Understanding the Exceptional Lie Group G_2

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Abstract: The exceptional Lie group G_2 can be viewed as the group of automorphisms on the octonians, the largest of the normed division algebras. In this paper I will attempt to illustrate the nature of G_2 and present a representation that emphasizes its symmetries. In particular, I derive a set of generators of G_2 that are related to each other by simple permutations of the coordinates.

Octonians

The octonians are an eight dimensional extension of the quaternions (which are in turn and extension of complex numbers) which are normed (the product of the norm is the norm of the product) and they are a division algebra (every non-zero element has an inverse.) There are two common ways of defining them.

■ By doubling

You can build the octonians by repeated use of the Cayley-Dickson construction. [3]

The complex numbers can be defined as an ordered pair of real numbers (x, y) with the multiplication rule

$$(a, b)(c, d) = (a c - b d,\ a d + b c)$$

This is equivalent to adding a new element i to the real numbers, where $i^2 = -1$. Complex numbers are commonly written in the form

$$(a, b) = a + i b$$

An important operation is complex conjugation, a reflection on the imaginary axis.

$$\overline{(a, b)} = (a, -b),\ \overline{a + i b} = a - i b$$

The quaternions can be defined as an ordered pair of complex numbers with the multiplication rule

$$(a, b)(c, d) = (a c - b \overline{d},\ \overline{a} d + b c)$$

This is equivalent to introducing a second imaginary element j where $j^2 = -1$, and $i j = -j i$, which implies a third imaginary element, $k = i j$. Quaternions are commonly written in the form

$$w + i x + j y + k z$$

Quaternion conjugation is a reflection in the 3D imaginary space.

$$\overline{(a, b)} = (\overline{a}, -b), \qquad \overline{w + i x + j y + k z} = w - i x - j y - k z$$

Finally, octonians can be defined as an ordered pair of quaternions with the multiplication rule

$$(a, b)(c, d) = (a c - \overline{d} b,\ a d + b \overline{c})$$

This is equivalent to introducing a forth imaginary element l, where $l^2 = -1$, which in turn implies three more:

$\mathbb{i}\mathbb{l}$, $\mathbb{j}\mathbb{l}$, and $\mathbb{k}\mathbb{l}$, which all anti-commute and which all square to -1. Octonians are commonly written in terms of two quaternions and $\mathbb{l}$.

$$(a,\ b) = a + b\,\mathbb{l}$$

Octonian conjugation is a reflection in the 7D imaginary space.

$$\overline{(a,\ b)} = (\overline{a}, -b), \qquad \overline{a + b\,\mathbb{l}} = \overline{a} - b\,\mathbb{l}$$

Octonians are non-associative. In particular

$$\mathbb{i}\,(\mathbb{j}\,\mathbb{l}) = -(\mathbb{i}\,\mathbb{j})\,\mathbb{l}$$

This is clearly necessary if we are to require that $(\mathbb{k}\mathbb{l})^2 = ((\mathbb{i}\,\mathbb{j})\,\mathbb{l})^2 = -1$.

It's important to note that these choices of basis are purely arbitrary. You can chose any purely imaginary octonian to be the generator of the complex numbers. You can chose any second imaginary octonian, as long as it's orthogonal to the first to be your $\mathbb{j}$, and get the quaternions. You can choose any third octonian, orthogonal to $\mathbb{i}$, $\mathbb{j}$, and $\mathbb{k}$, to be your $\mathbb{l}$, and generate the full octonians. *There are no preferred directions, other than the real axis.*

■ By 7-cycle multiplication rules

An equivalent way of defining the octonians (described in [2]) is by giving a basis, $\mathbb{i}_n$ for $n \in \{1..7\}$ with the multiplication rules

$$\mathbb{i}_n{}^2 = -1,$$
$$\mathbb{i}_n\,\mathbb{i}_{n+1} = -\mathbb{i}_{n+1}\,\mathbb{i}_n = \mathbb{i}_{n+3}$$
$$\mathbb{i}_{n+1}\,\mathbb{i}_{n+3} = -\mathbb{i}_{n+3}\,\mathbb{i}_{n+1} = \mathbb{i}_n$$
$$\mathbb{i}_{n+3}\,\mathbb{i}_n = -\mathbb{i}_n\,\mathbb{i}_{n+3} = \mathbb{i}_{n+1}$$

(where the subscripts cycle modulo 7). A general octonian can be written

$$x = a_0 + \sum_{n=1}^{7} a_n\,\mathbb{i}_n$$

and the octonian conjugate can be written

$$\overline{x} = a_0 - \sum_{n=1}^{7} a_n\,\mathbb{i}_n$$

There is a simple relationship between the doubling basis and the 7-cycle basis.

$$\mathbb{i} = \mathbb{i}_1,\ \mathbb{j} = \mathbb{i}_2,\ \mathbb{l} = \mathbb{i}_3,\ \mathbb{k} = \mathbb{i}_4,\ \mathbb{j}\mathbb{l} = \mathbb{i}_5,\ -\mathbb{k}\mathbb{l} = \mathbb{i}_6,\ \mathbb{i}\mathbb{l} = \mathbb{i}_7$$

