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Nonclassical Potential Symmetry Generators of Differential Equations

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Abstract. We determine the nonclassical potential symmetries for a number of equations that arise in the literature. A large number of these are obtained for some equations which only admit a single potential (classical) symmetry (e.g., the wave equation and the motion of waves through some medium). However, we show that some of the exact solutions invariant under the nonclassical potential symmetries are equivalent to known solutions but these solutions are not obtainable through the classical point or potential symmetries. The Korteweg–deVries equation, it is shown, does not admit nonclassical potential symmetries – as in the classical case.

Keyword: Nonclassical potential symmetries.

1. Introduction

It is well known that the symmetry group method plays an important role in the analysis of differential equations. It has been shown in [1–4], how one can use Lie symmetry group method for finding symmetry reductions of partial differential equations and this method is generally referred to as the classical method. By a classical or strong symmetry group of a system of partial differential equations we mean a continuous group of transformations acting on the space of independent and dependent variables which transforms solutions of the system to other solutions.

There have been several advances on the classical method for symmetry reductions of partial differential equations. Some of these are the nonclassical method of Bluman and Cole [5], the direct method of Clarkson and Kruskal [6] and its generalization to the differential constraint approach by Olver and Rosenau [7].

The nonclassical symmetries are the usual symmetries of the new system of PDEs obtained by appending to the original PDE, the invariant surface conditions sometimes called ‘side conditions’, which is a system of first order differential equations satisfied by all functions invariant under a certain vector field [8]. The first approach to these symmetries was made by Bluman and Cole [5]. In general, the number of determining equations for the nonclassical method is smaller than the classical method, so that it is difficult to find all possible solutions to the overdetermined system. By this method, a much wider class of groups is potentially available, and hence there is a possibility of further kinds of group-invariant solutions being found by the same reduction technique. The set of nonclassical symmetries do not form a Lie algebra [8].

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In [9], Bluman et al. introduced an algorithmic method which yields new classes of symmetries of a given PDE that are neither Lie point nor Lie–Bäcklund symmetries. They are nonlocal symmetries.

That is, by embedding a given system of PDEs in an auxiliary covering system with additional dependent variables, useful nonlocal symmetries can be found. A nonlocal symmetry of the given system is obtained, when a Lie point symmetry of the auxiliary system is acting on the space consisting of the independent and dependent variables of the given system and also the auxiliary variables. The auxiliary system is obtained by the replacement of the given PDE by an equivalent conservation law and the corresponding nonlocal symmetries are called 'potential symmetries' of the given system.

We now present the essence of the notions of potential and nonclassical symmetries [1, 8, 11] and in Section 2, we consider the notion of *nonclassical potential symmetries* as applied to some well-known examples.

Consider a scalar kth-order PDE $R\{x, u\}$, which is written in a conserved form

$$D_i[f^i(x, u, u_{(1)}, \dots, u_{(k-1)}] = 0, \tag{1.1}$$

with $n \geq 2$ independent variables $x = (x_1, \dots, x_n)$, a single dependent variable u and $u_{(j)}$, $j = 1, \dots, k-1$, is a collection of jth-order partial derivative: $u_{(1)} = (u_1, \dots, u_n)$, $u_{(2)} = (u_{11}, u_{12}, \dots, u_{nn}), \dots$, (e.g. for two independent variables t and x, $u_{(2)} = (u_{tt}, u_{tx}, u_{xx})$ and $u_{(3)} = (u_{ttt}, u_{ttx}, u_{txx}, u_{xxx})$) and

$$D_i = \frac{\partial}{\partial x_i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \cdots + u_{ii_1 i_2,\dots,i_{k-1}} \frac{\partial}{\partial u_{i_1 i_2,\dots,i_{k-1}}}, \quad i = 1, 2, \dots, n.$$

Since the PDE (1.1) is in a conserved form, there exist $(1/2)n(n-1)$ functions $\Psi^{i,j}$ components of an antisymmetric tensor ($i < j$), such that (1.1) can be expressed in the form [1]

$$\begin{aligned} &f^i(x, u, u_{(1)}, \dots, u_{(k-1)}) \\ &\quad = \sum_{i<j} (-1)^j \frac{\partial}{\partial x_j} \Psi^{i,j} + \sum_{j<i} (-1)^{i-1} \frac{\partial}{\partial x_j} \Psi^{j,i}, \quad i, j = 1, 2, \dots, n. \end{aligned} \tag{1.2}$$

