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Exterior differential system for cosmological G_2 perfect fluids and geodesic completeness

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Abstract. In this paper a new formalism based on exterior differential systems is derived for perfect-fluid spacetimes endowed with an Abelian orthogonally transitive G_2 group of motions acting on spacelike surfaces. This formulation allows simplifications of Einstein equations and it can be applied for different purposes. As an example a singularity-free metric is rederived in this framework. A sufficient condition for a diagonal metric to be geodesically complete is also provided.

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1. Introduction

Perfect-fluid spacetimes endowed with an Abelian G_2 group of isometries have been used for describing many different physical situations. When the group is acting on timelike surfaces, they have been used extensively for describing axisymmetric compact objects in stationary rotation (cf [1] for a review).

On the other hand, if the group acts on spacelike surfaces, the applications are different [2]. A classification of these spacetimes is given in [3]. They can model spacetimes where two plane gravitational waves are colliding (cf [4] for a review), but they are also useful for describing inhomogeneous cosmologies (cf [5, 6] for a review) in an attempt to cope with the inhomogeneity present in our Universe. An interesting feature of G_2 cosmologies is the possibility of avoiding initial and final singularities (cf [1, 7, 8] for a review) and therefore physics will be valid in the whole spacetime. These models satisfy the causality and energy conditions and just fail to contain trapped sets, according to the well known singularity theorems [9, 10].

In this paper we shall try to cope with two features concerning non-singular perfect-fluid orthogonally transitive G_2 cosmological models: first of all one has to devise a method for obtaining exact models and then one should check whether the solution is singular or not. Both of these will be the aim of this paper.

A new tetrad formalism based on differential forms will be introduced for deriving results about spatially inhomogeneous spacetimes. It will be shown that the methods initially devised for stationary axially symmetric spacetimes [11] are also useful when the group of motions acts on spacelike surfaces. The 1-forms that are used in this formalism will be shown to have

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kinematical meaning when considering spatial congruences. If the congruence corresponds to an invariantly defined quantity (for instance, in the cylindrical case, the axial Killing), the kinematical properties can be used to classify the solutions. Also, using the remaining gauge freedom, the tetrad can be adapted to the congruence so that the exterior system can be written in terms of 1-forms with an invariant and physical interpretation. Moreover, when one tries to obtain an exact model, it is generally useful to impose certain assumptions on these kinematical quantities (now with an invariant interpretation). As the approach is grounded on an exterior differential system, the coordinates can be chosen according to the ansatz that is performed, instead of fixing them from the beginning.

Assuming one has obtained a new cosmological model, it usually takes lengthy calculations to determine whether there are singularities in it. The question is settled if the curvature invariants are already singular, but if they are regular in the whole spacetime, there is *a priori* no reason to assume that every geodesic is complete. In fact, there are cosmologies with regular curvature invariants that are incomplete and therefore singular, despite no quantity becoming unboundedly large [9]. Therefore we deem it convenient to have a general result that may simplify the task of analysing the issue of geodesic completeness. This matter will be addressed for the diagonal case in this paper.

Let us describe in more detail the contents of the paper. In section 2 spacelike congruences in a general spacetime will be studied in order to achieve an interpretation for the mathematical quantities that will appear in the formalism. In the stationary axisymmetric case timelike congruences were considered and their tangent field could be considered as a velocity and therefore the interpretation was straightforward. Another way of interpreting spacelike congruences is due to Greenberg [12], but in this paper a different approach will be followed. In section 3 the formalism is written in terms of an exterior system of equations that include Cartan and Einstein field equations as well as their integrability conditions. The set of equations will be simplified taking advantage of the remaining gauge freedom. As an example of how the exterior system can be used for obtaining exact solutions the singularity-free model in [13] will be obtained within the formalism in section 4. The question of geodesic completeness of diagonal inhomogeneous cosmological models will be addressed in section 5 and a theorem will be derived as a sufficient condition for a model to be non-singular. This condition will be shown to be weak enough to comprise all known diagonal singularity-free inhomogeneous cosmological models in section 6.

2. Spacelike congruences

The kinematical properties of a timelike congruence can be defined by decomposing the covariant derivative of the timelike vector field defined by the congruence [14, 15]. An analysis of the spacelike congruences has been made by Greenberg [12]. In his analysis Greenberg introduces an observer moving with a 4-velocity w^a. Projecting the covariant derivative of the vector field defined by the congruence orthogonal to w^a he obtains 'spacelike' quantities characterizing the congruence. In this section we will follow a different approach to study the kinematical properties of a spacelike congruence. We will use a straightforward translation of the timelike congruence analysis. In this way we obtain kinematical properties not necessarily with a spacelike character, but we will find that these kinematical quantities can be used in a natural way to formulate a simplified differential form approach for spacetimes with two commuting spacelike Killing vectors.

Let us consider a congruence of spacelike curves,

$$x^\alpha = x^\alpha(y^a, \tau), \qquad a = 1, 2, 3,$$

where the three constants $y^a = c^a$ specify a particular curve and τ is the arc length. We can define a unit tangent vector,

$$n^\beta = \frac{dx^\beta}{d\tau}, \qquad n_\alpha n^\alpha = 1.$$

We will assume in the following that the congruence defines, at least locally, a vector field $n^\beta(x^\alpha)$.

Let us take a particular curve (C) in the congruence, specified by three constants (y^a), and a point p in the curve (C) characterized by an arc length τ. Now consider another curve in the congruence (C^*) near (C) and specified by constants $y^{a*} = y^a + \delta y^a$ and a point p^* in (C^*) with the same arc length (τ) as p. Then, up to first order, we have,

$$\delta x^\alpha = \frac{\partial x^\alpha}{\partial y^a}\delta y^a,$$

which, in general, is not orthogonal to n^α. In order to obtain a vector orthogonal to n^α we introduce the projector tensor

$$P^\alpha_\beta = g^\alpha_\beta - n^\alpha n_\beta,$$

such that $P^\alpha_\beta n^\beta = 0$. Then we can define

$$\delta_\perp x^\alpha = P^\alpha_\beta \delta x^\beta \qquad \left(\delta_\perp x^\alpha n_\alpha = 0\right).$$

Note that $\delta_\perp x^\alpha$ can be spacelike, timelike or null.

The rate of change of the connecting vector of two spacelike curves of the congruences allows us to characterize locally the congruence. First, let us consider the change in the 'modulus'. Assume $\delta_\perp x^\alpha$ is timelike or spacelike, then,

$$\epsilon(\delta l)^2 = g_{\alpha\beta}\delta_\perp x^\alpha \delta_\perp x^\beta, \qquad \epsilon = \pm 1.$$

The rate of change of δl along the congruence is

$$\frac{(\delta l)^{\cdot}}{\delta l} = \epsilon S_{\alpha\beta}\frac{\delta_\perp x^\alpha}{\delta l}\frac{\delta_\perp x^\beta}{\delta l} + \frac{1}{3}\Phi, \tag{1}$$

where we define,

$$\Phi \equiv n^\alpha_{\ ;\alpha} \tag{2}$$

$$\dot{n}_\alpha \equiv n_{\alpha;\beta}n^\beta \tag{3}$$

$$S_{\alpha\beta} \equiv n_{(\alpha;\beta)} - \dot{n}_{(\alpha}n_{\beta)} - \tfrac{1}{3}\Phi P_{\alpha\beta}, \tag{4}$$

where S is a trace-free symmetric tensor.

