The Korteweg - de Vries Equation, the Theory of Solitons and its Impact on Science

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1 History

It was in the year 1834 that a naval architect John Scott Russell made an observation, which much later gave rise to important developments in mathematics and physics. He was on horseback, riding along the Union Canal between Edinburgh and Glasgow, and he saw a boat rapidly drawn by a pair of horses, which however stopped suddenly and a "rounded, smooth and well defined heap of water" loosened from its prow, "continuing its course along the channel apparently without change of form and diminition of speed", rolling forward "at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height", [1]. John Scott Russell performed a large number of experiments in wave tanks, schematized below,[2].

Our naval architect obtained the following results.

i) There is an essential difference between the two instances, whether the elevation of the water to the left of the slide is higher or lower than that to the right of the slide: one or more "heaps" of water, (a) and (b), or an oscillatory wave, (c).
(a) and (b) are examples of solitary waves, called later solitons.

ii) it is possible that a solitary wave splits into a number of smaller solitary waves; this depends on the height of the excitation.

iii) Solitary waves propagate without change of form and without damping, worth mentioning. The velocity of propagation is proportional to its amplitude.

iv) When two solitons propagate from left to right and the larger soliton is behind the smaller one, then the latter will be overtaken by the former. After the collison both emerge without change of form, only their position is interchanged. They behave like colliding marbles exchanging their momentum.

v) A simple empirical formula was obtained for the speed of propagation of the solitary wave.

These results were debated and the question of the possible existence of this type of waves was discussed. Airy came to the conclusion that Scott Russel's wave could not exist and Stokes used the right equation, but drew the wrong conclusion. However, Boussinesq and Lord Rayleigh gave in 1871 respectively 1876 the mathematical proof of the existence of the solitary wave, an "a priori demonstration a posteriori". Nevertheless, the mathematical community, in so far as interested in the solitary wave, dit not agree unanimously upon the results of Boussinesq and Lord Rayleigh and this was for Korteweg and his student de Vries a motive to investigate again the existence of the wave. They published their results in 1895 in the Philosophical Magazine in a paper titled "On the Change of Form of Long Waves advancing in a Rectangular Canal and on a New Type of Long Stationary Waves", [3]. For reasons to be explained below this paper is nowadays on the top of the hit parade of the citation index. Korteweg and de Vries deduced in this paper their equation

$$\frac{\partial \eta}{\partial t} = \frac{3}{2}\sqrt{\frac{g}{\ell}}\frac{\partial}{\partial x}\left(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\sigma\frac{\partial^2 \eta}{\partial x^2}\right),$$

where $\ell + \eta(x,t)$ represents the elevation of the water surface above the bottom of the canal at time t and at a horizontal distance x from the origin of coordinates, g is the constant of gravity, α a small constant in connection with the velocity of the uniform motion given to the liquid and σ is a constant related to as well the capillary tension at the surface as to the fluid density. This equation is valid for waves with an amplitude small and with a wave length large in comparison with the depth ℓ of the canal.

¿From the evaluation of $\eta(x,t)$, being the height of the water surface above equilibrium, follows their conclusion that in a frictionless liquid there may exist absolutely stationary waves, [3].

The K.d.V.-equation is after some scaling usually written in the more attractable form

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \tag{1}$$

where $u(x,t)$ denotes the form of the wave. This equation has a nonlinear term $6u\frac{\partial u}{\partial x}$ accounting for the breaking of the wave – the higher a wave particle, the larger its velocity – and a dispersion term $\frac{\partial^3 u}{\partial x^3}$ accounting for the broadening of the wave profile. These two effects balance each other and give rise to the stable stationary behaviour. Substitution of a travelling wave solution $u(x,t) = f(x - ct)$ leads to periodic solutions; f appears to become the square of the elliptic function cn and that is why these travelling waves are called cnoidal waves.

Whenever the modulus of this elliptic function approaches 1 the soliton solution – a"heap of water" emerges, viz.

$$u(x,t) = -a\,\mathrm{sech}^2\{\sqrt{\frac{a}{2}}(x - 2at)\}, \tag{2}$$

However, when the modulus approaches zero one gets the sinusoidal wave, the one derived by Stokes. After the publication of the paper by Korteweg and de Vries the stationary wave was not considered as an important topic and it remained so untill about 1965, when Zabusky and Kruskal reported on waves in a collisionless plasma and the recurrence of initial states, [4].

Their point of departure was the K.d.V.-equation and besides the recurrence of the initial state of the system their numerical calculations confirmed also the experimental observations of Scott Russell.