Setting

$$\Psi^{i,j} = 0, \quad \text{for } j \neq i+1$$

and introducing

$$v^i = \Psi^{i,i+1}, \quad i = 1, 2, \dots, n-1,$$

the system (1.2) associated with $R\{x, u\}$ given by (1.1) becomes the following auxiliary system of PDEs, $S\{x, u, v\}$, given by

$$\begin{aligned} f^1 &= \frac{\partial}{\partial x_2} v^1, \quad j = 1, 2, \dots, k-1, \\ f^j &= (-1)^{j-1} \left[\frac{\partial}{\partial x_{j+1}} v^j + \frac{\partial}{\partial x_{j-1}} v^{j-1} \right], \quad 1 < j < n, \\ f^n &= (-1)^{n-1} \frac{\partial}{\partial x_{n-1}} v^{n-1}. \end{aligned} \tag{1.3}$$

In the sequel, we will consider e.g.s of a time and single space variables. To this end, we spell out the above formulation for the case $n = 2$ (see [1, 11]).

For $n = 2$, let

$$\begin{aligned} f^1 &= f(x, u, u_{(1)}, \dots, u_{(k-1)}), \\ f^2 &= -p(x, u, u_{(1)}, \dots, u_{(k-1)}), \end{aligned} \tag{1.4}$$

so that $R\{x, u\}$ becomes

$$D_1(f) - D_2(p) = 0. \tag{1.5}$$

By the introduction of the potential variable $v = v^1 = \Psi^{1,2}$, the auxiliary system $S\{x, u, v\}$ is written as

$$\frac{\partial v}{\partial x_2} = f, \quad \frac{\partial v}{\partial x_1} = p. \tag{1.6}$$

Assume now that the system $S\{x, u, v\}$ given by (1.6) admits a Lie point symmetry

$$X = \tau(x, u, v)\frac{\partial}{\partial x_1} + \xi(x, u, v)\frac{\partial}{\partial x_2} + \phi(x, u, v)\frac{\partial}{\partial u} + \zeta(x, u, v)\frac{\partial}{\partial v}, \tag{1.7}$$

which can be calculated by solving the determining equations for the exact symmetries

$$\begin{aligned} X^{[k-1]}\left(\frac{\partial v}{\partial x_2} - f\right)\Bigg|_{\left\{\frac{\partial v}{\partial x_2}=f, \frac{\partial v}{\partial x_1}=p\right\}} &= 0, \\ X^{[k-1]}\left(\frac{\partial v}{\partial x_2} - p\right)\Bigg|_{\left\{\frac{\partial v}{\partial x_2}=f, \frac{\partial v}{\partial x_1}=p\right\}} &= 0. \end{aligned} \tag{1.8}$$

If the infinitesimals τ, ξ, ϕ depend on v explicitly (the condition is not required for ζ), the generator (1.7) defines a nontrivial potential symmetry of $R\{x, u\}$.

Now in order to determine the *nonclassical* potential symmetries of the system of first-order differential equations (1.6), obtained from Equation (1.5), we also append the invariant surface conditions:

$$\begin{aligned} \tau u_t + \xi u_x - \phi &= 0, \\ \tau v_t + \xi v_x - \zeta &= 0, \end{aligned} \tag{1.9}$$

to the first-order differential equations (1.6). That is, the variables u_t, u_x, v_t, v_x found from (1.6) and (1.9) must be substituted into both Equations (1.8). Consequently, the nonclassical potential symmetries have joint characteristics of potential and nonclassical symmetries.

We note that nonclassical potential symmetry generators have not been studied previously. In the following examples we consider two cases: first when $\tau = 1$ and the second case when $\tau = 0$ and $\xi = 1$. These cases are the only cases that need to be considered [8].

2. Applications

Here, we consider, with a view to applications of the theory in section 1, some well-known examples which have been analysed previously, using the standard symmetry methods.

