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Structural Symmetries and Solutions of Nonlinear Mechanical Systems

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This paper draws attention to the role played by symmetry in the applied mathematics of mechanics and, in particular, to the way it influences both the nature of the solution and the way solutions can be found. It attempts to show that while geometrical symmetry can always be used to advantage when seeking solutions, it is the structural symmetry within the governing system of equations that has the most far reaching influence on the nature of solutions. While the effect of geometrical symmetry is easily appreciated, the structural symmetry of a system of differential equations is described in terms of Lie algebras so its occurrence is less obvious. Never the less, it is structural symmetry that has the most significant implications for the nature of solutions, including as it does as special cases, similarity solutions, travelling wave solutions, soliton behaviour, Bäcklund transformations and Hamiltonian systems.

MSC (1991): 35K22, 35Q53, 35Q51, 58D25, 70Hxx

1. Geometrical Symmetry

The most obvious way symmetry enters into physical problems is through their geometry. For example, when considering the steady-state heat equation, a two-dimensional circularly symmetric problem reduces Laplace's equation to a second order ordinary differential equation, so in this case symmetry has reduced a problem in two-dimensions to a problem in one dimension.

Geometrical symmetry coupled with symmetry of boundary conditions can also simplify the study of problems in regions containing lines about which the solution is symmetrical. This is illustrated in Fig. 1 which represents a two-dimensional temperature distribution problem in a square region inside of which is a square hole with the temperature $T = T_1$ on the inner boundary and the temperature $T = T_2$ on the external boundary. In this case, because of geometrical symmetry and symmetry of the boundary conditions, the solution in the entire region can be obtained by repeated reflection of the solution in the shaded octant. The boundary conditions to be applied across the dashed lines is the symmetry condition $\partial T/\partial n = 0$, where $\partial/\partial n$ is a derivative normal to a line of symmetry.

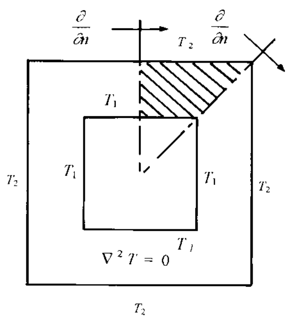

Fig. 1. Symmetrical region with symmetrical boundary conditions

2. Self-similar solutions

A different form of symmetry that is used to advantage in fluid and solid mechanics arises when working with self-similar solutions. The familiar idea involved here is that sometimes a partial differential equation can be re-expressed in terms of groups of independent variables that are fewer in number than the original set of independent variables. Typically, the number of dimensions involved can be reduced by one, and in higher dimensional problems this represents a considerable simplification. Solutions of this type can be regarded as part of the large class of symmetry groups, to which belong the transformation groups associated with translations, rotations, and scale transformations.

A simple example of how a combination of independent variables can simplify a partial differential equation is provided by considering the Riemann problem for the following nonlinear hyperbolic equation:

$$\frac{\partial u}{\partial t} + u^2 \frac{\partial u}{\partial x} = 0 \quad \text{with} \quad u(x, 0) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases} \tag{1}$$

If x has the dimension of length and t the dimension of time, the variable $\zeta = x/t$ has the dimensions of a speed. Changing from the variables x and t to the single variable ζ, and setting $U(\zeta) = u(x/t)$, transforms the partial differ-

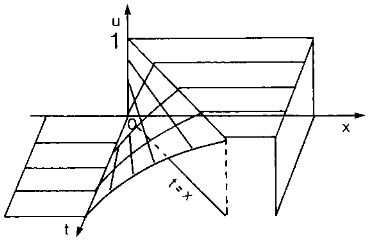

Fig. 2. Centered simple wave resolving a discontinuous initial condition

ential equation to

$$\frac{\mathrm{d}U}{\mathrm{d}\zeta}\,(U^2 - x/t) = 0\,, \tag{2}$$

showing that $\mathrm{d}U/\mathrm{d}\zeta = \text{constant}$ when $U = (x/t)^{1/2} = \zeta^{1/2}$. This shows that the solution U is constant along the lines $\zeta = \text{constant}$ emanating from the origin for $t > 0$. This conclusion, when taken together with the initial conditions, shows the solution in the half-plane $t > 0$ is given by

$$u(x, t) = \begin{cases} 0\,, & x/t < 0\,, \\ (x/t)^{1/2}\,, & 0 < x/t < 1\,, \\ 1\,, & x/t > 1\,. \end{cases} \tag{3}$$

This solution is illustrated in Fig. 2, from which it can be seen that the discontinuity in the initial conditions is resolved immediately and a smooth transition is provided between the two constant states. This solution is, of course, a *centered simple wave* of the type that occurs throughout fluid and solid mechanics. The solution is seen to be constant along each characteristic $\zeta = \text{constant}$ in the interval $0 \le \zeta \le 1$.

