# ABSOLUTE EQUIVALENCE OF EXTERIOR DIFFERENTIAL SYSTEMS

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This paper concerns an equivalence relation first defined by E. Cartan for certain systems of ordinary differential equations [2]. He called two systems absolutely equivalent if they had isomorphic prolongations. A similar concept was used in Cartan's theory of infinite groups [1]. We extend Cartan's definition to general exterior differential systems. For ordinary equations one has only normal prolongations, but in general it is necessary to define and study partial prolongations. This is done and absolute equivalence is defined in §1. In §2 an integer is found which is an absolute invariant and which may be calculated from any given involutive system. In §3 other invariants are found which are calculated from the sequence of normal prolongations of a system. Examples are given. All manifolds, functions and forms are complex analytic. We use the notations and definitions of [4]. We also deal only with systems which are *involutive* at each of their points.

## 1. Absolute equivalence

Let $D_p$ and $M$ be neighborhoods of 0 in $R^p = \{(x_1, \cdots, x_p)\}$ and $R^m = \{(y_1, \cdots, y_m)\}$, respectively. Let $D = D_p \times M$ Then $J^k(D_p, M)$ denotes the manifold of $k$-jets of maps on $D_p$ into $M$. The usual source and target projections are $\alpha$ and $\beta$, and

$$\rho = \alpha \times \beta : J^k(D_p, M) \rightarrow D.$$

If $f : D_p \rightarrow M$, let $\tilde{f} : D_p \rightarrow D$ be defined by $\tilde{f}(x) = (x, f(x))$. Let $\Omega$ be the module of 1-forms generated by $dx_1, \cdots, dx_p$ on $D_p$. We shall consistently use the same notation $\Omega$ for $\alpha^*\Omega$ on $D$ or $J^k(D_p, M)$. Let $(\Sigma, \Omega)$ be an exterior differential system on $D$ having independent variables $\Omega$. Denote by $(P^k\Sigma, \Omega)$ the $k^{\text{th}}$ prolongation of $(\Sigma, \Omega)$ on $J^k(D_p, M)$.

If $D' = D_p \times M'$ is a submanifold of $D$ by an imbedding $F$ such that $\alpha F = \alpha$, and if $F(D')$ contains the manifold of integral points of $(\Sigma, \Omega)$, the restriction of $(\Sigma, \Omega)$ to $D'$ is called an *admissible restriction* of $(\Sigma, \Omega)$.

If $F : D \rightarrow D$ and $f : D_p \rightarrow D_p$ are bi-analytic functions such that $\alpha F = f\alpha$, then the transformed system $(F^*\Sigma, f^*\Omega)$ is said to be a *transform* of $(\Sigma, \Omega)$.

DEFINITION 1. A system $(\Sigma_1, \Omega)$ on $D_1 = D_p \times M_1$ is a *partial prolongation* of $(\Sigma, \Omega)$ on $D$ if there exist maps

$$a : D_1 \rightarrow D \quad \text{and} \quad b : J^1(D_p, M) \rightarrow D_1$$

which satisfy:

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(1) $ab = \rho$ on integral points;

(2) $\alpha a = \alpha$, $\alpha b = \alpha$;

(3) $a^*(\Sigma) \subset \Sigma_1$, $b^*(\Sigma_1) \subset P\Sigma$;

(4) if $j_x^1(f_1)$ is an integral 1-jet of $(\Sigma_1, \Omega)$, then by (3), $j_x^1(\beta a(\tilde{f}_1))$ is an integral 1-jet of $(\Sigma, \Omega)$, and hence is an integral point of $(P\Sigma, \Omega)$. It is required that $bj_x^1(\beta a\tilde{f}_1) = \tilde{f}_1(x) = (x, f_1(x))$.

Proposition 1. *If* $(\Sigma_1, \Omega)$ *is a partial prolongation of* $(\Sigma, \Omega)$, *then* $(P\Sigma, \Omega)$ *is a partial prolongation of* $(\Sigma_1, \Omega)$.

*Proof.* Given $j_x^1(f_1)$ in $J^1(D_p, M_1)$ define

$$a'(j_x^1(f_1)) = j_x^1(\beta\alpha\tilde{f}_1).$$

We shall prove that (1), (2), (3), (4) of Definition 1 hold for $a'$ and $b$.

Condition (1) holds for $a'$ and $b$ because of (4). Condition (2) follows from the definitions of $a'$, $\alpha$, $\beta$. Since $a'$ maps integral points of $(P\Sigma_1, \Omega)$ into integral points of $(P\Sigma, \Omega)$,

$$a'^*(P\Sigma^{[0]}) \subset P\Sigma^{[0]}.$$

One may check that the generating 1-forms in $P\Sigma^{[1]}$ are carried by $a'^*$ into 1-forms in $P\Sigma_1^{[1]}$ by local coordinates.

