ON INFINITE PROLONGATIONS OF DIFFERENTIAL SYSTEMS[1]

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From any exterior differential system with independent variables, it will be shown how to construct, on an infinite dimensional space, an equivalent completely integrable differential system. Local solutions are discussed in the real or complex analytic case.

The calculus of exterior differential systems will be used freely [**1**; **2**]. Prolongations are clearly described in [**3**]. All functions and forms are assumed infinitely differentiable unless otherwise restricted.

1. **Infinite prolongations.** Let E^{n+m} be $n+m$-dimensional real/complex Euclidean space. Let Σ be any real/complex infinitely differentiable closed exterior differential system with dependent variables $z^1, \cdots, z^m$ and independent variables $x^1, \cdots, x^n$, defined on an open subset D of E^{n+m}.

The first prolongation, Σ', of Σ, can be accomplished as follows. Everywhere in Σ, replace dz^λ by $\sum_{j=1}^n z_j^\lambda dx^j$, $\lambda = 1, \cdots, m$, where z_j^λ are nm new variables. Using the exterior calculus, the coefficients of the independent terms in the resulting forms, together with the 0-forms of Σ, and the 1-forms $dz^\lambda - \sum_{j=1}^n z_j^\lambda dx^j$, together generate Σ'. (λ will always run from 1 to m, Latin letters from 1 to n; and the summation convention will be used.)

Thus, Σ' consists of

(1) a collection of 0-forms or functions, $\{\Phi_1^\alpha(x^i, z^\lambda, z_i^\lambda); \alpha = 1, \cdots, \alpha_1\}$ and their exterior derivatives, $\{d\Phi_1^\alpha; \alpha = 1, \cdots, \alpha_1\}$,

(2) the 1-forms $dz^\lambda - z_j^\lambda dx^j$ and their derivatives, $dz_j^\lambda \wedge dx^j$.

The second prolongation, Σ'', is obtained from Σ' by replacing dz_j^λ with $z_{jk}^\lambda dx^k$, where the $z_{jk}^\lambda = z_{kj}^\lambda$ are $mn(n-1)/2$ new variables. Σ'' consists of

(1) the 0-forms of Σ', $\{\Phi_1^\alpha; \alpha = 1, \cdots, \alpha_1\}$, and new functions $\{\Phi_2^\alpha; \alpha = 1, \cdots, \alpha_2\}$, together with the derivatives

$$\{d\Phi_2^\alpha; \alpha = 1, \cdots, \alpha_2\},$$

(2) $dz^\lambda - z_j^\lambda dx^j$, $dz_j^\lambda - z_{jk}^\lambda dx^k$, $dz_{jk}^\lambda \wedge dx^k$. It is important that $d\Phi_1^\alpha \equiv 0$ modulo Σ'', $\alpha = 1, \cdots, \alpha_1$, and $d(dz^\lambda - z_j^\lambda dx^j) \equiv 0$ modulo Σ''. The former is a consequence of the prolongation process, the latter of the symmetry $z_{jk}^\lambda = z_{kj}^\lambda$.

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$\Sigma^{(r)}$, the rth prolongation, will be a closed system in x^i, $z^\lambda, \cdots,$ $z^\lambda_{i_1 \cdots i_q}$, where $q=1, 2, \cdots, r$, and each $z^\lambda_{i_1 \cdots i_q}$ is symmetric in the lower indices. $\Sigma^{(r)}$ consists of

(1) 0-forms $\{\Phi_1^\alpha;\ \alpha=1, \cdots, \alpha_1\}$, $\{\Phi_2^\alpha;\ \alpha=1, \cdots, \alpha_2\}, \cdots,$ $\{\Phi_r^\alpha;\alpha=1, \cdots, \alpha_r\}$, and 1-forms $\{d\Phi_r^\alpha;\alpha=1, \cdots, \alpha_r\}$,

(2) $dz^\lambda - z^\lambda_j dx^j$, $dz^\lambda_{i_1} - z^\lambda_{i_1 j}dx^j, \cdots, dz^\lambda_{i_1 \cdots i_{r-1}} - z^\lambda_{i_1 \cdots i_{r-1} j}dx^j$, $dz^\lambda_{i_1 \cdots i_{r-1} j} \wedge dx^j$.