For information about *weak solutions* and *shocks* and the so-called entropy conditions used to select stable physical solutions from all possible weak solutions we refer to the work of Rozhdestvensky [1], Jeffrey [2], Smoller [3], and LeVeque [4], where many additional references are to be found.

3. Hyperbolic conservation laws

Symmetry in the structure of partial differential equations arises in many ways, one of which is when dealing with systems of *hyperbolic conservation laws* that are of fundamental importance throughout mechanics whenever wave propagation is involved. Many physical problems are described by the first order quasilinear system

$$A_0(U, \mathbf{x}, t)\,U_t + \sum_{i=1}^{n} A_i(U, \mathbf{x}, t)\,U_{x_i} + B(U, \mathbf{x}, t) = 0\,, \tag{4}$$

where U is an m element vector, $\mathbf{x}$ is a vector in $\mathbb{R}^n$ and the A_i are $m \times m$ matrices with elements depending on U, $\mathbf{x}$, and t.

This system will be *strictly hyperbolic in the t-direction* at a point P if the zeros $\lambda_1, \lambda_2, \ldots, \lambda_m$ of the characteristic polynomial

$$Q(P; \mathbf{v}, \boldsymbol{\lambda}) = \left|\sum_{i=1}^{n} v_i A_i(P) - \boldsymbol{\lambda} A_0(P)\right| \tag{5}$$

associated with the partial differential equation are all real and distinct for all choices of the unit space vector $\mathbf{v}$ at point P, and if the eigenvectors $\mathbf{r}_j$ satisfying

$$\sum_{i=1}^{n} (v_i A_i(P) - \lambda_j A_0(P))\,\mathbf{r}_j = 0 \quad \text{with} \quad j = 1, 2, \ldots, m \tag{6}$$

span the space E^m to which the eigenvectors belong.

These systems, and ones that can be reduced to this type by means of a suitable transformation, have many desirable m a t h e m a t i c a l p r o p e r t i e s, the two most important of which are:

1. If the system is symmetric hyperbolic in the t-direction at a point P, then it is symmetric hyperbolic in any direction T through P provided T lies inside a suitably narrow cone with its vertex at P and its axis parallel to the t-direction.

2. An initial value problem for a symmetric hyperbolic system is unique.

Combining the notion of hyperbolicity with the fact that a real symmetric matrix has real eigenvalues leads to the following useful result:

If $A_0(U, \mathbf{x}, t)$ is symmetric and positive definite, and the matrices $A_i(U, \mathbf{x}, t)$ are symmetric, then the system is symmetric hyperbolic.

The knowledge that the solution of an initial value problem for a symmetric hyperbolic systems is unique is of particular value when studying new systems about which little is known. This result has, for example, been of use in connection with the study of magnetohydrodynamics and magneto-acoustics.

4. Travelling wave solutions

A different form of symmetry arises when nonlinear partial differential equations possess travelling wave solutions, because a Galilean-type translation of the wave at time t_1 transforms it into an exactly similar wave at some other time t_2. Not all nonlinear evolution equations possess travelling wave solutions, because purely hyperbolic equations exhibit the gradient catastrophe in which the solution breaks down due to the formation of an infinite spatial derivative [2, 4, 5].

A prototype equation that exhibits travelling wave solutions is the *Burgers equation*

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = \nu\,\frac{\partial^2 u}{\partial x^2}\,, \qquad \nu > 0\,, \tag{7}$$

the left-hand side of which is seen to be a simple first order nonlinear hyperbolic equation the solution of which will develop an unbounded spatial derivative after some finite elapsed time, while the right-hand side contains a term that is dissipative provided $\nu > 0$. It is this dissipative type term that counteracts the unbounded growth of the derivative u_x that would occur if $\nu = 0$ and allows the equation to possess a travelling wave solution. This travelling wave solution, called the *Burgers shock wave*, is the solution with a Galilean-type translation property of the type mentioned previously. The Burgers shock wave solution of (7) subject to the boundary conditions $u = u_{-\infty}$ at $x = -\infty$ and $u = u_{+\infty}$ at $x = +\infty$ takes the form

$$u(x,\,t) = \frac{1}{2}\,(u_{-\infty} + u_{+\infty}) - \frac{1}{2}\,(u_{-\infty} - u_{+\infty})\,\tanh\left[\frac{(u_{-\infty} - u_{+\infty})}{4\nu}\,\zeta\right], \tag{8}$$

where

$$\zeta = x - \lambda t\,, \qquad \lambda = (1/2)\,(u_{-\infty} + u_{+\infty})\,, \quad \text{and} \quad u_{-\infty} - u_{+\infty} > 0\,.$$

The Burgers shock wave is illustrated in Fig. 3, and its speed of propagation is seen to be the average of the values of u at $x = \pm\infty$ [3, 5].

