$\nabla_r(g^{rs}\partial_s\phi) \equiv |\, g\,|^{-\frac{1}{2}}\partial_r(|\, g\,|^{\frac{1}{2}}g^{rs}\partial_s\phi) = 0, \qquad [g \equiv \det g_{rs}].$$

Multiply through by $|\, g\,|^{\frac{1}{2}}$; this proves the sufficiency.

Conversely, if equation (2.1) is self-adjoint, then in a suitable gauge $d^r = \partial_s g^{sr}$. From (2.4)′,

$$f_r = (1 - n/2)^{-1} g_{rs}(\partial_p g^{ps} - \partial_p g^{ps} - \tfrac{1}{2} g^{sq}\partial_q \log |\, g\,|) = (n-2)^{-1}\partial_r \log |\, g\,|,$$

so f_r is a gradient, which proves the necessity.

3. Curvature and the equivalence criterion. Two equations (2.1) are *equivalent* (by this we shall always mean equivalent under the product group $\mathfrak{G}_n \times \mathfrak{F}$) if and only if their intrinsic Weyl geometries are the same. For they both can be written in the form (2.2) with connections of the form (2.3)′ and this prescription is invariant against the transformations considered. It follows that the equivalence problem for these linear equations reduces to the equivalence problem for the Weyl geometry W_n. This closely parallels the usual treatment of the equivalence of quadratic differential forms with slight complications due to the fact that here we have a linear form adjoined and the gauge group as well as coordinate transformation group. It is a noteworthy fact that the equivalence-characterizing system of invariants of the quadratic form and that of the linear form are completely unified in the invariants of W_n. We sketch the proof below, followed by the theorem.

Let g_{pq} and f_r, functions of x^p, define the intrinsic geometry of the first equation, and $'g_{p'q'}$, $'f_{r'}$, functions of $x^{p'}$, that of the second equation. Then we ask whether there exist a gauge transformation $\lambda(x^{q'})$ and coordinate transformation $x^p = f^p(x^{q'})$ such that

(3.1) $\quad 'g_{p'q'} - \lambda g_{pq}A^p{}_{p'}A^q{}_{q'} = 0, \qquad$ (3.2) $\quad 'f_{p'} - f_p A^p{}_{p'} - \Lambda_{p'} = 0,$

where

(3.3) $\quad \partial_{p'}x^p = A^p{}_{p'} \qquad$ (3.3)′ $\quad \partial_{p'}\log\lambda = \Lambda_{p'}.$

Differentiating (3.1) and using (3.2), (3.3), and (3.3)′, we get on rearranging

(3.4) $\quad \Gamma_{p'q'}{}^{r'}A^r{}_{r'} - \Gamma_{pq}{}^r A^p{}_{p'}A^q{}_{q'} - \partial_{p'}A^r{}_{q'} = 0,$

where $\Gamma_{pq}{}^r$ (the linear connection of W_n) is short for

$$\Gamma_{pq}{}^r = C_{pq}{}^r - \tfrac{1}{2}(A^r{}_p f_q + A^r{}_q f_p - g_{pq} f^r)$$

and $\Gamma_{p'q'}{}^{r'}$ is the corresponding expression [2] in $'g_{p'q'}$ and $'f_{r'}$. Differentiating (3. 2) and using (3. 3) and 3. 4), we get on rearranging

$$(3.\,4)' \qquad \nabla_{q'}{}'f_{p'} - \nabla_q f_p A^q{}_{q'} A^p{}_{p'} - \nabla_{q'}\Lambda_{p'} = 0,$$

where ∇_q *from here on will mean covariant differentiation with respect to* $\Gamma_{pq}{}^r$, and $\nabla_{q'}$, correspondingly, with respect to $\Gamma_{p'q'}{}^{r'}$. The problem then reduces to solving the system of partial differential equations (3. 3), (3. 3)′, (3. 4), (3. 4)′ and the finite equations (3. 1), (3. 2) in the $(n+1)^2$ unknowns x^p, λ, $A^p{}_{p'}$, $\Lambda_{p'}$ as functions of $x^{p'}$.