EXAMPLE 2.1. We shall first find the nonclassical potential symmetries of the (1+1)-dimensional wave equation,

$$u_{tt} - u_{xx} = 0. \tag{2.1}$$

Writing in a conserved form the PDE (2.1) becomes

$$D_t(u_t) - D_x(u_x) = 0. \tag{2.2}$$

Now consider the corresponding (potential) system of first-order differential equations

$$v_t = u_x, \quad v_x = u_t. \tag{2.3}$$

Equation (2.3) admits the principle Lie algebra ∂_t, ∂_x, $u\partial_u + v\partial_v$ of which none is a potential symmetry of (2.1). We investigate the existence of nonclassical potential symmetries of (2.1). For the vector field (1.7), the invariant surface conditions are

$$\begin{aligned} \tau u_t + \xi u_x - \phi &= 0, \\ \tau v_t + \xi v_x - \zeta &= 0. \end{aligned} \tag{2.4}$$

From Equations (1.8), we obtain

$$X^{[1]}(v_t - u_x)|_{(2.3)} = 0,$$

i.e.,

$$\left(\tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial u} + \zeta\frac{\partial}{\partial v} + \phi^t\frac{\partial}{\partial u_t} + \phi^x\frac{\partial}{\partial u_x} + \zeta^t\frac{\partial}{\partial v_t} + \zeta^x\frac{\partial}{\partial v_x}\right)(v_t - u_x)|_{(2.3)} = 0,$$

i.e.,

$$(\zeta^t - \phi^x)|_{(2.3)} = 0,$$

which in expanded form becomes

$$\begin{aligned} \zeta_t \; &+ \; \zeta_u u_t + \zeta_v u_x - \tau_t u_x - \tau_u u_t u_x - \tau_v u_x^2 - \xi_t u_t - \xi_u u_t^2 - \xi_v u_t u_x - \phi_x \\ &- \; \phi_u u_x - \phi_v u_t + \tau_x u_t + \tau_u u_t u_x + \tau_v u_t^2 + \xi_x u_x + \xi_u u_x^2 + \xi_v u_t u_x = 0. \end{aligned} \tag{2.5}$$

Similarly,

$$X^{[1]}\,(v_x - u_t)|_{(2.3)} = 0,$$

i.e.,

$$\left(\zeta^x - \phi^t\right)|_{(2.3)} = 0,$$

which in expanded form becomes

$$\begin{aligned} \zeta_x \; &+ \; \zeta_u u_x + \zeta_v u_t - \tau_x u_x - \tau_u u_x^2 - \tau_v u_t u_x - \xi_x u_t - \xi_u u_t u_x - \xi_v u_t^2 - \phi_t \\ &- \; \phi_u u_t - \phi_v u_x + \tau_t u_t + \tau_u u_t^2 + \tau_v u_t u_x + \xi_t u_x + \xi_u u_t u_x + \xi_v u_x^2 = 0. \end{aligned} \tag{2.6}$$

Following the procedure and justification given in [8], we consider the following specific cases.

Case 1: $\tau = 1$
Now, from (2.4), we obtain

$$u_t = \frac{\phi - \xi\zeta}{1-\xi^2}, \quad u_x = \frac{\zeta - \xi\phi}{1-\xi^2}, \quad \xi \neq \pm 1. \tag{2.7}$$

Substituting (2.7) into (2.5) and (2.6) yields

$$\zeta_t + \frac{\zeta_u\phi - \xi\zeta\zeta_u}{1-\zeta^2} + \frac{\zeta_v\zeta - \zeta_v\xi\phi}{1-\xi^2} + \frac{\xi_t\xi\zeta - \phi\xi_t}{1-\xi^2} + \frac{2\phi\xi_u\xi\zeta - \xi_u\phi^2 - \xi_u\xi^2\zeta^2}{1-\xi^2} - \phi_x$$
$$+ \frac{\xi\phi_u\phi - \phi_u\zeta}{1-\xi^2} + \frac{\xi\phi_v\zeta - \phi_v\phi}{1-\xi^2} + \frac{\zeta\xi_x - \xi\xi_x\phi}{1-\xi^2} + \frac{\xi_u\zeta^2 - 2\xi_u\zeta\xi\phi + \xi^2\xi_u\phi^2}{1-\xi^2} = 0, \tag{2.8}$$

