ALGEBRAIC ASPECTS OF PROLONGATIONS

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Exterior differential systems and their prolongations were introduced by É. Cartan [2, pp. 585 ff.]. They have been studied by E. Kähler [3, pp. 50–51], Y. Matsushima [6], M. Kuranishi [4], [5] and É. Cartan himself [2, Chapter 6]. Two viewpoints seem to predominate in modern treatments. One approach is geometric [4], [6]. The prolonged system is defined on a submanifold of a Grassmann bundle. In [1], [5] the equivalence between exterior differential systems and partial differential equations is emphasized, as one uses for new variables the partials of the given dependent variables with respect to the independent variables. [5] uses jets to accomplish this.

In many of É. Cartan's applications of prolongations, however, a purely algebraic flavor prevails [1, pp. 116–119], [2, p. 585]. This is particularly true in infinite continuous groups [1, pp. 638–639 and the examples following]. The author seems to be merely introducing as many new variables as possible. Indeed, in [2, p. 1361] after defining prolongations according to the first method above, he states that this can be obtained by solving certain equations in the most general possible way, which is a purely algebraic problem.

It is our purpose to discuss this algebraic problem and show that É. Cartan's "normal prolongation" does indeed possess the maximal property among all possible prolongations. We begin as he did in [1, pp. 577–578], assuming that the system is of the form

$$d\theta^i \equiv a^i_{j\rho}\omega^j \wedge \pi^\rho \text{ modulo } (\theta^1, \cdots, \theta^m).$$

Then d can be considered linear over the ring of C^ω or C^∞ functions, since

$$d(f\theta^i) \equiv f d\theta^i \text{ modulo } (\theta^1, \cdots, \theta^m).$$

Hence the problem reduces to the study of linear transformations between certain modules.

A is a fixed commutative ring with identity element. All modules are unitary A-modules. I is a fixed module called the module of *independent variables.*

DEFINITION 1. A *differential system* (S, d, T) consists of two modules S and T together with a linear transformation $d: S \rightarrow I \otimes T$. T is said to be *minimal* when it contains no proper submodule T' such that $d(S) \subset I \otimes T'$.

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Let j: $I\otimes I \to I\wedge I$ be the defining epimorphism. i will denote the identity transformation on various modules.

DEFINITION 2. A *prolongation* of (S, d, T) is a differential system (T, δ, U) such that if $i\otimes\delta$: $I\otimes T\to I\otimes I\otimes U$, $j\otimes i$: $I\otimes I\otimes U\to (I\wedge I)\otimes U$, then $(j\otimes i)(i\otimes\delta)d=0$.

PROPOSITION 1. *If (T, δ, U) is a prolongation of (S, d, T) and $\phi: U\to U'$ is a linear transformation on U to a module U', then $(T, (i\otimes\phi)\delta, U')$ is a prolongation of (S, d, T).*

PROOF. This follows from commutativity in the diagram

$$\begin{array}{ccccccc} S & \xrightarrow{d} & I\otimes T & \xrightarrow{i\otimes\delta} & I\otimes I\otimes U & \xrightarrow{j\otimes i} & I\wedge I\otimes U \\ & & & & {\scriptstyle i\otimes\phi}\downarrow & & \downarrow{\scriptstyle i\otimes\phi} \\ & & & & I\otimes I\otimes U' & \xrightarrow{j\otimes i} & I\wedge I\otimes U'. \end{array}$$ Q.E.D.

Let U^*, I^*, T^*, etc., denote the dual modules to U, I, T, etc.

DEFINITION 3. The prolongation (T, ∂, U') is said to be *obtained* from the prolongation (T, δ, U) if there exists ϕ in $\operatorname{Hom}(U, U')$ so that $\partial=(i\otimes\phi)\delta$.

PROPOSITION 2. *Let V be a submodule of* $\operatorname{Hom}(T, I)$. *Then there exists a canonical linear transformation δ: $T\to I^{**}\otimes V^*$. If X and Y are any two modules and ϕ is in* $\operatorname{Hom}(X, Y)$, *ξ is in $X\otimes T$, λ is in $Y^*\otimes I^*$, and θ is in V, then*

$$\langle(\phi\otimes\delta)(\xi), \lambda\otimes\theta\rangle = \langle(\phi\otimes\theta)(\xi), \lambda\rangle. \tag{1}$$

PROOF. Given t in T, we define $\delta(t)$ to be that element of $I^{**}\otimes V^* =(I^*\otimes V)^*$ whose value on $\omega^*\otimes\theta$ in $I^*\otimes V$ is given by $\langle\delta(t), \omega^*\otimes\theta\rangle =\langle\theta(t), \omega^*\rangle$. Since this is bilinear in ω^* and θ, it defines an element of $(I^*\otimes V)^*$.

