Towards an Intuitive Understanding of Quaternion Rotations

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Abstract:

Although quaternions are generally deemed useful for representing three dimenional rotations and are often used in 3D redering software, most of the information on them is either extremely technical, or merely defines them without offering any intuitive understanding. In particular, while the expression for a general 4D quaternion rotation is often given, there is seldom any explanation of how to construct a particular desired rotation, other than blindly converting from the equivalent matrix.

In this paper I'd like to present my own attempts at an intuitive understanding. (Intuitive, at least for someone who is very familiar with complex numbers and is comfortable with linear algebra.) Although this by it's very nature has a hand waving quality, I hope that this kind of understanding will lead to the more effective application of quaternions.

Properties of 4D Rotations

Rotations in three dimensions are usually described as rotation about an axis — a particular vector. It is more accurate to say that rotation occurs in a plane, but in three dimensions it's easier to describe a plane by giving a vector perpendicular to the plane at the center of rotation.

That isn't sufficient in four dimensions. There is a two dimensional space perpendicular to any rotation plane, so to describe the rotation plane, we'll have to give two directions in it. In this paper I will describe the "x-y plane" or the "z-w plane" and so on. I'll eventually want to construct a rotation given any two non-parallel vectors.

In addition, since there are two dimensions perpendicular to any rotation plane, that allows us to have a second, completely independent rotation plane spinning at the same time. A general rotation will have two angles of rotation, as well as two vectors describing one rotation plane. The orientation of the second rotation plane can be derived from the first one, but only up to an overall sign. We can't tell which way the second plane is spinning without additional input, or at least a convention of some sort.

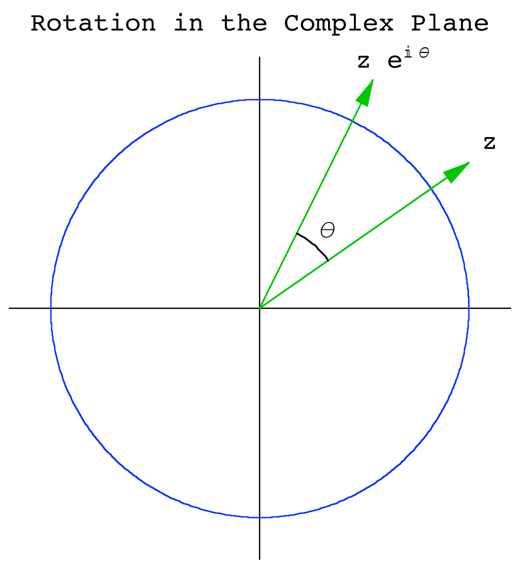
Review of Rotations in the Complex Plane

Quaternions in four dimensions can be viewed as a generalization of complex numbers in two dimensions. Complex rotations should be familiar to anyone with a Physics or Engineering background.

Given a complex number $z = x + i y$ (where x and y are real numbers and $i^2 = -1$), which can also be viewed as the point (x,y) on the complex plane, we can rotate that point about the origin by multiplying by a pure complex phase, such as $e^{i\theta} = \cos(\theta) + i \sin(\theta)$.

$$\begin{aligned} e^{i\theta} z &= (\cos(\theta) + i \sin(\theta))\,(x + i\,y) \\ &= x\cos(\theta) - y\sin(\theta) + i\,(y\cos(\theta) + x\sin(\theta)) \end{aligned} \tag{1}$$

which looks like a rotation by angle θ in the x-y plane.



The rotation itself is just another complex number, except that it sits on a circle of radius one. In fact, any complex number can be written in this polar form.

$$z = r\,e^{i\theta} = r\cos(\theta) + i\,r\sin(\theta) \tag{2}$$

The complex conjugate (replaceing i with -i) gives us a rotation in the opposite direction.

$$z^* = r\,e^{-i\theta} = r\cos(-\theta) + i\,r\sin(-\theta) = r\cos(\theta) - i\,r\sin(\theta) \tag{3}$$

A Review of Quaternions

■ Definition

A quaternion is a combination of four real numbers,

$$\mathbf{q} = w + \mathbf{i}\,x + \mathbf{j}\,y + \mathbf{k}\,z \tag{4}$$

where $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ (just like for complex numbers) and $\mathbf{i}\,\mathbf{j}\,\mathbf{k} = -1$. For any pair of them, $\mathbf{i}\,\mathbf{j} = -\mathbf{j}\,\mathbf{i}$ (they *anticommute*). In particular:

$$\begin{array}{lll} \mathbf{i}\,\mathbf{j} = \mathbf{k} & \mathbf{j}\,\mathbf{k} = \mathbf{i} & \mathbf{k}\,\mathbf{i} = \mathbf{j} \\ \mathbf{j}\,\mathbf{i} = -\mathbf{k} & \mathbf{k}\,\mathbf{j} = -\mathbf{i} & \mathbf{k}\,\mathbf{i} = -\mathbf{j} \end{array} \tag{5}$$

Physicists will recognize **i**, **j**, and **k** as similar to the Pauli matrices from Quantum Mechanics. Indeed, you can write them as 2x2 unitary matrices, but we won't need to do that here. All we need to know is that they're the axes of a four-dimensional space.

■ Polar Form

We can also write a quaternion in "polar form" in the same way we handled complex numbers.

$$\mathbf{q} = r\, e^{\hat{\mathbf{n}}\,\theta} \tag{6}$$

where r is a real number, θ is an angle in radians, and $\hat{\mathbf{n}}$ is a three dimensional unit vector. I will sometimes call $\hat{\mathbf{n}}$ an "axis vector" and will often treat it like any other quaternion.

$$\begin{array}{l} \hat{\mathbf{n}} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} \\ x^2 + y^2 + z^2 = 1 \end{array} \tag{7}$$

We can expand the polar form just as we did with complex numbers.

$$\mathbf{q} = r\, e^{\hat{\mathbf{n}}\,\theta} = r\,\cos(\theta) + r\,\hat{\mathbf{n}}\sin(\theta) \tag{8}$$

It may seen surprising that a complicated mix of **i**, **j**, and **k** like $\hat{\mathbf{n}}$ is should behave as if it were just **i** alone, but it shouldn't be. We tend to think that "**i** is along the x-axis" but really, one direction is just as good as any other. Any combination of **i**, **j**, and **k** (that still has length 1) is just as good as any other.

Exercise: $\hat{\mathbf{n}}^2 = ?$

Given an arbitrary quaternion **q**, you can calculate the polar form like this. (Where $\mathbf{q}^*$ is the quaternion equivalent of the complex conjugate.)

$$\begin{array}{l} r = |\,\mathbf{q}\,| = \sqrt{\mathbf{q}\,\mathbf{q}^*} \\ \cos(\theta) = \mathrm{Re}\left(\dfrac{\mathbf{q}}{r}\right) = \dfrac{\mathbf{q}+\mathbf{q}^*}{2\,r} \\ \sin(\theta) = \left|\dfrac{\mathbf{q}-\mathbf{q}^*}{2\,r}\right| = \dfrac{\sqrt{-(\mathbf{q}-\mathbf{q}^*)^2}}{2\,r} \\ \hat{\mathbf{n}} = \dfrac{\mathbf{q}-\mathbf{q}^*}{|\,\mathbf{q}-\mathbf{q}^*\,|} = \dfrac{\mathbf{q}-\mathbf{q}^*}{2\,r\,\sin(\theta)} \end{array} \tag{9}$$

Be careful of the case where **q** is real. Then $\theta = 0$ and $\hat{\mathbf{n}}$ is undefined. Also note that the sign of θ can be absorbed into $\hat{\mathbf{n}}$, so even though you can find θ in the range $-\pi < \theta \leq \pi$ given both the sine and the cosine (and many math libraries have a two argument arctangent function for that purpose), in the above we limited θ to $\theta \geq 0$. If we had wanted the full range of θ we could have limited the directions of $\hat{\mathbf{n}}$ instead.