and

$$\zeta_x + \frac{\zeta\zeta_u - \zeta_u\xi\phi}{1-\xi^2} + \frac{\zeta_v\phi - \zeta_v\xi\zeta}{1-\xi^2} + \frac{\xi_x\xi\zeta - \xi_x\phi}{1-\xi^2} + \frac{2\xi_v\phi\xi\zeta - \xi_v\phi^2 - \xi_v\xi^2\zeta^2}{1-\xi^2} - \phi_t$$
$$+ \frac{\phi_u\xi\zeta - \phi_u\phi}{1-\xi^2} + \frac{\phi_v\xi\phi - \phi_v\zeta}{1-\xi^2} + \frac{\xi_t\zeta - \xi_t\xi\phi}{1-\xi^2} + \frac{\xi_v\zeta^2 - 2\xi_v\zeta\xi\phi + \xi_v\xi^2\phi^2}{1-\xi^2} = 0. \tag{2.9}$$

Now, we will restrict ourselves to finding nonclassical potential symmetries (1.7) for Examples 2.1 and 2.2, for which $\xi = A(t,x)$, $\phi = B(t,x)u + C(t,x)v$, $\zeta = D(t,x)u + E(t,x)v$.

After substitutions and then separating by the coefficients of u and v in both Equations (2.8) and (2.9), we obtain the following two pairs of nonlinear equations:

$$u\colon \quad D_t + \frac{DB}{1-A^2} - \frac{AD^2}{1-A^2} + \frac{DE}{1-A^2} - \frac{ABE}{1-A^2} + \frac{AA_tD}{1-A^2} - \frac{A_tB}{1-A^2} - B_x$$
$$+ \frac{AB^2}{1-A^2} - \frac{BD}{1-A^2} + \frac{ACD}{1-A^2} - \frac{BC}{1-A^2} + \frac{A_xD}{1-A^2} - \frac{ABA_x}{1-A^2} = 0, \tag{2.10}$$

$$v\colon \quad E_t + \frac{DC}{1-A^2} - \frac{ADE}{1-A^2} + \frac{E^2}{1-A^2} - \frac{ACE}{1-A^2} + \frac{AA_tE}{1-A^2} - \frac{A_tC}{1-A^2} - C_x$$
$$+ \frac{ABC}{1-A^2} - \frac{BE}{1-A^2} + \frac{ACE}{1-A^2} - \frac{C^2}{1-A^2} + \frac{A_xE}{1-A^2} - \frac{AA_xC}{1-A^2} = 0 \tag{2.11}$$

and

$$u\colon \quad D_x + \frac{D^2}{1-A^2} - \frac{ABD}{1-A^2} + \frac{BE}{1-A^2} - \frac{ADE}{1-A^2} + \frac{AA_xD}{1-A^2} - \frac{A_xB}{1-A^2} - B_t$$
$$+ \frac{ABD}{1-A^2} - \frac{B^2}{1-A^2} + \frac{ABC}{1-A^2} - \frac{CD}{1-A^2} + \frac{A_tD}{1-A^2} - \frac{AA_tB}{1-A^2} = 0, \tag{2.12}$$

$$v\colon \quad E_x + \frac{DE}{1-A^2} - \frac{ACD}{1-A^2} + \frac{CE}{1-A^2} - \frac{AE^2}{1-A^2} + \frac{AA_xE}{1-A^2} - \frac{A_xC}{1-A^2} - C_t$$
$$+ \frac{ABE}{1-A^2} - \frac{BC}{1-A^2} + \frac{AC^2}{1-A^2} - \frac{CE}{1-A^2} + \frac{A_tE}{1-A^2} - \frac{AA_tC}{1-A^2} = 0. \tag{2.13}$$

Here, it is clear that obtaining precise forms of the functions A, B, C, D and E will be tedious. Hence, we choose some specific forms to demonstrate some of the nontrivial nonclassical potential symmetries that arise for which we later give some invariant solutions.