An advantage of the 7-cycle basis is that whenever you have a relationship between octonians, you can quickly generate seven other relationships by incrementing the indices. In particular (and I emphasize this because I'll be using it a lot), *any two octonians and their product, for example $\mathbb{i}_n$, $\mathbb{i}_{n+1}$, and $\mathbb{i}_{n+3}$* for any n*, are in the same quaternion subalgebra and are associative.*

■ What they do

The *norm* of a real, complex, quaternion, or octonian number can be written

$$[\![x]\!] = x\,\overline{x}$$

For octonians, that can be expanded as

$$[\![x]\!] = x\,\overline{x} = a_0{}^2 + \sum_{n=1}^{7} a_n{}^2$$

A *unit* is a number (real, complex, quaternion, or octonian) whose norm is 1. Any unit u can be written as

$$u = e^{\hat{n}\,\theta} = \cos(\theta) + \hat{n}\sin(\theta)$$

where $\hat{n}$ is a unit with zero real part and θ is a real angle of rotation. (In other words, Euler's formula generalizes to quaternions and octonians.) For complex numbers, multiplication by a unit, $e^{\mathrm{n}\theta}\,z$, is a simple rotation about the origin by angle θ. Multiplication by quaternion units is also a rotation. It's probably easiest to explain by showing the equivalent matrix.

$$e^{i\theta}\,x \;\to\; \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & \cos(\theta) & -\sin(\theta) \\ 0 & 0 & \sin(\theta) & \cos(\theta) \end{pmatrix} x$$

In words, left multiplying by $e^{i\theta}$ rotates by θ in the w-i plane and also in the j-k plane. Right multiplying give the opposite rotation in only the second plane.

$$x\,e^{i\theta} \;\to\; \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & \cos(\theta) & \sin(\theta) \\ 0 & 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix} x$$

We can split the difference in two ways

$$e^{i\theta/2}\,x\,e^{i\theta/2} \;\to\; \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} x$$

$$e^{i\theta/2}\,x\,e^{-i\theta/2} \;\to\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\theta) & \sin(\theta) \\ 0 & 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix} x$$

One can see that with these pieces one can build up completely general groups of rotations. In fact, give an arbitrary quaternion unit u, the operation $u\,x\,\overline{u}$ is a general 3D rotation, and given two arbitrary unit quaternions u and v, the operation $u\,x\,v$ is a general 4D rotation.

We can find the *generators of the Lie algebra* by taking the derivative near the identity (finding the *tangent space* as they say). The matrix you get is the same as if you'd set the angle to 90°. It gives you a much more compact version of the structure of the group.

$$i\,x \;\to\; \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} x,\; j\,x \;\to\; \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} x,\; k\,x \;\to\; \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} x$$

Note that the matrix forms of these operations obey the same algebra as the operators themselves.

The equivalent matrices for the octonians are similar. In the 7-cycle basis we have

$$i_1\,x \to \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},\; x\,i_1 \;\to\; \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

There are several interesting things to note.

1) i_1 rotates in the planes: w-1, 2-4, 3-7, and 5-6.

2) The rotation planes are related to the quaternion subalgebras which contain i_1 : 124, 713, and 561.

3) Right multiplying has the same rotation in the plane through the real axis as left multiplying does but the opposite rotation everywhere else.

Clearly $e^{\hat{n}\,\theta/2}\, x\, e^{\hat{n}\,\theta/2}$ is a rotation in the Real-$\hat{n}$ plane that is fixed everywhere else. Using that alone, we can build up completely general rotations in eight dimensions. Unlike in the quaternion case, the non-associativity prevents us from reducing that to multiplying by one or two octonians. However, it can be proven that (see Conway [2] section 8.4) any general rotation can be built up from at most seven left multiplications.

Elements and Generators of G_2

The group G_2 can be represented by the group of automorphisms on the octonians. That is, elements of G_2 are mappings from the octonians onto themselves which preserve multiplication. If g is an element of G_2, then

$$g(x)\, g(y) = g(x\, y)$$

It can be shown (see Cacciatori et. al. [ref. 1], section 2) that if g is a series of left handed multiplications by purely imaginary octonians, and if $g(1) = 1$, then $g \in G_2$.

Following ref. [1] we define

$$g_{\text{abc}}(x) = -((c\, b)\, a)\, (a\, (b\, (c\, x)))$$

where, a, b, c, and $(c\, b)\, a$ are all purely imaginary octonians. Clearly our choice of $(c\, b)\, a$ guarantees $g_{\text{abc}}(1) = 1$, so $g_{\text{abc}} \in G_2$. The condition that $(c\, b)\, a$ is imaginary means that (cb) $\perp$ a.

We next add a parameter to g_{abc} so we can find the tangents near the identity. Still following ref. [1] we consider $g_{a_t\,\text{bc}}$ where $a_t = c\cos(t) + a\sin(t)$, with the additional requirements that $a \perp b$, $b \perp c$, and $c \perp a$. Then by definition

$$C_{\text{abc}} = \frac{d}{\text{dt}}\Big|_{t=0}\, g_{a_t\,\text{bc}}$$

is an element of Lie(G_2).

To quote ref. [1], "With suitable choices of the elements a, b, and c among the elements of the canonical basis of $\mathbb{O}$ we can find a basis of this algebra." It's the "suitable choices" that I'm interested in. I think it's informative to examine all 210 (7 × 6 × 5) choices and see what's there.