Continued indefinitely, the result is an infinite sequence of 0- and 1-forms which may be written, grouping the 0-forms together, as a system Σ^* of

(1) 0-forms $\psi^\beta(x^i, z^\lambda, \cdots, z^\lambda_{i_1 \cdots i_q}, \cdots)$, $\beta=1, 2, \cdots$, each depending on an only finite number of variables,

(2) $dz^\lambda - z^\lambda_j dx^j, \cdots, dz^\lambda_{j_1 \cdots j_q} - z^\lambda_{j_1 \cdots j_q j}dx^j, \cdots$. The $z^\lambda_{j_1 \cdots j_q}$ are symmetric in lower indices and may assume any real/complex values, while the $(x^i, z^\lambda) \in D$. Each ψ^β is infinitely differentiable. Most important, Σ^* is *completely integrable*, i.e., for any 0- or 1-form $\omega \in \Sigma^*$, there is a finite subset, Σ^*_ω of Σ^*, such that

$$d\omega \equiv 0 \text{ modulo } \Sigma^*_\omega.$$

A *solution* of Σ^* consists of infinitely differentiable functions $z^\lambda(x^i), \cdots, z^\lambda_{i_1 \cdots i_q}(x^i), \cdots$, defined on a neighborhood N in E^n, so that $(x^i, z^\lambda(x^i)) \in D$ when $(x^i) \in N$, and which causes every form in Σ^* to vanish identically when z^λ, dz^λ, $z^\lambda_{i_1 \cdots i_q}$, and $dz^\lambda_{i_1 \cdots i_q}$ are replaced by $z^\lambda(x^i)$, $dz^\lambda(x^i)$, the differential, $z^\lambda_{i_1 \cdots i_q}(x^i)$, and $dz^\lambda_{i_1 \cdots i_q}(x^i)$, respectively. Vanishing of the forms in (2) in Σ^* is equivalent to

$$\text{(A)} \qquad \frac{\partial^q z^\lambda(x^i)}{\partial x^{j_1} \cdots \partial x^{j_q}} = z^\lambda_{j_1 \cdots j_q}(x^i).$$

There is a one-to-one correspondence between the solutions of Σ and $\Sigma^{(r)}$ [**3**], and it is easy to see that this property extends to Σ^*, the correspondence being given by (A).

2. **Analytic case.** Assume that the forms in Σ are real/complex analytic, i.e., all functions appearing as coefficients have this property. Then all forms in Σ^* are analytic. Let $(j_1, \cdots, j_q)$ be a sequence of q integers, $1 \leqq j_k \leqq n$. If, among these, there are exactly ρ_1 ones, ρ_2 twos, $\cdots$, and ρ_n n's, let $\rho(j_1, \cdots, j_q) = \rho_1!\rho_2! \cdots \rho_n!$.

A sequence $(u^i, v^\lambda, \cdots, v^\lambda_{j_1 \cdots j_q}, \cdots)$ of real/complex numbers, where the $v^\lambda_{j_1 \cdots j_q}$ are symmetric in the lower indices, is called a *zero* of Σ^* if, for all β, $\psi^\beta(u^i, v^\lambda, \cdots, v^\lambda_{j_1 \cdots j_q}, \cdots) = 0$.