If we define the unitary connecting vector,

$$e^\alpha \equiv \frac{\delta_\perp x^\alpha}{\delta l}.$$

Equation (1) can be written as follows:

$$\frac{(\delta l)^{\cdot}}{\delta l} = \epsilon S_{\alpha\beta}e^\alpha e^\beta + \frac{1}{3}\Phi. \tag{5}$$

On the other hand, for the rate of change of e^α we find,

$$P^\beta_{\ \alpha}\left(e^\alpha\right)^\cdot = \left(W^\beta_{\ \alpha} + S^\beta_{\ \alpha} - \epsilon S_{\mu\nu}e^\mu e^\nu \delta^\beta_{\ \alpha}\right)e^\alpha \tag{6}$$

where

$$W_{\alpha\beta} \equiv n_{[\alpha;\beta]} - \dot{n}_{[\alpha}n_{\beta]}.$$

If e^α is an eigenvector of $S_{\mu\nu}$ ($S^\mu_{\ \nu}e^\nu = \lambda e^\mu$) then the previous equations reduce to

$$P^\beta_{\ \alpha}\left(e^\alpha\right)^\cdot = W^\beta_{\ \alpha}e^\alpha. \tag{7}$$

It is interesting to note that if we know $\dot{n}_\alpha$, $S_{\alpha\beta}$, $W_{\alpha\beta}$, and Φ in a given point we can reconstruct the congruence locally using that

$$n_{\alpha;\beta} = \dot{n}_\alpha n_\beta + W_{\alpha\beta} + S_{\alpha\beta} + \tfrac{1}{3}\Phi P_{\alpha\beta}.$$

Some important properties are

$$P_{\alpha\beta}n^\beta = 0, \qquad \dot{n}_\alpha n^\alpha = 0, \qquad W_{\alpha\beta}n^\beta = 0 = S_{\alpha\beta}n^\beta.$$

In order to have a better characterization of the spacelike congruence we can study the eigenvalue problem for the trace-free symmetric three-dimensional tensor $S_{\alpha\beta}$ that can be formulated as follows:

$$(S_{\alpha\beta} - \lambda P_{\alpha\beta})v^\beta = 0.$$

As the quadratic forms defined by $S_{\alpha\beta}$ and $P_{\alpha\beta}$ are not definite forms, a standard analysis gives us the following different situations:

- λ_1, λ_2 and $\lambda_3 = -\lambda_1 - \lambda_2$ are three different real numbers. In this case we have one timelike eigenvector and two spacelike eigenvectors. They are mutually orthogonal.
- λ_1, $\bar{\lambda}_1$ are complex conjugates and $\lambda_2 = -2\mathrm{Re}\,(\lambda_1)$. In this case we have two complex conjugate eigenvectors, m and $\bar{m}$, corresponding to the complex eigenvalues and a real spacelike eigenvector orthogonal to m and $\bar{m}$. The complex eigenvector can be written as $m = a + \mathrm{i}b$, where a is timelike and b spacelike and they are mutually orthogonal and normalized to $-\frac{1}{2}$ and $\frac{1}{2}$, respectively.
- $\lambda_1 = \lambda_2 \neq 0$, $\lambda_3 = -2\lambda_1$ are real numbers. In this case there is a null eigenvector corresponding to the double eigenvalue and one spacelike eigenvector orthogonal to the plane containing the null eigenvector.
- $\lambda_1 = \lambda_2 = \lambda_3 = 0$. In this case there is a null eigenvector corresponding to the triple eigenvalue.

The interpretation of the kinematical properties of a spacelike congruence depends on the physical interpretation of the vector field n and the character of the eigenvalues and eigenvectors of $S_{\alpha\beta}$. Assume, for instance, that the three eigenvalues are real and different. Consequently, the three eigenvectors are mutually orthogonal and there is one timelike and two spacelike eigenvectors. In order to give an interpretation in this case, let us consider a curve (C) in the congruence and a point (P) in it corresponding to $\tau = \tau_0$. Assume that in the spacelike subspace in the orthogonal hyperspace to (C) in (P) we have a disc. Orthogonal to it there is a unit timelike vector. If we imagine different curves of the congruences crossing the border of the disc and the final point of the timelike vector, going from τ_0 to $\tau_0 + \delta\tau$, the disc changes to an ellipse and the direction and modulus of the timelike vector also changes. These changes are determined by the kinematical variables of the congruences Φ, $S_{\alpha\beta}$ and $W_{\alpha\beta}$, and the final effect can be decomposed in the following two steps.

Step 1 (Φ and $S_{\alpha\beta}$ effects). The influence of Φ and $S_{\alpha\beta}$ on the disc and timelike vector can be understood using equation (5). If e^α is an eigenvector of $S_{\alpha\beta}$ and verifies $S_{\alpha\beta}e^\beta = \lambda e^\alpha$ the equation reduces to

$$\frac{(\delta l)^{\cdot}}{\delta l} = \lambda + \frac{1}{3}\Phi.$$

Hence, the two directions indicated by the two spacelike eigenvectors will transform to the principal directions of the ellipse and the length of the principal axis will be obtained from the previous equation. The new direction of the timelike vector will be given by the direction of the timelike eigenvector and the modulus will be obtained from the previous equation. In this way we can think of the spacelike congruence as generating a 'virtual tube flux' where the transverse section is given by the ellipse, the velocity of the fluid by the timelike unit eigenvector of $S_{\alpha\beta}$ and its modulus can be interpreted as the density, whose changes are given by the previous equation (note that, in principle, the virtual fluid has nothing to do with any real perfect fluid; the physical interpretation of such a virtual fluid depends on the physical interpretation of the congruence).

Step 2 ($W_{\alpha\beta}$ effects). Finally, the effect of $W_{\alpha\beta}$ on the 'tube flux' is to 'bend and twist the tube'. This result follows from equation (7).

The only remaining kinematical variable is $\dot{n}_\alpha$ but this represents the curvature of the selected curve in the congruence.

In the case when $S_{\alpha\beta}$ has degenerate or complex eigenvalues the interpretation is not as straightforward. We will study those cases in the particular situation when the metric admits two Killing vectors.

2.1. Kinematical variables in a tetrad formalism

Using a tetrad adapted to n such that $\theta^2 \equiv n$,

$$\mathrm{d}s^2 = -\theta^0 \otimes \theta^0 + \theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2 + \theta^3 \otimes \theta^3,$$

the covariant derivative of n takes the following form:

$$n_{a;b} = -\gamma_{2ab},$$

where $\gamma^a{}_{bc} = -\theta^a_{i;j}e^i_b e^j_c$ (where $\{e_a\}$ is the orthonormal frame dual to $\{\theta^a\}$) are the Ricci rotation coefficients. The kinematical properties of the congruence of n read as follows:

$$\dot{n} = \gamma_{022}\theta^0 + \gamma_{122}\theta^1 + \gamma_{322}\theta^3 \tag{8}$$

$$\Phi = -\gamma_{020} + \gamma_{121} - \gamma_{233} \tag{9}$$

$$W_T = (\gamma_{021} - \gamma_{120})\theta^0 \wedge \theta^1 + (\gamma_{023} + \gamma_{230})\theta^0 \wedge \theta^3 + (\gamma_{123} + \gamma_{231})\theta^1 \wedge \theta^3 \tag{10}$$

$$\begin{aligned} S_T = {}& \left(\tfrac{2}{3}\gamma_{020} + \tfrac{1}{3}\gamma_{121} - \tfrac{1}{3}\gamma_{233}\right)\theta^0 \otimes \theta^0 + \left(\tfrac{1}{3}\gamma_{020} + \tfrac{2}{3}\gamma_{121} + \tfrac{1}{3}\gamma_{233}\right)\theta^1 \otimes \theta^1 \\ & + \left(\tfrac{1}{3}\gamma_{020} - \tfrac{1}{3}\gamma_{121} - \tfrac{2}{3}\gamma_{233}\right)\theta^3 \otimes \theta^3 + (\gamma_{021} + \gamma_{120})\theta^0 \otimes_s \theta^1 \\ & + (\gamma_{023} - \gamma_{230})\theta^0 \otimes_s \theta^3 + (\gamma_{123} - \gamma_{231)})\theta^1 \otimes_s \theta^3, \end{aligned} \tag{11}$$

where $\otimes_s$ is the symmetric part of the tensor product, that is, $v \otimes_s w = (v \otimes w + w \otimes v)/2$.