■ Manipulating Quaternions

Adding two quaternions is easy. Just add the components like you would a vector.

$$\mathbf{q}_1 + \mathbf{q}_2 = (w_1 + w_2) + \mathbf{i}\,(x_1 + x_2) + \mathbf{j}\,(y_1 + y_2) + \mathbf{k}\,(z_1 + z_2) \tag{10}$$

Multiplying two quaternions can look really hairy due to the multiplication rules of the **i**, **j**, and **k**.

$$\begin{aligned}\mathbf{q}_1\,\mathbf{q}_2 = {} & (w_1 w_2 - x_1 x_2 - y_1 \mathrm{i}_2 - z_1 z_2) + \mathbf{i}\,(w_2 x_1 + w_1 x_2 - y_2 z_1 + y_1 z_2) + \\ & \mathbf{j}\,(w_2 y_1 + w_1 y_2 + x_2 z_1 - x_1 z_2) + \mathbf{k}\,(-x_2 y_1 + x_1 y_2 + w_2 z_1 + w_1 z_2)\end{aligned} \tag{11}$$

It makes a little more sense if you write the quaternion in a "1+3" notation, [w, **v**], where **v** is just an ordinary 3-vector made from the real cooefficients of the **i**, **j**, and **k**. Then

$$\begin{aligned}\mathbf{q}_1\,\mathbf{q}_2 &= \left[w_1, \vec{\mathbf{v}}_1\right]\left[w_2, \vec{\mathbf{v}}_2\right] \\ &= \left[w_1 w_2 - \vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_2,\; w_1 \vec{\mathbf{v}}_2 + w_2 \vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_1 \times \vec{\mathbf{v}}_2\right]\end{aligned} \tag{12}$$

Note that when w_1 and w_2 are zero, then the quaternion product is just the vector dot product and vector cross product at the same time.

Finding the inverse of a quaternion is easiest in polar form. It's just like that of a complex number. We can interpret it as changing a rotation by angle θ to a rotation by -θ.

$$\mathbf{q}^{-1} = \left(r\, e^{\hat{\mathbf{n}}\theta}\right)^{-1} = \frac{1}{r}\, e^{-\hat{\mathbf{n}}\theta} = \frac{1}{r}\cos(\theta) - \frac{1}{r}\,\hat{\mathbf{n}}\sin(\theta) \tag{13}$$

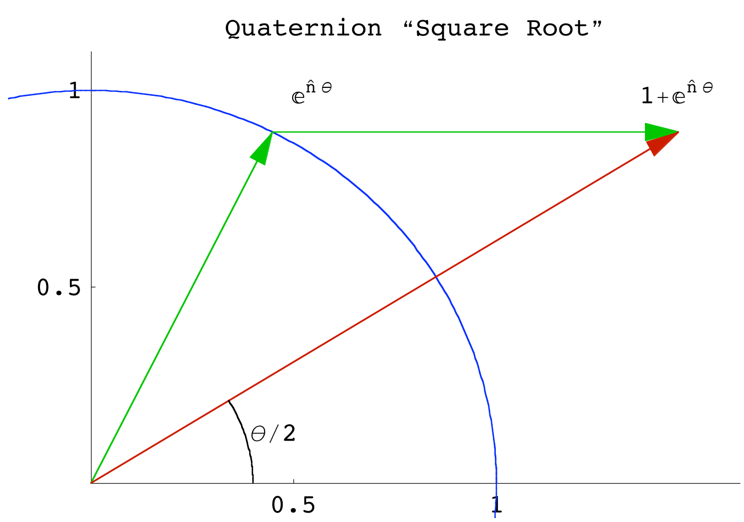
In particular, if **q** is a unit quaternion, the inverse is the same thing as the conjugate $\mathbf{q}^*$.

Exercise: Write down an expression for $\sqrt{\mathbf{q}}$ in polar form.

There is a clever way to find the "square root" (the half angle version, ignoring overall ±) of a unit quaternion: Add one and renormalize.

$$e^{\hat{\mathbf{n}}\theta/2} = \frac{1 + e^{\hat{\mathbf{n}}\theta}}{\left|\,1 + e^{\hat{\mathbf{n}}\theta}\,\right|} = \frac{1 + e^{\hat{\mathbf{n}}\theta}}{\sqrt{2\,(1 + \cos(\theta))}} \tag{14}$$

It's easy to see how this works if you draw a picture of the two quaternions being added. The total quaternion (in red) is half the angle from the real axis no matter what the starting quaternion is.



It's also easy to see that the quaternion square root of -1 is undefined. It could be any 3-space unit vector $\hat{\mathbf{n}}$. In such cases, you'll need some additional criteria to figure out which one you might want in a particular situation.

Quaternions as a “Complex-Like” Rotation

To build up to a general quaternion rotation, I’ll start by naively treating it just like a complex rotation. We’ll apply a pure rotation of the form $e^{\hat{\mathbf{n}}\,\theta}$ to a quaternion $\mathbf{q}$. The trouble is, the order of multiplication matters here. Do we multiply the phase from the left or the right? The correct answer is both. We split the difference, like this.

$$e^{\hat{\mathbf{n}}\,\theta/2}\;\mathbf{q}\;e^{\hat{\mathbf{n}}\,\theta/2} \tag{15}$$

To see what's going on, however, I'm going to write $\mathbf{q}$ as the sum of two parts: $\mathbf{q}_c$, the part which commutes with $\hat{\mathbf{n}}$ (the real part and the part that’s parallel with $\hat{\mathbf{n}}$) and $\mathbf{q}_a$, the part which anticommutes with $\hat{\mathbf{n}}$ (the part that is orthogonal to $\hat{\mathbf{n}}$). For example, if $\hat{\mathbf{n}} = \mathbf{i}$, then $\mathbf{q}_c = w + \mathbf{i}\,x$, and $\mathbf{q}_a = \mathbf{j}\,y + \mathbf{k}\,z$.

$$\begin{aligned} e^{\hat{\mathbf{n}}\,\theta/2}\;\mathbf{q}\;e^{\hat{\mathbf{n}}\,\theta/2} &= e^{\hat{\mathbf{n}}\,\theta/2}\;(\mathbf{q}_c + \mathbf{q}_a)\;\;e^{\hat{\mathbf{n}}\,\theta/2} \\ &= e^{\hat{\mathbf{n}}\,\theta/2}\;\mathbf{q}_c\;\;e^{\hat{\mathbf{n}}\,\theta/2} + e^{\hat{\mathbf{n}}\,\theta/2}\;\mathbf{q}_a\;\;e^{\hat{\mathbf{n}}\,\theta/2} \end{aligned} \tag{16}$$

The commuting part, $\mathbf{q}_c$, is easy. It passes right though one of the phases, allowing us to combine them. The anticommuting part, $\mathbf{q}_a$, is almost as easy. It passes through the phase *changing the sign of* $\hat{\mathbf{n}}$ *as it goes*, leaving us with:

$$e^{\hat{\mathbf{n}}\,\theta/2}\;e^{\hat{\mathbf{n}}\,\theta/2}\;\mathbf{q}_c\;+e^{\hat{\mathbf{n}}\,\theta/2}\;e^{-\hat{\mathbf{n}}\,\theta/2}\;\mathbf{q}_a\;=e^{\hat{\mathbf{n}}\,\theta}\;\mathbf{q}_c\;+\mathbf{q}_a \tag{17}$$

The interpretation of our transformation is clear:

> *The transformation* $e^{\hat{\mathbf{n}}\,\theta/2}\;\mathbf{q}\;e^{\hat{\mathbf{n}}\,\theta/2}$ *can be interpreted as a rotation of angle* θ *in the plane defined by* $\hat{\mathbf{n}}$ *and the “real axis”. I will call these “w-axis rotations”.*

