Decomposition of ODEs with an $\mathbf{sl}_2$ algebra of symmetries.

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Abstract. Viewing linear ODEs as D-modules provides an algebraic apparatus which can be used to embed aspects of the theory of representations of Lie algebras into the quest of decomposing and solving equations. In particular we shall see examples of how equations of high order may be solved in terms of a small symmetry algebra. Our main example will be $\mathbf{sl}_2$-equations.

EQUATIONS AS D-MODULES

Geometric image of equations in Jet space.

Consider a vector bundle $B \xrightarrow{\beta} \mathbf{R}$ of rank m with sections $C^\infty(\beta)$. The bundle $J^k(\beta) \xrightarrow{\pi_k} \mathbf{R}$ of k-jets of sections of β is of rank $m(k+1)$ over R, and is equipped with the *Cartan distribution*. A system of linear kth order ordinary differential equations is a linear subbundle

$$\mathscr{E} \xrightarrow{\alpha} \mathbf{R} \ \subset \ J^k(\beta) \xrightarrow{\pi_k} \mathbf{R} \tag{1}$$

of codimension m such that the *Cartan distribution* on $J^k(\beta)$ when restricted to $\mathscr{E}$, denoted $\mathscr{C}_{\mathscr{E}}$, is 1-dimensional, and projects to $\mathbf{R}$ without singularities. The $C^\infty(\mathbf{R})$-module of sections $C^\infty(\alpha)$ is free and of rank m. We have a *linear connection* in the bundle α,

$$\nabla : D(\mathbf{R}) \longrightarrow Der(C^\infty(\alpha)) \tag{2}$$

where $Der(C^\infty(\alpha))$ denotes derivations of $C^\infty(\alpha)$ over $\frac{d}{dx}$, i. e. $\mathbf{R}$-linear maps

$$\Delta : C^\infty(\alpha) \to C^\infty(\alpha) \quad \text{ with } \quad \Delta(fs) = f' \cdot s + f \cdot \Delta(s)\ , \tag{3}$$

for any $f \in A, s \in C^\infty(\alpha)$. ∇ is defined by the requirement that it lifts $\frac{d}{dx}$ on the base $\mathbf{R}$ to a generator $X \in D(\mathscr{E})$ of $\mathscr{C}_{\mathscr{E}}$ on $\mathscr{E}$. Constant sections of ∇ , i.e. sections s such that $\nabla_Y(s) = 0$ for all $Y \in D(\mathbf{R})$ are precisely the *integral curves* of $\mathscr{C}_E$.

Thus, to any linear $n \times n$ system

$$\underline{h}' + B(x)\underline{h} = 0 \tag{4}$$

there is the associated pair $(C^\infty(\alpha), \delta = \nabla_{\frac{d}{dx}})$ such that

$$ker\delta \cong \text{Solutions}(4) \tag{5}$$

Definition 1 *Let A be a commutative ring over a field k and δ_A a derivation of A. (E,δ) is a D*-module *over (A,δ_A) if E is a rank n free A-module, and δ is a derivation over δ_A. The latter means that δ is k-linear and satisfies a Leibniz property*

$$\delta(a\cdot e) = \delta_A(a)\cdot e + a\cdot\delta(e) \tag{6}$$

for $a \in A$, $e \in E$.

Hence, associated to any system (4) is a D-module (E,δ) over $(A = C^\infty(\mathbf{R}), \delta_A = \frac{d}{dx})$, with $\ker\delta \cong \text{Solutions}(4)$. We may treat these modules abstractly, without the geometric realization as sections in a jet sub bundle.

Algebraic properties.

D-modules over $(A = C^\infty(\mathbf{R}), \frac{d}{dx})$ form a monoidal category with the product being tensor product of modules over A, and morphisms being homomorphisms that commute with the respective δ-s. Given D-modules (E_1,δ_1) (E_2,δ_2) their product is $(E_1 \otimes_A E_2, \delta)$ with

$$\delta(e_1\otimes e_2) = \delta_1(e_1)\otimes e_2 + e_1 \otimes \delta_2(e_2) \tag{7}$$

on decomposable elements. This extends to symmetric and wedge products, hence we get induced D-modules $(S^n(E),\delta)$ and $(\bigwedge^l(E),\delta)$. Homomorphisms form a module $(Hom_A(E_1,E_2),\delta)$ with

$$\delta(F) \stackrel{def}{=} \delta\circ F - F\circ\delta \tag{8}$$

for $F \in Hom_A(E_1,E_2)$. We may define the dual module $E^* = Hom_A(E,A)$, and it yields exactly the adjoint equation of E. Denote the kernel $\ker\delta = E^\# \subset E$.