A more interesting case that involves symmetry in various different ways arises when both nonlinearity and dispersion are present and the effects balance, counteracting the development of an unbounded spatial derivative and allowing a travelling wave to exist. The prototype equation in this case is the *Korteweg de Vries* (KdV) *equation* [5, 6, 7],

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + \mu\,\frac{\partial^3 u}{\partial x^3} = 0\,. \tag{9}$$

The travelling wave solution for this equation, called the *KdV solitary wave solution*, is given by

$$u(x,\,t) = u_\infty + a\,\mathrm{sech}^2\left(\zeta\,\sqrt{\frac{a}{12\mu}}\right), \tag{10}$$

where

$$\zeta = x - \lambda t \quad \text{and} \quad \lambda = u_\infty + a/3\,.$$

This is a symmetrical uni-directional wave in the form of a pulse that, relative to the solution u_∞ at infinity, propagates with a speed proportional to the amplitude a. A typical KdV solitary wave is shown in Fig. 4.

The importance of the KdV equation that describes long waves in shallow water, and other physical phenomena, arises from the fact that it was the first equation to exhibit *soliton* behaviour. The name *soliton* refers to travelling wave solutions of certain special types of evolution equation with the property that after any number of interactions the waves return exactly to the form they had before the interactions occurred [5, 6, 8]. Examination of equation (10) shows that a wave of large amplitude moves faster than one of smaller amplitude, so if propagation is to the right, as

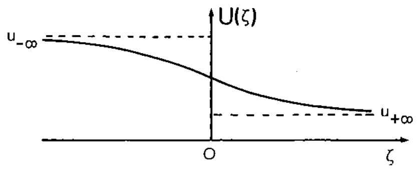

Fig. 3. The Burgers shock wave

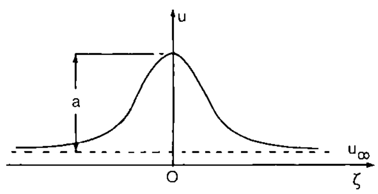

Fig. 4. KdV solitary wave

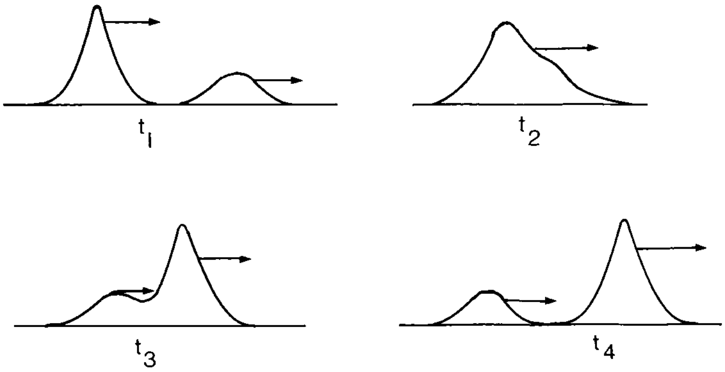

Fig. 5. KdV soliton interaction

the propagation is uni-directional a large amplitude wave to the left of a smaller amplitude one will overtake the smaller wave. Although the interaction is nonlinear, after the interaction the waves interchange their positions but recover their identical initial shapes, the only effect of the interaction being a shift in their positions relative to those they would have had if no interaction had taken place. Typical soliton behaviour is illustrated in Fig. 5. The shift in relative positions due to the interaction, called the *phase shift*, is shown in Fig. 6.

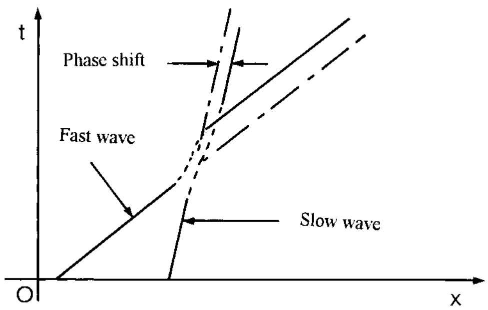

Fig. 6. Phase shift due to soliton interaction

KdV soliton behaviour was first found numerically [8], but now various methods are available for determining analytical solutions, like the inverse scattering transform and the AKN method [5, 7].

An example of soliton behaviour in mechanics due to Konno and Jeffrey [9] is the *loop soliton* that occurs in a long very flexible rod under tension governed by the equation

$$\frac{\partial^2 y}{\partial t\,\partial x} + \operatorname{sign}\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)\frac{\partial^2}{\partial x^2}\left[\frac{\partial^2 y/\partial x^2}{\left(1+(\partial y/\partial x)^2\right)^{3/2}}\right] = 0\,, \tag{11}$$

where t is the time, x is distance along the undistorted rod, y is the displacement of the rod normal to the x-axis and s is arc length measured along the rod. A typical single loop soliton is shown in Fig. 7.