In particular,

$e^{\mathbf{i}\,\theta/2}\;\mathbf{q}\;e^{\mathbf{i}\,\theta/2}$	rotates in the w-x plane
$e^{\mathbf{j}\,\theta/2}\;\mathbf{q}\;e^{\mathbf{j}\,\theta/2}$	rotates in the w-y plane
$e^{\mathbf{k}\,\theta/2}\;\mathbf{q}\;e^{\mathbf{k}\,\theta/2}$	rotates in the w-z plane

Quaternions as a 3D Rotation

Textbooks usually describe a quaternion rotation as $\mathbf{u}\;\mathbf{q}\;\mathbf{u}^{-1}$ where $\mathbf{u}$ is any unit quaternion. (Aside: In this case it doesn’t have to be a unit quaternion, since the inverse will cancel out any overall r scale factors, but I'll continue to assume that it is one.) Using the same techniques as in the previous section, we can see this is

$$\begin{aligned} e^{\hat{\mathbf{n}}\,\theta/2}\;\mathbf{q}\;e^{-\hat{\mathbf{n}}\,\theta/2} &= e^{\hat{\mathbf{n}}\,\theta/2}\;(\mathbf{q}_c + \mathbf{q}_a)\;\;e^{-\hat{\mathbf{n}}\,\theta/2} \\ &= e^{\hat{\mathbf{n}}\,\theta/2}\;\mathbf{q}_c\;\;e^{-\hat{\mathbf{n}}\,\theta/2} + e^{\hat{\mathbf{n}}\,\theta/2}\;\mathbf{q}_a\;\;e^{-\hat{\mathbf{n}}\,\theta/2} \end{aligned} \tag{18}$$

Now when we pull the phases through, the phases on the commuting part cancel, and we have

$$e^{\hat{\mathbf{n}}\,\theta/2}\;e^{-\hat{\mathbf{n}}\,\theta/2}\;\mathbf{q}_c\;+e^{\hat{\mathbf{n}}\,\theta/2}\;e^{\hat{\mathbf{n}}\,\theta/2}\;\mathbf{q}_a\;=\mathbf{q}_c\;+e^{\hat{\mathbf{n}}\,\theta}\;\mathbf{q}_a \tag{19}$$

It’s clear that the rotation is now only acting on $\mathbf{q}_a$, but in what way, exactly? I’m going to rewrite the anti-commuting part in a different way. For the sake of notation, I’m going to assume that $\hat{\mathbf{n}}=\mathbf{i}$, so that $\mathbf{q}_a$ can be written $\mathbf{j}\, y + \mathbf{k}\, z$. Then, I will insert a factor of $1 = \mathbf{j}(-\mathbf{j})$, and pull one $\mathbf{j}$ out to the end (changing the sign of $\mathbf{i}$ as it anticommutes.)

$$\begin{aligned} &e^{\mathbf{i}\,\theta/2}\, \mathbf{q}_a\, e^{-\mathbf{i}\,\theta/2} = \\ &e^{\mathbf{i}\,\theta/2}\, (\mathbf{j}\, y + \mathbf{k}\, z)\, (-\mathbf{j})\, \mathbf{j}\, e^{-\mathbf{i}\,\theta/2} = e^{\mathbf{i}\,\theta/2}\, (y + \mathbf{i}\, z)\, e^{\mathbf{i}\,\theta/2}\, \mathbf{j} \end{aligned} \tag{20}$$

And that looks like an ordinary complex rotation in the y-z plane (with a $\mathbf{j}$ tacked on the end to make the final result space-like.)

The interpretation of this transformation is clear:

> *The transformation* $e^{\hat{\mathbf{n}}\,\theta/2}\, \mathbf{q}\, e^{-\hat{\mathbf{n}}\,\theta/2}$ *can be interpreted as a rotation of angle* θ *about the axis* $\hat{\mathbf{n}}$ *leaving the real part unchanged. I will call this a "3-space rotation".*

In particular,

$e^{\mathbf{i}\,\theta/2}\, \mathbf{q}\, e^{-\mathbf{i}\,\theta/2}$	rotates in the y-z plane
$e^{\mathbf{j}\,\theta/2}\, \mathbf{q}\, e^{-\mathbf{j}\,\theta/2}$	rotates in the z-x plane
$e^{\mathbf{k}\,\theta/2}\, \mathbf{q}\, e^{-\mathbf{k}\,\theta/2}$	rotates in the x-y plane

It is worth noting that while a general 3D rotation like $\mathbf{u}\,\mathbf{q}\,\mathbf{u}^{-1}$ rotates in a plane in 3-space, the cooresponding "w-axis" rotation, $\mathbf{u}\,\mathbf{q}\,\mathbf{u}$, rotates in a plane perpendicular to that of $\mathbf{u}\,\mathbf{q}\,\mathbf{u}^{-1}$. From that we can see what quaternion multiplication really is: In the expression $\mathbf{u}\,\mathbf{q}$, the $\mathbf{u}$ transforms $\mathbf{q}$ by applying a rotation, by the same angle, in two orthogonal planes at once. At the same time, you can interpret it as $\mathbf{q}$ transforming $\mathbf{u}$**,** rotating $\mathbf{u}$ about two different planes by some different angle.

■ 3D Rotations Applied to a Polar Form

Here is a useful identity.

$$\mathbf{u}\, e^{\hat{\mathbf{n}}\,\theta}\, \mathbf{u}^{-1} = e^{\mathbf{u}\,\hat{\mathbf{n}}\,\mathbf{u}^{-1}\,\theta} \tag{21}$$

This makes intuitive sense — a 3-space rotation ought to just rotate the 3-space axis vector. It's easy to prove if we expand it this way.

$$\begin{aligned} \mathbf{u}\, e^{\hat{\mathbf{n}}\,\theta}\, \mathbf{u}^{-1} &= \mathbf{u}\, (\cos(\theta) + \hat{\mathbf{n}}\sin(\theta))\, \mathbf{u}^{-1} \\ &= \cos(\theta) + \mathbf{u}\,\hat{\mathbf{n}}\,\mathbf{u}^{-1}\sin(\theta) \\ &= e^{\mathbf{u}\,\hat{\mathbf{n}}\,\mathbf{u}^{-1}\,\theta} \end{aligned} \tag{22}$$

Note, however, that this trick does not work for rotations through the w-axis

$$\mathbf{u}\, e^{\hat{\mathbf{n}}\,\theta}\, \mathbf{u} \neq e^{\mathbf{u}\,\hat{\mathbf{n}}\,\mathbf{u}\theta} \tag{23}$$

since $\mathbf{u}\,\hat{\mathbf{n}}\,\mathbf{u}$ typically has a real component, and our polar form breaks down.

General 4D Rotations

The most general 4D rotation of a quaternion $\mathbf{q}$ can be written $\mathbf{u}_L\, \mathbf{q}\, \mathbf{u}_R$, which is multiplying on the left and on the right by arbitrary unit quaternions. So, given two quaternions, what rotation do they represent? What are the two rotation planes, and what is the angle of rotation for each one? I'll start by writing the rotation in polar form.

$$e^{\hat{\mathbf{n}}_1\,\theta_1}\, \mathbf{q}\, e^{\hat{\mathbf{n}}_2\,\theta_2} \tag{24}$$

First note that if $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$ are parallel or antiparallel ($\hat{\mathbf{n}}_1 = \pm\hat{\mathbf{n}}_2$), we already know what rotations they represent. The trick will be to rewrite $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$ as parallel and antiparallel parts. Define $\hat{\mathbf{n}}_i$ and $\hat{\mathbf{n}}_j$ as rotations of $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$ so that

$$\begin{aligned}\hat{\mathbf{n}}_1 &= \cos(\varphi)\hat{\mathbf{n}}_i + \sin(\varphi)\hat{\mathbf{n}}_j \\ \hat{\mathbf{n}}_2 &= \cos(\varphi)\hat{\mathbf{n}}_i - \sin(\varphi)\hat{\mathbf{n}}_j\end{aligned} \tag{25}$$