If $A = D = E = 0$ (the nonclassical potential symmetries are then of the form $(\partial/\partial t) + (Bu + Cv)(\partial/\partial u)$), then Equations (2.10), (2.11), (2.12) and (2.13) yield

$$B_x + BC = 0, \tag{2.14}$$

$$C_x + C^2 = 0, \tag{2.15}$$

$$B_t + B^2 = 0, \tag{2.16}$$

and

$$C_t + BC = 0. \tag{2.17}$$

Solving Equations (2.14–2.17), we obtain

$$B = \frac{c_1}{x + c_1 t + c_2}, \quad C = \frac{1}{x + c_1 t + c_2}. \tag{2.18}$$

Hence the nonclassical potential symmetries are

$$X_1 = \frac{\partial}{\partial t} + \left(\frac{u + v}{t + x}\right)\frac{\partial}{\partial u}, \tag{2.19}$$

$$X_2 = \frac{\partial}{\partial t} + \left(\frac{v}{x + 1}\right)\frac{\partial}{\partial u}. \tag{2.20}$$

If $A = B = E = 0$ and $C = D = 1$, we get

$$X_3 = \frac{\partial}{\partial t} + v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}. \tag{2.21}$$

If $A = 0$, $B = C = D = E = 1$, then the nonclassical potential symmetry is

$$X_4 = \frac{\partial}{\partial t} + (u + v)\frac{\partial}{\partial u} + (u + v)\frac{\partial}{\partial v}. \tag{2.22}$$

If $A = 1/2$, $B = D = E = 0$ and $C = 3/(4x - 2t + c_3)$, then the nonclassical potential symmetry is

$$X_5 = \frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x} + \left(\frac{3v}{4x - 2t + c_3}\right)\frac{\partial}{\partial u}. \tag{2.23}$$

Case 2: $\tau = 0$, $\xi = 1$

Then from (2.4) we obtain

$$u_x = \phi, \quad u_t = v_x = \zeta. \tag{2.24}$$

Substituting Equation (2.24) in (2.5) and (2.6) yields

$$\zeta_t + \zeta_u\zeta + \zeta_v\phi - \phi_x - \phi_u\phi - \phi_v\zeta = 0 \tag{2.25}$$

and

$$\zeta_x + \zeta_u\phi + \zeta_v\zeta - \phi_t - \phi_u\zeta - \phi_v\phi = 0. \tag{2.26}$$

After substitutions and then separating by the coefficients of u and v yield from Equations (2.25) and (2.26), the following nonlinear equations

$$u\colon\quad D_t + D^2 + BE - B_x - B^2 - CD = 0, \tag{2.27}$$

$$v\colon\quad E_t + DE + CE - C_x - BC - CE = 0, \tag{2.28}$$

$$u\colon\quad D_x + BD + DE - B_t - BD - BC = 0, \tag{2.29}$$

$$v\colon\quad E_x + CD + E^2 - C_t - BE - C^2 = 0. \tag{2.30}$$

If $D = E = 0$ and $B \neq 0$, then Equations (2.27), (2.28), (2.29) and (2.30) yield

$$B_x + B^2 = 0, \tag{2.31}$$

$$C_x + BC = 0, \tag{2.32}$$

$$B_t + BC = 0, \tag{2.33}$$

and

$$C_t + C^2 = 0. \tag{2.34}$$

Solving Equations (2.31–2.34), we obtain

$$B = \frac{c_4}{t + c_4x + c_5}, \quad C = \frac{1}{t + c_4x + c_5}.$$

Hence the nonclassical potential symmetries are

$$\begin{aligned} X_6 &= \frac{\partial}{\partial x} + \left(\frac{u+v}{t+x}\right)\frac{\partial}{\partial u}, \\ X_7 &= \frac{\partial}{\partial x} + \left(\frac{v}{t+1}\right)\frac{\partial}{\partial u}. \end{aligned}$$

If $B = C = D = E = 1$, then the nonclassical potential symmetry is

$$X_8 = \frac{\partial}{\partial x} + (u+v)\frac{\partial}{\partial u} + (u+v)\frac{\partial}{\partial v}.$$

If $B = E = 0$ and $C = D = 1$, then the nonclassical potential symmetry is

$$X_9 = \frac{\partial}{\partial x} + v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}.$$

Remark. We show that the nonclassical potential symmetries obtained above for the wave equation (2.1) give rise to group-invariant solutions of the wave equation.