210 Combinations

■ Overall sign

Our first identity is

$$C_{\text{abc}} = -C_{\text{cba}}$$

I know because I've tried them all, but you could probably prove it by carefully permuting the octonians in g_{abc}. (to do...)

■ Quaternion Subalgebras

If a, b, and c are part of a quaternion subalgebra (i_1, i_2, and i_4 for example) then $((c\, b)\, a)$ is real, and our conditions on the indices of g_{abc} aren't satisfied. Clearly this is a non-starter. Although C_{124} is not an element of Lie(G_2), it's still interesting. Compare it to right multiplying by i_2.

$$
x\, i_2 \to \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \; C_{124} \to \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}
$$

Note that C_{124} is just i_2 with the axes 1, 2, and 4 (along with the real axis) projected out. This is true for each of the 7-cycle equivalents, and for the 6 permutations of the indices:

When i_a, i_b, and i_c generate a quaternion algebra, then C_{abc} is equivalent to right multiplying by i_b and then projecting out the real, a, b, and c axes.

■ Complements to the Quaternion Subalgebras

Just as when we added $\mathbb{l}$ to the quaternions to get the additional basis elements $\mathbb{i}\mathbb{l}$, $\mathbb{j}\mathbb{l}$, and $\mathbb{k}\mathbb{l}$, we can multiply the indices of a quaternion C_{abc} by something outside the quaternion space, call it d, to get $C_{\text{ad, bd, cd}}$. It doesn't matter if you right or left multiply — d anticommutes with everything else and C_{abc} is an even function of each index.

There are four choices for the extra axis. C_{124}, for example, can be modified by i_3, i_5, i_6, or i_7, giving us the following elements of $\text{Lie}(G_2)$. (I'm going to start suppressing the first row and column of the matrices, since elements of G_2 always leave the real axis alone.)

$$
C_{13,23,43} = C_{756}, \; C_{15,25,45} = C_{637}, \; C_{16,26,46} = C_{573}, \; C_{17,27,47} = C_{365}
$$

$$
C_{756} \to \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \; C_{637} \to \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix},
$$

$$
C_{573} \to \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \; C_{365} \to \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}
$$

Curiously, the choice almost doesn't matter. We find that $C_{756} = C_{365} = -C_{637} = -C_{573}$. More than that, when we introduced i_a into C_{124}, it had the effect of reflecting the ath row and column. For i_3 and i_5 that had the effect of reversing the direction of the 3-5 plane rotation. For i_6 and i_7, it reversed the direction of the 6-7 plane rotation. Since these Cs all represent rotations in the same two planes, they all represent the same operation, up to an overall sign.

Constructing one of these "quaternion complements" tends to obscure the original quaternion basis. To avoid that, I'll introduce a new notation.

$$C_{\text{abcd}} = C_{\text{ad,bd,cd}}$$

■ Dependent Triplets

When we have four versions of the same thing with some extra signs, is there a preferred choice? Is there one that's easier to use?

Recall that C_{124} was a projection of i_2. What else is a projection of i_2? It turns out there are three of them, corresponding to the cyclic permutations $(i,\ i+1,\ i+3)$, $(i+3,\ i,\ i+1)$, and $(i+1,\ i+3,\ i)$, adjusted so that there is a '2' in the middle of each.

$$C_{124},\quad C_{523},\quad C_{726}$$

If we turn each of these into an element of Lie(G_2) with the corresponding i_{i-1} (which gives a more convenient choice of signs than i_{i+2}, i_{i+4}, or i_{i+5} do), we get

$$C_{1247} = C_{365},\ C_{5231} = C_{647},\ C_{7265} = C_{431}$$

or

$$C_{1247} \to \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$C_{5231} \to \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

$$C_{7265} \to \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Note that:

These were based on C_{a2c} and their 2nd row and column is zero.
Each has two of the three rotation planes that the i_2 matrix has.
Their sum is zero.
Their product is zero.
They commute with each other.
They are not orthogonal, but the difference between any two is orthogonal to the third.

The orthonormality condition is (from ref. [1], sec 2)

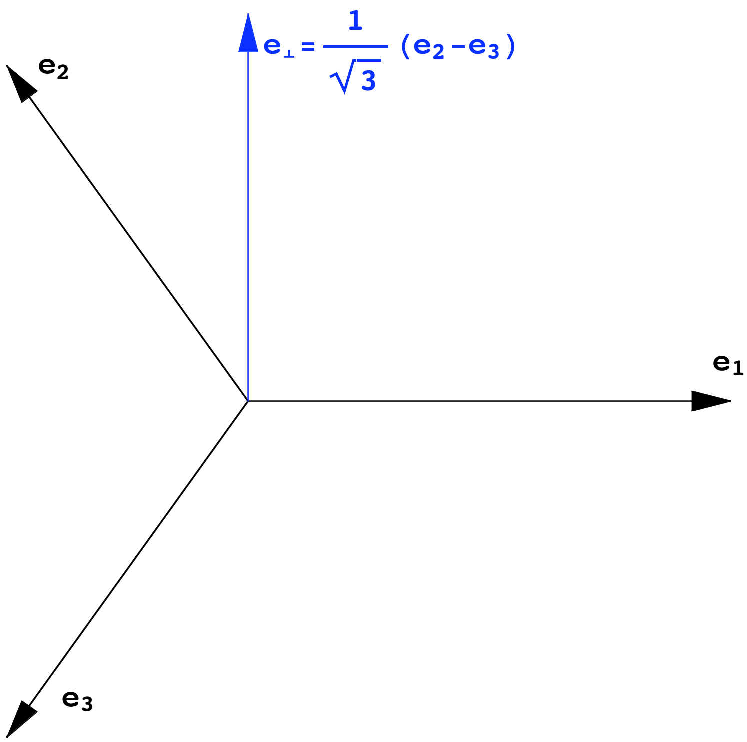
$$\text{Tr}(C_I\, C_J) = -4\,\delta_{IJ}$$

from which we find that (for example)

$$C_{1247} \text{ and } \frac{1}{\sqrt{3}}\left(C_{5231} - C_{7265}\right) = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

are orthogonal to each other and normalized.