3. Tetrad formalism for a spacetime endowed with two spacelike orthogonally transitive commuting Killing vectors

If a spacetime has two spacelike Killing vectors $\{\xi, \eta\}$ defining an Abelian orthogonally transitive G_2 group of motions we can choose the tetrad for the spacetime $\{\theta^0, \theta^1, \theta^2, \theta^3\}$,

$$\mathrm{d}s^2 = -\theta^0 \otimes \theta^0 + \theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2 + \theta^3 \otimes \theta^3,$$

such that θ^2 and θ^3 are in $\mathrm{lin}\{\xi, \eta\}$ and [11, 16]:

$$L_\xi \theta^0 = 0 = L_\eta \theta^0; \qquad L_\xi \theta^1 = 0 = L_\eta \theta^1,$$
$$L_\xi \theta^2 = 0 = L_\eta \theta^2; \qquad L_\xi \theta^3 = 0 = L_\eta \theta^3.$$

It is important to notice that there is a residual $SO(1,1)$ gauge in the $\{\theta^0\text{–}\theta^1\}$ subspace and also a $SO(2)$ gauge in the $\{\theta^2\text{–}\theta^3\}$ subspace.

For the tetrad presented above a family of independent non-vanishing Ricci rotation coefficients are the following:

$$\gamma_{010}, \gamma_{011}, \gamma_{022}, \gamma_{023} = \gamma_{032}, \gamma_{033}, \gamma_{122}, \gamma_{123} = \gamma_{132}, \gamma_{133}, \gamma_{230}, \gamma_{231}.$$

Then, the kinematical variables reduce to

$$\dot{n} = \gamma_{022}\theta^0 + \gamma_{122}\theta^1 \tag{12}$$
$$\Phi = 0 \tag{13}$$
$$W_T = (\gamma_{023} + \gamma_{230})\theta^0 \wedge \theta^3 + (\gamma_{123} + \gamma_{231})\theta^1 \wedge \theta^3 \tag{14}$$
$$S_T = (\gamma_{023} - \gamma_{230})\theta^0 \otimes_s \theta^3 + (\gamma_{123} - \gamma_{231)})\theta^1 \otimes_s \theta^3. \tag{15}$$

It is interesting to define 1-forms α, ω and σ,

$$\alpha = -\gamma_{022}\theta^0 - \gamma_{122}\theta^1 \tag{16}$$
$$\omega = -(\gamma_{023} + \gamma_{230})\theta^0 - (\gamma_{123} + \gamma_{231})\theta^1 \tag{17}$$
$$\sigma = (\gamma_{230} - \gamma_{023})\theta^0 + (\gamma_{231} - \gamma_{123})\theta^1 \tag{18}$$

such that the kinematical variables can be written as

$$\dot{n} = -\alpha$$
$$W_T = -\omega \wedge \theta^3$$
$$S_T = -\sigma \otimes_s \theta^3$$

so that α, ω and σ completely parametrize the kinematical variables.

The eigenvalue problem for S_T simplifies and there are three different cases. Defining $A \equiv \gamma_{023} - \gamma_{230}$ and $B \equiv \gamma_{123} - \gamma_{231}$ we find:

(a) $B = \epsilon A$ where $\epsilon^2 = 1$ (σ null).
In this case σ is a null form $\sigma = A(\theta^0 + \epsilon\theta^1)$. We have $\lambda_1 = \lambda_2 = \lambda_3 = 0$ whose eigenvector is proportional to σ.

(b) $B > A$ (σ spacelike).
In this case we have $\lambda_1 = 0$, $V_1 \propto *\sigma$ (timelike).

$$\lambda_2 = \sqrt{B^2 - A^2}, \qquad V_2 \propto \sigma + \sqrt{B^2 - A^2}\theta^3 \quad \text{(spacelike)},$$
$$\lambda_3 = -\sqrt{B^2 - A^2}, \qquad V_3 \propto \sigma - \sqrt{B^2 - A^2}\theta^3 \quad \text{(spacelike)}.$$

(c) $B < A$ (σ timelike).

In this case we have $\lambda_1 = 0$, $V_1 \propto *\sigma$ (spacelike).

$$\lambda_2 = \mathrm{i}\sqrt{A^2 - B^2}, \qquad V_2 \propto \sigma + \mathrm{i}\sqrt{A^2 - B^2}\theta^3,$$
$$\lambda_3 = -\mathrm{i}\sqrt{A^2 - B^2}, \qquad V_3 \propto \sigma - \mathrm{i}\sqrt{A^2 - B^2}\theta^3.$$

In order to complete the family of 1-forms that will be used to write an exterior system equivalent to the Einstein equations, we introduce two new 1-forms β and ν,

$$\beta = -(\gamma_{022} + \gamma_{033})\theta^0 - (\gamma_{122} + \gamma_{133})\theta^1 \tag{19}$$
$$\nu = \gamma_{010}\theta^0 + \gamma_{011}\theta^1. \tag{20}$$

The interpretation of these 1-forms will be found immediately below.

3.1. *Vanishing torsion equations*

With our choice of tetrad and variables, the vanishing torsion equations can be written as

$$\mathrm{d}\theta^0 = \nu \wedge \theta^1 \tag{21}$$
$$\mathrm{d}\theta^1 = \nu \wedge \theta^0 \tag{22}$$
$$\mathrm{d}\theta^2 = \alpha \wedge \theta^2 + \omega \wedge \theta^3 \tag{23}$$
$$\mathrm{d}\theta^3 = (\beta - \alpha) \wedge \theta^3 + \sigma \wedge \theta^2. \tag{24}$$

The meaning of ν and β is now clear, since ν is the connection form in the θ^0–θ^1 subspace, while $\mathrm{d}(\theta^2 \wedge \theta^3) = \beta \wedge \theta^2 \wedge \theta^3$ shows that β describes the expansion of the volume in the θ^2–θ^3 subspace†.

3.2. *First Bianchi identities*

The first Bianchi identities are the integrability conditions for the equations (21)–(24). By exterior differentiation of those equations we find

$$\mathrm{d}\beta = 0, \tag{25}$$
$$\mathrm{d}\Omega + \kappa \wedge \delta = 0, \tag{26}$$
$$\mathrm{d}\delta - \kappa \wedge \Omega = 0, \tag{27}$$
$$\mathrm{d}\kappa + \Omega \wedge \delta = 0, \tag{28}$$

where $\Omega \equiv \beta - 2\alpha$, $\delta \equiv \omega + \sigma$ and $\kappa \equiv \omega - \sigma$.

3.3. *Einstein field equations*

We consider the energy–momentum tensor of a perfect fluid, $T_{\alpha\nu} = (\mu + p)u_\alpha u_\nu + pg_{\alpha\nu}$ (where u_α is the velocity of the fluid, μ is the energy density and p the pressure of the fluid), as the source of the gravitational field. In appropriate units, the Einstein equations read as

$$R_{\alpha\nu} - \tfrac{1}{2}Rg_{\alpha\nu} = (\mu + p)u_\alpha u_\nu + pg_{\alpha\nu}$$

or, equivalently,

$$R_{\alpha\nu} = (\mu + p)u_\alpha u_\nu + \tfrac{1}{2}(\mu - p)g_{\alpha\nu}.$$

† Note that β is proportional to the differential of the transitivity surface area element, as defined in [17], and also is proportional to the differential of the function W in equation (15.3) in [2].

After writing the Ricci tensor in the orthonormal coframe in terms of the Ricci rotation coefficients and identifying the kinematical 1-forms and their derivatives, the Einstein field equations can be combined to produce the following equivalent exterior system†:

$$\mathrm{d}*\Omega + \beta\wedge *\Omega + \kappa\wedge *\delta = 0, \tag{29}$$

$$\mathrm{d}*\delta + \beta\wedge *\delta - \kappa\wedge *\Omega = 0, \tag{30}$$

$$\mathrm{d}*\beta + \beta\wedge *\beta = (\mu - p)u\wedge *u, \tag{31}$$

$$\mathrm{d}\nu - \tfrac{1}{4}\beta\wedge *\beta + \tfrac{1}{4}\Omega\wedge *\Omega + \tfrac{1}{4}\delta\wedge *\delta = -\tfrac{1}{2}(\mu + p)u\wedge *u, \tag{32}$$

$$\mathrm{d}*\tilde{\beta} + \tfrac{1}{2}\beta\wedge *\tilde{\beta} + \tfrac{1}{2}\Omega\wedge *\tilde{\Omega} + \tfrac{1}{2}\delta\wedge *\tilde{\delta} + 2\nu\wedge\tilde{\beta} = -(\mu + p)u\wedge *\tilde{u}, \tag{33}$$

$$\mathrm{d}\tilde{\beta} + \tfrac{1}{2}\beta\wedge\tilde{\beta} + \tfrac{1}{2}\Omega\wedge\tilde{\Omega} + \tfrac{1}{2}\delta\wedge\tilde{\delta} + 2\nu\wedge *\tilde{\beta} = -(\mu + p)u\wedge\tilde{u}, \tag{34}$$

where the tilde operation on 1-forms in the θ^0–θ^1 subspace is a reflection with respect to the direction determined by θ^1. If $\lambda = \lambda_0\theta^0 + \lambda_1\theta^1$, then $\tilde{\lambda} \equiv \lambda_0\theta^0 - \lambda_1\theta^1$. The $*$ operator is the Hodge dual in the 2-subspace θ^0–θ^1.