You can also write this as

$$\begin{aligned}\frac{1}{2}(\hat{\mathbf{n}}_1 + \hat{\mathbf{n}}_2) &= \frac{1}{2}\left|\hat{\mathbf{n}}_1 + \hat{\mathbf{n}}_2\right|\hat{\mathbf{n}}_i = \cos(\varphi)\,\hat{\mathbf{n}}_i \\ \frac{1}{2}(\hat{\mathbf{n}}_1 - \hat{\mathbf{n}}_2) &= \frac{1}{2}\left|\hat{\mathbf{n}}_1 - \hat{\mathbf{n}}_2\right|\hat{\mathbf{n}}_j = \sin(\varphi)\,\hat{\mathbf{n}}_j\end{aligned} \tag{26}$$

Note that $\hat{\mathbf{n}}_i$ and $\hat{\mathbf{n}}_j$ are orthonormal, unless $\hat{\mathbf{n}}_1 = \pm\hat{\mathbf{n}}_2$, in which case one of $\hat{\mathbf{n}}_i$ or $\hat{\mathbf{n}}_j$ is undefined. Also note that φ is the angle between $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_i$, or half the angle between $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$,

We can remove the explicit references to $\hat{\mathbf{n}}_j$ with a 3-space rotation about $\hat{\mathbf{n}}_k = \hat{\mathbf{n}}_i\,\hat{\mathbf{n}}_j$. Let $\mathbf{u} = e^{\hat{\mathbf{n}}_k \varphi/2}$. Then

$$\begin{aligned}\hat{\mathbf{n}}_1 &= \mathbf{u}\;\hat{\mathbf{n}}_i\;\mathbf{u}^{-1} \\ \hat{\mathbf{n}}_2 &= \mathbf{u}^{-1}\;\hat{\mathbf{n}}_i\;\mathbf{u}\end{aligned} \tag{27}$$

Note that you can calculate $\mathbf{u}$ without having to calculate φ.

$$\begin{aligned}\hat{\mathbf{n}}_1\,\hat{\mathbf{n}}_2 &= (\cos(\varphi)\,\hat{\mathbf{n}}_i + \sin(\varphi)\,\hat{\mathbf{n}}_j)\,(\cos(\varphi)\,\hat{\mathbf{n}}_i - \sin(\varphi)\,\hat{\mathbf{n}}_j) \\ &= -\cos(2\varphi) - \hat{\mathbf{n}}_k \sin(2\varphi) = -\mathbf{u}^4\end{aligned}$$

so $\mathbf{u} = (-\hat{\mathbf{n}}_1\,\hat{\mathbf{n}}_2)^{1/4}$.

Our original general rotation can then be written

$$\begin{aligned}e^{\hat{\mathbf{n}}_1 \theta_1}\;\mathbf{q}\;e^{\hat{\mathbf{n}}_2 \theta_2} &= \left(\mathbf{u}e^{\hat{\mathbf{n}}_i \theta_1}\;\mathbf{u}^{-1}\right)\;\mathbf{q}\;\left(\mathbf{u}^{-1}\;e^{\hat{\mathbf{n}}_i \theta_2}\;\mathbf{u}\right) \\ &= \left(\mathbf{u}\;\left(e^{\hat{\mathbf{n}}_i \theta_1}\;(\mathbf{u}^{-1}\;\mathbf{q}\;\mathbf{u}^{-1})\;e^{\hat{\mathbf{n}}_i \theta_2}\right)\;\mathbf{u}\right)\end{aligned} \tag{28}$$

When we rearrange the parenthesis, we can see a new interpretation of the transformation.

First we apply $\mathbf{u}^{-1}$ which tilts the plane of rotation by φ, so that it's aligned on the w-axis.

Then we apply a "complex-like" rotation in the w-$\hat{\mathbf{n}}_i$ plane and a space rotation in the $\hat{\mathbf{n}}_j$-$\hat{\mathbf{n}}_k$ plane (depending on the values of θ_1 and θ_2.)

Finally, we apply $\mathbf{u}$, to return the rotation plane to where it was.

To make this explicit, we write our arbitrary $\mathbf{q}$ in this basis

$$\begin{aligned}\mathbf{q}_w &= \mathbf{u}^2 \\ \mathbf{q}_i &= \mathbf{u}\;\;\hat{\mathbf{n}}_i\;\mathbf{u} = \hat{\mathbf{n}}_i \\ \mathbf{q}_j &= \mathbf{u}\;\;\hat{\mathbf{n}}_j\;\mathbf{u} = \hat{\mathbf{n}}_j \\ \mathbf{q}_k &= \mathbf{u}\;\;\hat{\mathbf{n}}_k\;\mathbf{u} = \mathbf{u}^2\;\hat{\mathbf{n}}_k\end{aligned} \tag{29}$$

so that

$\mathbf{q} = w\,\mathbf{q}_w + x\,\mathbf{q}_i + y\,\mathbf{q}_j + z\,\mathbf{q}_k$, or

$\mathbf{u}^{-1}\;\mathbf{q}\;\mathbf{u}^{-1} = w + x\,\hat{\mathbf{n}}_i + y\,\hat{\mathbf{n}}_j + z\,\hat{\mathbf{n}}_k$,

and then observe how the rotation acts on the pieces.

$$\begin{aligned} e^{\hat{\mathbf{n}}_1 \theta_1}\, \mathbf{q}\, e^{\hat{\mathbf{n}}_2 \theta_2} &= \left(\mathbf{u} \left(e^{\hat{\mathbf{n}}_i \theta_1} \left(w + x\,\hat{\mathbf{n}}_i + y\,\hat{\mathbf{n}}_j + z\,\hat{\mathbf{n}}_k\right) e^{\hat{\mathbf{n}}_i \theta_2}\right) \mathbf{u}\right) \\ &= \left(\mathbf{u} \left(e^{\hat{\mathbf{n}}_i \theta_1} \left(w + x\,\hat{\mathbf{n}}_i\right) e^{\hat{\mathbf{n}}_i \theta_2} + e^{\hat{\mathbf{n}}_i \theta_1} \left(y + z\,\hat{\mathbf{n}}_i\right) e^{-\hat{\mathbf{n}}_i \theta_2}\, \hat{\mathbf{n}}_j\right) \mathbf{u}\right) \\ &= \left(\mathbf{u} \left(e^{\hat{\mathbf{n}}_i (\theta_1+\theta_2)} \left(w + x\,\hat{\mathbf{n}}_i\right) + e^{\hat{\mathbf{n}}_i (\theta_1-\theta_2)} \left(y + z\,\hat{\mathbf{n}}_i\right) \hat{\mathbf{n}}_j\right) \mathbf{u}\right) \end{aligned} \quad (30)$$

At this point it's clear that the w and x parts are going to be rotated by an angle of $\theta_1 + \theta_2$, while the y and z parts are going to be rotated by an angle of $\theta_1 - \theta_2$. I'll leave the remaining algebra to the reader and jump to my conclusion.