In fact, these matrices act like three vectors on a plane, which are spaced 120° apart.



We can write down the corresponding rotation matrix by inspection. We want a rotation that maps

$$C_{1247} \to C_{5231} \to C_{7265} \to C_{1247}$$

To do that, it must map the indices according to

$$i_1 \to i_5 \to i_7 \to i_1, \qquad i_4 \to i_3 \to i_6 \to i_4$$

The matrix for that is obviously

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad S^3 = 1, \qquad S^{-1} = S^T = S^2$$

$S^T\ C_{1247}\ S\ =\ C_{5231}$ $\qquad S^T\ C_{5231}\ S\ =\ C_{7265}$ $\qquad S^T\ C_{7265}\ S\ =\ C_{1247}$

It's easy to write down a similar matrix for any other dependent triplet.

■ Counting to 210

We have examined them all. There are

3 cyclic permutations and 7 incremented version of C_{124} . That's 21 combinations which are not in Lie(G_2).
There are 4 ways to convert each of the above 21 into an element of Lie(G_2), though they are redundant.
Lastly, every $C_{\text{abc}}\ =\ -C_{\text{cba}}$

That's

3×7×(1+4)×2 = 210

Of these combinations, only 21 are significant. 42 are not even in Lie(G_2) and the rest are really copies of the 21, up to overall sign.

The 21 are not orthogonal. The 21 fall into seven groups of 3, and each group of 3 can be arranged as two orthogonal elements, so really there are 14 independent elements, which is the number of generators G_2 requires.

The Generators of Lie(G_2)

I will call my generators $C_{a,i}$ where $a \in \{1..7\}$ is the 2nd index in C_{abcd}, and $i \in \{1, 2, 3\}$ says which cyclic permutation of the subalgebra it belongs to. More explicitly

$$\begin{aligned} C_{a,1}\ &=\ C_{a+3,a,a+1,a-1} \\ C_{a,2}\ &=\ C_{a-1,a,a+2,a-2} \quad \text{all right hand indices incremented modulo 7} \\ C_{a,3}\ &=\ C_{a-2,a,a+1,a-3} \end{aligned}$$

These matrices are related by the 7-cycle and 3-cycle transformations (aren't they pretty?)

$$\Lambda\ =\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},\quad S_1\ =\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, S_{a+1}\ =\ \Lambda^T\ S_a\ \Lambda$$

$C_{a,i+1}\ =\ {S_a}^T\ C_{a,i}\ S_a\,,$ $\qquad C_{a+1,i}\ =\ \Lambda^T\ C_{a,i}\ \Lambda$

My actual matrices are (for $C_{\text{row, column}}$)

$$\begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&-1&0&0\\ 0&0&1&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0\\ 0&0&0&0&0&-1&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0\\ 0&0&0&0&0&0&1\\ 0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&-1&0&0&0&0 \end{pmatrix}$$

$$\begin{pmatrix} 0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ -1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&-1&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0\\ 0&0&-1&0&0&0&0\\ 0&0&0&0&0&0&-1\\ 0&0&0&0&0&1&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&0&-1&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&-1&0&0\\ 1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}$$

$$\begin{pmatrix} 0&0&0&0&0&0&-1\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0\\ -1&0&0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&-1&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&-1&0\\ 0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}$$

$$\begin{pmatrix} 0&1&0&0&0&0&0\\ -1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&-1&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&-1&0&0&0&0&0\\ 1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0\\ 0&0&0&0&-1&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&-1\\ 0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0 \end{pmatrix}$$

$$\begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&-1&0&0&0&0&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&0&0&0&-1&0\\ 0&0&-1&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&0\\ -1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0 \end{pmatrix}$$

$$\begin{pmatrix} 0&0&0&0&-1&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&-1&0&0&0&0\\ 1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&-1\\ 0&0&0&-1&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ -1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0 \end{pmatrix}$$

$$\begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&-1&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0\\ -1&0&0&0&0&0&0\\ 0&0&0&0&-1&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&-1&0&0&0&0\\ 0&0&0&0&0&1&0\\ 1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}$$

If we are going to build minimal orthogonal sets from these, we'll also need to define

$$D_{a,i} = \frac{1}{\sqrt{3}} (C_{a,i+1} - C_{a,i+2}) \quad (i \text{ index wraps around modulo 3})$$

so that $C_{a,i}$ and $D_{a,i}$ are an orthonormal set for any given a and i.