3.4. Integrability conditions

As u and the other physically relevant 1-forms have a vanishing Lie derivative with respect to the two Killing fields, the coefficients of the forms we have considered are independent of the variables in the θ^2–θ^3 subspace. The integrability conditions for the system of equations that we have considered above are either trivial or already incorporated in the system, except for the contracted Bianchi identity,

$$T^{\mu\nu}{}_{;\nu} = 0,$$

which in our case reduces to

$$\mathrm{d}u + \frac{1}{\mu + p}\mathrm{d}p\wedge u = 0, \tag{35}$$

$$\mathrm{d}*u + \left(\beta + \frac{1}{\mu + p}\mathrm{d}\mu\right)\wedge *u = 0, \tag{36}$$

and therefore the fluid is irrotational.

We also have to take into account the following constraint indicating that the velocity of the fluid is a unit timelike vector:

$$u\wedge *u = -\theta^0\wedge\theta^1.$$

3.5. Simplification of the equations

Recall that there is a remaining freedom of choice consisting in a rotation of θ^0 and θ^1 in their own plane, as well as a rotation of θ^2 and θ^3 in their own plane. This freedom can be used to simplify the differential system introduced above.

3.5.1. Gauge freedom in the θ^2–θ^3 subspace.

Under a rotation through an angle φ,

$$\begin{pmatrix}\hat{\theta^2}\\ \hat{\theta^3}\end{pmatrix} = \begin{pmatrix}\cos(\varphi) & \sin(\varphi)\\ -\sin(\varphi) & \cos(\varphi)\end{pmatrix}\begin{pmatrix}\theta^2\\ \theta^3\end{pmatrix} \tag{37}$$

† A similar calculation can be found in [18] for the stationary axisymmetric case.

the kinematical 1-forms transform as

$$\hat{\beta} = \beta, \tag{38}$$

$$\hat{\kappa} = \kappa + 2\,\mathrm{d}\varphi, \tag{39}$$

$$\begin{pmatrix} \hat{\Omega} \\ \hat{\delta} \end{pmatrix} = \begin{pmatrix} \cos(2\varphi) & -\sin(2\varphi) \\ \sin(2\varphi) & \cos(2\varphi) \end{pmatrix} \begin{pmatrix} \Omega \\ \delta \end{pmatrix}. \tag{40}$$

Without loss of generality, we can use this freedom in order to impose, for instance,

$$\hat{\kappa} = \epsilon\hat{\delta}; \qquad \left(\epsilon^2 = 1\right),$$

(note that $\kappa = \delta \Leftrightarrow \sigma = 0$ and $\kappa = -\delta \Leftrightarrow \omega = 0$) or,

$$\hat{\kappa} = \epsilon\hat{\Omega}; \qquad \left(\epsilon^2 = 1\right)$$

as the integrability condition for those equations derived from equation (39) are identically satisfied using the first Bianchi identities. A more general gauge-fixing condition is

$$\hat{\kappa} = \cos(2\varphi_0)\hat{\Omega} - \sin(2\varphi_0)\hat{\delta}; \qquad \varphi_0 = \text{constant},$$

and in the same way as in the previous cases the integrability conditions are satisfied using equations (25)–(28).

It is important, in order to characterize the solutions of the equations, to know whether a given solution is 'diagonal' or not. In an adapted gauge this is equivalent to $\delta = 0 = \kappa$. In a non-adapted gauge we get $\mathrm{d}\kappa = 0$ and $\Omega \wedge \delta = 0$. In other words, if we have a solution satisfying the previous equations, with an adequate transformation, the metric can be written in a diagonal form.

3.5.2. Gauge freedom in the θ^0–θ^1 subspace. The freedom in the θ^0–θ^1 subspace can be used to align θ^0 with a timelike kinematical form or θ^1 with a spacelike kinematical form; a good candidate seems to be u but a careful analysis shows that β is more useful. In this way we have to separate three different cases.

(i) *β null.* If β is a null form then we have

$$*\beta = \epsilon\beta; \qquad \epsilon^2 = 1.$$

Using this relation in equation (33) and adding equation (33) with equation (34) we find

$$\Omega \wedge (\tilde{\Omega} + \epsilon * \tilde{\Omega}) + \delta \wedge (\tilde{\delta} + \epsilon * \tilde{\delta}) + 4pu \wedge (\tilde{u} + \epsilon * \tilde{u}) = 0,$$

and introducing the relations,

$$\Omega = \Omega_0\theta^0 + \Omega_1\theta^1,$$
$$\delta = \delta_0\theta^0 + \delta_1\theta^1,$$
$$u = u_0\theta^0 + u_1\theta^1,$$

in the previous equation we find,

$$(\Omega_0 + \epsilon\Omega_1)^2 + (\delta_0 + \epsilon\delta_1)^2 + 4p(u_0 + \epsilon u_1)^2 = 0.$$

As a consequence if $p > 0$ we find that Ω, δ, and u are null forms. Hence, *if β null then either $p \leqslant 0$ or u is null.*

The cases with $p < 0$ or u null are not very physical for a perfect fluid and we will not consider them here. The only remaining case is $p = 0$ but from (31) and the fact that β is null we find $\mu = 0$ and then the solution represents a vacuum spacetime.

(ii) β *timelike.* If β is timelike we can align θ^0 with it. With this choice, the equations in (21)–(36) to be solved reduce to the system composed by

$$\beta = \mathrm{e}^{-Q}\theta^0$$
$$u \wedge *u = \mathrm{e}^{2Q}\beta \wedge *\beta,$$

plus the last two vanishing torsion equations, (23) and (24), the first Bianchi identities (25)–(28), the first three Einstein equations (29)–(31), the integrability conditions (35) and (36), and, finally,

$$\begin{aligned} \mathrm{d}Q = \tfrac{3}{4}\beta - \tfrac{1}{2}\mathrm{e}^{2Q}\big\{&(\mu - p)\beta + (\mu + p)[\langle\beta, u\rangle u + \langle\beta, *u\rangle * u] \\ &+\tfrac{1}{2}[\langle\beta, \Omega\rangle\Omega + \langle\beta, *\Omega\rangle * \Omega] + \tfrac{1}{2}[\langle\beta, \delta\rangle\, \delta + \langle\beta, *\delta\rangle * \delta]\big\}, \end{aligned} \tag{41}$$

where $\langle\,,\rangle$ denotes the scalar product and $\left\langle\theta^0, \theta^0\right\rangle = -1$, $\left\langle\theta^1, \theta^1\right\rangle = 1$, and $\left\langle\theta^0, \theta^1\right\rangle = 0 = \left\langle\theta^1, \theta^0\right\rangle$. It is important to notice that the connection ν can be solved algebraically from (33) and (34) and the ν thus obtained satisfies (32) identically, using the remaining equations of the system. Thus, we can forget about ν, which may be obtained trivially after we have solved the system presented above. A useful expression for ν is the following:

$$\nu = *\mathrm{d}Q - \left[1 - (\mu - p)\mathrm{e}^{2Q}\right] * \beta.$$

Also, it has to be noticed that for a generic 1-form α in the θ^0–θ^1-subspace we can write,

$$\alpha = -\langle\alpha, u\rangle u + \langle\alpha, *u\rangle * u,$$

and as a consequence,

$$\alpha_{Ru} = \langle\alpha, u\rangle u + \langle\alpha, *u\rangle * u,$$

can be interpreted as a reflection of α with respect to u. An analogous interpretation is possible for the similar expression changing u by Ω (or δ). Finally, let us note that the integrability condition for the equation (41) is identically satisfied using the rest of the system.