In order to prove (1), it suffices to suppose $\xi=x\otimes t$, $\lambda=y^*\otimes\omega^*$. Then

$$\begin{aligned} \langle(\phi\otimes\delta)(x\otimes t), y^*\otimes\omega^*\otimes\theta\rangle &= \langle\phi(x)\otimes\delta(t), y^*\otimes\omega^*\otimes\theta\rangle \\ &= \langle\phi(x), y^*\rangle\langle\delta(t), \omega^*\otimes\theta\rangle = \langle\phi(x), y^*\rangle\langle\theta(t), \omega^*\rangle \\ &= \langle(\phi\otimes\theta)(x\otimes t), y^*\otimes\omega^*\rangle. \end{aligned}$$ Q.E.D.

PROPOSITION 3. *Let $V=\{\theta\in\operatorname{Hom}(T, I) \mid j(i\otimes\theta)d=0\}$. If $I^{**}=I$, then (T, δ, V^*) is a prolongation of (S, d, T).*

PROOF. We must prove that for any s in S, $0=(j\otimes i)(i\otimes\delta)d(s) \in I\wedge I\otimes V^*=(I^*\wedge I^*\otimes V)^*$. Let $\mu\otimes\theta$ be an element of $I^*\wedge I^*\otimes V$. Then

$$\begin{aligned}\langle (j \otimes i)(i \otimes \delta)d(s), \mu \otimes \theta\rangle &= \langle (i \otimes \delta)d(s), j^*(\mu) \otimes \theta\rangle \\ &= \langle (i \otimes \theta)d(s), j^*(\mu)\rangle \qquad \text{by (1)} \\ &= \langle j(i \otimes \theta)d(s), \mu\rangle.\end{aligned}$$

However, $j(i\otimes\theta)d(s)=0$ by the definition of V. Q.E.D.

DEFINITION 4. (T, δ, V^*) is the *normal* prolongation.

THEOREM. *Assume that I has a finite basis. Let (T, ∂, W) be any minimal prolongation of (S, d, T) where $W=W^{**}$. Then (T, ∂, W) is obtained from the normal prolongation of (S, d, T).*

PROOF. There exists a canonical linear transformation ψ: $W^* \to \mathrm{Hom}(T, I)$ defined as follows. For w^* in W^*, t in T and ω^* in I^*, $\langle\psi(w^*)(t), \omega^*\rangle = \langle\partial(t), \omega^*\otimes w^*\rangle$. Since $I^{**}=I$, this is well-defined.

Suppose w_0^* is in $\ker\psi$. Then for all t in T, all ω^* in I^*, $\langle\partial(t), \omega^*\otimes w_0^*\rangle=0$. Let $W_1=\{w\in W \mid \langle w, w_0^*\rangle=0\}$. Let $\omega_1, \cdots, \omega_n$ be a basis of I, $\omega_1^*, \cdots, \omega_n^*$ the dual basis. Suppose $\xi=\sum a_j(\omega_j\otimes w_j)\in I\otimes W$ satisfies $\langle\xi, \omega^*\otimes w_0^*\rangle=0$ for every ω^* in I^*. When $\omega^*=\omega_k^*$, $\langle\xi, \omega_k^*\otimes w_0^*\rangle = a_k\langle w_j, w_0^*\rangle=0$. Hence ξ is in $I\otimes W_1$. Hence $\partial(T)\subset I\otimes W_1$. Since W is minimal, $W=W_1$, so $w_0^*=0$. Thus, $\ker\psi=0$, and we may consider $W^*\subset\mathrm{Hom}(T, I)$. Furthermore, under this identification, ∂ is the map δ of Proposition 2.

If θ is in W^*, μ is in $I^*\wedge I^*$ and s is in S, then since (T, ∂, W) is a prolongation of (S, d, T),

$$\begin{aligned}0 &= \langle (j \otimes i)(i \otimes \delta)d(s), \mu \otimes \theta\rangle = \langle (i \otimes \delta)d(s), j^*(\mu) \otimes \theta\rangle \\ &= \langle (i \otimes \theta)d(s), j^*(\mu)\rangle, \qquad \text{by Proposition 2} \\ &= \langle (j \otimes \theta\rangle d(s), \mu\rangle.\end{aligned}$$

Hence θ is in V, so $W^*\subset V$. The dual map to the injection i_n: $W^*\to V$ then satisfies $\partial=(i\otimes i_n^*)\delta$. Q.E.D.

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