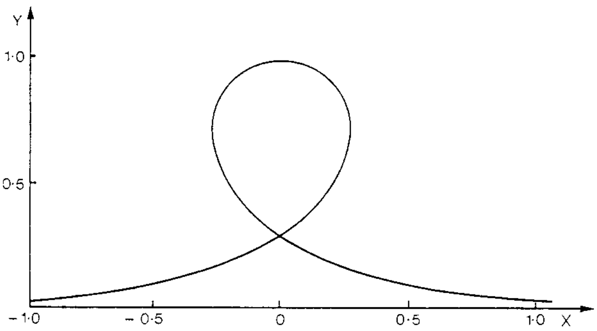

Fig. 7. Single loop soliton

Because of the way elastic energy is stored in a loop, a small loop soliton travels faster than a large loop soliton, so for propagation to the right, a small loop to the left of a large one will overtake the large one. Fig. 8 shows the analytical solution for the interaction of two loop solitons.

In this example the physical reason for the phase shift due to interaction is easy to understand, because the small loop takes time to travel around the large loop, and so after interaction it has not advanced as far to the right as would have been the case were no interaction to have taken place. Travelling wave solutions for some named equations are to be found in the Appendix at the end of the paper.

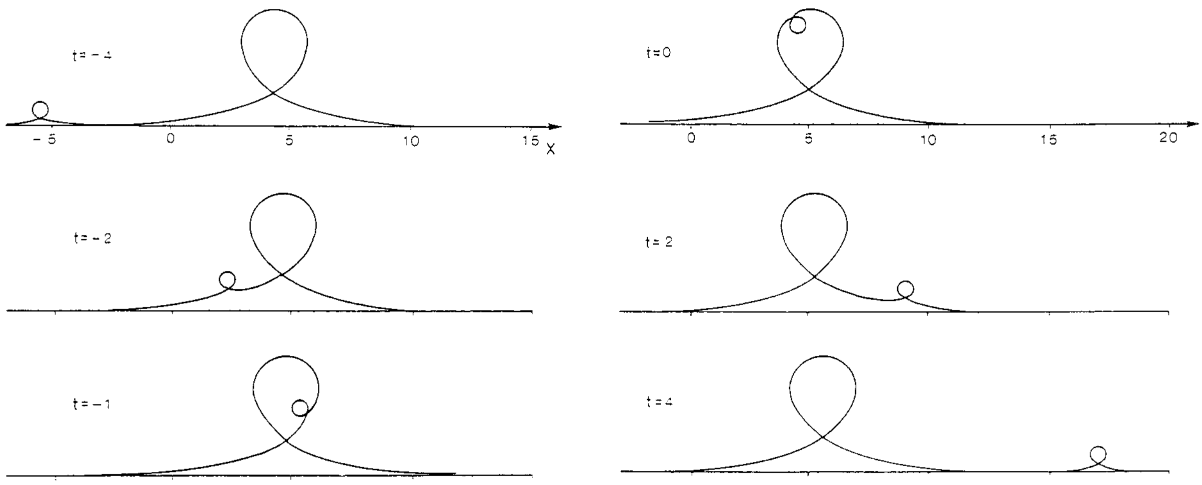

Fig. 8. Interaction of two loop solitons

5. Integrability and the Painlevé Test

Evolution equations with soliton solutions possess an infinite number of conservation laws, and this is related to what is called the *complete integrability* of the equation. A general question that arises is how to determine a priori whether a partial differential equation possesses soliton solutions. This question is closely connected with symmetries exhibited by the structure of the governing system of equations, and through this to the *integrability* of the system.

Before indicating how symmetries of the structure can be studied, it is necessary to outline the ideas underlying integrability, and to do so we now consider how to decide the integrability of the *Korteweg-deVries-Burgers* (KdVB) *equation*

$$\frac{\partial u}{\partial t} + 2au\,\frac{\partial u}{\partial x} + b\,\frac{\partial^2 u}{\partial x^2} + c\,\frac{\partial^3 u}{\partial x^3} = 0\,, \tag{12}$$

in which both a dissipative term ($b < 0$) and a dispersive term are present (c may be of either sign).

This can be considered as a prototype equation exhibiting both dissipation and dispersion, and such evolution equations arise when seeking the asymptotic behaviour of complicated systems governing physical processes in solid and fluid mechanics, as for example in nonlinear acoustics.

The existence of soliton solutions has been shown by WEISS, TABOR, and CARNEVALE [10] to be closely related to whether the governing equations possesses what is called the *Painlevé property.* The complete integrability of a partial differential equation, that is associated with the existence of soliton solutions, led to ABLOWITZ, RAMANI, and SEGUR [11] proposing the following conjecture:

Painlevé Test: *Every ordinary differential equation that arises as a similarity reduction of a completely integrable partial differential equation is of Painlevé type, sometimes after a transformation of variables.*

If this conjecture is true, it would provide a useful *necessary* condition when testing for complete integrability. Applications of the Painlevé Test made so far support the correctness of the conjecture.