(1) Firstly, we consider the symmetry generator X_3. The invariants of the generator X_3 are

$$\begin{aligned} y &= x, \\ z &= v^2 - u^2, \\ w &= -t + \log(v+u) - \frac{1}{2}\log(v^2 - u^2). \end{aligned} \tag{2.35}$$

Differentiating Equation (2.35b) w.r.t. t, we obtain $v = (uu_t)/u_x$. Now differentiating Equation (2.35c) w.r.t. t and then substituting $v = (uu_t)/u_x$ into the equation and solving for u yields $u = \alpha(t)\exp x$. Since, $v_t = u_x$, $v_x = u_t$, it follows that $\alpha''(t) - \alpha(t) = 0$. Hence we obtain

$$u(t,x) = (A\exp(x+t) + B\exp(x-t)),$$

where $A,\ B$ are constants.

(2) The symmetry generator X_4 has invariants

$$y = x, \quad z = v - u \quad \text{and} \quad w = -2t + \log(v+u).$$

Similar calculations yield

$$u(t,x) = \frac{1}{2}\exp(2t + 2x + k) - \frac{l}{2},$$

where $k,\ l$ are constants.

(3) The symmetry generators X_5 and X_6 give group-invariant solutions $u(t,x) = kx - l$, and $u(t,x) = k(4t - 2x) + l$, respectively, where l, k are constants.

EXAMPLE 2.2. Now we consider the wave equation which arises in the study of small transverse vibrations of a string with variable density, with wave speed $c(x) = x$ in an inhomogeneous medium, that is,

$$u_{tt} - x^2 u_{xx} = 0. \tag{2.36}$$

Writing in a conserved form the PDE (2.36) becomes,

$$D_t\left(\frac{1}{x^2}u_t\right) - D_x(u_x) = 0. \tag{2.37}$$

Then the system of first-order differential equations obtained from Equation (2.37) is

$$v_t = u_x, \quad v_x = \frac{1}{x^2}u_t, \tag{2.38}$$

Equation (2.38) admits the following four symmetries

$$\begin{aligned} \widetilde{X}_1 &= \frac{\partial}{\partial t}, \quad \widetilde{X}_2 = x\frac{\partial}{\partial x} - v\frac{\partial}{\partial v}, \\ \widetilde{X}_3 &= 2tx\frac{\partial}{\partial x} + 2\log x\frac{\partial}{\partial t} + [tu - xv]\frac{\partial}{\partial u} - [tv + (x)^{-1}u]\frac{\partial}{\partial v}, \\ \widetilde{X}_4 &= u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \end{aligned}$$

of which $\widetilde{X_3}$ is the only potential symmetry of (2.36) [1].

Following the procedure applied to Example 2.1, we investigate the existence of nonclassical potential symmetries of (2.36). A summary of results is provided below.

Case 1: $\tau = 1$
If $A = D = E = 0$, then we determine

$$B = \frac{c_1 x}{c_1 t x + c_2 x - 1}, \quad C = \frac{x}{c_1 t x + c_2 x - 1}.$$

Hence the nonclassical potential symmetries are

$$X_5 = \frac{\partial}{\partial t} + \left(\frac{xu + xv}{tx - 1}\right)\frac{\partial}{\partial u},$$

$$X_6 = \frac{\partial}{\partial t} + \left(\frac{xv}{x - 1}\right)\frac{\partial}{\partial u}.$$

If $A = B = D = E = 0$ and $c = -x$, then the nonclassical potential symmetry is

$$X_7 = \frac{\partial}{\partial t} - xv\frac{\partial}{\partial u}.$$

If $A = B = E = 0$, $c = x$ and $D = \frac{2}{x}$, then the nonclassical potential symmetry is

$$X_8 = \frac{\partial}{\partial t} + xv\frac{\partial}{\partial u} + \frac{2u}{x}\frac{\partial}{\partial v}.$$

Case 2: $\tau = 0, \ \xi = 1$
If $D = E = 0$, then we determine

$$B = \frac{c_3}{t + c_3 x + c_4}, \quad C = \frac{1}{t + c_3 x + c_4}.$$

Hence the nonclassical potential symmetries are

$$X_9 = \frac{\partial}{\partial x} + \left(\frac{u + v}{t + x}\right)\frac{\partial}{\partial u},$$

$$X_{10} = \frac{\partial}{\partial x} + \left(\frac{v}{t + 1}\right)\frac{\partial}{\partial u}.$$

If $B = E = 0$, $C = 1$ and $D = 1/x^2$ then

$$X_{11} = \frac{\partial}{\partial x} + v\frac{\partial}{\partial u} + \frac{u}{x^2}\frac{\partial}{\partial v}.$$

Remark. The group-invariant solutions corresponding to the nonclassical symmetries obtained above for the wave equation (2.36) are presented below.