The integrability conditions of (3. 3) are satisfied in virtue of (3. 4). The integrability conditions of (3. 4) are

$$(3.\,5) \qquad R_{p'q'r'}{}^{t'} A^t{}_{t'} = R_{pqr}{}^t A^p{}_{p'} A^q{}_{q'} A^r{}_{r'},$$

where $R_{pqr}{}^t$ (the curvature tensor of W_n) stands for

$$R_{pqr}{}^t \equiv -2\partial_{[p}\Gamma_{q]r}{}^t - 2\Gamma_{[p|s|}{}^t\Gamma_{q]r}{}^s$$

correspondingly for $R_{p'q'r'}{}^{t'}$ in terms of $\Gamma_{q'r'}{}^{t'}$, and the infinite sequence of equations obtained by repeated differentiation of (3. 5):

$$(3.\,6) \qquad \nabla_{s'} R_{p'q'r'}{}^{t'} A^t{}_{t'} = \nabla_s R_{pqr}{}^t A^p{}_{p'} A^q{}_{q'} A^r{}_{r'} A^s{}_{s'}, \cdots = \cdots, \text{etc.}$$

It is remarkable to note now that the integrability conditions of the other equations, those arising from the linear form, impose no new conditions. For, the integrability conditions of (3. 3)′ are, from (3. 4)′,

$$f_{q'p'} = f_{qp} A^q{}_{q'} A^p{}_{p'}, \qquad f_{qp} \equiv \nabla_{[q} f_{p]} = \partial_{[q} f_{p]},$$

correspondingly for $f_{q'p'}$ in terms of $'f_{p'}$, and the infinite sequence of equations arising from these by repeated differentiation. But multiplying (3. 5) through by $A^{s'}{}_t$ (the normalized cofactors of $A^t{}_{t'}$), contracting r' and s', and using the identities $R_{pqr}{}^r = nf_{pq}$, $\nabla_s R_{pqr}{}^r = n\nabla_s f_{pq}, \cdots$, we find that these integrability conditions are satisfied in virtue of (3. 5), (3. 6), $\cdots$. Moreover, the integrability conditions of (3. 4)′, with the aid of (3. 2), come out to be

$$(R_{p'q'r'}{}^{t'} A^t{}_{t'} - R_{pqr}{}^t A^p{}_{p'} A^q{}_{q'} A^r{}_{r'}) f_t = 0$$

and equations arising from these by repeated differentiation. But these are

[2] Note on notation: Γ instead of $'\Gamma$ is written because it is gauge-invariant (cf. the definition equation). The same remark applies to Δ_q, $R_{pqr}{}^t$, and f_{pq}.

satisfied in virtue of (3.5), (3.6), $\cdots$. Hence by the well-known theorem (cf. say, [6], Chap. 5, § 7) on systems of partial differential equations the equivalence theorem for the linear equations (2.1) reads as follows:

THEOREM 3. *Two equations* (2.1) *whose intrinsic geometries are characterized by the invariants* g_{pq}, f_r (*functions of* x^p) *and* $'g_{p'q'}$, $'f_{r'}$ (*functions of* $x^{p'}$) *respectively for* $n \neq 2$ *are equivalent if and only if there exists a positive integer* N *such that* a) *the sets of equations* (3.1), (3.2), *and the first* N *sets of equations* (3.5), (3.6), $\cdots$, *in the unknowns* x^p, λ, $A^p{}_{p'}$, $\Lambda_{p'}$ *as functions of* $x^{p'}$ *are compatible and* b) *all sets of solutions of these equations satisfy the* $(N+1)$-th *set of equations.*

Therewith, equations (2.1) are classified into equivalence classes.

The simplest equation (2.1) is the ordinary Laplacian equation

$$\delta^{rs}\partial^2{}_{rs}\phi = 0 \qquad [\delta^{rr} = \pm 1; \delta^{rs} = 0, r \neq s]. \tag{3.7}$$

(The term "Laplacian" here embraces all metric signatures.) Then the equivalence theorem gives us immediately the criterion that any equation (2.1) be reducible to this form:

COROLLARY. *An equation* (2.1) *is equivalent to the ordinary Laplacian equation if and only if its intrinsic geometry is flat.*

By *flat* is meant $R_{pqr}{}^t = 0$. (This implies both that the geometry is Riemannian and flat in the Riemannian sense.)

Another application of the complete set of invariants of the intrinsic geometry (in the Riemannian case) is the determination by algebraic means of whether the equation admits any "plane-wave" type solutions. Consider the sets of equations in the unknown $\xi_s(x^p)$

$$(3.8)\quad g^{rs}\xi_r\xi_s = 0, \qquad (3.9)\quad R_{pqt}{}^s\xi_s = 0,$$

$$(3.10)\quad \nabla_l R_{pqt}{}^s\xi_s = 0, \qquad (3.11)\quad \nabla^2{}_{ml} R_{pqt}{}^s\xi_s = 0, \cdots.$$

THEOREM 4. *In the Riemannian case* $f_{pq} = 0$, *there exist solutions of* (2.1) *of the plane-wave type if and only if there exists a positive integer* M *such that* a) *the first* M *of* (3.9), (3.10), $\cdots$ *are compatible for the unknown* $\xi_s(x^p)$ *and* b) *all solutions of these satisfy the* $(M+1)$-th *set of equations, finally,* c) *some solution of these satisfies* (3.8).