THEOREM. *Let* $(u^i, v^\lambda, \cdots, v^\lambda_{j_1 \cdots j_q}, \cdots)$ *be a zero of* Σ^*, *where*

$$\frac{\left| v^{\lambda}_{j_1\cdots j_q} \right| R_1^{\rho_1} \cdots R_n^{\rho_n}}{\rho(j_1, \cdots, j_q)}$$

is a bounded sequence for some positive real numbers $R_1, \cdots, R_n$, *and* $\rho(j_1, \cdots, j_q) = \rho_1! \cdots \rho_q!$. *Then there exists a unique solution,*

$$z^{\lambda}(x^{i}), \cdots, z^{\lambda}_{j_1\cdots j_q}(x^{i}), \cdots$$

of Σ^* *defined in a neighborhood of* (u^i) *and satisfying the initial conditions*

$$z^{\lambda}(u^{i}) = v^{\lambda}, \cdots, z^{\lambda}_{j_1\cdots j_q}(u^{i}) = v^{\lambda}_{j_1\cdots j_q}.$$

PROOF. Let

$$a^{\lambda}_{\rho_1\cdots\rho_n} = \frac{1}{\rho_1! \cdots \rho_n!} v^{\lambda}_{i_1\cdots i_{\rho_1+\cdots\rho_n}},$$

where $(i_1, \cdots, i_{\rho_1+\cdots+\rho_n})$ consists of ρ_1 ones, ρ_2 twos, $\cdots$, and ρ_n n's, if not all these are zero, and $a^{\lambda}_{0\cdots 0} = v^{\lambda}$. Let

$$\text{(B)}\quad \begin{aligned} z^{\lambda}(x^{i}) &= \sum_{\rho_1=0}^{\infty} \cdots \sum_{\rho_n=0}^{\infty} a^{\lambda}_{\rho_1\cdots\rho_n}(x^{1} - u^{1})^{\rho_1} \cdots (x^{n} - u^{n})^{\rho_n} \\ z^{\lambda}_{j_1\cdots j_q}(x^{i}) &= \frac{\partial^q z^{\lambda}(x^i)}{\partial x^{j_1} \cdots \partial x^{j_q}}. \end{aligned}$$

These functions converge for $|x^j - u^j| < R_j$, satisfy the initial conditions, and cause the 1-forms $dz^{\lambda} - z^{\lambda}_j dx^j, \cdots, dz^{\lambda}_{j_1\cdots j_q} - z^{\lambda}_{j_1\cdots j_q j} dx^j, \cdots$ to vanish.

Let $\bar{\psi}^{\beta}(x^i)$ denote ψ^{β} after $z^{\lambda}, \cdots, z^{\lambda}_{j_1\cdots j_q}, \cdots$ are replaced by the corresponding functions (B). By hypothesis, $\bar{\psi}^{\beta}(u^i) = 0$. Since Σ^* is completely integrable,

$$d\psi^{\beta} \equiv 0 \text{ modulo } \Sigma^*$$

hence,

$$\frac{\partial\psi^{\beta}}{\partial x^j} + \frac{\partial\psi^{\beta}}{\partial z^{\lambda}} z^{\lambda}_j + \cdots + \frac{\partial\psi^{\beta}}{\partial z^{\lambda}_{j_1\cdots j_q}} z^{\lambda}_{j_1\cdots j_q j} + \cdots \equiv 0 \text{ modulo } \{\psi^{\beta}\},$$

and the sums on both sides are finite. Hence, there are analytic functions $A^{\beta}_{j\gamma}(x^i)$ such that

$$\frac{\partial\bar{\psi}^{\beta}}{\partial x^j} = A^{\beta}_{j\gamma}\bar{\psi}^{\gamma}.$$

Therefore, $(\partial\Psi^\beta/\partial x^j)(u^i)=0$, all β. In the same way

$$\frac{\partial^2\Psi^\beta}{\partial x^j\partial x^k}=A^\beta_{j\gamma}\frac{\partial\Psi^\gamma}{\partial x^k}+\frac{\partial A^\beta_{j\gamma}}{\partial x^k}\Psi^\gamma,$$

implies $(\partial^2\Psi^\beta/\partial x^j\partial x^k)(u^i)=0$, etc. Thus, the Taylor's expansion of Ψ^β at (u^i) is zero. The initial conditions thus determine uniquely the $z^\lambda(x^i)$, hence the solution. Q.E.D.

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