In the same time there remained the still unsolved problem of Fermi, Pasta

and Ulam concerning the finite heat conductivity in solids. In order to obtain some insight into this problem the solid was modelled by a one-dimensional string of particles of equal mass and connected with each other by springs. Excitation of the string sets the particles into motion and the subsequent behaviour of these particles is described by a set of ordinary differential equations depending on the interaction force of the springs between neighbouring particles. To obtain statistical equilibrium and finite thermal conductivity Fermi expected that the spring force should be nonlinear, as was already suggested by Debije in 1914. However, the numerical results showed that the initial state of the string returned again after some time and this was against all physical expectations [5].
This recurrence of initial states is similar as that discovered later by Zabusky and Kruskal.
Toda analyzed in about 1967 the Fermi-Pasta-Ulam string using an exponential spring potential; he obtained periodic solutions in the form of elliptic functions for the periodic Toda chain and solitary wave solutions for the infinite chain; the latter involves again the sech^2 function (with discrete space variable); they have all the soliton properties as discovered by Scott Russell, [6].
This is not a miracle because the equations for the discrete Toda chain may be considered as a spatial discretization of the K.d.V.-equation.

2 The Korteweg - de Vries Equation as a Hamilton System

The Toda-chain with a finite number N of particles provides an example of a classical finite dimensional Hamilton system of ordinary differential equations. Toda showed that this system has N independent conserved quantities, mutually in involution and it is therefore according to the Liouville-Arnold theorem integrable. Since this system is a discretization of the K.d.V.-equation, an analysis of the latter calls for a translation of the "classical" Hamilton theory for a system of ordinary differential equations to a Hamilton theory for a partial differential equation such as the K.d.V.-equation. This is possible by replacing the Hamilton function H by a Hamilton functional $\mathcal{H}[u, u_x, u_{xx}, \ldots]$, the gradient ∇H by the variational derivative $\delta_u \mathcal{H}$ and the skew symmetric structure matrix by a skew adjoint differential operator, which may depend on the dependent variable u and its x-derivatives. The following important results have been obtained. The K.d.V.-equation ($m = 1$) can be embedded in a hierarchy of evolution equations which may be expressed in the Hamilton representation

$$\frac{\partial u}{\partial t_{2m+1}} = \frac{\partial}{\partial x}\delta_u \mathcal{H}_{2m+1},\ m = 1, 2, \ldots, \tag{3}$$

where the infinite number of functionals $\{\mathcal{H}_{2n+1}\}_{n=1}^{\infty}$ are conserved for *each* flow $\frac{\partial}{\partial_{2m+1}}$ Moreover, by a second Hamilton representation of this hierarchy it is easily deduced that the conserved functionals are in involution with respect to an appropriately defined Poisson bracket and hence the evolutionary vector fields $\{\frac{\partial}{\partial t_{2n+1}}\}_{n=1}^{\infty}$ commute. An important operator is the recursion operator R mapping the symmetry $\frac{\partial}{\partial t_{2n-1}}$ on $\frac{\partial}{\partial t_{2n+1}}$, [7].
The K.d.V. hierarchy is an example of an "integrable" system of evolution equations and this hierarchy is not the only one. A list of 39 different "integrable" systems together with their recursion operators is presented in [8].

3 Spectral Properties

All solutions $u(x, t_{2n+1})$ of any K.d.V.-equation out of the hierarchy with $u(x, t_{2n+1}) \to 0$ for $|x| \to \infty$ have the property that the discrete L_2-spectrum $\{p_j\}_{j=1}^{J}$ of the Schrödinger operator

$$\frac{d^2}{dx^2} - u(x, t_{2n+1}),$$

is independent of t_{2n+1}. The normalization coefficients of the eigenfunctions $\{\psi_j\}_{j=1}^{J}$, such that $||\psi_j|| = 1$ are denoted by $c_j(t_{2n+1})$ and depend on t_{2n+1}. Besides the discrete spectrum there is the continuous spectrum with generalized eigenfunctions $f_\pm(x, k) \approx e^{ikx}$ for $x \to \pm\infty$. The relation

$$f_+(x, -k) = T(k) f_-(x, k) - R(k, t_{2n+1}) f_+(x, k)$$

defines the transmission coefficient $T(k)$, which is independent of t_{2n+1}, and the reflection coefficient $R(k, t_{2n+1})$. An expansion of $\{T(k)\}^{-1}$ into powers of k^{-1} yields immediately the infinite number of conservation laws.
The potential $u(x, t_{2n+1})$ yields the spectral set
$S(t_{2n+1}) = [\{\lambda_j\}_{j=1}^{J}, \{c_j(t_{2n+1})\}_{j=1}^{J}, T(k), R(k, t_{2n+1})]$, but also conversely this

set $S(t_{2n+1})$ determines the potential $u(x, t_{2n+1})$. This important property and the time evolution of $S(t_{2n+1})$ provides a method to solve initial value problems. The initial value problem for the K.d.V.-equation with $u(x,0) = -N(N+1)\text{sech}^2 x, \quad N = 1,2,3,\ldots$, yields N eigenvalues $\{\lambda_j = j^2\}_{j=1}^N$ and $R(k, t_3) = 0$; the solution is a so-called N-soliton consisting of N separated solitary waves with amplitudes $\{-2j\}_{j=1}^N$ and velocities $\{4j\}_{j=1}^N$. The overtaking of slow solitons by faster solitons without change of form is now easily verified by considering the wave travelling from $x = -\infty$ to $x = +\infty$; there appears only a phase shift for every solitary wave, but the sum of all phase shifts does not change, [9].
A new solution can be constructed from a known solution by adding one or more eigenvalues to those corresponding with the known solution. This proces generates a so-called Bäcklund transformation, named after Bäcklund, who applied already in 1880 a related transformation in his theory of surfaces of constant negative curvature, [9]. A great step forward was made by Sato, taking together all time variables and introducing the τ-function,