■ Commutators

Matrices in the same row commute.

$$[C_{a,i}, C_{a,j}] = 0 \qquad \forall\ a \in \{1..7\},\ i, j \in \{1, 2, 3\}$$

A few commutators give a factor of ±2. These form su(2) subgroups.

$$[C_{a,1}, C_{a+1,2}] = 2\, C_{a+3,3} \qquad \forall\ a \in \{1..7\} \text{ and cyclic permutations of the } C_{a,i}\text{ s.}$$
$$[C_{a,1}, C_{a+2,2}] = 2\, C_{a+6,3}$$

For every su(2) based commutator like

$$[C_{a,1}, C_{b,2}] = 2\, C_{c,3}$$

there are several related commutators where you change the second index of one $C_{i,j}$.

$$[C_{a,1}, C_{b,1}] = -C_{c,3} \qquad [C_{a,1}, C_{b,3}] = -C_{c,3}$$
$$[C_{a,2}, C_{b,2}] = -C_{c,3} \qquad [C_{a,3}, C_{b,2}] = -C_{c,3}$$

Thus, changing only one of the 3-cycle indices gives the same commutator, except for a factor of $-1/2$. (Or, if you prefer, $\cos(120°)$, as if the matrix had been rotated a third of a circle.)

If you change both of the 3-cycle indices inside the commutator the third 3-cycle index will change as well. How it changes depends on the if the first two change in the same direction. If we have

$$[C_{a,1}, C_{b,2}] = 2\, C_{c,3}$$

then

$$[C_{a,1+1}, C_{b,2+1}] = -C_{c,3+1} \qquad [C_{a,1-1}, C_{b,2-1}] = -C_{c,3+1}$$
$$[C_{a,1+1}, C_{b,2-1}] = -C_{c,3-1} \qquad [C_{a,1-1}, C_{b,2+1}] = -C_{c,3-1}$$

or

$$[C_{a,2}, C_{b,3}] = -C_{c,1} \qquad [C_{a,3}, C_{b,1}] = -C_{c,1}$$
$$[C_{a,2}, C_{b,1}] = -C_{c,2} \qquad [C_{a,3}, C_{b,3}] = -C_{c,2}$$

Communting a C with a D is similar to rotating a C by 90°. (Just as "rotating" a introduced a factor of a, "rotating" a introduced a factor of a.) This commutator will be important when we go looking for su(3) subgroups of G_2.

$$[C_{a,1}, D_{b,2}] = [C_{a,1}, C_{b,3} - C_{b,1}] \big/ \sqrt{3}$$
$$= (-C_{c,3} + C_{c,3}) \big/ \sqrt{3}$$
$$= 0$$

Continuing in that vein we can rotate by any multiple of $\pi/6$.

$$[C_{a,1}, C_{b,2}] = 2\, C_{c,3} \cos(0) = 2\, C_{c,3}$$
$$[C_{a,1}, D_{b,2}] = 2\, C_{c,3} \cos(\pi/2) = 0$$
$$[C_{a,1}, C_{b,3}] = 2\, C_{c,3} \cos(2\pi/3) = -\, C_{c,3}$$
$$[C_{a,1}, D_{b,3}] = 2\, C_{c,3} \cos(7\pi/6) = -\sqrt{3}\ C_{c,3}$$
$$[C_{a,1}, C_{b,1}] = 2\, C_{c,3} \cos(4\pi/3) = -\, C_{c,3}$$
$$[C_{a,1}, D_{b,1}] = 2\, C_{c,3} \cos(11\pi/6) = \sqrt{3}\ C_{c,3}$$

The su(2) based $[D, D]$ commutators are (given $[C_{a,1}, C_{b,2}] = 2\,C_{c,3}$)

$$[D_{a,1}, D_{b,2}] = [C_{a,1+1} - C_{a,1-1}, C_{b,2+1} - C_{b,2-1}]/3$$
$$= [C_{a,1+1}, C_{b,2+1}] - [C_{a,1+1}, C_{b,2-1}] - [C_{a,1-1}, C_{b,2+1}] + [C_{a,1-1}, C_{b,2-1}]/3$$
$$= (-C_{c,3+1} + C_{c,3-1} + C_{c,3-1} - C_{c,3+1})/3$$
$$= -2\,D_{c,3}\big/\sqrt{3}$$

Thus, it's the same form as for $[C_{a,1}, C_{b,2}]$ but with a different factor.

Changing one of the 3-cycle indices is similar to the $[C, C]$ commutators.

$$[D_{a,1}, D_{b,1}] = D_{c,3}\big/\sqrt{3} \quad [D_{a,1}, D_{b,3}] = D_{c,3}\big/\sqrt{3}$$
$$[D_{a,2}, D_{b,2}] = D_{c,3}\big/\sqrt{3} \quad [D_{a,3}, D_{b,2}] = D_{c,3}\big/\sqrt{3}$$

Changing both of the 3-cycle indices gives you mixed terms. Because our notation is redundant, there are lots of ways of expressing the result. The following is simplest, in my opinion

$$[D_{a,1+1}, D_{b,2+1}] = -C_{c,1} - \tfrac{2}{3} C_{c,2} + \tfrac{2}{3} C_{c,3} = -C_{c,1} - \tfrac{2}{\sqrt{3}} D_{c,1}$$
$$[D_{a,1+1}, D_{b,2-1}] = +\tfrac{2}{3} C_{c,1} + C_{c,2} - \tfrac{2}{3} C_{c,3} = C_{c,2} - \tfrac{2}{\sqrt{3}} D_{c,2}$$
$$[D_{a,1-1}, D_{b,2+1}] = +\tfrac{2}{3} C_{c,1} + C_{c,2} - \tfrac{2}{3} C_{c,3} = C_{c,2} - \tfrac{2}{\sqrt{3}} D_{c,2}$$
$$[D_{a,1-1}, D_{b,2-1}] = -C_{c,1} - \tfrac{2}{3} C_{c,2} + \tfrac{2}{3} C_{c,3} = -C_{c,1} - \tfrac{2}{\sqrt{3}} D_{c,1}$$

Note that the third 3-cycle index moves in the opposite direction as the corresponding C commutators.