A general 4D rotation of quaternion $\mathbf{q}$*, given by* $e^{\hat{\mathbf{n}}_1 \theta_1}\, \mathbf{q}\, e^{\hat{\mathbf{n}}_2 \theta_2}$ *, is a rotation by* $\theta_1 + \theta_2$ *in the* $\mathbf{q}_w$ *-* $\mathbf{q}_i$ *plane, and a rotation by* $\theta_1 - \theta_2$ *in the* $\mathbf{q}_j$ *-*$\mathbf{q}_k$ *plane, where, if* $\hat{\mathbf{n}}_1 \neq \pm\hat{\mathbf{n}}_2$

$$\mathbf{q}_w = e^{\hat{\mathbf{n}}_k \varphi} = \left(-\hat{\mathbf{n}}_1\, \hat{\mathbf{n}}_2\right)^{1/2}$$

$$\mathbf{q}_i = \frac{\hat{\mathbf{n}}_1 + \hat{\mathbf{n}}_2}{|\hat{\mathbf{n}}_1 + \hat{\mathbf{n}}_2|} = \hat{\mathbf{n}}_i$$

$$\mathbf{q}_j = \frac{\hat{\mathbf{n}}_1 - \hat{\mathbf{n}}_2}{|\hat{\mathbf{n}}_1 - \hat{\mathbf{n}}_2|} = \hat{\mathbf{n}}_j$$

$$\mathbf{q}_k = e^{\hat{\mathbf{n}}_k \varphi}\, \hat{\mathbf{n}}_k = \mathbf{q}_w\, \hat{\mathbf{n}}_i\, \hat{\mathbf{n}}_j$$

If $\hat{\mathbf{n}}_1 = -\hat{\mathbf{n}}_2$ *then replace* $\theta_2 \to -\theta_2$ *and* $\hat{\mathbf{n}}_2 \to -\hat{\mathbf{n}}_2$ *, and then...*

If $\hat{\mathbf{n}}_1 = \hat{\mathbf{n}}_2$ *then* $\mathbf{q}_w = 1$, $\mathbf{q}_i = \hat{\mathbf{n}}_1$ *, and the second rotation is a 3-space rotation about the* $\hat{\mathbf{n}}_1$ *axis.*

■ Interpretation

For a general 4D rotation of the form $e^{\hat{\mathbf{n}}_1 \theta_1}\, \mathbf{q}\, e^{\hat{\mathbf{n}}_2 \theta_2}$, the following is true.

The rotation is in one plane leaving the other fixed, if and only if $\theta_1 = \pm\theta_2$ (modulo 2π of course).
If the angle between $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$ is 2φ then the planes of rotation are tilted by φ.
The planes of rotation intersect the 3D space along $\hat{\mathbf{n}}_1 \pm \hat{\mathbf{n}}_2$, which we have called $\hat{\mathbf{n}}_i$ and $\hat{\mathbf{n}}_j$.

■ Pictures (to do)

Rotation in an Particular Plane

Suppose we want to find a rotation that will rotation unit quaternion $\mathbf{q}_1$ into $\mathbf{q}_2$. Since each quaternion can also be a rotation, there is a very simple way of doing this — by multiplying by $\mathbf{q}_1$ and $\mathbf{q}_2$. In fact, there are lots of ways of doing this.

$$(\mathbf{q}_2\, \mathbf{q}_1^{-1})\mathbf{q}_1 = \mathbf{q}_1\, (\mathbf{q}_1^{-1}\, \mathbf{q}_2) = (\mathbf{q}_1^{-1})\, \mathbf{q}_1\, (\mathbf{q}_2) = (\mathbf{q}_2^{1/2}\, \mathbf{q}_1^{-1/2})\, \mathbf{q}_1\, (\mathbf{q}_1^{-1/2}\, \mathbf{q}_2^{1/2}) = \mathbf{q}_2 \quad (31)$$

While all of these rotations "work", in that they rotate $\mathbf{q}_1$ into $\mathbf{q}_2$, they each rotate the perpendicular plane in different ways. Let's break this down a bit. I'll write the quaternions in the 1+3 notation: $\mathbf{q}_n = \left[q_n, \vec{q}_n\right]$

$$\begin{aligned} \mathbf{q}_2\, \mathbf{q}_1^{-1} &= \left[q_2, \vec{q}_2\right]\left[q_1, -\vec{q}_1\right] = \left[q_1\, q_2 + \vec{q}_1 \cdot \vec{q}_2,\ q_1\, \vec{q}_2 - q_2\, \vec{q}_1 + \vec{q}_1 \times \vec{q}_2\right] \\ \mathbf{q}_1^{-1}\, \mathbf{q}_2 &= \left[q_1, -\vec{q}_1\right]\left[q_2, \vec{q}_2\right] = \left[q_1\, q_2 + \vec{q}_1 \cdot \vec{q}_2,\ q_1\, \vec{q}_2 - q_2\, \vec{q}_1 - \vec{q}_1 \times \vec{q}_2\right] \end{aligned} \quad (32)$$

The first thing to notice is that the real parts of these are symmetric in the indices 1 and 2, while the space parts are anti-symmetric. Thus, if you were to swap the indices, you'd turn this rotation into its conjugate, and therefore its inverse. This makes perfect sense, since rotation from $\mathbf{q}_1$ into $\mathbf{q}_2$ is the inverse of rotation from $\mathbf{q}_2$ into $\mathbf{q}_1$.

The second thing to notice is that the real parts are the 4D dot product between $\mathbf{q}_1$ and $\mathbf{q}_2$, and is therefore the cosine of the angle between them. This is what you'd expect for a rotation from $\mathbf{q}_1$ into $\mathbf{q}_2$.

Next, notice that the 3-space parts are different only in that the third term has a different sign.

$$\begin{aligned}&[0,\, q_1\, \vec{q}_2 - q_2\, \vec{q}_1] = q_1\, \mathbf{q}_2 - q_2\, \mathbf{q}_1 \;=\; |\, q_1\, \mathbf{q}_2 - q_2\, \mathbf{q}_1 \,|\, \hat{\mathbf{n}}_i \\ &[0,\, \pm\, \vec{q}_1 \times \vec{q}_2] = \pm\, |\, \vec{q}_1 \times \vec{q}_2 \,|\, \hat{\mathbf{n}}_j\end{aligned} \tag{33}$$

The first two terms is precisely the linear combination of $\mathbf{q}_1$ and $\mathbf{q}_2$ which has no real part. It's the direction along which the rotation plane intersects the 3-space. Earlier, we had called this direction $\hat{\mathbf{n}}_i$. The third term is the space direction that's orthogonal to the plane of $\mathbf{q}_1$ and $\mathbf{q}_2$, the direction along which the other, perpendicular, rotation plane intersects the 3-space. Earlier, we had called this direction $\hat{\mathbf{n}}_j$.

It is easy to see that these transformations have the same form as the more general transformation we found in the previous section. In fact, we can now write down by inspection the following expressions.

$$\begin{aligned}\mathbf{q}_2\, \mathbf{q}_1^{-1} &= \cos(\theta) + \sin(\theta)\, (\cos(\varphi)\, \hat{\mathbf{n}}_i + \sin(\varphi)\, \hat{\mathbf{n}}_j) = e^{\theta(\cos(\varphi)\, \hat{\mathbf{n}}_i + \sin(\varphi)\, \hat{\mathbf{n}}_j)} \\ \mathbf{q}_1^{-1}\, \mathbf{q}_2 &= \cos(\theta) + \sin(\theta)\, (\cos(\varphi)\, \hat{\mathbf{n}}_i - \sin(\varphi)\, \hat{\mathbf{n}}_j) = e^{\theta(\cos(\varphi)\, \hat{\mathbf{n}}_i - \sin(\varphi)\, \hat{\mathbf{n}}_j)}\end{aligned}$$

where θ is the angle between $\mathbf{q}_1$ and $\mathbf{q}_2$, and (34)

$$\begin{aligned}&q_1\, \mathbf{q}_2 - q_2\, \mathbf{q}_1 \;=\; |\, q_1\, \mathbf{q}_2 - q_2\, \mathbf{q}_1 \,|\, \hat{\mathbf{n}}_i = \sin(\theta)\cos(\varphi)\, \hat{\mathbf{n}}_i \\ &[0,\, \vec{q}_1 \times \vec{q}_2] = |\, \vec{q}_1 \times \vec{q}_2 \,|\, \hat{\mathbf{n}}_j = \sin(\theta)\sin(\varphi)\, \hat{\mathbf{n}}_j\end{aligned}$$

Claim: The rotation that takes the unit quaternion $\mathbf{q}_1$ into $\mathbf{q}_2$ while leaving the perpendicular rotation plane fixed can be written

$$(\mathbf{q}_2\, \mathbf{q}_1^{-1})^{1/2}\;\; \mathbf{q}\, (\mathbf{q}_1^{-1}\, \mathbf{q}_2)^{1/2} \tag{35}$$

We expand this as

$$e^{\frac{\theta}{2}\, (\cos(\varphi)\, \hat{\mathbf{n}}_i + \sin(\varphi)\, \hat{\mathbf{n}}_j)}\;\; \mathbf{q}\, e^{\frac{\theta}{2}\, (\cos(\varphi)\, \hat{\mathbf{n}}_i - \sin(\varphi)\, \hat{\mathbf{n}}_j)} \tag{36}$$

and use the results of the previous section to note that this is a rotation of angle θ in the plane given by

$$\begin{aligned}\mathbf{q}_w &= e^{\hat{\mathbf{n}}_k\, \varphi} = e^{\hat{\mathbf{n}}_i\, \hat{\mathbf{n}}_j\, \varphi} \\ \mathbf{q}_i &= \hat{\mathbf{n}}_i\end{aligned}$$

and a rotation of zero in the perpendicular plane.