Condition (4) is satisfied: if $j_x^1(\varphi)$ is an integral 1-jet on $J^1(D_p, M)$, $j_x^1(\beta b\tilde{\varphi})$ is an integral 1-jet of $(\Sigma_1, \Omega)$ by (3) in Definition 1. Thus, $j_x^1(\beta b\tilde{\varphi})$ in $J^1(D_p, M)$ is integral; hence

$$a'j_x^1(\beta b\tilde{\varphi}) = j_x^1[\beta a(\beta b\tilde{\varphi})^{\sim}] = j_x^1[\beta a(b\tilde{\varphi})] = j_x^1(\beta\rho\tilde{\varphi}) = j_x^1(\beta\tilde{\varphi}) = j_x^1(\varphi), \quad \text{Q.E.D.}$$

Proposition 2. *If* $(\Sigma_1, \Omega)$ *is a partial prolongation of* $(\Sigma, \Omega)$, *then there is a natural one-to-one correspondence between their solutions.*

*Proof.* If $f$ is a solution of $(\Sigma, \Omega)$, $\beta b(j^1(f)^{\sim})$ is a solution of $(\Sigma_1, \Omega)$ by (3) in Definition 1. If $f_1$ is a solution of $(\Sigma_1, \Omega)$, then $\beta a\tilde{f}_1$ is a solution of $(\Sigma, \Omega)$. Further,

$$\beta a[\beta b(j^1(f)^{\sim}]^{\sim} = \beta ab[j^1(f)^{\sim}] = \beta\rho(j^1(f)^{\sim}] = f,$$

while by (4) in Definition 1, $\beta b[j^1(\beta a\tilde{f}_1)^{\sim}] = f_1$, Q.E.D.

Definition 2. One exterior differential system is *absolutely equivalent* to a second if there exists a finite sequence of systems beginning with the first and ending with the second such that for each adjacent pair in the sequence, one is an admissible restriction, a transform or a partial prolongation of its neighbor.

*Remark.* The involutiveness assumption is not as restrictive as it may seem, for many non-involutive systems can be prolonged to be involutive by Kuranishi's Prolongation Theorem. If one system is a partial prolongation of a second they can be simultaneously prolonged to equivalent systems of which one is a prolongation of the second.

## 2. An absolute invariant of characters

We now assume that $x_1, \cdots, x_p, y_1, \cdots, y_m$ and $x_1, \cdots, x_p, y_1', \cdots, y_n'$ form regular coordinate systems for $(\Sigma, \Omega)$ and $(\Sigma_1, \Omega)$ at 0, respectively [4, Def. III. 4], and let $a(0) = 0$. Suppose $(\Sigma_1, \Omega)$ is a partial prolongation of $(\Sigma, \Omega)$ with respect to the maps $a$ and $b$ as in Definition 1. Let $H_p^m$ denote the set of all sequences of $m$ formal power series in $x_1, \cdots, x_p$. We may define a germ of infinite analytic mappings $\pi : H_p^m \to H_p^{m+pm}$ by

$$\pi(\xi^1, \cdots, \xi^m) = (\xi^1, \cdots, \xi^m, \partial\xi^1/\partial x^1, \cdots, \partial\xi^m/\partial x)$$

[4, Def. I. 10, 11]. If $\xi^1, \cdots, \xi^m$ define a solution of $(\Sigma, \Omega)$, $\pi(\xi^1, \cdots, \xi^m)$ will define a solution of $(P\Sigma, \Omega)$. The map

$$b : J^1(D_p, M) \to D_1$$

defines a germ of infinite analytic mappings $B : H_p^{m+pm} \to H_p^{m'}$. Let $\Phi$ be the composition $B \circ \pi$. Define $\Phi_1 : H_p^{m'} \to H_p^m$ by means of the map $a$. Then we have

PROPOSITION 3. *If $\xi \in H_p^m$ is a solution of $(\Sigma, \Omega)$, then $\Phi(\xi)$ is a solution of $(\Sigma_1, \Omega)$. If $\xi_1$ is a solution of $(\Sigma_1, \Omega)$ then $\Phi_1(\xi_1)$ is a solution of $(\Sigma, \Omega)$. Further, $\Phi\Phi_1$ and $\Phi_1\Phi$ are identity mappings on all solutions.*