As an example, here are all of the $[C_{1,i}, C_{2,j}]$ related commutators.

$[C_{1,1}, C_{2,1}] = -C_{4,3}$	$[C_{1,1}, D_{2,1}] = \sqrt{3}\,C_{4,3}$	$[D_{1,1}, D_{2,1}] = \frac{D_{4,3}}{\sqrt{3}}$
$[C_{1,1}, C_{2,2}] = 2\,C_{4,3}$	$[C_{1,1}, D_{2,2}] = 0$	$[D_{1,1}, D_{2,2}] = -\frac{2\,D_{4,3}}{\sqrt{3}}$
$[C_{1,1}, C_{2,3}] = -C_{4,3}$	$[C_{1,1}, D_{2,3}] = -\sqrt{3}\,C_{4,3}$	$[D_{1,1}, D_{2,3}] = \frac{D_{4,3}}{\sqrt{3}}$
$[C_{1,2}, C_{2,1}] = -C_{4,2}$	$[C_{1,2}, D_{2,1}] = -D_{4,2}$	$[D_{1,2}, D_{2,1}] = C_{4,2} - \frac{2\,D_{4,2}}{\sqrt{3}}$
$[C_{1,2}, C_{2,2}] = -C_{4,3}$	$[C_{1,2}, D_{2,2}] = -D_{4,3}$	$[D_{1,2}, D_{2,2}] = \frac{D_{4,3}}{\sqrt{3}}$
$[C_{1,2}, C_{2,3}] = -C_{4,1}$	$[C_{1,2}, D_{2,3}] = -D_{4,1}$	$[D_{1,2}, D_{2,3}] = -C_{4,1} - \frac{2\,D_{4,1}}{\sqrt{3}}$
$[C_{1,3}, C_{2,1}] = -C_{4,1}$	$[C_{1,3}, D_{2,1}] = D_{4,1}$	$[D_{1,3}, D_{2,1}] = -C_{4,1} - \frac{2\,D_{4,1}}{\sqrt{3}}$
$[C_{1,3}, C_{2,2}] = -C_{4,3}$	$[C_{1,3}, D_{2,2}] = D_{4,3}$	$[D_{1,3}, D_{2,2}] = \frac{D_{4,3}}{\sqrt{3}}$
$[C_{1,3}, C_{2,3}] = -C_{4,2}$	$[C_{1,3}, D_{2,3}] = D_{4,2}$	$[D_{1,3}, D_{2,3}] = C_{4,2} - \frac{2\,D_{4,2}}{\sqrt{3}}$

$[C_{2,1}, C_{4,1}] = -C_{1,3}$	$[C_{2,1}, D_{4,1}] = D_{1,3}$	$[D_{2,1}, D_{4,1}] = C_{1,3} - \frac{2 D_{1,3}}{\sqrt{3}}$
$[C_{2,1}, C_{4,2}] = -C_{1,2}$	$[C_{2,1}, D_{4,2}] = D_{1,2}$	$[D_{2,1}, D_{4,2}] = -C_{1,2} - \frac{2 D_{1,2}}{\sqrt{3}}$
$[C_{2,1}, C_{4,3}] = -C_{1,1}$	$[C_{2,1}, D_{4,3}] = D_{1,1}$	$[D_{2,1}, D_{4,3}] = \frac{D_{1,1}}{\sqrt{3}}$
$[C_{2,2}, C_{4,1}] = -C_{1,1}$	$[C_{2,2}, D_{4,1}] = -\sqrt{3}\, C_{1,1}$	$[D_{2,2}, D_{4,1}] = \frac{D_{1,1}}{\sqrt{3}}$
$[C_{2,2}, C_{4,2}] = -C_{1,1}$	$[C_{2,2}, D_{4,2}] = \sqrt{3}\, C_{1,1}$	$[D_{2,2}, D_{4,2}] = \frac{D_{1,1}}{\sqrt{3}}$
$[C_{2,2}, C_{4,3}] = 2\, C_{1,1}$	$[C_{2,2}, D_{4,3}] = 0$	$[D_{2,2}, D_{4,3}] = -\frac{2 D_{1,1}}{\sqrt{3}}$
$[C_{2,3}, C_{4,1}] = -C_{1,2}$	$[C_{2,3}, D_{4,1}] = -D_{1,2}$	$[D_{2,3}, D_{4,1}] = -C_{1,2} - \frac{2 D_{1,2}}{\sqrt{3}}$
$[C_{2,3}, C_{4,2}] = -C_{1,3}$	$[C_{2,3}, D_{4,2}] = -D_{1,3}$	$[D_{2,3}, D_{4,2}] = C_{1,3} - \frac{2 D_{1,3}}{\sqrt{3}}$
$[C_{2,3}, C_{4,3}] = -C_{1,1}$	$[C_{2,3}, D_{4,3}] = -D_{1,1}$	$[D_{2,3}, D_{4,3}] = \frac{D_{1,1}}{\sqrt{3}}$