$\phi = F(\int k_s dx^s)$ is defined to be a *plane-wave* if F is any twice differentiable function, $\int$ means the indefinite line integral, and k_s is a null parallel field: $\nabla_r k_s = 0$, $g^{rs}k_r k_s = 0$.

Proof. If and only if such an M exists does there exist a field k_s parallel with respect to the Riemannian space defined by W_n in the gauge for which $f_r = 0$ (cf. [7], § 23); k_s then is a solution of (3. 9), (3. 10), $\cdots$. If and only if one of these solutions also satisfies (3. 8) does there exist a null parallel field. Hence the conditions of the theorem are necessary and sufficient in order that there exist a plane-wave. But every plane-wave yields a solution of (2. 1). For in the gauge in which $f_r = 0$ we get from (2. 2)

$$g^{rs}\nabla^2{}_{rs}\{F(\int k_s dx^s)\} = F''g^{rs}k_r k_s = 0,$$

where F' means the derivative of F with respect to its argument.

4. The case $n = 2$. We add a few words on the anomalous case $n = 2$. The first part of Theorem 1 is still true, but no Weyl geometry W_2 can be associated with the equation. For although g_{pq} and F_r (given by (2. 4)) are tensors against $\mathfrak{G}_2$, under $\mathfrak{F}$ they transform as follows:

$$(4.1) \qquad 'g_{pq} = \lambda g_{pq}, \qquad 'F_p = F_p.$$

Hence the geometry is that of a class of conformally related Riemannian spaces V_2, on each of which the same Pfaffian is superimposed.

Theorem 2 holds in the form: Equation (2. 1) is self-adjoint if and only if F_r is a gradient.

Of course $\partial_{[s}F_{r]} = 0$ has not now any Riemannian interpretation. In proof, note that if the equation is self-adjoint, then in a coordinate and gauge frame in which it takes the form (2. 1)′, we have $F_r = \frac{1}{2}\partial_r \log |g|$ (cf. the proof of Theorem 2). Conversely, if F_r is a gradient, $\frac{1}{2}\partial_r \log h$, $h > 0$, say, perform the gauge transformation with $\lambda = h^{\frac{1}{2}} |g|^{-\frac{1}{2}}$ to a frame where $|'g| = h$, $'F_r = F_r = \frac{1}{2}\partial_r \log |'g|$. Then, as in the proof of Theorem 2, in this coordinate and gauge frame $'d^r = \partial_s {'g^{sr}}$ and hence the equation is self-adjoint.

Theorem 3 does not apply. In the present case $n = 2$ the equations we start from in the equivalence problem are

$$(4.2) \qquad 'g_{p'q'} - \lambda g_{pq}A^p{}_{p'}A^q{}_{q'} = 0; \qquad \partial_{p'}x^p = A^p{}_{p'}$$

$$(4.3) \qquad 'F_{p'} - F_p A^p{}_{p'} = 0.$$

Since any two V_2's of the same signature are conformal (cf. [7], § 28), the problem reduces to using the remaining freedom in the coordinate transformations satisfying (4. 2) to satisfy (4. 3) as well. This was treated at lenth by Cotton ([1], p. 236 *et seq.*) and also elsewhere in the literature.

From the facts that $F_r = 0$ for the ordinary Laplacian equation (3.7), F_r is gauge-invariant for $n = 2$, and any two V_2's of the same signature are conformal, we infer immediately the following

COROLLARY. *The vanishing of F_r is necessary and sufficient in order that the equation* (2.1) *be reducible to ordinary Laplacian form.*

Thus in the case of a definite metric, $F_r = 0$ is the criterion that the solution of (2.1) be reducible to the solution of the Beltrami differential equations.

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BIBLIOGRAPHY.

[1] E. Cotton, "Sur les invariants différentiels de quelques équations linéaires aux dérivées partielles du second ordre," *Annales de l'Ecole Normale Supérieure*, vol. 17 (1900), pp. 211-244.

[2] D. Struik and N. Weiner, "A relativistic theory of quanta," *Journal of Mathematics and Physics*, vol. 7 (1927), pp. 1-23.

[3] T. Levi-Cività, "Sulla trasformazione delle equazioni lineari a derivate parziali del secondo ordine," *Atti del Reale Istituto Veneto*, 3a ser. 15 tomo 72 (1912-13), pp. 1331-1357.

[4] J. Shouten, "La théorie projective de la relativité," *Annales de L'Institut Henri Poincaré,* vol. 5 fasc. 1 (1935), pp. 51-88.

[5] L. Eisenhart, "*Non-Riemannian Geometry,*" (1927).

[6] O. Veblen, "*Invariants of Quadratic Differential Forms,*" (1927).

[7] L. Eisenhart, "*Riemannian Geometry,*" (1926).