An application of the Painlevé Test is based on the fact that the singularities of analytic functions of several complex variables cannot be isolated, so that if $f = f(z_1, z_2, \ldots, z_n)$ is a meromorphic function of n complex variables, the singularities of f occur along analytic manifolds of dimension $2n - 2$. These manifolds are determined by conditions of the form

$$\Phi(z_1, z_2, \ldots, z_n) = 0\,, \tag{13}$$

where Φ is an analytic function of $z_1, z_2, \ldots, z_n$ in a neighbourhood of the manifold. If a partial differential equation possesses the Painlevé property this means, in effect, that the solutions of the equation are single valued about the movable singularity manifolds.

Using the approach of WEISS et al., in order to establish whether a partial differential equation possesses the Painlevé property it is necessary to proceed as follows.

Let the solution u of the partial differential equation be expressed as an infinite series of the form

$$u = \varphi^\alpha \sum_{j=0}^{\infty} u_j \varphi^j\,, \tag{14}$$

where α is a negative integer to be determined by comparing the lowest powers of φ corresponding to the nonlinear and linear terms in the evolution equation. For the KdVB equation it turns out that $\alpha = -2$. A recurrence relation is then obtained for the determination of the coefficients u_j, and for the KdVB equations this has the form

$$cu_j(j+1)\,(j-4)\,(j-6)\,\boldsymbol{\Phi}_x^2 = h(\boldsymbol{\Phi}_t,\, \boldsymbol{\Phi}_x, \ldots,\, u_0,\, u_1, \ldots,\, u_{j-1})\,, \tag{15}$$

where h is a nonlinear function of its arguments. What are called *resonances* occur when $h = 0$, so for the KdVB equation these resonances correspond to $j = -1$, 4, and 6, at which values the u_j become arbitrary. A partial differential equation possesses the Painlevé property if the recurrence relations are all consistent at each resonance.

In the case of the KdVB equation it can be shown that incompatibilities arise when the recurrence relation corresponding to $j = 6$ is considered. Thus the KdVB equation does not have the Painlevé property at the resonance $j = 6$. However, the Painlevé property does hold for particular functions $\boldsymbol{\Phi}$, and so the KdVB equation can be said to possess the Painlevé property *conditionally* [12].

The KdVB equation possesses a travelling wave solution, though as the equation is not completely integrable, according to the Painlevé Test this solution does not have the soliton property. The form of the KdVB travelling wave solution when the equation is written in the form

$$u_t + uu_x - au_{xx} + bu_{xxx} = 0\,, \tag{16}$$

is

$$u(x,\, t) = -\frac{3a^2}{25b}\exp(\xi)\,\cosh^{-2}(\xi/2)\,, \tag{17}$$

with

$$\xi = \frac{a}{5b}\left(x + \frac{6a^2}{25b}\,t\right). \tag{18}$$

By means of the Painlevé approach it is possible to examine the integrability of many nonlinear evolution equations, though the analysis can be complex and require the use of a computer symbolic algebra program to simplify the calculations. Using this approach the non-integrability of the *fifth order* KdV *equation*

$$\frac{\partial u}{\partial t} + au\,\frac{\partial u}{\partial x} + b\,\frac{\partial^3 u}{\partial x^3} + c\,\frac{\partial^5 u}{\partial x^5} = 0\,, \tag{19}$$

was established [13].

As the KdV equation is completely integrable, it is interesting to investigate the relationship between its periodic solutions and its sech^2 solutions [14]. To do this we start with the equation in the form

$$u_t - 6uu_x + u_{xxx} = 0\,. \tag{20}$$

A solution is now sought of the form due to Matsuno

$$u(x,\, t) = u_0 - 2\,\frac{\partial^2}{\partial x^2}\,(\ln\, f(x,\, t))\,, \tag{21}$$

with u_0 an arbitrary constant and F satisfies the Hirota bilinear form

$$(D_xD_t - 6u_0D_x^2 + D_x^4 + A)\; f\cdot f = 0\,, \tag{22}$$

in which $A = A(t)$ is an arbitrary function of integration and the D operators are the Hirota derivatives. The *Hirota bilinear operator* is defined as

$$\mathrm{D}_t^m\mathrm{D}_x^n(a\cdot b) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n a(x,\, t)\, b(x',\, t')|_{x'=x,\, t'=t}\,.$$

Dubrovin has shown that (22) has solutions

$$u(x,\, t) = U_0 - 2\,\frac{\partial^2}{\partial x^2}\,(\ln\,\theta(z))\,, \tag{23}$$

where $\theta(z)$, with $z = (z_1,\, z_2, \ldots,\, z_n)$ and $z_i = k_ix + \omega_{\mathrm{I}} + \alpha_{\mathrm{I}}$, is the n-dimensional theta function.