Let now $s_0, s_1, \cdots, s_{p-1}, s_p = m - s_0 - \cdots - s_{p-1}$ and $s_0', s_1', \cdots, s_p'$ be the characters of $(\Sigma, \Omega)$ and $(\Sigma_1, \Omega)$, respectively [3, Def. I. 7]. Let

$$H(S) = H_0^{s_0} + H_1^{s_1} + \cdots + H_p^{s_p},$$

and similarly for $H(S')$ using $s_0', \cdots, s_p'$. Then there exist convergent formal infinite analytic maps $F$ of $H(S)$ into $H_p^m$ and $F_1$ of $H_p^m$ into $H(S)$ such that $(F, F_1)$ form a solution mapping of $(\Sigma, \Omega)$ at $X^0$, the 0-function [4, Theorem III. 2]. Then if $\Phi$ and $\Phi_1$ denote the convergent formal infinite analytic mappings arising from the germs defined in Proposition 3, we have that $(\Phi F, F_1\Phi_1)$ constitute a solution mapping for $(\Sigma_1, \Omega)$. But then $H(S)$ and $H(S')$ are homeomorphic [4, Remark following Theorem III. 2]. Hence they have equal dimensions $(p, \sigma)$ where $\sigma$ is the last non-zero integer in the sequence $s_0, \cdots, s_p$ or $s_0', \cdots, s_p'$. This same relation holds for admissible restrictions; hence we have

THEOREM 1. *Let $(\Sigma, \Omega)$ and $(\Sigma_1, \Omega)$ be absolutely equivalent systems having characters $s_0, s_1, \cdots, s_p$ and $s_0', s_1', \cdots, s_p'$ respectively. Let $\sigma$ be the last non-zero integer in the sequence $s_0, s_1, \cdots, s_p$, and let $\sigma'$ be the similar member in $s_0', \cdots, s_p'$. Then $\sigma = \sigma'$.*

## 3. Other invariants

Let $M^k$ be the manifold of integral points in $J^k(D_p, M)$, so that

$$\dim M^k \le \dim J^k(D_p, M) = p + m\binom{p+k}{p-1} \le p + m(k+1)^{p-1}.$$

The restriction $(\Sigma_k, \Omega)$ of $(P^k\Sigma, \Omega)$ to $M^k$ is equivalent to $(P^k\Sigma, \Omega)$.

If $\Sigma_k^{[r]}$ denotes the module of $r$-forms in $\Sigma_k$ and $w$ is any 1-form of $\Omega$ let

$$L_k^r(w) = \{\theta \in \Sigma_k^{[r]} \mid \theta \wedge w = 0\}.$$

If $l_k^r(w)$ is the minimal dimension of $L_k^r(w)$ at any point of $M^k$, then $l_k^r(w) \leq \binom{v}{r}$ where $v = p + m(k+2)^{p-1}$. Hence $\lim_{k\to\infty}(1/k^{rp})l_k^r(w) = 0$.

The sequence $\{l_k^r(w) \mid k = 1, 2, \cdots\}$ is said to be of *characteristic* $(u^r(w), v^r(w))$ if $u^r(w)$ is the smallest non-negative integer for which

$$\lim_{k\to\infty} l_k^r(w)/k^{u^r(w)}$$

exists, and $v^r(w)$ is the limit.

DEFINITION 3. The integers

$$\begin{aligned} U^r &= \sup\{u^r(w) \mid w \in \Omega\}, \\ u^r &= \inf\{u^r(w) \mid w \in \Omega\}, \\ V^r &= \sup\{v^r(w) \mid w \in \Omega\}, \\ v^r &= \inf\{v^r(w) \mid w \in \Omega\}, \qquad r = 1, 2, \cdots, \end{aligned}$$

are the *characteristics* of $(\Sigma, \Omega)$.

LEMMA. *If $(\Sigma_1, \Omega)$ is a partial prolongation of $(\Sigma, \Omega)$ and if $\Sigma_{10}$ and $\Sigma_0$ denote their respective restrictions to the manifolds of integral points, then*

$$\dim\{\theta \in \Sigma_0^{[r]} \mid \theta \wedge w = 0\} \leq \dim\{\theta \in \Sigma_{10}^{[r]} \mid \theta \wedge w = 0\}.$$

*Proof.* By involutiveness, each integral point of $(\Sigma, \Omega)$ is contained in a solution $f$. Then $bj^1(f)$ is a solution of $(\Sigma_1, \Omega)$ and $abj^1(f) = \tilde{f}$. Hence $a$ maps the integral points of $(\Sigma_1, \Omega)$ onto those of $(\Sigma, \Omega)$. Now $\rho^*$ is one-to-one and $\rho = ab$ on the integral points. Hence $a^*$ is one-to-one on $\Sigma_0$ and $a^*(\Sigma_0^{[r]}) \subset \Sigma_{10}^{[r]}$, and the conclusion follows.