(iii) β *spacelike.* If β is spacelike we can align θ^1 with it. In a similar way as in the timelike case, the system can be reduced to (23)–(31), (35)–(36) and,

$$\beta = \mathrm{e}^{-Q}\theta^1$$
$$u \wedge *u = -\mathrm{e}^{2Q}\beta \wedge *\beta$$

$$\begin{aligned} \mathrm{d}Q = \tfrac{3}{4}\beta + \tfrac{1}{2}\mathrm{e}^{2Q}\big\{&(\mu - p)\beta + (\mu + p)[\langle\beta, u\rangle u + \langle\beta, *u\rangle * u] \\ &+\tfrac{1}{2}[\langle\beta, \Omega\rangle\Omega + \langle\beta, *\Omega\rangle * \Omega] + \tfrac{1}{2}[\langle\beta, \delta\rangle\delta + \langle\beta, *\delta\rangle * \delta]\big\}. \end{aligned} \tag{42}$$

The reasoning for timelike β is valid for the spacelike case. The 1-form ν can be written as

$$\nu = *\mathrm{d}Q - \left[1 + (\mu - p)\mathrm{e}^{2Q}\right] * \beta.$$

Again the integrability condition for the equation (42) is satisfied identically.

Components of the differential of Q in the lightcone and positivity. The spacelike or timelike character of β is essential in many senses [2, 17]. Here we present another one related with the gradient of the modulus of β, parametrized by e^{2Q}.

Hence, let us study the components of the differential of Q in the lightcone, that is, along the null geodesic directions on the θ^0–θ^1-subspace:

β timelike. We have,

$$\beta = \mathrm{e}^{-Q}\theta^0$$
$$\Omega = \Omega_0\theta^0 + \Omega_1\theta^1$$
$$\delta = \delta_0\theta^0 + \delta_1\theta^1$$
$$u = u_0\theta^0 + u_1\theta^1$$
$$\mathrm{d}Q = \partial_0 Q\theta^0 + \partial_1 Q\theta^1$$

and then we find

$$\mathrm{e}^{-Q}(\partial_0 \pm \partial_1)Q = \tfrac{3}{4}\mathrm{e}^{-2Q} - \tfrac{1}{2}(\mu - p) + \tfrac{1}{2}(\mu + p)(u_0 \pm u_1)^2 + \tfrac{1}{4}(\Omega_0 \pm \Omega_1)^2 + \tfrac{1}{4}(\delta_0 \pm \delta_1)^2.$$

β spacelike. The only new equation is

$$\beta = \mathrm{e}^{-Q}\theta^1$$

and then the new components are,

$$\mathrm{e}^{-Q}(\partial_1 \pm \partial_0)Q = \tfrac{3}{4}\mathrm{e}^{-2Q} + \tfrac{1}{2}(\mu - p) + \tfrac{1}{2}(\mu + p)(u_0 \pm u_1)^2 + \tfrac{1}{4}(\Omega_0 \pm \Omega_1)^2 + \tfrac{1}{4}(\delta_0 \pm \delta_1)^2.$$

We can see that the right-hand side of the equation for $(\partial_1 \pm \partial_0)Q$ is positive in the spacelike case (when the energy condition $\mu \geqslant |p|$ is satisfied) and has no definite sign in the timelike case. Therefore in the spacelike case the transitivity cylinders are not trapped surfaces.

4. An example

In order to show how the formalism works let us derive the non-singular solution in [13]. This non-singular cosmology is diagonal, therefore we can choose a gauge such that $\omega = \sigma = 0$. The 1-form β is spacelike as in every other regular inhomogeneous spacetime and θ^1 is chosen parallel to it as indicated in section 3.5.2.

Concerning the fluid, two ansätze are made. The velocity of the fluid is taken to be $-\theta^0$ and the fluid is stiff, that is, $p = \mu$. Then we find that $u = \mathrm{e}^{Q} * \beta$. Introducing this expression in (35) and (36) the resulting equations imply,

$$\mathrm{d}Q = -\tfrac{1}{2}\mathrm{d}\ln\mu + \beta. \tag{43}$$

The equations are written using a 1-form basis formed by the closed forms, α and β. Locally this means that we can write,

$$\alpha = \mathrm{d}u, \qquad \beta = \mathrm{d}v, \tag{44}$$

and therefore (43) can be integrated to yield,

$$\mu\mathrm{e}^{2Q} = a^2\mathrm{e}^{2v}, \tag{45}$$

where a is a constant.

Now we have to solve (29) and (30) (note that the only remaining equation (42) can be integrated by line integrals, when we have solved the rest of the equations, as its integrability condition is identically satisfied using the rest of the equations of the system). Then, we introduce two functions f, g of the chosen variables u, v to express the Hodge dual of β as

$$*\mathrm{d}v = \frac{\mathrm{d}u + f\,\mathrm{d}v}{g}, \tag{46}$$

that allows us to complete the dual of our 1-form local basis,

$$*\mathrm{d}u = \frac{-f\,\mathrm{d}u + \left(g^2 - f^2\right)\mathrm{d}v}{g}, \tag{47}$$

taking into account the properties of the Hodge dual.

Now (29) and (30) are equivalent to the following two partial differential equations:

$$g\frac{\partial f}{\partial u} - f\frac{\partial g}{\partial u} + \frac{\partial g}{\partial v} - g = 0, \tag{48}$$

$$-\frac{\partial f}{\partial v} + f\frac{\partial f}{\partial u} - g\frac{\partial g}{\partial u} = 0, \tag{49}$$

A simple solution of the system of equations can be obtained by requiring that f be a function of v. Then g can be integrated from (49),

$$g = \sqrt{h - 2u\frac{\mathrm{d}f}{\mathrm{d}v}}, \tag{50}$$

where h is a function of v. Introducing this expression for g in (48) we find

$$\left(-\frac{\mathrm{d}^2 f}{\mathrm{d}v^2} + 2\frac{\mathrm{d}f}{\mathrm{d}v}\right)u + \frac{1}{2}\frac{\mathrm{d}h}{\mathrm{d}v} - h + f\frac{\mathrm{d}f}{\mathrm{d}v} = 0, \tag{51}$$

which can be easily integrated to find

$$f = p + q\mathrm{e}^{2v}, \tag{52}$$

$$h = -2q^2\mathrm{e}^{4v} - 4\mathrm{e}^{2v}pqv + wq\mathrm{e}^{2v}, \tag{53}$$

where p, q and w are constants. This allows us to write down the function g,

$$g = \sqrt{-2q^2\mathrm{e}^{4v} - 4\mathrm{e}^{2v}pqv + wq\mathrm{e}^{2v} - 4qu\mathrm{e}^{2v}}. \tag{54}$$

The last equation to be integrated is (42). Using the expression for Q that has been obtained previously, it can be written as follows:

$$\frac{\mathrm{d}\mu}{\mu} = \left(2f^2 - 2g^2 - 2a^2\mathrm{e}^{2v}\right)\mathrm{d}v + (2 + 4f)\,\mathrm{d}u, \tag{55}$$

which allows us to obtain the following expression for the fluid energy density:

$$\mu = k\exp\left(2p^2v + \tfrac{3}{2}q^2\mathrm{e}^{4v} + 4\mathrm{e}^{2v}pqv - wq\mathrm{e}^{2v} - a^2\mathrm{e}^{2v} + 2u + 4pu + 4qu\mathrm{e}^{2v}\right), \tag{56}$$

where the constant k will be taken to be a^2 without loss of generality.

The metric can be written in a simple form if we perform a change of variables and the coordinate v is taken to be $\ln r$. If t is chosen accordingly so that the coordinates are isotropic,

$$v = \ln r, \qquad u = -\tfrac{1}{2}qr^2 - p\ln r - qt^2, \tag{57}$$

the fluid density takes the form,

$$\mu = a^2r^{-2p(p+1)}\exp\left(-\tfrac{1}{2}q^2r^4 - \left(a^2 + q + 2pq\right)r^2 - 2qt^2 - 4pqt^2 - 4q^2t^2r^2\right), \tag{58}$$

after eliminating unnecessary parameters.