Considering the case $n = 1$, dropping the suffix 1 and working with $\theta_3(z,\, q)$, leads to the result

$$f(x,\, t) = \theta_3(z,\, q) = \sum_{n=-\infty}^{\infty} q^{n^2}\,\mathrm{e}^{2inz}\,, \tag{24}$$

with $q = \mathrm{e}^{in\tau}$, and τ a complex parameter. Substitution of (24) into (22) then gives a result of the form

$$\sum_{n=-\infty}^{\infty} F_n\,\mathrm{e}^{2inz} = 0\,, \tag{25}$$

where the F_n involve k, ω, and τ. The linear independence of the functions e^{2inz} require that $F_n = 0$ for each n. An examination of the equations shows the F_n must satisfy the recurrence relation

$$F_n = q^{2(n-1)}F_{n-2}\,. \tag{26}$$

When n is even this result relates F_n to F_0, and when it is odd it relates F_n to F_1, so

$$F_{2m} = q^{2m^2} F_0 \quad \text{and} \quad F_{2m-1} = q^{2m(m-1)}\,, \tag{27}$$

for $m = 0, \pm 1, \pm 2, \ldots$. This result implies that $f = \theta_3$ is a solution of the bilinear form (22) if, and only if,

$$F_0 = F_1 = 0\,. \tag{28}$$

Because of this, the problem of finding the periodic solutions of the KdV equation is reduced to finding solutions of the two equations $F_0 = 0$ and $F_1 = 0$ involving ω and A, where u_0 represents the ambient level of the wave, k is its wave number, and α is its phase.

The equations corresponding to $F_0 = 0$ and $F_1 = 0$ are

$$B_0 A + B_1(5u_0k^2 - \omega k) + B_2k^4 = 0 \quad \text{and} \quad C_0A + C_1(6u_0k^2 - \omega k) + C_2k^4 = 0\,, \tag{29}$$

where the B_i and C_i depend only on q, and hence on τ.

Solving these equations leads to the solution

$$\theta_1(z, q) = -\theta_2(z + \pi/2, q)\,, \tag{30}$$

where

$$\theta_2(z, q) = \sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2} \exp[i(2n+1)\,z]\,. \tag{31}$$

This result can be used to show how the cnoidal wave solution of the KdV equation is representable as an infinite sum of equally spaced sech^2 functions, each with its peak at $x = 2n\pi$, for $n = 0, \pm 1, \pm 2, \ldots$. The formation of a cnoidal wave solution from the sech^2 solutions is illustrated in Fig. 9.

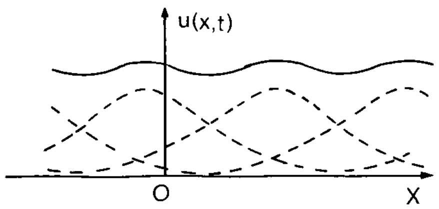

Fig. 9. Cnoidal wave solution formed from sech^2 solutions

From the viewpoint of soliton theory it is important to recognise that although the solitary wave solutions of the KdV equation are sech^2 functions, this periodic solution is not a sum of solitary waves. What has been obtained is a solitary wave *profile* rather than a sum of solitary waves. This follows because the speed of the cnoidal wave is equal to the speed of the sech^2 waves, but this speed is not the same as the speed of a solitary wave of the same amplitude. Waves of this type have been studied by PARKER [14], and the series (31) that is involved has been called an *imbricate series*, in the sense that the individual solutions overlap. This structure is also implied by a study of the structural symmetry of the equation that is revealed by group theoretical analysis.

6. Bäcklund transformations

Before mentioning Lie group theory, it is necessary to make some reference to Bäcklund transformations. These are remarkable transformations that enable the solution of a nonlinear partial differential equation to be used to find a solution of a different more complicated nonlinear partial differential equation, or sometimes another solution of the same partial differential equation. The name *Bäcklund transformation* is given to a transformation that relates a solution of one partial differential equation to the solution of a different partial differential, while the name *auto-Bäcklund transformation* is given to a transformation that relates solutions of the same partial differential equation. The simplest example is provided by the Cauchy-Riemann equations from complex analysis; namely, if the analytic function $f(z) = u + iv$, then $u_x = v_y$ and $u_y = -v_x$. This provides an auto-Bäcklund transformation because both u and v are harmonic functions, and given one solution u the other solution v can be found, and conversely.