Lemma 1 *Let (E,δ) be a D-module over $(C^\infty(\mathbf{R}), \frac{d}{dx})$. Then any basis of $E^\#$ over $\mathbf{R}$ is a basis of E over A.*

Proof Let $\{e_1,...,e_k\}$ be a basis of E with $\delta\underline{e} = B(x)^t\underline{e}$. Then the associated system is (4). Let $\underline{h}_1,...,\underline{h}_k$ be linear independent solutions of the system. Take $H(x) = [\underline{h}_1,...,\underline{h}_k]^t$. Then

$$\underline{\gamma} = H(x)\underline{e} \tag{9}$$

is a new basis of E over A, since $H(x)$ is the Wronskian of the system, and nondegenerate. Moreover, any basis of $E^\#$ over $\mathbf{R}$ is on the form in (9).

Note that taking the kernel commutes with the above algebraic operations, i.e. $(Hom_A(E_1,E_2))^\# = Hom_{\mathbf{R}}(E_1^\#,E_2^\#)$, $(E_1\otimes_A E_2)^\# = E_2 \otimes_{\mathbf{R}} E_2$ etc.

SYMMETRIES AND REPRESENTATIONS.

In the D-module setting a symmetry of an equation (E,δ) is an element $F \in (End_A(E))^\#$, and we denote the equation corresponding to the module $(End_A(E),\delta)$ the symmetry equation of E. Its solution generates linear symmetries of E. A Lie-algebra $\mathbf{g}$ is a *symmetry algebra* of E if there is a representation

$$\rho \,:\, \mathbf{g} \longrightarrow \mathrm{End}_A(E) \tag{10}$$

such that

$$\rho(g)\circ\delta \;=\; \delta\circ\rho(g) \quad \forall g\in\mathbf{g} \tag{11}$$

i.e. ρ maps $\mathbf{g}$ into δ-invariant endomorphisms of E, $\mathrm{End}_A(E)^{\#}\subset\mathrm{End}_A(E)$. Note that this yields a representation into the $\mathbf{R}$-vector space $E^{\#}$, and combining this with Lemma 1 enables us to make use of results from the theory of representations of Lie algebras into vector spaces, and extend this to our modules.

Theorem 1 *Any equation E with a representation of symmetries*

$$\mathbf{sl_2}(\mathbf{R})\rightarrow End_A(E)$$

is decomposable into a direct sum of D-modules

$$E=\bigoplus_{i=1}^{m}E_i \tag{12}$$

where each $E_i\cong S^{n_i}(M_i)$ for a rank 2 D-module M_i , and each E_i is an irreducible subrepresentation $\subset E$.

Proof The representation of $\mathbf{sl}_2$ into $V=E^{\#}$ decomposes into

$$V=\bigoplus_{i=1}^{m}V_i$$

where the V_i are irreducible subrepresentations of V, hence they must be isomorphic to symmetric powers $S^{n_i}(Y_i)$, where dim $Y_i=2$, see [1]. The E_i-s in the theorem are precisely the sub-D-modules in E spanned by V_i over A, and $E_i\cong S^{n_i}(M_i)$, M_i spanned by V_i over A. They are really sub-D-modules of E, and $\oplus E_i=E$ due to Lemma 1, which states that any basis of V over $\mathbf{R}$ is a basis of E over A.

Corollary 1 *(1) If mult(n_i) $=1$ for $i=1\ldots m$ equation E is solved directly by algebraic operations and integration of the base second order equations.*

(2) For each i such that mult(n_i) $=j$ the remaining problem is reduced to solving a new $j\times j$ system of first order order equations.

Note that Theorem 1 deals purely with existence and theory, we shall turn to see how second order Schrödinger equations can be viewed as model equations of the Lie algebra $\mathbf{sl}_2$, in the sense that the rank 2 modules of Theorem 1 correspond to such equations.

Symmetries as operators.

An alternative formulation of a linear symmetry of a kth order equation $L(y)=0$ is the following. $\Delta = b_1(x)+b_2(x)\partial + \dots + b_k(x)\partial^{k-1}$ is a symmetry if there is an operator ∇ such that

$$L\circ\Delta = \nabla\circ L \tag{13}$$

Δ in this case obviously maps $kerL$ into itself. Note that Δ is a symmetry iff the associated function $\phi = b_1 p_0 + \dots + b_k p_{k-1}$ in $J^k(\mathbf{R})$ is a generating function of a symmetry of the Cartan distribution on $\mathscr{E}\subset J^k(\mathbf{R})$.