$[C_{4,1}, C_{1,1}] = -C_{2,2}$	$[C_{4,1}, D_{1,1}] = -D_{2,2}$	$[D_{4,1}, D_{1,1}] = \frac{D_{2,2}}{\sqrt{3}}$
$[C_{4,1}, C_{1,2}] = -C_{2,3}$	$[C_{4,1}, D_{1,2}] = -D_{2,3}$	$[D_{4,1}, D_{1,2}] = -C_{2,3} - \frac{2 D_{2,3}}{\sqrt{3}}$
$[C_{4,1}, C_{1,3}] = -C_{2,1}$	$[C_{4,1}, D_{1,3}] = -D_{2,1}$	$[D_{4,1}, D_{1,3}] = C_{2,1} - \frac{2 D_{2,1}}{\sqrt{3}}$
$[C_{4,2}, C_{1,1}] = -C_{2,2}$	$[C_{4,2}, D_{1,1}] = D_{2,2}$	$[D_{4,2}, D_{1,1}] = \frac{D_{2,2}}{\sqrt{3}}$
$[C_{4,2}, C_{1,2}] = -C_{2,1}$	$[C_{4,2}, D_{1,2}] = D_{2,1}$	$[D_{4,2}, D_{1,2}] = C_{2,1} - \frac{2 D_{2,1}}{\sqrt{3}}$
$[C_{4,2}, C_{1,3}] = -C_{2,3}$	$[C_{4,2}, D_{1,3}] = D_{2,3}$	$[D_{4,2}, D_{1,3}] = -C_{2,3} - \frac{2 D_{2,3}}{\sqrt{3}}$
$[C_{4,3}, C_{1,1}] = 2\, C_{2,2}$	$[C_{4,3}, D_{1,1}] = 0$	$[D_{4,3}, D_{1,1}] = -\frac{2 D_{2,2}}{\sqrt{3}}$
$[C_{4,3}, C_{1,2}] = -C_{2,2}$	$[C_{4,3}, D_{1,2}] = -\sqrt{3}\, C_{2,2}$	$[D_{4,3}, D_{1,2}] = \frac{D_{2,2}}{\sqrt{3}}$
$[C_{4,3}, C_{1,3}] = -C_{2,2}$	$[C_{4,3}, D_{1,3}] = \sqrt{3}\, C_{2,2}$	$[D_{4,3}, D_{1,3}] = \frac{D_{2,2}}{\sqrt{3}}$

Converting to Other Notations

■ Pauli Matrices / su(2) Subgroups

The Pauli matrices [ref. 4, 6] obey the commutation relations

$$[\sigma_i, \sigma_j] = 2\, i\, \varepsilon_{\text{ijk}}\, \sigma_k$$

Note, however, that the Pauli matrices are hermitian, while our $C_{a,i}$ s are antihermitian. We need to find the commutators for the $i\, \sigma_i$ s.

$$[i\, \sigma_i, i\, \sigma_j] = -2\, \varepsilon_{\text{ijk}}\, (i\, \sigma_k)$$

It's easy to find a match.

$$i\, \sigma_1 = C_{4,3} \quad i\, \sigma_2 = C_{2,2} \quad i\, \sigma_3 = C_{1,1}$$

Note that these three $C_{a,i}$ s are exactly the matrices for which rows and columns 1, 2, and 4 are zero. These are actually 4x4 matrices, which is what you'd expect for a real representation of su(2).

Clearly this is not the only representation of su(2). We can cyclically permute $C_{4,3}$, $C_{2,2}$, and $C_{1,1}$ (giving three trivial choices), use their $D_{a,i}$ equivalents (in an anti-cyclic order), or we can increment the first indices of all three (giving seven distinct choices.)

■ Gell-Mann Matrices / su(3) Subgroups

Like the Pauli matrices, the Gell-Mann matrices [ref. 5, 6] are hermitian, so we are obliged to multiply them by i before comparing them to the $C_{\alpha,\beta}$ s. The first three are a copy of the Pauli matrices, so we start with

$$i\lambda_1 = C_{4,3} \qquad i\lambda_2 = C_{2,2} \qquad i\lambda_3 = C_{1,1}$$

It's an obvious guess that λ_8 is going to be built from the degenerate triplet that λ_3 belongs to. For example

$$i\lambda_8 = D_{1,1} = \frac{1}{\sqrt{3}}(C_{1,2} - C_{1,3})$$

commutes with $i\lambda_1$, $i\lambda_2$, and $i\lambda_3$ as required.