It is difficult to find Bäcklund transformations [15], but a typical example is provided by considering the Burgers equation

$$u_t + uu_x = \nu u_{xx}\,. \tag{32}$$

Let us suppose a solution $u(x, t)$ is known. Then if $\varphi(x, t)$ is defined as any solution of the linear partial differential equation

$$\varphi_t + h(x, t)\,\varphi_x = \nu\varphi_{xx}\,, \tag{33}$$

with $h(x, t)$ an arbitrary function, differentiation shows that $v(x, t)$ is also a solution of Burgers equation, where

$$v(x, t) = -2\nu(\varphi_x/\varphi) + h(x, t)\,. \tag{34}$$

This transformation is, of course, a slight generalization of the Hopf-Cole transformation [3] used to transform the Burgers equation into the heat equation. To see how this Bäcklund transformation generates new solutions we start with the null solution $u(x, t) \equiv 0$ and set $h(x, t) \equiv 0$ thereby reducing (33) to the heat equation

$$\varphi_t = v\varphi_{xx} \,. \tag{35}$$

Equation (35) has the well known solution

$$\varphi(x, t) = \exp[-x^2/4vt]/(4\pi vt)^{1/2} \,, \tag{36}$$

so another solution of Burgers equation is

$$v(x, t) = -2v(\varphi_x/\varphi) + h(x, t) \,,$$

or

$$v(x, t) = x/t \,. \tag{37}$$

Repeated applications of this process using each solution to generate another solution produces a family of solutions.

When a Bäcklund transformation is applied to the KdV equation, and a multisoliton interaction solution is used as the initial *seed* solution, an entire hierarchy of soliton superposition solutions is generated.

Bäcklund and auto-Bäcklund transformations are related to the structural symmetry properties of the equation, and information that helps find these transformations can be obtained by study of the group theoretic properties of the system.

7. Lie groups

It is now necessary to provide a brief introduction to the study of the structure of the partial differential equations using the notion of a *symmetry group*. This term is used to refer to a group of transformations that maps any solution to another solution of the system. In Lie theory such a group depends on continuous parameters and consists of either point transformations (point symmetries) acting on the system spaces of independent and dependent variables or, more generally, it involves *contact transformations* (contact symmetries) acting on the space that includes all first order derivatives of the dependent variables.

Briefly, a *Lie group* is a topological group in which the group operation and taking inverses are continuous functions, and in which the group manifold has a structure that allows calculus to be performed on it.

For example, it must be possible to assign a coordinate system in an open set about each point of the manifold in such a way that overlapping coordinate systems are analytically related. This means that if $\mathbf{a}$ and $\mathbf{b}$ are points of the group and the product $\mathbf{c} = \mathbf{a} \circ \mathbf{b}$, then the coordinates of $\mathbf{c}$ must be analytic functions of the coordinates of $\mathbf{a}$ and $\mathbf{b}$.

To illustrate ideas let us consider ordinary differential equations and the transformation

$$x^* = f(x, y; \varepsilon) \,, \qquad y^* = g(x, y; \varepsilon) \,, \tag{38}$$

where ε is a parameter. If a second transformation in the group is given by

$$x^{**} = f(x, y; \bar{\varepsilon}) \,, \qquad y^{**} = g(x, y; \bar{\varepsilon}) \,,$$

the group closure property requires that

$$x^{**} = f(x, y; \gamma) \,, \qquad y^{**} = g(x, y; \gamma) \,,$$

where the product parameter $\gamma = \Phi(\varepsilon, \bar{\varepsilon})$.

To apply these ideas to a k-th order ordinary differential equation

$$F(x, y, y', \ldots, y^{(k)}) = 0$$

so it is invariant under the one-parameter Lie group of transformations in (38), each derivative must transform in a particular way. From (38)

$$\begin{aligned} y'^* &= \frac{\mathrm{d}y^*}{\mathrm{d}x^*} = \frac{g_x + g_y y'}{f_x + f_y y'} \equiv \theta^1(x, y, y', \varepsilon) \,, \\ y''^* &= \frac{\mathrm{d}y'^*}{\mathrm{d}x^*} = \frac{\theta^1_x + \theta^1_y y' + \theta^1_{y'} y''}{f_x + f_y y'} \equiv \theta^2(x, y, y', y'', \varepsilon) \,, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y^{(k)^*} &= \ldots \equiv \theta^{(k)}(x, y, y', \ldots, y^{(k)}, \varepsilon) \,. \end{aligned} \tag{39}$$

This extension of the transformation to higher order derivatives is called a *prolongation*. Infinitesimally, this can be written

$$\mathbf{X} = \xi\partial_x + \eta\partial_y\,, \tag{40}$$

where

$$\xi(x,\,y) \equiv \left.\frac{\partial x^*}{\partial\varepsilon}\right|_{\varepsilon=0} \quad \text{and} \quad \eta(x,\,y) \equiv \left.\frac{\partial y^*}{\partial\varepsilon}\right|_{\varepsilon=0}\,. \tag{41}$$

Thus the *infinitesimal generator* $\mathbf{X}^{(k)}$ of the prolongation of $\mathbf{X}$ is

$$\mathbf{X}^{(k)} = \xi\partial_x + \eta\partial_y + \eta^{(1)}\partial_{y'} + \ldots + \eta^{(k)}\partial_{y^{(k)}} = \mathbf{X} + \eta^{(1)}\partial_{y'} + \ldots + \eta^{(k)}\,\partial_{y^{(k)}} = \mathbf{X}^{(k-1)} + \eta^{(k)}\partial_{y^{(k)}}\,. \tag{42}$$

This approach extends to partial differential equations and allows Lie groups and their infinitesimal generators to be prolonged to act on the space of independent variables, dependent variables, and derivatives of the dependent variables up to any finite order.