THEOREM 2. *Two absolutely equivalent systems have equal characteristics.*

*Proof.* For admissible restrictions or transforms, the theorem is not difficult to prove. If $(\Sigma_1, \Omega)$ on $D_1$ is a partial prolongation of $(\Sigma, \Omega)$ on $D$ then $(P^{k+1}\Sigma, \Omega)$ is a partial prolongation of $(P^k\Sigma_1, \Omega)$, which is a partial prolongation of $(P^k\Sigma, \Omega)$. Denoting the numbers $l_k^r(w)$, $u^r(w)$, $v^r(w)$ for the system $(\Sigma_1, \Omega)$ by $l_{1k}^r(w)$, $u_1^r(w)$, $v_1^r(w)$, it follows from the previous lemma that

$$l_k^r(w) \leq l_{1k}^r(w) \leq l_{k+1}^r(w)$$

for every $k$, $r$ and $w$. Hence

$$\lim_{k\to\infty} l_k^r(w)/k^u = \lim_{k\to\infty} l_{1k}^r(w)/k^u, \qquad \text{Q.E.D.}$$

*Example* 1. Let $\Sigma$ be generated by

$$\begin{aligned} \theta_1 &= dy_1 + y_2\,dx_1, & d\theta_1 &= dy_2 \wedge dx_1 \\ \theta_2 &= dy_3 + dx_2 + y_4\,dx_1, & d\theta_2 &= dy_4 \wedge dx_1 \end{aligned}$$

Here $s_0 = s_1 = 2$, so $\sigma = 2$. $\Sigma_k$ is generated by $\theta_1$, $\theta_2$, together with

$$\begin{aligned}
\theta_1' &= dy_2 + y_1'\,dx_1, & d\theta_1' &= dy_1' \wedge dx_1\\
\theta_2' &= dy_4 + y_2'\,dx_1, & d\theta_2' &= dy_2' \wedge dx_1\\
\theta_1'' &= dy_1' + y_1''\,dx_1, & d\theta_1'' &= dy_1'' \wedge dx_1\\
\theta_2'' &= dy_2' + y_2''\,dx_1, & d\theta_2'' &= dy_2'' \wedge dx_1\\
&\vdots & &\vdots\\
\theta_1^{(k)} &= dy_1^{(k-1)} + y_1^{(k)}\,dx_1, & d\theta_1^{(k)} &= dy_1^{(k)} \wedge dx_1\\
\theta_2^{(k)} &= dy_2^{(k-1)} + y_2^{(k)}\,dx_1, & d\theta_2^{(k)} &= dy_2^{(k)} \wedge dx_1 .
\end{aligned}$$

Here,

$$L_k^{[2]}(dx_2) = \Sigma_k^{[1]} \wedge dx_2 .$$

so

$$l_k^2(dx_2) = 2(k+1).$$

Hence $u^2(dx_2) = 2$, $v^2(dx_2) = 2$, so $u^2 \leq 2$, $v^2 \leq 2$.

*Example* 2. Let $\Sigma$ be generated by

$$\begin{aligned}
\theta_1 &= dy_1 + y_2\,dx_1, & d\theta_1 &= dy_2 \wedge dx_1\\
\theta_2 &= dy_3 + y_4\,dx_2, & d\theta_2 &= dy_4 \wedge dx_2 .
\end{aligned}$$

Again $s_0 = s_1 = \sigma = 2$. $\Sigma_k$ is generated by

$$\begin{aligned}
\theta_1' &= dy_2 + y_1'\,dx_1, & d\theta_1' &= dy_1' \wedge dx_1\\
\theta_2' &= dy_4 + y_2'\,dx_2, & d\theta_2' &= dy_2' \wedge dx_2\\
&\vdots & &\vdots\\
\theta_1^{(k)} &= dy_1^{(k-1)} + y_1^{(k)}\,dx_1, & d\theta_1^{(k)} &= dy_1^{(k)} \wedge dx_1\\
\theta_2^{(k)} &= dy_2^{(k-1)} + y_2^{(k)}\,dx_2, & d\theta_2^{(k)} &= dy_2^{(k)} \wedge dx_2 .
\end{aligned}$$

Now, $L_k^{[2]}(a_1\,dx_1 + a_2\,dx_2) = L_k^{[2]}(w)$ contains $\Sigma_k^{[1]} \wedge w$ and the forms
$a_1\,d\theta_1^{(h)} + a_2\,\theta_1^{(h+1)} \wedge dx_2$, $\quad a_2\,d\theta_2^{(h)} + a_1\,\theta_2^{(h+1)} \wedge dx_1$, $\quad h = 0, 1, \cdots, k-1$.
Hence $l_k^2(w) \geq 2(k+1) + 2k$, and consequently for this system $v^2 \geq 4$. Hence the systems in these examples are not absolutely equivalent. Yet they have the same $\sigma = 2$.

## References


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