The metric functions or the fluid density are singular unless $p = 0, -1$. Both cases are indeed the same solution just changing q for $-q$. In [13], q is named β and a^2 is α.

Note that, in principle, this metric could also have been obtained by the generation algorithm due to Wainwright *et al* [19].

5. Geodesic completeness

In this section we shall attempt to give a sufficient condition for a diagonal cylindrical G_2-cosmology to be non-singular. Once one encounters a cosmological model with regular curvature invariants it usually takes lengthy and tedious calculations to prove that the metric is causally geodesically complete. The condition we state here shall be large enough to comprise most known cylindrical non-singular cosmological models.

For diagonal cylindrical G_2-cosmology we can choose a gauge where $\omega = 0 = \sigma$ (note that there is a residual gauge, a rotation with $\varphi =$ constant). Equations (25) and (26) can be integrated, obtaining

$$\alpha = -\mathrm{d}f, \qquad \beta = \frac{\mathrm{d}\rho}{\rho}. \tag{59}$$

Using these expressions for α and β, as well as the fact that $\omega = 0 = \sigma$, we can integrate equations (23) and (24) to find

$$\theta^2 = \mathrm{e}^{-f}\,\mathrm{d}z,$$
$$\theta^3 = \rho\mathrm{e}^{f}\,\mathrm{d}\phi.$$

All known non-singular diagonal cylindrical G_2-cosmologies have a spacelike β. Hence, following the method described in section 3.5.2, we align θ^1 with β,

$$\theta^1 = \mathrm{e}^{Q}\frac{\mathrm{d}\rho}{\rho}. \tag{60}$$

As the cosmological fluid is irrotational, the velocity of the fluid can be written as

$$u = \mathrm{e}^{g}\,\mathrm{d}t,$$

using isotropic coordinates t and r such that,

$$*\mathrm{d}t = -\mathrm{d}r \qquad \text{and} \qquad *\mathrm{d}r = -\mathrm{d}t, \tag{61}$$

that provides the following expression for $\theta^0 = -{*}\theta^1$:

$$\theta^0 = \mathrm{e}^{Q}\left(\frac{\rho_r}{\rho}\,\mathrm{d}t + \frac{\rho_t}{\rho}\,\mathrm{d}r\right). \tag{62}$$

The constraint $u \wedge *u = -\mathrm{e}^{2Q}\beta \wedge *\beta$ imposes a relation between g and Q,

$$\mathrm{e}^{2Q} = \frac{\rho^2\mathrm{e}^{2g}}{\rho_r^2 - \rho_t^2}. \tag{63}$$

Using the previous results the metric takes the following form:

$$\mathrm{d}s^2 = \mathrm{e}^{2g(t,r)}\left\{-\mathrm{d}t^2 + \mathrm{d}r^2\right\} + \rho^2(t,r)\,\mathrm{e}^{2f(t,r)}\,\mathrm{d}\phi^2 + \mathrm{e}^{-2f(t,r)}\,\mathrm{d}z^2. \tag{64}$$

Hence, the calculations will be performed in a chart where t and r are isotropic coordinates for the subspace orthogonal to the Killing orbits and ϕ and z are coordinates adapted to the Killing fields. The usual ranges for these coordinates are chosen,

$$-\infty < t, \qquad z < \infty, \qquad 0 < r < \infty, \qquad 0 < \phi < 2\pi. \tag{65}$$

We shall assume from the beginning that the metric functions f, g, ρ are C^2 and that ρ is positive. Certainly a C^2 requirement is needed in order to avoid a singular Riemann tensor, but for geodesic completeness C^1 would be enough since just the affine connection is involved in the equations.

Following [2], if we are to have a regular symmetry axis on the locus where $\Delta = g(\xi, \xi) = 0$, then we have to impose that

$$\lim_{\rho\to 0} \frac{g(\mathrm{grad}\,\Delta, \mathrm{grad}\,\Delta)}{4\Delta} = 1, \tag{66}$$

on approaching the axis. In our case this means that if the metric functions f, g are C^1 at the axis,

$$\lim_{\rho\to 0} \mathrm{e}^{2\{f(t,r)+g(t,r)\}}\{\rho_r(t,r)^2 - \rho_t(t,r)^2\} = 1. \tag{67}$$

From now on we shall denote the derivatives with respect to r and t by subscripts.

There is no loss of generality in imposing that the axis is located on $r = 0$, since we are always free to choose a different set of isotropic coordinates T, R by performing a transformation,

$$T_t = R_r, \qquad T_r = R_t, \tag{68}$$

which amounts to taking a solution of the one-dimensional wave equation,

$$R_{tt} - R_{rr} = 0, \tag{69}$$

with a boundary condition $R = 0$ on the axis. This problem is underdetermined, since no initial condition has been imposed. For instance, one could take $R = r$ and $R_t = 0$ at $t = 0$.

In order to determine whether the metric is geodesically complete, we shall write the expressions for the geodesic equations,

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0, \tag{70}$$

where the dot denotes derivative with respect to the affine parameter of the geodesic.

In principle we would have to write a second-order system of four differential equations, but two of them become first order due to the existence of isometries. If $u = (\dot{t}, \dot{r}, \dot{\phi}, \dot{z})$ is the 4-velocity of the geodesic and ξ is a Killing field then $p = \xi \cdot u$ is a constant of geodesic motion. The following quantities are then conserved along geodesics:

$$L = \rho^{-2}(t,r)\,\mathrm{e}^{-2f(t,r)}\dot{\phi}, \tag{71}$$

$$P = \mathrm{e}^{2f(t,r)}\dot{z}, \tag{72}$$

respectively the angular momentum and the z component of the linear momentum of a unit mass test particle in free fall.

There is also a conserved quantity, δ, which takes the value zero for lightlike, one for timelike and minus one for spacelike geodesics. This quantity arises from the fact that the 4-velocity u is normalized when an affine parametrization is used. We shall consider just future causal geodesics,

$$\delta = \mathrm{e}^{2g(t,r)}\left\{\dot{t}^2 - \dot{r}^2\right\} - L^2\rho^{-2}(t,r)\,\mathrm{e}^{-2f(t,r)} - P^2\,\mathrm{e}^{2f(t,r)}. \tag{73}$$

The remaining second-order equations read,

$$\begin{aligned}\ddot{t} + g_t(t,r)\dot{t}^2 + 2g_r(t,r)\dot{t}\dot{r} + g_t(t,r)\dot{r}^2 - P^2\mathrm{e}^{2\{f(t,r)-g(t,r)\}}f_t(t,r)\\ +L^2\frac{\mathrm{e}^{-2\{f(t,r)+g(t,r)\}}}{\rho(t,r)^3}\{\rho_t(t,r) + \rho(t,r)f_t(t,r)\},\end{aligned} \tag{74}$$

$$\begin{aligned}\ddot{r} + g_r(t,r)\dot{t}^2 + 2g_t(t,r)\dot{t}\dot{r} + g_r(t,r)\dot{r}^2 + P^2\mathrm{e}^{2\{f(t,r)-g(t,r)\}}f_r(t,r)\\ -L^2\frac{\mathrm{e}^{-2\{f(t,r)+g(t,r)\}}}{\rho(t,r)^3}\{\rho_r(t,r) + \rho(t,r)f_r(t,r)\},\end{aligned} \tag{75}$$

after substituting the derivatives of the cyclic coordinates z, ϕ for their expressions in terms of the constants of geodesic motion.