$$q(t_1, t_3, t_5 \ldots) = \frac{\partial^2}{\partial t_1^2} \log \tau(t_1, t_3, t_5, \ldots),$$

where the variable x has been replaced by t_1 and where the function q satisfies all equations from the K.d.V. hierarchy.
The operators

$$X(k) = \exp\left[\sum_{n=0}^{\infty} (ik)^{2n+1} t_{2n+1}\right] \exp\left[\sum_{n=0}^{\infty} \frac{-1}{2n+1} (ik)^{-2n-1} \frac{\partial}{\partial t_{2n+1}}\right]$$

produce new solutions from known solutions, satisfying the whole K.d.V. hierarchy by

$$\tau_{\text{new}} = \{A(k)X(k) + B(k)X(-k)\}\tau_{\text{old}}.$$

The above mentioned N soliton-solution is obtained by applying this operator consecutively to $\tau_{\text{old}} \equiv 1$ with $k = i\sqrt{p}_j, \ j = 1,2,\ldots N$. The operators $X(k)$ are known as vertex operators and they form a representation of the infinite dimensional Lie-algebra $A_1^{(1)}$, the one-dimensional central extension of the loop algebra $s\ell(2,\mathbb{C}) \otimes \mathbb{C}(t, t^{-1})$. This connection of soliton equations and the representation of infinite Lie-algebras gives a deep understanding of the rich mathematical structure behind families of nonlinear equations of soliton type. For literature concerning this section we refer the reader to [9]–[13].

As already mentioned in section 2 there exist many other evolution equations with solutions of soliton type. The most well-known are the sine-Gordon equation

$$\frac{\partial^2 \varphi}{\partial u^2} - \frac{\partial^2 \varphi}{\partial v^2} = \sin \varphi$$

and the nonlinear Schrödinger equation

$$i\frac{\partial \psi}{\partial t} - \frac{\partial^2 \psi}{\partial x^2} - \psi|\psi|^2 = 0.$$

The first one appeared in the work of Bäcklund; it is the equation for the angle φ between the asymptotic directions on a surface of constant negative curvature with u and v orthogonal coordinates in the directions of minimal and maximal curvature. The nonlinear Schrödinger equation is the equation for the envelope of nonlinear waves satisfying

$$\frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial t^2} = \alpha\varphi - \beta\varphi^3,$$

where the right-hand side is an approximation of $\sin \varphi$ for $\alpha = 1,\ \beta = \frac{1}{6}$ and φ small.
Besides these continuous equations also their discrete counterparts, so-called lattice equations, should be noted. Research on e.g. magnetic spin chains led physicists to the study of the semi-discrete and fully discrete versions of the soliton equations, also in three and four dimensions. There is, of course, much similarity and the field is of great importance in theoretical physics. The reader is referred to [14].

4 Solitons in Physics and Engineering

Already in an early phase of the development of the theory there were many applications in physics and engineering. As to the K.d.V.-equation we have already mentioned surface waves, magnetohydrodynamic waves in a plasma, and the conduction of heat, but there are many many more applications. An interesting one is the occurrence of internal large solitons, very deep in the Andaman sea between Thailand and Sumatra and in the Sulu sea between the Philipines and Kalimantan; these solitons brought much dammage to oil platforms located in these regions, [11], [15].
Another concerns pressure waves in liquid-gas bubbles mixtures, causing

dammage to ship propellers, [15].
Finally, we mention the stability of the vortex in the red spot of Jupiter [15]. The sine-Gordon equation has been used in several models: the collision of elementary particles (Perring and Skyrme in 1962!), the propagation of a crystal dislocation and the propagation of magnetic flux in a Josephson-junction, a device for information processing systems [2].
The nonlinear Schrödinger equation has been applied in a model for the propagation of optical solitons in glasfibers; this is investigated by Hasegawa [15] and his theory is, due to the stability and the controllability of the optical pulses, of great significance in the theory of high speed communication. Mollenauer c.s. succeeded in transporting 20.10^9 soliton pulses per second over a distance of 14.000 km, [15].
The interested reader may consult a long list of many early applications in [16].

5 Recent Developments

This review is intended only to provide an impression of one of the great discoveries in nonlinear analysis, starting with the K.d.V.-equation. For developments, more recent than those reported above, we mention some sources giving new results. The processes of scattering and inverse scattering are generalized in [17] to the case of several space dimensions; in this reference is also discussed the connection between complete integrability of partial differential equations and the Painlevé equations.
Another source of literature is reference [18], where the impact of soliton theory on many areas of mathematics has been highlighted, such as in nonlinear analysis, algebraic structures, geometry, knot and braid theory, and quantum and statistical mechanics.
Finally, we mention reference [15], the proceedings of the International Symposium in Amsterdam to commemorate the centennial of the equation by and named after Korteweg and de Vries.

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