Model equations for $\mathbf{sl}_2$.

Consider an equation of Schrödinger type, with potential $W(x)$.

$$y'' + W(x)y = 0 \tag{14}$$

We recall that all irreducible representations of $\mathbf{sl_2}$ are symmetric powers of the standard two dimensional representation.
A general second order equation

$$y'' + fy' + gy = 0 \tag{15}$$

has associated D-module (E,δ) with primitive element basis $\{e_1 = e, e_2 = \delta e\}$ with $\delta^2 e = (g - f')e + (f)\delta e$. The induced equation $(\bigwedge^2(E^*),\delta)$ is first order, and $\delta(e_1^*\wedge e_2^*) = -f e_1^*\wedge e_2^*$. Thus, Schrödinger equations are precisely the second order equations with a generic δ-invariant volume form. Conditions for

$$\Delta = a_1 + a_2\partial \tag{16}$$

to be a symmetry of (14) in the sense of (13) are by direct calculation found to be that a_2 solves

$$z''' + 4Wz' + 2W'z = 0 \tag{17}$$

and that $a_1 = \frac{-a_2'}{2}$. Thus symmetries are given by generating functions that solve (17), i.e.

$$\Delta_a = -\frac{a'}{2} + a\partial\ ,\ \ a\in \mathrm{Sol}(17)$$

The symmetry equation actually coincides with the symmetric 2-power $S^2(E)$ of our basic equation (14), and we will see that it plays a special role in the whole hierarchy of $S^k(E)$ equations. Its solution space is isomorphic to $\mathbf{sl}_2$, where the bracket operation is defined by means of the corresponding operator commutators

$$[\Delta_a,\Delta_b] = \Delta_{<a,b>} \qquad \text{i.e.} \qquad <a,b> = ab' - a'b \tag{18}$$

From our basic equation we derive a hierarchy of new equations $S^k(E^*)$. Choosing to work with the dual modules $S^k(E^*)$ merely simplifies some calculations, and generates exactly the same equations as the module E, as follows:

$$\begin{array}{rcl} E^* & : & y'' + Wy = 0 \\ S^2(E^*) & : & y''' + 4Wy' + 2W'y = 0 \\ S^3(E^*) & : & y^{(4)} + 10Wy'' + 10W'y' + (9W^2 + 3W'')y = 0 \\ S^4(E^*) & : & y^{(5)} + 20Wy''' + 30W'y'' + [64W^2 + 18W'']y' + [64WW' + 4W''']y = 0 \\ & & \ldots \end{array}$$

For each k the kernel $S^k(E^*)^\#$ consists of elements

$$\theta_y = y\alpha_1 + y'\alpha_2 + \sum_{l=3}^{k+1} g_l(y)\alpha_l \quad , \quad y \in \mathrm{Sol}(k) \tag{19}$$

where $g_l = \frac{1}{l-1}[(k-l+3)W \cdot g_{l-2} + g'_{l-1}]$ for $l = 3,..,k+1$ and

$$\alpha_l = (e_1^*)^{k-l+1} \cdot (e_2^*)^{l-1}$$

is the standard symmetric product basis of $S^k(E^*)$ induced by $\{e_1^*, e_2^*\}$ in E^*, dual to the primitive element basis.

Proposition 1 *There is a unique skew-symmetric bracket*

$$[\,\cdot\,,\,\cdot\,] : S^m(E^*) \times S^n(E^*) \to S^{m+n-2}(E^*)$$

for $m, n \geq 1$ which is

(i) A-linear.
(ii) $[\,f \cdot g\,,\,h\,] = f \cdot [\,g\,,\,h\,] + g \cdot [\,f\,,\,h\,] \quad \forall f, g, h \in S^{\cdot}(E^*)$
(iii) $[\,\cdot\,,\,\cdot\,] = <\,\cdot\,,\,\cdot\,> = e_1 \wedge e_2 = \Omega$ *for $m = n = 1$.*

Proof The δ-invariant skew-symmetric form $\Omega = e_1 \wedge e_2 \in \bigwedge^2(E)^\#$ defines the base bracket $<\cdot,\cdot> = \Omega : E^* \times E^* \to A$, and the properties $(i)-(ii)$ determine its extension to $[\cdot\,,\,\cdot]$.