These equations can be written in a more compact form making use of the equation for δ,

$$\left\{e^{2g(t,r)}\dot{t}\right\}^{\cdot} - \frac{e^{-2g(t,r)}}{2}\left\{e^{2g(t,r)}\left[\delta + P^2 e^{2f(t,r)} + L^2\frac{e^{-2f(t,r)}}{\rho^2(t,r)}\right]\right\}_t = 0, \tag{76}$$

$$\left\{e^{2g(t,r)}\dot{r}\right\}^{\cdot} + \frac{e^{-2g(t,r)}}{2}\left\{e^{2g(t,r)}\left[\delta + P^2 e^{2f(t,r)} + L^2\frac{e^{-2f(t,r)}}{\rho^2(t,r)}\right]\right\}_r = 0. \tag{77}$$

An important family of geodesics are lightlike radial geodesics ($L = P = \delta = 0$), for which there is a constant k such that

$$\dot{t} = |\dot{r}| = k e^{-2g(t,r)}. \tag{78}$$

The derivative of the time coordinate must not grow too fast if these geodesics are complete. A sufficient condition is achieved by imposing that t does not grow faster than exponentially for large values of t. This amounts to the following condition on g:

$$g(t,r) \geqslant -\tfrac{1}{2}\ln|t+a| + b, \tag{79}$$

where a and b are constants. If this condition is fulfilled, t is defined for arbitrarily large values of the affine parameter.

The system of second-order equations (76) and (77) is shown to be equivalent to a system of first-order equations by introducing a new function ξ. Since,

$$\dot{t}^2 - \dot{r}^2 = e^{-2g(t,r)}\left\{\delta + L^2\frac{e^{-2f(t,r)}}{\rho^2(t,r)} + P^2 e^{2f(t,r)}\right\}, \tag{80}$$

it is tempting to parametrize $\dot{t}$, $\dot{r}$ by means of hyperbolic functions of ξ,

$$\dot{t} = e^{-g(t,r)}\sqrt{\delta + L^2\frac{e^{-2f(t,r)}}{\rho^2(t,r)} + P^2 e^{2f(t,r)}}\cosh\xi(t,r), \tag{81}$$

$$\dot{r} = e^{-g(t,r)}\sqrt{\delta + L^2\frac{e^{-2f(t,r)}}{\rho^2(t,r)} + P^2 e^{2f(t,r)}}\sinh\xi(t,r). \tag{82}$$

After introducing these expressions in (76) and (77), a first-order equation for ξ is easily obtained,

$$\dot{\xi}(t,r) = -e^{-2g(t,r)}\{F_t(t,r)\sinh\xi(t,r) + F_r(t,r)\cosh\xi(t,r)\}, \tag{83}$$

$$F(t,r) = e^{g(t,r)}\sqrt{\delta + L^2\frac{e^{-2f(t,r)}}{\rho^2(t,r)} + P^2 e^{2f(t,r)}}, \tag{84}$$

which expands into the following expression for $\dot{\xi}$:

$$\begin{aligned}\dot{\xi} = &-\frac{e^{-g}}{\sqrt{\delta + L^2\rho^{-2}e^{-2f} + P^2 e^{2f}}}\\ &\times\left\{\cosh\xi\left[\delta g_r + L^2\frac{e^{-2f}}{\rho^2}\left(g_r - f_r - \frac{\rho_r}{\rho}\right) + P^2 e^{2f}(g_r + f_r)\right]\right.\\ &\left.+\sinh\xi\left[\delta g_t + L^2\frac{e^{-2f}}{\rho^2}\left(g_t - f_t - \frac{\rho_t}{\rho}\right) + P^2 e^{2f}(g_t + f_t)\right]\right\},\end{aligned} \tag{85}$$

where the dependence of the functions on t, r has been omitted in order to avoid a large equation.

The equivalent equation for past-directed geodesics can be also obtained,

$$\dot{\xi}(t,r) = e^{-2g(t,r)}\{F_t(t,r)\sinh\xi(t,r) - F_r(t,r)\cosh\xi(t,r)\}. \tag{86}$$

Now we can check the equations to extract conditions for causal geodesics to be complete. No attention needs to be paid to the equations for $\dot{z}$ and $\dot{\phi}$, since they are just quadratures to be solved once t and r are known as functions of the affine parameter and cannot be singular unless the functions involved in the integration are singular and we have already imposed that the metric functions must be smooth.

We have to prevent the coordinates t, r from tending to infinity at a finite value of the affine parameter. Also the coordinate r must not tend to zero at a finite value of the affine parameter when the constant of geodesic motion L is non-zero. When L is zero, the denominators that depend on ρ disappear and the expressions for $\dot{t}$, $\dot{r}$ and $\dot{\xi}$ are not singular at $r = 0$.

First of all we shall exclude the possibility of having arbitrarily large values of $|\xi|$. There are two cases to consider.

- $\dot{\xi} > 0, \xi > 0$. We shall impose a condition in order to prevent this situation from lasting too long by requiring that $\dot{\xi}$ changes sign for large values of the time coordinate t. Since we may have geodesics with any of the constants of motion equal to zero, the terms that multiply them in the numerator of $\dot{\xi}$ must be treated independently. For instance, for δ, we need,

$$0 < \cosh\xi g_r + \sinh\xi g_t = \sinh\xi(g_r + g_t) + \mathrm{e}^{-\xi} g_r, \tag{87}$$

and similar conditions for L and P. These conditions can be fulfilled by requiring

$$\begin{aligned} & g_r + g_t > 0, \\ & (g - f - \ln\rho)_r + (g - f - \ln\rho)_t > 0, \\ & (g + f)_r + (g + f)_t > 0, \end{aligned} \tag{88}$$

for large values of t and increasing r. And for the remaining terms that multiply $\mathrm{e}^{-\xi}$, we shall have further to impose that

$$\begin{aligned} & g_r, \\ & (g - f - \ln\rho)_r, \\ & (g + f)_r, \end{aligned} \tag{89}$$

be either positive or at most of the same order as their respective terms in (88) so that $\mathrm{e}^{-\xi}$ makes them negligible, also for large values of t and increasing r.
With all these requirements $\dot{\xi}$ becomes negative before t may develop a singularity.
- $\dot{\xi} < 0, \xi < 0$. As in the previous case, we shall prevent ξ from growing unboundedly large. Since the unfriendly denominators depending on ρ, that are only dangerous in this case since r decreases, only appear when $L \neq 0$, we shall just require for non-zero L, that ξ becomes positive for large values of t. Again, by imposing such a restriction on the numerator of $\dot{\xi}$ one finds, for large values of t and decreasing r,

$$\begin{aligned} \delta\{g_t - g_r\} + L^2\frac{\mathrm{e}^{-2f}}{\rho^2}\{(g - f - \ln\rho)_t - (g - f - \ln\rho)_r\} \\ + P^2\mathrm{e}^{2f}\{(g + f)_t - (g + f)_r\} > 0 \end{aligned} \tag{90}$$

by making use of the identity $\cosh\xi = -\sinh\xi + \mathrm{e}^{\xi}$. Since $L \neq 0$, its term may help compensate for the others to achieve a positive numerator. The terms that multiply the exponential of ξ,

$$\delta g_r + L^2\frac{\mathrm{e}^{-2f}}{\rho^2}\left(g_r - f_r - \frac{\rho_r}{\rho}\right) + P^2\mathrm{e}^{2f}(g_r + f_r), \tag{91}$$

must be either negative or at most of the same order as the term in (90) for large values of t and decreasing r.

Finally, we shall prescribe a behaviour for $\dot{t}$ similar to that used for lightlike radial geodesics to avoid faster than exponential growth for large values of t,

$$\left.\begin{array}{l} g(t,r) \\ g(t,r)+f(t,r)+\ln\rho \\ g(t,r)-f(t,r) \end{array}\right\} \geqslant -\ln|t+a|+b, \tag{92}$$

the first of which is obviously weaker than the condition already required for lightlike radial geodesics. The constants a, b need not be the same for the three inequalities.

No similar restrictions need be imposed on $\dot{r}$, since this derivative cannot become singular before $\dot{t}$ does according to equation (73).

Similar results can be derived for past-directed causal geodesics by just replacing the sign of the derivatives with respect to t for the opposite in (88) and (90), for small values of the time coordinate. The other conditions remain unchanged for small values of t since they were obtained for $|\dot{t}|$.

According to [20], the spacetime is globally hyperbolic since from the derivation of the results it follows that every lightlike geodesic intersects once and only once every hypersurface of constant time and therefore they are Cauchy hypersurfaces.

These results can be summarized in the following theorem.

Theorem. *A cylindrically symmetric diagonal metric in the form (64) with C^2 metric functions f, g, ρ is future causally geodesically complete if conditions (79), (88)–(92) are fulfilled for large values of t.*

In addition, the spacetime is globally hyperbolic.