(1) The invariants of the symmetry generator X_8 are

$$y = x, \quad z = \frac{1}{2}x^2v^2 - u^2 \quad \textit{and} \quad w = -\sqrt{2}t + \log(xv + \sqrt{2}u) - \frac{1}{2}\log(x^2v^2 - 2u^2).$$

Similar calculations as we have done in Example 2.1 yield

$$u(t,x) = (A\exp(\sqrt{2}t) + B\exp(-\sqrt{2}t))x^2,$$

where $A,\ B$ are constants.

(2) The symmetry generator X_{11} gives the group-invariant solution

$$u(t,x) = \left(C_1 x^{(1+\sqrt{5})/2} + C_2 x^{(1-\sqrt{5})/2}\right)\exp t,$$

where $C_1,\ C_2$ are constants.

(3) The symmetry generators $X_5,\ X_6$ and X_7 all give the group-invariant solution $u(t,x) = kt - l$, where $k,\ l$ are constants.

EXAMPLE 2.3. Now we consider the Korteweg–deVries equation

$$u_t - u_{xxx} - uu_x = 0. \tag{2.39}$$

Writing in a conserved form the PDE (2.39) becomes,

$$D_t(u) - D_x(u_{xx} + \frac{1}{2}u^2) = 0. \tag{2.40}$$

Then the system of first-order differential equations

$$v_t = u_{xx} + \frac{1}{2}u^2, \quad v_x = u, \tag{2.41}$$

obtained from Equation (2.40), from which no potential symmetries for (2.39) are obtained (see [10]).

The calculations are straight forward and we obtain

Case 1: $\tau = 1$

$$\begin{aligned}
\xi &= A(t,x,v)u + B(t,x,v),\\
\phi &= A_x u^2 + \frac{1}{2}B_v u^2 + C(t,x,v)u + D(t,x,v),\\
\zeta &= \frac{1}{2}Au^2 + E(t,x,v).
\end{aligned}$$

There arises two subcases; $A = 0$ or $A = -1$.

Subcase $A = 0$: We obtain for the above case, $B_v = 0$, $C_v = 0$ and $D_v = 0$. Thus, ξ and ϕ are independent of v. Hence there are no nonclassical potential symmetries for Equation (2.39).

Subcase $A = -1$: For this case we obtain only one symmetry, that is,

$$X = \frac{\partial}{\partial t} - \frac{\partial}{\partial x},$$

which is not a potential symmetry.

Case 2: $\tau = 0,\ \xi = 1$
For this case by separation by powers of u_t leads to contradiction and hence, there are no nonclassical potential symmetries.

Remarks. One can consider other e.g.s such as Burgers' equation whose nonclassical symmetries with corresponding invariant solutions are presented in [8]. Also, it is known (see [1]) that Burgers' equation has potential symmetries. Thus, it would not be surprising that Burgers' equation has nonclassical potential symmetries in which case one can also investigate a similar situation for the heat equation by transforming Burgers' equation to the heat equation via the Hopf–Cole transformation (see [3]).

3. Conclusion

We have shown that equations may display interesting properties, when studied through their nonclassical potential symmetries. The combination of potential symmetries and nonclassical symmetries has not, previously, been considered and we conclude that additional information about equations may be obtained by studying the nonclassical potential symmetries. We demonstrated this for equations that are considered differently in the literature. It should be noted that the specific cases of $\tau = 1$ and arbitrary ξ (Case 1) or $\tau = 0$ and $\xi = 1$ (Case 2) is followed in response to the justification of these choices given in [8]. That is, exact 'invariant' solutions for (2.39) are not obtainable as was demonstrated in the above e.g.s. As in the case of potential symmetries, different conservation laws for a given system may produce different sets of these symmetries.

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