λ_4 and λ_5 are part of another su(2) subgroup, but their commutator is a mix of λ_3 and λ_8. We're looking for a commutator like

$$[i\lambda_4, i\lambda_5] = -2i\left(\frac{1}{2}\lambda_3 + \frac{\sqrt{3}}{2}\lambda_8\right) = -(C_{1,1} + C_{1,2} - C_{1,3}) = 2C_{1,3}$$

λ_6 and λ_7 are similar.

$$[i\lambda_6, i\lambda_7] = -2i\left(-\frac{1}{2}\lambda_3 + \frac{\sqrt{3}}{2}\lambda_8\right) = -(-C_{1,1} + C_{1,2} - C_{1,3}) = -2C_{1,2}$$

Looking through our collection of su(2) subgroups, we easily find a match.

$$[C_{5,1}, C_{6,2}] = 2\,C_{1,3} \text{ and } [C_{7,1}, C_{3,3}] = -2\,C_{1,2}$$

giving

$$i\lambda_1 = C_{4,3} \qquad i\lambda_2 = C_{2,2} \qquad i\lambda_3 = C_{1,1}$$
$$i\lambda_4 = C_{5,1} \qquad i\lambda_5 = C_{6,2}$$
$$i\lambda_6 = C_{7,1} \qquad i\lambda_7 = C_{3,3} \qquad i\lambda_8 = D_{1,1}$$

Note that these nine $C_{\alpha,\beta}$ s (counting $D_{1,1}$ as two matrices) are exactly the matrices for which the first row and column is zero. These are actually 6x6 matrices, which is what you'd expect for a real representation of su(3).

Clearly this is not the only representation of su(3). If we had chosen $i\lambda_8 = -D_{1,1}$ for example, that would have swapped the rolls of (λ_4, λ_5) with (λ_6, λ_7). And as always we can get seven distinct representations by incrementing the indices.

■ Generators of G_2

Cacciatori, et. al. [ref. 1, sec. 2] list their version of the generators of G_2, calling them C_1 through C_{14}. They use the doubling basis, so to compare them to ours we first have to convert them to the 7-cycle basis. (That conversion is not unique, since the choice of $\mathbb{l}$ is not unique, but I'll use the same conventions as above.) After that it's easy to show that

$$C_3 = C_{1,1} \qquad C_8 = -D_{1,1}$$
$$C_1 = -C_{4,3} \qquad C_9 = -D_{4,3}$$
$$C_2 = C_{2,2} \qquad C_{10} = D_{2,2}$$
$$C_4 = C_{7,1} \qquad C_{11} = -D_{7,1}$$
$$C_5 = C_{3,3} \qquad C_{12} = D_{3,3}$$
$$C_6 = -C_{6,2} \qquad C_{13} = -D_{6,2}$$
$$C_7 = -C_{5,1} \qquad C_{14} = -D_{5,1}$$

An Intuitive Description of G_2

The thing that makes rotations in four dimensions significantly different from three, is that you can have rotations in two completely independent planes at once. Multiplying by a unit quaternion, such as $e^{\hat{n}\theta}$ where $\hat{n}$ is a pure imaginary unit quaternion, not only rotates in the plane of $\hat{n}$ and the real axis (call that rotation R_0) but also in the imaginary plane perpendicular to $\hat{n}$ (call that rotation R_1). Left and right multiplication are different in how they generate those rotations.

$$e^{\hat{n}\theta}\, x \to R_0\, R_1, \qquad \text{and} \qquad x\, e^{\hat{n}\theta} \to R_0\, R_1^{-1}$$

You could, in a sloppy, intuitive, handwaving kind of way, think of $R_0\, R_1^{-1}$ as an operation that takes rotation out of one plane and moves it to the other, at least with regards to left multiplication. You can easily use a right multiply to move all the rotation onto one plane. Thus

$$e^{\hat{n}\theta}\, x\, e^{\hat{n}\theta} \to R_0^{\,2}, \qquad \text{and} \qquad e^{\hat{n}\theta}\, x\, e^{-\hat{n}\theta} \to R_1^{\,2}$$

In eight dimensions a general rotation will have four completely independent rotation planes. Multiplying by a unit octonian, which we will also write $e^{\hat{n}\theta}$, can be written as a product of four plane rotations.

$$e^{\hat{n}\theta}\, x \to R_0\, R_1\, R_2\, R_3, \text{ and } \qquad x\, e^{\hat{n}\theta} \to R_0\, R_1^{-1}\, R_2^{-1}\, R_3^{-1}$$

We can isolate R_0 just as we did for quaternions, but the other part is more complicated.

$$e^{\hat{n}\theta}\, x\, e^{\hat{n}\theta} \to R_0^{\,2}, \qquad \text{and} \qquad e^{\hat{n}\theta}\, x\, e^{-\hat{n}\theta} \to R_1^{\,2}\, R_2^{\,2}\, R_3^{\,2}$$

We know that the octonians can generate all possible eight dimensional rotations, so it must be possible to isolate R_1, R_2, and R_3. The elements of G_2 are precisely the tools we need. G_2 contains operations equivalent to

$R_1\, R_2^{-1}$, $R_2\, R_3^{-1}$, and $R_3\, R_1^{-1}$

Thus, for any unit vector $e^{\hat{n}\theta}$ (which has 7 parameters) there is a 2D subspace of G_2 spanned by

$R_1\, R_2^{-1}$ and $\frac{1}{\sqrt{3}}\left(R_2\, R_3^{-1} - R_3\, R_1^{-1}\right)$

which are rotations in 8-dimensions which you can't get with a single octonian multiplication, or even one left and one right octonian multiplication. Similarly, given any rotation, you can use elements of G_2 to compensate for the rotation in two of three imaginary planes, leaving a 4D rotation which can always be described by one left and one right multiplication. (I'm being overly sloppy here. Can I do better?)

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