As a consequence, the apparently intractable nonlinear conditions of group invariance of a given system of partial differential equations reduces to a set of linear homogeneous equations that determine the infinitesimal generators of the group. As this system of equations is in the form of an overdetermined system of linear homogeneous partial differential equations, it is usually possible to determine the infinitesimal generators in closed form.

If the system of partial differential equations is invariant under a Lie group of point transformations it is possible to find similarity solutions. When applied to boundary value problems it is necessary to find a subgroup of the full group of partial differential equations that leaves the boundary curves and boundary conditions invariant.

Noether's theorem establishes a one-one relationship between conservation laws and symmetry generators for equations derivable from a variational principle, and recent developments have been connected with Hamiltonian systems. For example, the connection with the KdV equation can be seen by setting $u = \Phi_x$ in the equation

$$u_t - 6uu_x + u_{xxx} = 0 \tag{43}$$

to obtain

$$\Phi_{xt} - 6\Phi_x\Phi_{xx} + \Phi_{xxxx} = 0\,. \tag{44}$$

This last equation has associated with it the *Lagrangian*

$$L = \tfrac{1}{2}\,\Phi_x\Phi_t - \Phi_x^3 - \tfrac{1}{2}\,\Phi_{xx}^2\,. \tag{45}$$

Noether's theorem establishes a one-one relationship between conservation laws and symmetry generators, and when applied to the KdV equation, included amongst the laws generated are the laws of conservation of mass, momentum, and energy. The law for the conservation of energy can be used in a *Hamiltonian formulation* by writing

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left\{\frac{\delta}{\delta u}\left(u^3 + \frac{1}{2}\,u_x^2\right)\right\}, \tag{46}$$

where $\delta/\delta u$ is the *variational* or *Frechet derivative*.

More information about group theoretic approaches to ordinary and partial differential equations is to be found in the work of OVSIANNIKOV [16], BLUMAN and KUMEI [17, 18], HARRISON and ESTABROOK [19], STEPHANI [20], and OLVER [21].

Appendix: Travelling Wave Solutions of Some Named Nonlinear Evolution Equations

Generalised Burgers Equation

$$u_t + u^n u_x = vu_{xx}\,, \qquad u(\infty) = 0\,, \qquad a = [u(-\infty) - u(\infty)]/2 \quad (a > 0)\,,$$

$$u(x,\,t) = \left\{\left(\frac{n+1}{2}\right)\,c\left(1 - \tanh\left[nc\left(\frac{x-ct}{2a}\right)\right]\right)\right\}^{1/n} \quad \text{with} \quad c = u(-\infty)^n/(n+1)\,.$$

Kuramoto-Sivashinsky Equation

$$u_t + uu_x - au_{xx} - bu_{xxx} = 0\,, \qquad ab > 0\,, \qquad u(x,\,t) = \tfrac{15}{19}\,ak(9f - 11f^3 - 2)\,,$$

where

$$f = \mathrm{th}\left[\frac{k}{2}\left(x - \frac{30}{19}\,akt\right)\right] \quad \text{and} \quad k = \pm\sqrt{\frac{11a}{19b}}\,.$$

Modified KdV Equation

$$u_t + u^2u_x + bu_{xxx} = 0\,, \qquad u(x,\,t) = \sqrt{6c}\,\mathrm{sech}\left[\sqrt{\frac{c}{b}}\,(x - ct)\right] \quad \text{with} \quad c > 0, \quad b > 0\,.$$

Sine-Gordon Equation

$$u_{xx} - u_{tt} = a^2 \sin u\,, \qquad u(x,\,t) = 4 \arctan \exp\left[\pm a\, \frac{x - ct}{\sqrt{1 - c^2}}\right], \qquad c^2 < 1\,.$$

KdVB Equation

$$u_t + uu_x - au_{xx} + bu_{xxx} = 0\,, \qquad u(x,\,t) = -\frac{3a^2}{25b} \exp(\xi) \cosh^{-2}(\xi/2)\,,$$

with $\xi = \dfrac{a}{5b}\left(x + \dfrac{6a^2}{25b}\, t\right)$.

Kadomtsev-Petviashilli (KP) Equation

$$(u_t + 6uu_x + bu_{xxx})_x + +\varepsilon y_{yy} = 0\,, \qquad u(x,\,t) = \frac{c}{2} \cosh^{-2}\left[\frac{\sqrt{c}}{2a\sqrt{b}}\,(ax + by - \omega t)\right], \qquad cb > 0\,,$$

with $c = (a\omega - \varepsilon b^2)/a^2$.

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