We immediately observe that Ω being δ-invariant implies that so is the extended bracket $[\cdot\,,\,\cdot]$, and by restriction we get

Proposition 2 *The bracket in* **Proposisition 1** *restricts to an* **R**-*linear bracket*

$$[\,\cdot\,,\,\cdot\,] \;:\; S^m(V) \times S^n(V) \;\longrightarrow\; S^{m+n-2}(V)$$

where $V = (E^*)^\#$. This is equivalent to a bracket

$$[\,\cdot\,,\,\cdot\,] : \mathrm{Sol}(m) \times \mathrm{Sol}(n) \to \mathrm{Sol}(m+n-2)$$

where

$$[\,\theta_y\,,\,\theta_z\,] \;=\; \theta_{[y,z]}$$

We are now ready to observe that solutions of the $S^2(E^*)$ equation (17) produce symmetries of *all* equations $S^k(E^*)$, and not only E^*.

Theorem 2 *Any solution* $a \in \mathrm{Sol}(S^2(E^*))$ *produces a symmetry*

$$\mathscr{O}^m_{\theta_a} \stackrel{\text{def}}{=} [\ \theta_a\,,\ \cdot\ ]\ :\ S^m(E^*) \longrightarrow S^m(E^*)$$

The corresponding symmetry *operator* is

$$\mathscr{O}\ =\ \mathscr{O}^m_a\ :\ \mathrm{Sol}(m) \longrightarrow \mathrm{Sol}(m) \tag{20}$$

with the correspondence

$$\mathscr{O}^m_{\theta_a}(\beta_y)\ =\ \beta_{\mathscr{O}^m_a(y)}$$

The precise expression is

$$\mathscr{O}^m_a\ =\ \frac{1}{2}(\,-ma' + 2a\partial\,) \tag{21}$$

for any $m \geq 1, a \in \mathrm{Sol}(S^2(E^*))$.

Now, let u, v span the solution space of the base equation E^*.

Theorem 3 *For any* $k \leq 1$ *the symmetries of* $S^k(E^*)$

$$X_+\ =\ -\frac{1}{2c}\,\mathscr{O}^k_{v^2}\ ,\ \ X_-\ =\ \frac{1}{2c}\,\mathscr{O}^k_{u^2} \quad \text{and} \quad H\ =\ \frac{1}{c}\mathscr{O}^k_{uv} \tag{22}$$

where $c =< u, v >\in \mathbf{R}$ *constitute a basis of the* $\mathbf{sl_2}$*-algebra of symmetries* $\cong Sol(S^2(E^*))$ *with commutators*

$$[X_+, X_-] = H\ ,\ \ [H, X_+] = 2X_+\ ,\ \ [H, X_-] = -2X_- \tag{23}$$

$S^k(E^*)$ *decomposes into rank* 1 *sub-D-modules corresponding to different eigenvalues of* H

$$S^k(E^*)\ =<\theta_{u^k}>_A\ \oplus\ <\theta_{u^{k-1}v}>_A\ \oplus \ldots \oplus <\theta_{v^k}>_A$$

$-k, -k+2, \ldots, k-2, k$ *respectively, and the action is precisely* $S^k(V_0)$*, with* V_0 *being the standard representation of* $\mathbf{sl_2}$.

Thus, the action of $(S^2(E^*)^{\#}, [\cdot\,,\cdot])$ into $V \subset E^*$ is precisely the standard representation of $\mathbf{sl_2}$, and the Schrödinger equations serve as models for the second order base modules in Theorem 1.

Given an equation with an $\mathbf{sl_2}$ action, the first step should be to identify the standard basis of the action, as in Theorem 3, and decompose the module according to the eigenvalues of the diagonal element H. If, as in part (1) of Corollary 1, the multiplicities of the n_i-s are one, this is enough to identify the irreducible subrepresentations $E_i \cong S^{n_i}(M_i)$. Then to identify the corresponding base module M_i it is sufficient to take $w \in E_i$ with $w \in \ker X_+ \cap Eig_{n_i}(H)$ and recover the potential $W_i(x)$ of the corresponding

Schrödinger equation in terms of derivatives of fractions of coefficients of w and $X_- w$. Any $w \in (\ker X_+ \cap Eig_{n_i}(H)) \cap A \cdot \ker \delta$ will generate a sub-D-module $\cong S^{n_i}(M_i)$. For $mult(n_i) > 1$ we should thus add the requirement $\delta w = \lambda(x) w$ to find a generating element. The base second order equations can be integrated, they have a three dimensional symmetry algebra, and solutions of M_i produce all solutions of $S^{n_i}(M_i)$, which are again solutions of E.

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