A fact that is worth mentioning is the relation for the orbit of a future-directed geodesic,

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \tanh\xi. \tag{93}$$

This means that, for events (t_0, r_0), (t, r), $r_0 < r$, $t_0 < t$, on the geodesic we have

$$r \leqslant r_0 + (t - t_0), \tag{94}$$

and if $r_0 > r$, $t_0 < t$, then,

$$r \geqslant r_0 - (t - t_0). \tag{95}$$

Furthermore, the orbit lies above its tangent lines while $\xi, \dot{\xi} > 0$ and, therefore, we obtain a lower bound. For $r_0 < r$, $t_0 < t$,

$$r \geqslant r_0 + \tanh\xi(t_0, r_0)\cdot(t - t_0), \tag{96}$$

And similarly, for $\xi, \dot{\xi} < 0$, $r_0 > r$, $t_0 < t$,

$$r \leqslant r_0 + \tanh\xi(t_0, r_0)\cdot(t - t_0). \tag{97}$$

6. Examples

In this section we shall show explicitly how the theorem works with all known diagonal non-singular models.

- Senovilla [21]. This is the first known non-singular cosmological model in the literature. It describes a universe in a radiation-dominated epoch. Its geodesic completeness is proven in [22]. The metric functions for this model are

$$\begin{aligned} g(t,r) &= 2\ln\cosh(at) + \ln\cosh(3ar), \\ f(t,r) &= \ln\cosh(at) + \tfrac{1}{3}\ln\cosh(3ar), \\ \rho(t,r) &= \frac{1}{3a}\cosh(at)\sinh(3ar)\cosh^{-2/3}(3ar), \end{aligned} \tag{98}$$

where a is a positive constant related to the maximum of the pressure.
Since they are even functions of t we need not worry about past-directed geodesics.
All three functions in (79) and (92) are positive except for constants and the term $\ln(\sinh(3ar))$ in the third function. For increasing r, it does not mean a problem. For decreasing r, it is bounded by (95),

$$\ln(\sinh(3ar)) \geqslant \ln(\sinh(3a|t - r_0 + t_0|)), \tag{99}$$

and therefore they fulfil this condition.
The first and third conditions in (88) are always satisfied for positive t and the same happens with the second for large values of r.
The expressions in (89) do not depend on t and are always positive, except the second one, which requires large values of r.
All three terms in (90) independently are positive for small values of r and positive t.
Concerning (91), the term as a whole is negative for small r and positive t.
Hence this spacetime is geodesically complete.

- Ruiz–Senovilla [23]. This family includes the previous one as a subcase for $K = 1$ and $n = 3$. The metric is not written in isotropic form in the original paper, but it can be cast in that form by the change of variable,

$$\begin{aligned} r &= \frac{1}{na}\int_1^{\cosh(naR)} \frac{\mathrm{d}x}{P(x)}, \\ P(x) &= \sqrt{x^2 + (K-1)x^{(2n-1)/n} - K}, \end{aligned} \tag{100}$$

where $P(x)$ is a growing function for $x > 1$. We shall need for our reasoning that

$$\frac{\mathrm{d}R(r)}{\mathrm{d}r} = \frac{P(\cosh(naR(r)))}{\sinh(naR(r))}, \tag{101}$$

which tends to a positive constant when r tends to zero, to one when r tends to infinity and is always positive for positive r,

$$\begin{aligned} g(t,r) &= \tfrac{1}{2}(1+n)\ln\cosh(at) + \tfrac{1}{2}(n-1)\ln\cosh(naR(r)), \\ f(t,r) &= \frac{n-1}{2}\ln\cosh(at) + \frac{n-1}{2n}\ln\cosh(naR(r)), \\ \rho(t,r) &= \frac{1}{naL}\cosh(at)\cosh^{(1-n)/n}(naR(r))P\big(\cosh^2(naR(r))\big), \\ L &= K - \frac{K-1}{2n}. \end{aligned} \tag{102}$$

The ranges of the parameters are $n \geqslant 3$, $K > 0$ and a is again an arbitrary positive constant and every metric function is even in the coordinate time t.
Every function in (79) and (92) is either positive or involves a term that behaves as $\ln r$ for small r and therefore can be bounded like in the previous example.
The expressions in (88) are all positive for large r and positive t as happened in the Senovilla subcase. The first and third ones are positive for all r.
The radial derivatives in (89) are again positive for large values of r. The first and third ones are positive for all r.
Again the three terms in (90) are positive for large t and small values of r.
Finally, the angular momentum term in (91) is unboundedly negative for small r while the others are bounded and therefore the whole term is negative for small values of the radial coordinate.
Hence the whole set of conditions is fulfilled and the family is geodesically complete.

- Mars [24]. This is a family of non-singular models with no equation of state and depending on two parameters a, related to the maximum density of perfect fluid, and $c \geqslant \frac{3}{4}$†,

$$\begin{aligned} g(t,r) &= -\tfrac{1}{2}\ln\cosh(2at) + \tfrac{1}{2}c\cosh^2(2at)\sinh^2(2ar), \\ f(t,r) &= \tfrac{1}{2}\ln\cosh(2at) - \ln\cosh(ar), \\ \rho(t,r) &= \frac{1}{2a}\frac{\sinh(2ar)}{\cosh(2at)}. \end{aligned} \tag{103}$$

  Again the metric functions are even in the time coordinate t.
  The negative terms in (79) and (92) are counteracted by the exponential growth of the c terms for large r and t.
  Concerning (88), the three expressions are positive for positive t except for bounded terms in t and r.
  The same sort of behaviour is to be found in (89) but for all values of t.
  The terms in the expression of (90) are all bounded or tend to zero except for $2a\coth(2ra)$. Therefore, this term is positive for small r. The reasoning why the terms on $\sinh(ar)$ tend to zero faster than those with $\cosh(at)$ is explained in the next example.
  Finally, the expression in (91) is negative for small values of r since every term tends to zero or is bounded except for the $-2a\coth(2ra)$ term.
  Therefore, this family is also geodesically complete.
- Fernández-Jambrina. This non-separable metric, that has been obtained in section 4, corresponds to a spacetime filled with a stiff perfect fluid. Both parameters, α and β, are positive. α is related to the maximum of the density of the fluid. The metric functions are

$$\begin{aligned} g(t,r) &= \tfrac{1}{4}\beta^2 r^4 + \tfrac{1}{2}(\alpha+\beta)r^2 + \beta t^2 + 2\beta^2 r^2 t^2, \\ f(t,r) &= \tfrac{1}{2}\beta\left(r^2 + 2t^2\right), \qquad \rho(t,r) = r, \end{aligned} \tag{104}$$

  and again they are even functions of time.
  The three functions in (79) and (92) are positive, except for a constant and a term $\ln r$ to which applies the same reasoning used for Senovilla's metric and so this condition is trivially satisfied.
  The first and third expressions in (88) are positive for positive values of t. The second expression requires in addition large values of r.
  The same happens for (89), regardless this time of the sign of t.
  The terms in (90) are all either positive and growing linearly with t or as $1/r$ or tending to zero. Note that the terms polynomic in r tend to zero although they may be multiplied by powers of t since according to (97) r decreases to zero faster than t grows to infinity.
  Finally, all of the terms in (91) tend to zero except $-1/r$, which makes the expression negative for small values of r.
  Therefore these spacetimes are geodesically complete.

7. Summary

In this paper a new approach for G_2 inhomogeneous cosmological models has been derived. The description is grounded on kinematical properties of spacelike congruences that have been encoded in several 1-forms. The exterior differential system, equivalent to Einstein equations, has been submitted to different kinds of simplifications. Based on this formalism several consequences are drawn. It is shown that when the transitivity surfaces are null a perfect fluid is not admissible. Also, when these surfaces are spacelike they are not trapped sets.

† Note that there is a misprint in the original paper.

As an application to non-singular cosmological models a solution is reobtained within the formalism. A theorem is written stating a sufficient condition for diagonal models to be singularity-free and globally hyperbolic. All known non-singular diagonal perfect-fluid spacetimes fit within this framework.

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