(Aside: The literature describes something called a "Slerp" (**s**pherical **l**inear int**erp**olation) that will smoothly transform $\mathbf{q}_1$ into $\mathbf{q}_2$. They let $\mathbf{q}(t) \;=\; (\mathbf{q}_2\, \mathbf{q}_1^{-1})^t\, \mathbf{q}_1$, which is $\mathbf{q}_1$ when $t = 0$ and $\mathbf{q}_2$ when $t = 1$. Unfortunately, this only multiplies on one side, so it also rotates the plane orthogonal to $\mathbf{q}_1$ and $\mathbf{q}_2$. This isn't actually wrong, since the Slerp is only intended for interpolating between two points on a hypersphere — it was not ment to be a rotation operator, even if it looks like one in quaterion form. Still, if we wanted a slerp-like rotation in only one plane, it's clear what we need: $\mathbf{q}(t) \;=\; (\mathbf{q}_2\, \mathbf{q}_1^{-1})^{t/2}\, \mathbf{q}_1\, (\mathbf{q}_1^{-1}\, \mathbf{q}_2)^{t/2}$)

It's useful to look at eq. 35 for two special cases. First, if $\mathbf{q}_1$ and $\mathbf{q}_2$ lie in a plane that goes through the w-axis, then they have the same axis vector and therefore they commute. In particular $\mathbf{q}_2\, \mathbf{q}_1^{-1} = \mathbf{q}_1^{-1}\, \mathbf{q}_2$ and our rotation has the form $\mathbf{u}\; \mathbf{q}\; \mathbf{u}$ as you'd expect. Second, if $\mathbf{q}_1$ and $\mathbf{q}_2$ are both pure space vectors (having zero real components) then $\mathbf{q}_1^{-1} = -\,\mathbf{q}_1$ and $\mathbf{q}_2\, \mathbf{q}_1^{-1} = -\,\mathbf{q}_2\, \mathbf{q}_1 = \mathbf{q}_2^{-1}\, \mathbf{q}_1 = (\mathbf{q}_1^{-1}\, \mathbf{q}_2)^{-1}$, and therefore our rotation has the form $\mathbf{u}\; \mathbf{q}\; \mathbf{u}^{-1}$.

Swapping Rotation Planes

Given a rotation in two orthogonal rotation planes, can we find a rotation that swaps the rolls of those planes? This is surprisingly easy, given the machinery we already have. Suppose our rotation is represented by

$$e^{\hat{\mathbf{n}}_1 \theta_1} \mathbf{q} \; e^{\hat{\mathbf{n}}_2 \theta_2} \tag{37}$$

We already know that the tilt of one of the rotation planes is given by (assuming $\hat{\mathbf{n}}_1 \neq \pm\hat{\mathbf{n}}_2$ and $\theta_1, \; \theta_2 \neq 0$.)

$$\mathbf{q}_w = e^{\hat{\mathbf{n}}_k \varphi} = (-\hat{\mathbf{n}}_1 \, \hat{\mathbf{n}}_2)^{1/2} \tag{38}$$

All we need to do is spin this by 90°, which is just multiplying by $\hat{\mathbf{n}}_k$, though we'll probably want to do it symmetrically.

$$\begin{gathered} \mathbf{u} = \hat{\mathbf{n}}_k^{1/2} = (1 + \hat{\mathbf{n}}_k) \big/ \sqrt{2} \\ \left(\mathbf{u}^{-1} \, e^{\hat{\mathbf{n}}_1 \theta_1} \, \mathbf{u}\right) \mathbf{q} \; (\mathbf{u} \;\; e^{\hat{\mathbf{n}}_2 \theta_2} \, \mathbf{u}^{-1}) \end{gathered} \tag{39}$$

If $\hat{\mathbf{n}}_1 = \pm\hat{\mathbf{n}}_2$, or one of $\theta_1, \; \theta_2 = 0$, then $\hat{\mathbf{n}}_k$ is undefined. You can choose any $\hat{\mathbf{n}}_k$ which is perpendicular to whichever of $\hat{\mathbf{n}}_1$ or $\hat{\mathbf{n}}_1$ you can interpret.

Rotating a Triangle

Rotating from $\mathbf{q}_1$ to $\mathbf{q}_2$ only partially specifies a rotation. The rotation in the plane perpendicular to q1 and q2 is still undetermined. Rotation a pair of quaternions while keeping the angle between them fixed doesn't completely specify the 4D rotation, but it's the next step. This is equivalent to spining a rigid triangle around one corner to some other arbitrary rotation.

Suppose we want to rotate a pair of quaternions, $\mathbf{q}_1$ and $\mathbf{p}_1$, into another pair, $\mathbf{q}_2$ and $\mathbf{p}_2$. For the sake of argument, let's first consider a brute force approach. We know how to find a rotation from $\mathbf{q}_1$ to $\mathbf{q}_2$. If we describe it in polar form

$$e^{\hat{\mathbf{n}}_1 \frac{\theta}{2}} \mathbf{q} \;\; e^{\hat{\mathbf{n}}_2 \frac{\theta}{2}} \tag{40}$$

we can then tweek it to add a rotation by θ' in the plane perpendicular to $\mathbf{q}_1$ and $\mathbf{q}_2$.

$$e^{\hat{\mathbf{n}}_1 \frac{\theta+\theta'}{2}} \mathbf{q} \;\; e^{\hat{\mathbf{n}}_2 \frac{\theta-\theta'}{2}} \tag{41}$$

It's obvious that θ' is the angle between $e^{\hat{\mathbf{n}}_1 \frac{\theta}{2}} \mathbf{p}_1 \, e^{\hat{\mathbf{n}}_2 \frac{\theta}{2}}$ and $\mathbf{p}_2$ *after* being projected onto the plane perpendicular to $\mathbf{q}_1$ and $\mathbf{q}_2$. Not only is this messy, it breaks when $\mathbf{q}_1 = \mathbf{q}_2$. In that case, there is an entire 3D space that $\mathbf{p}_1$ and $\mathbf{p}_2$ can lie in.

Let's back up a bit. Remember that there is a simple way to rotate $\mathbf{q}_1$ into $\mathbf{q}_2$, if we don't care about the perpendicular plane.

$$(\mathbf{q}_2 \, \mathbf{q}_1^{-1}) \, \mathbf{q} \tag{42}$$

If we look at this as a two step process — $\mathbf{q}_1^{-1}$ takes $\mathbf{q}_1$ to the real axis before we apply $\mathbf{q}_2$ — then we can insert a completely general 3D rotation about the real axis which won't interfere with finally getting to $\mathbf{q}_2$.

$$(\mathbf{q}_2 \, \mathbf{u} \, \mathbf{q}_1^{-1}) \, \mathbf{q} \; (\mathbf{u}^{-1}) \tag{43}$$

We can find out what $\mathbf{u}$ is by requiring that our rotation take $\mathbf{p}_1$ to $\mathbf{p}_2$.

$$(\mathbf{q}_2\, \mathbf{u}\, \mathbf{q}_1^{-1})\, \mathbf{p}_1\, (\mathbf{u}^{-1}) = \mathbf{p}_2 \tag{44}$$

or

$$\mathbf{u}\, (\mathbf{q}_1^{-1}\, \mathbf{p}_1)\, \mathbf{u}^{-1} = \mathbf{q}_2^{-1}\, \mathbf{p}_2 \tag{45}$$

So $\mathbf{u}$ is the pure space rotation that takes $\mathbf{q}_1^{-1}\, \mathbf{p}_1$ into $\mathbf{q}_2^{-1}\, \mathbf{p}_2$. (We know it's a pure space rotation since $\mathbf{q}_1^{-1}\, \mathbf{p}_1$ and $\mathbf{q}_2^{-1}\, \mathbf{p}_2$ have the same real component: the cosine of the angle between each $\mathbf{p}$ and $\mathbf{q}$.) We've reduced a 4D problem rotating two quaternions to a 3D problem rotating one quaternion.

We know how to rotate one quaternion. We can use eq. 35 if we're careful to project our vectors onto the 3D space first. The projections are easy to calculate: zero the real part and renormalize. In fact this is exactly the axis vector of their polar form.

Let $\hat{\mathbf{n}}_1 =$ the axis vector of $\mathbf{q}_1^{-1}\, \mathbf{p}_1$, and
$\hat{\mathbf{n}}_2 =$ the axis vector of $\mathbf{q}_2^{-1}\, \mathbf{p}_2$.

So our constraint (eq. 45) looks like

$$\mathbf{u}\, \hat{\mathbf{n}}_1\, \mathbf{u}^{-1} = \hat{\mathbf{n}}_2$$

There is an intuitive interpretation of this $\mathbf{u}$. $\hat{\mathbf{n}}_1$ is a 90° rotation in the plane of the initial triangle. (Note that right-multiplying by $\mathbf{q}_1^{-1}\, \mathbf{p}_1$ will rotate $\mathbf{q}_1$ into $\mathbf{p}_1$.) Similarly, $\hat{\mathbf{n}}_2$ is a 90° rotation in the plane of the final triangle. In fact, if the $\mathbf{p}$s and $\mathbf{q}$s are both pure space vectors, then $\hat{\mathbf{n}} \sim \mathbf{p} \times \mathbf{q}$, the normal vector to the triangle. What our $\mathbf{u}$ is really doing then, is rotating the "normal vector" of the first triangle onto the "normal vector" of the second. The "normal vector" of a triangle isn't defined in 4D, but these $\hat{\mathbf{n}}_i$ s are the 4D equivalent.

We can find $\mathbf{u}$ using eq. 35,

$$\mathbf{u} = (-\, \hat{\mathbf{n}}_2\, \hat{\mathbf{n}}_1)^{1/2}$$

and our rotation now looks like

$$\left(\mathbf{q}_2 (-\, \hat{\mathbf{n}}_2\, \hat{\mathbf{n}}_1)^{1/2}\, \mathbf{q}_1^{-1}\right) \mathbf{q}\, (-\, \hat{\mathbf{n}}_2\, \hat{\mathbf{n}}_1)^{-1/2} \tag{46}$$

There's one thing you have to watch out for. When $\hat{\mathbf{n}}_1 = -\, \hat{\mathbf{n}}_2$, $\mathbf{u} = \sqrt{-1}$ which isn't uniquely defined. This happens when the triangle is flipped over by 180°. All we need to do is set $\mathbf{u}$ to any space direction that's perpendicular to $\hat{\mathbf{n}}_1$.

Rotating a Tetrahedron (not yet ready)

Matrices

It is useful to be able to convert between a pair of quaternions representing a rotation, and a 4x4 matrix representing the same rotation.

$$\mathbf{u}_L\, \mathbf{q}\, \mathbf{u}_R \quad \leftrightarrow \quad Rq = \begin{pmatrix} r_{00} & r_{01} & r_{02} & r_{03} \\ r_{10} & r_{11} & r_{12} & r_{13} \\ r_{20} & r_{21} & r_{22} & r_{23} \\ r_{30} & r_{31} & r_{32} & r_{33} \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} \tag{55}$$

This is fairly easy to do, if you remember that rotations take one set of orthonormal basis vectors into another.

■ Quaternions ⟶ Matrix

Each column of the matrix is the result of transforming one of our basis vectors, which in quaternion form are {**1**, **i**, **j**, **k**}. Given that, we can practically write down our final matrix without any effort.

$$R = \begin{pmatrix} (\mathbf{u}_L\ \mathbf{1}\ \mathbf{u}_R)_0 & (\mathbf{u}_L\ \mathbf{i}\ \mathbf{u}_R)_0 & (\mathbf{u}_L\ \mathbf{j}\ \mathbf{u}_R)_0 & (\mathbf{u}_L\ \mathbf{k}\ \mathbf{u}_R)_0 \\ (\mathbf{u}_L\ \mathbf{1}\ \mathbf{u}_R)_1 & (\mathbf{u}_L\ \mathbf{i}\ \mathbf{u}_R)_1 & (\mathbf{u}_L\ \mathbf{j}\ \mathbf{u}_R)_1 & (\mathbf{u}_L\ \mathbf{k}\ \mathbf{u}_R)_1 \\ (\mathbf{u}_L\ \mathbf{1}\ \mathbf{u}_R)_2 & (\mathbf{u}_L\ \mathbf{i}\ \mathbf{u}_R)_2 & (\mathbf{u}_L\ \mathbf{j}\ \mathbf{u}_R)_2 & (\mathbf{u}_L\ \mathbf{k}\ \mathbf{u}_R)_2 \\ (\mathbf{u}_L\ \mathbf{1}\ \mathbf{u}_R)_3 & (\mathbf{u}_L\ \mathbf{i}\ \mathbf{u}_R)_3 & (\mathbf{u}_L\ \mathbf{j}\ \mathbf{u}_R)_3 & (\mathbf{u}_L\ \mathbf{k}\ \mathbf{u}_R)_3 \end{pmatrix} \tag{56}$$

Let $\mathbf{u}_L = (a, b, c, d)$ and $\mathbf{u}_R = (e, f, g, h)$. The explicit form of our matrix is then.

$$R = \begin{pmatrix} ae - bf - cg - dh & -be - af - dg + ch & -ce + df - ag - bh & -de - cf + bg - ah \\ be + af - dg + ch & ae - bf + cg + dh & -de - cf - bg + ah & ce - df - ag - bh \\ ce + df + ag - bh & de - cf - bg - ah & ae + bf - cg + dh & -be + af - dg - ch \\ de - cf + bg + ah & -ce - df + ag - bh & be - af - dg - ch & ae + bf + cg - dh \end{pmatrix} \tag{57}$$

■ Matrix ⟶ Quaternions by Diagonalization

To go the other way, we first interpret each column of the matrix R as a quaternion.

$$\begin{aligned} \mathbf{r}_0 &= (r_{00}, r_{10}, r_{20}, r_{30}) \\ \mathbf{r}_1 &= (r_{01}, r_{11}, r_{21}, r_{31}) \\ \mathbf{r}_2 &= (r_{02}, r_{12}, r_{22}, r_{32}) \\ \mathbf{r}_3 &= (r_{03}, r_{13}, r_{23}, r_{33}) \end{aligned} \tag{58}$$

We'll work it column by column. What transformation will take **1** to $\mathbf{r}_0$? It's $\mathbf{r}_0$ itself. We could either right multiply, left multiply, or even some mix of the two. I'll choose the left multiply. Let $\mathbf{u}_1 = \mathbf{r}_0$ Our tentative transformation is then $\mathbf{u}_1\ \mathbf{q}\ \mathbf{1}$.

For the next column, we ask what transformation will take **i** to $\mathbf{r}_1$ before $\mathbf{u}_1$ has been applied? Clearly it's $\mathbf{u}_1^{-1}\ \mathbf{r}_1\ \mathbf{i}^{-1}$. (That is, undo the **i**, and then apply the $\mathbf{r}_1$, then undo the $\mathbf{u}_1$.) We can't apply that to either the left or right without clobbering the transformation on the first column, so we split it across both sides. Let $\mathbf{u}_2 = (\mathbf{u}_1^{-1}\ \mathbf{r}_1\ \mathbf{i}^{-1})^{1/2}$. Our tentative transformation is now $\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{q}\ \mathbf{u}_2^{-1}$.

(There is a special case when $\mathbf{u}_2^2 = -1$, where the square root is poorly defined. This only happens when the diagonal of the matrix (the **i** component) is -1. To clear it, we can rotate by 180° about any axis in the **j**-**k** plane. $\mathbf{u}_2 = \mathbf{j}$ will work.)

(There is another interpretation for the $\mathbf{i}^{-1}$ in the last transformation. We know that simple rotation matrices have cosine of the angles along the diagonal, and that quaternion rotations have the cosine in the real component. What's an easy way to move the cosine in $(\mathbf{r}_1)_1$ to the real component $(\mathbf{r}_1)_0$? Multiply by **-i**.)

What's left of our rotation is just a 2-D rotation about the x-axis. We could just read the sine and cosine of the rotation off of the 3rd row of the remaining matrix (which is $\mathbf{u}_2^{-1}\ \mathbf{u}_1^{-1}\ \mathbf{r}_2\ \mathbf{u}_2$, the transformed version of $\mathbf{r}_2$), but let's do it formally. What transformation will take **j** to $\mathbf{r}_2$ before $\mathbf{u}_1$ and $\mathbf{u}_2$ have been applied? Let $\mathbf{u}_3 = (\mathbf{u}_2^{-1}\ \mathbf{u}_1^{-1}\ \mathbf{r}_2\ \mathbf{u}_2\ \mathbf{j}^{-1})^{1/2}$. Our final transformation is now $\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3\ \mathbf{q}\ \mathbf{u}_3^{-1}\ \mathbf{u}_2^{-1}$.

(Again there is a special case where $\mathbf{u}_3^2 = -1$. This time it's due to a -1 on the **j** component. To clear it, we must rotate by 180° about the **i** axis, so $\mathbf{u}_3 = \mathbf{i}$. We can't use the **k** axis: it would clobber the second column of the matrix which we had cleared with $\mathbf{u}_2$.)

To summarize:

$$\begin{aligned}
\mathbf{u}_1 &= \mathbf{r}_0 \\
\mathbf{u}_2 &= (\mathbf{u}_1^{-1}\,\mathbf{r}_1\,\mathbf{i}^{-1})^{1/2} \\
\mathbf{u}_3 &= ((\mathbf{u}_1\,\mathbf{u}_2)^{-1}\,\mathbf{r}_2\,\mathbf{u}_2\,\mathbf{j}^{-1})^{1/2} \\
\mathbf{u}_L &= \mathbf{u}_1\,\mathbf{u}_2\,\mathbf{u}_3 \\
\mathbf{u}_R &= (\mathbf{u}_2\,\mathbf{u}_3)^{-1}
\end{aligned} \tag{59}$$

(While this is correct and (somewhat) intuitive, it is computationally intensive. Can I simplify?)

■ Matrix ⟶ Quaternions by Geometry

Just use the Rotate a Tetrahedron method described above. Since a matrix rotates {1, i, j, k} into the columns of the matrix, and since any column of the matrix can be determined by the other three, we can choose any three of the four columns as our tetrahedron. Since the matrix columns are also orthogonal, we don't have to project the quaternions onto the 3-space. ($\mathbf{p}_i = \hat{\mathbf{p}}_i$ and $\mathbf{r}_i = \hat{\mathbf{r}}_i = \bar{\mathbf{r}}_i$ in the notation of that section.)

The resulting algorithm is actually quite similar to the "by Diagonalization" method above. (For the "simple, two-step" version at least.) It's just the interpretation that's different.

Summary

We've reviewed the nature of rotations in four dimensions, of rotations in the complex plane, and the mathematics of quaternions. Then we've described how rotations can act as rotations in certain limited planes before attacking the general case. We've shown how to calculate the quaternion pair that describes rotation in particular planes, and how to calculate the planes given the quaternion pair.

A general 4D rotation of quaternion $\mathbf{q}$, *given by* $e^{\hat{\mathbf{n}}_1 \theta_1}\,\mathbf{q}\,e^{\hat{\mathbf{n}}_2 \theta_2}$, *is a rotation by* $\theta_1 + \theta_2$ *in the* $\mathbf{q}_w$-$\mathbf{q}_i$ *plane, and a rotation by* $\theta_1 - \theta_2$ *in the* $\mathbf{q}_j$-$\mathbf{q}_k$ *plane, where, if* $\hat{\mathbf{n}}_1 \neq \pm\hat{\mathbf{n}}_2$

$$\begin{aligned}
\mathbf{q}_w &= e^{\hat{\mathbf{n}}_k \varphi} = (-\hat{\mathbf{n}}_1\,\hat{\mathbf{n}}_2)^{1/2} \\
\mathbf{q}_i &= \frac{\hat{\mathbf{n}}_1 + \hat{\mathbf{n}}_2}{|\hat{\mathbf{n}}_1 + \hat{\mathbf{n}}_2|} = \hat{\mathbf{n}}_i \\
\mathbf{q}_j &= \frac{\hat{\mathbf{n}}_1 - \hat{\mathbf{n}}_2}{|\hat{\mathbf{n}}_1 - \hat{\mathbf{n}}_2|} = \hat{\mathbf{n}}_j \\
\mathbf{q}_k &= e^{\hat{\mathbf{n}}_k \varphi}\,\hat{\mathbf{n}}_k = \mathbf{q}_w\,\hat{\mathbf{n}}_i\,\hat{\mathbf{n}}_j
\end{aligned}$$

If $\hat{\mathbf{n}}_1 = -\hat{\mathbf{n}}_2$ *then replace* $\theta_2 \to -\theta_2$ *and* $\hat{\mathbf{n}}_2 \to -\hat{\mathbf{n}}_2$, *and then...*

If $\hat{\mathbf{n}}_1 = \hat{\mathbf{n}}_2$ *then* $\mathbf{q}_w = 1$, $\mathbf{q}_i = \hat{\mathbf{n}}_1$, *and the second rotation is a 3-space rotation about the* $\hat{\mathbf{n}}_1$ *axis.*

Some properties of this rotation are:

The rotation planes intersect the 3-space along the directions $\hat{\mathbf{n}}_i$ and $\hat{\mathbf{n}}_k$.

The rotation planes form angles φ and $\varphi+\pi/2$ with the w-axis.

Some special cases of particular interest are:

If $\theta_1 = \pm\theta_2$ then the rotation is in one plane, leaving the orthogonal plane fixed.

If $\theta_1 = \theta_2 = \theta/2$ and $\hat{\mathbf{n}}_1 = \hat{\mathbf{n}}_2$ then the rotation plane passes through the w-axis.

If $\theta_1 = \theta_2 = \theta/2$ and $\hat{\mathbf{n}}_1 = -\hat{\mathbf{n}}_2$ then the rotation plane is entirely within the 3-space, perpendicular with the w-axis.

The rotation that takes $\mathbf{q}_1$ into $\mathbf{q}_2$ while leaving the perpendicular rotation plane fixed can be written

$$(\mathbf{q}_2\,\mathbf{q}_1^{-1})^{1/2}\;\mathbf{q}\,(\mathbf{q}_1^{-1}\,\mathbf{q}_2)^{1/2}$$

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